NONLINEAR MECHANICAL SYSTEMS

MECHANISMS AND LAGRANGE’S EQUATIONS

Start with the uncoupled members of the mechanism

\( \mathbf{x} \) uncoupled coordinates (orientations, locations of mass centers) with respect to a non-accelerating (inertial) reference frame

\( \mathbf{v} \) velocities

\( \mathbf{p} \) momenta

\( \mathbf{F} \) forces

As usual, these four fundamental quantities are related

\[ \frac{d\mathbf{x}}{dt} = \mathbf{v} \]

\[ \frac{d\mathbf{p}}{dt} = \mathbf{F} \]
Constitutive equation for kinetic energy storage (inertia)

$M$ diagonal matrix of inertial parameters (masses, moments of inertia, e.g. about mass centers)

$$p = Mv$$

Kinetic co-energy

$$E_k^* = \int p^t dv = \frac{1}{2} v^t Mv = E_k^*(v)$$

by definition

$$p = \partial E_k^*/\partial v$$
**ASIDE:**

The underlying mechanics is fundamentally independent of choice of coordinates.

Therefore, this may be regarded as a tensor equation.

By the usual conventions:

- $\mathbf{v}$ is a contravariant rank 1 tensor (vector)
- $\mathbf{M}$ is a twice co-variant rank 2 tensor
- $\mathbf{p}$ is covariant rank 1 tensor (vector)

These observations will be useful when consider transformations of variables.
Next consider the kinematically coupled mechanism

\( \theta \) generalized coordinates (or configuration variables)
- a (non-unique) set of variables sufficient to define the (mechanism) configuration uniquely

\( \omega \) generalized velocities
- the time derivative of generalized coordinates.

\[
\frac{d\theta}{dt} = \omega
\]

\( \tau \) generalized forces (moments or torques)

\( \eta \) generalized momenta

The relation between generalized forces and momenta requires care. Read on ...
Relation between coordinates:

\[ x = L(\theta) \]

Relation between velocities:

\[ \frac{dx}{dt} = \left( \frac{\partial L(\theta)}{\partial \theta} \right) \frac{d\theta}{dt} \]

\[ v = J(\theta)\omega \]

where \( J(\theta) = \left( \frac{\partial L(\theta)}{\partial \theta} \right) \)
Relation between forces:
May be derived from power continuity (a differential statement of energy conservation).

Power:

\[ P = \tau t \omega \]

Aside:
A common source of error is mis-identification of generalized forces. The relation between power, generalized force and generalized velocity is a useful definition of generalized forces.

Power continuity:

\[ P = \tau t \omega = F t v = F t J(\theta) \omega \]
This must be true for all values of \( \omega \), therefore

\[ \tau = J(\theta) t F \]
Relation between kinetic co-energies:
(by substitution)

\[ E^*_k = \frac{1}{2} \omega^t J(\theta)^tMJ(\theta)\omega \]

Kinetic energy in generalized coordinates is a quadratic form in velocity. The kernel of the quadratic form is the **inertia tensor**.

\[ I(\theta) = J(\theta)^tMJ(\theta) \]

\[ E^*_k = \frac{1}{2} \omega^t I(\theta)\omega \]

Note that kinetic co-energy, which previously was a function of velocity alone, is now a function of velocity and position.

\[ E^*_k = E^*_k(\theta, \omega) \]

This is the main reason why (to paraphrase Prof. Stephen Crandall) “mechanics is hard for humans”.
Relation between momenta:
Generalized momenta are defined as before.
\[ \eta = \frac{\partial E_k^*}{\partial \omega} \]
\[ \eta = I(\theta)\omega = J(\theta)^t MJ(\theta)\omega \]
\[ \eta = J(\theta)^t Mv = J(\theta)^t p \]
\[ \eta = J(\theta)^t p \]

**KEY POINT:**

Generalized force is *not* the derivative of generalized momentum
\[ d\eta/dt \neq \tau \]
Differentiate the relation between momenta

\[
d\eta/dt = J(\theta)^t \, d\mathbf{p}/dt + \omega^t \left[ \partial J(\theta)^t / \partial \theta \right] \mathbf{p}
\]

\[
d\eta/dt = J(\theta)^t \, \mathbf{F} + \omega^t \left[ \partial J(\theta)^t / \partial \theta \right] \mathbf{M} J(\theta) \omega
\]

The second term appears to be related to the kinetic co-energy. It is:

\[
\partial E_k^*/\partial \theta = \frac{1}{2} \, \omega^t \left[ \partial J(\theta)^t / \partial \theta \right] \mathbf{M} J(\theta) \omega + \frac{1}{2} \, \omega^t J(\theta)^t \mathbf{M} \left[ \partial J(\theta) / \partial \theta \right] \omega
\]

\[
\partial E_k^*/\partial \theta = \omega^t \left[ \partial J(\theta)^t / \partial \theta \right] \mathbf{M} J(\theta) \omega
\]

\[
d\eta/dt = \tau + \partial E_k^*/\partial \theta
\]

This is Lagrange's equation

\[
d\eta/dt - \partial E_k^*/\partial \theta = \tau
\]
This may be more familiar in expanded form.

Identify kinetic co-energy with the Lagrangian, L(θ,ω)

\[ L(\theta,\omega) = E_k^*(\theta,\omega) \]

\[
\frac{d}{dt}\left[ \frac{\partial E_k^*}{\partial \omega} \right] - \frac{\partial E_k^*}{\partial \theta} = \tau
\]

\[
\frac{d}{dt}\left[ \frac{\partial L}{\partial \omega} \right] - \frac{\partial L}{\partial \theta} = \tau
\]
A scalar example:

\[ x = L(\theta) \]
\[ v = J(\theta)\omega \]
\[ \tau = J(\theta)F \]
\[ E_k^* = \frac{1}{2} \, mv^2 = \frac{1}{2} \, mJ(\theta)^2\omega^2 \]
\[ \eta = \frac{\partial E_k^*}{\partial \omega} = mJ(\theta)^2\omega = J(\theta)mv = J(\theta)p \]
\[ \frac{d\eta}{dt} = J(\theta)\frac{dp}{dt} + \left[ \frac{\partial J(\theta)}{\partial \theta} \right] \omega p \]
\[ \frac{d\eta}{dt} = \tau + \left[ \frac{\partial J(\theta)}{\partial \theta} \right] \omega mJ(\theta)\omega \]
\[ \frac{\partial E_k^*}{\partial \theta} = mJ(\theta)\omega^2 \frac{\partial J(\theta)}{\partial \theta} \]
\[ \frac{d\eta}{dt} = \tau + \frac{\partial E_k^*}{\partial \theta} \]
### SUMMARIZING:

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<td>( \mathbf{I}(\theta) = \mathbf{J}(\theta)^t \mathbf{M} \mathbf{J}(\theta) )</td>
<td>( \mathbf{I}(\theta) )</td>
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<tr>
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<td>( \eta = \mathbf{I}(\theta)\omega )</td>
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<td>( \frac{1}{2} \mathbf{v}^t \mathbf{M} \mathbf{v} )</td>
<td>( \frac{1}{2} \mathbf{\omega}^t \mathbf{I}(\theta)\omega )</td>
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The key ideas behind the Lagrangian formulation:

1. incorporate kinematic constraints directly
2. write momentum balance equation in terms of a state function, the kinetic co-energy

Advantages:

1. velocities are easily identified and kinetic co-energy is easily computed.
2. there is no need to write explicit expressions for the forces of constraint.
Disadvantages:

1. The kinetic co-energy is a quadratic form whose kernel typically contains trigonometric functions of sums of coordinates. Differentiating trigonometric functions of sums of coordinates breeds terms very rapidly. The Lagrangian approach requires a partial derivative of the co-energy followed by a total derivative of the co-energy. The algebraic complexity of the result can be staggering.

2. The Lagrangian approach is fundamentally a 2° form. To achieve the 1° form required for numerical integration (and most mathematical analysis) the inertia tensor must be inverted. This is anything but trivial.