Musculoskeletal Dynamics

Space Biomedical Engineering
& Life Support

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Part II

Musculoskeletal Dynamics

Free Body Diagram of Link i

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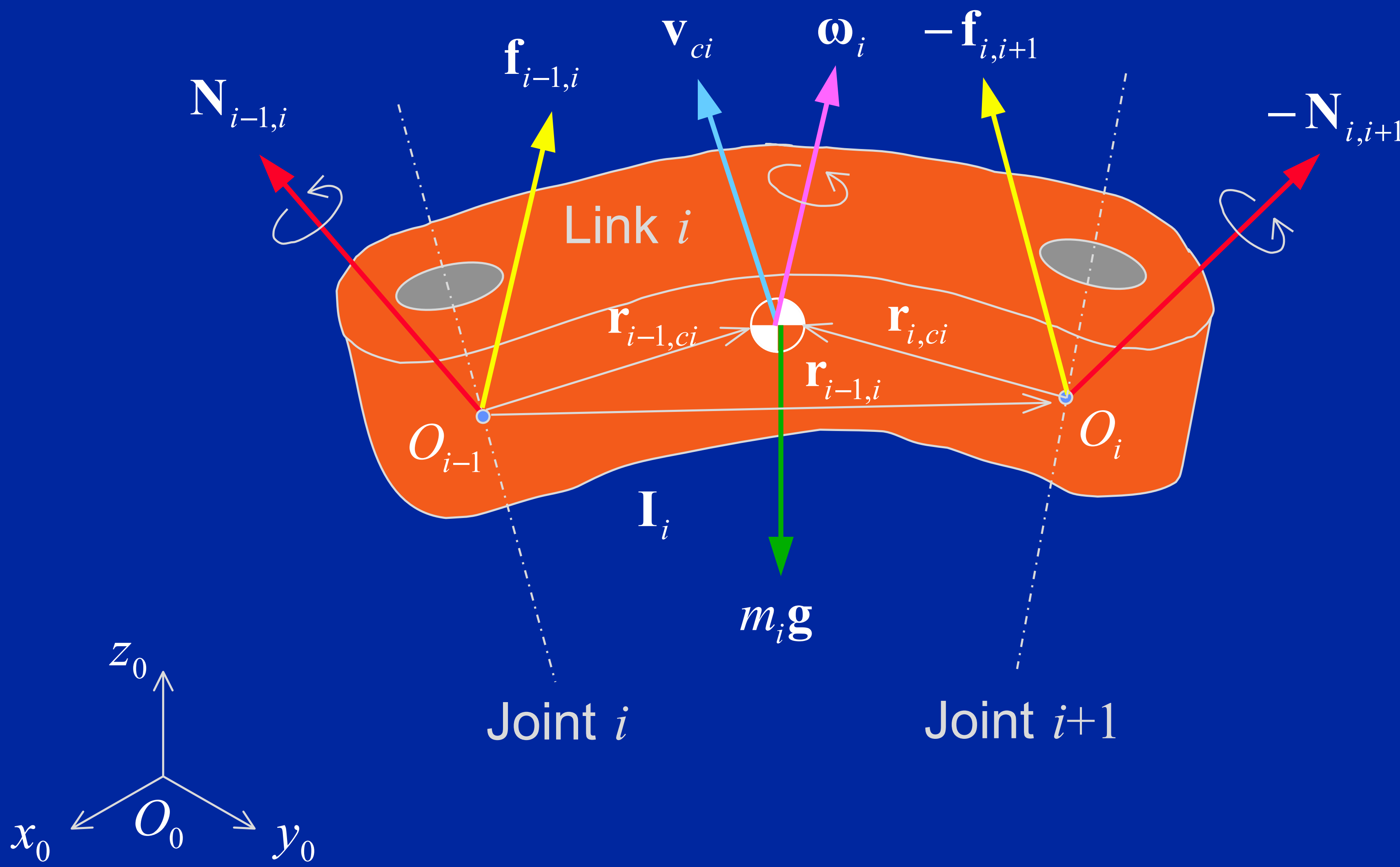

Lagrangian Formulation of Equations of Motion

- Describes the behavior of a dynamic system in terms of work and energy stored in the system
- Constraint forces are automatically eliminated (an advantage and a disadvantage), called the “closed form” dynamic equations
- Equations are derived systematically (easier to use)

$q_1, \cdots, q_n =$ generalized coordinates of a dynamic system

$\mathbf{T} =$ total kinetic energy

$\mathbf{U} =$ total potential energy

Define Lagrangian : $L(q_i, \dot{q}_i) = \mathbf{T} - \mathbf{U}$

Equations of motion are then derived from :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i \qquad i = 1, \cdots, n \qquad (2\text{-}1)$$

$Q_i =$ generalized force corresponding to generalized coord q_i

Lagrangian Formulation

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- Compute the velocity and angular velocity of an individual link i (think of the link as an end effector with coord sys at the link c.m.)

$$\mathbf{v}_{ci} = \mathbf{J}_{L1}^{(i)}\dot{q}_1 + \ldots + \mathbf{J}_{Li}^{(i)}\dot{q}_i = \mathbf{J}_{L}^{(i)}\dot{\mathbf{q}}$$
$$\boldsymbol{\omega}_{ci} = \mathbf{J}_{A1}^{(i)}\dot{q}_1 + \ldots + \mathbf{J}_{Ai}^{(i)}\dot{q}_i = \mathbf{J}_{A}^{(i)}\dot{\mathbf{q}} \qquad (2\text{-}2)$$

where $\mathbf{J}_{Lj}^{(i)}$ and $\mathbf{J}_{Aj}^{(i)}$ are the j-th column vectors of the 3xn Jacobian matrices $\mathbf{J}_{L}^{(i)}$ and $\mathbf{J}_{A}^{(i)}$ for the linear and angular velocities of link i, i.e.,

$$\mathbf{J}_{L}^{(i)} = [\mathbf{J}_{L1}^{(i)} \quad \ldots \quad \mathbf{J}_{Li}^{(i)} \quad \mathbf{0} \quad \ldots \quad \mathbf{0}]$$
$$\mathbf{J}_{A}^{(i)} = [\mathbf{J}_{A1}^{(i)} \quad \ldots \quad \mathbf{J}_{Ai}^{(i)} \quad \mathbf{0} \quad \ldots \quad \mathbf{0}] \qquad (2\text{-}3)$$

Note: Since the motion of link i depends on only joints 1 through i, the column vectors are set to zero for $j \geq i$

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- Each column vector is given by:

$$\mathbf{J}_{Lj}^{(i)} = \begin{cases} \mathbf{b}_{j-1} \times \mathbf{r}_{j-1,ci} & \text{(revolute jt)} \\ \mathbf{b}_{j-1} & \text{(prismatic jt)} \end{cases}$$

$\mathbf{r}_{j-1,ci}$ = Position vector of centroid of link i wrt inboard link coordinate frame

(2-4)

$$\mathbf{J}_{Aj}^{(i)} = \begin{cases} \mathbf{b}_{j-1} & \text{(revolute jt)} \\ \mathbf{0} & \text{(prismatic jt)} \end{cases}$$

$\mathbf{b}_{j-1}$ = 3x1 unit vector along joint axis j-1

$$\mathbf{T} = \sum_{i=1}^{n} \left(\tfrac{1}{2} m_i \mathbf{v}_{ci}^T \mathbf{v}_{ci} + \tfrac{1}{2} \boldsymbol{\omega}_i^T \mathbf{I}_i \boldsymbol{\omega}_i \right) \qquad (2\text{-}5)$$

where

m_i = mass of link i

$\mathbf{I}_i = 3 \times 3$ inertia tensor at the centroid, wrt **base coord frame**

Note: $\mathbf{I}_i$ varies with the orientation of the link wrt the base coord frame

Lagrangian Formulation

$$\mathbf{T} = \tfrac{1}{2}\sum_{i=1}^{n}\left(m_i\dot{\mathbf{q}}^T\mathbf{J}_L^{(i)^T}\mathbf{J}_L^{(i)}\dot{\mathbf{q}} + \dot{\mathbf{q}}^T\mathbf{J}_A^{(i)^T}\mathbf{I}_i\mathbf{J}_A^{(i)}\dot{\mathbf{q}}\right) = \tfrac{1}{2}\dot{\mathbf{q}}^T\mathbf{H}\dot{\mathbf{q}} \qquad (2\text{-}6)$$

where

$$\mathbf{H} = \sum_{i=1}^{n}\left(m_i\mathbf{J}_L^{(i)^T}\mathbf{J}_L^{(i)} + \mathbf{J}_A^{(i)^T}\mathbf{I}_i\mathbf{J}_A^{(i)}\right) = \text{system inertia tensor } (n\times n) \qquad (2\text{-}7)$$

($\mathbf{H}$ is symmetric positive definite)

Note: $\mathbf{I}_i$ can be obtained from $\bar{\mathbf{I}}_i$, the inertia tensor defined relative to the coord frame fixed to the link, using

$$\mathbf{I}_i = \mathbf{R}_i^0\bar{\mathbf{I}}_i\mathbf{R}_i^{0^T} \qquad (2\text{-}8)$$

Lagrangian Formulation

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Potential Energy

$$\mathbf{U} = \sum_{i=1}^{n} m_i \mathbf{g}^T \mathbf{r}_{0,ci} \qquad (2\text{-}9)$$

Generalized Forces

$$\mathbf{Q} = \boldsymbol{\tau} + \mathbf{J}^T \mathbf{F}_{ext} \qquad (2\text{-}10)$$

$\boldsymbol{\tau}$ = joint torques $\quad \mathbf{J}^T \mathbf{F}_{ext}$ = external forces and moments

Lagrange's Equations of Motion (see Asada & Slotine for derivation)

$$\underbrace{\sum_{j=1}^{n} H_{ij} \ddot{q}_j}_{\text{Inertia torques}} + \underbrace{\sum_{j=1}^{n} \sum_{k=1}^{n} h_{ijk} \dot{q}_j \dot{q}_k}_{\text{Centrifugal/ Coriolis trqs}} + \underbrace{G_i}_{\text{Gravity torque}} = \underbrace{Q_i}_{\text{Generalized Forces}} \qquad (i = 1, \cdots, n) \qquad (2\text{-}11)$$

$$h_{ijk} = \frac{\partial H_{ij}}{\partial q_k} - \frac{1}{2} \frac{\partial H_{jk}}{\partial q_i}$$

$$G_i = \sum_{j=1}^{n} m_j \mathbf{g}^T \mathbf{J}_{Li}^{(j)}$$

Example: 2 dof planar arm

- Velocities of centroids c_1 and c_2

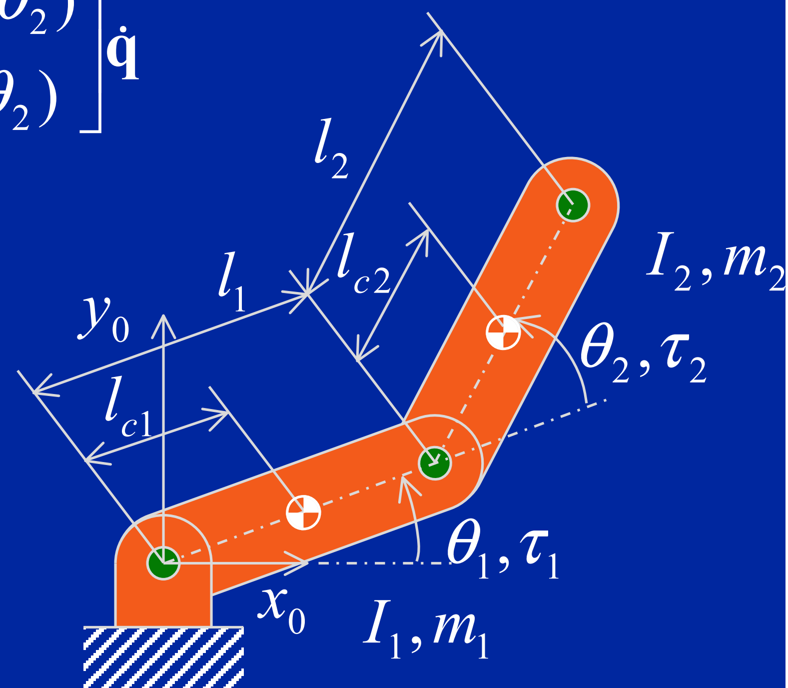
$$\mathbf{v}_{c1} = \begin{bmatrix} -l_{c1}\sin\theta_1 & 0 \\ l_{c1}\cos\theta_1 & 0 \end{bmatrix}\dot{\mathbf{q}}$$

$$\mathbf{v}_{c2} = \begin{bmatrix} -l_1\sin\theta_1 - l_{c2}\sin(\theta_1+\theta_2) & -l_{c2}\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_{c2}\cos(\theta_1+\theta_2) & l_{c2}\cos(\theta_1+\theta_2) \end{bmatrix}\dot{\mathbf{q}}$$

- These 2x2 matrices are the $\mathbf{J}_L^{(i)}$
- $\mathbf{J}_A^{(i)}$ associated with the angular velocities are 1x2 row vectors in this planar case

$$\omega_1 = \dot{\theta}_1 = [1 \quad 0]\dot{\mathbf{q}}$$

$$\omega_2 = \dot{\theta}_1 + \dot{\theta}_2 = [1 \quad 1]\dot{\mathbf{q}}$$



Example: 2 dof planar arm

- Substituting the linear and angular Jacobians into eqn (2-7) gives

$$\mathbf{H} = \begin{bmatrix} m_1 l_{c1}^{\ 2} + I_1 + m_2(l_1^{\ 2} + l_{c2}^{\ 2} + 2 l_1 l_{c2} \cos\theta_2) + I_2 & m_2 l_1 l_{c2} \cos\theta_2 + m_2 l_{c2}^{\ 2} + I_2 \\ m_2 l_1 l_{c2} \cos\theta_2 + m_2 l_{c2}^{\ 2} + I_2 & m_2 l_{c2}^{\ 2} + I_2 \end{bmatrix} \dot{\mathbf{q}}$$

- Centrifugal / Coriolis term coefficients

$$h_{111} = 0, \qquad h_{122} = -m_2 l_1 l_{c2} \sin\theta_2, \qquad h_{112} + h_{121} = -2 m_2 l_1 l_{c2} \sin\theta_2$$

$$h_{211} = m_2 l_1 l_{c2} \sin\theta_2, \qquad h_{222} = 0, \qquad h_{212} + h_{221} = 0$$

- Gravity terms

$$G_1 = \mathbf{g}^T [m_1 \mathbf{J}_{L1}^{(1)} + m_2 \mathbf{J}_{L1}^{(2)}]$$

$$G_2 = \mathbf{g}^T [m_1 \mathbf{J}_{L2}^{(1)} + m_2 \mathbf{J}_{L2}^{(2)}]$$

- Substituting the above into (2-11) gives

$$H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 + h_{122}\dot{\theta}_2^{\ 2} + (h_{112} + h_{121})\dot{\theta}_1\dot{\theta}_2 + G_1 = \tau_1$$

$$H_{22}\ddot{\theta}_2 + H_{12}\ddot{\theta}_1 + h_{211}\dot{\theta}_1^{\ 2} + G_2 = \tau_2$$