Leakage Flux

The model of the electrical transformer developed above assumed that the magnetic flux was common to both coils. Unfortunately, a real transformer is not so simple. A more realistic model should describe flux leakage, the fact that not all of the magnetic flux is shared by both coils. This phenomenon is equivalent to having an additional magnetic path in the system which does not link both coils, as shown in the following sketch.

As the flux is no longer common to both coils, a zero junction is required in the bond graph between the two gyrators. Each of the three fluxes is distinct and corresponds to a magnetic storage element, a capacitance. A bond graph is as follows.

Note that with the causal assignment shown, only two of the three capacitances in the magnetic domain are independent. Therefore two fluxes are required to specify the energy stored in the magnetic field.

This condition is known as Maxwell's reciprocity condition.

For example, consider the magnetic field in a transformer. If we make no assumptions about the causality of the inputs from the coils we are free to assign integral causality to each of the three capacitors, as follows.
But be careful! Although it may appear that we have three independent energy storage elements, the energy stored in the magnetic field is still characterised by only two displacements (magnetic fluxes). This can be seen by writing the constitutive equations for the field.

\[ F_1 = R_1 \phi_1 + R_3 (\phi_1 + \phi_2) \]
\[ F_2 = R_2 \phi_2 + R_3 (\phi_1 + \phi_2) \]

These equations may be written in vector form as follows.

\[
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
= \begin{bmatrix}
R_1 + R_3 & R_3 \\
R_3 & R_2 + R_3
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix}
\]

or

\[ F = R \phi \]

This is a two-port linear example of the general constitutive relation \( e = \Phi(q) \) for a multiport capacitor. Note that the matrix \( R \) is symmetric. This is a necessary consequence of the fact that the magnetic field stores energy.

The energy stored in this field is defined by

\[ E - E_0 = \int F^t \, d\phi = \int \phi^t R^t d\phi = (\phi^t R \phi)/2 \]

The stored energy is a quadratic form in the displacement vector. Expanding the quadratic form:

\[ E - E_0 = \{(R_1 + R_3)\phi_1^2 + 2 R_3 \phi_1 \phi_2 + (R_1 + R_3)\phi_2^2\}/2 \]

Thus only two fluxes, \( \phi_1 \) and \( \phi_2 \), are required to characterise the energy stored in the magnetic field. This two-port C field has integral causality on both ports.
If this C field were part of a larger dynamic system, then, when state equations were derived it would give rise to two independent state variables, and no more. In general, an n-port C field can give rise to no more than n state variables.

**MULTIPORT INERTIAS**

Now consider the representation of the transformer entirely in the electrical domain. The electromagnetic bond graph is as follows.

![Electrical Bond Graph](image)

To reflect the C field onto one side of the transformer we use the gyrator equations for each coil in the constitutive equations for the magnetic field.

\[
i_1 = \frac{R_1 \lambda_1/N_1 + R_3 (\lambda_1/N_1 + \lambda_2/N_2)}{N_1}
\]

\[
i_1 = \frac{R_2 \lambda_2/N_2 + R_3 (\lambda_1/N_1 + \lambda_2/N_2)}{N_2}
\]

Thus we get the following bond graph in the electrical domain.

![Electrical Bond Graph](image)

As before, although the three inertias have integral causality, only two independent flux linkages are required to characterise the energy stored in this network. The network of inertias can be represented as a two-port inertia. The bond graph symbol is as follows.

![Bond Graph Symbol](image)

In general, a multiport inertia is the dual of a multiport capacitor. It is defined as any object for which the flow vector is a single-valued (integrable) function of the momentum vector.

\[
f = \Psi(p)
\]

The vector function \( \Psi(\cdot) \) is the constitutive relation for the inertia.

A multiport inertia stores energy just as a one-port inertia does. The stored energy is characterised by the momentum vector. In general, the constitutive relation of the multiport inertia permits the flows on one port to be related to the momenta on any and
all of the other ports, but because the multiport inertia stores energy, this cross-coupling must be reciprocal or symmetric. That is,

\[ E - E_0 = \int f e \, dt = \int f t \, dp = \int \Psi(p)^t \, dp = E(p) \]

Therefore the constitutive relation, \( \Psi(\cdot) \), must have zero curl.

\[ \int f = 0 \]

or

\[ \int \Psi = 0 \]

Returning to the example, the relation between currents and flux linkages can be written in matrix notation as follows.

\[
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix} =
\begin{bmatrix}
\frac{R_1+R_3}{N_1^2} & \frac{R_3}{N_1N_2} \\
\frac{R_3}{N_1N_2} & \frac{R_2+R_3}{N_2^2}
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2
\end{bmatrix}
\]

This relation is commonly written in the inverse form as follows.

\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2
\end{bmatrix} =
\begin{bmatrix}
L_1 & M_{12} \\
M_{12} & L_2
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix}
\]

where \( L_1 \) is the self-inductance (or simply inductance) of one coil, \( L_2 \) is the self-inductance of the other coil, and \( M_{12} \) is the mutual inductance between the two coils, the cross coupling which is possible in a multiport inertia. Because the magnetic field which gives rise to the inductance stores energy, this matrix must be symmetric.

mutual inductance ...

other parasitic effects ...?