Chapter 7

Radiation from Charged Particle Interaction with Matter

7.1 Bremsstrahlung

When charged particles collide, they accelerate in each other's electric field. As a result, they radiate electromagnetic waves. This type of radiation occurs when a fast electron slows down by collisions, and so it has acquired the German name Bremsstrahlung ("braking radiation").

7.1.1 Radiation in Collisions, Non-relativistic.

We have analysed collisions of charged particles in some detail in previous chapters, ignoring the possibility of radiation. The orbit of the projectile is, classically, a hyperbola. However, as an approximation, albeit one that will break down if the impact parameter, b, is small enough, we can ignore the curvature of the orbit and take the collision to occur with the projectile travelling along a straight line. This "straight-line-collision" approach we adopted previously as an approximation for calculating the energy transfer to a simple harmonic oscillator in a collision. Our present approach follows a parallel argument.

As it passes the target, the projectile experiences the field of the target, which accelerates it. When the projectile is far away from the target, either before or after the collision, the acceleration becomes negligible. Therefore, the projectile has experienced an "impulse", a brief period of acceleration. We can estimate the duration of that impulse as being the time it takes the projectile to travel a distance of approximately b, namely $\tau = b/v_0$, where v_0 is the incoming projectile velocity. On average the impulse is perpendicular to the projectile velocity.

The total energy radiated in this impulse is given by our previous formula (4.85) for the instantaneous radiated power by an accelerated charge,

$$P' = \frac{q_1^2}{4\pi\epsilon_0}\frac{2}{3c}\frac{\dot{v}^2}{c^2} \quad , \tag{7.1}$$

integrated over the duration of the impulse, τ. Taking the characteristic value of the acceleration as given by the electric field force at the closest approach,

$$\dot{\mathbf{v}} = E/m_1 = \frac{q_1 q_2}{4\pi\epsilon_0 b^2 m_1} \quad , \tag{7.2}$$

we derive an estimate of the radiated energy

$$W \approx P'\tau \approx \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{2}{3c^3} \frac{1}{m_1^2 b^4} \frac{b}{v_0} = \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{2}{3c^3} \frac{1}{m_1^2 v_0} \frac{1}{b^3} \tag{7.3}$$

This is the energy radiated in a single collision with impact parameter b. To obtain the energy radiated per unit length we multiply by the density of targets and integrate over impact parameters to obtain

$$\frac{dW}{d\ell} = n_2 \int \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{2}{3c^3} \frac{1}{m_1^2 v_0} \frac{1}{b^3} 2\pi b db = n_2 \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{4\pi}{3c^3} \frac{1}{m_1^2 v_0} \left[\frac{1}{b}\right]_{b_{min}}^{b_{max}} \quad . \tag{7.4}$$

Notice that in this case, there is no need to invoke an upper limit to the integration, b_{max}. We can perfectly well let b_{max} tend to infinity without any divergence of the integral. The same is not true of the lower limit. We will either have to invoke the limit on the classical impact parameter, b_{90}, where our straight-line approximation breaks down, or, more likely the usual quantum limit where the wave nature of the projectile becomes important, at

$$b_{min} = b_q = \frac{\hbar}{m_1 v_0} \quad . \tag{7.5}$$

With this quantum cut-off for b_{min} and infinity for b_{max}, the energy radiated becomes

$$\frac{dW}{d\ell} = n_2 \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{4\pi}{3c^3} \frac{1}{m_1 \hbar} \quad . \tag{7.6}$$

7.1.2 Bremsstrahlung from light or heavy particles

So far we have treated the collision maintaining generality in the projectile and targets but have considered the radiation only from the projectile. Now we need to discuss what types of collisions give rise to significant bremsstrahlung. Equation (7.6) helps this discussion.

First we see that the projectile velocity does not enter into the formula. The projectile *mass*, however, is a very important effect. Light projectiles like electrons or positrons are far more efficient radiaters (by the inverse mass ratio) than protons or heavy nuclei, because their acceleration is so much greater.

That said, however, we realize that if a heavy projectile is colliding with a free electron target, then the electron target will experience an acceleration and give rise to radiation. This radiation from the target-particle acceleration is given by the same expression as before except with the charge and mass of the particles exchanged:

$$\frac{dW}{d\ell} = n_2 \frac{q_2^4 q_1^2}{(4\pi\epsilon_0)^3} \frac{4\pi}{3c^3} \frac{1}{m_2 \hbar} \quad . \tag{7.7}$$

Second, concerning targets, there are two effects that tend to cause the nuclei to dominate as targets in producing bremsstrahlung. The first effect is plain in eq (7.6). It is that the radiation is proportional to $q_2^2 \propto Z^2$, which for heavy atoms is a factor Z larger than the increase in the radiation caused by there being Z electrons per atom. The second effect that causes electron-electron collisions to be inefficient in producing bremsstrahlung is that the radiated electric fields caused by accelerations of the projectile electron and the target electron cancel each other.

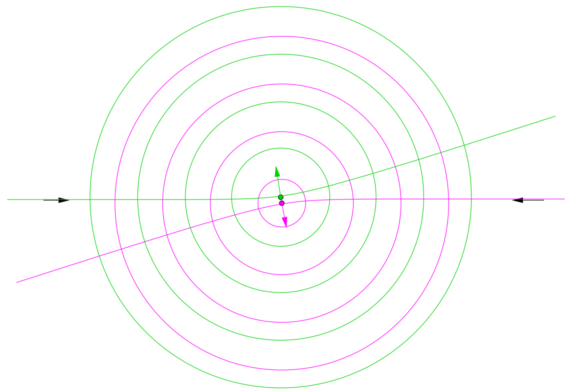

Figure 7.1: Electron-electron bremsstrahlung radiation wavefronts are out of phase and interfere destructively when the collision is close compared with the wavelength.

In electron-electron collisions, the accelerations of the projectile and the target are equal and opposite. Therefore they tend to give rise to equal and opposite radiated electric fields, which need to be coherently added together to obtain the total field. The far fields will actually cancel provided that there is only a small phase difference arising from the difference in the exact positions of the projectile and target electrons. That phase difference is roughly kb, where k is the relevant wave-number of the emitted radiation, and b is the impact parameter. However, the characteristic wave-number is given by

$$k = \frac{\omega}{c} \approx \frac{1}{c\tau} \approx \frac{v_0}{cb} \quad . \tag{7.8}$$

Therefore $kb \approx v_0/c$, in other words, the contributions from the projectile and target will cancel because $kb \ll 1$ if the incoming velocity is substantially less than the speed of light. Electron-electron bremsstrahlung is important only for relativistic electrons. Notice, though, that electron-positron bremsstrahlung does *not* produce this field cancellation, so it can be significant even in the non-relativistic case.

For the predominant case of electron-nucleus bremsstrahlung we can write eq (7.6) using the definitions of the fine structure constant $\alpha = e^2/4\pi\epsilon_0\hbar c$ and the classical electron radius $r_e = e^2/4\pi\epsilon_0 m_e c^2$ as

$$\frac{dW}{d\ell} = n_2 Z^2 m_e c^2 \alpha \frac{4\pi}{3} r_e^2 \quad . \tag{7.9}$$

7.1.3 Comparison of Bremsstrahlung and Collisional Energy Loss

The question now arises of the relative importance of bremsstrahlung in calculating the energy loss of an energetic particle in matter. This is determined by the ratio of the radiated energy per unit length, eq(7.6), to the collisional energy loss, eq(6.45). For non-relativistic bremstrahlung from collisions with nuclei, so that $n_2 = n_a$, this ratio is

$$\left|\frac{dW}{dK}\right| = Z_1^2 Z_a \alpha \frac{m_e v_0^2}{m_1 c^2} \frac{1}{3 \ln \Lambda} \quad , \tag{7.10}$$

using the definition of the fine structure constant, α, and denoting the atomic number of the nuclei as Z_a.

We see immediately, that bremsstrahlung in non-relativistic collisions is *never* an important contributor to the total energy loss, because even for electron collisions with the heaviest elements, $Z_2\alpha \sim 92/137 \approx 0.67$ and dW/dK is much smaller than one because of the factors v_0^2/c^2 and $1/3\ln\Lambda$.

If the projectile is a heavy particle, then the radiation from nuclear collisions is totally negligible, because of the mass ratio. One might be concerned then about radiation arising from the acceleration of the atomic electrons by the passing heavy particle. However, this can never exceed the energy transferred to the electrons in the collision, since the acceleration transfers the collisional energy as well as giving rise to radiation. Formally taking the ratio of eq(7.7) to the collisional loss we get the same expression as eq(7.10) except with m_1 replaced by the electron mass, thus confirming that bremsstrahlung is negligible in non-relativistic energy loss of heavy particles as well as electrons.

We shall see, nevertheless, that electron-nucleus bremsstrahlung can become important for relativistic electrons.

7.1.4 Spectral Distribution

We may want to calculate the spectrum of the electromagnetic radiation. It arises as a result of the impulse shape. For a single collision, the radiation's frequency spectrum will reflect the frequency spectrum of the impulse. An infinitely sharp impulse has a uniform frequency spectrum out to infinite frequency. This accelerating impulse has a duration $\tau = b/v_0$, and consequently has an approximately uniform spectrum only out to an angular frequency $\omega \approx 1/\tau$.

Return therefore to the expression (7.3) for the radiated energy in a single collision with impact parameter b. This energy is spread over a total spectral width of approximately $1/\tau$ so the energy spectral power density is

$$\frac{dW}{d\omega} \approx W\tau = \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{2}{3c^3} \frac{1}{m_1^2 v_0^2 b^2} \quad . \tag{7.11}$$

This is the energy spectrum radiated in a single collision of specified impact parameter. If we want to obtain the energy radiated per unit length, then as usual, we need to multiply by the target density and integrate $2\pi b db$ over all impact parameters, which gives a logarithmic dependence:

$$\frac{d^2W}{d\ell d\omega} = n_2 \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{4\pi}{3c^3} \frac{1}{m_1^2 v_0^2} \ln \left| \frac{b_{max}}{b_{min}} \right| \quad . \tag{7.12}$$

The b_{min} will arise because of the wave nature of the projectile, provided that the corresponding $b_{min} = \hbar/m_1 v_0$ is greater than b_{90}. For any fixed value of the photon frequency, ω, the maximum impact parameter at which this formula is appropriate is that parameter for which $\tau = b/v_0 \approx 1/\omega$, since, as we have already discussed, for larger values of b the power spectrum falls off rapidly by virtue of the Fourier spectrum of the time variation of the electric field. Thus

$$\Lambda = \frac{b_{max}}{b_{min}} \approx \frac{2m_1 v_0^2}{\hbar\omega} \quad ,$$

(and the extra factor of 2 is a fudge to make the numerical value come out right). Actually, since some energy and momentum is carried away by the photon radiated, the speed is not simply v_0 both before and after the collision. We could recognize that fact by substituting the average value of the velocity $\frac{1}{2}\{v_0 + \sqrt{[2(K - \hbar\omega)/m_1]}\}$ instead of v_0 in this logarithmic argument, where K is the initial kinetic energy. If we do so, and also replace (arbitrarily, but then we have only done an estimate, not a proper calculation) the factor 4π with 16 in the coefficient we obtain:

$$\frac{d^2W}{d\ell d\omega} = n_2 \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{16}{3c^3} \frac{1}{m_1^2 v_0^2} \ln \left| \frac{(\sqrt{K} + \sqrt{K - \hbar\omega})^2}{\hbar\omega} \right| \quad . \tag{7.13}$$

This expression is precisely what is obtained by a non-relativistic quantum mechanical calculation based on the Born approximation, first performed by Bethe and Heitler, 1934.

7.1.5 Bremsstrahlung from Relativistic Electrons

It is not straightforward to obtain estimates for bremsstrahlung from relativistic electrons. A key reason is that since the photon energy emitted extends from zero up to the electron's incident energy, we have to deal with photons having energies comparable to the electron rest mass or more, and hence carrying away momentum that is critical in the scattering process. One way to think about this process is to regard bremsstrahlung as the scattering of "Virtual Photons" associated with the field of the nucleus.

This approach, also known as the Weizsäcker-Williams method, after its earliest proponents, considers bremsstrahlung in the frame of reference in which the electron is stationary, and the ion moves past the electron. The electron feels a time-varying electric field of the ion, whose spectrum we have already discussed in the context of collisional energy transfer and the oscillator strength. This time-varying field (at least for velocities near the speed of light) can be approximated as a spectrum of plane waves. These are the virtual quanta.

The virtual quanta encounter the (initially) stationary electron. They can then be scattered by it, by the process of Compton scattering. In just the same way as photon momentum alters the Compton scattering process relative to the non-relativistic Thomson scattering, the bremsstrahlung is affected by the photon momentum and electron recoil. Because the total Compton scattering cross-section falls off inversely proportional to the photon energy

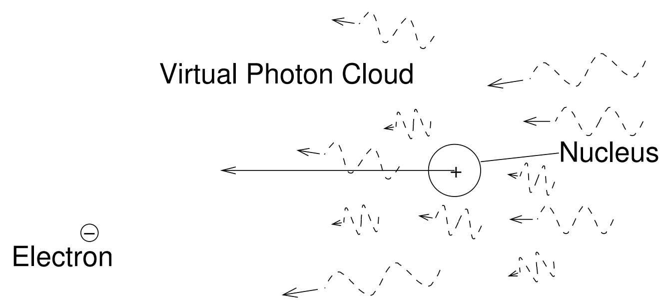

Figure 7.2: In the rest frame of the electron, the electric field of the nucleus is regarded as a "cloud" of virtual photons with a spectrum of energies, which scatter from the electron.

for energetic photons, [and in fact $\sigma_c \approx \sigma_T(3/4)(m_e c^2)/(\hbar\omega)$ for $m_e c^2 \ll \hbar\omega$, Jackson p 697], in this frame of reference, the scattered virtual photons (bremsstrahlung photons) are predominantly such that $\hbar\omega < m_e c^2$.

Recall that the virtual photon energy spectral density is essentially independent of the velocity [see section on straight-line collisions].

For relativistic velocities, the rate of scattering of photons of all energies is thus roughly constant, independent of collision energy, because it consists of a constant rate (of order the Thomson cross-section) of scattering of a constant photon spectral density up to a constant spectral limit ($m_e c^2$). The relativistic Doppler effect upshifts the majority of these photons in the laboratory frame to much higher energies, producing a spectrum extending up to $\gamma m_e c^2$, the electron energy in the lab frame. As illustrated schematically in figure 7.3. The *energy* loss rate is thus approximately proportional to the collision energy, because it consists of a constant rate of photon scattering but with energies on average proportional to the collision energy, $\gamma m_e c^2$.

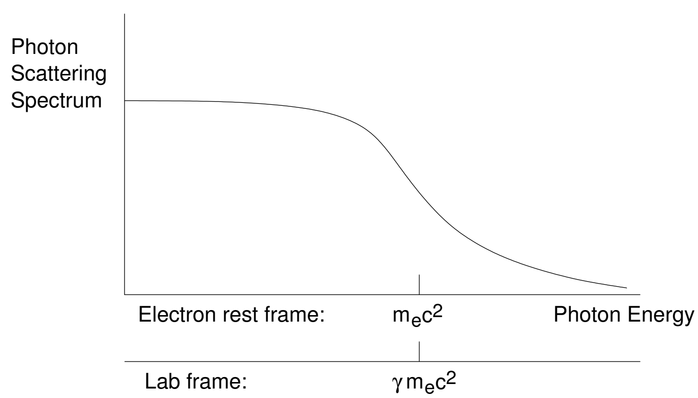

Figure 7.3: Photon scattering spectrum in the lab frame can be thought of approximately as a spectum flat to $m_e c^2$ in the electron rest frame, Doppler upshifted to $\gamma m_e c^2$ in the lab frame.

This qualitative argument indicates that we should expect the bremsstrahlung energy-loss spectrum in the bulk of the relevant photon energy range to have a value given approximately by the same formula as the non-relativistic case (7.13), although with a different value of the logarithmic factor. The full relativistic formula can be written [Jackson eq 15.34]

$$\frac{d^2W}{d\ell d(\hbar\omega)} \approx n_2 \frac{q_1^4 q_2^2}{(4\pi\epsilon_0)^3} \frac{16}{3c^3\hbar} \frac{1}{m_1^2 v_0^2} \ln \left| \frac{2\gamma\gamma' m_e c^2}{\hbar\omega} \right| = n_2 Z_1^4 Z_2^2 \alpha r_e^2 \left(\frac{m_e}{m_1}\right) \frac{16}{3} \left(\frac{m_e c^2}{m_1 v_0^2}\right) \ln\Lambda \quad , \tag{7.14}$$

where $\gamma' = \gamma - \hbar\omega/m_e c^2$ is the relativistic gamma factor of the electron after the photon has been emitted, and $v_0 \approx c$, since this is a relativistic collision.

It is possible to write a universal expression for the photon energy spectrum per unit length, applicable for all energies. The quantum-mechanical Born-approximation calculations for electron projectiles yields this expression in the form [Heitler p 250]

$$\frac{d^2W}{d\ell d(\hbar\omega)} = n_2 Z_2^2 \alpha r_e^2 \frac{\gamma}{\gamma - 1} B \quad , \tag{7.15}$$

where B is a dimensionless function of the ratio $\hbar\omega/m_e c^2(\gamma - 1)$, that replaces the factor $(16/3)\ln\Lambda$. It is dependent on collision energy (i.e. $(\gamma - 1)m_e c^2$) and photon energy, but weakly so. The form of this function B is shown in figure 7.4. It has a value of order 15 over most of the photon spectrum. One can readily verify that this expression has the correct scaling with velocity at both low and high electron energy.

Figure 7.4: The spectral shape of the bremsstrahlung spectrum (from Heitler).

The magnitude of the cross-section is given by the term

$$\alpha r_e^2 = 0.580 \times 10^{-31}\ \mathrm{m}^{-2} = 0.580\ \text{millibarn} \quad . \tag{7.16}$$

(A barn is 10^{-28} m^{-2}).

7.1.6 Screening and Total radiative loss

We need to account for the screening of the nuclear potential by surrounding electrons of the atom when the collisions are distant.

The "Thomas-Fermi" potential is an approximation to the screened nuclear potential that can be approximated as

$$\phi = \frac{Ze}{4\pi\epsilon_0 r}\exp(-r/a) \quad , \tag{7.17}$$

with the characteristic length $a \approx 1.4 a_0 Z^{-1/3}$. This form of screening is identical to what applies to Coulomb collisional energy loss etc.

It is most important at low photon energy (relative to the incident energy) because the distant collisions are most effective there. It reduces the cross-section (or power radiated) because it essentially lowers the maximum effective impact parameter to $\sim a$. The solid curves on the figure are the screened estimates (for lead). The dotted curves are the unscreened. Clearly there is a big difference.

Estimates of the screening effect can be obtained by putting b_{max} equal to a instead of v_0/ω, resulting in a logarithmic factor that for non-relativistic collisions is

$$\Lambda = \frac{b_{max}}{b_{min}} \approx \frac{m_1 v_0 a}{\hbar} = \frac{m_1 v_0 1.4 a_0 Z^{-1/3}}{\hbar} = \left(\frac{1.4\beta}{\alpha Z^{1/3}}\frac{m_1}{m_e}\right) \quad , \tag{7.18}$$

where the final form follows from

$$\frac{\hbar}{a_0} = m_e c \alpha \tag{7.19}$$

Actually screening effects are most important not for non-relativistic collisions but for relativistic collisions. For relativistic collisions, we replace the characterisic maximum impact parameter $2\gamma\gamma' c/\omega$ with a if a is smaller, so that screening is important. It will be if

$$\left(\frac{\omega}{2\gamma^2 c}\right)\left(\frac{1.4 a_0}{Z^{1/3}}\right) < 1 \tag{7.20}$$

This inequality will apply over the entire frequency range up to the maximum possible photon energy $\hbar\omega = \gamma m_1 c^2$ if the incident energy satisfies:

$$\frac{m_1 c^2}{2\gamma c\hbar}\frac{1.4 a_0}{Z^{1/3}} = \frac{0.7}{\alpha\gamma Z^{1/3}}\frac{m_1}{m_e} < 1 \quad , \tag{7.21}$$

using eq (7.19) again. This criterion is $\gamma > 196/Z^{1/3}$ for electron projectiles. When it is satisfied, the collisions are said to be in the range of "complete screening", and the logarithmic factor becomes $\ln\Lambda \approx \ln(233/Z^{1/3})$. [Jackson p722, although our calculation would make it $\ln(192/Z^{1/3})$].

For non-relativistic electrons, the radiative energy loss is always negligible compared with the collisional loss. This is not the case for strongly relativistic electrons because the total bremsstrahlung power loss, for the roughly constant spectral power, is proportional to the total width of the spectrum, i.e. to the collision energy.

Taking the completely screened cross-section case, in which the logarithmic term and $\gamma/(\gamma-1)$ are approximately constant, the total spectrally integrated energy loss rate is given by

$$\frac{dW}{d\ell} = \int \frac{d^2W}{d\ell d(\hbar\omega)} d(\hbar\omega) = n_2 Z_2^2 \alpha r_e^2 \left(\frac{16}{3}\right) \ln\left|\frac{233}{Z^{1/3}}\right| \gamma m_e c^2 \tag{7.22}$$

So writing $K = \gamma m_e c^2$ for the total electron energy, we get a slowing down equation

$$-\frac{dK}{d\ell} = K n_2 Z_2^2 \alpha r_e^2 \left(\frac{16}{3}\right) \ln\left|\frac{233}{Z^{1/3}}\right| \tag{7.23}$$

If we compare this to the slowing down rate due to collisional effects (excluding bremsstrahlung) we find that these rates, whose dependence on the nuclear charge, Z are different, are equal when $\gamma \approx 200$ for air and $\gamma \approx 20$ for lead.

When bremsstrahlung loss predominates over collisional loss, the energy is sufficient for the screening to be complete. Then the slowing down rate is constant. That is, the energy loss equation reduces approximately to

$$-\frac{dK}{d\ell} = K/\lambda \tag{7.24}$$

with exponentially decaying solutions $K \propto \exp(-\ell/\lambda)$ having characteristic length:

$$\lambda = \left[n_2 Z_2^2 \alpha r_e^2 \left(\frac{16}{3}\right) \ln\left|\frac{233}{Z^{1/3}}\right| \right]^{-1} \tag{7.25}$$

The expressions most quoted are slightly different [Heitler, and subsequently Evans] replacing as follows in the completely screened limit:

$$\left(\frac{16}{3}\right) \ln\left|\frac{233}{Z^{1/3}}\right| \rightarrow 4\ln\left|\frac{183}{Z^{1/3}}\right| + \frac{2}{9} = B \tag{7.26}$$

although the difference is small, within uncertainties of the whole approximate approach.

7.1.7 Thick target Bremsstrahlung.

Remarks not typed up.

7.2 Čerenkov Radiation

Maxwell's equations with a dielectric medium:

$$\nabla \wedge \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t} \quad , \quad \nabla \wedge \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{7.27}$$

The current consists of partly the medium polarization

$$\mathbf{j}_{\text{medium}} = \epsilon_0 \frac{\partial \mathbf{P}}{\partial t} \tag{7.28}$$

and partly "external" currents, $\mathbf{j}_x$, like the particle moving through it. We combine the polarization current into the $\frac{\partial \mathbf{E}}{\partial t}$ term as

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j}_x + \mu_0 \frac{\partial \mathbf{D}}{\partial t} = \mu_0 \mathbf{j}_x + \frac{1}{c^2} \frac{\partial}{\partial t} (\epsilon \mathbf{E}) \tag{7.29}$$

where ϵ = dielectric constant = relative permittivity. Eliminate $\mathbf{B}$:

$$- \nabla \wedge (\nabla \wedge \mathbf{B}) = \mu_0 \frac{\partial \mathbf{j}_x}{\partial t} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} (\epsilon \mathbf{E}) \tag{7.30}$$

or

$$- \nabla (\nabla . \mathbf{E}) + \nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} (\epsilon \mathbf{E}) = \mu_0 \frac{\partial \mathbf{j}_x}{\partial t} \tag{7.31}$$

This is now a wave-equation but with a source on the right-hand side. A helpful way to think about Cerenkov radiation is then to regard the current of the swift particle as coupling to oscillators consisting of plane waves propagating with wave-vector $\mathbf{k}$ and frequency ω. Because of the dielectric medium the wave "oscillators" satisfy $k^2 c^2 = \epsilon \omega^2$. This is the standard result that the refractive index of a wave in a dielectric is

$$\frac{kc}{\omega} = \epsilon^{1/2} = N \tag{7.32}$$

The wave velocity is $\frac{\omega}{k} = c/\epsilon^{1/2}$, i.e. the waves travel slower than c. This allows the particle to couple to the oscillators <u>resonantly</u>. We saw previously (oscillator strength calculation) that it is resonance that is required [$|E(\omega)|^2$ is what gives the energy transfer]. For resonance with a wave $\propto \exp i(\mathbf{k}.\mathbf{x} - \omega t)$, we require a uniformly moving particle (i.e. one without an intrinsic oscillating frequency) to move such that the phase of the wave is constant at the particle. Particle position is $\mathbf{r} = \mathbf{v}t$ (+ constant) so resonance is

$$\text{constant} = \mathbf{k}.\mathbf{r} - \omega t = (\mathbf{k}.\mathbf{v} - \omega)\, t \tag{7.33}$$

i.e. $\mathbf{k}.\mathbf{v} = \omega$. So if we choose a specific frequency ω, we need to satisfy simultaneously

1. $k = \frac{\omega}{c} \epsilon^{1/2}$ (wave dispersion relation)
2. $\mathbf{k}.\mathbf{v} = \omega$ (resonance with particle)

Graphically: For a particle moving faster than wave phase velocity ($v > c/\epsilon^{1/2}$) solutions exist because $\frac{\omega}{v} < \frac{\omega}{c} \epsilon^{1/2}$, otherwise <u>not</u>. Cerenkov radiation requires "superluminary" velocity. Also angle between $\mathbf{k}$ and $\mathbf{v}$ is simply given by

$$\cos \theta = \frac{\omega}{kv} = \frac{c}{v \epsilon^{1/2}} \tag{7.34}$$

If ϵ is independent of ω, the result is to form an optical "shock front" All electromagnetic

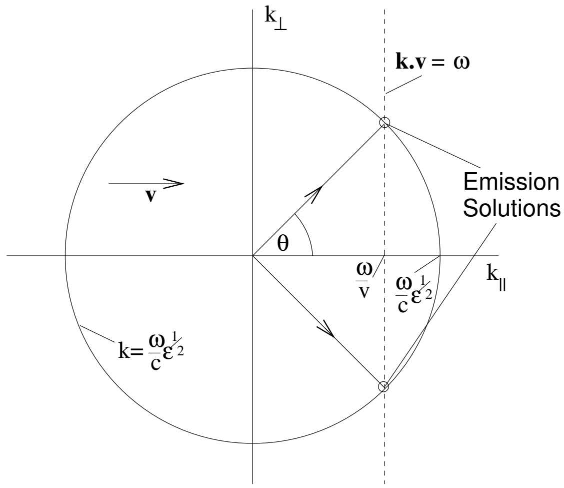

Figure 7.5: k-coordinates for satisfying resonance and the dispersion relation

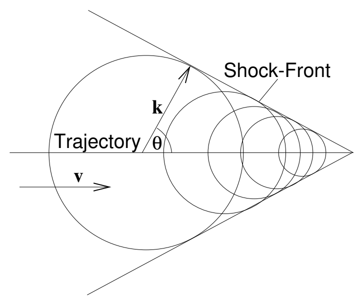

Figure 7.6: Shock-Front arising from coherent addition of waves from all along the particle trajectory.

wave fronts add coherently along the shock front, leading to a singularity. Actually if ϵ is $> \frac{c^2}{v^2}$ for all frequencies, then an infinite amount of energy is radiated per unit length. This is a reflection of the singularity at the shock front. The variation of ϵ with frequency is crucial for proper treatment of Cerenkov emission. Optical materials have refractive index that does vary with frequency (prism splits spectrum of white light). Resonance in atoms of medium is usually in UV. Radiation can occur for all frequencies up to the resonance (different resonance from Cerenkov) and down to the place where $\frac{v}{c} = \frac{1}{\epsilon^{1/2}}$. Variation $\epsilon(\omega)$ removes the singularity, gives a spectral variation and finite spectral range.

7.2.1 Coupling Strength

We are interested in transverse waves

$$\mathbf{k}.\mathbf{E} = 0 \quad (\Rightarrow \nabla.\mathbf{E} = 0) \tag{7.35}$$

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\epsilon\mathbf{E}) = -k^2\mathbf{E} + \frac{\omega^2\epsilon}{c^2}\mathbf{E} = \mu_0\frac{\partial \mathbf{j}_x}{\partial t} \tag{7.36}$$

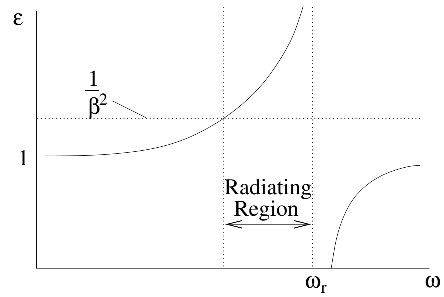

Figure 7.7: Typical variation of the relative permittivity of a transparent material.

$\mathbf{E}$ is perpendicular to $\mathbf{k}$. If $\mathbf{v}$ is in x-direction, then, $\mathbf{k} = k(\cos\theta, \sin\theta)$

$$\mathbf{E} = E(\sin\theta, \cos\theta, 0) \quad \text{or} \tag{7.37}$$
$$= E(0, \quad 0, \quad 1) \quad \text{are possible polarizations} \tag{7.38}$$

But coupling to the wave is determined by the vector $\frac{\partial \mathbf{j}_x}{\partial t}$. For moving point particle

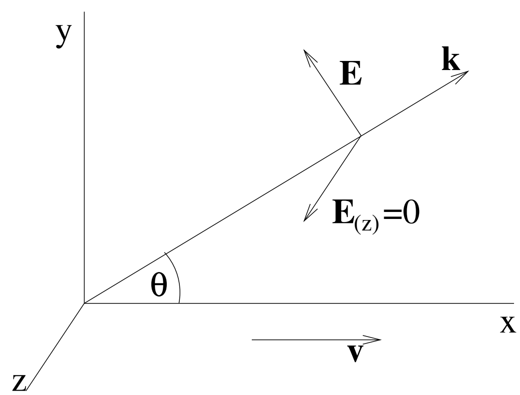

Figure 7.8: Polarization of Čerenkov emission is purely in the plane of emission. Coupling to $\mathbf{E}_{(z)}$ is zero.

$\mathbf{j}_x = q\mathbf{v}\delta(\mathbf{x} - \mathbf{v}t)$ which has only x-component. Hence

1. It does not couple at all to E_z polarization.
2. Coupling to the in-plane polarization, $\mathbf{E} = \mathbf{E}(\sin\theta, \cos\theta)$ is proportional to $\frac{\mathbf{E.j}}{Ej}$ i.e. $\sin\theta$

Final point note that driver is $\frac{\partial \mathbf{j}_x}{\partial t}$ so that since the spectrum of j_x is flat, because it is a delta function in time, the spectrum of the drive term is $\propto (i)\omega$. All of this can be made rigorous. The result is that the energy radiated per unit length of path is

$$\frac{dW}{d\ell} = \frac{q_1^2}{4\pi\epsilon_0} \frac{1}{c^2} \int_{\epsilon(\omega) > \frac{1}{\beta^2}} \left(1 - \frac{c^2}{v^2 \epsilon(\omega)}\right) \omega \, d\omega \quad \text{[Frank, Tamm 1937]} \tag{7.39}$$

and we can identify the terms as

$$1 - \frac{c^2}{v^2\epsilon} = 1 - \cos^2\theta = \sin^2\theta \tag{7.40}$$

i.e. the coupling dependence on radiation angle. Squared because energy goes like the square of the electric field.

$$\omega \text{ arising from } \quad \frac{\partial}{\partial t} j_x. \tag{7.41}$$

This equation also gives the frequency spectrum of the radiated power (the integrand) but it is non-zero only for frequencies such that $\epsilon > \frac{c^2}{v^2}$ or $v >$ phase velocity $\frac{c}{\epsilon^{1/2}}$. Energy emitted per unit length is estimated by putting $1 - \frac{c^2}{v^2\epsilon}$ equal to an average value and so

$$\int_{\omega_1}^{\omega_2} \left(1 - \frac{c^2}{v^2\epsilon}\right) \omega \; d\omega \simeq \frac{1}{2}\left[\omega_2^2 - \omega_1^2\right]\left(1 - \frac{c^2}{v^2\bar{\epsilon}}\right) \tag{7.42}$$

where $\omega_{2,1}$ are the upper and lower limits of spectral region of emission.

$$\begin{aligned}
\frac{dW}{d\ell} &= \frac{q_1^2}{4\pi\epsilon_0}\frac{1}{c^2}\frac{1}{2}\left[\omega_2^2 - \omega_1^2\right]\left(1 - \frac{c^2}{v^2\bar{\epsilon}}\right) \\
&= \alpha\frac{\omega_2}{c}\frac{1}{2}\left[\hbar\omega_2 - \frac{\hbar\omega_1^2}{\omega_2}\right]\left(1 - \frac{c^2}{v^2\epsilon}\right) \\
&\simeq \alpha\frac{\omega_2}{c}\frac{1}{2}\hbar\omega_2\left(1 - \frac{c^2}{v^2\bar{\epsilon}}\right) \quad \text{if } \omega_1 << \omega_2. \\
&= \alpha\frac{\pi}{\lambda_2}\hbar\omega_2\left(1 - \frac{c^2}{v^2\bar{\epsilon}}\right) \qquad \left(\frac{\lambda}{2\pi} = \frac{c}{\omega}\right)
\end{aligned} \tag{7.43}$$

The rough value of this energy per unit length can be estimated noting that the resonance (where $\bar{\epsilon} \to \infty$) in the optical response of glasses is generally near $\lambda_2 = 100$ nm $\Rightarrow \hbar\omega_2 = 12eV$, and near that resonance $\frac{c^2}{v^2\bar{\epsilon}} \to 0$ so

$$\frac{dW}{d\ell} \sim \alpha\pi\frac{1}{10^{-7}}.12 \text{ eV/m} = 2.7 \times 10^6 \text{eV/m} \tag{7.44}$$

This is tiny in comparison with the rate of loss of energy by other processes. The number of photons emitted per unit length is even easier

$$\frac{dN}{d\ell} \simeq \frac{q_1^2}{4\pi\epsilon_0}\frac{1}{\hbar c^2}\left[\omega_2 - \omega_1\right]\left[1 - \frac{c^2}{v^2\bar{\epsilon}}\right] \tag{7.45}$$

$$= \alpha 2\pi\left[\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right]\left[1 - \frac{c^2}{v^2\bar{\epsilon}}\right] \tag{7.46}$$

$$\simeq \alpha 2\pi\frac{1}{\lambda_2} \qquad (\text{for } \omega_2 >> \omega_1). \tag{7.47}$$

[And the photon spectral distribution is

$$\frac{dN}{d\ell d\omega} = \alpha\frac{1}{c}\sin^2\theta = \alpha\frac{1}{c}\left[1 - \frac{c^2}{v^2\epsilon}\right]. \quad] \tag{7.48}$$

Rough estimate of photons (total) / length for $\lambda_2 \simeq 100$ nm:

$$\frac{dN}{d\ell} \simeq 5 \times 10^5 \text{ photons/m} \tag{7.49}$$

Optical range ($\lambda \simeq 400 - 600$ nm) contains about $(\frac{1}{4} - \frac{1}{6}) = 0.083$ times as many $(\times \sin^2 \theta)$ so can be as little as $\frac{1}{100}$ of this total ~ 5 photons/mm.

7.2.2 Energy Spectrum

$$\text{Energy Spectrum proportional to} \qquad \omega \left(1 - \frac{c^2}{v^2 \epsilon}\right) \tag{7.50}$$

is

1. broad and smooth.
2. larger at larger ω (smaller λ) "blue" because
   (a) ω factor
   (b) ϵ increase with $\omega \Rightarrow 1 - \frac{c^2}{v^2 \epsilon}$ increases

Result: Bluish-White light. Observed by Marie Curie 1910. Studied in detail by Cerenkov 1935. Explained Frank & Tamm 1937 (classical). Used for detectors starting mid 1940s.