Chapter 6

Collisions of Charged Particles

The interactions of a moving charged particle with any surrounding matter are governed by the properties of collisions. We will usually call the incident particle the "projectile" and the components of the matter with which it is interacting the "target-particles" or just the "targets". The simplest situation one might imagine is that the matter consisted of free charged particles, electrons and nuclei. This is exactly the situation that applies if the matter with which the particle is interacting is a plasma. It might be thought that in this case, the mutual interaction of the target-particles themselves could be ignored, and the collisions treated as if they were all simple two-body collisions. This is not quite true because of the long-range nature of the electromagnetic force, as we shall see, but it is possible, nevertheless, to treat the collisions as two-body, but correct for the influence of the other target particles in this process.

In interactions with the atoms of solids, liquids or (neutral) gases, the fact that the target electrons are bound to the nuclei of their atom is obviously, in the end, important to the interaction processes. The atoms themselves can usually be treated ignoring the interactions between them, at least for projectiles with substantial kinetic energy. The simplest approximate analysis goes further, and starts from the highly simplified view that the electrons can be treated initially *ignoring* the force binding them to atoms. The corrections to this approach are naturally substantial, and the treatment cannot always yield accurate results. Nevertheless it represents a kind of baseline that more accurate calculations and measurements can be compared with.

The nuclei of the target are important in collisions with plasmas. However, in interactions with neutral atoms, direct electromagnetic interaction with the nucleus requires the projectile to penetrate the shielding of the orbiting electrons in the atom. Only particles with very high momentum can do that. Therefore the electrons of the target are usually the most important to consider, and tend to dominate the energy loss.

The topic of atomic collisions is an immense and complex one, in which quantum mechanics naturally plays a crucial role. It would take us far beyond the present intention to attempt a proper introduction to this topic. Two simplifying factors enable us, nevertheless, to develop this aspect of electromagnetic interactions in enough detail for many practical purposes. The first factor is that the details of atomic structure become far less influential in collisions at energies much higher than the binding energies of atoms (which is about ten electron volts or so). The second is that even when quantum effects *are* important in the collisions, approximate formulas with wide applicability, but ignoring the details of particular atomic species, can be obtained by semi-classical arguments. The quantum corrections are then applied in a way that seems somewhat *ad hoc*, but often represents the way the earliest calculations were done, and gives simple analytic formulas.

6.1 Elastic Collisions

6.1.1 Reference Frames and Collision Angles

Consider an idealized non-relativistic collision of two interacting particles, subscripts 1 and 2, with positions $\mathbf{r}_{1,2}$ and velocities $\mathbf{v}_{1,2}$, which are not acted on by any forces other than their mutual interactions and which experience no changes in internal energy, so the collision is elastic. Their total (combined) momentum, $m_1\mathbf{v}_1 + m_2\mathbf{v}_2$, is constant, so that their center-of-mass,

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \quad , \tag{6.1}$$

moves at a constant velocity, the center-of-mass velocity:

$$\mathbf{V} \equiv \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \quad . \tag{6.2}$$

It is helpful also to introduce the notation

$$m_r \equiv \frac{m_1 m_2}{(m_1 + m_2)} \tag{6.3}$$

for what is called the "reduced mass". In terms of this quantity and the relative position vector $\mathbf{r} \equiv \mathbf{r_1} - \mathbf{r_2}$, the positions of the particles can be written:

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_r}{m_1}\mathbf{r} \qquad \mathbf{r}_2 = \mathbf{R} - \frac{m_r}{m_2}\mathbf{r} \quad , \tag{6.4}$$

and their velocities:

$$\mathbf{v}_1 = \mathbf{V} + \frac{m_r}{m_1}\mathbf{v} \qquad \mathbf{v}_2 = \mathbf{V} - \frac{m_r}{m_2}\mathbf{v} \quad , \tag{6.5}$$

where $\mathbf{v} \equiv \dot{\mathbf{r}}$ is the relative velocity.

Some of our calculations need to be done in the center-of-mass frame of reference, in which $\mathbf{R}$ is stationary. Others need to be done in the lab frame or other frames of reference, for example in which one or other of the particles is initially stationary. The angles of vectors in these frames are important. The directions of all position vectors and of all velocity *differences* are the same in all inertial frames. However the directions of *velocities* are not the same in different frames.

For example, consider a collision illustrated in Fig 6.1. Collisions can be considered in a single plane-of-scattering which is perpendicular to the angular momentum of the system, itself a constant. The angle of scattering, which we denote χ is just the angle between the initial direction of the relative velocity $\mathbf{v}$ and its final direction, $\mathbf{v}'$. This angle is different in different reference frames. Call the angle in the center-of-mass frame χ_c. By conservation of energy, the final relative velocity $\mathbf{v}'$ has absolute magnitude equal to that of the initial relative velocity, v_0. So the final velocity can be written in component form, in the center of mass frame, as

$$\mathbf{v}' = v_0(\cos\chi_c, \sin\chi_c) \quad , \tag{6.6}$$

where we have chosen the initial relative velocity direction for the x-axis.

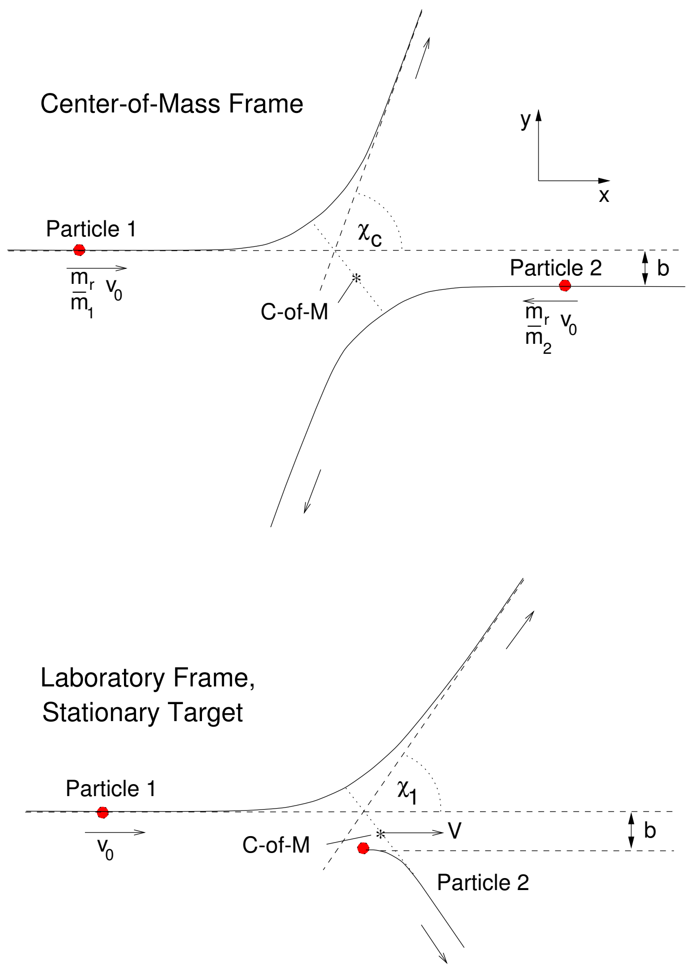

Figure 6.1: Collisions in center-of-mass and laboratory frames.

Substituting into eq(6.5) we find the final velocities in the lab frame to be given by

$$\mathbf{v}_1' = \frac{m_r}{m_1}\mathbf{v}' + \mathbf{V} \quad = \quad \left(\frac{m_r}{m_1}v_0\cos\chi_c + V, \frac{m_r}{m_1}v_0\sin\chi_c\right) \quad , \tag{6.7}$$

$$\mathbf{v}_2' = -\frac{m_r}{m_2}\mathbf{v}' + \mathbf{V} \quad = \quad \left(-\frac{m_r}{m_2}v_0\cos\chi_c + V, -\frac{m_r}{m_2}v_0\sin\chi_c\right) \quad . \tag{6.8}$$

The angle in the lab frame of the final velocity of particle 1 to its initial velocity (which is in the x-direction) , χ_1 say, is then just given by the ratio of the components of the final velocity,

$$\cot\chi_1 = \frac{\frac{m_r}{m_1}v_0\cos\chi_c + V}{\frac{m_r}{m_1}v_0\sin\chi_c} = \cot\chi_c + \frac{V}{v_0}\frac{m_1}{m_r}\csc\chi_c \quad . \tag{6.9}$$

For the specific case when particle 2 is a *stationary target*, with initial lab-frame velocity zero, the center-of-mass velocity is $V = m_1 v_0/(m_1+m_2) = (m_r/m_2)v_0$ and so

$$\cot\chi_1 = \cot\chi_c + \frac{m_1}{m_2}\csc\chi_c \quad . \tag{6.10}$$

We often want to know how much energy or momentum is transferred from an incident projectile, (particle 1) to an initially stationary target (particle 2). Clearly from eq(6.8) we can obtain these quantities in terms of the scattering angle χ_c. So, the change in the x-momentum of particle 1 is simply

$$m_1\left(\frac{m_r}{m_1}v_0\cos\chi_c + V\right) - mv_0 = m_r v_0\left(\cos\chi_c + \frac{m_1}{m_2}\right) \tag{6.11}$$

and the final recoil energy of particle 2 (which is the energy lost by particle 1) is

$$\begin{aligned} Q \equiv \frac{1}{2}m_2\left[\left(-\frac{m_r}{m_2}v_0\cos\chi_c + V\right)^2 + \left(-\frac{m_r}{m_2}v_0\sin\chi_c\right)^2\right] &= \\ \frac{1}{2}m_2\left(\frac{m_r}{m_2}v_0\right)^2\left[(-\cos\chi_c+1)^2 + \sin^2\chi_c\right] &= \\ \frac{1}{2}\frac{m_r^2}{m_2}v_0^2 2(1-\cos\chi_c) &= \frac{1}{2}\frac{m_r^2}{m_2}v_0^2 4\sin^2(\frac{\chi_c}{2}) \quad . \end{aligned} \tag{6.12}$$

Notice that the maximum possible energy transfer, which occurs when $\chi_c = 180°$, is

$$Q_{\max} = \frac{4m_r^2}{m_1 m_2}\tfrac{1}{2}m_1 v_0^2 \quad . \tag{6.13}$$

All of these relations are completely independent of the nature of the interaction between the particles, since we have invoked only conservation of momentum and energy.

Impact Parameter and Cross-section

By definition, the cross-section, σ, for any specified collision process, when a particle is passing though a density n_2 of targets, is that quantity which makes the number of such collisions per unit path length equal to $n_2\sigma$.[1] Sometimes a continuum of types of collision is under consideration. For example we can consider collisions giving rise to different scattering angles (χ) to be distinct. In that case, we speak in terms of **differential cross-sections**, and define the differential cross-section $\frac{d\sigma}{d\chi}$ (for example) as being that quantity such that the number of collisions within an angle element $d\chi$ per unit path length is

$$n_2\frac{d\sigma}{d\chi}d\chi \quad .$$

[1] An alternative definition can be invoked, equivalent to this first definition but in the frame of reference in which the single particle (1) is stationary and the particles of density n_2 are moving.

Sometimes other authors use different notation for the differential cross-section, for example $\sigma(\chi)$. However, our notation, with which we are familiar from calculus, is highly suggestive and the cross-sections obey natural rules for differentials implied by the notation.

For classical collisions, the **impact parameter**, b, shown in Fig 6.1, is a convenient parameter by which to characterize the collision. It is the distance of closest approach that would occur for the colliding particles if they just followed their initial straight-line trajectories. Alternatively, the impact parameter can be considered to be a measure of the angular momentum of the system in the center-of-mass frame, which is $m_r v_0 b$.

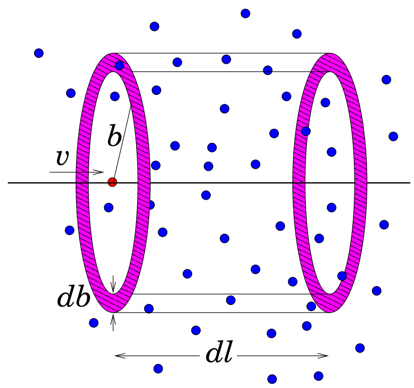

Figure 6.2: Differential volume for counting the number of collisions in length $d\ell$ with impact parameter b.

The differential cross-section with respect to the impact parameter is defined purely by geometry. As illustrated in Fig 6.2, one can think of the projectile (particle 1) as dragging along with itself an anulus of radius b and thickness db as it moves along a distance $d\ell$ of its path length. This anulus drags out a volume $d\ell 2\pi b db$, and the number of targets that are in this volume, and hence have been encountered in the impact parameter element db at b in this path-length is $n_2 d\ell 2\pi b db$. Consequently, from our definition, the differential cross-section for scattering at impact parameter b is

$$\frac{d\sigma}{db} = 2\pi b \quad . \tag{6.14}$$

Notice that the integral of this quantity over all impact parameters (i.e. $0 < b < \infty$) will certainly diverge, because it considers the projectile to be colliding with all the target particles it passes, no matter how far away they are. Therefore the total number of "collisions" of all possible types, per unit length in an infinite target medium is infinite. This mathematical singularity in the "total cross-section" points out the need to define more closely what constitutes a collision, and alerts us to the fact that for collisions governed by interactions of infinite range, such as the forces between charged particles, we shall have to define our collisions in such a way as to account for some effective termination of the impact-parameter integration.[2] This termination, which is often expressed approximately as a cut off of the

[2]It also shows the fundamental incoherence of the notion of the total number of collisions per unit length and concepts that depend on it such as the average change in some parameter *per collision*, which some authors unfortunately employ.

impact parameter integration at a maximum $b_{\max}$, will be governed by consideration of the particle parameter whose change due to collisions we are trying to calculate. For example, the momentum or energy change in the collision may become negligible for $b > b_{\max}$.

There is usually a one-to-one relationship between the impact parameter and the angle of scattering and hence with the energy transfer, Q, given by eq(6.12). Consequently the differential cross-section with respect to energy transfer, scattering angle and impact parameter are all related thus:

$$\frac{d\sigma}{dQ} = \frac{d\sigma}{d\chi_c}\left|\frac{d\chi_c}{dQ}\right| = \frac{d\sigma}{db}\left|\frac{db}{d\chi_c}\right|\left|\frac{d\chi_c}{dQ}\right| \tag{6.15}$$

If we are concerned with a quantity such as the energy of the projectile, which is changing because of collisions, and the change in each collision is an amount $Q(b)$ that depends on the impact parameter, then the total rate of change per unit length due to all possible types of collisions is obtained as

$$n_2 \int Q d\sigma = n_2 \int Q\, 2\pi b db \tag{6.16}$$

6.1.2 Classical Coulomb Collisions

The exact relationship between the impact parameter, b, and the scattering angle is determined by the force field existing between the colliding particles. For electromagnetic interactions of charged particles, the fundamental force is the Coulomb interaction between the forces, an inverse square law. As Isaac Newton showed, the orbit of a particle moving under an inverse square law force is a conic section; that is, an ellipse for closed orbits or a hyperbola for the open orbits relevant to collisions.

Elementary analysis shows that the resulting scattering angle χ_c for a collision with impact parameter b is given by

$$\cot\left(\frac{\chi_c}{2}\right) = \frac{b}{b_{90}} \quad , \tag{6.17}$$

where, for particles of charge q_1 and q_2 and initial collision velocity v_0 the quantity b_{90} is given by

$$b_{90} \equiv \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{m_r v_0^2} \quad . \tag{6.18}$$

Clearly from eq(6.17), b_{90} is the impact parameter at which the scattering angle in the center of mass frame is 90°. Trignometric identities allow us to deduce immediately from eq(6.17) that

$$\sin^2(\chi_c/2) = \frac{1}{1+(b/b_{90})^2} \quad \text{and} \quad \frac{db}{d\chi_c} = -\frac{b_{90}}{2}\csc^2(\chi_c/2) \quad . \tag{6.19}$$

So that the energy transfer in a collision (see eq(6.12)) is

$$Q = \frac{1}{2}\frac{m_r^2}{m_2}v_0^2 4\frac{1}{1+(b/b_{90})^2} \tag{6.20}$$

and the rate of transfer of energy per unit length for a particle of energy $K \equiv \frac{1}{2}m_1 v_0^2$ colliding with stationary targets is

$$-\frac{dK}{d\ell} = n_2 \frac{m_1 v_0^2}{2}\frac{4m_r^2}{m_1 m_2}\int \frac{2\pi b db}{1+(b/b_{90})^2} = n_2 K \frac{4m_r^2}{m_1 m_2}\pi b_{90}^2 \ln[1+(b_{\max}/b_{90})^2] \quad , \tag{6.21}$$

where the upper limit of the b-integration, $b_{\max}$, which prevents the integral diverging, will be discussed in a moment. One way to think of this equation is to regard the quantity $\pi b_{90}^2 \frac{4m_r^2}{m_1 m_2} \ln[1 + (b_{\max}/b_{90})^2]$ as an effective collision cross-section for total energy loss. When multiplied by the density n_2 of targets it gives the inverse scale-length for energy loss, $d \ln K/d\ell$.

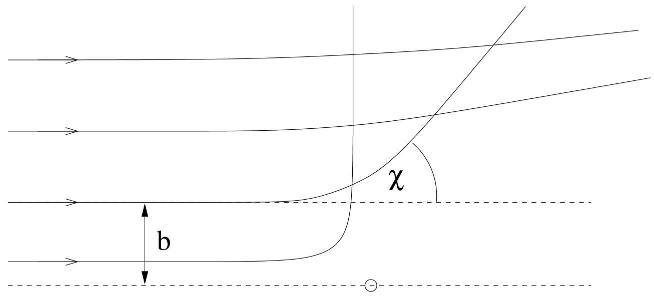

Figure 6.3: Scattering angle and impact parameter shown schematically for different Coulomb collisions.

The integral over impact parameters diverges if we extend it to infinite b. This is because the inverse square law has essentially infinite range. As a result, the dominant contribution to the energy loss cross-section comes from distant collisions, in which $b \gg b_{90}$, and hence the scattering angle is small. Several different physical effects can enter at large impact parameters to change the effective force-law and prevent the divergence. We will treat these effects separately in later sections, but in almost all cases, the exact value of the upper limit is not a very strong quantitative effect on the cross-section because $b_{\max}/b_{90}$ is large and appears inside a logarithmic term that may be written approximately $\ln(b_{\max}/b_{90})$, which therefore varies very slowly with $b_{\max}$. Many treatments adopt a small-angle approximation for the differential cross-section earlier in the derivation, leading to an expression $Q \propto 1/b^2$ and an integral that diverges both at small b and at large b. Such treatments then need to invoke a $b_{\min}$ cut-off of the integration, justifying it on the basis of a breakdown of the approximation, and naturally adopt b_{90} as that cut-off in this classical case. The resulting expression is then essentially identical to ours, which was obtained more rigorously. There are, in some circumstances, important physical effects that require us to cut-off the integration at small b *even before* b_{90} is reached. In those cases we simply replace the term $\ln[1 + (b_{\max}/b_{90})^2]$ with $2\ln(b_{\max}/b_{\min})$.

6.2 Inelastic Collisions

The effects that give rise to the cut-off of the Coulomb logarithm are primarily associated with the presence of other particles and forces in the system. If the target particles experience the force-field of another nearby particle, such as will be the case if the targets are electrons bound to the nuclei to form the atoms of a target material, then the dynamics of their binding gives rise to a cut-off. One way to think of this effect is to regard the electrons as behaving as if they were free only in collisions in which the energy transfer from the projectile is larger than their binding energy in the atom. Distant, small angle, collisions transfer less energy.

A cut-off $b_{\max}$ should be applied at that impact parameter where the energy transfer is equal to approximately the binding energy.

Alternatively, and more physically, one can regard these collisions as being with a composite target system, the atom, in which there is a transfer of energy inelastically to the system, the energy being partially taken up in the ionization or excitation energy of the atom. Clearly, a fully rigorous calculation of such collisions requires the quantum structure of the atom to be considered, and so is intrinsically quantum-mechanical. Nevertheless, semi-classical calculations, taking quantum effects into account in a somewhat ad hoc manner, give substantial insight into the governing principles and, in fact, are able to give quantitatively correct forms for the cross-sections and energy loss.

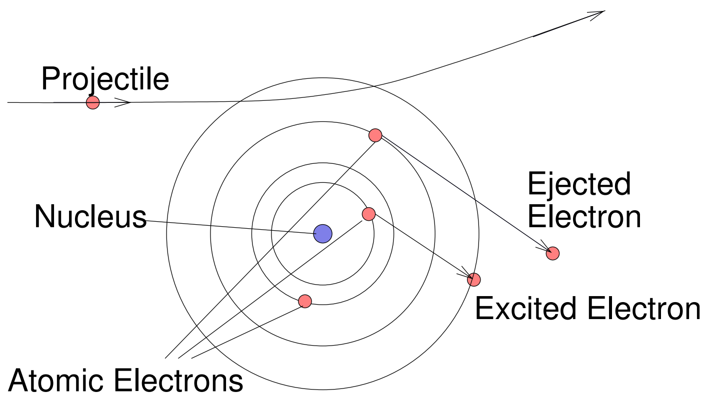

Figure 6.4: Collisions with an atomic system can excite or eject electrons from the atom.

6.2.1 Energy transfer to an oscillating particle

An approach to the problem of collisions with bound particles that can be treated classically, and becomes the basis for a quantum description, is to approximate the system as a charge bound in a simple harmonic potential well. Because we are mostly interested in large impact parameters, we regard the electric field of the projectile as uniform at the atom and then ask the question, in the encounter of the projectile with this oscillating electron, how much energy does the oscillator gain as a result of the fluctuating electric field of the passing projectile.

So consider a simple oscillating particle in a uniform electric field, $E(t)$. Its position x is governed by the equation

$$\ddot{x} + \omega^2 x = \frac{q}{m} E(t) \quad . \tag{6.22}$$

We solve this equation in the time range (t_1, t_2), with some assumed initial condition at t_1 so as to determine the energy gained by the particle at time t_2. This solution is readily obtained using what is called the "one-sided Green's function" as follows. The solutions to the homogeneous problem (the equation with zero right hand side) are $\sin \omega t$ and $\cos \omega t$. The Green's function is constructed as

$$H(t, \tau) = (\sin \omega t \cos \omega \tau - \cos \omega t \sin \omega \tau)/\omega \tag{6.23}$$

and the general solution is then

$$x(t) = A \sin \omega t + B \cos \omega t + \int_{t_1}^{t} H(t,\tau) \frac{q}{m} E(\tau) d\tau \quad , \tag{6.24}$$

where A and B are constants determined by the initial conditions. Actually, we don't need to solve for A and B because when we calculate the energy of the oscillator, averaged over an oscillator period, A and B make exactly the same contribution at the end of the integration as they did at the beginning and any cross-terms between them and H average to zero. [The point about the cross terms is not really obvious, but in the interests of time we won't prove it.] Therefore, the gain in energy is determined just by the integral term and we can simply set $A = B = 0$. Then at time t_2 the solution may be written

$$\frac{\omega m}{q} x(t_2) = \sin \omega t_2 \int_{t_1}^{t_2} \cos \omega \tau E(\tau) d\tau - \cos \omega t_2 \int_{t_1}^{t_2} \sin \omega \tau E(\tau) d\tau \quad . \tag{6.25}$$

When this expression is differentiated, the terms arising from the differentials of the limits cancel and we get

$$\frac{\omega m}{q} \dot{x}(t_2) = \omega \cos \omega t_2 \int_{t_1}^{t_2} \cos \omega \tau E(\tau) d\tau + \omega \sin \omega t_2 \int_{t_1}^{t_2} \sin \omega \tau E(\tau) d\tau \quad . \tag{6.26}$$

So the total (kinetic plus potential) energy in the oscillator can then rapidly be evaluated as

$$\begin{aligned} \frac{1}{2} m(\omega^2 x + \dot{x}^2) &= \frac{q^2}{2m} \left[\left(\int_{t_1}^{t_2} \cos \omega \tau E(\tau) d\tau \right)^2 + \left(\int_{t_1}^{t_2} \sin \omega \tau E(\tau) d\tau \right)^2 \right] \\ &= \frac{q^2}{2m} |E(\omega)|^2 \quad , \end{aligned} \tag{6.27}$$

with the Fourier transform of the electric field written

$$E(\omega) = \int \exp(i \omega \tau) E(\tau) d\tau \quad . \tag{6.28}$$

We did this integration over a finite time, which avoids some mathematical difficulties, but we can now readily let $t_1 \to -\infty$ and $t_2 \to \infty$ and obtain the full domain Fourier integral. We have obtained the important general result that the energy transferred to a harmonic oscillator is proportional to the Fourier transform of the electric field evaluated at the resonant frequency of the oscillator, eq(6.27).

6.2.2 Straight-Line Collision

We are interested mostly in small-angle collisions, because, as we previously noted, they dominate the behavior, especially at the cut-off, $b_{\max}$. We approximate the orbit of the projectile in this case as a straight-line. Then, as illustrated in Fig 6.5, the electric field at

Straight-Line Collision

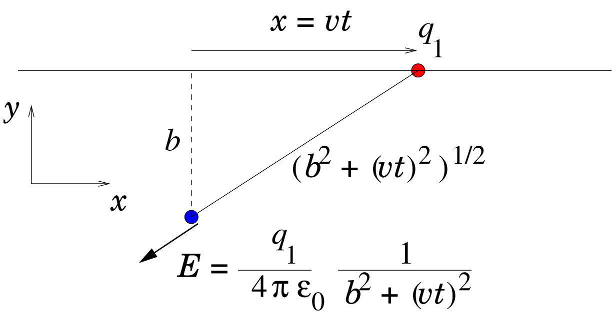

Figure 6.5: The approximation of a straight orbit gives a simple expression for the electric field as a function of time.

the atom is just that due to a charge moving past at an impact parameter b and a constant speed. For a non-relativistic speed v the components of the electric field as a function of time are then

$$E_x(t) = \frac{-q_1}{4\pi\epsilon_0}\frac{vt}{(b^2+v^2t^2)^{3/2}} \quad \text{and} \quad E_y(t) = \frac{-q_1}{4\pi\epsilon_0}\frac{b}{(b^2+v^2t^2)^{3/2}} \quad . \tag{6.29}$$

the relativistic forms are qualitatively similar, and were calculated previously in section 4.2, see eq(4.40)

$$E_x(t) = \frac{-q_1}{4\pi\epsilon_0}\frac{\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}} \quad \text{and} \quad E_y(t) = \frac{-q_1}{4\pi\epsilon_0}\frac{\gamma b}{(b^2+\gamma^2v^2t^2)^{3/2}} \quad , \tag{6.30}$$

where γ is the relativistic factor $(1 - v^2/c^2)^{-1/2}$. The field components are plotted as a function of time in Fig 6.6. Clearly, by inspection of Fig 6.6, and eq(6.28) there will be a qualitative change in the behaviour of the Fourier transform of $E(t)$ and hence the energy transfer for $\omega b/\gamma v \gg 1$ compared with $\omega b/\gamma v \ll 1$. The characteristic time duration of the collision is $\sim b/\gamma v$. If this is much shorter than the characteristic oscillator time, $1/\omega$, we can take $\omega \approx 0$ and obtain by elementary integration

$$E_y(\omega) = \frac{-2q_1}{4\pi\epsilon_0 bv} \quad . \tag{6.31}$$

Because $E_x(t)$ is antisymmetric, $E_x(\omega) = 0$ in this small impact parameter limit. In the opposite limit, that is for collisions in which b is so large that $\omega b/\gamma v \gg 1$, $E(\omega)$ will be small because in eq(6.28) there are many oscillations of the factor $\exp(-i\omega t)$ within the smooth variation of $E(t)$. Thus we see that in collisions with a simple harmonic oscillator of frequency ω, there is a natural cut-off to the energy transfer at a maximum impact parameter

$$b_{\max} \approx \frac{\gamma v}{\omega} \quad . \tag{6.32}$$

Substituting eq(6.31) into eq(6.27), and restoring our notation of subscript 2 for the target and subscript 0 for the incident velocity, we obtain the energy transfer in a straight-line collision as

$$Q(b) = \frac{q_1^2 q_2^2}{(4\pi\epsilon_0)^2}\frac{2}{m_2 v_0^2 b^2} = \frac{1}{2}\frac{m_r^2}{m_2}v_0^2\, 4\left(\frac{b_{90}}{b}\right)^2 \tag{6.33}$$

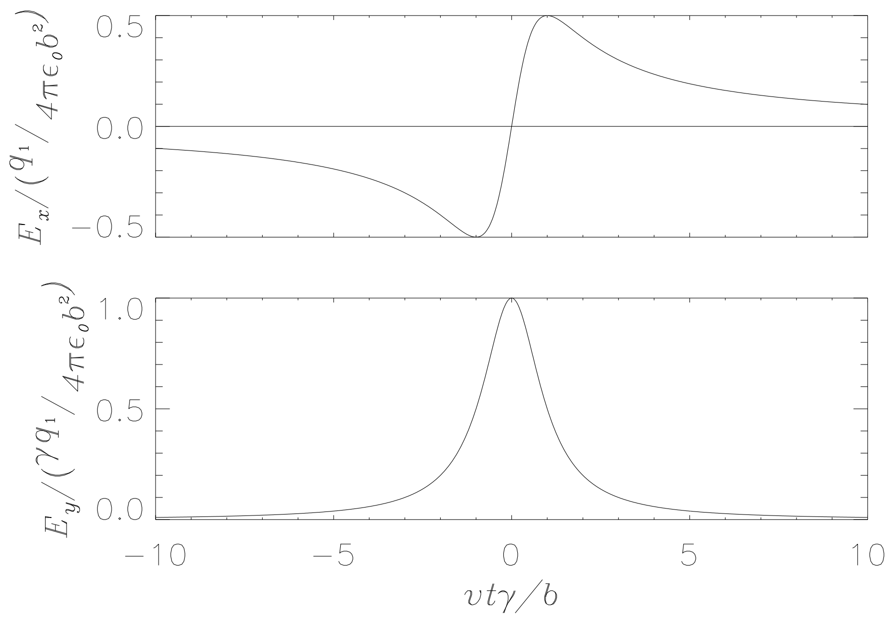

Figure 6.6: The electric field components in a straight-line collision.

Notice that this is essentially the same expression as in eq(6.20) for the energy transfer to a free electron, except that the lower impact-parameter cut-off is not present here because of the assumption of a straight-line orbit for the projectile, which is unjustified at small impact parameters. The rate of energy loss is then obtained, as before, by integration over impact parameters from the minimum to the maximum corresponding to the limits of applicability of eq(6.31)

$$-\frac{dK}{d\ell} = n_2 K\pi b_{90}^2 \frac{m_r^2}{m_1 m_2} 8 \ln\left|\frac{b_{\max}}{b_{\min}}\right| = n_2 \left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_2 v_0^2} \ln\left|\frac{\gamma v_0}{\omega b_{90}}\right| \quad . \tag{6.34}$$

6.2.3 Classical Energy Loss Rate Formula

One final consideration is needed before we have an energy loss formula useful for practical purposes. We have to have some way of applying the idealized harmonic oscillator calculation to actual atoms. An atom in general has a number Z, say, of electrons bound to the nucleus. Each electron may act as a target oscillator for energy transfer, and actually each electron may act as one of an infinite set of oscillators, corresponding to each of its possible quantum transitions. Energy transitions of magnitude $\mathcal{E}_i$ correspond to oscillators of frequency $\omega_i = \mathcal{E}_i/\hbar$, of course. To the ith transition may be assigned an *oscillator strength*, f_i, defined as the ratio of the actual rate of energy absorption by that transition to that of a corresponding harmonic oscillator. The semi-classical argument is then that each electron spends some fraction of its time behaving as if it were each of the possible oscillators, and consequently $\sum f_i = Z$. There is a more rigorous theorem in quantum physics called the (Thomas-Reiche-Kuhn) f-sum rule which states that the sum of all possible transition oscillator strengths from a specific level is equal to the number of electrons in the level. If this were applied blindly to all the electrons of the atom, it would give the same equation.

To obtain the total energy loss rate arising from collisions with a density of atoms n_a, whose atomic number is Z, we add up the contributions from all the possible transitions, weighted by the oscillator strength of that transition. Thus we obtain for the logarithmic term:

$$\sum_i f_i \ln\left|\frac{\gamma v_0}{\omega_i b_{90}}\right| = Z\ln\left|\frac{\gamma v_0}{b_{90}}\right| - \sum f_i \ln\omega_i = Z\ln\left|\frac{\gamma v_0}{\langle\omega\rangle b_{90}}\right| \quad , \tag{6.35}$$

where we have defined a kind of average oscillator frequency $\langle\omega\rangle$ by the equation

$$Z\ln\langle\omega\rangle \equiv \sum_i f_i \ln\omega_i \quad . \tag{6.36}$$

The total classical energy loss rate is then

$$\frac{dK}{d\ell} = n_a\left(\frac{q_1 e}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_e v_0^2} Z\ln\Lambda \quad , \tag{6.37}$$

where we have substituted electron charge and mass, and for brevity denoted the argument of the logarithm by

$$\Lambda = \frac{\gamma v_0}{\langle\omega\rangle b_{90}} \quad . \tag{6.38}$$

Actually, it turns out to be possible to evaluate the Fourier transforms of the relativistic fields in eq(6.30) in closed form and carry through the integration of the modified Bessel functions thus obtained [ref6.2.3]. When that is done, two very small corrections to our formula appear. The argument of the logarithm is multiplied by the factor 1.123 and an additional relativistic term is added, equivalent to the replacement

$$\ln\Lambda \to \ln\left|\frac{1.123\gamma v_0}{\langle\omega\rangle b_{90}}\right| - \frac{v_0^2}{2c^2} \quad . \tag{6.39}$$

Neither of these corrections is quantitatively significant. The result was first obtained by Bohr in 1913, prior to the development of quantum mechanics. It is hardly complete as it stands, since the average $\langle\omega\rangle$ has to be estimated. However, because $\langle\omega\rangle$ appears only in the logarithm, even a rough estimate, for example setting $\hbar\langle\omega\rangle$ equal to the atom's ionization potential, will give a useful quantitative formula for the energy loss.

6.2.4 Quantum effects on close collisions

For the classical minimum impact parameter b_{90} to be applicable requires that the particles of the collision behave as point particles down to that impact parameter. However, quantum mechanics teaches us that particles do not behave like perfect points. The Heizenberg uncertainty principle states that the particle is localized only within a position uncertainty Δx if its momentum uncertainty is Δp such that $\Delta x\Delta p \approx \hbar$. Alternatively one can say that a particle with momentum $p = \gamma m v$ behaves like a wave with wave-vector $k = p/\hbar$. Or again, one can say that orbital angular momentum is quantized in indivisible units of $\hbar$. All of these are ways of indicating that in collisions the effective position of a particle is spread out over a distance of order $\hbar/p$. Consequently, quantum effects prevent us from extending the classical integration over impact parameters below a value of

$$b_q \approx \frac{\hbar}{\gamma m v} \quad . \tag{6.40}$$

The classical b_{90} lower impact parameter cut-off will be applicable only if

$$\frac{b_{90}}{b_q} \approx \frac{q_1 q_2}{4\pi\epsilon_0 \hbar v} = \frac{q_1 q_2}{e^2}\alpha\frac{c}{v} > 1 \quad , \tag{6.41}$$

where α is the fine structure constant, approximately 1/137, This criterion is a requirement that the collision velocity with electron targets should be less than $Z_1 c/137$.

In practice this means that electrons with energy greater than 1.9 keV, protons with energy greater than 3.5 MeV, or alpha particles with energy greater than 55 MeV will *not* be appropriately treated using the classical lower impact parameter cut off. Instead, an approximation to the quantum-mechanical result may be obtained by simply cutting off the impact parameter integration at b_q rather than b_{90}. If we choose[3] $b_q = \hbar/2\gamma m_e v$, then in collisions of heavy particles with atoms, for which $m_r = m_e$,

$$\ln\left|\frac{b_{\max}}{b_{\min}}\right| = \ln\left|\frac{2\gamma^2 m_e v_0^2}{\hbar\langle\omega\rangle}\right| \quad . \tag{6.42}$$

This value is then consistent with that obtained for the relativistic case using a quantum scattering treatment and the first Born approximation, by Bethe (1930),

$$-\frac{dK}{d\ell} = n_a\left(\frac{q_1 e}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_e v_0^2} Z\left(\ln\left|\frac{2\gamma^2 m_e v_0^2}{\hbar\langle\omega\rangle}\right| - \frac{v_0^2}{c^2}\right) \quad , \tag{6.43}$$

where again the final term, v_0^2/c^2, which we have not derived, is at most a small correction.

If the projectile is an electron or positron, then the quantum cut-off must be estimated in the center-of-mass frame, and the expression becomes

$$-\frac{dK}{d\ell} = n_a\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_e v_0^2} Z\left(\ln\left|\left(\frac{\gamma+1}{2}\right)^{1/2}\frac{(\gamma-1)m_e c^2}{\hbar\langle\omega\rangle}\right| - \frac{v_0^2}{2c^2}\right) \quad . \tag{6.44}$$

6.2.5 Values of the Stopping Power

We have so far left open the question of what value to take for $\hbar\langle\omega\rangle$. Bloch (1933) showed from an analysis of the Thomas-Fermi model of the electron charge distribution in an atom that one would expect that $\hbar\langle\omega\rangle \propto Z$. In recognition of the work of Bethe and Bloch, eq 6.43 is often referred to as the Bethe-Bloch formula. The formula is often written as

$$-\frac{dK}{d\ell} = n_a\left(\frac{q_1 e}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_e v_0^2} B \quad , \tag{6.45}$$

[3]The factor of 2 here is our only real artifice. It gives the argument of the logarithm equal to that obtained by a full quantum calculation

with the quantity B, called the "atomic stopping number", corresponding to the factor

$$Z\left(\ln\left|\frac{2\gamma^2 m_e v_0^2}{\hbar\langle\omega\rangle}\right| - \frac{v_0^2}{c^2}\right) \quad .$$

Also B/Z is then called the "stopping power" per (atomic) electron, recognizing that an atom has Z electrons. The stopping power is determined from experiments, and the appropriate value to use for $\hbar\langle\omega\rangle$ is determined from those measurements.

A complication that we have not discussed arises because our treatment has assumed that the orbital velocity of the electrons in the atom can be ignored relative to the velocity of the incident particle. This is not the case when dealing with the inner shell electrons of high-Z atoms or very low incident-energy projectiles. Then a reduction in the stopping number occurs because (for example) the (innermost) K-shell electrons are ineffective in removing the projectile's energy. This effect is numerically compensated by substracting a correction term C_K so that

$$B = Z\left(\ln\left|\frac{2\gamma^2 m_e v_0^2}{\hbar\langle\omega\rangle}\right| - \frac{v_0^2}{c^2}\right) - C_K \quad . \tag{6.46}$$

In this form, the value of $\hbar\langle\omega\rangle$ is empirically determined to be about $11.5 \times Z$ eV, and C_K is a function of the quantity $\xi \equiv (c^2/v_0^2)(Z-0.3)^2\alpha^2$ (which represents the squared ratio of the K-shell velocity to the projectile velocity). A simple approximate form for C_K is

$$C_K(\xi) = \frac{2.3\xi}{1+1.3\xi^2} \quad ,$$

correct to within 10% from $\xi = 0$ to $\xi = 2$. It tends to zero at high projectile energy and peaks at about unity at low velocity, where $\xi \approx 1$. These and many other details have been reviewed by Evans (1955).

6.2.6 Effects of surrounding particles on distant collisions

Let us return now to our primitive energy loss rate calculation, eq 6.34 which may be considered in the form

$$\frac{dK}{d\ell} = n_2\left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_2 v_0^2}\ln\left|\frac{b_{\max}}{b_{\min}}\right| \quad . \tag{6.47}$$

In the preceding sections we have been discussing appropriate choices of $b_{\min}$ based either on the classical effects of large scattering angles (giving b_{90}) or on quantum-mechanical effects of the de Broglie wave-length of the projectile/target combination. We also discussed the appropriate $b_{\max}$ based on the effects of the binding of electron targets to their nuclei. However, another effect can sometimes be more important than the atomic binding structure in determining $b_{\max}$, namely the influence of surrounding particles.

We have tacitly assumed so far that the interaction of the projectile and any specific target can be treated *ignoring* the effects of the other targets in the vicinity. We have calculated the projectile/target interaction in isolation and then presumed that we can add up the effects of all the different targets via a simple impact-parameter integration. This may not be the case. For example, it definitely is not the case when the electrons of the target are *unbound*; or in other words for a plasma target. In that case there is no intrinsic cut-off to the the collision integral arising from the oscillator effects introduced in section 6.2.1 and the effect of the nearby particles essentially always determines $b_{\max}$. Even in collisions with atomic matter, especially for relativistic electrons, the effect of nearby particles can significantly lower the energy transfer rate. In the atomic collision context the corrections are often referred to as the "density effect" because they are most significant for high-density matter.

It is still the case that transfer of energy to the target arises from the electric field produced by the incident projectile. However, what we need to do is to account for the influence of the other particles in the target medium on the electric field that the projectile produces at a specific target. Expressed in this way, it is immediately clear that what we need is to take account of the *dielectric properties* of the target medium. The individual particles of the medium respond to the influence of charge (the projectile in this case) so as to alter the electric field in the medium from what it would otherwise have been. This is exactly what we mean by the dielectric response of the medium.

Of course, though, it is not the steady-state dielectric response that we require but the response at the high frequencies of interest in the collisions. Moreover, when we think about a target medium consisting of a density of idealized oscillators, as we did before, it is the properties of those oscillators themselves that determines the dielectric response at frequencies close to their resonant frequencies. Thus the dielectric response and the energy-loss collisional response are not two *separate* properties of the medium; they are intimately connected.

The idealized oscillator model can be generalized to discuss a medium with any relative dielectric permittivity $\epsilon(\omega)$ having a resonant form ($\epsilon - 1 \propto (\omega - \omega_i)^{-1}$), and an expression for the rate of loss of energy of an incident projectile to this resonance can then be obtained. Fermi (1940) first gave the following formula, which would take us too long to rederive, for the energy loss attributable to collisions with impact parameter greater than a as

$$\left.\frac{dK}{d\ell}\right|_{b>a} = \frac{2}{\pi}\frac{q_1^2}{4\pi\epsilon_0 v_0^2}\Re\int_0^\infty is^* K_1(s^*)K_0(s)\left(\frac{1}{\epsilon(\omega)} - \beta^2\right)d\omega \quad , \qquad (6.48)$$

where $\Re$ denotes real part, $\beta = v_0/c$, K_1 and K_2 are modified Bessel functions, and their argument is s such that

$$s^2 \equiv \frac{a^2\omega^2}{v_0^2}[1 - \beta^2\epsilon(\omega)] \quad . \qquad (6.49)$$

It can be shown, but not trivially, [Jackson] that this $dK/d\ell$ reduces to the the Bohr expression (eq 6.39) if the $\beta^2\epsilon(\omega)$ term in s is neglected.

Rather than pursue the topic for the atomic case, let us consider a simple argument for a plasma. The dielectric constant for a (magnetic field-free) plasma at high frequency is

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \quad , \qquad (6.50)$$

where

$$\omega_p \equiv \frac{n_e e^2}{m_e \epsilon_0}$$

is called the plasma frequency. Therefore when the field frequency of interest is less than ω_p the dielectric constant is negative and wave electric fields no longer propagate in the medium; instead they decay exponentially with distance from their source. In collisions, as we have seen before, the frequency of the interaction electric field is approximately v_0/b. Therefore, for impact parameter, b, greater than v_0/ω_p we would expect that the effectiveness of the collisions would fall off because of the dielectric effects. Applying this value for $b_{\max}$ we obtain an energy loss rate expression corresponding to eq 6.37 as

$$\frac{dK}{d\ell} = n_e \left(\frac{q_1 e}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_e v_0^2} \ln\Lambda = \frac{q_1^2}{4\pi\epsilon_0} \frac{\omega_p^2}{v_0^2} \ln\Lambda \tag{6.51}$$

but with Λ given approximately by

$$\Lambda = \frac{v_0}{\omega_p b_{90}} \quad . \tag{6.52}$$

What we have done, in effect then, is to replace the value $b_{\max} = \gamma v_0/\langle\omega\rangle$ in the definition of Λ, eq (6.38) with

$$b_{\max} = \frac{v_0}{\omega_p} \quad . \tag{6.53}$$

The factor by which the logarithmic argument Λ of the Bethe-Bloch formula is multiplied is therefore $\gamma\omega_p/\langle\omega\rangle$. But the density effect can only lower the absorption rate so we should more properly have used $b_{\max} = \min(v_0/\omega_p,\ \gamma v_0/\langle\omega\rangle)$. The electrons behave as if they are free when $\omega > \omega_{ij} \sim \langle\omega\rangle$. Hence plasma-like, i.e. free-electron, behaviour occurs only when $\omega_p > \langle\omega\rangle$, which is when the plasma expression for $b_{\max}$ applies, because it is the smaller.

A rough estimate of the ratio of $\omega_p/\langle\omega\rangle$ may be obtained by taking the density of atoms in a solid to be about 10^{30} m^{-3}, and the electron density to be Z times that. Then

$$\hbar\omega_p \approx 37Z \quad \text{eV}. \tag{6.54}$$

For medium weight solid elements, $\hbar\langle\omega\rangle \sim 11Z$ eV so we expect the plasma effect to be slightly noticeable since on this basis $\omega_p/\langle\omega\rangle > 1$. The question is a little more complicated than this, though because not all the electrons are going to behave as if free so we have somewhat over estimated the density of the electrons that behave as if free. In extreme relativistic cases, $\gamma \gg 1$ the plasma (density) effect will always dominate.

6.3 Angular Scattering from Nuclei

Up to this point we have been discussing the energy loss of the projectile and have focussed on its interactions with electrons. This focus on electron targets is entirely appropriate for calculating energy loss because, as illustrated by eq (6.21) or (6.34) the rate of energy loss is, classically, inversely proportional to the mass of the target particle[4]. Therefore the loss of energy is in fact predominantly to the light particles, electrons, and this predominance

[4]This proportionality can be traced to the inverse dependence of the energy transfer in a collision on m_2, but only because cancellation of reduced mass factors occurs in the product Qb_{90}^2.

depends only on the elementary dynamics of collisions. However, in addition to losing energy, the projectile also generally experiences angular scattering in the direction of its velocity. If this angular scattering is our concern, as it was in Rutherford's original experiments on the angular scattering of alpha particles which established that the nucleus is far smaller than the atom, then collisions with the heavy particles in our scattering medium, the nuclei of the atoms or the ions of a plasma, are important. This process, illustrated in Figure 6.7, is often called "elastic scattering", although the expression may be considered somewhat misleading in that some energy is lost by the projectile in the collision, and the process is no more elastic than a collision with a free electron, for instance.

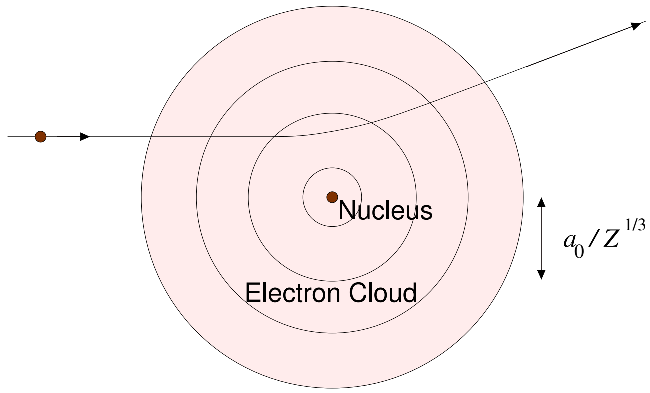

Figure 6.7: Angular scattering from nuclei occurs only if the impact parameter is less than the size of the electron cloud.

Qualitatively, the relative importance of energy loss and angular scattering can be grasped by imagining the difference between a ping-pong ball colliding with a random arrangement of snooker balls, or a snooker ball colliding with a random arrangement of ping-pong balls. In the first case, the light projectile will bounce around changing its direction of motion many times before losing its energy; while in the second case, the heavy projectile will plough through the light targets, losing energy faster than its direction is deflected.

The angular scattering of a particle in a classical coulomb collision is governed by the Rutherford formula for the differential scattering cross-section per unit solid angle at a scattering angle *in the center of mass frame*, χ_c,

$$\frac{d\sigma}{d\Omega} = \frac{b_{90}^2}{4\sin^4 \chi_c/2} \quad . \tag{6.55}$$

This formula may readily be derived from the considerations in section 6.1.1. It shows that the predominant scattering is through small angles. Those small angles arise from large impact parameters. There are some collisions, of course, which arise from small impact parameters, close to b_{90}, that give rise to large scattering angles, but these are far fewer in number than the small-angle collisions; so by the time the probability of scattering by a large angle is significant, multiple scatterings by small angles will have caused a kind of diffusion of the direction of the particles in perpendicular velocity. Figure 6.8 illustrates an idealized situation, in which the projectile loss of energy is taken as zero, so its velocity vector has constant magnitude and moves on a sphere. Taking the initial direction to be along the z-axis, each small-angle collision causes a random step to be taken in the (v_x,v_y) plane.

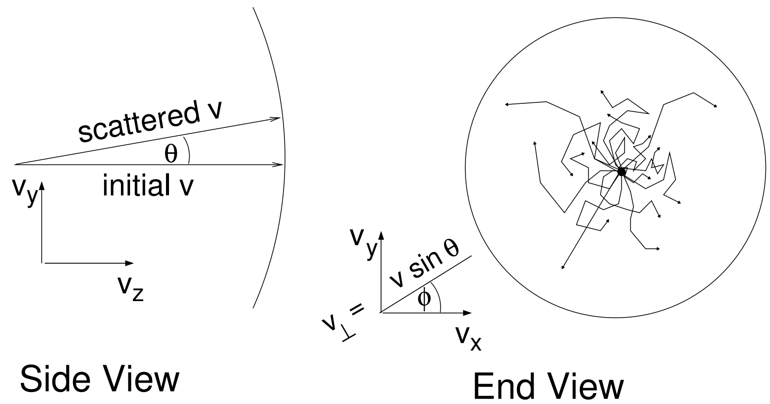

Figure 6.8: Multiple small-angle Coulomb collisions cause a diffusive "random walk" of the angle of the projectile velocity, or equivalently its perpendicular components.

Setting the large-angle collisions aside for a moment, we can treat the total angular scattering experienced by a projectile passing through a finite length of scattering path as the result of many small scatterings each of which has random direction and magnitude, governed by the fact that $\cot(\chi_c/2) = b/b_{90}$ (eq 6.17). Although we cannot calculate for any individual projectile what its final angle will be, we can treat the whole process statistically, by presuming there to be many small-angle scatterings. Actually, this calculation requires not the Rutherford differential cross-section per unit solid-angle Ω, but the differential cross-section per unit scattering angle χ_c,

$$\frac{d\sigma}{d\chi_c} = \left|\frac{db}{d\chi_c}\right| \frac{d\sigma}{db} = \frac{b_{90}}{2} \csc^2(\chi_c/2)\, 2\pi b = \pi b_{90}^2 \csc^2(\chi_c/2) \cot(\chi_c/2) \quad , \tag{6.56}$$

which result is obtained immediately from our previous formulae.

The *mean* scattering angle experienced by the projectiles is always zero, because there is equal probability of scattering at negative and positive angles; the scattering is isotropic in the (v_x,v_y) plane. The spread of scattering angles is quantified by the *mean square* scattering angle, which is non-zero and can be evaluated as follows. Succeeding collisions are statistically independent of each other. The final value of v_x is given by the sum of the steps in v_x at each of the individual collisions. (Similarly for v_y.) We therefore make use of the basic statistical theorem that the variance (which is the mean-square value for a zero-mean random variable) of the sum of independent random variables is the sum of the variances. We perform this sum by dividing the collisions into appropriate ranges of scattering angle $d\chi_c$ and azimuthal angle $d\phi$. The number of steps per unit path length belonging in these ranges is

$$dN = n_2 \frac{d\sigma}{d\chi_c} d\chi_c \frac{d\phi}{2\pi} \tag{6.57}$$

and the change in v_x that such collisions cause is

$$\delta v_x = (m_r/m_1) v_0 \sin\chi_c \cos\phi \quad . \tag{6.58}$$

Here, the quantity $(m_r/m_1)v_0$ is the initial (and final) speed of the incident particle (1) in the center-of-mass frame. Consequently, the total variance of v_x per unit path length arising from all possible types of collisions is

$$\frac{d\langle v_x^2\rangle}{d\ell} = \int (\delta v_x)^2 dN = \int\int n_2 v_0^2 (m_r/m_1)^2 \sin^2\chi_c \cos^2\phi \frac{d\sigma}{d\chi_c} d\chi_c \frac{d\phi}{2\pi} \quad . \tag{6.59}$$

Performing the integration over azimuthal angle, ϕ, and substituting for the differential cross-section from eq (6.56) we get

$$\begin{aligned} \frac{d\langle v_x^2\rangle}{d\ell} &= \frac{1}{2} n_2 v_0^2 (m_r/m_1)^2 \int \sin^2\chi_c \frac{d\sigma}{d\chi_c} d\chi_c \\ &= \frac{1}{2} n_2 v_0^2 (m_r/m_1)^2 \pi b_{90}^2 \int \sin^2\chi_c \csc^2(\chi_c/2)\cot(\chi_c/2) d\chi_c \, . \end{aligned} \tag{6.60}$$

The final integral may be transformed using trignometric identities, becoming

$$\int \sin^2\chi_c \csc^2(\chi_c/2)\cot(\chi_c/2)d\chi_c = 8\int \frac{1}{s} - s ds \qquad (s \equiv \sin\chi_c/2). \tag{6.61}$$

The upper limit of the integral is $s = 1$. The singularity of this expression at zero lower limit of s shows again the now-familiar need for a cut-off of the collision integral at large impact-parameter (small χ_c or s). That cut-off and eq(6.19) make the value of the integral $8(\ln|b_{\max}/b_{90}| - \frac{1}{2})$, where $b_{\max}$ is the maximum impact parameter, and the $\frac{1}{2}$ term should be dropped since it is an artifact of the approximation implied by our use of eq(6.58). In the case of scattering by a plasma, the relevant impact-parameter cut-off is the length beyond which the collective interactions in the plasma screen out the electric field of individual nuclei. This distance is called the Debye length. When the scattering is from neutral atoms, the relevant cut-off length corresponds to the size of the atom, because for impact parameters larger than the atom the projectile sees the whole atom, neutral because of its electrons, rather than a bare nucleus.

An identical treatment governs the y-component v_y, and consequently the square of the total transverse velocity $v_\perp^2 = v_x^2 + v_y^2$ evolves as

$$\frac{d\langle v_\perp^2\rangle}{d\ell} = n_2 v_0^2 (m_r/m_1)^2 \, \pi b_{90}^2 \, 8\ln|b_{\max}/b_{90}| \quad , \tag{6.62}$$

with $b_{\max}$ approximately the size of the atom. For small angles $\theta \approx v_\perp/v$ and so this equation can be written in terms of the angle of the scattered velocity direction:

$$\frac{d\langle\theta^2\rangle}{d\ell} = n_2 \, (m_r/m_1)^2 \pi b_{90}^2 \, 8\ln|b_{\max}/b_{90}| \quad . \tag{6.63}$$

After a finite path length ℓ, there is a distribution of $\mathbf{v}_\perp$ with variance

$$\langle v_x^2\rangle = \langle v_y^2\rangle = \ell n_2 v_0^2 (m_r/m_1)^2 \, \pi b_{90}^2 \, 4\ln|b_{\max}/b_{90}| \quad , \tag{6.64}$$

which we assume is still small compared to v_0^2 so that small-angle approximations remain valid. Because this distribution arises from many independent scatterings, it becomes Gaussian (following the Central Limit theorem of statistics):

$$f(v_x, v_y) = \left(\frac{1}{2\pi\langle v_x^2\rangle}\right) \exp\left\{-\frac{(v_x^2 + v_y^2)}{2\langle v_x^2\rangle}\right\} \quad , \tag{6.65}$$

with $\langle v_x^2\rangle$ given by eq(6.64). We may alternatively regard the Gaussian shape as arising because the particle distribution is experiencing a *diffusion* of velocity from an initial localized distribution (delta function) at $\mathbf{v}_\perp = 0$. The solution of the diffusion equation in this case is this Gaussian.

The maximum impact parameter (minimum χ_c) is determined by the shielding of the nucleus by its atomic electrons. Only for impact parameters small compared to the atom size, will the projectile see the bare nucleus because then it penetrates deep inside the electron shielding cloud. So b_{max} is approximately the radius of the electron cloud surrounding the nucleus. This is generally taken to have a characteristic size approximately[5] $a_0/Z^{1/3}$.

There is no mathematical compulsion to cut off the upper limit of the χ_c integral short of $\chi_c = \pi$, that is $s = 1$. However, if very energetic particles are involved, the value of b_{90}, which is inversely proportional to particle energy, becomes very small, eventually so small that it is smaller than the size of the nucleus. In that case, the large-angle scattering is affected by the structure of the nucleus itself and the upper limit is affected. Of course, this is the basis for experimental high-energy physics investigations of nuclear structure by electron scattering, but it requires electron energies greater than roughly $Ze^2/(4\pi\epsilon_0 r_n)$ ($\approx Z$ MeV), where r_n is the nuclear radius, of order 10^{-15} m, and Z its nuclear charge.

6.4 Summary

Collisions of charged particles are governed by the long range Coulomb force. The range of that force is limited by one of several different processes, depending on the exact physical situation to a maximum impact parameter b_{max}. A minimum impact parameter for the process is also needed if approximations such as that the collision has a straight-line trajectory are made, or if quantum effects are important. Table 6.1 gives a summary of the situations discussed.

[5] See M.Born "Atomic Physics" 8th ed., Blackie p199, for a derivation of the Thomas-Fermi distribution of electron density around an atom based on the Pauli exclusion principle and a continuum approximation.

	Impact parameters		Stopping Power (per electron)
Collision Type	$b_{\rm min}$	$b_{\rm max}$	$\ln\Lambda = B/Z$
Classical Coulomb	b_{90}	$\gamma v_0/\omega$	$\ln \frac{\gamma v_0}{\omega b_{90}}$
Classical energy loss to atoms			$\ln\lvert\frac{1.123\gamma v_0}{\langle\omega\rangle b_{90}}\rvert - \frac{v_0^2}{2c^2}$
Quantum ion loss to atoms	$\sim \hbar/\gamma m v$	$\gamma v_0/\omega$	$\ln\lvert\frac{2\gamma^2 m_e v_0^2}{\hbar\langle\omega\rangle}\rvert - \frac{v_0^2}{c^2}$
Corrected for inner shell effects			$\ln\lvert\frac{2\gamma^2 m_e v_0^2}{\hbar\langle\omega\rangle}\rvert - \frac{v_0^2}{c^2} - C_k/Z$
Quantum electron loss to atoms	$\sim \hbar/\gamma m v$	$\gamma v_0/\omega$	$\ln\lvert\left(\frac{\gamma+1}{2}\right)^{1/2}\frac{(\gamma-1)m_e c^2}{\hbar\langle\omega\rangle}\rvert - \frac{v_0^2}{2c^2}$
Density effect (non-rel. plasma)	b_{90}	v_0/ω_p	$\ln\lvert\frac{v_0}{\omega_p b_{90}}\rvert$
Angular scattering from nucleus	b_{90}	$\sim a_0/Z^{1/3}$	$\ln\lvert a_0/Z^{1/3}b_{90}\rvert$

Table 6.1: Summary of collision calculations.

In collisions of the projectile particle 1, initial velocity v_0, with particles of type 2, density n_2, the rate of loss of kinetic energy K per unit pathlength ℓ is given by

$$\frac{dK}{d\ell} = -K\ n_2\pi b_{90}^2\frac{m_r^2}{m_1 m_2}8\ln\Lambda = -n_2\left(\frac{q_1q_2}{4\pi\epsilon_0}\right)^2\frac{4\pi}{m_2v_0^2}\ln\Lambda$$

and the angular scattering from nuclei by

$$\frac{d\langle v_\perp^2\rangle}{d\ell} = v_0^2\ n_2\pi b_{90}^2\frac{m_r^2}{m_1^2}8\ln\Lambda = n_2\left(\frac{q_1q_2}{4\pi\epsilon_0}\right)^2\frac{8\pi}{m_1^2v_0^2}\ln\Lambda$$

with the $\ln\Lambda$ values indicated. See eqs(6.18) and (6.3) for other definitions.