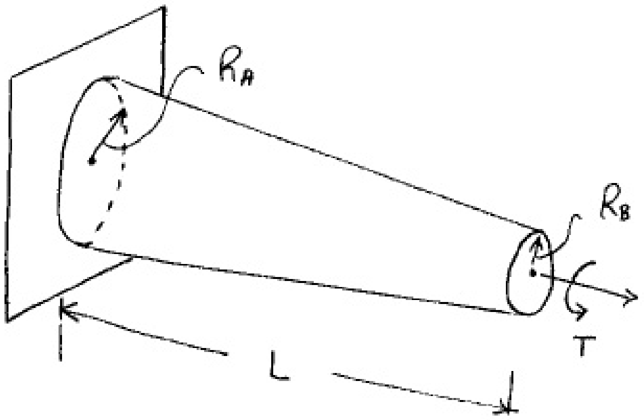
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering

2.001 - Mechanics of Materials I
Spring, 2003

Solutions for Final Exam

Problem 1



$$R = R(x) = R_A + \left(\frac{R_B - R_A}{L}\right)x$$

$$I_p = \frac{\pi}{2}R^4 = \frac{\pi}{2}\left[R_A + \left(\frac{R_B - R_A}{L}\right)x\right]^4$$

ϕ = angle of twist

$$\frac{d\phi}{dx} = \frac{\gamma}{r} \quad \text{where } \gamma = \frac{\sigma_{xy}}{G}$$

$$\sigma_{xy} = \frac{T \cdot r}{I_p}$$

$$\Rightarrow \frac{d\phi}{dx} = \frac{T}{GI_p}$$

$$\phi = \int \frac{T}{GI_p}\,dx$$

at $x = L$ $\phi = \Phi$ = twist at the end

$$\Rightarrow \Phi = \int_0^L \frac{T}{G\frac{\pi}{2}\left[R_A + \frac{R_B - R_A}{L}x\right]^4}\,dx$$

$$\Phi = \frac{2T}{\pi G}\int_0^L \frac{dx}{\left(R_A + \frac{R_B - R_A}{L}x\right)^4}$$

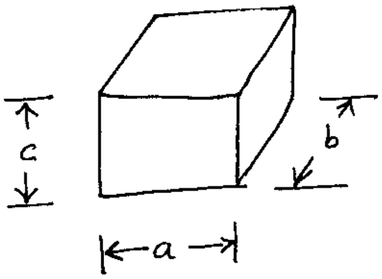
$$= \frac{2T}{G\pi} \cdot \frac{L}{3(R_B - R_A)}\left[\frac{-1}{R_B^3} + \frac{1}{R_A^3}\right]$$

$$= \frac{2TL}{G\pi \cdot 3(R_B - R_A)}\left[\frac{R_B^3 - R_A^3}{R_A^3 \cdot R_B^3}\right]$$

$$K_\theta = \frac{T}{\Phi}$$

$$= \frac{3G\pi(R_B - R_A)R_A^3 \cdot R_B^3}{2L(R_B^3 - R_A^3)}.$$

Problem 2



$$\sigma_x = \sigma_y = \sigma_z = -p$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_x = \frac{1}{E}\left[-p + 2\nu p\right]$$

$$= \frac{-p(1-2\nu)}{E} = \varepsilon_y = \varepsilon_z = \varepsilon$$

Initial Volume $V_i = abc$.

Final volume $V_f = (a+\Delta a)(b+\Delta b)(c+\Delta c)$

$$\Delta a = \varepsilon a \implies a + \Delta a = a(1+\varepsilon)$$

$$\Delta b = \varepsilon b \implies b + \Delta b = b(1+\varepsilon)$$

$$\Delta c = \varepsilon c \implies c + \Delta c = c(1+\varepsilon).$$

$$\implies V_f = a(1+\varepsilon)\cdot b(1+\varepsilon)\cdot c(1+\varepsilon)$$

$$= abc(1+\varepsilon)^3$$

$$= abc(1+\varepsilon)(1+\varepsilon^2+2\varepsilon)$$

$$= abc(1+\cancel{\varepsilon^2}+2\varepsilon+\varepsilon+\cancel{\varepsilon^3}+\cancel{2\varepsilon^2})$$ drop higher order terms since $\varepsilon \ll 1$

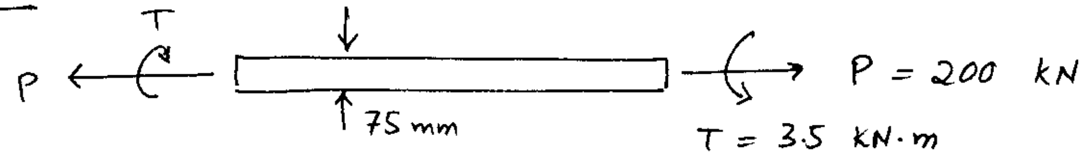
$$V_f = abc(1+3\varepsilon)$$

$$\implies \frac{\Delta V}{V} = \frac{V_f - V_i}{V_i} = \frac{abc(1+3\varepsilon) - abc}{abc}$$

$$= \frac{\cancel{abc}[\cancel{1}+3\varepsilon-\cancel{1}]}{\cancel{abc}} = 3\varepsilon$$

$$\frac{\Delta V}{V} = -\frac{3p(1-2\nu)}{E}.$$

Problem 3



$P = 200 \times 10^3$ N

$T = 3.5 \times 10^6$ N·mm

$A = \frac{\pi}{4} d^2 = \frac{\pi}{4}(75)^2$

$= 4418$ mm^2

$I_p = \frac{\pi}{32} d^4 = \frac{\pi}{32}(75)^4$

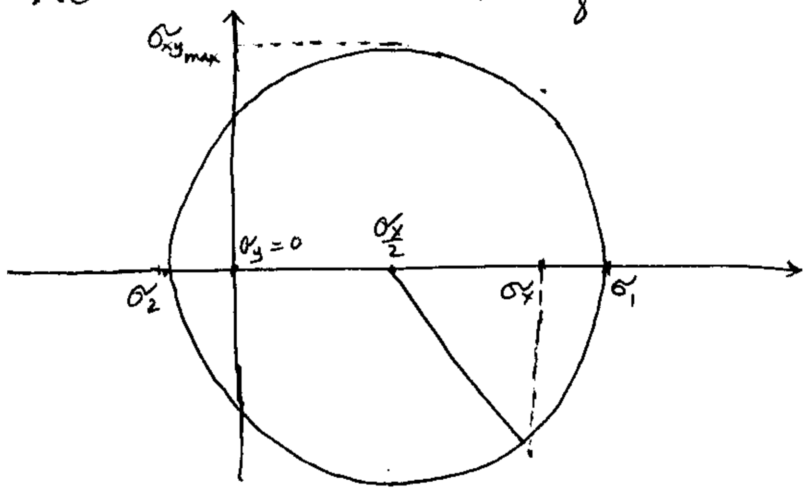
$= 3.106 \times 10^6$ mm^4

$$\sigma_x = \frac{P}{A} = \frac{200 \times 10^3}{4418}$$

$$= 45.27 \text{ N/mm}^2$$

$$\sigma_{xy\,r=37.5} = \frac{T \times 37.5}{I_p} = \frac{3.5 \times 10^6 \times 37.5}{3.106 \times 10^6} = 42.26 \text{ N/mm}^2$$

We now make use of the Mohr's Circle.



R = Radius of the circle

$$= \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \sigma_{xy}^2}$$

$$= \sqrt{\left(\frac{45.27}{2}\right)^2 + 42.26^2}$$

$$= 47.94.$$

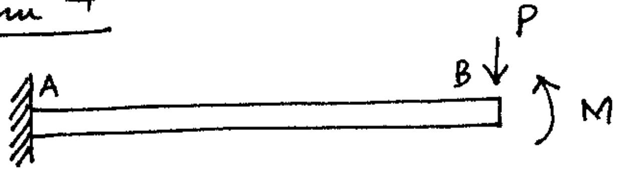
$$\sigma_1 = \frac{\sigma_x}{2} + R = \frac{45.27}{2} + 47.94 = 70.58 \text{ N/mm}^2$$

$$\sigma_2 = \frac{\sigma_x}{2} - R = \frac{45.27}{2} - 47.94 = -25.31 \text{ N/mm}^2.$$

$$\sigma_{xy\,max} = R = 47.94 \text{ N/mm}^2$$

Problem 4

a)

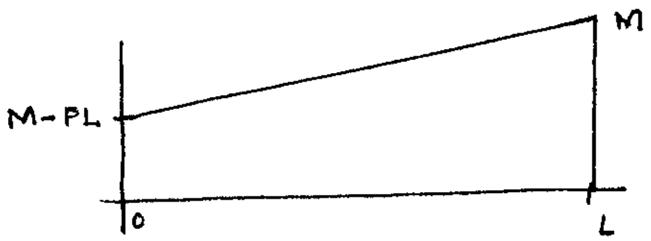


$$R_y^A - P = 0 \implies R_y^A = P$$

$$^{A}M + P.L - M = 0 \implies {}^{A}M = M - PL.$$

$$M(x) = {}^{A}M + R_y^A \cdot x.$$

$$= M - PL + Px = M - P(L-x)$$



b) Calculate the elastic curve

$$EI\,u'' = M(x) = M - PL + Px.$$

$$EI\,u' = Mx - PLx + \frac{Px^2}{2} + C_1$$

$$u'\Big|_{x=0} = 0 \implies C_1 = 0$$

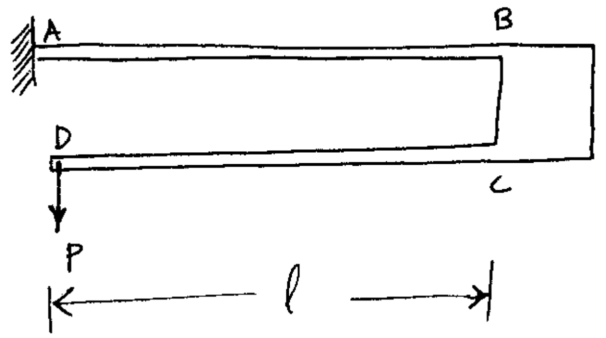
$$EI\,u = \frac{Mx^2}{2} - \frac{PLx^2}{2} + \frac{Px^3}{6} + C_2$$

$$u\Big|_{x=0} = 0 \implies C_2 = 0.$$

$$\implies \quad u = \frac{1}{EI}\left[\frac{Mx^2}{2} - \frac{PLx^2}{2} + \frac{Px^3}{6}\right].$$

$$u' = \frac{1}{EI}\left[Mx - PLx + \frac{Px^2}{2}\right]$$

Problem 5 Calculate the deflection at point D.



At Point B $\quad M = P.l$.

$$u_B = \frac{1}{EI}\left[\frac{Pl^3}{2} - \frac{Pl^3}{2} + \frac{Pl^3}{6}\right]$$

$$= \frac{Pl^3}{6EI} \quad \text{upwards !!}$$

$$u_B' = \frac{1}{EI}\left[Pl^2 - Pl^2 + \frac{Pl^2}{2}\right]$$

$$= \frac{Pl^2}{2EI}$$

Displacement of point D.

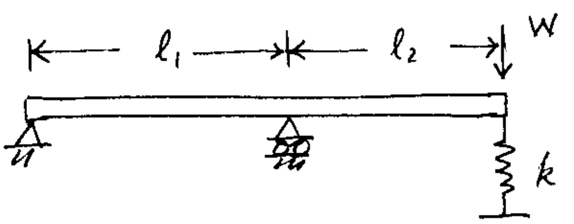
$$u_D = u_B - u_B' l - \frac{Pl^3}{3EI}$$

$$= \frac{Pl^3}{6EI} - \frac{Pl^3}{2EI} - \frac{Pl^3}{3EI}$$

$$= \frac{Pl^3 - 3Pl^3 - 2Pl^3}{6EI}$$

$$= -\frac{2}{3}\frac{Pl^3}{EI}$$

Problem 6



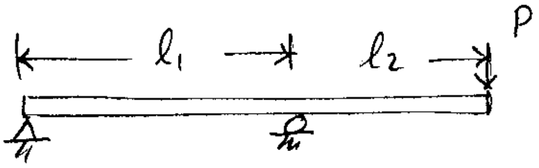
Calculate deflection of point c.

Let δ = deflection at point c.

F = Force in spring $= k\delta$ (acting upwards)

W = Weight acting downwards.

Let $P = W - F = W - k\delta$.

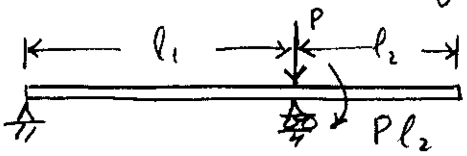


$$\delta = \frac{Pl_2^3}{EI} - \theta_B l_2$$

$$\begin{pmatrix} \text{clockwise } \theta = -ve \\ \text{anticlockwise } \theta = +ve \end{pmatrix}.$$

To calculate θ_B

consider the following beam, which equivalent for calculating response in l_1 only.



$$\theta_B = -\frac{Ml_1}{3EI} \quad \text{from table.}$$

$$= -\frac{Pl_2 l_1}{3EI}$$

$$\Rightarrow \quad \delta = \frac{Pl_2^3}{3EI} - \left(\frac{-Pl_2 l_1}{3EI}\right) l_2$$

$$= \frac{Pl_2^3}{3EI} + \frac{Pl_2^2 l_1}{3EI} = \frac{l_2^2}{3EI}\left[Wl_2 - \delta k l_2 + Wl_1 - \delta k l_1\right]$$

$$= \frac{Wl_2^2}{3EI}(l_2 + l_1) - \frac{\delta k l_2^2}{3EI}(l_2 + l_1).$$

$$\delta\left(1 + \frac{kl_2^2(l_2+l_1)}{3EI}\right) = Wl_2^2(l_2 + l_1)/3EI$$

$$\Longrightarrow \quad \delta = Wl_2^2(l_2+l_1)\Big/\left(3EI + kl_2^2(l_2+l_1)\right).$$