Essays on Coordination, Cooperation, and Learning

by

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Abstract

This thesis is a collection of essays on coordination and learning in dynamic cooperation games.

Chapter One begins by establishing results which are required in order to extend the global games approach to settings where the game structure is endogenous. In particular it shows that the selection argument of Carlsson and van Damme (1993) holds uniformly over appropriately controlled families of games. It also discusses selection results when the game lacks dominance regions.

Chapter Two uses these results to investigate the impact of miscoordination fear in a class of dynamic cooperation games with exit. More specifically, it explores the effect of small amounts of private information on a class of dynamic cooperation games with exit. It is shown that lack of common knowledge creates a fear of miscoordination which pushes players away from the full-information Pareto frontier. Unlike in one-shot two-by-two games, the global games information structure does not yield equilibrium uniqueness, however, by making it harder to coordinate, it does reduce the range of equilibria and gives bite to the notion of local dominance solvability. Finally, Chapter Two provides a simple criterion for the robustness of cooperation to miscoordination fear, and shows it can yield predictions that are qualitatively different from those obtained by focusing on Pareto efficient equilibria under full information.

Finally Chapter Three studies how economic agents learn to cooperate when the details of what cooperation means are ambiguous. It considers a dynamic game in which one player's cost for the cooperative action is private information. From the perspective of the other player, this cost is an unknown but stationary function of observable states of the world. Initially, because of information asymmetries, full cooperation can be sustained only at the cost of inefficient punishment. As players gain common experience, however, the uninformed player may learn how to predict her partner's cost, thereby resolving informational asymmetries. Once learning has occurred, players can sustain cooperation more efficiently and reduce the partnership's sensitivity to adverse economic conditions. Nevertheless, because inducing information revelation has an efficiency cost, it may sometimes be optimal for the uninformed player to remain uninformed even though that limits the amount of cooperation that can be sustained in equilibrium.

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This work is dedicated to my family
whom I cannot begin to thank for the love and support
I have received over the years.

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Ma jeunesse ne fut qu'un ténébreux orage,
Traversé çà et là par de brillants soleils
– Charles Baudelaire

This has been a formative but often lonely experience. Still, as I reflect upon these years at MIT, I am stunned by the wealth of warmth and support that I have received from my advisors, my friends, and my family. I could not have managed without them.

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Chapter One: Uniform selection in global games.

Abstract

This paper brings together results which are required in order to extend the global games approach to settings where the game structure is endogenous. In particular it shows that the selection argument of Carlsson and van Damme [2] holds uniformly over appropriately controlled families of games. It also discusses selection results when the game lacks dominance regions.

KEYWORDS: global games, equilibrium selection, uniform selection, endogenous games
JEL CLASSIFICATION CODES : C72, C73

1 Introduction

The global games framework, first proposed in Carlsson and van Damme [2], has been widely applied as a selection device in 2×2 coordination games[1]. Their first result is that if players have an information structure resulting from independent additive observational noise, then, the set of rationalizable strategies shrinks to a unique equilibrium as the amplitude of the noise goes to 0. Their second result is that the selected equilibrium is in fact to play the risk-dominant strategy studied in Harsanyi and Selten [4].

While it is true that global games have been widely used in applied work, there have been only few attempts to use the information structure of Carlsson and van Damme [2] in models of greater complexity than one shot two actions coordination games. A notable exception is Frankel, Morris and Pauzner [3] which proves selection results for a class supermodular games in which the actions and the state of the world belong to the real line. Adding layers of decision making on top of a 2×2 coordination problem is another tempting direction in which to extend the results of Carlsson and van Damme [2]. The main hurdle towards that goal is that in such models, the payoffs of the coordination game will be endogenously determined: to use selection results in this setting, we need them to hold uniformly over the class of possible payoffs. Up to now however, available selection results all take the payoff structure as given; in other words, they hold pointwise while we need uniform selection. An example makes this point clearer.

Consider the problem of a principal trying to get her two agents to cooperate. The game has three periods: at time $t = 1$, the principal can invest in some capital k at a positive increasing cost $c(k)$. At time $t = 2$, the two agents observe k perfectly and then play a global game $\Gamma(\theta, k, \sigma)$ with actions $\{cooperate, defect\}$, where θ is the noisily observed state of the world, σ the amplitude of the noise and capital k parameterizes players' payoffs. At time

[1]See Morris and Shin [7] for a literature review.

$t = 3$, the principal gets a payoff depending on whether the agents cooperated or not. What is the optimal capital stock k^*_σ the principal should purchase? How do k^*_σ and the principal's payoff vary as σ goes to zero?

To solve her optimization problem, the principal must form some belief about her agents' behavior. When the state of the world is common knowledge, because of multiplicity of equilibria, this problem is not well defined. This motivates the use of a global game information structure. The typical global games selection results state that for a given capital stock k, the agents will cooperate if and only if the state of the world is above some threshold $\theta_\sigma(k)$ and that as σ goes to zero, $\theta_\sigma(k)$ converges to the risk-dominant threshold $\theta^{RD}(k)$. Uniqueness of equilibrium makes the principal's problem well defined. Assume there is an optimal amount of capital k^*_0 the principal would choose if there was no noise and the agents used the risk-dominant threshold $\theta^{RD}(k)$. Is it true that k^*_σ converges to k^*_0 as σ goes to zero? Is it true that the principal's payoff is continuous in σ?

The answer to these questions is affirmative, however as the counter-example of Figure 1 shows, pointwise convergence of $\theta_\sigma(k)$ is not sufficient for these results to hold. In this counter-example the cooperation threshold $\theta_\sigma(k)$ converges pointwise to a constant threshold equal to $\frac{1}{2}$, but does not converge uniformly. In fact, there is always a capital stock such that the players' cooperation threshold is $\frac{1}{4}$. If agents behaved according to Figure 1, the principal might choose a capital stock $k^*_\sigma = 1 - \frac{\sigma}{2}$ for all $\sigma > 0$, but at the limit she would choose $k^*_0 = 0$ since capital is costly. To show that in fact k^*_σ does converges to k^*_0, we need to prove uniform convergence of $\theta_\sigma(k)$ over the set of possible capital stocks as σ goes to 0.

The goal of this paper is to provide uniform selection results over general families of payoffs. The purpose being to give results that are widely applicable, Section 2 defines general classes of payoffs over which we will prove uniform selection. Section 3, which constitutes the core of the paper, proves the main selection results. Finally, because the usual dominance assumption might not hold when payoffs are endogenous, Section 4 discusses selection when there are no dominance regions.

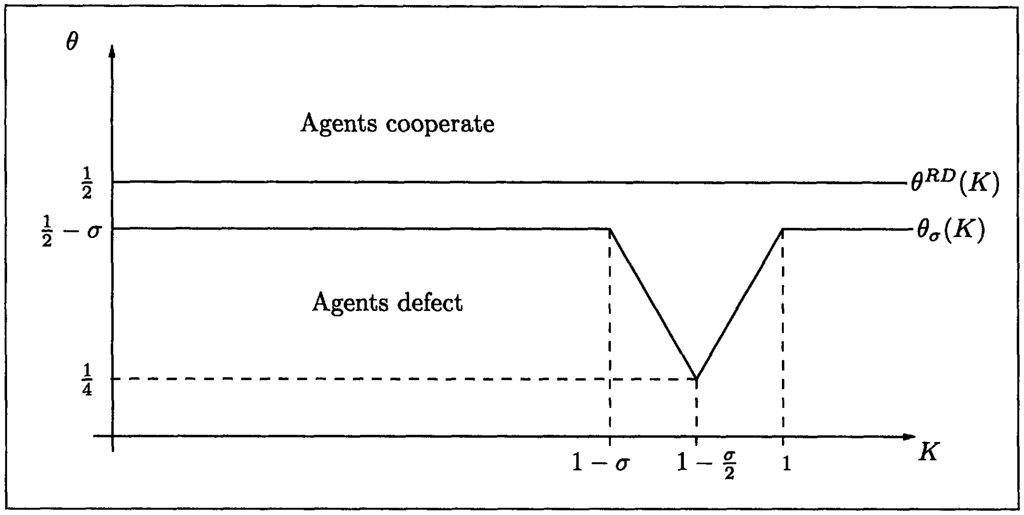

Figure 1: A sequence of thresholds converging pointwise but not uniformly. Agents cooperate above the threshold and defect below.

2 Choosing an appropriate payoff class.

This section introduces the class of games that we will be studying. Keeping with the framework of Carlsson and van Damme [2], the paper focuses on two actions two players games. However, all results presented in further sections extend to symmetric games with two actions and a continuum of players, such as those reviewed by Morris and Shin [7]. The extension is presented in the appendix.

We consider 2×2 games, with players $i \in \{1, 2\}$, actions $a \in \{C, D\}$ and payoffs that depend continuously on a state of nature $\theta \in I$, where I is an interval of $\mathbb{R}$. Payoffs are denoted by

	C	**D**
C	$w^i_{11}(\theta)$	$w^i_{12}(\theta)$
D	$w^i_{21}(\theta)$	$w^i_{22}(\theta)$

where i is the row player. Both players get signals $x_i = \theta + \sigma\varepsilon_i$, where ε_1 and ε_2 are independent random variables with support $[-1,1]$, and θ is a random variable with a C^1 distribution f_θ and convex support.

Let $G(\theta)$ denote the game with *perfect information* at state θ and let Γ_σ be the global game with *noisy information.* Denote by w the payoff structure $(w^i_{11}, w^i_{12}, w^i_{21}, w^i_{22})_{i\in\{1,2\}}$. Pure strategies are functions $s : \mathbb{R} \mapsto \{C, D\}$. For completeness mixed strategies will be succinctly considered. By allowing players to privately observe independent random variables $\tilde{u}$ uniformly distributed on $[0,1]$, mixed strategies can be viewed as functions $s : \mathbb{R} \times [0,1] \mapsto \{C, D\}$. To eliminate multiple representations, one imposes the constraint that for all $x \in \mathbb{R}$, and $(u, u') \in [0,1]^2$, whenever $u < u'$, then $\{s(x,u) = C \Rightarrow s(x,u') = C\}$. Pure strategies are also mixed strategies, which do not depend on the random variable $\tilde{u}$.

In order to apply global games techniques to endogenous payoff structures we need to prove selection results holding uniformly over some family $\mathcal{W}$ of possible payoff functions. Rather than dealing with the problem on a case by case basis, the goal of this paper is to prove uniform selection results holding for general classes of payoffs. However, choosing an appropriate reference class of payoff functions $\mathcal{W}$ is delicate. While well behaved classes allow for simple proofs and fewer cases, they also limit the applicability of the results. This section defines and motivates the fairly general reference class of payoff functions we will use. Generality comes at some notational cost and we must first introduce a few assumptions and definitions.

Assumption 1. *For any state of the world θ, game $G(\theta)$ has pure strategy equilibria. The set of equilibria is either $\{(C,C)\}$,$\{(D,D)\}$ or $\{(C,C),(D,D)\}$.*

This assumption rules out games of "matching pennies" and ensures that there exists a fixed order on actions such that for all states of the world, the game $G(\theta)$ is either dominance solvable or has increasing differences in actions with respect to the aforementioned order.

Note that this is equivalent to assuming that there is a fixed order on $\{C, D\}$ such that for all $\theta \in I$, game $G(\theta)$ satisfies the single-crossing condition of Milgrom and Shannon [6] with respect to the chosen order on $\{C, D\}$.

Assumption 2 (increasing differences in the state of the world). *The game has increasing differences in θ:*

$\forall i \in \{1, 2\}$, both $a^i(\theta) \equiv w^i_{12}(\theta) - w^i_{22}(\theta)$ and $b^i(\theta) \equiv w^i_{11}(\theta) - w^i_{21}(\theta)$ are strictly increasing in θ.

Assumption 3 (Dominance regions). *Let w be a payoff structure satisfying Assumptions 1 and 2. There exist thresholds $\underline{\theta}_i$ and $\overline{\theta}_i$ solutions to*

$$w^i_{12}(\overline{\theta}_i) - w^i_{22}(\overline{\theta}_i) = 0 \quad \textit{and} \quad w^i_{11}(\underline{\theta}_i) - w^i_{21}(\underline{\theta}_i) = 0.$$

Together, Assumptions 1, 2, and 3 insure that whenever $G(\theta)$ has multiple equilibria, either $(-\infty, \theta]$ is included in the risk-dominance region of (D, D) or $[\theta, \infty)$ is included in the risk-dominance region of (C, C). This is the unidimensional equivalent of Carlsson and van Damme's assumption that all states of the world should be connected to dominance regions by a continuous path that is included in the risk-dominant region of one equilibrium. Note that Assumption 1 implies that we must have $\underline{\theta}_i \leq \overline{\theta}_i$.

Definition 1 (differences in actions). *Given a payoff structure w, we define $h^i_w(\theta) = b^i_w(\theta) - a^i_w(\theta) = w^i_{11} - w^i_{12} - w^i_{21} + w^i_{22}$.*

Whenever $h^i_w(\theta) > 0$, then the game has strictly increasing differences in actions for player i at θ, that is, $w^i_{11} - w^i_{21} > w^i_{12} - w^i_{22}$. Supermodularity requires that for all $\theta \in I$, $h^i_w(\theta) \geq 0$ should hold. While Assumption 1 does imply increasing differences in actions over some intermediate range of states, assuming full supermodularity is in fact quite restrictive.

Consider for instance the symmetric game :

	C	D
C	θ	$\gamma\theta$
D	M	$M/2$

where payoffs are given for the row player, $\gamma < 1/2$, and $\theta \in \mathbb{R}$. This game satisfies Assumption 1 but not supermodularity. To include such games, we will not assume supermodularity and content ourselves with the weaker Assumption 1.

Definition 2 (modulus of continuity). *A function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a modulus of continuity if and only if it is continuous, strictly increasing and $\rho(0) = 0$.*

Let $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ and $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ be two normed vector spaces. A funtion $g : \mathcal{A} \to \mathcal{B}$ has a modulus of continuity ρ if and only if

$$\forall (x, y) \in \mathcal{A}^2, \quad |g(x) - g(y)|_{\mathcal{B}} \leq \rho(|x - y|_{\mathcal{A}}).$$

We will require payoff functions to share a common modulus of continuity. We know from the Arzelà-Ascoli theorem[2] that this is in fact a compactness assumption. It is less restrictive than assuming that payoff functions are Lipschitz continuous with a common rate R. Because utility functions commonly used in economics typically satisfy the Inada conditions, they are not Lipschitz continuous; however they do admit a modulus of continuity. This is why we deal with this greater level of generality.

Definition 3 (rates). *Let g be some function from I to $\mathbb{R}$. We define the upper and lower*

[2]For a reference, see James Munkres, *Topology*, Prentice Hall, 2000.

rates of g at θ by,

$$\begin{aligned}\frac{\partial^+ g}{\partial \theta}(\theta) &= \limsup_{\theta' \to \theta} \frac{g(\theta') - g(\theta)}{\theta' - \theta} \\ \frac{\partial^- g}{\partial \theta}(\theta) &= \liminf_{\theta' \to \theta} \frac{g(\theta') - g(\theta)}{\theta' - \theta}.\end{aligned}$$

Those rates are always well defined although they might take infinite values. The lower and upper rates of g coincide at θ if and only if g is differentiable at θ.

Definition 4 (Reference payoff class). *Consider $r, \kappa > 0$, $\hbar \in \mathbb{R}$, a compact set $K \subset \mathbb{R}$ and a modulus of continuity ρ. We denote by $\mathcal{W}_{\rho,\kappa,\hbar,r,K}$ the class of payoff structures w such that*

1. *w satisfies Assumptions 1, 2 and 3*
2. *$\forall i \in \{1,2\}$, a^i_w and b^i_w have lower rates greater than $\kappa > 0$ over $[\underline{\theta}_i(w) - r, \overline{\theta}_i(w) + r]$*
3. *$\forall i \in \{1,2\}$, $[\underline{\theta}_i(w), \overline{\theta}_i(w)] \subset K$*
4. *payoff functions corresponding to w have a modulus of continuity ρ*
5. *$\forall i \in \{1,2\}$, $\forall \theta \in [\underline{\theta}_i - r, \overline{\theta}_i + r]$, $h^i_w(\theta) \geq \hbar$.*

This definition may seem odd, in particular requirement 5. The class of payoffs it describes is in fact fairly general. More restrictive classes could be described in simpler ways, but some natural applications force us to deal with that level of generality. The following lemma gives an equivalent definition of classes $\mathcal{W}_{\rho,\kappa,\hbar,r,K}$ when $\hbar > 0$.

Lemma 1 (equivalent characterization). *For $d > 0$ and $\nu > 0$, we define the class $\Lambda_{\rho,\nu,\kappa,d,K}$ of payoff structures w satisfying,*

1. *w satisfies Assumptions 1, 2 and 3*
2. *$\forall i \in \{1,2\}$, a^i_w and b^i_w have lower rates greater than $\kappa > 0$ over $[\underline{\theta}_i(w) - \nu, \overline{\theta}_i(w) + \nu]$*

3. $\forall i \in \{1,2\}$, $[\underline{\theta}_i(w), \overline{\theta}_i(w)] \subset K$

4. *payoff functions corresponding to w have a modulus of continuity ρ*

5. $\forall i \in \{1,2\}$, $\overline{\theta}_i(w) - \underline{\theta}_i(w) > d$.

We have $\Lambda_{\rho,\nu,\kappa,d,K} \subset \mathcal{W}_{\rho,\kappa,\hbar,r,K}$ with $\hbar = d\kappa/2$ and $r = \min\{\rho^{-1}(d\kappa/8), \nu\}$. Inversely for all $\hbar > 0$, $\mathcal{W}_{\rho,\kappa,\hbar,r,K} \subset \Lambda_{\rho,\nu,\kappa,d,K}$ with $\nu = r$ and $d = \rho^{-1}(\hbar/4)$.

Proof. For the first inclusion: we have by definition $b^i_w(\underline{\theta}_i) = 0$ and $a^i_w(\overline{\theta}_i) = 0$. Since b^i_w and a^i_w both have lower rates greater than κ, this implies that for all w,

$$\begin{aligned}\forall \theta \in [\underline{\theta}_i, \overline{\theta}_i], \quad h^i_w(\theta) &= b^i_w(\theta) - a^i_w(\theta) \geq b^i_w(\theta) - b^i_w(\underline{\theta}_i) + a^i_w(\overline{\theta}_i) - a^i_w(\theta) \\ &\geq \kappa(\theta - \underline{\theta}_i) + \kappa(\overline{\theta}_i - \theta) \geq d\kappa.\end{aligned} \tag{1}$$

Since all components of w have a modulus of continuity ρ, h^i_w has a modulus of continuity 4ρ. This implies that whenever $|x - y| \leq \rho^{-1}(d\kappa/8)$, then $|h^i_w(x) - h^i_w(y)| < d\kappa/2$. For any $\theta \in [\underline{\theta}_i - \rho^{-1}(d\kappa/8), \overline{\theta}_i + \rho^{-1}(d\kappa/8)]$, there exists $\tilde{\theta}$ such that $\tilde{\theta} \in [\underline{\theta}_i, \overline{\theta}_i]$ and $|\theta - \tilde{\theta}| < d\kappa/2$. Using inequality (1) we get that,

$$\forall \theta \in [\underline{\theta}_i - \rho^{-1}(d\kappa/8)\ ,\ \overline{\theta}_i + \rho^{-1}(d\kappa/8)], \quad h^i_w(\theta) \geq h^i_w(\tilde{\theta}) - |h^i_w(\tilde{\theta}) - h^i_w(\theta)| \geq d\kappa/2$$

which gives us the first inclusion.

For the second inclusion: by definition, we know that $b^i_w(\underline{\theta}_i) = a^i_w(\overline{\theta}_i) = 0$. Since for all payoffs w, b^i_w and a^i_w share a modulus of continuity 2ρ, then for $\theta \in [\underline{\theta}_i(w) - r, \overline{\theta}_i(w) + r]$, we must have $h^i_w(\theta) = b^i_w(\theta) - a^i_w(\theta) < 4\rho(|\overline{\theta}_i(w) - \underline{\theta}_i(w)|)$. Since by assumption, $h^i_w(\theta) \geq \hbar$, this implies

$$|\overline{\theta}_i(w) - \underline{\theta}_i(w)| > \rho^{-1}(\hbar/4).$$

Finally, Assumption 1 implies that $\overline{\theta}_i(w) - \underline{\theta}_i(w) \geq 0$. This gives us the second inclusion. $\square$

3 Uniform selection

In this section we prove the two main results of the paper:

Joint selection: There exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$, and all payoff structures w in $\mathcal{W}_{\rho,\kappa,\hbar,r,K}$, the game $\Gamma_\sigma(w)$ has a unique rationalizable pair of strategies.

Uniform convergence: The selected equilibrium converges uniformly over $\mathcal{W}_{\rho,\kappa,\hbar,r,K}$ to the risk-dominant equilibrium associated with each payoff profile.

The logic of the proof is the following: we first place ourselves in settings where equilibrium multiplicity is non-trivial; we then prove that for a noise term σ small enough, Assumption 1 implies that the game Γ_σ exhibits monotone best response and has extreme monotone Nash equilibria; finally we prove joint and uniform selection results by showing that these extreme equilibria are characterized by two real equations whose solutions are well behaved in the underlying payoff structure.

3.1 The regular case

Assumption 1 implies that $\overline{\theta}_i - \underline{\theta}_i \geq 0$. We distinguish proof techniques depending on whether $\min_{i\in\{1,2\}}\{\overline{\theta}_i - \underline{\theta}_i\} > 0$ or $\min_{i\in\{1,2\}}\{\overline{\theta}_i - \underline{\theta}_i\} = 0$. The first case, which we will refer to as the *regular case*, features a non-trivial coordination problem under complete information. The second case, which we refer to as the *non-regular case*, features no such coordination problem. Naturally, we are mostly interested in the effect of a global games information structure in the regular case, however when payoffs are endogenous, we may not be able to exclude the non-regular case. This means we will need to prove uniform selection in both cases.

We will say that an entire family of payoffs $\mathcal{W}$ is regular whenever there exists $\Lambda_{\rho,\nu,\kappa,d,K}$ such that $\mathcal{W} \subset \Lambda_{\rho,\nu,\kappa,d,K}$. Lemma 1 implies that $\mathcal{W}_{\rho,\kappa,\hbar,r,K}$ is regular if and only if $\hbar > 0$.

Because selection happens for different reasons in the two cases, we will treat regular and non-regular payoff classes separately. The reason why we have to make such a distinction will become clearer in Section 3.3. The non-regular case is presented in the appendix.

3.2 Structural results

In this section we consider a payoff class $\mathcal{W}_{\rho,\kappa,\hbar,r,K}$ and show that there exists $\overline{\sigma} > 0$, such that for all $\sigma \in (0, \overline{\sigma})$, and all $w \in \mathcal{W}_{\rho,\kappa,\hbar,r,K}$, the set of rationalizable strategies of $\Gamma_\sigma(w)$ is bounded by extreme monotone Nash equilibria.

To do this, we first define a natural order on strategies denoted by $\preccurlyeq$. We then show that for σ small enough, the best response mapping is increasing with respect to $\preccurlyeq$, and preserves the monotonicity of strategies.

Note that since we do not assume supermodularity, the results of van Zandt and Vives [8] for Bayesian games do not apply. Also, although there is a consistent order over $\{C, D\}$ such that for all $\theta \in I$, $G(\theta)$ satisfies the single-crossing property of Milgrom and Shannon [6], this does not imply that the Bayesian game Γ_σ satisfies the single-crossing property. This is why we cannot apply the results of Milgrom and Shannon [6], and must limit the amplitude σ of the noise term.

Definition 5 (monotone strategies). *A strategy s is said to be monotone if it is a pure strategy, and admits a threshold x_s such that,*

$$x < x_s \Rightarrow s(x) = D \quad \textit{and} \quad x > x_s \Rightarrow s(x) = C.$$

A monotone strategy of threshold x will be denoted s_x and inversely, the threshold of a monotone strategy s will be denoted x_s.

Definition 6 (ordered strategies). *Let $\preccurlyeq$ denote the partial order on pure an mixed strategies* *defined by*

$$s \preccurlyeq s' \iff \forall (x,u) \in \mathbb{R} \times [0,1], \quad s(x,u) = C \Rightarrow s'(x,u) = C.$$

Given a pair of strategies (s, s') such that $s \preccurlyeq s'$, we denote by $[s, s']$ the set of all strategies s'' such that $s \preccurlyeq s'' \preccurlyeq s'$.

Let BR^i denote the best response mapping of player i.

Lemma 2 (rationalizable strategies). *For all $\sigma \geq 0$ and any strategy s, $BR^i(s) \in [s_{\underline{\theta}_i-\sigma}, s_{\overline{\theta}_i+\sigma}]$. Moreover, a rationalizable strategy s has to belong to $\cap_{i\in\{1,2\}}[s_{\underline{\theta}_i-2\sigma}, s_{\overline{\theta}_i+2\sigma}]$.*

Proof. For the first part: whenever she gets a signal $x < \underline{\theta}_i - \sigma$, player i knows that D is dominant in all possible games $G(\theta)$ given her signal. Thus it is dominant for her to play D. Similarly, whenever $x > \overline{\theta}_i + \sigma$, it is dominant for her to play C in all possible games $G(\theta)$.

For the second part: a rationalizable strategy s is a best response of i to some strategy s_{-i} which is itself a best response of player $-i$ to an other strategy of i. The first part of the lemma implies that, $s \in [s_{\underline{\theta}_i-\sigma}, s_{\overline{\theta}_i+\sigma}]$ and $s_{-i} \in [s_{\underline{\theta}_{-i}-\sigma}, s_{\overline{\theta}_{-i}+\sigma}]$. Thus, whenever she gets a signal $x < \underline{\theta}_{-i} - 2\sigma$, player i knows that player $-i$ will play D. Because w is regular, Assumption 1 implies player i must choose to coordinate on D. Respectively, when she gets a signal $x > \underline{\theta}_{-i} + 2\sigma$, player i knows that player $-i$ will play C and coordinates on C. This proves the result. $\square$

Given her opponent's strategy s and her signal x_i, player i's expected payoffs upon cooperation an defection are

$$\Pi_C(x_i, s) = \mathbf{E}\left[w^i_{12} + \{w^i_{11} - w^i_{12}\}\mathbf{1}_{s=C} | x_i, s\right] \tag{2}$$

$$\Pi_D(x_i, s) = \mathbf{E}\left[w^i_{22} + \{w^i_{21} - w^i_{22}\}\mathbf{1}_{s=C} | x_i, s\right]. \tag{3}$$

Define $\Delta_i^w(x_i, s)$ by $\Delta_i^w(x_i, s) \equiv \Pi_C(x_i, s) - \Pi_D(x_i, s)$. Player i's best response to s is C whenever $\Delta_i^w(x_i, s) > 0$.

Theorem 1 (monotone best response). *For all $w \in \mathcal{W}_{\rho,\kappa,\hbar,r,K}$ and $\sigma \in [0, r/2]$, the game $\Gamma_\sigma(w)$ has monotone best response. That is, for any s and s',*

$$s' \preccurlyeq s \quad \Rightarrow \quad BR_w^i(s') \preccurlyeq BR_w^i(s).$$

Proof. Player i's best response is to cooperate if and only if $\Delta_i^w(x_i, s) > 0$. For all $x \in [\underline{\theta}_i - r/2, \overline{\theta}_i + r/2]$, we must have $\theta \in [\underline{\theta}_i - r, \overline{\theta}_i + r]$. This implies $h^i(\theta) \geq \hbar \geq 0$. Thus, when $s' \preccurlyeq s$, $\forall x \in [\underline{\theta}_i - r/2, \overline{\theta}_i + r/2]$,

$$\begin{aligned}
\Delta_i^w(x, s) &= \mathbf{E}[w_{12}^i - w_{22}^i + \underbrace{(w_{11}^i - w_{21}^i - w_{12}^i + w_{22}^i)}_{=h^i(\theta)\geq 0} \mathbf{1}_{s=C} | x] \\
&\geq \mathbf{E}[w_{12}^i - w_{22}^i + (w_{11}^i - w_{21}^i - w_{12}^i + w_{22}^i) \mathbf{1}_{s'=C} | x] \;\geq\; \Delta_i^w(x, s').
\end{aligned}$$

The best response is to play C if and only if $\Delta_i^w > 0$. Thus whenever $x \in [\underline{\theta}_i - r/2, \overline{\theta}_i + r/2]$, $BR^i(s')(x) = C \Rightarrow BR^i(s)(x) = C$. Finally we know from Lemma 2 that if $x < \underline{\theta}_i - r/2 \leq \underline{\theta}_i - \sigma$, then $BR^i(s')(x) = BR^i(s)(x) = D$. Similarly if, $x > \overline{\theta}_i + r/2 \geq \overline{\theta}_i + \sigma$ then $BR^i(s')(x) = BR^i(s)(x) = C$. This concludes the proof, $BR^i(s') \preccurlyeq BR^i(s)$. □

Lemma 3 (monotone strategies). *For all $w \in \mathcal{W}_{\rho,\kappa,\hbar,r,K}$ and $\sigma < r/2$, whenever s is a monotone strategy, then $BR_w^i(s)$ is also monotone. Moreover, the threshold associated to $BR_w^i(s)$ is a continuous function of threshold x_s.*

Proof. Consider the function $\Delta_i^w(x_i, s)$. A best response to s is characterized by the solutions of equation $\Delta_i^w(x_i, s) = 0$. Thus, it suffices to show that when s is monotone, Δ_i^w is increasing in its first argument, x_i.

$$\Delta_i^w(x_i, s) = \mathbf{E}[(w_{12}^i - w_{22}^i)\mathbf{1}_{x_{-i} < x_s} + (w_{11}^i - w_{21}^i)\mathbf{1}_{x_{-i} > x_s} | x_i, s]. \tag{4}$$

We have already defined $a^i(\theta) = w^i_{12}(\theta) - w^i_{22}(\theta)$ and $b^i(\theta) = w^i_{11}(\theta) - w^i_{21}(\theta)$. Denote F_ε the cumulative distribution function of ε and $G_\varepsilon = 1 - F_\varepsilon$. We can write

$$\Delta^w_i(x_i, s) = \int_{-\infty}^{+\infty} \underbrace{\left[a^i(\theta)F_\varepsilon\left(\frac{x_s - \theta}{\sigma}\right) + b^i(\theta)G_\varepsilon\left(\frac{x_s - \theta}{\sigma}\right)\right]}_{\equiv\phi(x_s,\theta)} f(\theta|x_i)\mathrm{d}\theta. \tag{5}$$

Since x_i is the potential threshold of the best response to some other strategy, Lemma 2 implies that we can restrict our attention to $x_i \in [\underline{\theta}_i - r/2, \overline{\theta}_i + r/2]$. This implies that the conditional density $f(\theta|x_i)$ in equation (5) only puts mass on $\theta \in [\underline{\theta}_i - r, \overline{\theta}_i + r]$. For such a θ we know that $h^i(\theta) = b^i(\theta) - a^i(\theta) \geq \hbar \geq 0$. Let us show it implies that the function ϕ, defined in equation (5), is strictly increasing in θ. We have

$$\frac{\partial^- \phi}{\partial \theta} \geq \frac{\partial^- a^i}{\partial \theta} F_\varepsilon + \frac{\partial^- b^i}{\partial \theta} G_\varepsilon + (b^i - a^i) f_\varepsilon.$$

Assumption 2 insures that the first two terms are strictly positive. We just noted that the third term is weakly positive. Finally, note that the conditional distribution $f(\theta|x_i)$ is increasing in x_i in the sense of first order stochastic dominance. Because ϕ is strictly increasing in θ, this implies that Δ^w_i is strictly increasing in x_i. Therefore, $BR^i_w(s)$ is monotone.

For the second part of the lemma, note that equation (5) implies that Δ^w_i is continuous in x_i and x_s. The continuity and strict monotonicity of Δ^w_i imply that the solution $x_i(x_s)$ to $\Delta^w_i(x_i, x_s) = 0$ is continuous in x_s[3]. □

Corollary 1 (extreme strategies). *For all $w \in \mathcal{W}_{\rho,\kappa,\hbar,r,K}$ and $\sigma \in [0, r/2]$, the set of rationalizable strategies of game $\Gamma_\sigma(w)$ is bounded by extreme monotone equilibria.*

Proof. Let R_i denote the set of rationalizable strategies of player i. R_i is the biggest fixed set of $BR^i \circ BR^{-i}$. We know from Lemma 1 that $BR^i \circ BR^{-i}$ is monotonically increasing

[3]Note that when s is a monotone strategy of threshold x_s we use the notation $\Delta^w_i(x_i, x_s) \equiv \Delta^w_i(x_i, s)$.

with respect to the partial order $\preccurlyeq$. Thus we can entirely replicate the construction given by Milgrom and Roberts [5] and Vives [9] for supermodular games.

Begin with the set of all possible strategies, that is, the interval $S_0 = [\mathbf{D}, \mathbf{C}]$, where $\mathbf{D}$ and $\mathbf{C}$ respectively correspond to "always defect" and "always cooperate". We have,

$$BR^i \circ BR^{-i}(S_0) \subset [BR^i \circ BR^{-i}(\mathbf{D}), BR^i \circ BR^{-i}(\mathbf{C})] \subset S_0.$$

By iteratively applying the mapping $BR^i \circ BR^{-i}$, we obtain that for all $k \in \mathbb{N}$, $R^i \subset [(BR^i \circ BR^{-i})^k(\mathbf{D}), (BR^i \circ BR^{-i})^k(\mathbf{C})]$. The extreme strategies are monotone and their thresholds form increasing and decreasing sequences. Monotonicity and boundedness implies these sequences admit limits with respect to convergence in probability. By construction and by continuity of $BR^i \circ BR^{-i}$ with respect to convergence in probability, these limits are Nash equilibria. □

Using this result, the rest of the analysis can now focus on threshold form monotone strategies. Note that monotone strategies are pure strategies.

3.3 Selection in the regular case

In this section we prove *joint selection* and *uniform convergence* in the regular case. Consider a class of payoffs $\mathcal{W}_{\rho,\kappa,\hbar,r,K}$ with $\hbar > 0$. We want to characterize the set of rationalizable strategies of global games $\Gamma_\sigma(w)$ with $w \in \mathcal{W}_{\rho,\kappa,\hbar,r,K}$ and σ small. The first step is to use Corollary 1, which implies that when $\sigma \in [0, r/2]$, we only need to study monotone Nash equilibria to prove that there is a unique rationalizable equilibrium. A monotone equilibrium is characterized by a pair of thresholds (x_i, x_{-i}) such that,

$$\Delta_i^w(x_i, x_{-i}, \sigma) = 0, \quad \text{for } i \in \{1, 2\}. \tag{6}$$

This is in fact the main insight that this paper takes from the global games approach: in the presence of observational noise, players' payoffs must be continuous in their signal. This implies that at a threshold point, players must be indifferent between their two actions, which gives us extra restrictions that equilibria must satisfy. To study equilibrium selection, it is equivalent to study the behavior of the set of indifference equations (6). We have

$$\Delta_i^w(x_i, x_{-i}, \sigma) = \int_{-\infty}^{+\infty} \left[a^i(\theta) F_\varepsilon\left(\frac{x_{-i}-\theta}{\sigma}\right) + b^i(\theta) G_\varepsilon\left(\frac{x_{-i}-\theta}{\sigma}\right)\right] \frac{f_\varepsilon\left(\frac{x_i-\theta}{\sigma}\right) f_\theta(\theta)}{\int_{-\infty}^{+\infty} f_\varepsilon\left(\frac{x_i-\theta}{\sigma}\right) f_\theta(\theta) \mathrm{d}\theta} \mathrm{d}\theta. \tag{7}$$

Let us do the change in variable $u = \frac{x_i-\theta}{\sigma}$. Noting that f_ε only puts mass on the $[-1,1]$ interval, we obtain

$$\begin{aligned} \Delta_i^w(x_i, x_{-i}, \sigma) &= \int_{-1}^{+1} \left[a^i(x_i - \sigma u) F_\varepsilon\left(u + \frac{x_{-i}-x_i}{\sigma}\right)\right. \\ &+ \left. b^i(x_i - \sigma u) G_\varepsilon\left(u + \frac{x_{-i}-x_i}{\sigma}\right)\right] \frac{f_\varepsilon(u) f_\theta(x_i - \sigma u)}{\int_{-1}^{1} f_\varepsilon(u) f_\theta(x_i - \sigma u) \mathrm{d}u} \mathrm{d}u. \end{aligned}$$

The above expression has a $(x_i - x_{-i})/\sigma$ term which blows up as σ goes to 0. In order to have functions that have a continuous limit as σ goes to 0, we define α by $x_{-i} = x_i + \alpha\sigma$ and abuse notations slightly by denoting $\Delta_i^w(x_i, \alpha, \sigma) \equiv \Delta_i^w(x_i, x_{-i}, \sigma)$. We obtain

$$\begin{aligned} \Delta_i^w(x_i, \alpha, \sigma) &= \int_{-1}^{+1} \left[a^i(x_i - \sigma u) F_\varepsilon(u + \alpha)\right. \\ &+ \left. b^i(x_i - \sigma u) G_\varepsilon(u + \alpha)\right] \frac{f_\varepsilon(u) f_\theta(x_i - \sigma u)}{\int_{-1}^{1} f_\varepsilon(u) f_\theta(x_i - \sigma u) \mathrm{d}u} \mathrm{d}u. \end{aligned} \tag{8}$$

We can now think of a monotone equilibrium as a pair (x_i, α), such that $\Delta_i^w(x_i, \alpha, \sigma) = \Delta_{-i}^w(x_i + \alpha\sigma, -\alpha, \sigma) = 0$. The essence of our proof technique is to show that this equation has a unique solution and that it is well behaved in w. To do this we need to understand how Δ_i^w varies with x_i, α and σ.

More precisely we show that : over a specific range for parameters (x_i, α) which is convex and includes any threshold equilibrium for σ small enough, $\Delta_i^w(x_i, \alpha, \sigma)$ is strictly increasing in x_i and strictly decreasing in α with rates bounded away from 0 independently of either σ or w; as σ goes to 0, $\Delta_i^w(x, \alpha, \sigma)$ converges uniformly over $(x, \alpha) \in \mathbb{R}^2$ at a rate that depends only on $\mathcal{W}_{\rho,\kappa,\hbar,r,K}$.

Lemma 4. *For any payoff structure w and any σ, all monotone equilibria (x_i, α) of game $\Gamma_\sigma(w)$ are such that $\alpha \in [-2, 2]$.*

Proof. Consider a potential equilibrium threshold x_i. If player $-i$ gets a signal $x < x_i - 2\sigma$ or $x > x_i + 2\sigma$, then she knows for sure what player i does. Because w is regular, Assumption 1 implies player $-i$ must choose the same action. This implies that if (x_i, α) is an equilibrium, then $\alpha \in [-2, 2]$.

□

Lemma 5. *Given ρ, κ, r and K, there exists $\overline{\sigma} > 0$ small enough such that for all $\hbar > 0$, $\sigma \in (0, \overline{\sigma})$, $\alpha \in [-2, 2]$ and $w \in \mathcal{W}_{\rho,\kappa,\hbar,r,K}$*

$$\forall\, x_i \;\in\; [\underline{\theta}_i(w) - r, \overline{\theta}_i(w) + r], \quad \frac{\partial \Delta_i^w}{\partial x_i}(x_i, \alpha, \sigma) \geq \kappa/2 > 0.$$

Proof. Define

$$\begin{aligned} \xi(x_i, \alpha, u) &= a^i(x_i - \sigma u) F_\varepsilon(u + \alpha) + b^i(x_i - \sigma u) G_\varepsilon(u + \alpha) & (9) \\ \Psi(x_i, u) &= \frac{f_\varepsilon(u) f_\theta(x_i - \sigma u)}{\int_{-1}^{1} f_\varepsilon(u) f_\theta(x_i - \sigma u) \mathrm{d}u}. & (10) \end{aligned}$$

Note that we know from equation (7) that Δ_i^w is differentiable in x_i. We have

$$\frac{\partial \Delta_i^w}{\partial x_i} \geq \int_{-1}^{1} \left[\frac{\partial^- \xi}{\partial x_i} \Psi + \xi \frac{\partial \Psi}{\partial x_i} \right] \mathrm{d}u. \qquad (11)$$

From equation (9) and by definition of $\mathcal{W}_{\rho,\kappa,\hbar,r,K}$, we obtain that

$$\frac{\partial^- \xi}{\partial x_i} \geq \frac{\partial^- a_w^i}{\partial \theta} F_\varepsilon + \frac{\partial^- b_w^i}{\partial \theta} G_\varepsilon \geq \kappa. \tag{12}$$

Moreover, we have

$$\frac{\partial \Psi}{\partial x_i} = \frac{f_\varepsilon f_\theta'}{\int_{-1}^{1} f_\varepsilon(u) f_\theta(x_i - \sigma u) \mathrm{d}u} - \frac{f_\varepsilon f_\theta \int_{-1}^{1} f_\varepsilon(u) f_\theta'(x_i - \sigma u) \mathrm{d}u}{\left[\int_{-1}^{1} f_\varepsilon(u) f_\theta(x_i - \sigma u) \mathrm{d}u\right]^2}. \tag{13}$$

Note that equation (13) does not depend on the particular payoff structure w. Let us show that as σ goes to zero, $\frac{\partial \Psi}{\partial x_i}$ converges to 0 uniformly over K. Indeed, for any continuous function g, $\int_{-1}^{1} f_\varepsilon(u) g(x_i - \sigma u) \mathrm{d}u$ converges uniformly to $g(x_i)$ over any compact set. By assumption, the density f_θ is bounded away from 0 over the range $[\underline{\theta}_i(w) - r, \overline{\theta}_i(w) + r]$. This implies that uniformly over $[\underline{\theta}_i(w) - r, \overline{\theta}_i(w) + r]$,

$$\lim_{\sigma \to 0} \frac{\partial \Psi}{\partial x_i} = \frac{f_\varepsilon(u) f_\theta'(x_i)}{f_\theta(x_i)} - \frac{f_\varepsilon(u) f_\theta(x_i) f_\theta'(x_i)}{f_\theta(x_i)^2} = 0. \tag{14}$$

Since all payoffs w have a common modulus of continuity, and because $a_w^i(\overline{\theta}_i) = b_w^i(\underline{\theta}_i) = 0$, there exists a constant $M \in \mathbb{R}$ such that for all $w \in \mathcal{W}_{\rho,\kappa,\hbar,r,K}$ and all $\theta \in K$,

$$|\xi_w(\theta, \alpha, u)| < |a_w^i(\theta)| + |b_w^i(\theta)| < M. \tag{15}$$

Equations (14) and (15) imply there exists $\overline{\sigma}$ small enough such that whenever $\sigma \in (0, \overline{\sigma})$, then $\left|\xi \frac{\partial \Psi}{\partial x_i}\right| \leq \kappa/2$. This and equation (12) imply that over K, $\frac{\partial \Delta_i^w}{\partial x_i} > \kappa/2$. $\square$

Lemma 6. *For all $\sigma < r/2$, $x_i \in [\underline{\theta}_i(w) - r/2, \overline{\theta}_i(w) + r/2]$ and $w \in \mathcal{W}_{\rho,\kappa,\hbar,r,K}$,*

$$\forall\, \alpha \in [-2, 2], \quad \frac{\partial \Delta_i^w(x_i, \alpha, \sigma)}{\partial \alpha} \leq -\hbar \int_{-1}^{1} f_\varepsilon(u + \alpha) \Psi(x_i, u) du \leq 0.$$

Proof. We know from equation (8) that Δ_i^w is differentiable in α. We have,

$$\frac{\partial \Delta_i^w}{\partial \alpha} \leq \int_{-1}^{1} \frac{\partial^+ \xi}{\partial \alpha} \Psi \mathrm{d}u.$$

Moreover,

$$\begin{aligned} \frac{\partial^+ \xi}{\partial \alpha}(x_i, \alpha, u) &= \left[a^i(x_i - \sigma u) - b^i(x_i - \sigma u)\right] f_\varepsilon(u + \alpha) \\ &= -h_w^i(x_i - \sigma u) f_\varepsilon(u + \alpha). \end{aligned}$$

By assumption, for all $\theta \in [\underline{\theta}_i(w) - r, \overline{\theta}_i(w) + r]$ we have $h_w^i(\theta) \geq \hbar$. This yields,

$$\frac{\partial \xi}{\partial \alpha}(x_i, \alpha, u) \leq -\hbar f_\varepsilon(u + \alpha). \tag{16}$$

Integrating over $[-1, 1]$, we obtain

$$\frac{\partial \Delta_i^w(x_i, \alpha, \sigma)}{\partial \alpha} \leq -\hbar \int_{-1}^{1} f_\varepsilon(u + \alpha) \Psi(x_i, u) \mathrm{d}u < 0.$$

□

We can now state our first selection result. It says that for all σ less than some $\overline{\sigma}$ small enough, selection happens jointly for all games with payoffs in $\mathcal{W}_{\rho,\kappa,\hbar,r,K}$. It does not discuss how the selected equilibria behave as σ goes to 0. This will be the object of Theorem 3.

Theorem 2 (joint selection). *There exists $\overline{\sigma} > 0$ sufficiently small such that for all $\sigma \in (0, \overline{\sigma})$ and $w \in \mathcal{W}_{\rho,\kappa,\hbar,r,K}$, all global games $\Gamma_\sigma(w)$ have a unique pair of rationalizable strategies.*

Proof. Take $\overline{\sigma}$ such that Lemmas 5 and 6 hold. We know from Theorem 1 that the set of rationalizable strategies is bound by monotone equilibria, so it suffices to show there is a unique monotone equilibrium. Such an equilibrium is characterized by a pair (x_i, α) such that, $\Delta_i^w(x_i, \alpha, \sigma) = \Delta_{-i}^w(x_i + \alpha\sigma, -\alpha, \sigma) = 0$. From Lemmas 5 and 6 we know that Δ_i^w is strictly increasing in x_i and weakly decreasing in α. Thus, the first equilibrium condition $\Delta_i^w(x_i, \alpha) = 0$ implicitly defines a function $\alpha(x_i)$ that is weakly increasing in x_i. Replace that in the other equilibrium condition: x_i is such that $\Delta_{-i}^w(x_i + \alpha(x_i)\sigma, -\alpha(x_i), \sigma) = 0$. Define $\zeta(x_i, \sigma, w) \equiv \Delta_{-i}^w(x_i + \alpha(x_i)\sigma, -\alpha(x_i), \sigma)$. Lemmas 5 and 6 imply that this function is strictly increasing in x_i which implies there is at most a unique value x_i satisfying $\zeta(x_i, \sigma, w) = 0$. Existence results from Assumption 3. Using Corollary 1 we conclude that there is a unique pair of rationalizable strategies. $\square$

Lemma 7. *Pick r, ρ and K, then there exists $N > 0$ such that for all κ, $\hbar$ and $w \in \mathcal{W}_{\rho,\kappa,\hbar,r,K}$,*

$$|\Delta_i^w(x, \alpha, \sigma) - \Delta_i^w(x, \alpha, 0)| \leq N \max\{\rho(\sigma), \sigma\}.$$

Proof. All payoff functions w in $\mathcal{W}_{\rho,\kappa,\hbar,r,K}$ share a common modulus of continuity. Moreover, we know from equation (15) that there exists M such that for all $w \in \mathcal{W}_{\rho,\kappa,\hbar,r,K}$, we have $|a_w^i| + |b_w^i| < M$ over K. Denoting by $||\cdot||_\infty$ the supremum norm, this implies that

$$|\Delta_i^w(x, \alpha, \sigma) - \Delta_i^w(x, \alpha, 0)| \leq 4\rho(\sigma) + M||f_\theta'||_\infty \sigma.$$

This shows there exists $N > 0$ independent of w such that $|\Delta_i^w(x, \alpha, \sigma) - \Delta_i^w(x, \alpha, 0)| \leq N \max\{\rho(\sigma), \sigma\}$. $\square$

Without loss of generality, we can always assume that $\rho(\sigma) \geq \sigma$. Indeed if a function has a modulus of continuity ρ, it has a modulus of continuity $\tilde{\rho}$ for all $\tilde{\rho}$ greater than ρ.

We now want to show that the unique equilibrium thresholds selected in Theorem 2 converge towards the risk-dominance threshold uniformly over $\mathcal{W}_{\rho,\kappa,\hbar,r,K}$. We know from Lemma 1 that it is equivalent to consider payoff classes $\Lambda_{\rho,\nu,\kappa,d,K}$ and $\mathcal{W}_{\rho,\kappa,\hbar,r,K}$ when $\hbar > 0$. Here we switch to working with a payoff class $\Lambda_{\rho,\nu,\kappa,d,K}$ because it simplifies notations.

Lemma 8. *Given, ρ, κ and ν, there exist strictly increasing continuous functions η, λ and τ mapping $\mathbb{R}^+$ into $\mathbb{R}^+$, such that for any family of payoff structures $\Lambda_{\rho,\nu,\kappa,d,K}$, whenever $\sigma < \tau(d)$, any monotone equilibrium (x_i, α) of $\Gamma_\sigma(w)$ with $w \in \Lambda_{\rho,\nu,\kappa,d,K}$ is such that,*

1. $\alpha \in [-2 + \lambda(d), 2 - \lambda(d)]$

2. $\forall x_i \in [\underline{\theta}_i - \nu, \overline{\theta}_i + \nu]$, $\forall \alpha \in [-2 + \lambda(d), 2 - \lambda(d)]$,

$$\frac{\partial \Delta_i^w(x_i, \alpha, \sigma)}{\partial \alpha} < -\eta(d)$$

3. *Denote by $\alpha(x_i, w)$ the implicit function solving $\Delta_i^w(x_i, \alpha, \sigma) = 0$. For all x_i in $[\underline{\theta}_i - \nu, \overline{\theta}_i + \nu]$, $\alpha(x_i, w)$ is $(\frac{4}{\eta(d)})$-Lipschitz in w, with respect to the norm on payoff structures defined by, $||w - \tilde{w}|| \equiv \max_{i,j,k \in \{1,2\}^3} ||w_{jk}^i - \tilde{w}_{jk}^i||_\infty$.*

Proof. A monotone equilibrium is characterized by a pair (x_i, α) such that $\Delta_i^w(x_i, \alpha, \sigma) = \Delta_{-i}^w(x_i + \alpha\sigma, -\alpha, \sigma) = 0$. We know from Lemma 2 that whenever $\sigma < \nu$ $x_i \in \cap_{i \in \{1,2\}} [\underline{\theta}_i - \nu, \overline{\theta}_i + \nu]$. Moreover, we must have $\alpha \in [-2, 2]$.

Let us first show the tighter bounds on equilibrium values of α. Define $\chi_\sigma^w(x_i, \alpha) = \Delta_i^w(x_i, \alpha, \sigma) - \Delta_{-i}^w(x_i + \alpha\sigma, -\alpha, \sigma)$. If (x_i, α) is an equilibrium, then $\chi_\sigma^w(x_i, \alpha) = 0$. At the limit where $\sigma = 0$, we have,

$$\begin{aligned} \chi_0^w(x_i, \alpha) \;=\; & \int_{-1}^{1} \Big[a_w^i(x_i) F_\varepsilon(u + \alpha) + b_w^i(x_i) G_\varepsilon(u + \alpha) \\ & - a_w^{-i}(x_i) F_\varepsilon(u - \alpha) - b_w^{-i}(x_i) G_\varepsilon(u - \alpha) \Big] f_\varepsilon(u) \mathrm{d}u \end{aligned} \tag{17}$$

Which yields,

$$\chi_0^w(x_i, -2) = b_w^i(x_i) - a_w^{-i}(x_i) \geq d\kappa \tag{18}$$

$$\chi_0^w(x_i, 2) = a_w^i(x_i) - b_w^{-i}(x_i) \leq -d\kappa \tag{19}$$

Over this range, we know there exists M dependent only on ρ and K such that $|b_w^i| + |a_w^{-i}| < M$. Moreover, f_ε is bounded over $[-1, 1]$; thus we conclude from equation (17), that there exists a constant $Q > 0$ such that $\chi_0^w(x_i, \alpha)$ is Q-Lipschitz in α. This and equations (18) and (19) imply that,

$$\begin{aligned}
\chi_0^w(x_i, \alpha) &\geq d\kappa - Q(\alpha + 2) \\
\chi_0^w(x_i, \alpha) &\leq -d\kappa + Q(2 - \alpha)
\end{aligned}$$

Finally, using Lemma 7, we know there exists N depending only on ρ and K such that

$$\chi_\sigma^w(x_i, \alpha) \geq d\kappa - Q(\alpha + 2) - N\rho(\sigma) \tag{20}$$

$$\chi_\sigma^w(x_i, \alpha) \leq -d\kappa + Q(2 - \alpha) + N\rho(\sigma) \tag{21}$$

Using equations (20) and (21) and the fact that at an equilibrium (x_i, α), we must have $\chi_\sigma^w(x_i, \alpha) = 0$, we obtain that whenever (x_i, α) is an equilibrium, then we must have

$$\alpha \in \left[-2 + \frac{d\kappa - N\rho(\sigma)}{Q}, 2 - \frac{d\kappa - N\rho(\sigma)}{Q}\right]. \tag{22}$$

From Lemma 6, we know that,

$$\frac{\partial \Delta_i^w(x_i, \alpha, \sigma)}{\partial \alpha} \leq -\hbar \int_{-1}^{1} f_\varepsilon(u + \alpha)\Psi(x_i, u)\mathrm{d}u$$

We know from equation (10) and the fact that f_θ is C^1, that there exists some constant, $C > 0$ depending only on f_ε and f_θ such that $|\Psi(x_i, u) - f_\varepsilon(u)| \leq C\sigma$. Thus, over the range $\left[-2 + \frac{d\kappa - N\rho(\sigma)}{Q}, 2 - \frac{d\kappa - N\rho(\sigma)}{Q}\right]$,

$$\frac{\partial \Delta_i^w(x_i, \alpha, \sigma)}{\partial \alpha} \leq -d\kappa \int_{-1}^{1} f_\varepsilon(u + \alpha) f_\varepsilon(u)\mathrm{d}u + C\sigma. \tag{23}$$

Define for any $\lambda > 0$,

$$m(\lambda) = \min_{\alpha \in [-2+\lambda, 2-\lambda]} \int_{-1}^{1} f_\varepsilon(u+\alpha) f_\varepsilon(u) \mathrm{d}u.$$

Because f_ε is strictly positive over $(-1,1)$, m is positive and strictly increasing in λ. We now define $\tau(d)$ by,

$$\tau(d) \equiv \{\rho^{-1}(d\kappa/2N)\} \wedge \{d\kappa \times m(d\kappa/2Q)/2C\}.$$

For all $\sigma < \tau(d)$, we obtain that equilibrium values of α are such that,

$$\alpha \in [-2 + d\kappa/2Q, 2 - d\kappa/2Q] \equiv [-2 + \lambda(d), 2 - \lambda(d)].$$

And over that range,

$$\frac{\partial \Delta_i^w(x_i, \alpha, \sigma)}{\partial \alpha} < -d\kappa \times m(d\kappa/2Q)/2 \equiv -\eta(d). \tag{24}$$

Finally, given α and x_i and σ, the function $\Delta_i^w(x_i, \alpha, \sigma)$ is 4-Lipschitz in w. This and equation (24) yield that $\alpha(x_i, w, \sigma)$ is $(\frac{4}{\eta(d)})$-Lipschitz in w. □

We can now state the main result of the paper.

Theorem 3 (uniform convergence). *Consider a class of payoffs $\Lambda_{\rho,\nu,\kappa,d,K}$. We know from Theorem 2 that for $\sigma \in (0, \overline{\sigma})$, all games $\Gamma_\sigma(w)$ have a unique pair of rationalizable strategies, with thresholds $(x_i(w,\sigma), x_{-i}(w,\sigma))$. As σ goes to 0, the equilibrium threshold $x_i(w,\sigma)$ converges uniformly over $\Lambda_{\rho,\nu,\kappa,d,K}$ to the risk-dominant threshold. More precisely, for any $\sigma \in (0, \overline{\sigma})$,*

$$|x_i(w,\sigma) - x_i(w,0)| \le \frac{2}{\kappa}\left[4\rho\left(\frac{\sigma N \rho(\sigma)}{\eta(d)}\right) + 2M||f_\varepsilon||_\infty \frac{N\rho(\sigma)}{\eta(d)}\right] \tag{25}$$

where we use the constant N and the functions ρ and η defined in Lemmas 7 and 8.

Note that if we had not restricted ourselves to the regular case, inequality (25) would not be defined since in a non-regular case $d = 0$.

Proof. First, from Lemmas 7 and 8, we know that $|\Delta_i^w(x, \alpha, \sigma) - \Delta_i^w(x, \alpha, 0)| < N\rho(\sigma)$ and that $\frac{\partial \Delta}{\partial \alpha} < -\eta(d)$. This implies that the solution $\alpha(x, \sigma)$ to $\Delta_i^w(x, \alpha, \sigma) = 0$ must satisfy

$$|\alpha(x_i, w, \sigma) - \alpha(x_i, w, 0)| \leq \frac{N\rho(\sigma)}{\eta(d)}$$

Recalling the definition, $\zeta(x, w, \sigma) \equiv \Delta_{-i}(x + \alpha(x, w, \sigma)\sigma, -\alpha(x, w, \sigma), \sigma)$, and using the majoration of inequality (15), we obtain after simple manipulations,

$$|\zeta(x_i, w, \sigma) - \zeta(x_i, w, 0)| \leq 4\rho\left(\frac{\sigma N\rho(\sigma)}{\eta(d)}\right) + 2M||f_\varepsilon||_\infty \frac{N\rho(\sigma)}{\eta(d)}$$

From Lemma 5 we know that $\frac{\partial^- \zeta}{\partial x} > \kappa/2$. This yields

$$|x_i(w, \sigma) - x_i(w, 0)| \leq \frac{2}{\kappa}\left[4\rho\left(\frac{\sigma N\rho(\sigma)}{\eta(d)}\right) + 2M||f_\varepsilon||_\infty \frac{N\rho(\sigma)}{\eta(d)}\right].$$

□

Theorem 3 implies that in the example given in the introduction, the principal's optimal level of capital stock for σ positive does converge to the optimal capital stock in the risk-dominant equilibrium, as long as the family of payoffs indexed by k belongs to some regular class $\Lambda_{\rho,\nu,\kappa,d,K}$.

The next two theorems deal with the continuity of the selected equilibrium with respect to the payoff structure. These continuity results are useful in applications, for instance to ensure the existence of maxima. Theorem 5, can be used to apply global games selection recursively and thus combine the Abreu, Pearce and Stachetti (1990) stationary approach to dynamic games and a global games information structure.

Theorem 4 (continuous selection). *Consider a class of payoff structures $\Lambda_{\rho,\nu,\kappa,d,K}$, the functions $\eta(d)$, $\lambda(d)$ and $\tau(d)$ defined in Lemma 8 and the joint selection threshold $\overline{\sigma}$ of Theorem 2. Then, for $\sigma \in (\overline{\sigma} \wedge \tau(d))$, the selected equilibrium threshold $x_i(w,\sigma)$ is a continuous function of the payoff structure w. More strongly, x_i has a modulus of continuity in w defined by the inequality*

$$|x_i(w,\sigma) - x_i(\tilde{w},\sigma)| \leq \frac{2}{\kappa}\left[\left(4 + 2M||f_\varepsilon||_\infty \frac{1}{\eta(d)}\right)||w-\tilde{w}|| + 4\rho\left(\frac{4\sigma||w-\tilde{w}||}{\eta(d)}\right)\right] \tag{26}$$

where $||\cdot||_\infty$ denotes the sup. norm, and $||w-\tilde{w}|| \equiv \max_{i,j,k\in\{1,2\}^3}||w^i_{jk} - \tilde{w}^i_{jk}||_\infty$.

Proof. $\Delta^w_i(x_i,\alpha,\sigma)$ is continuous in w. In fact it is 4−Lipschitz in w. This and Lemma 8 imply that the implicit function $\alpha(x_i,w)$ is strictly increasing in x_i and $(\frac{4}{\eta(d)})$-Lipschitz in w. Thus the function $\zeta : x_i \mapsto \Delta^w_{-i}(x_i + \alpha(x_i)\sigma, -\alpha(x_i),\sigma)$ is continuous in the payoffs and strictly increasing in x_i. This implies that the solution to $\Delta^w_i(x_i + \alpha(x_i)\sigma, -\alpha(x_i),\sigma) = 0$ is a continuous function of the payoffs.

To get the announced modulus of continuity, note that from Lemma 5 we know that the function $\zeta(x_i,\sigma,w) = \Delta^w_{-i}(x_i + \alpha(x_i)\sigma, -\alpha(x_i),\sigma)$ has a lower rate in x_i greater than $\kappa/2$. Moreover, because $\alpha(x_i,w)$ is $(\frac{4}{\eta(d)})$-Lipschitz in w, we have

$$|\zeta(x_i,w) - \zeta(x_i,\tilde{w})| \leq \left(4 + 2M||f_\varepsilon||_\infty \frac{1}{\eta(d)}\right)||w-\tilde{w}|| + 4\rho\left(\frac{4\sigma||w-\tilde{w}||}{\eta(d)}\right)$$

This implies that the implicit function $x_i(w)$ solution to $\zeta(x_i,w) = 0$ has the modulus of continuity defined by inequality (26). □

So far we considered general payoff classes. In applications, however, it is frequent for payoff classes to be parameterized by some finite vector of real numbers. The following theorem describes how the selected equilibrium varies as a function of such a parameter.

Theorem 5 (parametrized payoffs). *Consider a family of payoffs $w_k \in \Lambda_{\rho,\nu,\kappa,d,K}$ parametrized by $k \in \mathbb{R}$ and such that there exists $H \in \mathbb{R}$ satisfying*

$$\forall i \in \{1,2\}, \quad \left|\frac{\frac{\partial^+ a^i}{\partial k}}{\frac{\partial^- a^i}{\partial \theta}}\right| < H \quad and \quad \left|\frac{\frac{\partial^+ b^i}{\partial k}}{\frac{\partial^- b^i}{\partial \theta}}\right| < H$$

Then, there exists $\overline{\sigma} > 0$ such that for all $\sigma < \overline{\sigma}$, the unique equilibrium $(x_i(k,\sigma), x_{-i}(k,\sigma))$ of $\Gamma_\sigma(w_k)$ is $2H$-Lipschitz in k.

Proof. An equilibrium (x_k, α_k) is a solution to the pair of equations, $\Delta_i^w(x_k, \alpha_k, \sigma, k) = \Delta_{-i}(x_k + \alpha_k\sigma, -\alpha_k, \sigma, k) = 0$. Assume temporarily that the functions involved are infinitely differentiable in all arguments. Denote $\alpha(k,x)$ the solution to $\Delta_i^w(x,\alpha,\sigma,k) = 0$. Differentiate this equation.

$$\begin{aligned} d\Delta_i^w &= \int_{-1}^{1} \left(\left[\frac{\partial a^i}{\partial k}F_\varepsilon + \frac{\partial b^i}{\partial k}G_\varepsilon\right] dk + \left[\frac{\partial a^i}{\partial \theta}F_\varepsilon + \frac{\partial b^i}{\partial \theta}G_\varepsilon\right] dx + [a^i - b^i]f_\varepsilon\, d\alpha \right) \Psi_\sigma(u|x)du \\ &+ \int_{-1}^{1} \left(a^i F_\varepsilon + b^i G_\varepsilon\right) \frac{\partial \Psi_\sigma(u|x)}{\partial x} du\, dx = 0 \end{aligned} \tag{27}$$

We showed in Lemma 5 that $\frac{\partial \Psi_\sigma(u|x)}{\partial x}$ converged uniformly to 0 over K as σ goes to 0. Moreover, $\frac{\partial a^i}{\partial \theta}F_\varepsilon + \frac{\partial b^i}{\partial \theta}G_\varepsilon > \kappa > 0$, thus for σ small enough,

$$\left|\int_{-1}^{1} \left(a^i F_\varepsilon + b^i G_\varepsilon\right) \frac{\partial \Psi_\sigma(u|x)}{\partial x} du\right| < \frac{1}{2}\int_{-1}^{1} \left[\frac{\partial a^i}{\partial \theta}F_\varepsilon + \frac{\partial b^i}{\partial \theta}G_\varepsilon\right] \Psi_\sigma(u|x)du$$

Using equation (27) and the fact that if $a, b, c, d > 0$ are such that $a/c < m$ and $b/d < m$ then $\frac{a+b}{c+d} < m$, we obtain, by considering the integral as a limit of sums, that

$$\left|\frac{\frac{\partial \alpha}{\partial k}}{\frac{\partial \alpha}{\partial x}}\right| \le 2 \left|\frac{\int_{-1}^{1} \left[\frac{\partial a^i}{\partial k}F_\varepsilon + \frac{\partial b^i}{\partial k}G_\varepsilon\right] \Psi_\sigma(u|x)du}{\int_{-1}^{1} \left[\frac{\partial a^i}{\partial \theta}F_\varepsilon + \frac{\partial b^i}{\partial \theta}G_\varepsilon\right] \Psi_\sigma(u|x)du}\right| \le 2H. \tag{28}$$

Similarly, we know that x_k is the unique solution of $\zeta_k(x) = 0$. Differentiate that equation,

$$d\zeta = \int_{-1}^{1} \left[\left(\frac{\partial a^{-i}}{\partial k} + \sigma \frac{\partial a^{-i}}{\partial \theta} \frac{\partial \alpha}{\partial k} \right) F_\varepsilon + \left(\frac{\partial b^{-i}}{\partial k} + \sigma \frac{\partial b^{-i}}{\partial \theta} \frac{\partial \alpha}{\partial k} \right) G_\varepsilon \right] \Psi_\sigma(u|x) du\, dk$$
$$+ \int_{-1}^{1} \left(\left[\frac{\partial a^{-i}}{\partial \theta} (1 + \sigma \frac{\partial \alpha}{\partial x}) F_\varepsilon + \frac{\partial b^{-i}}{\partial \theta} (1 + \sigma \frac{\partial \alpha}{\partial x}) G_\varepsilon \right] \Psi_\sigma(u|x) + \left(a^{-i} F_\varepsilon + b^{-i} G_\varepsilon \right) \frac{\partial \Psi_\sigma(u|x)}{\partial x} \right) du\, dx = 0$$

Using the above equation, it is possible to express $\frac{\partial x}{\partial k}$ as a ratio involving $\frac{\partial a^{-i}}{\partial \theta}$, $\frac{\partial b^{-i}}{\partial \theta}$ and $\frac{\partial \alpha}{\partial x}$. By exploiting this expression of $\frac{\partial x}{\partial k}$, inequality (28) and the fact that $\frac{\partial \alpha}{\partial x} \geq 0$, it results from simple but tedious algebra, which we skip, that,

$$\left| \frac{\partial x}{\partial k} \right| \leq 2H. \tag{29}$$

Finally, since the Lipschitz rate of x with respect to k is independent of the smoothness of the equilibrium equations, we can use the fact that smooth functions are dense to conclude that the $2H$-Lipschitz continuity of x holds at the limit, even when none of the involved functions are differentiable. □

Extending the result to a multidimensional k poses no difficulty. One can simply use Theorem 5 iteratively along each dimension.

4 Games without dominance regions

The view of global games fostered in this paper is that observational noise imposes that in equilibrium, players' payoffs should be continuous in the state of the world. More precisely, if x^* is such that player i switches from playing D to playing C around $x*$, then at $x*$ player i must be indifferent between playing C and D. This gives additional indifference conditions that an equilibrium with switching must satisfy. This obviously restricts the set of equilibria with switching compared with the full information case. We have shown that these additional restrictions can be used to prove uniqueness.

From this perspective, the role of dominance regions is simply to insure that any equilibrium *must* have a switch point. Thus, it should be possible to get selection results over the class of equilibria that admit at least one switch point even when the global game does not admit dominance regions. This is what we show next. It is a useful result for endogenous global games when the existence of dominance regions cannot be assumed.

Definition 7. *Consider a payoff structure w satisfying Assumptions 1 and 2, but not necessarily Assumption 3. We say that w admits a finite risk-dominance threshold whenever both (C, C) and (D, D) have non-empty risk-dominant regions.*

Theorem 6 (selection in games without dominance regions). *Consider a payoff structure w satisfying Assumptions 1, 2, without dominance regions, but such that there is a finite risk-dominant threshold. Then, there exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$, the global game Γ_σ admits only three Nash equilibria: cooperating always, defecting always and a monotone equilibrium with switching from defection to cooperation at a threshold converging to the risk-dominance threshold as σ goes to 0.*

Proof. Always defecting and always cooperating are clearly equilibria. Now for the third equilibrium, consider once again the function $\Delta_i^w(x, s_{-i})$, which we will denote $\Delta_i^w(x, x_{-i})$ whenever strategy s_{-i} is monotone with threshold x_{-i}. We know that for σ small enough, $\Delta_i^w(x, y)$ is strictly increasing in x and strictly decreasing in y. Denoting $x_{-i}(x)$ the solution of equation $\Delta_i^w(x, x_{-i}) = 0$, we also showed in the proof of Theorem 2 that, $\zeta_{-i}(x) \equiv \Delta_{-i}(x_{-i}(x), x)$ is strictly increasing in x. Because there is a finite risk-dominant threshold, we know that for σ small enough there is a solution to equation $\Delta_{-i}(x_{-i}(x), x) = 0$. Thus, we do have at least one monotone equilibrium with switching, which we know must converge to the risk-dominant equilibrium as σ goes to 0. Let us show it is the only equilibrium with switching.

Denote (x_i^*, x_{-i}^*) the thresholds of the monotone equilibrium defined above. Consider a potential equilibrium (s_i, s_{-i}) involving switching. Denote $\underline{x}_i$ and $\overline{x}_i$ the infimum and supremum of switch points of player i when she uses strategy s_i. We will show that $\underline{x}_i = \overline{x}_i = x_i^*$.

Since player i must be indifferent on either side of a switch point, we must have $\Delta_i^w(\underline{x}_i, s_{-i}) = 0$. Since there are no dominance regions, Assumption 1 implies that h_w^i is positive over the entire state space I. Noting that $s_{-i} \preccurlyeq s_{\underline{x}_{-i}}$, we infer from Theorem 1 that $\Delta_i^w(\underline{x}_i, \underline{x}_{-i}) \geq \Delta_i^w(\underline{x}_i, s_{-i}) = 0$. Since $\Delta_i^w(x_i, x_{-i})$ is decreasing in x_{-i}, this implies that $\underline{x}_{-i} \leq x_{-i}(\underline{x}_i)$. Since $\Delta_{-i}^w(x_{-i}, x_i)$ is increasing in its first argument we obtain,

$$\zeta_{-i}(\underline{x}_i) \geq \Delta_{-i}^w(x_{-i}(\underline{x}_i), \underline{x}_i) \geq \Delta_{-i}^w(\underline{x}_{-i}, \underline{x}_i) \geq \Delta_{-i}^w(\underline{x}_{-i}, s_i) \geq 0 = \zeta_{-i}(x_i^*)$$

Which implies that $\underline{x}_i \geq x_i^*$ since ζ_{-i} is increasing.

Similarly we can prove that $\overline{x}_i \leq x_i^*$. These two inequalities imply $\underline{x}_i = \overline{x}_i = x_i^*$. □

5 Conclusion

The approach of global games taken in this paper stresses the fact that when players observe noisy signals, their payoffs should be continuous in their signals. This gives additional constraints on equilibria which can be exploited to prove uniform selection results over fairly general classes of payoffs. This view of global games also permits to describe the set of Nash equilibria when the game lacks dominance regions.

These uniform selection results are handy tools for applied economists wishing to use the global games approach to study more intricate game structures. For instance, in the agency problem presented in the introduction, we can now conclude that the limit of the principal's behavior is indeed the best response to the limit of the agents' behavior.

Finally, because these uniform selection results hold over general classes of payoffs, they also allow to apply global games selection recursively and may prove useful to extend the use of such an information structure to dynamic games.

Appendix A: uniform selection in the non-regular case

The regular case is the case for which selection is a relevant question under full information. Indeed, under full information, the non-regular case is dominance solvable. However, one might worry that private information induces unexpected behavior. In order to have a framework that imposes the least number of constraints on the payoff structure, we prove uniform selection results that hold in the non-regular case as well.

Theorem 7 (uniform convergence). *Consider some class of payoffs $\mathcal{W} = \mathcal{W}_{\rho,\kappa,\hbar,r,K}$ with $\hbar = 0$. Then as σ goes to 0, the set or rationalizable strategies of $\Gamma_\sigma(w)$ converges uniformly over $\mathcal{W}$ to the risk-dominant equilibrium.*

Proof. We first note that $\mathcal{W} = \bigcup_{d>0} \Lambda_{\rho,\nu,\kappa,d,K} \cup (\mathcal{W} \setminus \bigcup_{d>0} \Lambda_{\rho,\nu,\kappa,d,K})$. We know that if $w \in (\mathcal{W} \setminus \bigcup_{d>0} \Lambda_{\rho,\nu,\kappa,d,K})$, then, w is non-regular and any selection problem is spurious because the players' zones of non-dominance do not overlap. Let x_0 be the unique equilibrium threshold under common knowledge. Any rationalizable strategy of $\Gamma_\sigma(w)$ belongs to $[s_{x_0-2\sigma}, s_{x_0+2\sigma}]$.

Now let us consider $w \in \bigcup_{d>0} \Lambda_{\rho,\nu,\kappa,d,K}$. From Lemma 2, we know that if (x_i, x_{-i}) is an equilibrium, then $x_i \in \bigcap_{i\in\{1,2\}} [s_{\underline{\theta}_i-2\sigma}, s_{\overline{\theta}_i+2\sigma}]$. By definition, the risk-dominant threshold belongs to the same set. This implies that if $\overline{\theta}_i(w) - \underline{\theta}_i(w) < d$, then

$$|x_i(w,\sigma) - x_i(w,0)| < d + 2\sigma. \tag{30}$$

Using inequality (25), obtained for $d > 0$ in Theorem 3, we get that,

$$|x_i(w,\sigma) - x_i(w,0)| < \frac{2}{\kappa}\left[4\rho\left(\frac{\sigma N\rho(\sigma)}{\eta(d)}\right) + 2M||f_\varepsilon||_\infty \frac{N\rho(\sigma)}{\eta(d)}\right]. \tag{31}$$

Define $d(\sigma) \equiv \eta^{-1}\left(\sqrt{\rho(\sigma)}\right)$. Note that $d(\cdot)$ is continuous, strictly increasing and satisfies $d(0) = 0$. Using inequality (30) for $d < d(\sigma)$ and inequality (31) for $d \geq d(\sigma)$, one obtains,

$$|x_i(w,\sigma) - x_i(w,0)| < \max\left\{\eta^{-1}\left(\sqrt{\rho(\sigma)}\right) + 2\sigma, \frac{2}{\kappa}\left[4\rho\left(\sigma N\sqrt{\rho(\sigma)}\right) + 2M||f_\varepsilon||_\infty N\sqrt{\rho(\sigma)}\right]\right\}$$

As σ goes to 0, this uniform upper bound goes to 0 as well. $\square$

Theorem 8 (continuous selection). *Consider some class of payoffs $\mathcal{W} = \mathcal{W}_{\rho,\kappa,\hbar,r,K}$ with $\hbar = 0$. Then there exists $\overline{\sigma} > 0$ such that for all $\sigma < \overline{\sigma}$, the equilibrium $(x_i(w,\sigma), x_{-i}(w,\sigma))$ is continuous in w over $\mathcal{W}$.*

Proof. Note that $\mathcal{W} = \overline{\bigcup_{h>0}\mathcal{W}_{\rho,\kappa,\hbar,r,K}}$. We know from Theorem 4 that $(x_i(w,\sigma), x_{-i}(w,\sigma))$ is continuous in w over $\bigcup_{h>0}\mathcal{W}_{\rho,\kappa,\hbar,r,K}$. For any $w \notin \bigcup_{h>0}\mathcal{W}_{\rho,\kappa,\hbar,r,K}$ pick a sequence $\{w_n\}_{n\geq 0} \subset \bigcup_{h>0}\mathcal{W}_{\rho,\kappa,\hbar,r,K}$, converging uniformly towards w. The payoff structures $\{w_n\}_{n\in\mathbb{N}}$ are associated with a sequence of equilibria $\{(x_i^n, x_{-i}^n)\}_{n\geq 0}$ all satisfying $\Delta_i^{w_n}(x_i^n, x_{-i}^n) = \Delta_{-i}^{w_n}(x_{-i}^n, x_i^n) = 0$. By compactness one can extract converging sequences from $\{(x_i^n, x_{-i}^n)\}_{n\geq 0}$. By continuity of Δ_w^i, any limit $(x_i^\infty, x_{-i}^\infty)$ must satisfy, $\Delta_i^w(x_i^\infty, x_{-i}^\infty) = \Delta_{-i}^w(x_{-i}^\infty, x_i^\infty) = 0$. This means that $(x_i^\infty, x_{-i}^\infty)$ is an equilibrium of $\Gamma_\sigma(w)$. We know from Theorem 2 that for σ small enough, we have joint selection over $\mathcal{W}$. This implies that all converging subsequences of $\{(x_i^n, x_{-i}^n)\}_{n\geq 0}$ converges to the unique equilibrium of $\Gamma_\sigma(w)$. This gives continuity over $\mathcal{W}$. $\square$

Appendix B: Extension to games with a continuum of players

Here we briefly outline why results presented in Section 3 still hold in symmetric games with a continuum of agents.

We consider games with a continuum of agents indexed by $t \in [0,1]$. Each player has an action set $\{C, D\}$. All decisions are taken simultaneously. Let us denote by q the proportion of players choosing to play C. Players have identical payoffs which depend on their own action $a \in \{C, D\}$, the aggregate outcome q, and a state of the world θ, with convex support $I \subset \mathbb{R}$, and a C^1 density f_θ. Let these payoffs be denoted by $U_C(q,\theta)$ and $U_D(q,\theta)$. Before taking action, player t gets a signal $x_t = \theta + \sigma\varepsilon_t$, where ε_t has support in $[0,1]$ and all draws are independent. We denote $\Gamma_\sigma(u)$ this global game.

We define the class of game structures $\mathcal{H}_{k,\rho}$ as follows,

Definition 8. *Given a modulus of continuity ρ and a number $k > 0$, we denote by $\mathcal{H}_{k,\rho}$ the class of payoff structures U such that,*

1. *For all $q \in [0,1]$, the functions $U_C(q,\cdot)$ and $U_D(q,\cdot)$ have a modulus of continuity rho with respect to θ.*

2. *The mapping $m : (q,\theta) \mapsto U_C(q,\theta) - U_D(q,\theta)$ is strictly increasing in both θ and q with lower rates greater than k.*

3. *There exist $\underline{\theta}$ and $\overline{\theta}$ in I such that $m(0,\overline{\theta}) > 0$ and $m(1,\underline{\theta}) < 0$.*

To simplify analysis, we assume that the games are fully supermodular. As in the case of two player games it is possible to work under the weaker assumption that the game has strictly increasing differences in the state of the world, and that at all states θ, the perfect information version of the game that players face is either dominance solvable or exhibits increasing differences in q. For the sake of simplicity, this appendix will not deal with that level of generality.

Because we have assumed that payoff structures were supermodular, it follows that game $\Gamma_\sigma(U)$ has extreme Nash equilibria which are monotone and symmetric. Each of these equilibria takes a threshold form, meaning that there is a threshold x such that player t chooses C when $x_t > x$ and D when $x_t < x$.

Consider the incentives of player t when other players use threshold x. The proportion of people choosing C is $q = P\left[\varepsilon > \frac{x-\theta}{\sigma}\right]$. Thus payoffs are given by

$$\Pi_C(x_t, x, \sigma) = \int_I U_C\left(P\left[\varepsilon > \frac{x-\theta}{\sigma}\right], \theta\right) f(\theta|x_t)d\theta \tag{32}$$

$$\Pi_D(x_t, x, \sigma) = \int_I U_D\left(P\left[\varepsilon > \frac{x-\theta}{\sigma}\right], \theta\right) f(\theta|x_t)d\theta. \tag{33}$$

For x to be an equilibrium threshold, it must be that player is indifferent between C and D when $x_t = x$. Thus equilibrium is characterized by the equation

$$\Delta^U(x, x, \sigma) \equiv \int_I m\left(P\left[\varepsilon > \frac{x-\theta}{\sigma}\right], \theta\right) f(\theta|x_t)d\theta = 0. \tag{34}$$

Do the change in variable $u = \frac{x-\theta}{\sigma}$, equation (34) becomes,

$$\Delta^U(x, x, \sigma) \equiv \int_{-1}^{1} m\left(P\left[\varepsilon > u\right], x - \sigma u\right) \frac{f_\varepsilon(u) f_\theta(x-\sigma u)}{\int_{-1}^{1} f_\varepsilon(u) f_\theta(x-\sigma u)du} du = 0. \tag{35}$$

As in section 3, we define Ψ_σ by

$$\Psi_\sigma(x, u) = \frac{f_\varepsilon(u) f_\theta(x-\sigma u)}{\int_{-1}^{1} f_\varepsilon(u) f_\theta(x-\sigma u)du}$$

We have already shown using standard results for convolution products that, over any compact, $\Psi_\sigma(u, x)$ converges uniformly to $f_\varepsilon(u)$ as σ goes to 0 and that $\frac{\partial \Psi}{\partial x}$ converges uniformly to 0.

We also have that $\frac{\partial m}{\partial x} > k$. Finally because we consider only symmetric games, we know that the α term that we needed to consider in section 3 is equal to zero.

This implies that uniformly over any compact,

$$\lim_{\sigma \to 0} \Delta(x, x, \sigma) = \int_{-1}^{1} m\left(P\left[\varepsilon > u\right], x\right) f_{\varepsilon}(u) du = \int_{-1}^{1} m(u, x) du$$

and that the solutions of the equation $\Delta(x, x, \sigma) = 0$ converge to the risk-dominant equilibrium, that is, the solution of equation $\int_{-1}^{1} m(u, x) du = 0$.

More precisely Theorems 2 and 3 extend as follows,

Theorem 9. *Consider a class of payoffs $\mathcal{H}_{k,\rho}$. There exists $\overline{\sigma}$ such that for all $\sigma < \overline{\sigma}$, and all $U \in \mathcal{H}_{k,\rho}$, all games $\Gamma_{\sigma}(U)$ have a unique pair of rationalizable strategies, with threshold $x(U, \sigma)$. As σ goes to 0, the equilibrium threshold $x(U, \sigma)$ converges uniformly over $\mathcal{H}_{k,\rho}$ to the risk-dominant equilibrium.*

Proof. The proofs are identical to those of Theorems 2 and 3. Joint selection is proven by showing that for some $\overline{\sigma}$, $\sigma < \overline{\sigma}$ implies that $\Delta(x, x, \sigma)$ is strictly in x. Uniform convergence results from the fact that the rate of convergence of $x(U, \sigma)$ has an upper bound that depends only on k and ρ. $\square$

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Chapter Two: Fear of Miscoordination and the Robustness of Cooperation in Dynamic Global Games with Exit

Abstract

This paper develops a framework to assess the impact of miscoordination fear on agents' ability to sustain dynamic cooperation. Building on theoretical insights from Carlsson and van Damme (1993), it explores the effect of small amounts of private information on a class of dynamic cooperation games with exit. It is shown that lack of common knowledge creates a fear of miscoordination which pushes players away from the full-information Pareto frontier. Unlike in one-shot two-by-two games, the global games information structure does not yield equilibrium uniqueness, however, by making it harder to coordinate, it does reduce the range of equilibria and gives bite to the notion of local dominance solvability. Finally, the paper provides a simple criterion for the robustness of cooperation to miscoordination fear, and shows it can yield predictions that are qualitatively different from those obtained by focusing on Pareto efficient equilibria under full information.

KEYWORDS: cooperation, strategic risk, miscoordination risk, global games, dynamic games, exit games, rationalizability, local strong rationalizability, local dominance solvability. JEL CLASSIFICATION CODES: C72, C73

1 Introduction

The folk theorem for repeated games teaches us that even though short run incentives may lead to suboptimal outcomes, continued interaction can allow players to sustain efficient cooperation when promises of future benefits are large enough. One notable property of repeated games is that they admit a large set of equilibria. As a consequence, much of the applied work using dynamic cooperation games focuses on Pareto optimal equilibria for the purpose of deriving comparative statics. One may however worry that using Pareto efficiency as a selection criterion overestimates the players' ability to coordinate. Indeed, there exists a substantial experimental literature on coordination failure in one-shot coordination games[1] indicating that empirically, Pareto efficiency is not a fully satisfying selection criterion and that risk-dominance, in the sense of Harsanyi and Selten (1988), is often a better predictor of experimental outcomes. The work of Carlsson and van Damme (1993) sheds theoretical light on these empirical findings by showing that the Pareto efficiency criterion relies heavily on common knowledge and that for a natural family of small departures from full information, the risk-dominant action will be the unique rationalizable outcome.

This paper uses the information structure of Carlsson and van Damme (1993) to model miscoordination risk in a class of games with exit that replicates much of the intuition underlying repeated games, while being simple enough to study the effects of small amounts of private information. The exit games considered are two-player games with infinite horizon and positive discount rate, in which players decide each period whether they want to stay or exit. Under the global games information structure, in each period t, players' payoffs are affected by an i.i.d. state of the world w_t, on which players make noisy observations.

The paper's main result is a characterization of rationalizable strategies as players' signals become arbitrarily precise. Although the likelihood of miscoordination becomes vanishingly small as signals get more precise, the ghost of miscoordination is enough to push players away from the Pareto efficient frontier. The set of surviving equilibria – which are inter-

[1]See for instance Cooper, DeJong, Forsythe, and Ross (1990) or Battalio, Samuelson, and Van Huyck (2001).

preted as those equilibria that are robust to miscoordination risk – depends both on the magnitude of miscoordination losses and on the distribution of states of the world w_t. Unlike the case of one-shot coordination games studied by Carlsson and van Damme (1993) and Frankel, Morris and Pauzner (2003), the global games information structure does not yield unique selection in infinite horizon games. However, the dominance solvability of static global games does carry over in the weaker form of local dominance solvability. As players' signals get arbitrarily precise it is possible to characterize local dominance solvability explicitly for a focal class of equilibria. This allows us to identify equilibria that are robust to strategic uncertainty in addition to being robust to miscoordination fear. Finally, the paper provides a simple criterion for cooperation to be robust in games with approximately constant payoffs, and shows how taking into account the impact of miscoordination fear on cooperation can yield predictions that are qualitatively different from those obtained by focusing on full-information Pareto-efficient equilibria. This is illustrated in an applied model which investigates the question of how wealth affects people's ability to cooperate.

From a methodological perspective, the paper shows how the Abreu, Pearce, and Stacchetti (1990) approach to dynamic games can be used to study the impact of a global games information structure in a broader set of circumstances than one-shot coordination games. The approach has two steps: the first step is to recognize that one-shot action profiles in a perfect Bayesian equilibrium must be Nash equilibria of an augmented one-shot game incorporating continuation values; the second step is to apply global games selection results that hold uniformly over the family of possible augmented games, and derive a fixed point equation for possible continuation values. This approach can accommodate the introduction of an observable Markovian state variable and auto-correlated states of the world.

This paper contributes to the literature on the effect of private information in infinite horizon cooperation games. Since Green and Porter (1984), Abreu, Pearce, and Stacchetti (1986), and Radner, Myerson, and Maskin (1986), much of this literature[2] has focused on the issue of imperfect monitoring of other players' actions and on the amount of inefficient punishment that must occur on an equilibrium path. In this paper however, actions are

[2]See for instance Fudenberg, Levine, and Maskin (1994), Compte (1998), or Kandori (2003)

observable. It is the players' assessment of the state of the world that is private information. Interestingly, this form of private information prevents the players from attaining the full-information Pareto frontier even as the players' assessments become arbitrarily precise.

This paper also fits in the growing literature on dynamic global games. Much of this literature however avoids intertemporal incentives. Levin (2001) studies a global game with overlapping generations. Chamley (1999), Morris and Shin (1999), and Angeletos, Hellwig and Pavan (2006) consider various models of dynamic regime change, but assume a discount rate equal to zero, and focus on the endogenous information dynamics that result from agents observing others' actions and new signals of the state of the world. In this sense, these models are models of dynamic herds rather than models of repeated interaction. Closer to the topic of this paper is Giannitsarou and Toxvaerd (2003), which extends results from Frankel, Morris, and Pauzner (2003) and discusses equilibrium uniqueness in a family of finite, dynamic, supermodular global games. From the perspective of the present paper, which is concerned with infinite horizon games, their uniqueness result is akin to equilibrium uniqueness in a finitely repeated dominance solvable game. Finally, in two papers that do not rely on private noisy signals as the source of miscoordination, but carry a very similar intuition, Burdzy, Frankel, and Pauzner (2001), and Frankel and Pauzner (2000) obtain full selection for a model in which players' actions have inertia and fundamentals follow a random walk. However, their unique selection result hinges strongly on the random walk assumption and does not rule out multiplicity in settings where fundamentals follow different processes.

The paper is organized as follows. Section 2 presents the setup. Section 3 is the core of the paper and proves selection and local dominance solvability results. It illustrates how tools developed for one-shot global games can be applied to study perfect Bayesian equilibria in dynamic games. Section 4 applies the results of Section 3 and makes the case that the model of miscoordination fear proposed in this paper is practical and can yield qualitatively new comparative statics. Section 5 concludes. Proofs are contained in Appendix A, unless mentioned otherwise. The results of Section 3 are extended to non-stationary games in Appendix B.

2 Stationary exit games

2.1 The setup

Consider an infinite-horizon game with discrete time $t \in \{1, \dots, +\infty\}$ and two players $i \in \{1, 2\}$ with discount rate β. The two players act simultaneously and can take two actions: $\mathcal{A} = \{Stay, Exit\}$. Payoffs are indexed by a state of the world $w_t \in \mathbb{R}$, which is independently drawn each period. Given the state of the world w_t, player i expects flow payoffs,

	S	E
S	$g^i(w_t)$	$W^i_{12}(w_t)$
E	$W^i_{21}(w_t)$	$W^i_{22}(w_t)$

where i is the row player. States of the world $\{w_t\}_{t \in \{1,\dots,\infty\}}$ form an i.i.d. sequence of real numbers drawn from a distribution with density f, c.d.f. F and convex support $I \subseteq \mathbb{R}$. All payoffs, $g^i, W^i_{12}, W^i_{21}, W^i_{22}$ are continuous in w_t.

At time t, the state of the world w_t is unknown, but each player gets a signal $x_{i,t}$ of the form

$$x_{i,t} = w_t + \sigma \varepsilon_{i,t}$$

where $\{\varepsilon_{i,t}\}_{i \in \{1,2\}\,,\, t \geq 1}$ is an i.i.d. sequence of independent random variables taking values in the interval $[-1, 1]$. For simplicity w_t is ex-post observable[3].

Whenever there is an exit, the game ends and players get a continuation value equal to zero. This is without loss of generality since termination payoffs can be included in the flow-payoffs upon exit W^i_{12}, W^i_{21} and W^i_{22}. For all $\sigma \geq 0$, let Γ_σ denote this dynamic game with imperfect information. Note that Γ_0 corresponds to the game with full information. The paper is concerned with equilibria of Γ_σ with σ strictly positive but arbitrarily small.

[3]Note that the analysis that follows would hold if players' final payoffs were shifted by some idiosyncratic noise $\eta_{i,t}$ independent of all other random variables and with zero expectation.

2.2 Example: a partnership game

As a benchmark, consider the following – extremely simple – partnership game. Flow payoffs are symmetric and given by,

	S	E
S	w_t	$w_t - C + \beta V_E$
E	$b + V_E$	V_E

where payoffs are given for the row player only, and $C > b \geq 0$. Parameter w_t is the expected return from putting effort in the partnership at time t; C represents the diminished value of being in the partnership when the other player walks out; parameter b (which can be set to 0) represents a potential benefit from cheating on a cooperating partner; and V_E is the present value of the players' constant outside option. States of the world w_t are drawn from a distribution with density f and support $\mathbb{R}$. We assume that $\mathbf{E}|w_t| < \infty$ and $V_E > 0$.

As a benchmark, let us study subgame perfect equilibria under full information. Whenever $w_t \leq (1-\beta)V_E + C$, playing (E, E) is a possible equilibrium outcome. Similarly, there exists a lowest value $\underline{w}$ of w_t for which (S, S) can be an equilibrium play. This cooperation threshold is associated with the greatest equilibrium continuation value $\overline{V}$. The following equations characterize $\overline{V}$ and $\underline{w}$:

$$\underline{w} + \beta\overline{V} = b + V_E \tag{1}$$

$$\overline{V} = \mathbf{E}\left[(w_t + \beta\overline{V})\mathbf{1}_{w_t > \underline{w}}\right] + F(\underline{w})V_E. \tag{2}$$

Whenever w_t belongs to $[\underline{w}, (1-\beta)V_E + C]$, any symmetric pair of actions is an equilibrium play and any pair of actions is rationalizable. In fact, within these bounds, any symmetric pair of actions can be an outcome of a Markovian equilibrium, and hence, any action is rationalizable by a Markovian strategy. When w_t is greater than $(1-\beta)V_E + C$, staying is the dominant action. When w_t is smaller than $\underline{w}$, exit is the dominant action.

Under full information, the criterion of Pareto efficiency would imply that players coordinate on using $\underline{w}$ as their threshold for cooperation, independently of C, which does not enter equations (1) and (2). Is this prediction robust when players' assessments of w_t are private? If not, what equilibria are robust to such a departure from common knowledge? How do these robust equilibria move with respect to C? Section 3 develops tools to answer such questions for a variety of games.

2.3 Assumptions

To exploit existing results on one-shot global games, we make a few assumptions which essentially ensure that the assumptions of Carlsson and van Damme (1993) hold for the family of one-shot stage games augmented with the players' potential continuation values. While it is possible to find weaker conditions under which the results of Section 3 will hold, the assumptions given here have the advantage that they can be checked in a straightforward way from primitives.

Assumption 1 (boundedness) *Let m_i and M_i respectively denote the min-max and maximum values of player i in the full information game Γ_0. Both m_i and M_i are finite.*

This assumption is typically unrestrictive but is still important given that in many natural examples, w_t will have unbounded support. The min-max value m_i will appear again in Assumptions 4 and 5, while M_i will be used in Assumption 2.

In the partnership example of Section 2.2, we have $m_i = m_{-i} = \mathbf{E}\max\{V_E, w_t - C + \beta V_E\}$ and $M_i = M_{-i} = M$ where M satisfies $M = \mathbf{E}\max\{w_t + \beta M, b + V_E\}$.

Assumption 2 (dominance) *There exist $\underline{w}$ and $\overline{w}$ such that for all $i \in \{1,2\}$,*

$$g^i(\underline{w}) + \beta M_i - W^i_{21}(\underline{w}) < 0 \quad \textit{and} \quad W^i_{12}(\underline{w}) - W^i_{22}(\underline{w}) < 0 \quad \textit{(Exit dominant)}$$
$$\textit{and} \quad W^i_{12}(\overline{w}) - W^i_{22}(\overline{w}) > 0 \quad \textit{and} \quad g^i(\overline{w}) + \beta m_i - W^i_{21}(\overline{w}) > 0 \quad \textit{(Staying dominant).}$$

Assumption 3 (increasing differences in the state of the world) *For all $i \in \{1,2\}$, $g^i(w_t) - W^i_{21}(w_t)$ and $W^i_{12}(w_t) - W^i_{22}(w_t)$ are strictly increasing over $w_t \in [\underline{w}, \overline{w}]$, with a slope* *greater than some real number $r > 0$.*

Note that the assumption that $W_{12}^i - W_{22}^i$ is strictly increasing in the state of the world may rule out examples in which staying yields a constant zero payoff when the other player exits.

Definition 1 *For any functions $V_i, V_{-i} : \mathbb{R} \to \mathbb{R}$, let $G(V_i, V_{-i}, w_t)$ denote the full information one-shot game*

	S	E
S	$g^i(w_t) + \beta V_i(w_t)$	$W_{12}^i(w_t)$
E	$W_{21}^i(w_t)$	$W_{22}^i(w_t)$

where i is the row player. Let $\Psi_\sigma(V_i, V_{-i})$ denote the corresponding one-shot global game in which players observe signals $x_{i,t} = w_t + \sigma\varepsilon_{i,t}$.

Assumption 4 (equilibrium symmetry) *For all states of the world w_t, $G(m_i, m_{-i}, w_t)$ has a pure strategy Nash equilibrium and all pure equilibria belong to $\{(S,S),(E,E)\}$.*

Recall that m_i is player i's min-max value in the game with full information Γ_0. If Assumption 4 is satisfied, then for any function $\mathbf{V} = (V_i, V_{-i})$ taking values in $[m_i, +\infty) \times [m_{-i}, +\infty)$, the game $G(\mathbf{V}, w_t)$ also has a pure strategy equilibrium, and its pure equilibria also belong to $\{(S,S),(E,E)\}$. Indeed, whether (E,E) is an equilibrium or not does not depend on the value of (V_i, V_{-i}), and if (S,S) is an equilibrium when $\mathbf{V} = (m_i, m_{-i})$, then it is also an equilibrium when the continuation values of player i and $-i$ are respectively greater than m_i and m_{-i}.

Note that when Assumptions 2 and 3 hold, Assumption 4 is equivalent to the condition that for all $i \in \{1,2\}$, at the state w_i such that $W_{12}^i(w_i) - W_{22}^i(w_i) = 0$, we have $g^i(w_i) + \beta m_i - W_{21}^i(w_i) > 0$ and $g^{-i}(w_i) + \beta m_{-i} - W_{21}^{-i}(w_i) > 0$. Assumption 4 holds for the partnership game since $C > b$.

Together, Assumptions 3 and 4 insure that at any state of the world w and for any pair of individually rational continuation values $\mathbf{V}$, either (S,S) or (E,E) is the risk-dominant equilibrium of $G(\mathbf{V}, w)$, and that there is a unique risk-dominant threshold $x^{RD}(\mathbf{V})$ —

(S,S) being risk-dominant above this threshold and (E,E) being risk-dominant below. In conjunction with Assumption 2 this is in fact the unidimensional version of Carlsson and van Damme's assumption that states of the world should be connected to dominance regions by a path that is entirely contained in the risk-dominance region of one of the equilibria.

Definition 2 *For any function $V : \mathbb{R} \to \mathbb{R}$, and $w \in \mathbb{R}$, define $A_i(V,w)$ and $B_i(w)$ by,*

$$A_i(V,w) = g^i(w) + \beta V(w) - W^i_{12}(w) \quad \textit{and} \quad B_i(w) = W^i_{21}(w) - W^i_{22}(w).$$

Assumption 5 (staying is good) *For all players $i \in \{1,2\}$ and all states of the world $w \in [\underline{w}, \overline{w}]$, $A_i(m_i,w) \geq 0$ and $B_i(w) \geq 0$.*

Recall that $[\underline{w}, \overline{w}]$ corresponds to states of the world where there need not be a dominant action. Assumption 5 is restrictive but not unreasonable: it means that under full information, at a state $w \in [\underline{w}, \overline{w}]$ with no clearly dominant action, player i is weakly better off whenever player $-i$ stays, independently of her own action.

The partnership game of Section 2.2 satisfies this assumption since for all $w \in \mathbb{R}$, $A_i(m_i,w) = C + \beta(m_i - V_E) > 0$ and $B_i(w) = b \geq 0$.

2.4 Solution concepts

Because of exit, at any decision point, a history $h_{i,t}$ is characterized by a sequence of past signals and past outcomes: $h_{i,t} \equiv \{x_{i,1}, \ldots, x_{i,t} \ ; \ w_{i,1}, \ldots, w_{i,t-1}\}$. Let $\mathcal{H}$ denote the set of all such sequences. A pure strategy is a mapping $s : \mathcal{H} \mapsto \{S, E\}$. Denote by Ω the set of pure strategies. For any set of strategies $S \subset \Omega$, let $\Delta(S)$ denote the set of probability distributions over S that have a countable support. The two main solution concepts we will be using are *perfect Bayesian equilibrium* and *sequential rationalizability.* To define these concepts formally, it is convenient to denote by $h^0_{i,t} \equiv \{x_{i,1}, \ldots, x_{i,t-1} \ ; \ w_{i,1}, \ldots, w_{i,t-1}\}$ the histories before players receive period t's signal but after actions of period $t-1$ have been taken. A strategy s_{-i} of player $-i$, conditional on the history $h^0_{-i,t}$ having been observed, will be denoted $s_{-i|h^0_{-i,t}}$. A conditional strategy $s_{-i|h^0_{-i,t}}$ and player i's conditional belief $\mu_{|h^0_{i,t}}$ over

$h^0_{-i,t}$ induce a mixed strategy denoted by $(s_{-i|h^0_{-i,t}}, \mu_{|h^0_{i,t}})$. Player i's sequential best-response correspondence, denoted by $BR_{i,\sigma}$, is defined as follows.

Definition 3 (sequential best-response) *$\forall\, s_{-i} \in \Omega$, $s_i \in BR_{i,\sigma}(s_{-i})$ if and only if:*

(i) At any history $h^0_{i,t}$ that is attainable given s_{-i} and s_i, the conditional strategy $s_{i|h^0_{i,t}}$ is a best-reply of player i to the mixed strategy $(s_{-i|h^0_{-i,t}}, \mu_{|h^0_{i,t}})$, where conditional beliefs $\mu_{|h^0_{i,t}}$ over $h^0_{-i,t}$ are obtained by Bayesian updating;

(ii) At any history $h^0_{i,t}$ that is not attainable given s_{-i} and s_i, $s_{i|h^0_{i,t}}$ is a best-reply of player i to a mixed strategy $(s_{-i|h^0_{-i,t}}, \mu_{|h^0_{i,t}})$ for some *(any) conditional beliefs $\mu_{|h^0_{i,t}}$ over $h^0_{-i,t}$.*

With this definition of sequential best-response, a strategy s_i of player i is associated with a perfect Bayesian equilibrium of Γ_σ if and only if, $s_i \in BR_{i,\sigma} \circ BR_{-i,\sigma}(s_i)$. Sequential rationalizability is defined as follows.

Definition 4 (sequential rationalizability) *A strategy s_i belongs to the set of sequentially rationalizable strategies R_i of player i if and only if*

$$s_i \in \bigcap_{n \in \mathbb{N}} (BR^{\Delta}_{i,\sigma} \circ BR^{\Delta}_{-i,\sigma})^n(\Omega)$$

where $BR^{\Delta}_{i,\sigma} \equiv BR_{i,\sigma} \circ \Delta$.

Given strategies s_i, s_{-i} and beliefs upon unattainable histories, let $V_i(h_{i,t})$ denote the value player i expects from playing the game at history $h_{i,t}$. Pairs of value functions will be denoted $\mathbf{V} \equiv (V_i, V_{-i})$.

3 Selection and local dominance solvability

The first class of results presented in this section aims at characterizing the extent to which lack of common knowledge and the fear of miscoordination prevent players from achieving

Pareto efficient levels of cooperation. It is shown that payoffs upon miscoordination influence equilibrium selection, although in equilibrium, as σ goes to 0, miscoordination happens with a vanishing probability. Theorem 2 characterizes the limit set of equilibrium values explicitly. Section 4.1 applies these results to the partnership game introduced in Section 2.2 and derives simple comparative statics that do not hold when focusing on Pareto efficient equilibria under full information.

The second class of results given in this section explores how the dominance solvability result of Carlsson and van Damme (1993) extends to dynamic games. Since dynamic global games can admit multiple equilibria, it would seem that these results do not carry over. However, Section 3.4 shows that the global games structure gives bite to the notion of local dominance solvability, extensively discussed in Guesnerie (2002). Theorem 3 shows that for the class of exit games defined in Section 2, local dominance solvability, which is a high dimensional property of sets of strategies, is asymptotically characterized by the stability of the fixed points of an easily computable, increasing mapping from $\mathbb{R}$ to $\mathbb{R}$. This will allow us to discuss the issue of robustness of equilibria to strategic uncertainty.

3.1 General methodology

A useful methodological insight of this paper is to recognize that tools from equilibrium selection in one-shot global games can be exploited to study the impact of a global games information structure in dynamic games. Using the dynamic programming approach to dynamic games developed in Abreu, Pearce, and Stacchetti (1990), actions prescribed by a perfect Bayesian equilibrium of a dynamic game must be outcomes of a Nash equilibrium in the Bayesian one-shot game that incorporates the players' continuation values. The idea is to apply global games selection results to a family of such augmented games in order to characterize the equilibrium continuation values of the dynamic game.

The main difficulty is that the selection results of Carlsson and van Damme (1993) only hold pointwise – they take the payoff structure as given – while selection results will need to hold uniformly[4] to apply dynamic programming techniques. For this reason, the present paper draws on results from Chassang (2006) which show that selection does happen uniformly over equicontinuous[5] families of two-by-two games satisfying the assumptions of Carlsson and van Damme (1993). Mostly, we will use the following implication of Theorems 2, 3 and 4 of Chassang (2006).

Lemma 1 (uniform selection) *For any compact subset $\mathcal{V} \subset \mathbb{R}^2$, consider the family of one-shot global games $\Psi_\sigma(\mathbf{V})$ indexed by $\mathbf{V} \in \mathcal{V}$. If for all $\mathbf{V} \in \mathcal{V}$ the full information one-shot game $G(\mathbf{V}, w)$ has pure equilibria which are all symmetric and admits dominance regions with respect to w, then under Assumptions 2 and 3*

(i) There exists $\overline{\sigma}$ such that for all $\sigma \in (0, \overline{\sigma})$, all one-shot global games $\Psi_\sigma(\mathbf{V})$, indexed by values $\mathbf{V} \in \mathcal{V}$, have a unique rationalizable equilibrium;

(ii) This equilibrium takes a threshold form[6] with thresholds denoted by $\mathbf{x}^*_\sigma(\mathbf{V}) \in \mathbb{R}^2$. The mapping $\mathbf{x}^*_\sigma(\cdot)$ is continuous over $\mathcal{V}$;

(iii) As σ goes to 0, each component of $\mathbf{x}^*_\sigma(\mathbf{V})$ converges uniformly over $V \in \mathcal{V}$ to the risk-dominance threshold of $\Psi_0(\mathbf{V})$, denoted by $x^{RD}(\mathbf{V})$.

The analysis will proceed as follows. Section 3.2 shows that for an appropriate order over strategies, the game Γ_σ exhibits a restricted form of monotone best response which suffices to show that the set of sequentially rationalizable strategies is bounded by extreme Markovian equilibria. Section 3.3 characterizes the continuation values of Markovian equilibria by iteratively applying selection results on one-shot global games to families of augmented stage games. Section 3.4 shows how the dominance solvability of one-shot global games can be used to characterize the local dominance solvability of equilibria of the dynamic game Γ_σ.

[4] The reason for this will become clear in Section 3.3.

[5] More precisely, families of games whose associated family of payoff functions is equicontinuous.

[6] A strategy s_i in game $\Psi_\sigma(\mathbf{V})$ takes a threshold form if and only if there exists $x \in \mathbb{R}$ such that almost surely, $s_i(x_i) = S$ if and only if $x_i \geq x$.

3.2 Monotone best response and rationalizability

This section exploits assumptions of Section 2.3 and the exit game structure to prove simplifying structural properties on Γ_σ. In particular, it shows that for σ small enough, the set of rationalizable strategies of Γ_σ is bounded by extreme Markovian equilibria.

Definition 5 (Markovian strategies) *A strategy s_i is said to be Markovian if $s_i(h_{i,t})$ depends only on player i's current signal, $x_{i,t}$.*

A Markovian strategy s_i is said to take a threshold form if there exists a constant value x such that for almost all $x_{i,t} \geq x$, s_i prescribes player i to stay, and for almost all $x_{i,t} < x$, s_i prescribes player i to exit. The threshold of a threshold form strategy s will be denoted x_s and a strategy of threshold x will be denoted s_x.

Definition 6 *We define a partial order $\preceq$ on pure strategies by*

$$s' \preceq s \iff \{a.s. \forall h \in \mathcal{H},\ s'(h) = Stay \Rightarrow s(h) = Stay\}.$$

In other words, a strategy s is greater than s' with respect to $\preceq$ if and only if players stay more under strategy s. Consider a strategy s_{-i} of player $-i$ and a history $h_{i,t}$ observed by player i. From the perspective of player i, the one period action profile of player $-i$ is a mapping from player $-i$'s current signal to lotteries over $\{stay, exit\}$, which we denote by $a_{-i|h_{i,t}} : \mathbb{R} \to \Delta\{stay, exit\}$. The order $\preceq$ on dynamic strategies extends to one-shot action profiles as follows:

$$a' \preceq a \iff \{a.s. \forall x \in \mathbb{R},\ Prob[a'(x) = Stay] \leq Prob[a(x) = Stay]\}.$$

Note that if s_{-i} is Markovian, then $a_{-i|h_{i,t}}$ is effectively a mapping from $\mathbb{R}$ to $\{stay, exit\}$. For any mapping V_i that maps player i's current signal, $x_{i,t} \in \mathbb{R}$, to a continuation value $V_i(x_{i,t})$, and any mapping $a_{-i} : \mathbb{R} \to \Delta\{stay, exit\}$, one can define $BR_{i,\sigma}(a_{-i}, V_i)$, as the one period best response correspondence of player i when she expects a continuation value V_i and player $-i$ uses an action profile a_{-i}. Given a continuation value function V_i, the expected payoffs upon staying and exit – respectively denoted by $\Pi^i_S(V_i)$ and Π^i_E – are

$$\Pi^i_S(V_i) = \mathbf{E}\Big[W^i_{12}(w) + \{g^i(w) + \beta V_i(h_{i,t}, w) - W^i_{12}(w)\}\mathbf{1}_{s_{-i}=S}|h_{i,t}, s_{-i}\Big] \tag{3}$$

$$\Pi^i_E = \mathbf{E}\Big[W^i_{22}(w) + \{W^i_{21}(w) - W^i_{22}(w)\}\mathbf{1}_{s_{-i}=S}|h_{i,t}, s_{-i}\Big]. \tag{4}$$

Lemma 2 *For any one-shot action profile a_{-i} and value function V_i, the one-shot best-response correspondence $BR_{i,\sigma}(a_{-i}, V_i)$ admits a lowest and a highest element with respect to $\preceq$. These highest and lowest elements are respectively denoted $BR^H_{i,\sigma}(a_{-i}, V_i)$ and $BR^L_{i,\sigma}(a_{-i}, V_i)$.*

Proof: An action profile a_i belongs to the set of one-shot best-replies $BR_{i,\sigma}(a_{-i}, V_i)$ if and only if a_i prescribes S when $\Pi^i_S(V_i) > \Pi^i_E$ and prescribes E when $\Pi^i_S(V_i) < \Pi^i_E$. Because ties are possible $BR_{i,\sigma}(a_{-i}, V_i)$ need not be a singleton. However, by breaking the ties consistently in favor of either S or E, one can construct strategies a^H_i and a^L_i that are respectively the greatest and smallest elements of $BR_{i,\sigma}(a_{-i}, V_i)$ with respect to $\preceq$. ■

Lemma 3 *There exists $\overline{\sigma} > 0$ and $\nu > 0$ such that for all constant functions V_i taking value in $[m_i - \nu, M_i + \nu]$, and all $\sigma \in (0, \overline{\sigma})$, $BR^H_{i,\sigma}(a_{-i}, V_i)$ and $BR^L_{i,\sigma}(a_{-i}, V_i)$ are increasing in a_{-i} with respect to $\preceq$.*

The proof of this result exploits the fact that Assumption 4 implies a family of single-crossing conditions already identified in Milgrom and Shannon (1994). Note that the results of Athey (2002) do not apply directly since the conditions on distributions they require are only satisfied at the limit where σ is equal to 0.

Lemma 4 *Consider continuation value functions V and V' such that for all $h_{i,t} \in \mathcal{H}$, $V(h_{i,t}) \leq V'(h_{i,t})$. Then, for any a_{-i},*

$$BR^H_{i,\sigma}(a_{-i}, V) \preceq BR^H_{i,\sigma}(a_{-i}, V') \quad \textit{and} \quad BR^L_{i,\sigma}(a_{-i}, V) \preceq BR^L_{i,\sigma}(a_{-i}, V').$$

Proof: The result is proven for the greatest one-shot best-reply $BR^H_{i,\sigma}$. Player i chooses S over E whenever $\Pi^i_S(V_i) \geq \Pi^i_E$. As equation (3) shows, $\Pi^i_S(V_i)$ is increasing in V_i, while Π^i_E

does not depend on V_i. This yields that $BR^H_{i,\sigma}(a_{-i}, V) \preceq BR^H_{i,\sigma}(a_{-i}, V')$. The same proof applies for the lowest one-shot best-reply. ■

Lemma 5 *Whenever s_{-i} is a Markovian strategy, $BR_{i,\sigma}(s_{-i})$ admits a lowest and a highest element with respect to $\preceq$. These strategies are Markovian and are respectively denoted $BR^L_{i,\sigma}(s_{-i})$ and $BR^H_{i,\sigma}(s_{-i})$.*

Proof: Let V be the value player i obtains from best replying to s_{-i}. Since s_{-i} is Markovian, at any history $h^0_{-i,t}$ the conditional strategy $s_{-i|h^0_{-i,t}}$ is identical to s_{-i}, and the value player i expects conditional on $h^0_{i,t}$ is always V. Hence, $s_i \in BR_{i,\sigma}(s_{-i})$ if and only if action profiles prescribed by s_i at a history $h^0_{i,t}$ belong to $BR_{i,\sigma}(s_{-i}, V)$, where s_{-i} is identified with its one-shot action profile. Since $BR_{i,\sigma}(s_{-i}, V)$ admits highest and lowest elements a^H_i and a^L_i, the Markovian strategies s^H_i and s^L_i respectively associated with the one-shot profiles a^H_i and a^L_i are the greatest and a smallest elements of $BR_{i,\sigma}(s_{-i})$ with respect to $\preceq$. ■

We now show that game Γ_σ exhibits monotone best response as long as there is a Markovian strategy on one side of the inequality.

Proposition 1 (restricted monotone best response) *There exists $\overline{\sigma}$ such that for all $\sigma \in (0, \overline{\sigma})$, whenever s_{-i} is a Markovian strategy, then, for **all** strategies s'_{-i},*

$$s'_{-i} \preceq s_{-i} \Rightarrow \left\{\forall s'' \in BR_{i,\sigma}(s'_{-i}),\ s'' \preceq BR^H_{i,\sigma}(s_{-i})\right\}$$
$$\textit{and} \quad s_{-i} \preceq s'_{-i} \Rightarrow \left\{\forall s'' \in BR_{i,\sigma}(s'_{-i}),\ BR^L_{i,\sigma}(s_{-i}) \preceq s''\right\}.$$

Proof: Let us show the first implication. Consider s_{-i} a Markovian strategy and s'_{-i} such that $s'_{-i} \preceq s_{-i}$. Define V_i and V'_i the continuation value functions respectively associated to player i's best response to s_{-i} and s'_{-i}. Note that since s_{-i} is Markovian, V_i is a constant function. Assumption 5, that "staying is good", implies that at all histories $h_{i,t}$, $V'_i(h_{i,t}) \leq V_i(h_{i,t})$. From Lemma 4, we have that

$$BR^H_{i,\sigma}(a'_{-i}, V'_i(h_{i,t})) \preceq BR^H_{i,\sigma}(a'_{-i}, V_i(h_{i,t})). \tag{5}$$

Since $V_i(h_{i,t})$ is constant we want to apply Lemma 3. For this, let us show that $a'_{-i|h_{i,t}} \preceq a_{-i|h_{i,t}}$. This follows directly from s_{-i} being Markovian, and the fact that $s'_{-i} \preceq s_{-i}$. Indeed, whenever $Prob\{a'_{-i|h_{i,t}} = stay\} > 0$, we must have $Prob\{a_{-i|h_{i,t}} = stay\} = 1$. Applying Lemma 3 yields that

$$BR^H_{i,\sigma}(a'_{-i}, V_i(h_{i,t})) \preceq BR^H_{i,\sigma}(a_{-i}, V_i(h_{i,t})). \tag{6}$$

Combining equations (5) and (6) we obtain that indeed, for all $s'' \in BR_{i,\sigma}(s'_{-i})$, $s'' \preceq BR^H_{i,\sigma}(s_{-i})$. An identical proof holds for the other inequality. ■

Proposition 1 will allow us to prove the existence of extreme threshold-form equilibria. For this we will use the following lemma which shows that for σ small enough, the best response to a threshold-form strategy is unique and takes a threshold form.

Lemma 6 *There exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$ and any $x \in \mathbb{R}$, there exists $x' \in \mathbb{R}$ such that $BR_{i,\sigma}(s_x) = \{s_{x'}\}$, i.e. the best response to a threshold form Markovian strategy is a unique threshold form Markovian strategy. Moreover, x' is continuous in x.*

Theorem 1 (extreme strategies) *There exists $\overline{\sigma} > 0$ such that for all $\sigma < \overline{\sigma}$, sequentially rationalizable strategies of Γ_σ are bounded by a highest and lowest Markovian Nash equilibria, respectively denoted by $\mathbf{s}^H_\sigma = (s^H_{i,\sigma}, s^H_{-i,\sigma})$ and $\mathbf{s}^L_\sigma = (s^L_{i,\sigma}, s^L_{-i,\sigma})$.*

Those equilibria take threshold forms : for all $i \in \{1,2\}$ and $j \in \{H, L\}$, there exists $x^j_{i,\sigma}$ such that $s^j_{i,\sigma}$ prescribes player i to stay if and only if $x_{i,t} \geq x^j_{i,\sigma}$.

Indeed, although Γ_σ is not supermodular, Proposition 1 is sufficient for the construction of Milgrom and Roberts (1990) to hold. The first step is to note that the strategies corresponding to staying always, and exiting always are threshold form Markovian strategies that bound the set of possible strategies. The idea is then to apply the best response mappings iteratively to these extreme strategies. A formal proof is given in Appendix A.

Let us denote by $\mathbf{x}^H_\sigma$ and $\mathbf{x}^L_\sigma$ the pairs of thresholds respectively associated with the highest and lowest equilibria with respect to $\preceq$. Note that $\mathbf{s}^L_\sigma \preceq \mathbf{s}^H_\sigma$, but $\mathbf{x}^L_\sigma \geq \mathbf{x}^H_\sigma$. Let $\mathbf{V}^H_\sigma$ and $\mathbf{V}^L_\sigma$ be the value pairs respectively associated with $\mathbf{s}^H_\sigma$ and $\mathbf{s}^L_\sigma$.

Lemma 7 *$\mathbf{s}_\sigma^H$ and $\mathbf{s}_\sigma^L$ are respectively associated with the highest and lowest possible pairs of rationalizable value functions, $\mathbf{V}_\sigma^H$ and $\mathbf{V}_\sigma^L$. More precisely, if s_{-i} is a rationalizable strategy, the value function $V_{i,\sigma}$ associated with player i's best reply to s_{-i} is such that at all histories $h_{i,t}$, $V_{i,\sigma}^L \leq V_{i,\sigma}(h_{i,t}) \leq V_{i,\sigma}^H$.*

Proposition 1 and Theorem 1 are the main benefits of using an exit game structure. They also provide a first justification for why we are specifically interested in Markovian equilibria: they provide tight bounds for rationalizable behavior. This focus will be further justified in Section 3.4.

3.3 Dynamic selection

We can now state the first selection result of the paper. It shows that continuation values associated with Markovian equilibria of Γ_σ must be fixed points of a mapping $\phi_\sigma(\cdot)$ that converges uniformly to an easily computable mapping Φ from $\mathbb{R}^2$ to $\mathbb{R}^2$. This provides explicit bounds for the set of rationalizable value functions and shows that the set of Markovian equilibria – which is a continuum under full information – typically shrinks to a finite number of elements under a global games information structure.

Theorem 2 *Under Assumptions 1, 2, 3, 4 and 5 there exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$, there exists a continuous mapping $\phi_\sigma(\cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, mapping value pairs to value pairs such that,*

(i) $\mathbf{V}_\sigma^L$ and $\mathbf{V}_\sigma^H$ are the lowest and highest fixed points of $\phi_\sigma(\cdot)$;

(ii) A vector of continuation values is supported by a Markovian equilibrium if and only if it is a fixed point of $\phi_\sigma(\cdot)$;

(iii) As σ goes to 0, the family of functions $\phi_\sigma(\cdot)$ converges uniformly over any compact set of $\mathbb{R}^2$ to an increasing mapping $\Phi : \mathbb{R}^2 \mapsto \mathbb{R}^2$ defined by

$$(7)\quad \Phi(V_i, V_{-i}) = \begin{pmatrix} \mathbf{E}_w \left[(g^i + \beta V_i) \, \mathbf{1}_{w > x^{RD}(V_i, V_{-i})} + W_{22}^i(w) \mathbf{1}_{w < x^{RD}(V_i, V_{-i})} \right] \\ \mathbf{E}_w \left[(g^{-i} + \beta V_{-i}) \, \mathbf{1}_{w > x^{RD}(V_i, V_{-i})} + W_{22}^{-i}(w) \mathbf{1}_{w < x^{RD}(V_i, V_{-i})} \right] \end{pmatrix}$$

where $x^{RD}(V_i, V_{-i})$ is the risk-dominant threshold of the one-shot game $\Psi_0(V_i, V_{-i})$.

Proof: For any fixed σ, any Markovian equilibrium of Γ_σ is associated with a vector of constant continuation values $\mathbf{V}_\sigma = (V_{i,\sigma}, V_{-i,\sigma})$. By continuity of the min-max values, for any $\nu > 0$, there exists $\overline{\sigma} > 0$, such that for all $\sigma \in (0, \overline{\sigma})$, $V_{i,\sigma} \in [m_i - \nu, M_i]$. Stationarity implies that equilibrium actions at any time t must form a Nash equilibrium of the one-shot game

$$\begin{array}{c|ll} & S & E \\ \hline S & g^i(w_t) + \beta V_{i,\sigma} & W^i_{12}(w_t) \\ E & W^i_{21}(w_t) & W^i_{22}(w_t) \end{array}$$

where i is the row player and players get signals $x_{i,t} = w_t + \sigma\varepsilon_{i,t}$. All such one-shot games $\Psi_\sigma(\mathbf{V})$, indexed by $\mathbf{V} \in [m_i - \nu, M_i] \times [m_{-i} - \nu, M_{-i}]$ and $\sigma > 0$ have a global game structure à la Carlsson and van Damme (1993).

Assumption 4 implies that there exists $\nu > 0$ such that for all $\mathbf{V} \in [m_i - \nu, M_i] \times [m_{-i} - \nu, M_{-i}]$ and all $w \in I$, the one-shot game $G(\mathbf{V}, w)$ admits pure equilibria and they are all symmetric. Hence, Lemma 1 (uniform selection) implies that the following are true

1. There exists $\overline{\sigma}$ such that for all $\sigma \in (0, \overline{\sigma})$ and $V \in [m_i - \nu, M_i] \times [m_{-i} - \nu, M_{-i}]$, the game $\Psi_\sigma(\mathbf{V})$ has a unique pair of rationalizable strategy. These strategies take a threshold-form and the associated pair of thresholds is denoted by $\mathbf{x}^*_\sigma(\mathbf{V})$;

2. The pair of thresholds $\mathbf{x}^*_\sigma(\mathbf{V})$ is continuous in $\mathbf{V}$;

3. As σ goes to 0, $\mathbf{x}^*_\sigma(\mathbf{V})$ converges to the risk dominant threshold $x^{RD}(\mathbf{V})$ uniformly over $\mathbf{V} \in [m_i - \nu, M_i] \times [m_{-i} - \nu, M_{-i}]$.

The first result, joint selection, implies that there is a unique expected vector of values from playing game $\Psi_\sigma(\mathbf{V})$, which we denote $\phi_\sigma(\mathbf{V})$. The other two results imply that $\phi_\sigma(\mathbf{V})$ is continuous in $\mathbf{V}$, and that as σ goes to 0, $\phi_\sigma(\mathbf{V})$ converges uniformly over $\mathbf{V} \in \times_{i\in\{1,2\}}[m_i - \nu, M_i]$ to the vector of values $\Phi(\mathbf{V})$ players expect from using the risk-dominant strategy under full information.

Stationarity implies that the value vector $\mathbf{V}$ of any Markovian equilibrium of Γ_σ must satisfy the fixed point equation $\mathbf{V} = \phi_\sigma(\mathbf{V})$. Reciprocally, any vector of values $\mathbf{V}$ satisfying $\mathbf{V} = \phi_\sigma(\mathbf{V})$ is supported by the Markovian equilibrium in which players play the unique equilibrium of game $\Psi_\sigma(\mathbf{V})$ each period. This gives us (*ii*).

Furthermore, we know that the equilibrium strategies of game $\Psi_\sigma(\mathbf{V})$ converge to the risk-dominant strategy as σ goes to 0. This allows us to compute explicitly the limit function Φ. Because the risk-dominance threshold is decreasing in the continuation value, and using Assumption 5, it follows that Φ is increasing in $\mathbf{V}$. This proves (*iii*).

Finally, (*i*) is a straightforward implication of (*ii*). Values associated with Markovian equilibria of Γ_σ are the fixed points of $\phi_\sigma(\cdot)$. Hence the highest and lowest values associated with Markovian equilibria are also the highest and lowest fixed points of $\phi_\sigma(\cdot)$. ■

Theorem 2 states that extreme equilibria of games Γ_σ are characterized by the extreme fixed points of an operator $\phi_\sigma(\cdot)$ that converges uniformly to an explicit operator Φ as σ goes to 0. To show that the mapping Φ gives us a precise description of Markovian equilibria of Γ_σ however, we must show that the uniform convergence of the mapping $\phi_\sigma(\cdot)$ implies the convergence of its fixed points. This corresponds to the upper- and lower-hemicontinuity of fixed points of ϕ_σ.

The first important property we consider is upper-hemicontinuity. The next lemma states that fixed points of $\phi_\sigma(\cdot)$ converge to a subset of fixed points of Φ as σ goes to 0. In that sense, considering fixed points of Φ is sufficient: we do not need to worry about other equilibria.

Lemma 8 (upper-hemicontinuity) *The set of fixed points of $\phi_\sigma(\cdot)$ is upper-hemicontinuous at $\sigma = 0$. That is, for any sequence of positive numbers $\{\sigma_n\}_{n\in\mathbb{N}}$ converging to 0, whenever $\{\mathbf{V}_n\}_{n\in\mathbb{N}} = \{(V_{i,\sigma_n}, V_{-i,\sigma_n})\}_{n\in\mathbb{N}}$ is a converging sequence of fixed points of $\phi_{\sigma_n}(\cdot)$, the sequence $\{\mathbf{V}_n\}_{n\in\mathbb{N}}$ converges to a fixed point $\mathbf{V}$ of Φ.*

Theorem 2 and Lemma 8 imply that whenever Φ has a unique fixed point, the set of rationalizable strategies of game Γ_σ converges to a single pair of strategies as σ goes to 0. Section 4.2 will exploit that property to define a robustness criterion for cooperation in games with approximately constant payoffs.

As another illustration, Lemma 9 shows that conditional on continuation values belonging to some bounded set, whenever the states of the world have sufficient variance, then, equilibrium is unique. Let $||\cdot||_1$ denote the norm on $\mathbb{R}^2$ defined by $||\mathbf{V}||_1 = |V_i| + |V_{-i}|$.

Lemma 9 (uniqueness) *Let K be a bounded interval of $\mathbb{R}$. Under the maintained constraint that individually rational values V_i belong to K, there exists a constant $\eta > 0$, that depends only on payoff functions, such that whenever the distribution of states of the world f satisfies $\max_{[\underline{w},\overline{w}]} f < \eta$, then Φ is a contraction mapping with rate $\delta < 1$ with respect to the norm on vectors $||\cdot||_1$. That is, for all V_i taking values in K, $||\Phi(\mathbf{V}) - \Phi(\mathbf{V}')||_1 \leq \delta ||\mathbf{V} - \mathbf{V}'||_1$.*

The question is now what happens when Φ has multiple fixed points (see Section 4.1 for examples)? Does the game Γ_σ have multiple equilibria? This is not a trivial question. If all fixed points of Φ are indeed associated to equilibria of Γ_σ for σ small, this shows that while a global games information structure may yield uniqueness in static settings, this does not hold anymore when players have an infinite horizon. This question is closely related to the problem of lower-hemicontinuity: when is it that a fixed point of Φ is associated with a sequence of fixed points of $\phi_\sigma(\cdot)$ as σ goes to 0? This is the point of Proposition 2.

So far we have been characterizing Markovian equilibria by their continuation values. For the remainder of this section, it becomes convenient to characterize Markovian equilibria by their cooperation threshold. This is authorized by the following lemma.

Lemma 10 (threshold-form Markovian equilibria) *There exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$, all Markovian equilibria of Γ_σ take a threshold form.*

Furthermore, if $(x_{i,\sigma}, x_{-i,\sigma})$ is a pair of equilibrium thresholds, then $|x_{i,\sigma} - x_{-i,\sigma}| \leq 2\sigma$.

Proof: Consider a Markovian equilibrium of Γ_σ denoted by (s_i, s_{-i}). This Markovian equilibrium is associated to a pair of values (V_i, V_{-i}). The one-shot action profile (a_i, a_{-i}) associated with (s_i, s_{-i}) has to be a Nash equilibrium of the global game $\Psi_\sigma(V_i, V_{-i})$. Lemma 1 implies that there exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$, and all $\mathbf{V} \in [m_i, M_i] \times [m_{-i}, M_{-i}]$, the game $\Psi_\sigma(\mathbf{V})$ has a unique Nash equilibrium. Furthermore, this unique equilibrium takes a threshold form. This proves the first part of the lemma: for σ small enough, all Markovian equilibria of Γ_σ take a threshold form. The second part of the lemma is a direct application of Lemma 4 of Chassang (2006). ■

Note that the second part of this lemma shows that as σ goes to 0, Markovian equilibria of Γ_σ are asymptotically symmetric, so that the likelihood of actual miscoordination vanishes. This illustrates that the ghost of miscoordination, rather than miscoordination itself is enough to drive players away from efficient behavior.

Definition 7 *For all $\sigma \geq 0$, let $BRV_{i,\sigma}(x)$ denote the value that player i gets in game Γ_σ from best replying to a player $-i$ using a threshold form strategy s_x.*

For σ small enough for Lemma 10 to hold, denote by $\mathbf{x}^*_\sigma(\mathbf{V})$ the unique rationalizable pair of strategies of game $\Psi_\sigma(\mathbf{V})$. Note that $\mathbf{x}^*_\sigma(\mathbf{V})$ belongs to $\mathbb{R}^2$, while the risk-dominant threshold $x^{RD}(\mathbf{V})$ of game $\Psi_0(\mathbf{V})$ belongs to $\mathbb{R}$.

For any pair of thresholds $\mathbf{x} \in \mathbb{R}^2$, define $\xi_\sigma(\mathbf{x}) \equiv \mathbf{x}^*_\sigma(BRV_{i,\sigma}(x_{-i}), BRV_{-i,\sigma}(x_i))$ and $\xi(\mathbf{x}) \equiv x^{RD}(BRV_{i,0}(x_{-i}), BRV_{-i,0}(x_i))$. When $x \in \mathbb{R}$, $\xi(x)$ will be used to denote $\xi(x,x)$.

Lemma 11 (properties of ξ_σ) *There exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$, ξ_σ is a well defined, continuous mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. Furthermore, the following properties hold:*

(i) A pair of strategies (s_i, s_{-i}) is a Markovian equilibrium of Γ_σ if and only if it takes a threshold form and the associated pair of thresholds, $\mathbf{x} = (x_i, x_{-i})$, satisfies $\mathbf{x} = \xi_\sigma(\mathbf{x})$;

(ii) As σ goes to 0, $\xi_\sigma(\mathbf{x})$ converges uniformly over $\mathbf{x} \in \mathbb{R}^2$ to the symmetric pair $(\xi(\mathbf{x}), \xi(\mathbf{x}))$;

(iii) The mapping $\xi : \{\mathbb{R} \to \mathbb{R},\ x \mapsto \xi(x)\}$ is weakly increasing.

Note that point (i) of Lemma 11 implies that there is a bijection between fixed points of $\xi_\sigma(\cdot)$ and fixed points of $\phi_\sigma(\cdot)$.

Definition 8 (non-singular fixed points) *A fixed point x of ξ is non-singular if and only* *if there exists $\epsilon > 0$ such that either*

$$\begin{aligned} &\forall y \ \in [x-\epsilon, x),\ \xi(y) < y \quad \textit{and} \quad \forall y \ \in (x, x+\epsilon],\ \xi(y) > y \\ \textit{or} \quad &\forall y \ \in [x-\epsilon, x),\ \xi(y) > y \quad \textit{and} \quad \forall y \ \in (x, x+\epsilon],\ \xi(y) < y. \end{aligned}$$

In other terms x is non-singular whenever ξ cuts strictly through the 45^o line at x.

Proposition 2 (lower hemicontinuity) *Whenever x is a non-singular fixed point of ξ, then, for any sequence of positive numbers $\{\sigma_n\}_{n\in\mathbb{N}}$ converging to 0, there exists a sequence of fixed points of ξ_{σ_n}, $\{\mathbf{x}_n\}_{n\in\mathbb{N}} = \{(x_{i,\sigma_n}, x_{-i,\sigma_n})\}_{n\in\mathbb{N}}$, converging to (x, x) as n goes to infinity.*

This shows that all non-singular fixed point of ξ are the limit of threshold-form equilibria of the game Γ_σ as σ goes to 0. This shows that all equilibria (s_x, s_x) of Γ_0, with x a non-singular fixed point of ξ, are robust to miscoordination risk. Theorem 3 will enrich this result by showing that robustness to miscoordination risk implies robustness to strategic risk only if x is a *stable* non-singular fixed point of ξ. The next lemma shows that for an appropriate distance on payoff structures, fixed points of ξ are generically non-singular.

Definition 9 (topology on C^1 payoff structures) *A C^1 payoff structure π is a 9-tuple of C^1 functions $\pi = \times_{i\in\{1,2\}}(g^i, W^i_{12}, W^i_{21}, W^i_{22}) \times F$, that satisfies the assumptions of Section 2.3. Let Π^1 denote the set of C^1 payoff structures. The distance $||\cdot||_{\Pi^1}$ over payoff structures is defined as,*

$$||\pi - \tilde{\pi}||_{\Pi^1} = \sum_{l\in\{1,\dots,9\}} ||\pi_l - \tilde{\pi}_l||_\infty + \left|\left|\frac{\partial \pi_l}{\partial w} - \frac{\partial \tilde{\pi}_l}{\partial w}\right|\right|_\infty$$

where $||\cdot||_\infty$ denotes the supremum norm.

Lemma 12 (generic non-singularity) *There exists a subset P of Π^1 that is open and dense in Π^1 with respect to $||\cdot||_{\Pi^1}$ and such that whenever $\pi \in P$, the fixed points of ξ are all non-singular.*

Proposition 2 and Lemma 12 imply that typically, all fixed points of Φ are indeed associated with Markovian equilibria of Γ_σ as σ goes to 0. This shows that a global games information structure need not always yield full-fledged selection[7]. In that sense coordination in dynamic games is qualitatively different from coordination in one-shot games.

Although dominance solvability is clearly an attractive feature of one-shot global games, the possibility of multiplicity should not be considered a negative result in this context. As the example of Section 4.4 shows, trigger equilibria in a fully repeated global game are also equilibria of a related exit game in which payoffs upon exit are those obtained from reverting to the one-shot Nash equilibrium. In that setting, one can show that the one-shot Nash equilibrium is always an equilibrium of this exit game. If dynamic global games with exit were always dominance solvable, this would imply that the one-shot Nash is the only equilibrium in trigger strategies that is robust to private noisy assessments of the state of the world. From that perspective, the fact that a global games information structure does not always imply dominance solvability is reassuring.

Furthermore, Section 3.4 shows that the dominance solvability of one shot global games does survive in dynamic exit games, albeit in a weaker form. While equilibria may not be globally uniquely rationalizable, it is shown that the global games information structure can make them locally uniquely rationalizable.

3.4 Local dominance solvability, stability, and strategic uncertainty.

Local dominance solvability, discussed at length by Guesnerie (2002) in a macroeconomic context, can be viewed as an intermediary notion between Nash equilibrium and dominance solvability. For any two-player game, consider a set of strategies $\mathcal{Z}$ of player i and a strategy $s \in \mathcal{Z}$. The game is said to be locally dominance solvable at s with respect to $\mathcal{Z}$ whenever the sequence[8] $\{(BR_i^{\Delta} \circ BR_{-i}^{\Delta})^n(\mathcal{Z})\}_{n\in\mathbb{N}}$ converges to $\{s\}$ as n goes to infinity. In this case, we say that s is locally strongly rationalizable with respect to $\mathcal{Z}$. Equivalently, the game is said to be locally dominance solvable at s with respect to $\mathcal{Z}$ if and only if s is the only

[7]See Section 4.1 for examples.
[8]Recall that $BR_i^{\Delta} = BR_i \circ \Delta$.

rationalizable outcome when it is common knowledge among players that player i uses a strategy that belongs to $\mathcal{Z}$. From this perspective, s is a strict Nash equilibrium if and only if it is locally strongly rationalizable with respect to itself, and s is the unique rationalizable strategy of player i if and only if it is locally strongly rationalizable with respect to the set of all possible strategies.

The purpose of local dominance solvability is to be a middle ground between Nash equilibrium, which may not be demanding enough, and dominance solvability, which may be too demanding. The approach of Guesnerie (2002) is to introduce a topology on strategies and then define a strategy s as locally strongly rationalizable – without reference to any set – whenever there exists a neighborhood $\mathcal{N}$ of s such that s is locally strongly rationalizable with respect to $\mathcal{N}$. This effectively defines a stability criterion with respect to iterated best response. The object of this section is to characterize both the stability of Markovian equilibria of Γ_σ, and the size of their basin of attraction. We must first define a topology on strategies.

Definition 10 (balls in the "noise" topology) *Consider two histories h_t and h'_t. These are vectors of real numbers of length $2t-1$. Hence, we can define the distance $d(h_t, h'_t) = ||h_t - h'_t||_\infty$ and the Lebesgue measure λ over histories of length t. For any strategy s and any $\delta > 0$, the ball $\mathcal{B}_\delta(s)$ of center s and radius δ is defined as*

$$\mathcal{B}_\delta(s) \equiv \{s' \mid a.s.\ \forall h_t \in \mathcal{H}, \quad \lambda(\{h'_t \mid d(h'_t, h_t) < \delta \;\; and \;\; s(h'_t) = s'(h_t)\}) > 0\}.$$

A neighborhood $\mathcal{N}$ of a strategy s is a set that contains a ball of center s and radius $\delta > 0$.

Note that the choice of topology is not innocuous. Depending on the topology, the same equilibrium may be locally strongly rationalizable or not. In the topology defined above, a ball $\mathcal{B}_\delta(s_i)$ corresponds to the set of strategies an uninformed observer might deem possible when observing perfectly the moves of player i but observing a version of player i's signal that is garbled by a noise term of maximum amplitude δ. Alternatively, one can view a ball $\mathcal{B}_\delta(s_i)$ as the set of strategies deemed possible by a player getting a description of s_i that potentially misclassifies histories that differ by less than δ. In this sense, this topology is appropriate to discuss strategic uncertainty[9]. A strategy s will be locally strongly rationalizable with respect to balls in this topology whenever it is robust to small amounts of doubt regarding the players' common understanding of s.

Under full information, exit games admit no locally strongly rationalizable strategies because given any equilibrium, it is always possible to find another equilibrium that is arbitrarily close. The rest of this section shows that as σ goes to 0, the dominance solvability result of Carlsson and van Damme (1993) for one-shot two-by-two games translates into local dominance solvability for exit games. Furthermore, it is shown that as σ goes to 0, the local strong rationalizability of Markovian equilibria of Γ_σ – which is a stability property in the space of strategies – is asymptotically characterized by the stability of the increasing mapping $\xi : \mathbb{R} \to \mathbb{R}$ introduced in Definition 7. Let us first define asymptotic local dominance solvability formally.

Definition 11 (Asymptotic Local Dominance Solvability) *Consider a pair of strategies (s_i, s_{-i}). We say that the family of games $\{\Gamma_\sigma\}_{\sigma>0}$ is asymptotically locally dominance solvable (ALDS) at (s_i, s_{-i}) if there exist neighborhoods of s_i and s_{-i}, denoted $\mathcal{N}_i$ and $\mathcal{N}_{-i}$ such that $\forall i \in \{1, 2\}$,*

$$\lim_{\sigma\to 0}\lim_{n\to\infty}(BR^{\Delta}_{i,\sigma} \circ BR^{\Delta}_{-i,\sigma})^n(\mathcal{N}_i) = \{s_i\} \tag{8}$$

$$\text{and} \quad \lim_{\sigma\to 0}\lim_{n\to\infty} BR^{\Delta}_{-i,\sigma} \circ (BR^{\Delta}_{i,\sigma} \circ BR^{\Delta}_{-i,\sigma})^n(\mathcal{N}_i) = \{s_{-i}\} \tag{9}$$

The basin of attraction of (s_i, s_{-i}) is the greatest neighborhood $\mathcal{N}_i \times \mathcal{N}_{-i}$ of (s_i, s_{-i}) such that equations (8) and (9) hold.

The central result of this section is that the asymptotic local dominance solvability and the basin of attraction of Markovian threshold form equilibria are largely characterized by the stability and basins of attraction of fixed points of the mapping $\xi : \mathbb{R} \mapsto \mathbb{R}$.

[9] In fact the size of the balls $\mathcal{B}_\delta(s)$ for $\delta > 0$ is a good measure of the strategic uncertainty inherent to a strategy s. For instance, if $s = s_x$ then $B_\delta(s) = [s_{x+\delta}, s_{x-\delta}]$, while if s is defined by $s(w) = Stay$ if and only if $int[w/\delta]$ is an even number, then, $B_\delta(s)$ is the set of all possible strategies.

Proposition 3 is the key step to characterize local dominance solvability. It shows that whenever x is a stable fixed point of ξ, then for σ small enough, the first step of iterated best-response shrinks neighborhoods of s_x. Using the partial monotone best-response result of Proposition 1 this will allow us to prove asymptotic local dominance solvability.

Proposition 3 *Consider a stable fixed point x of ξ and y in the basin of attraction of x. If $y < x$, then there exists $x' \leq y$ and $\overline{\sigma} > 0$ such that x' belongs to the basin of attraction of x and, for all $\sigma \in (0, \overline{\sigma})$ and $i \in \{1, 2\}$, we have*[10] $BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'}) \preceq s_{x'}$.

Similarly, if $y > x$, there exists $x'' \geq y$ and $\overline{\sigma}$ such that x'' belongs to the basin of attraction of x and for all $\sigma \in (0, \overline{\sigma})$ and $i \in \{1, 2\}$, $s_{x''} \preceq BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x''})$.

Proposition 3 is a key step to understand the impact of a global games information structure on local dominance solvability, which is why a proof in the case of symmetric games is given here. It is instructive of how dominance solvability results for one-shot global games can be exploited in dynamic games. The proof in the case of asymmetric games is more delicate but follows the same intuition. It is given in Appendix A.

Proof (symmetric games): This proof applies to the case where players have the same payoff functions. This implies that $BR_{i,\sigma} = BR_{-i,\sigma} = BR_\sigma$. Let us show the first part of the lemma.

Pick $\overline{\sigma}$ small enough such that Proposition 1 applies. Then, for all $\sigma \in (0, \overline{\sigma})$, it is sufficient to prove that there exists $x' \leq y$ such that $BR_\sigma(s_{x'}) \preceq s_{x'}$. Let $BR_\sigma(a, V)$ denote the one-shot best response of a player expecting a continuation value V and facing a one-shot action profile a. Pick any[11] $x' \leq y$ that belongs to the basin of attraction of x. It must be that $\xi(x') > x'$. Using the fact that for Markovian strategies, one-shot action profiles are equivalent to full-fledged strategies, we can write, $BR_\sigma(s_{x'}) = BR_\sigma\left(s_{x'}, BRV_\sigma(x')\right)$.

The idea is to use this formulation to apply dominance solvability results from one-shot global games. From Lemma 1, we know that for $\overline{\sigma}$ small enough, all games $\Psi_\sigma(\mathbf{V})$, with $\sigma \in (0, \overline{\sigma})$ and $\mathbf{V} \in [m_i - \nu, M_i] \times [m_{-i} - \nu, M_{-i}]$, are dominance solvable. This and Lemma

[10]Recall that if a and b are thresholds such that $a > b$ then the corresponding strategies satisfy $s_a \preceq s_b$.

[11]In the case of asymmetric payoffs the choice of an appropriate x' becomes relevant, which is why the proposition allows for this extra degree of freedom.

3 implies that when n goes to infinity, the sequence $\{(BR_\sigma(\cdot, BRV_\sigma(x')))^n(s_{x'})\}_{n\in\mathbb{N}}$ converges monotonously to the unique rationalizable equilibrium of the one-shot global game[12] $\Psi_\sigma(BRV_\sigma(x'))$. This equilibrium is associated to the threshold $x^*_\sigma(BRV_\sigma(x')) = \xi_\sigma(x')$. We know that ξ_σ converges uniformly to ξ and that $\xi(x') > x'$. This implies that for σ small enough, $\xi_\sigma(x') > x'$. This implies that the sequence $\{(BR_\sigma(\cdot, BRV_\sigma(x')))^n(s_{x'})\}_{n\in\mathbb{N}}$ is decreasing with respect to $\preceq$. Hence, we must have $BR_\sigma(s_{x'}) \preceq s_{x'}$. This proves the first part of the lemma. The second part results from an entirely symmetric reasoning. ■

We can now prove the main result of this section. It states that asymptotically, basins of attraction of Markovian strategies are largely characterized by the basins of attraction of ξ.

Theorem 3 (Asymptotic Local Dominance Solvability) *Consider any symmetric pair of threshold form strategies (s_x, s_x). Whenever x is a stable fixed point of ξ, then the family $\{\Gamma_\sigma\}_{\sigma>0}$ is ALDS at (s_x, s_x).*

More strongly, if an interval $[y, z]$ is included in the basin of attraction of x with respect to ξ, and $x \in (y, z)$, then, $[s_z, s_y]^2$ is included in the basin of attraction of (s_x, s_x) with respect to asymptotic local dominance.

Proof: The second part of the theorem implies the first one. We prove the second part directly. Using Proposition 3, we know there exist $\overline{\sigma}$, $x_- \leq y$ and $x_+ \geq z$, with $[x_-, x_+]$ included in the basin of attraction of x, such that for all $\sigma \in (0, \overline{\sigma})$, and $i \in \{1, 2\}$,

$$BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x^-}) \preceq s_{x^-} \quad \text{and} \quad s_{x^+} \preceq BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x^+}).$$

These inequalities and Proposition 1 imply by iteration that for all $n \in \mathbb{N}$,

$$\begin{aligned}\left(BR^{\Delta}_{i,\sigma} \circ BR^{\Delta}_{-i,\sigma}\right)^n ([s_{x^+}, s_{x^-}]) \;&\subset\; [(BR_{i,\sigma} \circ BR_{-i,\sigma})^n (s_{x^+}), (BR_{i,\sigma} \circ BR_{-i,\sigma})^n (s_{x^-})] \\ &\subset\; [(BR_{i,\sigma} \circ BR_{-i,\sigma})^{n-1} (s_{x^+}), (BR_{i,\sigma} \circ BR_{-i,\sigma})^{n-1} (s_{x^-})] \\ &\subset\; \cdots \subset [s_{x^+}, s_{x^-}]\end{aligned}$$

[12]Given that we are considering symmetric games, the arguments of many previously defined functions become redundant. Such redundant arguments are dropped in all relevant cases.

Consider the decreasing sequence $\{(BR_{i,\sigma} \circ BR_{-i,\sigma})^n (s_{x^-})\}_{n \in \mathbb{N}}$. As n goes to ∞, it must converge to a threshold form strategy with threshold $x_{i,\sigma}^- \in [x_-, x_+]$. Moreover $(s_{x_{i,\sigma}^-}, BR_{-i,\sigma}(s_{x_{i,\sigma}^-}))$ must be a Markovian threshold form equilibrium of Γ_σ. Lemma 8 implies that as σ goes to 0, any converging subsequence of $\{(x_{i,\sigma}^-, x_{-i,\sigma}^-)\}_{\sigma>0}$ must converge to a symmetric pair $(\tilde{x}, \tilde{x})$ such that $\tilde{x}$ is a fixed point of ξ and $\tilde{x} \in [x_-, x_+]$. The only fixed point of ξ in $[x_-, x_+]$ is x. This implies that as σ goes to 0, $x_{i,\sigma}^-$ must converge to x. Similarly, as n goes to ∞, the sequence $(BR_{i,\sigma} \circ BR_{-i,\sigma})^n (s_{x^+})$ converges to a threshold strategy with a threshold $x_{i,\sigma}^+$ that converges to x as σ goes to 0. This concludes the proof. ■

The value of this result lies in the fact that the stability of strategies with respect to a complex iterated best response mapping is characterized by the stability of fixed points of a simple[13] mapping ξ from $\mathbb{R}$ to $\mathbb{R}$.

It is also interesting to note that the closure of basins of attraction of an increasing mapping is a partition of $\mathbb{R}$. In other words, any value $x \in \mathbb{R}$ is either a fixed point of ξ, or belongs to the basin of attraction of a fixed point of ξ. This implies that if a Markovian equilibrium is associated to a threshold x that is an unstable fixed point of ξ, then s_x is asymptotically unstable with respect to iterated best response. As σ goes to 0, arbitrarily small amounts of pessimism or optimism will push players' behavior away from s_x. Hence, a Markovian equilibrium associated to a fixed point x of ξ will be robust to strategic uncertainty if and only if x is a stable fixed point of ξ. The basin of attraction of x with respect to ξ measures the amount of strategic uncertainty that can be introduced before the players' behavior is perturbed away.

Finally, this result restricts possible non-Markovian strategies: asymptotically, there can be no non-Markovian equilibrium that is strictly contained within two consecutive Markovian equilibria with respect to $\preceq$.

[13]Computations can be further simplified by considering the mapping $\zeta : \mathbb{R} \mapsto \mathbb{R}$ defined by, $\zeta(x) = x^{RD}(NV_i(x), NV_{-i}(x))$, where $NV_i(x) \equiv \frac{1}{1-\beta Prob(w>x)} \mathbf{E}\left[g^i + (W_{22}^i - g^i)\mathbf{1}_{x>w}\right]$. Computing ζ is simpler than computing ξ and both functions coincide around their respective fixed points. See Lemma 17 in Appendix A for more details.

4 Applications

This section makes the case that the class of exit games introduced in Section 2 provides a practical framework to model miscoordination risk, and yields predictions that are qualitatively different from those obtained by focusing on Pareto-efficient equilibria under full information. Section 4.1 revisits the partnership game of Section 2.2 and shows how under private information, miscoordination fear can drive players to immediate exit even though the likelihood of miscoordination is vanishing. Section 4.2 shows how results from Section 3 can be used to define a simple criterion for the robustness of equilibria to miscoordination fear in exit games with approximately constant payoffs. As an example, Section 4.3 explores how wealth affects agents' ability to cooperate and shows that taking into account miscoordination fear yields predictions that are both intuitive and qualitatively distinct from those obtained under full information. Finally, Section 4.4 uses the example of repeated Cournot competition to show how exit games can be used to study the properties of trigger equilibria in repeated games with noisily observed states.

4.1 Miscoordination fear in the partnership game

Consider the partnership game introduced in Section 2.2. Flow payoffs are symmetric and given by

	S	E
S	w_t	$w_t - C + \beta V_E$
E	$b + V_E$	V_E

where payoffs are given for the row player only and $C > b \geq 0$. Under full information there exists a Pareto dominant equilibrium, of value $\overline{V}$, in which players stay whenever the state w_t is greater than a minimum threshold $\underline{w}$ defined by

$$\begin{aligned} \underline{w} + \beta\overline{V} &= b + V_E \\ \overline{V} &= \mathbf{E}\left[(w_t + \beta\overline{V})\mathbf{1}_{w_t > \underline{w}}\right] + F(\underline{w})V_E. \end{aligned}$$

If we use Pareto efficiency as a selection criterion in this full-information game, then the magnitude of C has no impact on players' behavior. Let us show this is not the case anymore when players privately assess the state of the world.

From Theorem 2, we know that the extreme equilibria of Γ_σ are asymptotically characterized by the fixed points of the mapping Φ. Hence, we are interested in the comparative statics of extreme fixed points of Φ with respect to C. Because the game is symmetric, fixed points of Φ will be symmetric and Φ can be restricted to a mapping from $\mathbb{R}$ to $\mathbb{R}$. The risk dominant threshold of the augmented game $\Psi_0(V)$ is given by the equation

$$x^{RD}(V) + \beta V - b - V_E = V_E - x^{RD}(V) + C - \beta V_E$$

so that $x^{RD}(V) = (1-\beta)V_E + \frac{b+C}{2} + \beta\frac{V_E - V}{2}$. The mapping Φ is defined by,

$$\forall\, V \in [V_E, +\infty), \quad \Phi(V) = V_E + \int_{w\in\mathbb{R}} (w + \beta V - V_E)\mathbf{1}_{w > x^{RD}(V)} f(w)\mathrm{d}w. \tag{10}$$

Figure 1 summarizes simulations of Φ in which f is a Gaussian distribution of parameters (μ, η^2). In the cases represented in Figure 1, $V_E = 5$, $\beta = 0.7$, $C = 3$, $b = 1$ and $\mu = 3$. As Figure 1(a) shows in the case of $\eta = 1$, private assessments of the state of the world can dramatically reduce the set of rationalizable strategies. The range of equilibrium values shrinks from the interval $[5.3, 9.9]$ under full-information to the singleton $\{7.4\}$ once players' fear of miscoordination is taken into account. Interestingly, as the standard-error η diminishes, the set of equilibria that are robust to miscoordination risk changes a lot even though extreme equilibrium values under full-information vary very little. Figure 1(b) corresponds to the case $\eta = 0.2$. Under full information, the set of equilibrium values is $[5.1, 9.8]$ and does not differ from the case $\eta = 1$ by much. However, unlike the case $\eta = 1$, the game with private information now exhibits multiple asymptotic equilibria: one middle equilibrium that is unstable with respect to iterated best-reply, and two extreme equilibria associated with values 5.2 and 9.5, that are stable with respect to iterated best-reply. Note that the two extreme equilibrium values under miscoordination risk are actually very close to the extreme equilibrium values under full information[14].

One can also derive comparative statics with respect to C directly from expression (10). Indeed, we have, $\frac{\partial \Phi(V)}{\partial C} = -f(x^{RD}(V))\left(x^{RD}(V) + \beta V - V_E\right)$. Since $x^{RD}(V)$ is the risk-dominant threshold of $\Psi_0(V)$, it must be that for $w = x^{RD}(V)$, staying is a strict Nash equilibrium of the game $G(V,w)$. Hence, we obtain that $(x^{RD}(V) + \beta V - V_E) > 0$. This shows that $\frac{\partial \Phi(V)}{\partial C} < 0$. Since Φ is an increasing mapping, downward shifts of Φ also shift its extreme fixed points downwards. Hence, we conclude that the extreme fixed points of Φ are strictly decreasing in C. Under a global games information structure, worsening the payoffs upon miscoordination diminishes the players' ability to cooperate, even though the probability of actual miscoordination is vanishingly small.

In fact, as C goes to $+\infty$, $x(V)$ goes uniformly to $-\infty$ over any compact. This implies that over any compact, $\Phi(V)$ converges uniformly to the constant V_E. Since we know that independently of C, fixed points of Φ must belong to $[V_E, \overline{V}]$, this implies that as $\lim_{C\to\infty}\lim_{\sigma\to 0} V_\sigma^H(C) = V_E$, and immediate exit is asymptotically the only rationalizable strategy that is robust to miscoordination fear. Given that having C go to $+\infty$ does not affect the Pareto efficient equilibrium of the full information game, this shows in a stark way how modeling miscoordination fear can generate new predictions. Section 4.3 illustrates this point in a richer economic context by using the robustness criterion developed in Section 4.2.

4.2 Robustness of cooperation in games with constant payoffs

This section considers the limit where the distribution f of states of the world w becomes arbitrarily concentrated around a particular state w_0. Interestingly, at the limit, the sensitivity of cooperation to miscoordination risk only depends on the payoffs at w_0. This allows us to define a simple explicit criterion for the robustness of cooperation in games with constant payoffs. The example of Section 4.3 will exploit that criterion to investigate the effect of wealth on agents' ability to cooperate.

[14]This could potentially suggest that as the underlying distribution f becomes concentrated around a particular state of the world w_0 (here $w_0 = \mu$), miscoordination risk has no impact on the sustainability of cooperation. Section 4.2 shows that this is not the case.

Definition 12 (global games extension) *Consider a vector of payoff functions*

$$\gamma = (g^i, W_{12}^i, W_{21}^i, W_{22}^i) \times (g^{-i}, W_{12}^{-i}, W_{21}^{-i}, W_{22}^{-i})$$

and a sequence $\{f_n\}_{n\in\mathbb{N}}$ *of density functions with convex support, converging weakly to a Dirac mass at* w_0 *when* n *goes to infinity. The sequence of game structures* $\pi_n = (\gamma, f_n)$ *is said to be a global games extension of the full information game with constant payoffs* $\gamma(w_0)$ *whenever, for all* $n \in \mathbb{N}$*, the payoff structure* π_n *satisfies Assumptions 1, 2, 3, 4 and 5.*

Note that a game with constant payoffs $\gamma(w_0)$ can admit multiple global game extensions, which can have different payoff functions and different densities.

Assumptions 2 and 3 only make sense in a global game context, Assumptions 1, 4, and 5 however naturally extend to games with constant payoffs. Indeed Assumption 1 is trivially satisfied, and Assumptions 4, 5 are required to hold only when the state of the world is w_0.

Lemma 13 *Any exit game with certain flow-payoffs* $\gamma(w_0)$ *satisfying Assumptions 4 and 5 admits a global game extension.*

Consider a global game extension $\{\pi_n\}_{n\in\mathbb{N}}$ of some game with constant payoffs $\gamma(w_0)$, and $\{\Phi_n\}_{n\in\mathbb{N}}$ the associated value mappings. Let $\mathbf{V}_n^H$ and $\mathbf{V}_n^L$ denote the highest and lowest fixed points of Φ_n. Denote by $\mathbf{V}^H$ the vector of values obtained by players if they stayed every period in game $\gamma(w_0)$ and $\mathbf{V}^L = (W_{22}^i(w_0), W_{22}^{-i}(w_0))$ the values they would obtain upon immediate exit.

The case of greatest interest - considered in the remainder of this section - is the one in which staying is a Nash equilibrium whenever players expect continuation values $\mathbf{V}^H$, but exit is the only Nash equilibrium when players expect continuation values $\mathbf{V}^L$. As before, we denote by $G(\mathbf{V}, w)$ the full information augmented game

	S	E
S	$g^i(w_t) + \beta V_i$	$W_{12}^i(w_t)$
E	$W_{21}^i(w_t)$	$W_{22}^i(w_t)$

where i is the row player.

The next proposition shows that in games with approximately constant payoffs, the robustness of cooperation to miscoordination fear is entirely characterized by the payoffs at w_0.

Proposition 4 (robustness to miscoordination fear) *Whenever staying is the risk-dominant equilibrium of game $G(\mathbf{V}^H, w_0)$, as n goes to infinity, $\mathbf{V}_n^L$ converges to $\mathbf{V}^L$ and $\mathbf{V}_n^H$ converges to $\mathbf{V}^H$.*

Whenever exit is the risk-dominant equilibrium of game $G(\mathbf{V}^H, w_0)$, as n goes to infinity, $\mathbf{V}_n^L$ converges to $\mathbf{V}^L$ and $\mathbf{V}_n^H$ converges to $\mathbf{V}^L$.

Proof: Denote by $\mathbf{V}_n(x)$ the vector of values players would obtain by best replying to s_x under full information for the payoff structure π_n.

Let us first prove that $\mathbf{V}_n^L$ always converges to $\mathbf{V}^L$ as n goes to ∞. Since $\mathbf{V}^L$ is the value of immediate exit, it is clear that $\liminf \mathbf{V}_n^L \geq \mathbf{V}^L$. Since staying is not an equilibrium action when players expect continuation values $\mathbf{V}^L$, it must be that there exists $\tau > 0$ such that $x^{RD}(\mathbf{V}^L) > w_0 + \tau$. By continuity of x^{RD} this implies that there exists $\delta > 0$ such that for all $\mathbf{V}$ satisfying $||\mathbf{V} - \mathbf{V}^L||_\infty < \delta$, we have $x^{RD}(\mathbf{V}) > w_0 + \tau/2$. Convergence of f_n to a Dirac mass at w_0 implies that there exists N such that for all $n \geq N$, $||\mathbf{V}_n(w_0 + \tau/2) - \mathbf{V}^L||_\infty < \delta$. This implies that for all $n \geq N$, $\xi_n(w_0 + \tau/2) \geq w_0 + \tau/2$. Hence ξ_n must have a fixed point above $w_0 + \tau/2$. Since f_n converges to a Dirac mass at w_0 the value pair associated with such an equilibrium converges to $\mathbf{V}^L$ as n goes to infinity. Hence, $\lim \mathbf{V}_n^L = \mathbf{V}^L$.

Assume that staying is risk-dominant in $G(\mathbf{V}^H, w_0)$. This means that there exists $\tau > 0$ such that $x^{RD}(\mathbf{V}^H) < w_0 - \tau$. By continuity of x^{RD} this implies that there exists $\delta > 0$ such that for all $\mathbf{V}$ satisfying $||\mathbf{V} - \mathbf{V}^H||_\infty < \delta$, we have $x^{RD}(\mathbf{V}) < w_0 - \tau/2$. Convergence of f_n to a Dirac mass at w_0 implies that there exists N such that for all $n \geq N$, $||\mathbf{V}_n(w_0 - \tau/2) - \mathbf{V}^H||_\infty < \delta$. This implies that for all $n \geq N$, $\xi_n(w_0 - \tau/2) \leq w_0 - \tau/2$. Hence ξ_n must have a fixed point below $w_0 - \tau/2$. This and the convergence of f_n to a Dirac mass at w_0 implies that as n goes to infinity, $\mathbf{V}_n^H$ converges to $\mathbf{V}^H$.

Assume now that exit is risk-dominant in $G(\mathbf{V}^H, w_0)$. This means that there exists $\tau > 0$ such that $x^{RD}(\mathbf{V}^H) > w_0 + \tau$. By continuity of x^{RD} this implies that there exists $\delta > 0$ such that for all $\mathbf{V}$ satisfying $\mathbf{V} < \mathbf{V}^H + \delta$, we have $x^{RD}(\mathbf{V}) > w_0 + \tau/2$. Convergence of f_n to a

Dirac mass at w_0 implies that there exists N such that for all $n \geq N$, $\mathbf{V}_n(+\infty) < \mathbf{V}^H + \delta$. This implies that for all $n \geq N$, and all $x \in \mathbb{R}$, $\xi_n(x) > w_0 + \tau/2$. Hence all fixed points of ξ_n are above $w_0 + \tau/2$. This and the convergence of f_n to a Dirac mass at w_0 implies that as n goes to infinity, $\mathbf{V}_n^H$ converges to $\mathbf{V}^L$. ■

Because this result does not depend on the particular global games extension of the game with constant payoffs $\gamma(w_0)$, Proposition 4 can be used to define a simple robustness criterion for cooperation in exit games with constant payoffs. According to this criterion, whenever the continuation value associated with full cooperation is high enough for staying to be risk-dominant in the augmented one-shot game, then full cooperation is robust to the fear of miscoordination. However if exit is the risk-dominant equilibrium in the augmented game, then immediate exit is the only robust equilibrium of the game with constant payoffs $\gamma(w_0)$. Section 4.3 provides an illustration of how the robustness criterion of Proposition 4 can yield predictions that are qualitatively different from those obtained when focusing on Pareto-efficient equilibria of the full-information game.

4.3 Wealth, miscoordination fear, and cooperation

This section investigates whether wealth facilitates cooperation or not. It is shown that taking into account the players' fear of miscoordination generates qualitatively new insights about the forces that affect players' ability to cooperate. In the exit game considered here, two symmetric players can cooperate on a project which increases their regular income I by an amount Π. Each player can either cooperate (Stay) or defect (Exit). When both players stay, the life of the project is extended by one period, otherwise the project dies next period and the players get their baseline stream of income. More precisely, we consider the symmetric exit game with the following, constant, flow payoffs

	S	E
S	$u(I+\Pi)$	$u(I-L)+\frac{\beta}{1-\beta}u(I)$
E	$u(I+G)+\frac{\beta}{1-\beta}u(I)$	$u(I)+\frac{\beta}{1-\beta}u(I)$

where payoffs are given for the row player, $G > \Pi > 0$, $L > 0$, $I \geq L$, and u is a concave twice differentiable utility function defined over $(0, +\infty)$.

In this game, the value of full cooperation is $V^H = \frac{1}{1-\beta}u(I+\Pi)$ while the value of immediate exit is $V^L = \frac{1}{1-\beta}u(I)$. Under full information, full cooperation will be sustainable if and only if

$$\text{(11)} \qquad \frac{\beta}{1-\beta}\left[u(I+\Pi) - u(I)\right] \geq u(I+G) - u(I+\Pi)$$

which is equivalent to $g(I) \equiv \frac{u(I+G)-u(I+\Pi)}{u(I+\Pi)-u(I)} \leq \frac{\beta}{1-\beta}$.

Proposition 5 (wealth makes cooperation harder under full information) *Whenever u exhibits (strictly) decreasing absolute risk aversion ($r \equiv -\frac{u''}{u'}$ decreasing), then g is (strictly) increasing in I.*

Decreasing absolute risk aversion is a standard property of utility functions. For instance it is satisfied for the class of CRRA functions $u(x) = \rho(x^\rho - 1)$, with $\rho \in (-\infty, 1)$. Hence Proposition 5 implies that for natural utility functions, focusing on the Pareto efficient outcome of the game with full information yields the prediction that wealth makes it harder to cooperate. While this prediction is not entirely counter-intuitive – it simply states that the rich just cannot be bothered to cooperate – the fact that it holds for all feasible levels of wealth is rather surprising. This result, however, does not hold anymore once we consider the impact of miscoordination fear.

The game defined above satisfies the conditions of Lemma 13, hence, it admits a global game extension, and we can use the robustness criterion of Proposition 4. Cooperation is robust to miscoordination fear if and only if staying is the risk-dominant action in the augmented symmetric one-shot game

	S	E
S	$u(I+\Pi) + \beta V^H$	$u(I-L) + \frac{\beta}{1-\beta}u(I)$
E	$u(w+G) + \frac{\beta}{1-\beta}u(I)$	$u(I) + \frac{\beta}{1-\beta}u(I)$.

Because the game is symmetric, staying will be risk-dominant if and only if

$$\text{(12)} \qquad \frac{\beta}{1-\beta}\underbrace{[u(I+\Pi)-u(I)]}_{\text{value of coop.}} \geq \underbrace{u(I+G)-u(I+\Pi)}_{\text{dev. tempt.}} + \underbrace{u(I)-u(I-L)}_{\text{miscoord. loss}}$$

which is equivalent to $h(I) \equiv g(I) + \frac{u(I-D)-u(I-L)}{u(I+\Pi)-u(I)} \leq \frac{\beta}{1-\beta}$.

Condition (12) has an intuitive interpretation: cooperation is robust to miscoordination risk if and only if the value of continued cooperation is greater than the sum of the deviation temptation and the miscoordination loss. For the same reason that $g(I)$ is increasing in I when u exhibits decreasing absolute risk aversion (DARA), the second term of h is decreasing in I when u is DARA, and hence, the monotonicity of h is unclear. The forces of deviation temptation and miscoordination fear push in opposite directions. The following proposition shows that when the loss L upon miscoordination is large enough, the prediction of Proposition 5 is entirely overturned: wealth facilitates cooperation at every income level.

Proposition 6 (wealth facilitates cooperation under miscoordination fear) *Whenever $L \geq G$ and the coefficient of absolute risk aversion $r \equiv -\frac{u''}{u'}$ is decreasing and (strictly) convex over $(0, \infty)$, then h is (strictly) decreasing for $I \in (L, +\infty)$.*

For DARA utility functions, it is quite natural for r to be convex. It simply states that the players' risk tolerance is increasing, but at a diminishing rate. This property is satisfied, for instance, for all CRRA functions. Proposition 6 implies that when strategic risk is significant enough, the impact of wealth on miscoordination fear always dominates the impact of wealth on the deviation temptation. Moreover, even when r is not convex, or $L < G$, the following lemma shows that at least for the very poor, wealth facilitates cooperation once the players' fear of miscoordination is taken into account.

Lemma 14 *Consider any concave function u such that $\lim_{x\to 0} u'(x) = +\infty$, then there exists $I^* > L$ such that h is strictly decreasing over the range (L, I^*).*

Because the miscoordination loss looms very large for the poor, they are particularly wary of miscoordination risk and choose not to purse projects that require them to rely on a partner.

This example shows how taking into account the robustness of cooperation to miscoordination fear can yield new comparative statics that are qualitatively different from those obtained by focusing on Pareto efficiency in the full-information game.

4.4 Trigger strategies in a game of repeated Cournot competition

This section uses the example of repeated Cournot competition to illustrate the point that selection results from Section 3 can be used to study trigger strategy equilibria in two-by-two repeated games with noisy assessments. In particular, the class of perfect Bayesian equilibria supported by trigger strategies can be mapped into the class of subgame perfect equilibria of the exit game in which players get the repeated Nash continuation value upon exit.

In each period, two firms $i \in \{1, 2\}$ can produce a quantity of good $Q_i \in \{Q, (1+\rho)Q\}$. The additional cost of producing ρQ units is $C > 0$. The unit price of the good is $P_t = \frac{D_t}{Q_1+Q_2}$, where D_t represents the strength of demand[15]. Parameters ρ, C, and Q are common knowledge, positive, and fixed in time. The intensity of demand, $\{D_t\}_{t\in\mathbb{N}}$, is an i.i.d. sequence of positive numbers drawn from some distribution f_D with c.d.f. F_D and support $[0, +\infty)$. Each firm gets a signal of current demand strength, $x_{i,t} = D_t + \sigma\varepsilon_{i,t}$. Each player's production decision is ex-post observable. Firms are risk neutral.

We say that a firm cooperates when its production is Q and defects when its production is $(1+\rho)Q$. Under full information, one-shot payoffs (for the row player) are given by

	Coop.	*Defect*
Coop.	$\frac{1}{2}D_t$	$\frac{1}{2+\rho}D_t$
Defect	$\frac{1+\rho}{2+\rho}D_t - C$	$\frac{1}{2}D_t - C.$

Clearly, for any D_t, cooperation is the efficient outcome of this one-shot game. Define $D^{NE} = 2\frac{2+\rho}{\rho}C$. Whenever $D_t > D^{NE}$, then defection is a dominant strategy. Inversely, whenever $D_t < D^{NE}$, then cooperation is a dominant strategy. Hence, this one shot game is dominance solvable. Denote V^{NE} the value of playing this one-shot Nash equilibrium

[15]Note that this particular functional form facilitates the analysis by making the total value of sales constant.

repeatedly under full information. Because this game exhibits increasing differences with respect to the state of the world $w_t \equiv -D_t$, it satisfies the assumptions of Carlsson and van Damme (1993). Hence, for σ small enough, this game is also dominance solvable under a global games information structure. Denote by V_σ^{NE} the value of repeatedly playing the one-shot Nash equilibrium under a global games information structure. Note that as σ goes to 0, V_σ^{NE} converges to V^{NE}.

The question is whether repeated interaction allows firms to sustain greater cooperation under trigger equilibria. Under trigger strategies players revert to repeatedly playing the one shot Nash equilibrium following any defection. In between periods, players also have the option to return to the repeated one-shot equilibrium. This insures that players always expect a continuation value weakly greater than V_σ^{NE}. Any trigger strategy equilibrium must be an equilibrium of the exit game Γ_σ with flow payoffs

$$\begin{array}{c|ll} & S & E \\ \hline S & \frac{1}{2}D_t & \frac{1}{2+\rho}D_t + \beta V_\sigma^{NE} \\ E & \frac{1+\rho}{2+\rho}D_t - C + \beta V_\sigma^{NE} & \frac{1}{2}D_t - C + \beta V_\sigma^{NE}. \end{array}$$

Because in this game payoffs upon exit are indexed by σ, we are not exactly in the framework of Section 2.3, the game, however, satisfies the more general assumptions of Appendix C: since players have the option to exit in-between periods, rational players expect values V greater than V_σ^{NE} and the equilibrium symmetry assumption holds since the one-shot game with common knowledge has only symmetric equilibria; increasing differences in the state of the world holds with respect to $-D_t$; dominance holds since staying is dominant for D_t close enough to 0, and exit is dominant for any D_t high enough; finally, the assumption that "staying is good" holds because

$$A_i(D_t, V_\sigma^{NE}) = \frac{\rho}{2(2+\rho)}D_t = B_i(\gamma_t) \geq 0.$$

As σ goes to 0, rationalizable strategies of game Γ_σ are bounded by extreme Markovian equilibria[16] whose continuation values converge to the highest and lowest fixed points of

$$\Phi(V) = \frac{1}{2}\mathbf{E}[D_t] - C + \beta V^{NE} + (C + \beta V - \beta V^{NE}) F_D(x^{RD}(V))$$

Where $x^{RD}(V) = \frac{2+\rho}{\rho}(2C + \beta V - \beta V^{NE})$. It is interesting to note that $\Phi(V^{NE}) = V^{NE}$, hence asymptotically, the one-shot Nash equilibrium is always an equilibrium of Γ_σ. Furthermore $\xi(D^{NE}) = D^{NE}$, independently of the particular distribution of states of the world.

We are particularly interested in the case where there can be multiple equilibria, and in their asymptotic stability properties. The asymptotic local strong rationalizability of Markovian equilibria of Γ_σ is characterized by the stability properties of fixed points of the mapping ξ. Because we are considering symmetric games, the stability of fixed points of ξ is equivalent to the stability of fixed points of Φ.

Lemma 15 *Define $g = \log f_D$. Then whenever $\partial^2 g/\partial D^2 \leq 0$ and $\partial^3 g/\partial D^3 \leq 0$, the mapping Φ is S-shaped.*

Notably, this lemma covers the case of exponential distributions and truncated Gaussian distributions. For such distributions, Φ will admit at most three fixed points.

Figure 2 presents various simulation in the case where f_D follows a truncated Gaussian of parameters (μ, η^2). In all cases, $C = 2, \rho = 1, \eta = 1$, and $\beta = 0.7$. The different cases correspond to different values of μ. Low values of μ put greater weight on states D_t that make cooperation easier to sustain while high values of μ make cooperation typically harder to sustain. For all simulations $D^{NE} = 12$ is an equilibrium threshold. Note that fixed points of ξ below D^{NE} do not correspond to equilibria, since they are associated with continuation values less than V^{NE} which players can opt out of. For $\mu = 8$, Figure 2(a) shows there is a unique rationalizable strategy with threshold D^{NE}. In this case, players already cooperate most of the time in the one-shot Nash equilibrium. Hence incremental

[16] Note that in this example, because the game exhibits increasing differences in $-D_t$ rather than D_t, for a given cooperation threshold x, players cooperate when D_t is less than x rather than greater than x. Hence the greatest equilibrium with respect to $\preceq$ is the one which is associated to the greatest cooperation threshold rather than the smallest cooperation threshold.

amounts of cooperation bring only very little gain in utility and cannot be self-sustained. Figure 2(b) corresponds to the case where $\mu = 12$. It is particularly interesting because only two of the fixed points of ξ correspond to Markovian equilibria. From Theorem 3, we know that the lowest one – corresponding to the one-shot Nash equilibrium – is unstable with respect to iterated best response, while the highest one is stable. This highest equilibrium is the only stable equilibrium of the game. Hence this higher equilibrium can be viewed as the natural outcome: any small amount of optimism will lead players to coordinate on the high cooperation equilibrium. Finally, Figure 2(c) corresponds to the case where $\mu = 15$ which puts greater weight on states of the world that make it difficult to sustain cooperation. There are three equilibria, D^{NE} being the lowest. In this case both extreme equilibria are stable.

5 Conclusion

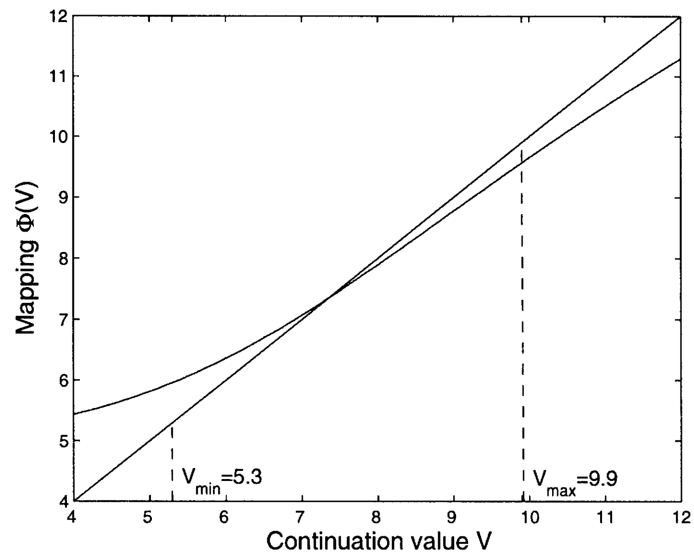
This paper provides a framework to model miscoordination fear in dynamic games. In particular it analyzes the robustness of cooperation to small amounts of observational noise in a class of dynamic games with exit. In equilibrium, this departure from common knowledge generates a fear of miscoordination that pushes players away from the full information Pareto efficient frontier, even though actual miscoordination happens with a vanishing probability. Payoffs upon miscoordination, which play no role when considering the Pareto efficient frontier under full information, determine the extent of the efficiency loss. The greater the loss upon miscoordination, the further will players be pushed away from the full information Pareto frontier.

The first step of the analysis is to show that for cooperation games with exit, the set of rationalizable strategies is bounded by extreme Markovian equilibria. The second step uses the dynamic programming approach to subgame perfection of Abreu, Pearce, and Stacchetti (1990) to recursively apply selection results in one-shot global games. As players' signals become increasingly correlated, this yields a fixed point equation for continuation values associated with Markovian equilibria. Whenever this mapping has a unique fixed point, the set of rationalizable strategies of the game with perturbed information converges to a singleton as signals become arbitrarily precise. However, unlike in one-shot two-by-two games, infinite horizon exit games can still admit multiple equilibria under the global game information structure.

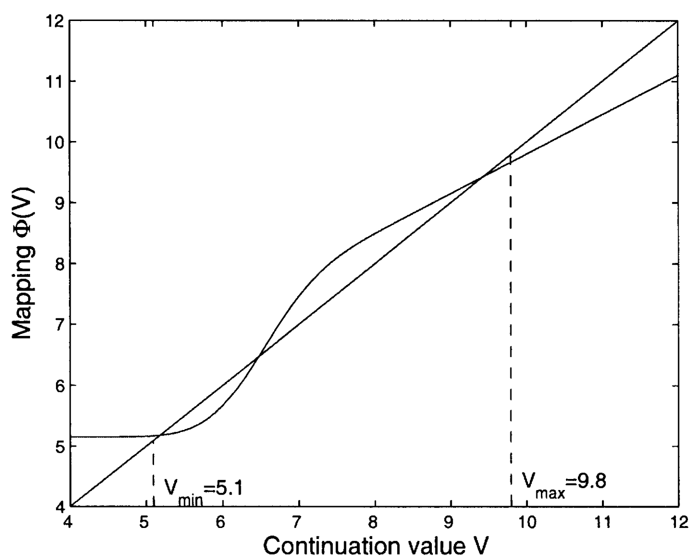
The dominance solvability of one-shot global games carries over in the weaker form of local dominance solvability, which can be interpreted both as a stability property and as a form of robustness to strategic uncertainty. As noise vanishes, the local dominance solvability and basins of attraction of Markovian equilibria are characterized by the stability of fixed points of an explicitly computable increasing mapping from $\mathbb{R}$ to $\mathbb{R}$. The greater the basin of attraction of an equilibrium s, the more robust it is to strategic uncertainty.

Finally, by considering various examples, the paper makes the case that this framework is simple and flexible enough to be used for applied purposes, and that it provides new insights about cooperation that could not be obtained by focusing on Pareto efficiency under full information. In particular, the model can be used to define a robustness criterion for cooperation in exit games with constant payoffs: whenever staying is the risk-dominant strategy of the one-shot game augmented with the players' continuation values, cooperation is robust to any global game extension; whenever defection is the risk-dominant strategy of the one-shot augmented game, then for any global game extension, the set of rationalizable strategies shrinks to immediate exit. This criterion can be readily used in applied games and provides insights on the determinants of cooperation that are qualitatively different from those obtained under full-information.

Figures

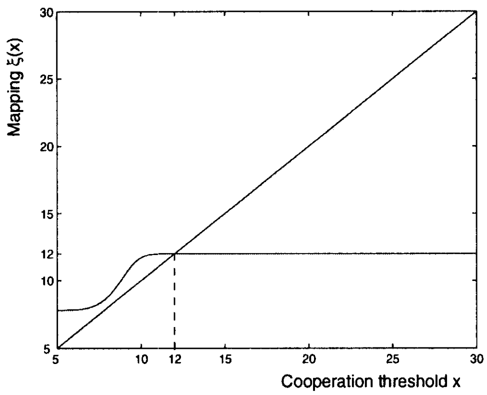


(a) Unique fixed point: $\eta = 1$.

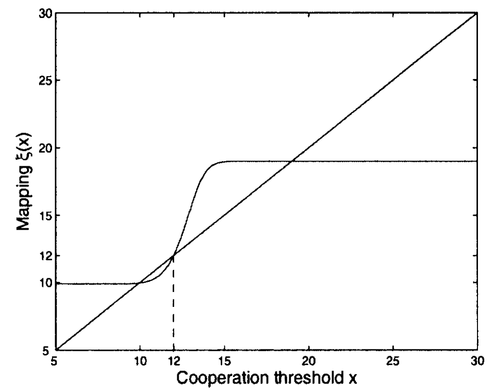


(b) Multiple fixed points: $\eta = 0.2$.

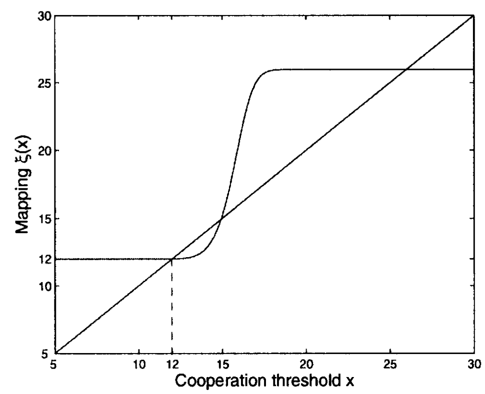
Figure 1: Equilibria of the partnership game depending on η. $V_E = 5$, $\beta = 0.7$, $\mu = 3$, $C = 3$ and $b = 1$



(a) Unique stable equilibrium: $\mu = 8$.



(b) One unstable and one stable equilibria: $\mu =$ 12.



(c) One unstable and two stable equilibria: $\mu =$ 15.

Figure 2: Trigger equilibria in a game of repeated Cournot competition depending on μ. $C = 2$, $\beta = 0.7$, $\rho = 1$, $\eta = 1$. Fixed points of ξ below $x = 12$ do not correspond to equilibria.

Appendix A: Proofs

Proof of Lemma 1: This is a direct application of Theorems 2, 3 and 4 of Chassang (2006). ■

Proof of Lemma 3: This is a direct application of Theorem 1 of Chassang (2006). ■

Proof of Lemma 6: Consider $s' \in BR_{i,\sigma}(s_x)$, and denote V the value player i expects from best-responding. The one-shot action profile a' induced by s' must belong to $BR_{i,\sigma}(s_x, V)$. Lemma 3 of Chassang (2006) implies that there exists $\overline{\sigma}$ such that for all $\sigma \in (0, \overline{\sigma})$, there is a unique such one-shot best-reply. It takes a threshold form $a_{x'}$ and the threshold x' is continuous in both x and V. This concludes the proof. ■

Proof of Theorem 1: This is a corollary of Proposition 1. The methodology of Milgrom and Roberts (1990) and Vives (1990) applies almost directly. Let R_i denote the set of rationalizable strategies of player i , U the set of all possible strategies and, $\mathbf{S}$ and $\mathbf{E}$ the strategies corresponding to "always staying" and "always exiting". Define $BR = BR_i \circ BR_{-i}$. R_i is the largest set of strategies such that $R_i \subset BR(R_i)$ and $R_{-i} = BR_{-i}(R_i)$.

Noting that $U = [\mathbf{E}, \mathbf{S}]$, since $\mathbf{S}$ and $\mathbf{E}$ are Markovian, Proposition 1 implies that $BR(U) \subset [BR(\mathbf{E}), BR(\mathbf{S})]$. Since the best response to a Markovian strategy is Markovian, we know that $BR(\mathbf{E})$ and $BR(\mathbf{S})$ are Markovian. This implies we can iterate forward. For all $k \in \mathbb{N}$, we obtain that $R_i \subset [BR^k(\mathbf{E}), BR^k(\mathbf{S})]$. Because $\{BR^k(\mathbf{E})\}_{k\in\mathbb{N}}$ and $\{BR^k(\mathbf{S})\}_{k\in\mathbb{N}}$ are monotone sequences of Markovian strategies, they are equivalent to monotone sequences of indicator functions specifying for all state of the world $w \in \mathbb{R}$ whether the player should stay or exit. As k goes to infinity, these sequences converge in probability to limits $BR^\infty(\mathbf{E})$ and $BR^\infty(\mathbf{S})$. We get that, $R_i \subset [BR^\infty(\mathbf{E}), BR^\infty(\mathbf{S})]$. Denote by s_i^L and s_i^H these extreme strategies (omitting the σ subscript for simplicity). By continuity of the best response mapping with respect to convergence in probability, we have $s_i^H = BR(s_i^H)$ and $s_i^L = BR(s_i^L)$, so that the pairs of strategies $(s_i^H, BR_{-i}(s_i^H))$ and $(s_i^L, BR_{-i}(s_i^L))$ are Nash equilibria in addition to being rationalizable.

It is easy to check that whenever s is a threshold form Markovian strategy, then Assumption 3 implies that $BR_i(s)$ is also Markovian and takes a threshold form. Since $\mathbf{E}$ and $\mathbf{S}$ take threshold forms, then by induction, extreme strategies also take a threshold form. ■

Proof of Lemma 7: Consider the highest equilibrium s_σ^H. For any rationalizable strategy s_{-i}, $s_{-i} \preceq s_{-i}^H$. Assumption 5, implies that player i gets a higher value from best-replying against $s_{-i,\sigma}^H$ than s_{-i}. Thus $V_i \leq V_{i,\sigma}^H$ in the functional sense. Identical reasoning yields the other inequality. ■

Proof of Lemma 8: Since $\mathbf{V}_n$ converges to $\mathbf{V}$ and Φ is continuous, for all $\tau > 0$, there exists N_1 such that for all $n \geq N_1$

$$||\Phi(\mathbf{V}) - \mathbf{V}||_\infty \leq ||\Phi(\mathbf{V}_n) - \mathbf{V}_n||_\infty + \tau/2.$$

Since $\phi_{\sigma_n}(\cdot)$ converges uniformly to Φ and $\mathbf{V}_n$ is a fixed point of ϕ_{σ_n}, there exists N_2 such that for all $n \geq N_2$, $||\Phi(\mathbf{V}_n) - \mathbf{V}_n||_\infty \leq \tau/2$. This yields that $||\Phi(\mathbf{V}) - \mathbf{V}|| \leq \tau$ for all $\tau > 0$. Hence, $\mathbf{V}$ must be a fixed point of Φ. ■

Proof of Lemma 9: Indeed, it results from expression (7) that

$$||\Phi(V)-\Phi(V')||_1 \leq \beta||V-V'||_1+||f||_\infty \sum_{i\in\{1,2\}} ||g_{11}^i+\beta V_i-W_{22}^i||_\infty \left|\left|\frac{\partial x^{RD}}{\partial V_i} + \frac{\partial x^{RD}}{\partial V_{-i}}\right|\right|_\infty ||V-V'||_1$$

Since $\sum_{i\in\{1,2\}} ||g_{11}^i + \beta V_i - W_{22}^i||_\infty ||\frac{\partial x^{RD}}{\partial V_i} + \frac{\partial x^{RD}}{\partial V_{-i}}||_\infty$ is finite, for any $\delta > \beta$, there exists $||f||_\infty$ small enough such that $||\Phi(V) - \Phi(V')||_1 \leq \delta||V - V'||_1$. ■

Proof of Lemma 11: $BRV_{i,\sigma}(x)$ is continuous in x since it is the maximum of a bounded function continuous in x. In conjunction with Theorems 2 and 4 of Chassang (2006), this yields the first part of the lemma.

Lemma 10 implies that for σ small enough all Markovian equilibria must take a threshold form. Such an equilibrium is associated with values (V_i, V_{-i}) and thresholds $\mathbf{x} = (x_i, x_{-i})$ which must satisfy $V_i = BRV_{i,\sigma}(x_{-i})$ and $(x_i, x_{-i}) = \mathbf{x}_\sigma^*(V_i, V_{-i})$. Hence, Markovian thresholds must satisfy $\mathbf{x} = \xi_\sigma(\mathbf{x})$. Inversely, if a vector x satisfies $\mathbf{x} = \xi_\sigma(\mathbf{x})$, then the values $\mathbf{V} = (V_i, V_{-i})$ defined by $V_i = BRV_{i,\sigma}(x_{-i})$ must satisfy, $\mathbf{V} = \phi(\sigma, \mathbf{V})$ and hence, using Theorem 2, values $\mathbf{V}$ support a Markovian equilibrium with thresholds $\mathbf{x}$. This gives us the second part of the lemma.

The third part of the lemma is an almost immediate consequence of Theorem 3 of Chassang (2006). One only needs to show that $BRV_{i,\sigma}(\cdot)$ converges uniformly to $BRV_{i,0}(\cdot)$ as σ goes to 0. Indeed, because states of the world have a bounded density and payoffs are Lipschitz, the best response when $\sigma = 0$ is almost optimal when $\sigma > 0$ and small and vice-versa. Hence, there exists $K > 0$ such that for all $x \in \mathbb{R}$, $|BRV_{i,0}(x) - BRV_{i,\sigma}(x)| < K\sigma$.

The fourth part of the lemma is a consequence of the fact that $x^{RD}(\mathbf{V})$ is decreasing in $\mathbf{V}$ and Assumption 5 which implies that $BRV_{i,0}(x)$ is decreasing in x. ■

Proof of Proposition 2: A direct proof can be given but it is faster to use the local dominance solvability property that will be proven in Theorem 3. For any $x \in \mathbb{R}$, $BR_{i,\sigma} \circ BR_{-i,\sigma}(s_x)$ takes a threshold form, $s_{x'}$. Define $\chi_\sigma(\cdot)$ by $\chi_\sigma(x) = x'$. For σ small enough, Lemma 6 and Proposition 1 imply that χ_σ is continuous and increasing. By definition of χ_σ, s_x is a threshold form Markovian equilibrium of Γ_σ if and only if $\chi_\sigma(x) = x$.

Consider a non singular fixed point of ξ denoted by x. Indeed, Assume that x is a stable fixed point – that is ξ cuts the 45^o line from below – then Theorem 3 implies that, for all $\tau > 0$, there exists $\overline{\sigma} > 0$ and $\eta \in (0, \tau)$ such that for all $\sigma \in (0, \overline{\sigma})$, the interval $[x - \eta, x + \eta]$ is stable by χ_σ. Since χ_σ is continuous and increasing, this implies that it has a fixed point belonging to $[x - \eta, x + \eta]$. This proves the lower hemicontinuity of stable fixed points of ξ.

Assume that x is unstable. Then for any $\tau > 0$, there exists $\eta \in (0, \tau)$ such that $x - \eta$ and $x + \eta$ respectively belong to the basins of attraction of a lower and a higher fixed point.

Proposition 3 implies that there exist η' and η'' in $(0, \eta)$ such that $\chi_\sigma(x - \eta') < x - \eta'$ and $\chi_\sigma(x + \eta'') > x + \eta''$. Since χ_σ is continuous, this implies that it admits a fixed point within $[x - \eta', x + \eta'']$. This proves the lower hemicontinuity of unstable non-singular points of ξ. ■

Lemma 16 *Consider x a fixed point of ξ. Then there exists $\eta > 0$ and $\overline{\sigma} > 0$ such that for all $\sigma \in [0, \overline{\sigma})$, $x' \in [x - \eta, x + \eta]$, and $i \in \{1, 2\}$ there exists $x'' \in \mathbb{R}$ such that $BR_{i,\sigma}(s'_x) = \{s_{x''}\}$ and $|x'' - x'| < 2\sigma$.*

Proof of Lemma 16: Since, x is a fixed point of ξ, it must be that at $w = x$, both (E, E) and (S, S) are strict Nash equilibria of the game $G(BRV_{i,0}(x), BRV_{-i,0}(x), w)$. Since $BRV_{i,\sigma}(x')$ is continuous in σ and x', and payoffs are continuous in w, there exist $\eta > 0$ and $\overline{\sigma} < \eta/4$ such that for all $\sigma < \overline{\sigma}$ and $x' \in [x - \eta, x + \eta]$, then for all $w \in [x' - \sigma, x' + \sigma]$, both (E, E) and (S, S) are strict Nash equilibria of $G(BRV_{i,\sigma}(x'), BRV_{-i,\sigma}(x'), w)$.

Take $\sigma < \overline{\sigma}$ and $x' \in [x - \eta/2, x + \eta/2]$. Assumption 3 implies that for any σ, the best reply to a threshold form strategy is also a threshold for strategy. This implies that indeed $BR_{i,\sigma}(x')$ takes the form $s_{x''}$. Let us show that$|x'' - x'| < 2\sigma$. When she gets a signal $x_{i,t} < x' - 2\sigma$, player i knows for sure that player $-i$ will be playing E. From the definition of η, we know that (E, E) is an equilibrium of $G(BRV_{i,\sigma}(x'), BRV_{-i,\sigma}(x'), w)$ for all values of w consistent with a signal value $x_{i,t}$. Thus, it must be that player i's best reply is to play E as well. Inversely, when she gets a signal $x_{i,t} > x' + 2\sigma$, player i knows that player $-i$ will play S, and her best reply is to Stay as well. This implies that $|x'' - x'| < 2\sigma$. ■

Lemma 17 *Define the function $\zeta : \mathbb{R} \mapsto \mathbb{R}$ by, $\zeta(x) = x^{RD}(NV_i(x), NV_{-i}(x))$, where $NV_i(x) \equiv \frac{1}{1-\beta Prob(w>x)}\mathbf{E}\left[g^i + (W^i_{22} - g^i)\mathbf{1}_{x>w}\right]$ is the value player i obtains when both players naively follow the threshold strategy s_x.*

Then, x is a fixed point of ξ if and only if it is a fixed point of ζ. Furthermore, for any fixed point x, there exists $\eta > 0$ such that for all $x' \in [x - \eta, x + \eta]$, $\zeta(x') = \xi(x')$.

Proof of Lemma 17: Lemma 16 implies that for $\sigma = 0$, whenever x is a fixed point of ξ, then there exists $\eta > 0$ such that for all $x' \in [x - \eta, x + \eta]$, $BR_i(s_{x'}) = s_{x'}$. Hence, for all $x' \in [x - \eta, x + \eta]$, $BRV_i(x') = NV_i(x')$ and thus, $\xi(x') = \zeta(x')$.

Moreover whenever x satisfies $\zeta(x) = x$, then s_x is a threshold form Markovian equilibrium of the full information game Γ_0, which implies that $BRV_i(x) = NV_i(x)$. Thus $\xi(x) = \zeta(x) = x$ and x is also a fixed point of ζ. ■

Proof of Lemma 12: Let us first show that the set P of payoff structures such that ξ has a finite number of fixed points and has a derivative different from 1 at each of these fixed points is open in Π_1. From Lemma 17, we know that x is a non-singular fixed point of ξ if and only if it is a non-singular fixed point of ζ. One can compute ζ explicitly : $\zeta(x) = x^{RD}(NV_i(x), NV_{-i}(x))$, where $NV_i(x) \equiv \frac{1}{1-\beta Prob(w>x)}\mathbf{E}\left[g^i + (W^i_{22} - g^i)\mathbf{1}_{x>w}\right]$. Since

$x^{RD}(V_i, V_{-i})$ is defined as the locally unique solution of the C^1 equation

$$Q_\pi(x, V_i, V_{-i}) \equiv \prod_{i\in\{1,2\}} (g^i + \beta V_i - W^i_{21}) - \prod_{i\in\{1,2\}} (W^i_{22} - W^i_{12}) = 0$$

the implicit function theorem implies that $\frac{\partial x^{RD}}{\partial V_i}$ exists and is a continuous expression of the derivatives of the payoff functions $(g^i, W^i_{12}, W^i_{21}, W^i_{22})_{i\in\{1,2\}}$. Hence ζ admits a derivative,

$$\begin{aligned}\frac{\partial \zeta}{\partial x} = \sum_{i\in\{1,2\}} \ \frac{\partial x^{RD}}{\partial V_i}\Bigg(&\frac{1}{1-\beta Prob(w>x)} f_w(x^{RD})\left(W^i_{22}(x^{RD}) - g^i(x^{RD})\right)\\ &-\frac{\beta f(x)}{(1-\beta Prob(w>x))^2}\mathbf{E}[g^i + (W^i_{22} - g^i)\mathbf{1}_{x>w}]\Bigg).\end{aligned}$$

This derivative is continuous with respect to x and continuous in the payoff structure with respect to $||\cdot||_{\Pi_1}$. Assume that for a payoff structure π, the mapping ξ has a finite number of fixed points and has a derivative that is different from 1 at all it's fixed points. Then there exists $\eta > 0$ such that for any fixed point x, $\frac{\partial \xi}{\partial x}$ is either less than $1-\eta$ over $[x-\eta, x+\eta]$ or greater than $1+\eta$ over $[x-\eta, x+\eta]$. A payoff structure $\tilde{\pi}$ close enough to π, is associated with a mapping $\tilde{\xi}$ such that all fixed points $\tilde{x}$ of $\tilde{\xi}$ belong to $[x-\eta, x+\eta]$ and such that its derivative over $[x-\eta, x+\eta]$ is either greater than $1+\eta/2$ or lower than $1-\eta/2$. This implies that all payoff structures close enough to π are also associated with mappings ξ that have a finite number of fixed points and have a derivative different from 1 at each of these fixed points.

Let us now show that P is dense in Π_1. Consider a payoff structure π and $\nu > 0$. We know from Assumption 2 that fixed points of ξ are restricted to a compact region $[\underline{x}, \overline{x}]$. By Weierstrass's Theorem, there exist uniform polynomial approximations of the derivative of the vector of functions π over $[\underline{x}-1, \overline{x}+1]$. Hence, one can find a payoff structure $\tilde{\pi}$ such that $||\pi - \tilde{\pi}||_{\Pi^1} < \nu/2$, π, $\tilde{\pi}$ coincide over the complementary of $[\underline{x}, \overline{x}]$, and $\tilde{\pi}$ is polynomial over $[\underline{x}, \overline{x}]$. By Lemma 18, this implies that the mapping ζ is analytic over $[\underline{x}, \overline{x}]$. Let us now define the family of payoffs $\tilde{\pi}^\delta$ by,

$$\begin{aligned}\forall w \in \mathbb{R}, \quad & \tilde{g}^{i,\delta}(w) \equiv \tilde{g}^i(w)\\ & \tilde{W}^{i,\delta}_{22}(w) = \tilde{W}^i_{22}(w)\\ & \tilde{W}^{i,\delta}_{12}(w) = \tilde{W}^i_{12}(w-\delta) + \tilde{g}^i(w) - \tilde{g}^i(w-\delta)\\ & \tilde{W}^{i,\delta}_{21}(w) = \tilde{W}^i_{21}(w-\delta) + \tilde{W}^i_{22}(w) - \tilde{W}^i_{22}(w-\delta)\end{aligned}$$

This new payoff structure is such that for any δ, and any $x \in \mathbb{R}$, $\tilde{\zeta}^\delta(x) = \tilde{\zeta}(x) + \delta$. Note that for δ small enough, $\tilde{\pi}^\delta$ is arbitrarily close to $\tilde{\pi}$. More over $\tilde{\zeta}^\delta$ is analytic in x. Whenever $\delta_1 \neq \delta_2$ then $\tilde{\zeta}^{\delta_1}$ and $\tilde{\zeta}^{\delta_2}$ have strictly different fixed points. Assume that for every $\delta \in (0, \nu)$, there exists a fixed point x^δ of $\tilde{\zeta}^\delta$ such that $\tilde{\zeta}^\delta$ is singular at x^δ. this implies that the derivative of ζ is equal to 1 an infinte number of times in a compact set. Since the derivative of ζ is analytic, this would imply that it is identically equal to 1 over $[\underline{x}-1, \overline{x}+1]$. Since ζ has a fixed point in $[\underline{x}, \overline{x}]$, this would imply that ζ is equal to the identity over $[\underline{x}-1, \overline{x}+1]$ which contradicts the fact that fixed points of ζ belong to $[\underline{x}, \overline{x}]$. ■

Lemma 18 (Analyticity of ζ) *Whenever functions of the payoff structure*

$$\pi = \times_{i\in\{1,2\}}(g^i, W_{12}^i, W_{21}^i, W_{22}^i) \times F_w$$

are polynomial over the range $[\underline{w}, \overline{w}]$ then the mapping ζ is analytic.

Proof: We give the proof for the stationary case. First, note that $x^{RD}(V_i, V_{-i})$ is a simple root of the polynomial $Q(w) = \Pi_{i\in\{1,2\}}(g^i(w) + \beta V_i - W_{21}^i(w)) - \Pi_{i\in\{1,2\}}(W_{22}^i(w) - W_{12}^i(w))$. Indeed, $Q(w)$ is strictly decreasing at x^{RD}. A simple root of a polynomial is jointly analytic in the polynomial's coefficients. This implies that $x^{RD}(V_i, V_{-i})$ is analytic in (V_i, V_{-i}). Further more the functions $V_i(x)$ and $V_{-i}(x)$ can be computed explicitly:

$$V_i(x) = \frac{1}{1-\beta P(w>x)}\mathbf{E}[W_{22}^i(x) + (g^i - W_{22}^i)\mathbf{1}_{w>x}].$$

Clearly, $V_i(x)$ is analytic in x. Since the composition of analytic functions is analytic, this implies that ζ is analytic in x. ■

Proof of Proposition 3: Let us prove the first part of the proposition. Define $ba_-(x) = \inf\{\tilde{x} < x \mid \forall y \in [\tilde{x}, x], \xi(y) > y\}$. Because x is stable, $ba_-(x)$ is well defined. We distinguish two cases, either $ba_-(x) = -\infty$ or $ba_-(x) \in \mathbb{R}$.

If $ba_-(x) = -\infty$, any $x' < x$ belongs to the basin of attraction of x. Assumption 2 implies that there exists $\underline{x}$ such that for all $\sigma < 1$, $BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{-\infty}) \preceq s_{\underline{x}}$. Pick any $x' < \min\{x, \underline{x}\}$. Using the monotonicity implied by Proposition 1, we conclude that there exists $\overline{\sigma} > 0$ such that for all $\sigma < \overline{\sigma}$, $BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'}) \preceq BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{-\infty}) \preceq s_{x'}$.

Consider now the case where $ba_-(x) \in \mathbb{R}$. Then by continuity of ξ, we have that $\xi(ba_-(x)) = ba_{-x}$. From Lemma 16 we know that there exist $\eta > 0$ and $\overline{\sigma}$ such that for all $x' \in [ba_-(x) - \eta, ba_-(x) + \eta]$, and $i \in \{1,2\}$, $BR_{i,\sigma}(s_{x'}) = s_{x_i''}$ with $|x_i'' - x'| < 2\sigma$. By definition, we must have $y > ba_-(x)$. Thus we can pick $x' \in (ba_-(x), ba_-(x) + \eta)$ such that $x' < \min\{x, y\}$. We have that $\xi(x') > x'$. By continuity of ξ there exists $\tilde{x}'$ such that $\tilde{x}' < x'$ and $\xi(\tilde{x}') > x'$. Using the notation $BR_{i,\sigma}(a, V)$ to denote the best reply of player i to a one shot action profile a and continuation value V, and using the fact that one-shot action profile are identical to Markovian strategies, we obtain,

$$BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'}) = BR_{i,\sigma}\left(BR_{-i}(s_{x'}, BRV_{-i,\sigma}(x')), BRV_{i,\sigma}(BR_{-i,\sigma}(s_{x'}))\right) \tag{13}$$

We know that $|x''_{-i} - x'| \leq 2\sigma$. Thus there exists $\overline{\sigma}$ small enough such that $BR_{-i,\sigma}(s_{x'}) \preceq s_{\tilde{x}'}$. Joint with Assumption 5, this implies that, $BRV_{i,\sigma}(BR_{-i,\sigma}(s_{x'})) \leq BRV_{i,\sigma}(\tilde{x}')$. Furthermore, $\tilde{x}' < x'$ implies that $BRV_{i,\sigma}(x') \leq BRV_{i,\sigma}(\tilde{x}')$. Hence, using inequality (13), and the fact that for $i \in \{1,2\}$, $BR_i(a,V)$ is increasing in a and V with respect to $\preceq$, we obtain

$$\begin{aligned} BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'}) &\preceq BR_{i,\sigma}\left(BR_{-i}(s_{x'}, BRV_{-i,\sigma}(\tilde{x}')), BRV_{i,\sigma}(\tilde{x}')\right) \\ &\preceq BR_{i,\sigma}\left(\cdot, BRV_{i,\sigma}(\tilde{x}')\right) \circ BR_{-i,\sigma}\left(\cdot, BRV_{-i,\sigma}(\tilde{x}')\right)(s_{x'}) \end{aligned} \tag{14}$$

We know from Theorem 2 of Chassang (2006) that there exists $\overline{\sigma}$ small enough such that for all $\sigma \in (0, \overline{\sigma})$ and all $(V_i, V_{-i}) \in [m_i, M_i] \times [m_{-i}, M_{-i}]$, the game $\Psi_\sigma(V_i, V_{-i})$ has a unique rationalizable pair of strategies $\mathbf{x}^*_\sigma(V_i, V_{-i})$.

We know from Theorem 3 of Chassang (2006) that $\mathbf{x}^*_\sigma(V_i, V_{-i})$ converges uniformly to $x^{RD}(V_i, V_{-i})$ as σ goes to 0. This implies that $\mathbf{x}^*_\sigma(BRV_{i,\sigma}(\tilde{x}'), BRV_{-i,\sigma}(\tilde{x}'))$ converges to $(\xi(x'), \xi(x'))$ as σ goes to 0. Since $x' < \xi(\tilde{x}')$, it implies there exists $\overline{\sigma}$ such that for all $\sigma < \overline{\sigma}$, $x' < x^*_\sigma(BRV_{i,\sigma}(\tilde{x}'), BRV_{-i,\sigma}(\tilde{x}'))$.

The fact that $\Psi_\sigma(BRV_{i,\sigma}(\tilde{x}'), BRV_{-i,\sigma}(\tilde{x}'))$ has a unique rationalizable strategy and the monotonicity of Lemma 3 imply that the sequence of threshold form strategies

$$\left(BR_{i,\sigma}\left(\cdot, BRV_{i,\sigma}(\tilde{x}')\right) \circ BR_{-i}\left(\cdot, BRV_{-i,\sigma}(\tilde{x}')\right)\right)^n (s_{x'}), \quad \text{for } n \in \mathbb{N},$$

converges monotonously to the Markovian strategy of threshold $x^*_\sigma(BRV_{i,\sigma}(x'), BRV_{-i,\sigma}(x'))$. Since $x' < x^*_\sigma(BRV_{i,\sigma}(\tilde{x}'), BRV_{-i,\sigma}(\tilde{x}'))$, the sequence must be decreasing with respect to $\preceq$. Thus $BR_{i,\sigma}\left(\cdot, BRV_{i,\sigma}(\tilde{x}')\right) \circ BR_{-i,\sigma}\left(\cdot, BRV_{-i,\sigma}(\tilde{x}')\right)(s_{x'}) \preceq s_{x'}$. Using inequality (14), this yields that indeed $BR_{i,\sigma} \circ BR_{-i,\sigma}(s'_x) \preceq s_{x'}$.

The second part of the lemma results from a symmetric reasoning, switching all inequalities. ■

Proof of Lemma 13: Set $w_0 = 0$. Denote m_i and M_i the bounds on value implied by Assumption 1. For $i \in \{1,2\}$, define $\lambda_i = \min\{1, \frac{\gamma^i_{22}(0) - \gamma^i_{12}(0)}{\gamma^i_{21}(0) - m_i - \gamma^i_{11}(0)}\}$. Note that $\lambda_i > 0$. For any $w \in \mathbb{R}$ and $\imath \in \{1,2\}$, define $\gamma^i(w)$ by,

$$\gamma^i_{21}(w) = \gamma^i_{21}(0)\ ; \quad \gamma^i_{22}(w) = \gamma^i_{22}(0)\ ; \quad \gamma^i_{11}(w) = \gamma^i_{11}(0) + w$$

$$\gamma^i_{12}(w) = \begin{cases} \gamma^i_{12}(0) + \lambda_i w & \text{when } w > 0 \\ \gamma^i_{12}(0) + \lambda_i^{-1} w & \text{when } w < 0 \end{cases}$$

Assume that Assumption 5 holds strictly, more precisely, that, $A(0, m_i) > 0$. Then whenever, f_n is close enough to a Dirac mass at 0, there exists a lower bound m^n_i arbitrarily close to m_i. Pick any sequence $\{f_n\}_{n\in\mathbb{N}}$ with support $\mathbb{R}$ and weakly converging to a Dirac mass at 0 as n goes to ∞. Then there exists N such that for all $n > N$, $A(0, m^n_i) > 0$. Since for all w, $A(w, m^n_i) \geq A(0, m^n_i)$, this implies that for all n, $\pi_n = (\gamma, f_n)$ satisfies Assumption 5. Assumptions 1, 4, 3 and 2 are easily checked. Hence $\{(\gamma, f_n)\}_{n>N}$ is a global game extension of $\gamma(0)$.

When $A(0, m_i) = 0$, then the sequence f_n has to be chosen appropriately skewed to the right so that $m^n_i \geq m_i$. This can clearly be done, since by skewing f_n to the right, we can give value to staying by guaranteeing future cooperation in dominant states. This essentially puts us in the former case, and for such a sequence $\{f_n\}_{n\in\mathbb{N}}$, $\{(\gamma, f_n)\}_{n>N}$ is a global game extension of $\gamma(0)$. ■

Proof of Proposition 5: We have that,

$$\frac{\partial \log g(I)}{\partial I} = \frac{u'(I+G)-u'(I+\Pi)}{u(I+G)-u(I+\Pi)} - \frac{u'(I+\Pi)-u'(I)}{u(I+\Pi)-u(I)} \tag{15}$$
$$= \frac{\int_{I+\Pi}^{I+G} u''(x)dx}{\int_{I+\Pi}^{I+G} u'(x)dx} - \frac{\int_{I}^{I+\Pi} u''(x)dx}{\int_{I}^{I+\Pi} u'(x)dx}$$

Consider the following lemma.

Lemma 19 *For any $n \in \{1,\dots,+\infty\}$, consider sequences $\{a_1, b_1, \dots, a_n, b_n\}$ and $\{a'_1, b'_1, \dots, a'_n, b'_n\}$ such that for all $k \in \{1,\dots,n\}$, $b_k > 0$, $b'_k > 0$, and*

$$\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \cdots \frac{a_n}{b_n} \leq \frac{a'_1}{b'_1} \leq \frac{a'_2}{b'_2} \leq \cdots \leq \frac{a'_n}{b'_n}$$

then we have that

$$\frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} \leq \frac{a'_1+\cdots+a'_n}{b'_1+\cdots+b'_n}.$$

For any $n \geq 1$, and $k \in \{0,\dots,n\}$, consider the wealth levels $x_k^n = I+\Pi+\frac{k}{n}(G-\Pi)$ and $y_k^n = I+\frac{k}{n}\Pi$. Lemma 19 applies to the numbers $a_k = u''(y_k^n)$, $b_k = u'(y_k^n$, $a'_k = u''(x_k^n)$, and $b'_k = u'(x_k^n)$. This yields that,

$$\frac{\frac{G-\Pi}{n}\sum_{k=0}^n u''(x_k^n)}{\frac{G-\Pi}{n}\sum_{k=0}^n u'(x_k^n)} \geq \frac{\frac{\Pi}{n}\sum_{k=0}^n u''(y_k^n)}{\frac{\Pi}{n}\sum_{k=0}^n u'(y_k^n)}.$$

Letting n go to infinity and using equation (15) yields that $\frac{\partial \log g(I)}{\partial I} \geq 0$. The proof can be easily adapted to show that the inequality holds strictly whenever u exhibits strictly diminishing absolute risk aversion. ■

Proof of Lemma 19: The property obviously holds for $n=1$. Let us show it holds for $n = 2$. $\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \frac{a'_1}{b'_1} \leq \frac{a'_2}{b'_2}$ implies the four inequalitites $a_k b'_l \leq a'_k b_l$ for $(k,l) \in \{1,2\}^2$. Summing these inequalities and dividing both sides of the resulting inequality by $(b_1+b_2)(b'_1+b'_2)$ yields the result.

We prove by induction the property for $n \geq 2$. Assume it holds for $n-1$, then by applying it to the subsequences $(a_1, b_1, \dots, a_{n-1}, b_{n-1})$ and $(a'_2, b'_1, \dots, a'_n, b'_n)$ yields that

$$\frac{a_1+\cdots+a_{n-1}}{b_1+\cdots+b_{n-1}} \leq \frac{a_n}{b_n} \leq \frac{a'_1}{b'_1} \leq \frac{a'_1+\cdots+a'_n}{b'_1+\cdots+b'_n}.$$

We can again apply the property for $n = 2$ to this last inequality. It yields that

$$\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \frac{a_1' + \cdots + a_n'}{b_1' + \cdots + b_n'}$$

which concludes the proof. ■

Proof of Proposition 6: We have,

$$\frac{\partial \log h(I)}{\partial I} = \frac{u'(I+G) - u'(I-L)}{u(I+G) - u(I-L)} - \frac{u'(I+\Pi) - u'(I)}{u(I+\Pi) - u(I)}.$$

Define

$$d(I) \equiv \frac{u'(I+G) - u'(I-L)}{u(I+G) - u(I-L)} = \frac{A_1 + A_2 + A_3}{B_1 + B_2 + B_3}$$

where

$$\begin{array}{ll} A_1 = u'(I+G) - u'(I+\Pi) & B_1 = u(I+G) - u(I+\Pi) \\ A_2 = u'(I+\Pi) - u'(I) & B_2 = u(I+\Pi) - u(I) \\ A_3 = u'(I) - u'(I-L) & B_3 = u(I) - u(I-L) \end{array}$$

Proving that $\frac{\partial \log h(I)}{\partial I} < 0$ boils down to showing that $\frac{A_1+A_2+A_3}{B_1+B_2+B_3} < \frac{A_2}{B_2}$. We have

$$\text{(16)} \qquad \frac{A_1 + A_2 + A_3}{B_1 + B_2 + B_3} = \frac{A_1}{B_1}\frac{B_1}{B_1 + B_2 + B_3} + \frac{A_2}{B_2}\frac{B_2}{B_1 + B_2 + B_3} + \frac{A_3}{B_3}\frac{B_3}{B_1 + B_2 + B_3}.$$

We know from the proof of Proposition 5, that $\frac{A_1}{B_1} \geq \frac{A_2}{B_2} \geq \frac{A_3}{B_3}$. Since by assumption $L \geq G$, we have $B_3 \geq B_1$. These last two inequalities and equation (16) imply that to prove $\frac{A_1+A_2+A_3}{B_1+B_2+B_3} \leq \frac{A_2}{B_2}$, it is sufficient to show that $\frac{1}{2}\left(\frac{A_1}{B_1} + \frac{A_3}{B_3}\right) < 2\frac{A_2}{B_2}$. We know from the proof of Proposition 5 that

$$\text{(17)} \qquad \frac{A_2}{B_2} \geq \frac{u''(I)}{u'(I)}.$$

Consider the following lemma.

Lemma 20 *For any $n \in \mathbb{N}$, consider a sequence of numbers $\{a_1, b_1, a_2, b_2, \ldots, a_{2n+1}, b_{2n+1}\}$ such that $b_1 \geq b_2 \geq \cdots \geq b_{2n+1} > 0$,*

$$\frac{a_1}{b_1} \leq \cdots \leq \frac{a_{2n+1}}{b_{2n+1}}$$

and for all $i \in \{1, \ldots, k\}$ then, we have that

$$\frac{a_1 + a_2 + \cdots + a_{2n+1}}{b_1 + b_2 + \cdots + b_{2n+1}} \leq \frac{a_n}{b_n}.$$

By considering integrals as the limit of sums as in the proof of Proposition 5, this lemma and the concavity of $\frac{u''}{u'}$ imply that

$$\frac{A_1}{B_1} = \frac{u'(I+G)-u'(I+\Pi)}{u(I+G)-u(I+\Pi)} \leq \frac{u''\left(I+\frac{G-\Pi}{2}\right)}{u'\left(I+\frac{G-\Pi}{2}\right)} \tag{18}$$

$$\frac{A_3}{B_3} = \frac{u'(I)-u'(I-L)}{u(I)-u(I-L)} \leq \frac{u''\left(I-\frac{L}{2}\right)}{u'\left(I-\frac{L}{2}\right)}. \tag{19}$$

Hence, by using inequalities (17), (18), and (19), we obtain that,

$$\frac{A_1}{B_1}+\frac{A_3}{B_3}-2\frac{A_2}{B_2} \leq 2r(I)-r\left(I+(G-\Pi)/2\right)-r\left(I-L/2\right).$$

Since $L \geq G > G-\Pi$, and r is strictly convex, this implies that indeed

$$\frac{A_1}{B_1}+\frac{A_3}{B_3}-2\frac{A_2}{B_2} < 0.$$

This implies that $\frac{\partial \log h(I)}{\partial I} < 0$, and concludes the proof. ■

Proof of Lemma 20: We can write,

$$\frac{a_1+a_2+\cdots+a_{2n+1}}{b_1+b_2+\cdots+b_{2n+1}} = \sum_{i=1}^{n} \frac{b_{n-i}+b_{n+i}}{\sum_{j=1}^{n} b_{n-j}+b_{n+j}} \left[\frac{a_{n-i}}{b_{n-i}}\frac{b_{n-i}}{b_{n-i}+b_{n+i}} + \frac{a_{n+i}}{b_{n+i}}\frac{b_{n-i}}{b_{n-i}+b_{n+i}}\right]. \tag{20}$$

By assumption, we know that $\frac{a_{n-i}}{b_{n-i}} \leq \frac{a_{n+i}}{b_{n+i}}$, and $b_{n-i} \geq b_{n+i} > 0$. This yields that

$$\frac{a_{n-i}}{b_{n-i}}\frac{b_{n-i}}{b_{n-i}+b_{n+i}} + \frac{a_{n+i}}{b_{n+i}}\frac{b_{n-i}}{b_{n-i}+b_{n+i}} \leq \frac{1}{2}\left(\frac{a_{n-i}}{b_{n-i}}+\frac{a_{n+i}}{b_{n+i}}\right).$$

Using the assumption that $\frac{1}{2}\left(\frac{a_{n-i}}{b_{n-i}}+\frac{a_{n+i}}{b_{n+i}}\right) \leq \frac{a_n}{b_n}$ and reinjecting in expression (20) yields that indeed,

$$\frac{a_1+a_2+\cdots+a_{2n+1}}{b_1+b_2+\cdots+b_{2n+1}} \leq \frac{a_n}{b_n}$$

which concludes the proof. ■

Appendix B: Extension to non-stationary games

From a methodological perspective, this paper shows how selection results holding for one-shot global games can be exploited to derive insights on the impact of a global game information structure in dynamic games. Because the key step of the approach is to recognize that actions in dynamic subgame perfect equilibria must be Nash equilibria in a one shot global game with augmented payoffs, there is hope that this methodology can be scaled – at least in part – to study the impact of a global game information structure on a variety of other games. This appendix extends the results of Section 3 to non-stationary exit games.

B.1 The setup

There are two players $i \in \{1,2\}$, time is discrete $t \in \{1,\ldots,\infty\}$, players have discount rate β and there are two actions $A = \{Stay, Exit\}$. In addition, payoffs are indexed by a state of the world $w_t \in \mathbb{R}$, which is independently drawn each period, and by a state variable $k_t \in K \subset \mathbb{R}^d$. We will discuss different processes for k_t. Given the state of the world, player i expects flow payoffs,

	S	E
S	$g^i(w_t,k_t)$	$W^i_{12}(w_t,k_t)$
E	$W^i_{21}(w_t,k_t)$	$W^i_{22}(w_t,k_t)$

where i is the row player. States of the world $\{w_t\}_{t\in\{1,\ldots,\infty\}}$ form an i.i.d. sequence of real numbers drawn from a distribution with density f_w, c.d.f. F and convex support $I \subset \mathbb{R}$. All payoffs, $g^i, W^i_{12}, W^i_{21}, W^i_{22}$ are continuous in w_t and k_t.

The state variable k_t is perfectly observable at the beginning of period t and common knowledge. The state of the world w_t is unknown but players get signals $x_{i,t} = w_t + \sigma\varepsilon_{i,t}$, where $\{\varepsilon_{i,t}\}_{i,t}$ is a sequence of independent random variables lying in the interval $[-1,1]$.

We allow for the possibility of players' final payoffs to be shifted by some idiosyncratic noise $\eta_{i,t}$ independent of everything else. That makes the true state of the world unobservable ex-post, but it is also more realistic and adds no difficulty. Let us denote $r_{i,t} = g(w_{i,t}) + \eta_{i,t}$ the realized payoff obtained when both players stay.

Whenever there is an exit, the game stops and we assume without loss of generality that players get a zero continuation value. Because of exit, any history $h_{i,t}$ is characterized by a sequence of past signals and past outcomes: $h_{i,t} \equiv \{x_{i,1},\ldots,x_{i,t}\ ;\ r_{i,1},\ldots,r_{i,t-1}\ ,\ k_1,\ldots,k_t\}$. Let us denote $\mathcal{H}$ the set of all such sequences. We will denote $V_i(h_{i,t})$ the value of playing the game starting at history $h_{i,t}$ from the perspective of player i. Note that these continuation values are endogenously determined and will depend on players strategies.

Assumption 6 (control) *There are finite bounds on the value of continuation $V_i \in [m_i, M_i]$.*

For instance one could take the max-min and maximum values. Those bounds have to be proven for each particular case. In the partnership game of Section 2, we had $m_i = 0$ and $M_i = \frac{1}{1-\beta}\mathbf{E}\max\{V_E, w_t\}$. The tighter bounds, the easier it will be to show that the selection results of Section 4 apply, however these bounds are mainly needed to insure compactness.

Definition 13 *For any pair of functions $(V_i, V_{-i}) : \mathbb{R}\times\mathbb{R}^d \mapsto \mathbb{R}^2$, we denote $G(V_i, V_{-i}, w_t, k_t)$ the full information one-shot game,*

	S	E
S	$g^i(w_t,k_t) + \beta V_i(w_t,k_t)$	$W^i_{12}(w_t,k_t)$
E	$W^i_{21}(w_t,k_t)$	$W^i_{22}(w_t,k_t)$

We denote $\Psi_\sigma(V_i, V_{-i})$ the corresponding global game in which players get signals $x_{i,t} = w_t + \sigma\varepsilon_{i,t}$.

Assumption 7 (symmetry) *For all states of the world w_t, and capital stock k_t, $G(m_i, m_{-i}, w_t, k_t)$ has a pure strategy equilibrium. All pure strategy equilibria belong to $\{(S,S),(E,E)\}$.*

Note that if Assumption 7 is satisfied, then for any $\mathbf{V}$ taking values in $[m_i, M_i] \times [m_{-i}, M_{-i}]$, the game $G(V_i, V_{-i}, w_t, k_t)$ also has a pure strategy equilibrium, and its pure strategy equilibria also belong to $\{(S,S),(E,E)\}$.

Assumption 8 (increasing differences in the state of the world) *For all $k \in \mathbb{R}^d$ and $i \in \{1,2\}$, $g^i(w_t,k) - W^i_{21}(w_t,k)$ and $W^i_{12}(w_t,k) - W^i_{22}(w_t,k)$ are strictly increasing in w_t with a rate greater than some $r > 0$ independent of k.*

Together, Assumptions 7 and 8 respectively insure that at any state of the world, either (S,S) or (E,E) is a risk-dominant equilibrium and that there is a unique risk-dominant threshold x^{RD} – (S,S) being risk-dominant above this threshold and (E,E) being risk-dominant below. This is the unidimensional version of Carlsson and van Damme's assumption that states of the world should be connected to dominance regions by a path that is entirely contained in the risk-dominance region of either of the equilibria.

Assumption 9 (dominance) *There exists $\underline{w}$ such that for all k, $g^i(\underline{w},k)+\beta M_i - W^i_{21}(\underline{w},k) < 0$ and $\overline{w}$ such that $W^i_{12}(\overline{w},k) - W^i_{22}(\overline{w},k) > 0$*

Definition 14 *For any function $V : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ and $w \in \mathbb{R}$, we define $A_i(V,w,k)$ and $B_i(w,k)$ by,*

$$A_i(V_i,w,k) = g^i(w,k) + \beta V_i(w,k) - W^i_{12}(w,k) \quad \textit{and} \quad B_i(w,k) = W^i_{21}(w,k) - W^i_{22}(w,k)$$

Take as given the strategy s_{-i} of player $-i$ and let V_i be some continuation value for player i. When choosing to stay or exit, player i expects payoffs,

$$\begin{aligned} \Pi^i_S(V_i,k) &= \mathbf{E}\Big[W^i_{12}(w,k) + \{\underbrace{g^i(w,k) + \beta V_i(h_{i,t},w,k) - W^i_{12}(w,k)}_{\equiv A_i(V_i,w,k)}\}\mathbf{1}_{s_{-i}=S}|h_{i,t},s_{-i}\Big] \\ \Pi^i_E(k) &= \mathbf{E}\Big[W^i_{22}(w,k) + \{\underbrace{W^i_{21}(w,k) - W^i_{22}(w,k)}_{\equiv B_i(w,k)}\}\mathbf{1}_{s_{-i}=S}|h_{i,t},s_{-i}\Big] \end{aligned}$$

Player i's best response is to choose S if and only if $\Pi_S > \Pi_E$.

Assumption 10 (staying is good) *For all players $i \in \{1,2\}$, all states of the world w and all capital stocks k, we have, $A_i(m_i,w,k) \geq 0$ and $B_i(w,k) \geq 0$.*

Finally we make a compactness assumptions for technical reasons.

Assumption 11 *All exogenously given payoffs, $g^i(w,k), W^i_{12}(w,k), W^i_{21}(w,k)$ and $W^i_{22}(w,k)$ are Lipschitz in w with a rate r independent of k.*

By the Azrelà-Ascoli theorem, this assumption guarantees that the set of payoff functions indexed by k and mapping w to real numbers is compact. Such compactness is required for global game selection to occur at a speed that is independent of the state variable k_t.

B.2 Markovian state variables

In this section we consider the case where k_t follows a Markov chain over a countable set of states $\mathcal{S} \subset K \subset \mathbb{R}^d$, where K is compact. Denote by $h(\cdot, k_t)$ the distribution of k_{t+1} conditional on k_t. We assume that for all $k \in K$, $h(\cdot, k)$ is bounded and continuous in k with respect to the supremum norm over functions $||\cdot||_\infty$.

B.2.1 General results

Lemma 21 *There exists $\overline{\sigma}$ such that for all $\sigma < \overline{\sigma}$, whenever s_{-i} is a Markovian strategy, then, for **all** strategies s'_{-i},*

$$s'_{-i} \preceq s_{-i} \Rightarrow BR_i(s'_{-i}) \preceq BR_i(s_{-i}) \quad \text{and} \quad s_{-i} \preceq s'_{-i} \Rightarrow BR_i(s_{-i}) \preceq BR_i(s'_{-i})$$

Lemma 22 (extreme strategies) *Under Assumptions 6, 7, 8 and 10, there exists $\overline{\sigma} > 0$ small enough such that for all $\sigma < \overline{\sigma}$, rationalizable strategies of Γ_σ are bounded by extreme Markovian Nash equilibria. Those equilibria take threshold forms : for any state variable k, there exists threshold $\{x_{i,k}\}_{k \in \mathcal{S}}$ such that player i chooses to stay at when the state variable is k if and only if her signal is above $\{x_{i,k}\}$.*

Let us denote by x^H_σ and x^L_σ the thresholds associated with the highest and lowest equilibria of Γ_σ. From Assumption 10 we obtain that x^H_σ and x^L_σ are respectively associated with the highest and lowest possible pairs of rationalizable value functions, V^H_σ and V^L_σ. More precisely, if s_{-i} is a rationalizable strategy, the value function V_i associated with player i's best reply is such that at all histories $h_{i,t}$, $V_i^L(h_{i,t}) < V_i(h_{i,t}) < V_i^H(h_{i,t})$.

Theorem 4 *Under Assumptions 6, 7, 8, 9, 10, and 11 there exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$, there exists a continuous operator $\phi_\sigma(\cdot)$ mapping value functions onto value functions such that,*

(i) $\mathbf{V}^L_\sigma(\cdot)$ *and* $\mathbf{V}^H_\sigma(\cdot)$ *are the lowest and highest fixed points of* $\phi_\sigma(\cdot)$.

(ii) A vector of continuation value functions is supported by a Markovian equilibrium if and only if it is a fixed point of $\phi_\sigma(\cdot)$.

(iii) As σ goes to 0, the family of operators $\phi_\sigma(\cdot)$ converges uniformly over any bounded family of functions to an increasing operator Φ defined by

$$\Phi(V_i, V_{-i})(k_t) = \begin{pmatrix} \mathbf{E}\left[W^i_{22}(w) + (g^i_{11} + \beta V_i(k_{t+1} - W^i_{22}(w)))\, \mathbf{1}_{w > x^{RD}(V_i(k_{t+1}, V_{-i}(k_{t+1}))} \mid k_t\right] \\ \mathbf{E}\left[W^{-i}_{22}(w) + \left(g^{-i}_{11} + \beta V_{-i}(k_{t+1} - V^{-i}_{22}(w))\right) \mathbf{1}_{w > x^{RD}(V_i(k_{t+1}), V_{-i}(k_{t+1}))} \mid k_t\right] \end{pmatrix}$$

Where $x^{RD}(V_i(k_{t+1}), V_{-i}(k_{t+1}), k_t)$ is the risk dominant threshold of the 2×2 global game $\Psi(V_i(k_{t+1}), V_{-i}(k_{t+1}, k_t))$.

Lemma 23 (upper hemicontinuity) *Denote $\mathbf{V}^H$ and $\mathbf{V}^L$ the highest and lowest fixed points of Φ. Consider any family $\{\mathbf{V}_\sigma\}_{\sigma>0}$ of fixed points of $\phi(\sigma, \cdot)$. Then,*

$$\limsup_{\sigma \to 0} \mathbf{V}_\sigma \ll \mathbf{V}^H \quad \text{and} \quad \liminf_{\sigma \to 0} \mathbf{V}_\sigma \gg \mathbf{V}^L.$$

Where the lim sup *and* lim inf *are taken component by component.*

The definition of ξ and ζ for real numbers is extended to mappings $x : K \mapsto \mathbb{R}$.

Definition 15 *For any mapping $x : K \mapsto \mathbb{R}$ we define the mappings ξ and ζ by,*

$$\begin{aligned} \forall k_t \in K, \quad \xi(x)(k_t) &= x^{RD}(BRV_i(x, k_{t+1}), BRV_{-i}(x, k_{t+1}), k_t) \\ \zeta(x)(k_t) &= x^{RD}(NV_i(x, k_{t+1}), NV_{-i}(x, k_{t+1}), k_t). \end{aligned}$$

Definition 16 (non-singular extreme fixed points) *An extreme fixed point x of an increasing mapping $g : \mathbb{R}^{\mathcal{S}} \mapsto \mathbb{R}^{\mathcal{S}}$ is said to be non-singular if and only if,*

1. *It is strongly isolated among fixed points of g in the sense that there exists $\delta > 0$ such that whenever, for all $n \in \mathbb{N}$, $y_n \in \mathbb{B}_{\delta, ||\cdot||_\infty}(x)$ and $\lim_{n\to\infty} ||g(y_n) - y_n||_\infty = 0$, then $\lim_{n\to\infty} ||y_n - x||_\infty = 0$*
2. *For all $\delta > 0$ there exists $\eta \in (0, \delta)$, and $v \in R^{\mathcal{S}}$ such that for all $k \in \mathcal{S}$, $v(k) \in (\eta, \delta)$ and for all $k \in \mathcal{S}$, we have:*

$$g(x+v)(k) < x + v - \eta/2 \quad \text{and} \quad g(x-v)(k) > x - v + \eta/2$$

Lemma 24 (lower hemicontinuity) *Assume the extreme fixed points of ξ, x^H and x^L (by convention $x^H \ll x^L$), are non-singular. Denote by $\mathbf{x}_\sigma^H$ (resp. $\mathbf{x}_\sigma^L$) the threshold function associated to the highest (resp. lowest) equilibrium of Γ_σ. Then, we have,*

$$\lim_{\sigma \to 0} ||\mathbf{x}_\sigma^H - (x^H, x^H)||_\infty = 0 \quad \text{and} \quad \lim_{\sigma \to 0} ||\mathbf{x}_\sigma^L - (x^L, x^L)||_\infty = 0.$$

Theorem 5 (ALSR of extreme equilibria) *Whenever the extreme fixed points x^H and x^L of ξ are non-singular, the strategies (s_{x^H}, s_{x^H}) and (s_{x^L}, s_{x^L}) are ALSR.*

Lemma 25 *Assume that $\mathcal{S}$ is finite and consider a C^1 payoff structure π. Then at an extreme fixed point x of ξ, the greatest eigenvalue λ_{max} of $d\xi$ is weakly less than one.*

Whenever λ_{max} is strictly less than one, then x is non-singular.

Lemma 26 (generic non-singularity) *Whenever $\mathcal{S}$ is finite, there exists a subset P of C^1 payoff structures that is open and dense in Π_1 with respect to $||\cdot||_{\Pi_1}$, and such that for any $\pi \in P$, the extreme fixed points of the associated mapping ξ are non-sigular.*

B.3 Auto-correlation

Because we might want to introduce autocorrelation in the states of the world, we may want to consider state variables following a recurrence equation of the form $k_{t+1} = f(k_t, w_t)$, where f is a deterministic function. In this formulation, w_t is the innovation and k_t is an observable sufficient statistic for past innovations.

Compared with the previous section, the main difficulty comes from the fact that because next period's capital stock depends on the realization of the state of the world, the continuation values at time t will now depend on the state of the world w_t. Because the uniform global games selection results we use require that all payoff functions be continuous and share a common modulus of continuity (see Chassang (2006) for more details), we must prove that equilibrium value functions indexed by the current capital stock are equicontinuous in w_t. More precisely we will to show they are increasing in w_t and Lipschitz with a rate independent of k. Let us denote by $\Psi_\sigma(V_i, V_{-i}, k_t)$ the global game,

	S	E
S	$g_i(w_t, k_t) + \beta V_i(k_{t+1})$	$W_{12}^i(w_t, k_t)$
E	$W_{21}^i(w_t, k_t)$	$W_{22}^i(w_t, k_t)$

where i is the row player. In addition to the assumptions of Section B.1, we need to make a few more technical assumptions. These assumptions make the statement of theorems somewhat tedious, but they are fairly general so that our selection result will in fact be easily applicable.

Assumption 12 (increasing differences in capital stock) *For all $w_t \in \mathbb{R}$ and $i \in \{1,2\}$, $g^i(w_t, k) - W_{21}^i(w_t, k)$ and $W_{12}^i(w_t, k) - W_{22}^i(w_t, k)$ are increasing in k.*

Assumption 13 (capital is good) *For $i \in \{1,2\}$, and $w \in \mathbb{R}$ we assume that g^i, V_{12}^i, V_{21}^i and V_{22}^i are weakly increasing in k.*

Definition 17 (iterated capital stock) *For all $n \in \mathbb{N}$ and $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$, we define by induction the iterated capital stock $f_n(k, \mathbf{w})$ as follows,*

$$f_1(k, w_1) = f(k, w_1) \quad \textit{and} \quad f_n(k, \mathbf{w}) = f\left(f_{n-1}\left(k, (w_1, \ldots, w_{n-1})\right), w_n\right)$$

In other words, $f_n(k_t, \mathbf{w}) = k_{t+n} \mid \mathbf{w}$.

Assumption 14 *We make four assumptions on the process of k_t and how it affects payoffs.*

1. *f is increasing in both arguments.*
2. *there exists $H \in \mathbb{R}$ such that*

$$\left|\frac{\frac{\partial f}{\partial k}}{\frac{\partial f}{\partial w}}\right| < H \, , \quad \left|\frac{\frac{\partial (g^i - W_{21}^i)}{\partial k}}{\frac{\partial (g^i - W_{21}^i)}{\partial w}}\right| < H \quad \textit{and} \quad \left|\frac{\frac{\partial (W_{12}^i - W_{22}^i)}{\partial k}}{\frac{\partial (W_{12}^i - W_{22}^i)}{\partial w}}\right| < H$$

3. There exists $n* \in \mathbb{N}$ and $\mu < \frac{1}{\beta}$ such that,

$$\forall k \in \mathbb{R}^d, \quad \mathbf{E}_w \left[\frac{\partial f_{n*}}{\partial k} \right] < \mu^{n*}$$

4.

$$\forall n \in \mathbb{N}, k \in \mathbb{R}^d, \quad \mathbf{E}_w \left[\frac{\partial f_n}{\partial k} \right] < +\infty$$

Lemma 27 (joint selection) *Define $\mathcal{V}_R$ the set of functions mapping $\mathbb{R}^d$ into $\mathbb{R}$ that are weakly increasing and Lipschitz continuous with rate R. For all R, there exists $\overline{\sigma} > 0$ such that for all $\sigma < \overline{\sigma}$, for any $\mathbf{V} = (V_i, V_{-i}) \in \mathcal{V}_R^2$, the global game $\Psi_\sigma(V_i, V_{-i}, k_t)$ has a unique pair of rationalizable strategies $(x_i(\mathbf{V}, k_t), x_{-i}(\mathbf{V}, k_t))$.*

Proof: This result is a direct application of Theorem 2 of Chassang (2005). ■

Given a class $\mathcal{V}_R$, for σ small enough, Lemma 27 allows us to define a mapping $\phi(\mathbf{V}, \sigma)$ which maps any vector of value functions from $\mathcal{V}_R$ into the value of playing game $\Psi_\sigma(V_i, V_{-i}, \cdot)$ for each player. Function ϕ maps value functions into value functions. In order to combine the Abreu, Pearce and Stacchetti (1990) approach and a global games selection argument, we need to find a class $\mathcal{V}_R$ stable by ϕ. Lemmas 28 and 29 show that $\phi_\sigma(\cdot)$ maps $\mathcal{V}_R$ into $\mathcal{V}_{R^\#}$ for some $R^\#$ independent of σ. Finally Lemma 30 shows that for some R big enough, all iterated images of $\mathcal{V}_R$ by $\phi_\sigma(\cdot)$ are subsets of some fixed $\mathcal{V}_{R^\#}$ with $R^\#$ independent of σ.

Lemma 28 (Lipschitz continuity in k of the selected equilibrium) *Pick $\mathbf{V} \in \mathcal{V}_R^2$ and $\overline{\sigma}$ such that Lemma 27 applies. Then there exists $\rho > 0$, independent of R, such that for all $\sigma < \overline{\sigma}$, the uniquely selected equilibrium of $\Psi_\sigma(\mathbf{V})$, $(x_i(\mathbf{V}, k), x_{-i}(\mathbf{V}, k))$ is Lipschitz in k with rate ρ.*

Lemma 29 (stability of Lipschitz continuity) *If $\mathbf{V}$ belongs to $\mathcal{V}_R$ for some $R > 0$, then $\mathbf{V}^\#$, defined by $\mathbf{V}^\#(k) = \phi(\mathbf{V}, k, \sigma)$, belongs to $\mathcal{V}_{R^\#}$ for some $R^\# > 0$.*

Lemma 29 allows us to define iterations of $\phi(\cdot, \sigma)$ as follows:

Definition 18 *Pick $n \in \mathbb{N}$. For $\mathbf{V}$ and σ small enough, we define by induction $\phi_n(\mathbf{V}, k, \sigma)$ by,*

$$\phi_1(\mathbf{V}, k, \sigma) = \phi(\mathbf{V}, k, \sigma) \quad \textit{and} \quad \phi_n(\mathbf{V}, k_t, \sigma) = \phi(\mathbf{E}\phi_{n-1}(\mathbf{V}, k_{t+1}, \sigma), k_t, \sigma)$$

Using Lemmas 27 and 29 we know that for any $R > 0$ and $n \in \mathbb{N}$ there exists $\overline{\sigma}_{n,R}$ such that for all $\sigma < \overline{\sigma}_{n,R}$, $\phi_n(\mathbf{V}, k, \sigma)$ is well defined for all $\mathbf{V} \in \mathcal{V}_R^2$ and all $k \in \mathbb{R}^d$.

Lemma 30 (stable Lipschitz class) *Take the integer $n*$ defined in Assumption 14. There exists $\overline{\sigma} > 0$ and R such that for all $\sigma < \overline{\sigma}$, $\phi_{n*}(\mathcal{V}_R^2, \sigma) \subset \mathcal{V}_R^2$*

Moreover, for all $n \in \mathbb{N}$ and $\sigma < \overline{\sigma}$, $\phi_n(\mathbf{V}, k, \sigma)$ is well defined for $\mathbf{V} \in \mathcal{V}_R$ and that there exists $R^\# > 0$ such that for all $n \in \mathbb{N}$, $\phi_n(\mathcal{V}_R^2, k, \sigma) \subset \mathcal{V}_{R^\#}^2$

The last tricky step is to prove that to obtain tight bounds on the set of rationalizable strategies, it is enough to study strategies corresponding to value functions in $\mathcal{V}_R$. To prove this, we prove that the set of rationalizable strategies is bounded by extreme Markovian Nash equilibria that can be obtained by iteratively applying the mapping $\phi_\sigma(\cdot)$ to vectors of constant value functions.

Lemma 31 *Pick $\overline{\sigma}$ and R such that Lemma 30 holds. Denote $\mathcal{V} = \mathcal{V}_i \times \mathcal{V}_{-i}$ the set of rationalizable continuation values of Γ_σ. Pick any vectors of value functions $\mathbf{V}^L$ and $\mathbf{V}^H$ in $\mathcal{V}_R$, then for all $\sigma < \overline{\sigma}$,*

$$\{\mathcal{V} \subset [\mathbf{V}^L, \mathbf{V}^H]\} \Rightarrow \{\mathcal{V} \subset [\phi(\mathbf{V}^L, \sigma), \phi(\mathbf{V}^H, \sigma)]\}$$

Theorem 6 *Under Assumptions 6, 7, 8, 9, 10, 11, 12, 13 and 14, there exists $\overline{\sigma} > 0$ such that for all $\sigma < \overline{\sigma}$, the rationalizable strategies of game Γ_σ are bounded by extreme Nash equilibria associated with extreme value functions $\mathbf{V}_\sigma^H(\cdot)$ and $\mathbf{V}_\sigma^L(\cdot)$. Moreover, there exists a continuously increasing operator $\phi_\sigma(\cdot)$ mapping value functions into value functions, such that,*

1. *$\mathbf{V}_\sigma^L(\cdot)$ and $\mathbf{V}_\sigma^H(\cdot)$ are the lowest and highest fixed points of $\phi_\sigma(\cdot)$.*

2. *As σ goes to 0, the family of operators $\phi_\sigma(\cdot)$ converges uniformly over any class $\mathcal{V}_R$ to a function Φ defined by*

$$\Phi(V_i, V_{-i})(k_t) = \begin{pmatrix} \mathbf{E}\left[W_{22}^i(w) + (g_{11}^i + \beta V_i(k_{t+1}) - W_{22}^i(w))\, \mathbf{1}_{w > x^{RD}(V_i(f(k,w)), V_{-i}(f(k,w)))}\right] \\ \mathbf{E}\left[W_{22}^{-i}(w) + \left(g_{11}^{-i} + \beta V_{-i}(k_{t+1}) - W_{22}^{-i}(w)\right) \mathbf{1}_{w > x^{RD}(V_i(k_{t+1}), V_{-i}(k_{t+1}))}\right] \end{pmatrix}$$

Where $x^{RD}\left(V_i(k_{t+1}), V_{-i}(k_{t+1})\right)$ is the risk-dominant threshold of the 2×2 global game $\Psi\left(V_i(k_{t+1}), V_{-i}(k_{t+1})\right)$.

Whenever the fixed points of Φ are isolated with respect to the uniform norm, then as σ goes to 0, uniform convergence of $\phi_\sigma(\cdot)$ implies that $\mathbf{V}_\sigma^H(\cdot)$ and $\mathbf{V}_\sigma^L(\cdot)$ converge to the highest and lowest fixed points of Φ with respect to the uniform norm.

B.4 Proofs for Appendix B

Proof of Lemma 21: Consider s_{-i} a Markovian strategy and s'_{-i} such that $s'_{-i} \preceq s_{-i}$. Define V_i and V'_i the continuation value functions respectively associated to player i's best response to s_{-i} and s'_{-i}. Assumption 5, that "staying is good", implies that at all histories $h_{i,t}$, $V'_i(h_{i,t}) < V_i(h_{i,t})$. At any history $h_{i,t}$, the best-reply action profiles of player i are $BR_i(a_{-i}, V_i(h_{i,t}), \sigma)$ and $BR_i(a'_{-i}, V'_i(h_{i,t}), \sigma)$. From Lemma 4, we have that

$$BR_i(a'_{-i}, V'_i(h_{i,t}), \sigma) \preceq BR_i(a'_{-i}, V_i(h_{i,t}), \sigma) \tag{21}$$

Since s_{-i} is Markovian, $V_i(h_{i,t})$ is constant. Thus Lemma 3 implies that

$$BR_i(a'_{-i}, V_i(h_{i,t}), \sigma) \preceq BR_i(a_{-i}, V_i(h_{i,t}), \sigma) \tag{22}$$

Combining equations (21) and (22) we obtain that indeed, $BR_i(s'_{-i}) \preceq BR_i(s_{-i})$. An identical proof holds for the other inequality. ∎

Proof of Lemma 22: This a corollary of Lemma 21. The proof, again drawing on the methodology of Milgrom and Roberts (1990) and Vives (1990) is identical to that of Theorem 1. ∎

Proof of Theorem 4: The proof is almost identical to that of Theorem 2. Selection in one-shot global games is applied to the augmented game associated with each capital stock. See the proof of Theorem 2 for more details. ∎

Proof of Lemma 23: There exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ going to 0, such that $\{\mathbf{V}_{\sigma_n}\}_{n\in\mathbb{N}}$ converges weakly to $\limsup_{\sigma\to 0}\mathbf{V}_\sigma \equiv \mathbf{V}^*$. Let us show that V^* is a fixed point of Φ. Indeed for every k_t, $\mathbf{V}_\sigma$ satisfies $\mathbf{V}_\sigma(k_t) = \phi(k_t, \mathbf{V}_\sigma(k_{t+1}), \sigma)$. Since $\phi_\sigma(\cdot)$ converges to Φ, and Φ is continuous in $\mathbf{V}$, the equation must hold at the limit. This implies that indeed $\mathbf{V}^*$ is a fixed point of Φ which proves the right side of the inequality. An symmetric proof gives the left side. ∎

Proof of Lemma 24: This is a direct implication of Theorem 5. Indeed a ball centered on a threshold form strategy and of radius ρ with respect to the topology on strategies corresponds to a ball centered on the threshold x and radius ρ with respect to the supremum distance. ∎

Proof of Theorem 5: The proof is very similar to that of Proposition 3 and Theorem 3. Let x denote an extreme fixed point of ξ. By assumption, for any $\delta > 0$, there exists $\eta > 0$ and $x'' \in \mathcal{B}_\delta(x)$ such that for all $k \in \mathcal{S}$, $x''(k) > x(k)+2\eta$ and $\xi(x'')(k) \ll x''(k)-\eta''$. This implies there exists $x' \in [\xi(x''), x'']$ such that for all $k \in \mathcal{S}$, $\xi(x'')(k) + \eta/2 < \xi(x')(k) < x''(k) - \eta/2$. Let us now show that for σ small enough, for all $k \in \mathcal{S}$, $BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'})(k) \prec s_{x'}(k)$. We have,

$$BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'})(k) = BR_{i,\sigma}(\cdot, BRV_{i,\sigma}(BR_{-i,\sigma}(x'), k), k) \circ BR_{i,\sigma}(\cdot, BRV_{-i,\sigma}(x'), k)(x').$$

For η small enough, we know that $|BR_{-i,\sigma}(x')(k) - x'(k)| < 2\sigma$. Hence, for σ small enough, $\forall k \in \mathcal{S}$, $BR_{-i,\sigma}(x')(k) \prec x''(k)$. This implies that,

$$BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'})(k) \prec BR_{i,\sigma}(\cdot, BRV_{i,\sigma}(x'', k), k) \circ BR_{i,\sigma}(\cdot, BRV_{-i,\sigma}(x'', k), k)(x').$$

From Theorem 3 of Chassang (2006), we know that $x^*_\sigma(BRV_{i,\sigma}(x'', k), BRV_{-i,\sigma}(x'', k), k)$ converges uniformly to $\xi(x'')(k)$, hence for σ small enough,

$$x^*_\sigma(BRV_{i,\sigma}(x'', k), BRV_{-i,\sigma}(x'', k), k) > x'(k). \tag{23}$$

From Theorem 2 of Chassang (2006), we know that for σ small enough and for all $k \in \mathcal{S}$, the sequence $[BR_{i,\sigma}(\cdot, BRV_{i,\sigma}(x'', k), k) \circ BR_{i,\sigma}(\cdot, BRV_{-i,\sigma}(x'', k), k)]^p\,(x')(k)$ converges monotonously to $x^*_\sigma(BRV_{i,\sigma}(x'', k), BRV_{-i,\sigma}(x'', k), k)$ as p goes to infinity. Using 23, this implies that

$$\forall k \in \mathcal{S}, \quad BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'})(k) \prec s_{x'}(k). \tag{24}$$

Since $BR_{i,\sigma} \circ BR_{-i,\sigma}$ is monotonous, this implies that as q goes to infinity, the sequence $\{BR_{i,\sigma} \circ BR_{-i,\sigma}\}^q(s_{x'})$ converges weakly to a strategy of threshold $x^*_{i,\sigma}$ such that $x^*_{i,\sigma} \in \mathcal{B}_{\delta,||\cdot||_\infty}(x)$. Note that $(x^*_{i,\sigma}, BR_{-i,\sigma}(x^*_{i,\sigma}))$ is a fixed point of ξ_σ. Let us now show that $x^*_{i,\sigma}$ converges to x as σ goes to 0.

We know that ξ_σ converges uniformly to (ξ, ξ). Indeed, $BRV_{i,\sigma}(y, k)$ converges uniformly over $(y, k) \in \mathbb{R}^{\mathcal{S}} \times \mathcal{S}$ to $BRV(y, k)$ as σ goes to 0, and Theorem 4 of Chassang (2006) implies that $x^*_\sigma(V_i, V_{-i}, k)$ is Lipschitz-continuous in $\mathbf{V}$ with a rate independent of k. This implies that for any $\nu > 0$, there exists $\overline{\sigma}$ such that for all $\sigma \in (0, \overline{\sigma})$,

$$||\xi(x^*_\sigma) - x^*_\sigma||_\infty \leq ||\xi(x^*_\sigma) - \xi_\sigma(x^*_\sigma) + \xi_\sigma(x^*_\sigma) - x^*_\sigma||_\infty \leq ||\xi(x^*_\sigma) - \xi_\sigma(x^*_\sigma)||_\infty \leq \nu.$$

Since x is an isolated fixed point of x, this implies that $\lim_{\sigma \to 0} ||x^*_\sigma - x||_\infty = 0$.

To prove ALSR, note that one can construct $\tilde{x}$ with the same properties as x', but strictly below x rather than strictly above. Then $[s_{x'}, s_{\tilde{x}}]$ is a neighborhood of s_x for which

$$\lim_{\sigma \to 0} \lim_{n \to \infty} [BR_{i,\sigma} \circ BR_{-i,\sigma}]^n([s_{x'}, s_{\tilde{x}}]) = s_x.$$

This shows that indeed, (s_x, s_x) is ALSR. ■

Proof of Lemma 25: ξ and ζ coincide around their fixed points. Since $\mathcal{S}$ is finite, ζ is really a mapping from $\mathbb{R}^p$ to $\mathbb{R}^p$ with $p \in \mathbb{N}$. Whenever $\pi \in \Pi_1$, ζ will be differentiable. Because ξ and ζ coincide around their fixed points, this implies that ξ is differentiable around its fixed points. Denote by x the highest fixed point of ξ. Since ξ is strictly increasing around its fixed points, the Perron-Frobenius theorem applies to $d\xi_{|x}$. It states that $d\xi_{|x}$ admits a greatest eigenvalue $\lambda_{max} > 0$ associated with an eigenvector v with strictly positive components. Assume that $\lambda_{max} > 1$. Then, for some $\delta > 0$ small enough, we will get that $\xi(x + \delta v) \gg x + \delta v$. This implies that ξ admits a fixed point $x' \gg x + \delta v$, which contradicts the fact that x is the highest fixed point of ξ. Hence, it must be that $\lambda_{max} \leq 1$.

For the second part of the lemma, $\lambda_{max} < 1$ implies that, there exists $\delta > 0$ such that for all $\eta \in (0, \delta)$, $\xi(x+\eta v) \ll x+\eta v - \frac{1-\lambda_{max}}{2}\eta v$ and $\xi(x-\eta v) \gg x-\eta v+\frac{1-\lambda_{max}}{2}\eta v$. Now consider any y such that $y \in [x - \delta v, x + \delta v]$. There exists $\eta \in (-\delta, \delta)$ such that y is weakly less than $x + \eta v$, and y and $x + \eta v$ share one strictly positive coordinate $y(k)$. Assume without loss of generality that $\eta > 0$. Since $\xi(x + \eta v) \ll x + \eta v - \frac{1-\lambda_{max}}{2}\eta v$, and ξ is increasing, this implies that $\xi(y)(k) < y(k)$. Hence, y cannot be a fixed point of ξ, proving that x is isolated. This concludes the proof. ■

Proof of Lemma 26: Let us consider the set off payoff functions such that at its extreme fixed points, ζ admits an eigenvalue $\rho \in (0, 1)$ associated with an eigenvector with strictly positive coordinates. Note that since ζ is increasing around its fixed points, this property clearly implies that ζ is contracting around its extreme fixed points. Denote $\mathcal{P}$ the set of payoff structures satisfying this property.

Denote x the highest fixed point of ζ. From Lemma 25, we already know that the Jacobian of ζ at x admits a largest eigenvalue $\lambda_{max} \in (0, 1]$, associated with an eigenvector v whose components are strictly positive.

Let us show that $\mathcal{P}$ is open. Pick $\pi \in \mathcal{P}$. At x^*, $\det(\mathrm{d}\zeta(x^*) - Id)$ is strictly different from 0. By continuity of $\mathrm{d}\zeta$, this implies that there exists a ball of center x^* and radius $\eta > 0$ and $\nu > 0$ such that for all $x \in \mathcal{B}_\eta(x^*)$, $|\det(\mathrm{d}\zeta - Id)| > \nu$. There exists $\mu > 0$ such that for all payoff structures $\tilde{\pi}$ within distance μ of π, the extreme fixed points of $\tilde{\zeta}$ are within distance η of x^* and $|\det(\mathrm{d}\zeta - Id)| > \nu/2$ over $\mathcal{B}_\eta(x^*)$. We already know that the greatest eigenvalue of $\tilde{\zeta}$ is weakly less than 1. This implies it is strictly less than one and proves that $\mathcal{P}$ is open.

We now show that $\mathcal{P}$ is dense. Since the intersection of dense open sets is dense and open, we proceed separately for the highest and lowest fixed points. The set of C^2 payoff structures in Π_1 that strictly satisfy Assumption 10 is dense in Π_1. Pick such a payoff structure π, and denote by x the highest fixed point of the associated mapping ζ. For any vector $u \in \mathbb{R}^{\mathcal{S}}$, consider the payoff structure π^u defined by,

$$\begin{aligned}
\forall (w,k) \in \mathbb{R} \times K, \quad & \tilde{g}^{i,u}(w,k) \equiv g^i(w,k) \\
& W_{22}^{i,u}(w,k) = W_{22}^i(w,k) \\
& W_{12}^{i,u}(w,k) = W_{12}^i(w - u(k), k) + g^i(w,k) - g^i(w - u(k), k) \\
& W_{21}^{i,u}(w) = W_{21}^i(w - u(k), k) + W_{22}^i(w,k) - W_{22}^i(w - u(k), k)
\end{aligned}$$

There exists $\delta > 0$ such that for all u satisfying $||u||_\infty < \delta$, π^u satisfies Assumptions 6, 7, 8, 9, and 10. Moreover, for $||u||_\infty$ small enough, π^u is arbitrarily close to π in the sense of $||\cdot||_{\Pi_1}$. Also, for any u, $\zeta^u = \zeta + u$. Assume that there exists $\eta > 0$ such that for all u satisfying $||u||_\infty < \eta$, at the highest fixed point of ζ^u, $\mathrm{d}\zeta$ has a largest eigenvalue equal to 1. We will show it leads to a contradiction. More precisely, let us show that if this is true, then x cannot be the highest fixed point of ζ. First note that ζ is C^2 and that Assumption 2 implies potential fixed points of ζ^u belong to a compact L. Hence, there exists $H > 0$ such that for all $u \in \mathbb{R}^{\mathcal{S}}$, and $y \in L$, $| < u, \mathrm{d}^2\zeta u > | \leq H < u, u >$. For all $n \in \mathbb{N}$, define the sequences $\{x_m^n\}_{m \in \{0,\dots,n\}}$ and $\{u_m^n\}_{m \in \{0,\dots,n\}}$ such that for all m, x_m^n is the highest fixed point of $\zeta^{u_m^n}$ and $||u_m^n||_\infty < \eta$, as follows:

1. $x_0^n \equiv x$ and $u_0^n = 0$
2. For all $m \in \{0, \dots, n-1\}$, by assumption, at x_m^n, $\mathrm{d}\zeta$ has a largest eigenvalue equal to 1. By the Perron-Frobenius theorem, this largest eigenvalue is associated to an eigenvector v with strictly positive coordinates. Pick a representant such that $||v||_\infty = 1$ and define $u_{m+1}^n = \zeta^{u_m^n}(x_m^n + \eta v/n) - x_m^n - \eta v/n + u_m^n$.
3. Define x_{m+1}^n as the highest fixed point of $\zeta^{u_{m+1}^n}$.

First note that $x_m^n + \eta v/n$ is a fixed point of $\zeta^{u_{m+1}^n}$, hence, $x_{m+1}^n \gg x_m^n + \eta v/n$. Second, since v was picked as an eigenvector associated to 1 at each stage, we have that $||u_{m+1}^n - u_m^n||_\infty =$

$||\zeta^{u_m^n}(x_m^n + \eta v/n) - x_m^n - \eta v/n||_\infty \leq H\eta^2/n^2$. Hence we obtain,

$$\begin{aligned}
\zeta(x_n^n) - \zeta(x) &= \sum_{m\in\{0,\ldots,n-1\}} \zeta(x_{m+1}^n) - \zeta(x_m^n) \\
&= \sum_{m\in\{0,\ldots,n-1\}} \zeta^{u_{m+1}^n}(x_{m+1}^n) - \zeta^{u_m^n}(x_m^n) + \sum_{m\in\{0,\ldots,n-1\}} u_m^n - u_{m+1}^n \\
&= x_n^n - x + \sum_{m\in\{0,\ldots,n-1\}} u_m^n - u_{m+1}^n
\end{aligned}$$

which yields

$$||\zeta(x_n^n) - x_n^n||_\infty \leq ||\sum_{m\in\{0,\ldots,n-1\}} u_m^n - u_{m+1}^n||_\infty \leq H\eta^2/n.$$

Consider $e \in \mathbb{R}^S$ the vector whose components are all equal to 1. By construction, $x_n^n \in [x, x+\eta e] \setminus [x, x+\frac{\eta e}{|S|}]$. Extract a converging sequence from $\{x_n^n\}_{n\in\mathbb{N}}$. Its limit x' is such that $x' \gg x + \frac{\eta e}{2|S|}$ and satisfies $\zeta(x') = x'$. This contradicts the fact that x is the greatest fixed point of ζ. Hence for any $\delta > 0$, there exists u satisfying $||u||_\infty < \delta$ such that at the highest fixed point of ζ^u, $d\zeta^u$ has a largest eigenvalue strictly less than 1. ■

Proof of Lemma 28: This is a direct application of Theorem 5 from Chassang (2006). We refer to that paper for details. This theorem holds under conditions which, in this particular case, boil down to showing there exists a constant C such that for all k, w, k', w',

$$\frac{\Delta_{k,k'}V_i(f(k,w))}{\Delta_{w,w'}V_i(f(k,w))} < C$$

Where for any function u, $\Delta_{s,s'}u(s) \equiv \frac{||u(s')-u(s)||}{||s'-s||}$.

Assumption 14 was specifically introduced to prove this inequality. Assume temporarily that V_i is differentiable. Then using the fact that $(V \circ f)' = V' \circ f \times f'$, we get that

$$\left|\frac{\frac{\partial V_i(f(k,w))}{\partial k}}{\frac{\partial V_i(f(k,w))}{\partial w}}\right| < H \tag{25}$$

Noting that we have $\frac{\partial V_i(f(k,w))}{\partial w} > 0$ and using the inequality

$$\forall a,b,c,d > 0, \quad \frac{a}{b} < m \ , \ \frac{c}{d} < m \Rightarrow \frac{a+c}{b+d} < m$$

we get by integration of the numerator and denominator of equation (25) that,

$$\frac{\Delta_{k,k'}V_i(f(k,w))}{\Delta_{w,w'}V_i(f(k,w))} < H$$

This result does not depend on the smoothness of V_i. Thus using the density of differentiable functions, we know it holds for any V_i in $\mathcal{V}_R$. Thus Theorem 5 of Chassang (2005) applies. The uniquely selected equilibrium $(x_i(\mathbf{V},k), x_{-i}(\mathbf{V},k))$ is Lipschitz in k with a rate ρ independent of R. ■

Proof of Lemma 29: $\mathbf{V}^{\#}(k) = (V_i^{\#}(k), V_{-i}^{\#}(k))$. Assume – temporarily – that $V_i^{\#}$ and V_i are differentiable, denoting x_i and x_{-i} the equilibrium strategies, we have

$$\begin{aligned}
\frac{\partial V_i^{\#}}{\partial k} &= \mathbf{E}\left[\left(\frac{\partial g_{11}^i}{\partial k}+\beta\frac{\partial V_i}{\partial k}\frac{\partial f}{\partial k}\right)\mathbf{1}_{s_i>x_i}\mathbf{1}_{s_{-i}>x_{-i}}+\frac{\partial W_{12}^i}{\partial k}\mathbf{1}_{s_i>x_i}\mathbf{1}_{s_{-i}<x_{-i}}+\frac{\partial W_{21}^i}{\partial k}\mathbf{1}_{s_i<x_i}\mathbf{1}_{s_{-i}>x_{-i}}+\frac{\partial W_{22}^i}{\partial k}\mathbf{1}_{s_i<x_i}\mathbf{1}_{s_{-i}<x_{-i}}\right] \\
&+ \mathbf{E}\left[\frac{\partial x_i}{\partial k} f_{s_i}(x_i)\left((W_{22}^i - W_{12}^i)\mathbf{1}_{s_{-i}>x_{-i}} + (W_{21}^i - g_{11}^i - \beta W_i)\mathbf{1}_{s_{-i}>x_{-i}}\right)\right] \\
&+ \mathbf{E}\left[\frac{\partial x_{-i}}{\partial k} f_{s_{-i}}(x_{-i})\left((W_{22}^i - W_{12}^i)\mathbf{1}_{s_i>x_i} + (W_{21}^i - g_{11}^i - \beta V_i)\mathbf{1}_{s_i>x_i}\right)\right]
\end{aligned}$$

Using Assumption 11, Lemma 28 and the fact that $V_i \in \mathcal{V}_R$, we conclude there exist absolute constants C_1, C_2 such that,

$$\left|\frac{\partial V_i^{\#}}{\partial k}\right| \leq C_1 + C_2 R$$

This inequality doesn't depend on the smoothness of either V_i or $V_i^{\#}$. Using the density of smooth functions we conclude it holds generally.

Finally, note that $V_i^{\#}$ is increasing in k. This results directly from Assumptions 12 and 13. Increasing k increases cooperation directly because of Assumption 12, and indirectly because Assumption 13 implies that more capital increases continuation values. ■

Proof of Lemma 30: We know that weak monotonicity is maintained, the difficulty is to show that Lipschitz continuity is maintained with a stable rate. Pick $\mathbf{V} \in \mathcal{V}_R$. We can express ϕ_{n*} explicitly.

$$\begin{aligned}
\phi_{n*}^i(\mathbf{V},k,\sigma) &= \mathbf{E}\Bigg[\sum_{t=1}^{n*-1}\beta^t\prod_{q=1}^{t-1}\mathbf{1}_{s_{i,q}>x_{i,q}}\mathbf{1}_{s_{-i,q}>x_{-i,q}}\big(g^i\mathbf{1}_{s_{i,t}>x_{i,t}}\mathbf{1}_{s_{-i,t}>x_{-i,t}} \\
&+ W_{12}^i\mathbf{1}_{s_{i,q}>x_{i,q}}\mathbf{1}_{s_{-i,q}<x_{-i,q}} + W_{21}^i\mathbf{1}_{s_{i,q}<x_{i,q}}\mathbf{1}_{s_{-i,q}>x_{-i,q}} + W_{12}^i\mathbf{1}_{s_{i,q}<x_{i,q}}\mathbf{1}_{s_{-i,q}<x_{-i,q}}\big)\Bigg] \qquad (26) \\
&+ \mathbf{E}\left[\beta^{n*}\prod_{q=1}^{n*}\mathbf{1}_{s_{i,q}>x_{i,q}}\mathbf{1}_{s_{-i,q}>x_{-i,q}}V(f_{n*}(k,\mathbf{w}))\right]
\end{aligned}$$

Assume temporarily, that all functions involved are differentiable with respect to k. Using Assumptions 11, 14, Lemma 28, and the fact that $\mathbf{V} \in \mathcal{V}_R^2$, equation (26) yields after some manipulation an inequality of the form,

$$\left|\frac{\partial \phi_{n*}}{\partial k}\right| \leq C + \beta^{n*}\mathbf{E}\left[\frac{\partial f_{n*}}{\partial k}\right]R \leq C + (\beta\mu)^{n*}R$$

Therefore, if we pick $R \geq \frac{C}{1-(\beta\mu)^{n*}}$, then $\mathcal{V}_R$ is stable via ϕ_{n*}. The second part of the lemma follows directly using this result and Lemma 29. ■

Proof of Lemma 31: Consider a maximal rationalizable action profile $a^i_{m,1}$. It is a best response to some action profile a^{-i} and some rationalizable continuation value V_i. This implies that $a^i_{m,1} \preceq BR_i(a^{-i}, V_i^H)$. Moreover, since $V_i^H \in \mathcal{V}_R$, we know from Lemma 3 of Chassang (2005) that $BR_i(\cdot, V_i^H)$ is monotone in strategies. Thus there exists a maximal rationalizable action $a^{-i}_{m,1}$ such that $a^i_{m,1} \preceq BR_i(a^{-i}_{m,1}, V_i^H)$. For $i \in \{1,2\}$, we define $\overline{BR}_i(\cdot) \equiv BR_i(\cdot, V_i^H)$. By iterating the former reasoning, we get a sequence of maximal actions $\{a^i_{m,q}\}_{q\in\mathbb{N}}$, such that $a^i_{m,1} \preceq (\overline{BR}_i \circ \overline{BR}_{-i})^q(a^i_{m,q})$. Taking q to infinity, this implies that rationalizable actions are smaller than the unique rationalizable strategy of $\Psi_\sigma(\mathbf{V}^H)$. Because of Assumption 10, this also implies that the value associated with any rationalizable action is less than the value of playing the unique equilibrium of $\Psi_\sigma(\mathbf{V}^H)$. This shows that $\mathcal{V} \preceq \phi(\mathbf{V}^H, \sigma)$. An identical proof holds for the lower bound. ■

Proof of Theorem 6: To prove the existence of extreme equilibria, we use Lemma 31 iteratively. Pick R and $\overline{\sigma}$ such that Lemma 30 holds. Denote $\mathcal{V}$ the set of rationalizable value functions. Begin by setting $\mathbf{V}^H_{\sigma,0} = (M_i, M_{-i})$ and $\mathbf{V}^L_{\sigma,0} = (m_i, m_{-i})$. We must have $\mathcal{V} \subset [\mathbf{V}^L_{\sigma,0}, \mathbf{V}^H_{\sigma,0}]$. Since $\mathbf{V}^H_{\sigma,0}$ and $\mathbf{V}^L_{\sigma,0}$ belong to $\mathcal{V}_R^2$, Lemma 31 implies that

$$\mathcal{V} \subset [\phi(\mathbf{V}^L_{\sigma,0}, \sigma), \phi(\mathbf{V}^H_{\sigma,0}, \sigma)] \subset [\mathbf{V}^L_{\sigma,0}, \mathbf{V}^H_{\sigma,0}]$$

From Lemma 30, we know that all functions $\phi_n(\mathbf{V}^L_{\sigma,0}, \sigma)$ and $\phi_n(\mathbf{V}^H_{\sigma,0}, \sigma)$ are Lipschitz with rate $R^{\#}$ so that we can keep applying $\phi_\sigma(\cdot)$ iteratively. Using Lemma 31 and the monotonicity of $\phi(\cdot, \sigma)$ at each step, we get, that for all $q \in \mathbb{N}$,

$$\mathcal{V} \subset [\phi_q(\mathbf{V}^L_{\sigma,0}, \sigma), \phi_q(\mathbf{V}^H_{\sigma,0}, \sigma)] \subset [\phi_{q-1}(\mathbf{V}^L_{\sigma,0}, \sigma), \phi_{q-1}(\mathbf{V}^H_{\sigma,0}, \sigma)] \subset \cdots \subset [\mathbf{V}^L_{\sigma,0}, \mathbf{V}^H_{\sigma,0}]$$

The sequences $\left\{\phi_q(\mathbf{V}^L_{\sigma,0}, \sigma)\right\}_{q\in\mathbb{N}}$ and $\left\{\phi_q(\mathbf{V}^H_{\sigma,0}, \sigma)\right\}_{q\in\mathbb{N}}$ are respectively increasing and decreasing. Moreover these are sequences of bounded functions with a fixed Lipschitz rate $R^{\#}$. Thus, by Ascoli's Theorem, they converge uniformly to value functions $\mathbf{V}^L_{\sigma,\infty}$ and $\mathbf{V}^H_{\sigma,\infty}$ with Lipschitz rate $R^{\#}$. Using Theorem 4 of Chassang (2005), we know that $\phi_\sigma(\cdot)$ is continuous over $\mathcal{V}_{R^{\#}}$ endowed with the uniform norm. This implies that $\mathbf{V}^H_{\sigma,\infty}$ and $\mathbf{V}^L_{\sigma,\infty}$ satisfy,

$$\mathbf{V}^H_{\sigma,\infty} = \phi(\mathbf{V}^H_{\sigma,\infty}, \sigma) \quad \text{and} \quad \mathbf{V}^L_{\sigma,\infty} = \phi(\mathbf{V}^L_{\sigma,\infty}, \sigma)$$

This implies that $\mathbf{V}^L_{\sigma,\infty}$ and $\mathbf{V}^H_{\sigma,\infty}$ sustain extreme Markovian Nash equilibria in which players respectively play the unique equilibria of $\Psi_\sigma(\mathbf{V}^L_{\sigma,\infty})$ and $\Psi_\sigma(\mathbf{V}^H_{\sigma,\infty})$. Finally, we know from Theorem 3 of Chassang (2005) that $\phi_\sigma(\cdot)$ converges uniformly towards Φ over $\mathcal{V}_{R^{\#}}$. ■

Appendix C: Alternative assumptions

This section describes assumptions generalizing those of Section 2.3, and under which the analysis of Section 3 still holds step by step. The analysis is not repeated here. Note that these assumptions accommodate the possibility of exit payoffs also being indexed by σ as in Section 4.4.

Consider an exit game with flow payoffs γ_σ (now indexed by σ)

$$\begin{array}{c|ll} & S & E \\ \hline S & g^i_\sigma(w_t) & W^i_{12,\sigma}(w_t) \\ E & W^i_{21,\sigma}(w_t) & W^i_{22,\sigma}(w_t). \end{array}$$

Denote by $G(\mathbf{V}, w, \sigma)$ the associated one-shot full-information game augmented with continuation $\mathbf{V}$, and by Γ_σ the exit game with payoffs indexed by σ and information $x_{i,t} = w_t + \sigma\varepsilon_{i,t}$.

Assumption 0' (Compactness of payoff structures) *There exists $\overline{\sigma} > 0$ such that for all $\sigma \in [0, \overline{\sigma}]$ all payoff structures γ_σ share a common modulus of continuity in w and converge to γ_0 with respect to the supremum norm $||\cdot||_\infty$ as σ goes to 0.*

Assumption 1' (Bounded values) *Denote by $m_{i,\sigma}$ and $M_{i,\sigma}$ the min-max and maximum values of player i in game Γ_σ. There exist finite bounds m and M such that for all σ, $m \leq m_{i,\sigma}$, and $M_{i,\sigma} \leq M$.*

Assumption 2' (Dominance) *There exist $\overline{\sigma} > 0$, $\underline{w}$ and $\overline{w}$ such that for all $\sigma \in [0, \overline{\sigma}]$ and all $i \in \{1, 2\}$,*

$$\begin{aligned} g^i_\sigma(\underline{w}) + \beta M_{i,\sigma} - W^i_{21,\sigma}(\underline{w}) < 0 \quad &\textit{and} \quad W^i_{12,\sigma}(\underline{w}) - W^i_{22,\sigma}(\underline{w}) < 0 \\ \textit{and} \quad W^i_{12,\sigma}(\overline{w}) - W^i_{22,\sigma}(\overline{w}) > 0 \quad &\textit{and} \quad g^i_\sigma(\overline{w}) + \beta m_{i,\sigma} - W^i_{21,\sigma}(\overline{w}) > 0. \end{aligned}$$

Assumption 3' (Increasing differences in the state of the world) *There exists $\overline{\sigma}$ such that for all $\sigma \in [0, \overline{\sigma}]$ and all $i \in \{1, 2\}$, $g^i_\sigma(w_t) - W^i_{21,\sigma}(w_t)$ and $W^i_{12,\sigma}(w_t) - W^i_{22,\sigma}(w_t)$ are strictly increasing over $w_t \in [\underline{w}, \overline{w}]$, with a slope greater than some real number $r > 0$*

Assumption 4' (Equilibrium symmetry) *For all states of the world w_t, $G(m_{i,\sigma}, m_{-i,\sigma}, w_t, \sigma)$ has a pure strategy equilibrium. All pure equilibria belong to $\{(S, S), (E, E)\}$.*

Assumption 5' (Staying is good) *For all players $i \in \{1, 2\}$ and all states of the world $w \in [\underline{w}, \overline{w}]$, $A_i(m_{i,\sigma}, w, \sigma) \geq 0$ and $B_i(w, \sigma) \geq 0$.*

Under these assumptions, the analysis of Section 3 goes through step by step.

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Chapter Three: Learning How to Cooperate when Contingencies are Ambiguous

Abstract

This paper studies how economic agents learn to cooperate when the details of what cooperation means are ambiguous. It considers a dynamic game in which one player's cost for the cooperative action is private information. From the perspective of the other player, this cost is an unknown but stationary function of observable states of the world. Initially, because of information asymmetries, full cooperation can be sustained only at the cost of inefficient punishment. As players gain common experience, however, the uninformed player may learn how to predict her partner's cost, thereby resolving informational asymmetries. Once learning has occurred, players can sustain cooperation more efficiently and reduce the partnership's sensitivity to adverse economic conditions. Nevertheless, because inducing information revelation has an efficiency cost, it may sometimes be optimal for the uninformed player to remain uninformed even though that limits the amount of cooperation that can be sustained in equilibrium.

KEYWORDS: cooperation, private information, learning, ambiguity, common understanding, indescribability.
JEL classification codes: C72, C73, D23

1 Introduction

Real-life partnerships are difficult to build and difficult to maintain. Breakdowns in cooperation and inefficient rigidities – which are unnecessary in full-information cooperation games – seem to be the rule rather than the exception. As a plausible explanation for these observations, private information has been recognized as an important hurdle to cooperation. In their seminal work on repeated games with private information, Green and Porter (1984) consider a model with unobservable actions (later extended by Abreu, Pearce and Stacchetti (1986, 1990) and Fudenberg, Levine, and Maskin (1994)) and show that inefficient breakdown must happen on any equilibrium path in order to induce cooperation. More recently, Athey and Bagwell (2001), Levin (2003) and Athey, Bagwell and Sanchirico (2004) have considered models in which it is the players' cost for the cooperative action that is the source of informational asymmetry. They find that when costs are private, cooperation does not necessarily require inefficient breakdown, but may involve inefficient rigidity.

The papers mentioned above are concerned with the difficulties of maintaining cooperation. This paper focuses on the particular hurdles involved in building cooperative relationships. Similarly to Athey and Bagwell (2001), Levin (2003) and Athey, Bagwell, and Sanchirico (2004), the players' cost for cooperation is initially private. The idea is to introduce the possibility of learning and to take into account the fact that as they gain common experience, players also become better judges of their partner's particular economic circumstances. For instance, knowing the history of disputes may help a workers' union determine when wage cuts implemented by management are necessary or not. The paper focuses on how learning dynamics interact with the patterns of cooperation.

The model considers two players engaged in an infinite horizon cooperation game which gives them asymmetric roles. Each period proceeds as follows: Player 1 first decides whether to terminate the partnership or not (stay or exit); when Player 1 stays, Player 2 gets a profit π while Player 1 incurs a cost of effort κ. Then, Player 2 has the option to take a cooperative action (C) which has a variable cost c and gives Player 1 a benefit b. As in Athey and Bagwell (2001), Levin (2003) or Athey, Bagwell, and Sanchirico (2004), the distribution of c is known by both players, but only Player 2 observes its realization perfectly. The modeling innovation of this paper is that although only Player 2 observes c directly, Player 1 knows that it is a stationary function $c(w)$ of states of the world w which are observable by both players. Some states w may have clear payoff implications: then it will be obvious to Player 1 whether Player 2 should cooperate or not. Other states may be ambiguous, in the sense that: (*i*) these states cannot be distinguished in a language that is common to Player 1 and Player 2, (*ii*) Player 1 does not know which states correspond to a low cost of cooperation and which correspond to a high cost of cooperation. These ambiguous states represent a hurdle for cooperation because Player 2 can be tempted to exploit Player 1's confusion in order to avoid paying the cost c. However, because Player 2's cost structure is stationary, learning is possible and ambiguity can be resolved in equilibrium as Player 1 infers $c(w)$ from Player 2's actions.

Because Player 2 can exploit Player 1's confusion only temporarily, learning will be costless when players are patient enough. For intermediate degrees of patience, however, inducing information revelation will come at an efficiency cost. More precisely, on the path of a Pareto-efficient equilibrium, and while learning takes place, the partnership will be sensitive to adverse economic events: it will terminate with positive probability if an ambiguous state where cooperation is very costly occurs. Once ambiguity has been resolved, however, the partnership becomes resilient in the sense that it survives negative shocks that would have caused termination earlier on. Nevertheless, the cost of resolving ambiguity can be prohibitively high, and it may be optimal for Player 1 to remain uninformed.

Because the distribution of c is common knowledge, this model exhibits neither "good" nor "bad" types: players agree on the general principle of cooperation, but need to work through the details of implementation. As a consequence, the impact of ambiguity on cooperation is tightly related to indescribability: when ex ante communication about ambiguous states is possible, the truthful revelation of Player 2's cost function $c(\cdot)$ via cheap-talk imposes no constraints on equilibrium continuation values. In other words, the game with asymmetric information and describable states has the same Pareto frontier as the game with full information. Hence, in this model it is both private information and indescribability – the lack of a precise enough common language – that make it delicate to build a cooperative arrangement.

As noted above, this paper is related to the literature on cooperation games with private costs. Athey and Bagwell (2001) and Athey, Bagwell, and Sanchirico (2004) consider a game of repeated Bertrand competition where the firms' production costs are private, and analyze efficient collusion schemes under symmetric and asymmetric strategies. Levin (2003) considers a dynamic principal-agent relationship where the agent's cost of effort is private information and highlights the tension between efficient flexibility and incentive compatibility. From the perspective of learning, the current paper is also related to the contributions of Watson (1999, 2002), which consider a cooperation game with "cooperative" and "uncooperative" types and show how delaying full cooperation emerges as a mechanism to efficiently sort out "uncooperative" types. In the current paper, there are no "cooperative" and "uncooperative" types: players agree on the broad features of cooperation. It is the specific details of implementation that are ambiguous and need to be elucidated.

The paper is also related to the literature on the relevance of indescribable contingencies initiated by Maskin and Tirole (1999) and developed in Hart and Moore (1999), Segal (1999) and Battigalli and Maggi (2002). Here, because indescribability implies that Player 1 does not observe the cost of Player 2 when the state is ambiguous, the results of Maskin and Tirole (1999) on the irrelevance of describability do not apply. In the dynamic game studied here, if costs were observable, indescribability would indeed have no impact. However, because indescribability creates confusion, it does affect the players' ability to cooperate. In the particular game considered in this paper, there are only two ambiguous states and indescribability implicitly results from a coarseness of language. The work of Al-Najjar, Anderloni and Felli (2006) shows how indescribable events can be modelled even for rich languages. However, their construction relies on a large enough space of states, and analyzing the dynamics of learning for a such a rich state space is beyond the scope of this paper.

The paper is structured as follows: Section 2 describes the model. Section 3 studies the joint dynamics of cooperation and learning. Section 4 generalizes the notion of ambiguity and proposes a model of empathy building. Section 5 concludes. Appendix A examines the robustness of results to specific modeling assumptions and presents an extension to three-states ambiguity. Proofs are contained in Appendix B unless mentioned otherwise.

2 The Framework: ambiguity, indescribability, and cooperation

2.1 The game

Consider a game with two players $i \in \{1,2\}$, infinite horizon $t \in \{1,\ldots,\infty\}$, and discount rate β. Each period t consists of the following four subperiods:

1. Player 1 chooses to stay (S) at cost $\kappa > 0$ or exit (E) at zero cost. If Player 1 exits, the game ends and both players get zero continuation values.

2. If Player 1 has chosen to stay, an i.i.d. state of the world w is drawn from $\{\underline{w}, \overline{w}, w_a^1, w_a^2\}$ with respective probabilities $\{\underline{p}, \overline{p}, p_a, p_a\}$, where $p_a = \frac{1-\underline{p}-\overline{p}}{2}$.

3. Player 2 observes the state w and her private cost of cooperation $c(w)$. Player 1 observes only w. Both players observe some payoff-irrelevant random variable x_t, uniformly distributed over $[0,1]$, that allows for public randomizations.

4. Player 2 decides whether to cooperate (C) or defect (D). Conditional on Player 1 having

stayed in stage 1, players' payoffs at the end of stage 4 are

	C	D
Player 2	$\pi - c(w)$	π
Player 1	$b - \kappa$	$-\kappa$

where π, b, and κ are common knowledge, strictly positive, and $b > \kappa$.

Ambiguity. The players' differing knowledge of the cost structure $c(\cdot)$ is the key element of this environment: the observability of a state is distinguished from the observability of the cost of cooperation that this state implies. More precisely, when the state w belongs to $\{\underline{w}, \overline{w}\}$, the cost $c(w)$ is common knowledge for the players. However, when w belongs to $\{w_a^1, w_a^2\}$, the state w is observable by both players, but the associated cost $c(w)$ is perfectly known only to Player 2. Player 1 merely knows that $(c(w_a^1), c(w_a^2))$ is equal to (c_L, c_H) with probability 1/2 and equal to (c_H, c_L) with probability 1/2, where $c_H > c_L > 0$. The states $\underline{w}$ and $\overline{w}$, which have clear implications for the cost of cooperation, are contrasted with the ambiguous states w_a^1 and w_a^2, which are harder to interpret. The game with ambiguous information is denoted by Γ_A. The game with full information, in which c is perfectly observable by both players in all states, is denoted by Γ_{FI}. The object of this paper is to study perfect Bayesian equilibria of both Γ_{FI} and Γ_A. Note that although they act at different moments, both players observe the same histories from period $t = 1$ on.

Definition 1 (histories) *Denote by d_t Player 1's decision to stay or not and a_t Player 2's decision to cooperate or not at time t. We respectively define by*

$$\begin{aligned} h_t^1 &\equiv \{d_1, w_1, x_1, a_1, \ldots, d_{t-1}, w_{t-1}, x_{t-1}, a_{t-1}\} \\ \textit{and} \quad h_t^2 &\equiv \{d_1, w_1, x_1, a_1, \ldots, d_{t-1}, w_{t-1}, x_{t-1}, a_{t-1}, d_t, w_t, x_t\} \end{aligned}$$

the histories respectively observed by Player 1 and Player 2 when they make their decisions *in period t.*

The set of all possible histories is denoted by $\mathcal{H}$. Pure strategies of Player 1 are mappings $s_{P_1} : \mathcal{H} \to \{S, E\}$ and pure strategies of Player are mappings $s_{P_2} : \mathcal{H} \to \{C, D\}$.

Note that there is no aggregate uncertainty about the amount of cooperation that could be sustained under full information. This game represents a situation where players agree on the possibility of cooperation but are trying to work out the details of when cooperation should occur or not.

Because the cost function is stationary, the observability of states allows learning: if at some ambiguous state, the equilibrium behavior of Player 2 is different when her cost is c_L than when her cost is c_H, Player 1 will learn the mapping $c(\cdot)$ perfectly and the continuation game will involve no informational asymmetry.

2.2 Interpretation

The game presented in Section 2.1, although very stylized, can be interpreted as a rough model for a variety of economic settings where the detailed contingencies in which cooperation should occur are ambiguous. For example, the game Γ_A can be used to model the problem of a union (Player 1) having to decide when wage cuts implemented by management (Player 2) are acceptable or not. It could also be a model of shareholders (Player 1) trying to decide when it is tolerable that the firm's CEO (Player 2) does not give out dividends. It could also be a model of a firm (Player 1) deciding to relocate its production if the local government (Player 2) raises unjustified taxes too frequently. One might also view the game Γ_A as a description of a public good provision problem in which a public authority (Player 1) is trying to find out whether a citizen (Player 2) is not contributing because she is short of money, or because she is being selfish. Finally this game can also be understood as a model of a lender/borrower relationship in which the lender (Player 1) decides whether or not she should tolerate delayed repayment by the borrower (Player 2), or demand liquidation.

In this paper "ambiguity" refers to a setting in which players agree on the principle of cooperation but do not know the details of when and how cooperation should be implemented. The main result of this paper is that although there is no uncertainty about the sustainability of cooperation under full-information, the inability to communicate about states of the world may limit the partnership to inefficient cooperative agreements. In particular while learning occurs, the partnership will be sensitive to negative economic shocks and inefficient termination may happen when an ambiguous state with cost c_H occurs. As ambiguity is resolved, the partnership becomes resilient to adverse circumstances and players can sustain efficient cooperation. However, because inducing information revelation has an efficiency cost of its own, under some conditions it will be optimal not to resolve informational asymmetries.

2.3 The role of indescribability

The game presented in Section 2.1 does not allow for communication. The assumption is that ambiguous states are recognizable but cannot be described using words from the players' common language. This indescribability is a key element of the setup. Indeed, Proposition 1 shows that if players have the ability to communicate ex ante about ambiguous states, then the Pareto frontier under ambiguity is the same as the Pareto frontier under full information.

More precisely, assume that at time $t = 0$, before the game described in Section 2.1 begins, Player 2 can send Player 1 a message that is either "w_a^1 is the low cost state" or "w_a^1 is the high cost state". In that setting, the following proposition holds.

Proposition 1 (communication and ambiguity) *If Player 2 can engage in ex ante communication, then ambiguity is innocuous. More formally, the Pareto frontier of the game with ambiguity is the same as the Pareto frontier of the game with full information: Player 2 can be induced to reveal her type without constraining continuation values.*

The intuition for this result is straightforward. Pick an equilibrium under full information, and let Player 1 play according to that equilibrium, taking Player 2's declaration at face value.

Given Player 1's behavior, ex ante, Player 2 does not benefit from Player 1's confusion and information revelation is incentive compatible.

Section 3 studies the joint dynamics of cooperation and learning when communication is impossible and Player 1 must learn from Player 2's actions. Incentive compatibility constraints will now need to hold ex post rather than ex ante. In particular, when she is at a state where she is supposed to cooperate, Player 2 may be tempted to delay cooperation and temporarily exploit Player 1's confusion. As Section 3 will show, Player 1 may need to use inefficient exit in order to induce Player 2 to cooperate at ambiguous states.

3 Learning how to cooperate

This section explores the joint dynamics of learning and cooperation under ambiguity. Section 3.1 studies the problem of optimal information revelation. It shows that when learning occurs, efficient cooperative equilibria will be non-stationary: early on, while ambiguity has not been resolved, negative economic shocks will cause the partnership to breakdown with positive probability; once players have built common understanding, however, the partnership will be resilient to such shocks and inefficient breakdowns become unnecessary. Because information revelation comes at an efficiency cost, Section 3.2 explores whether it may be optimal for Player 1 to remain uninformed. Depending on parameter values, both under-cooperation and over-cooperation on the part of Player 2 may be optimal.

3.1 The joint dynamics of learning and cooperation

This section studies game Γ_A under the assumption that $c(\underline{w}) = c(w_a^1) = c_L > 0$, and $c(\overline{w}) = c(w_a^2) = c_H = +\infty$ (where the fact that w_a^1 is the ambiguous state with cost c_L is of course unknown to Player 1). States w fall into four categories, depending on whether cooperation is possible or not (c equal to c_L or $c_H = +\infty$), and whether that information is private or not (w in $\{\underline{w}, \overline{w}\}$ or $\{w_a^1, w_a^2\}$). Section 3.1.1 characterizes the Pareto frontier under full information. Section 3.1.2 gives sufficient and necessary conditions for information revelation to be costly and characterizes optimal equilibria among the class of ambiguity-resolving equilibria.

3.1.1 The Pareto frontier under full information.

This section characterizes the Pareto frontier of the full-information game Γ_{FI}. In particular, it shows that on the full-information Pareto-efficient frontier, Player 1 should never exit in equilibrium.

Proposition 2 (no exit) *Under full information, the Pareto frontier is such that:*

1. *either it is reduced to a unique equilibrium for which Player 1 exits with certainty in period $t = 1$*
2. *or, Player 1 always chooses to stay following an action of Player 2 that is possible on the equilibrium path.*

Lemma 1 *Conditional on the existence of equilibria that are not reduced to immediate exit, any pair of values on the full-information Pareto frontier can be attained by equilibria such that*

1. *On the equilibrium path, Player 1 always stays;*
2. *Player 2 never cooperates when her cost is $c_H = +\infty$;*
3. *If she has never cooperated when her cost was c_L before, Player 2 cooperates with probability r_1^L at a state with cost c_L. If she has cooperated when her cost was c_L before, Player 2 cooperates with probability $r_2^L \leq r_1^L$ when her cost is c_L.*

Consider one of the equilibria described in Lemma 1. For a given rate of cooperation r_2^L, after the first time Player 2's cost is c_L and she cooperated, the long run continuation values are

$$V_{P_2}^{r_2^L} = \frac{1}{1-\beta}\left(\pi - (\underline{p} + p_a) r_2^L c_L\right) \tag{1}$$

$$V_{P_1}^{r_2^L} = \frac{1}{1-\beta}\left(-\kappa + (\underline{p} + p_a) r_2^L b\right). \tag{2}$$

For $r_2^L > 0$, this cooperation scheme will be sustainable in equilibrium if and only if

$$V_{P_1}^{r_2^L} \geq 0, \quad \text{and} \quad \beta V_{P_2}^{r_2^L} \geq c_L \tag{3}$$

These two conditions ensure that staying and cooperating are respectively incentive compatible for Player 1 and Player 2. Denote by $r_{min} \equiv \frac{\kappa}{b(\underline{p}+p_a)}$ the minimum constant cooperation rate required from Player 2 to induce Player 1 to stay. There are three cases:

1. If $r_{min} \in \left[0, \frac{\underline{p}}{\underline{p}+p_a}\right]$, then if Player 2 cooperates frequently enough at the unambiguous state $\underline{w}$, Player 1 can be induced to stay.

2. If $r_{min} \in \left[\frac{\underline{p}}{\underline{p}+p_a}, 1\right]$, then Player 2 must cooperate at both the unambiguous state $\underline{w}$ and the ambiguous state with low cost (state w_a^1 by convention) to induce Player 1 to stay.

3. If $r_{min} > 1$, then Player 2 cannot get Player 1 to stay, independently of how much she cooperates when her cost is c_L.

It is incentive compatible for Player 2 to cooperate at the minimum required rate r_{min} if and only if

$$\frac{1}{1-\beta}\left(\pi - (\underline{p} + p_a) r_{min} c_L\right) \geq \frac{c_L}{\beta}. \tag{4}$$

It is assumed that condition (4) holds and $r_{min} < 1$, so that there exists a cooperative equilibrium under full information. The maximum value attainable by Player 2 is

$$V_{P_2}^{max} = V_{P_2}^{r_{min}} = \frac{1}{1-\beta}\left(\pi - (\underline{p} + p_a) r_{min} c_L\right). \tag{5}$$

Define r_{max} as the maximum $r \in [0, 1]$ such that,

$$\frac{1}{1-\beta}\left(\pi - (\underline{p} + p_a) r c_L\right) \geq \frac{c_L}{\beta}. \tag{6}$$

Ambiguous states are relevant only if $r_{max} > \frac{\underline{p}}{\underline{p}+p_a}$. Otherwise, the Pareto frontier under full information can be spanned by equilibria such that Player 2 cooperates only at the unambiguous state $\underline{w}$. The maximum continuation value Player 1 can expect after Player 2 has cooperated at a state with cost c_L is

$$V_{P_1}^{max} = V_{P_1}^{r_{max}} = \frac{1}{1-\beta}\left(-\kappa + \left(\underline{p} + p_a\right) r_{max} b\right). \tag{7}$$

3.1.2 Optimal ambiguity-resolving strategies

This section considers strategies that induce revelation of Player 2's type in equilibrium. Since $c_H = +\infty$, revelation will occur whenever Player 2 cooperates at an ambiguous state. Because misbehavior (i.e. failing to cooperate in the ambiguous state when $c = c_L$) can be detected in finite time, inducing revelation need not always be costly. Proposition 3 and Lemma 5 provide conditions under which resolving ambiguity will be costly and characterize the extent of the efficiency loss. Proposition 4 characterizes optimal revelation inducing strategies.

Definition 2 *Consider a pair of equilibrium strategies (s_{P_1}, s_{P_2}) and some history h_t at which Player 2 can take action. A history h_t is a* ***revelation stage*** *if it is the first history at which the two types of Player 2 – with respective cost structures $(c(w_a^1), c(w_a^2))$ equal to (c_H, c_L) and (c_L, c_H) – take different actions. A revelation stage h_t is said to be* ***conclusive*** *(resp. inconclusive) when Player 2's type is such that she chooses to cooperate (resp. Player 2 chooses not to cooperate).*

A history h_{t+s} weakly following a revelation stage h_t is called a ***confirmation stage*** *if it is the first history subsequent to h_t at which Player 2 cooperates at an ambiguous state.*

In equilibrium, Player 1 will know Player 2's type after a revelation stage. Still, conclusive and inconclusive revelation stages differ in the sense that after an inconclusive revelation, Player 1 may worry that Player 2 simply misrepresented her cost to avoid cooperating in the short run. For this reason, inconclusive revelations may have to be followed by inefficient exit.

Proposition 3 (costly revelation) *Assume that some cooperation is feasible under full information and define*

$$V_{P_2}^{Liar} \equiv \max\left\{\frac{1}{1-(\overline{p}+p_a)\beta}\pi\ ;\ \frac{1}{1-(\underline{p}+\overline{p}+p_a)\beta}(\pi-\underline{p}c_L)\right\}. \tag{8}$$

If

$$\beta(V_{P_2}^{max} - V_{P_2}^{Liar}) < c_L \tag{9}$$

then, the revelation of Player 2's type is necessarily costly: no equilibrium of the game with ambiguity that involves revelation on the equilibrium path can be on the full-information Pareto frontier.

The value $V_{P_2}^{Liar}$ corresponds to the min-max value that Player 2 can attain when she chooses not to cooperate although her cost is low, and Player 1 never exits following actions that are consistent with Player 2 being truthful[1]. Under this constraint, the only way Player 1 can detect misbehavior is when Player 2 does not cooperate at both ambiguous states. The time multiplier comes from the fact that when there is no inefficient exit, Player 2 knows that she will keep getting benefits from the partnership at least until the other ambiguous state comes up.

As Proposition 3 suggests and Lemma 5 confirms, when Player 2's surplus is sufficiently high, revelation is incentive compatible at no cost: because it is possible to discover past misbehavior in finite time, when Player 2 gets enough surplus, she will reveal her type rather

[1]In other terms, $V_{P_2}^{Liar}$ is the min-max value of Player 2 when she misbehaves and Player 1 uses a strategy that would coincide with a Pareto-efficient equilibrium under full information.

take the chance of causing the partnership breakdown in the near future. This result is consistent with the usual folk theorem intuition[2]. When Player 2 gets only moderate surplus however, she can be tempted to misbehave and exploit Player 1's confusion for a while. In such a case, revelation will be induced only by having Player 1 exit on the equilibrium path, which is inefficient.

The rest of this section focuses on the case where the revelation of Player 2's type is costly, and characterizes the dynamics of optimal, ambiguity resolving, strategies.

Lemma 2 (revelation stages) *On the Pareto frontier of game Γ_A, revelation will occur only at a history for which the current state w_t is ambiguous, i.e. $w_t \in \{w_a^1, w_a^2\}$.*

Lemma 3 (forgiveness) *For any Pareto-efficient equilibrium of game Γ_A, once Player 2 has cooperated at an ambiguous state, then players behave according to some Pareto-efficient equilibrium of the game with full information Γ_{FI}.*

In other terms, if inefficient behavior must occur to induce revelation, it should not happen any more once Player 2 has cooperated at an ambiguous state. Indeed, the sole purpose of inefficient exit is to diminish the expected continuation value of Player 2 upon misbehavior. Once confirmation occurs, Player 1 knows for sure that Player 2 cannot have misbehaved, and hence there is no need for further punishment.

The next lemma concerns optimal behavior following an inconclusive revelation. It shows that optimally, Player 2 should cooperate as much as is incentive compatible in order to reduce Player 2's incentive to misrepresent her cost structure.

Lemma 4 (maximum cooperation) *Consider a Pareto efficient equilibrium (s_{P_1}, s_{P_2}) of game Γ_A that involves costly revelation. If revelation occurs at state w_a^i with $i \in \{1, 2\}$, then*

1. *Player 2 must cooperate with probability 1 the first time state w_a^{-i} or $\underline{w}$ comes up;*

[2]See for instance Fudenberg, Levine, and Maskin (1994).

2. *There exists an equilibrium of* Γ_A *that gives players the same ex-ante values as* (s_{P_1}, s_{P_2}) *and in which following 1) inconclusive revelation and 2) cooperation at a state* $w \in \{w_a^{-i},\ \underline{w}\}$*, Player 2 cooperates at a rate* r_{max} *at all further states* $w \in \{w_a^{-i},\ \underline{w}\}$*.*

The rationale for maximum cooperation is that it minimizes the time that Player 1 needs to identify for sure whether or not Player 2 has misbehaved. After an inconclusive revelation, maximum cooperation will tend to bias surplus division in favor of Player 1, but this can be undone by delaying the revelation stage itself. Finally, note that when revelation is not costly, equilibria on the Pareto frontier need not exhibit maximum cooperation. However, such equilibria are sufficient to describe the Pareto frontier. Using Lemmas 2, 3, and 4, we can now characterize optimal revelation-inducing equilibria.

Proposition 4 (early dispute, fast forgiveness) *Consider any equilibrium of* Γ_A *that involves revelation. There exists a weakly Pareto-dominating equilibrium in which exit – if it happens at all – needs to happen only in the period immediately following an inconclusive revelation of Player 2's type.*

In other term, there is no option value in delaying confrontation. Delaying exit may allow Player 2 to confirm its type in the meanwhile, but also makes the crisis more violent if Player 2 isn't able to dispel doubts. Proposition 4 states that these two forces exactly offset each other.

Proposition 4 shows that optimal ambiguity-resolving equilibria take a simple form. Some histories are designated as revelation stages. At a revelation stage, Player 2 either cooperates or defects. If Player 2 cooperates then players play some Pareto-efficient equilibrium of the full-information continuation game. If Player 2 defects then Player 1 exits with some probability in the next period. When exit does not happen, players follow some Pareto-efficient equilibrium of the full-information continuation game. The probability of exit following an inconclusive revelation is given by Lemma 5.

Lemma 5 *Let (s_{P_1}, s_{P_2}) be a Pareto-efficient equilibrium with early dispute as defined in Proposition 4. Consider a revelation stage h at which Player 2 would get a continuation value $V_{P_2}^{Coop}$ upon cooperation. Following an inconclusive revelation, Player 1 must exit with probability*

$$1 - q(V_{P_2}^{Coop}) = 1 - \min\left\{\frac{\beta V_{P_2}^{Coop} - c_L}{\beta V_{P_2}^{Liar}}\ ,\ 1\right\}. \tag{10}$$

Lemma 5 implies that the likelihood of exit following an inconclusive revelation is decreasing in Player 2's continuation value upon cooperation, $V_{P_2}^{Coop}$. This means that transferring surplus from the informed player to the uninformed player increases inefficiency: because it increases Player 2's incentives to misrepresent its cost, Player 1 must exit with greater probability following an inconclusive revelation in order to keep truthful revelation incentive compatible. In fact, Section 3.2 will show that in some cases, the inefficiency can be so large that Player 1 would rather have Player 2 cooperate only in the unambiguous state $\underline{w}$, rather than provide the costly incentives needed to induce revelation.

This section has shown that when states are indescribable, ambiguity is at the source of a monitoring problem which generates inefficiencies. When ambiguous states occur, indescribability creates an information asymmetry that Player 2 is tempted to exploit. Because misbehavior on the part of Player 2 can be detected in finite time, ambiguity can be resolved at no efficiency cost when the surplus is large enough. However, because Player 2 can benefit from the partnership for multiple periods after she has misbehaved, Player 1 must resort to inefficient exit following an inconclusive revelation when the surplus is moderate.

Optimal revealing equilibria take a simple form that is reminiscent of the bang-bang property of Abreu, Pearce, and Stacchetti (1990): some histories are selected as revelation stages. If revelation is conclusive, the continuation game is played according to some Pareto-efficient full-information equilibrium. If revelation is inconclusive, Player 1 exits with some probability. If exit does not happen, then players fall back to playing some Pareto-efficient equilibrium of the full-information game. While learning occurs, the partnership will be sensitive to adverse economic shocks and termination will happen with positive probability if an ambiguous state with high cost occurs. Once learning has happened, however, players behave according to a Pareto-efficient full information equilibrium which does not involve inefficient exit. In other words, once players have reached a common understanding, the partnership becomes resilient to adverse shocks.

This section was restricted to revealing equilibria. However some cooperation can be sustained in game Γ_A without revealing Player 2's type. There are two ways to achieve this: Player 2 can cooperate only in the unambiguous state $\underline{w}$ — this corresponds to under-cooperation; or Player 2 can cooperate in both states w_a^1, w_a^2 with the same probability — this corresponds to over-cooperation. The levels of cooperation achieved in revealing and non-revealing equilibria follow different dynamic patterns: revealing equilibria will initially exhibit some inefficient termination, later followed by greater cooperation, while non-revealing equilibria will sustain constant intermediate levels of cooperation. In that sense, resolving ambiguity is an investment in relational capital which comes at the immediate cost of inefficient exit but yields the future benefit of efficient cooperation. Section 3.2 studies the relative benefits of revelation and non-revelation and establishes that non-revelation, in the form of either under- and over-cooperation, may sometimes be optimal.

3.2 Optimal learning

Section 3.1.2 showed that reducing informational asymmetries comes at an efficiency cost. Given these costs, this section considers whether there should be any information revelation on the Pareto frontier. It is shown that depending on the costs and benefits of revelation, it can be optimal for Player 2 to under-cooperate or over-cooperate.

3.2.1 Under-cooperation

This section maintains the assumption that $c(\underline{w}) = c(w_a^1) = c_L > 0$, and $c(w_a^2) = c(\overline{w}) = c_H = +\infty$. Consider the region of the parameter space such that,

$$r_{min} \in \left[0, \frac{\underline{p}}{\underline{p} + p_a}\right] \quad \text{and} \quad r_{max} \in \left(\frac{\underline{p}}{\underline{p} + p_a}, 1\right). \tag{11}$$

When condition (11) is satisfied, Player 2 needs to cooperate only in the unambiguous state $\underline{w}$ to induce Player 1 to stay. Furthermore, under full information, it is incentive compatible for Player 2 to cooperate with probability 1 in the unambiguous state $\underline{w}$ and with some probability strictly less than 1 in the ambiguous state with low cost w_a^1. This section shows that under ambiguity, the efficiency cost of revelation can be so prohibitive that for all equilibria on the Pareto frontier of Γ_A, Player 2 should cooperate only in the unambiguous state $\underline{w}$. In this case, we say that *under-cooperation* is optimal.

Proposition 5 (sufficient condition for optimal under-cooperation) *Whenever*

$$b + \frac{\beta}{1-\beta} p_a b \leq \frac{\beta}{1-\beta}(\underline{p}b - \kappa), \tag{12}$$

all equilibria on the Pareto frontier of the game with ambiguity involve under-cooperation.

Under this condition, on the Pareto frontier of the game Γ_A, Player 2 will never cooperate at an ambiguous state: even from the perspective of Player 1, the cost of inducing revelation is greater than the potential benefit of full cooperation.

According to condition (12), under-cooperation will be optimal whenever the greatest possible increase in benefit Player 1 could get from full cooperation, $b + \frac{\beta}{1-\beta} p_a b$, is smaller than the value, $\frac{\beta}{1-\beta}(\underline{p}b - \kappa)$, it expects from having Player 2 cooperate whenever the state is $\underline{w}$. This will be the case when the unambiguous state $\underline{w}$ occurs with greater frequency than the ambiguous state w_a^1, and players' discount rate β is close enough to 1.

3.2.2 Over-cooperation

Section 3.2.1 showed that the efficiency cost of revelation can be so prohibitive that Pareto optimal equilibria of game Γ_A simply do not involve cooperation at an ambiguous state. Over-cooperation – meaning that Player 2 cooperates at both ambiguous states with equal probability – is another possible non-revealing strategy. So far, over-cooperation was ruled out by the assumption that $c(w_a^2) = c_H = \infty$. In order to study potential over-cooperation, this section considers the case where $c(w_a^1) = c(\underline{w}) = c_L$, $c(w_a^2) = c_H < +\infty$, and $c(\overline{w}) = \infty$. It may now be possible for Player 2 to cooperate at an ambiguous state with cost c_H. The section begins by extending Propositions 2 and 3: there must be no exit on the Pareto frontier of game Γ_{FI}, while under ambiguity revelation may require inefficient exit on the equilibrium path .

Lemma 6 (no exit) *The Pareto frontier under full information is such that:*

1. *either it is reduced to a unique equilibrium for which Player 1 exits with certainty in period $t = 1$*
2. *or Player 1 always chooses to stay following an action of Player 2 that is possible on the equilibrium path.*

Lemma 7 (sufficient condition for costly revelation) *Define $\underline{V}_{P_2}^{Liar} \equiv \frac{1}{1-\overline{p}\beta}\pi$. Whenever*

$$\beta(V_{P_2}^{max} - \underline{V}_{P_2}^{Liar}) < c_L \tag{13}$$

revelation is necessarily costly. In other terms, an equilibrium of the game with ambiguity that involves revelation cannot lie on the Pareto frontier of the game with perfect information.

When the inefficiency cost of revelation is high, non-revealing equilibria may be optimal. An equilibrium will be non-revealing whenever Player 2 cooperates at the same rate in both ambiguous states. Given a value V_{P_1} that Player 1 can obtain in a full information equilibrium, there may or may not be a non-revealing equilibrium that grants Player 1 the same utility. Proposition 6 provides conditions under which, if there exists a non-revealing equilibrium that grants Player 1 value V_{P_1}, then Pareto-efficient equilibria of Γ_A that give value V_{P_1} to Player 1 are non-revealing.

Proposition 6 (sufficient condition for optimal over-cooperation) *Assume that Player 1 can be granted value V_{P_1} in a non-revealing equilibrium of game Γ_A. Whenever*

$$\frac{\beta}{1-\beta}p_a(c_H - c_L) < c_L - \beta(V_{P_2}^{max} - \underline{V}_{P_2}^{Liar}), \tag{14}$$

the most efficient equilibrium of Γ_A that gives value V_{P_1} to Player 1 is non-revealing.

Note that condition (14) will be satisfied whenever c_L is high enough for condition (13) to hold and $c_H - c_L$ is small enough. Proposition 6 is particularly interesting when $r_{min} \in \left(\frac{\underline{p}}{\underline{p}+p_a}, 1\right)$, since in that case Player 2 must cooperate in an ambiguous state in order to induce Player 1 to stay. In that case, non-revelation means that Player 2 would rather over-cooperate than go through the potentially inefficient process of information revelation. Intuitively, revelation generates inefficiencies because it creates a temptation for Player 2 to misrepresent her cost and avoid cooperating in the short term. Because such misbehavior cannot be detected immediately, Player 1 must resort to inefficient exit in order to provide adequate incentives. Propositions 5 and 6 show that these efficiency costs can be greater that the gains of using a precise cooperative arrangement that distinguishes between ambiguous states.

4 A model of empathy building

The simple model of ambiguity presented in Section 2, the fact that there are only two states of the world results in a very stark learning process: whenever Player 2 cooperates at an ambiguous state, Player 1 learns the cost structure $c(w)$ perfectly. This section suggests a framework in which to model richer information structures. The model presented in what follows is perhaps more about what might be called empathy-building than ambiguity-resolution: Player 1 can observe a number of signals that may or may not be good predictors of Player 2's cost for cooperation. As players gain common experience, Player 1 learns about the relative predictive power of the various signals she observes, and can start using them to provide flexible contingent incentives for cooperation. Section 4.1 describes the model. Section 4.2 gives some examples of information structures that fit in this framework and shows that it is a generalization of the ambiguity framework. Section 4.3 shows that the essential properties of the ambiguity framework still hold.

4.1 The framework

Consider a game with two players, infinite horizon, $t \in \{1, \dots, +\infty\}$, and discount rate β. The timing of each period is similar to that of Section 2:

1. Player 1 decides to either stay (S) at a cost κ or exit (E) at zero cost. If Player 1 exits, the game ends and both players get zero continuation values.

2. If Player 1 has stayed, an i.i.d. random variable $\tilde{c}$ is drawn. It takes values in $\{c_L, c_H\}$ with respective probabilities $(p, 1-p)$.

3. Player 2 observes her cost of cooperation $\tilde{c}$. Player 1 observes a signal s of $\tilde{c}$. Both players observe a payoff-irrelevant random variable that allows public randomizations. Players can also exchange messages belonging to {*"cooperation"*, *"no cooperation"*}.

4. Player 2 decides to cooperate (C) or defect (D). Conditional on Player 1 having stayed in the first stage, final payoffs are given by

	C	D
Player 1	$\pi - \tilde{c}$	π
Player 2	$b - \kappa$	$-\kappa$

where π, b, and κ are common knowledge and strictly positive, and it is common knowledge that the cost of cooperation $\tilde{c}$ is either or c_L or c_H, with $0 < c_L < c_H = +\infty$.

Empathy. At the beginning of each period t, Player 1 can choose to watch one signal s from a family $\mathcal{S}$ of possible signals. Each of these signals is a real-valued i.i.d. stochastic processe $(s_t)_{t\in\mathbb{N}}$, normalized to share an identical distribution f, but that may or may not be good predictors of $\tilde{c}$. The precision of a signal is given by the conditional distribution $D_s(x) = Prob(\tilde{c} = c_H \mid s = x)$. When Player 1 has watched signals $\{s_1, \dots, s_n\}$ and chooses to draw a new signal, the precision of the new signal is drawn according to a probability distribution $f_{\mathcal{S}\setminus\{s_1,\dots,s_n\}}$ on $[0,1]^{\mathbb{R}}$ (the set of functions mapping $\mathbb{R}$ into $[0,1]$). Which signal she observes is private information to Player 1. This game will be denoted by $\Gamma_{\mathcal{S}}$.

In this framework, Player 1 may not be able learn how to perfectly predict the cost structure $c(w)$ immediately. In particular, when $\mathcal{S}$ is large, finding a good signal of Player 2's behavior will take time. As players gain common experience, Player 1 will be able to better understand Player 2's cost of cooperation. This greater empathy will allow for the provision of more efficient contingent incentives.

4.2 Some examples

This section gives a few examples of the empathy framework presented above. In particular it shows that this framework generalizes the game with ambiguity Γ_A defined in Section 2.

4.2.1 Ambiguity

This section shows that the game with ambiguity Γ_A is a specific case of the framework presented in Section 4.1. Let $\tilde{c}$ take values c_L and c_H with respective probabilities $p = \underline{p} + p_a$ and $1 - p = \overline{p} + p_a$. The set of possible signals contains only two signals, $\mathcal{S} = \{s_-, s_+\}$ taking values in $\{-2,-1,1,2\}$ with respective probabilities $\{\underline{p}, p_a, p_a, \overline{p}\}$ such that

$$Prob(\tilde{c} = c_H | s_-) = \begin{cases} 1 & \text{if} \quad s_- = 2 \\ 0 & \text{if} \quad s_- = 1 \\ 1 & \text{if} \quad s_- = -1 \\ 0 & \text{if} \quad s_- = -2 \end{cases} \quad \text{and} \quad Prob(\tilde{c} = c_H | s_+) = \begin{cases} 1 & \text{if} \quad s_- = 2 \\ 1 & \text{if} \quad s_- = 1 \\ 0 & \text{if} \quad s_- = -1 \\ 0 & \text{if} \quad s_- = -2 \end{cases}$$

States $\underline{w}$ and $\overline{w}$ can respectively be defined as the events $\{s_+ = 2 \cup s_- = 2\}$ and $\{s_+ = -2 \cup s_- = -2\}$, while states w_a^1 and w_a^2 respectively correspond to events $\{s_+ = 1 \cup s_- = -1\}$, and $\{s_+ = -1 \cup s_- = 1\}$.

4.2.2 Finding a reliable signal

The set of signals $\mathcal{S}$ can also be infinite. This would naturally be the case if states of the world were infinite dimensional vectors, and Player 1 could pay attention only to a particular dimension of the state of the world. This can be modeled as follows: $\mathcal{S}$ is an infinite, countable, set of i.i.d. stochastic processes s such that for all $t \geq 1$, s_t takes values in $\{-1, 1\}$, with respective probabilities $(p, 1-p)$. The precision of a signal s is entirely characterized by the value $d_s = Prob(\tilde{c} = c_H \mid s = 1)$. Each time Player 1 draws a new signal from $\mathcal{S}$, its precision d_s is drawn from some constant distribution f over $[0, 1]$.

4.3 Indescribability and costly revelation

This section shows that the class of games described in Section 4.1 satisfies the basic properties of the ambiguity framework.

Lemma 8 (no exit) *Under full information the Pareto frontier is such that:*

1. *either it is reduced to a unique equilibrium for which Player 1 exits with certainty in period $t = 1$*

2. or, Player 1 always chooses to stay following an action of Player 2 that is possible on the equilibrium path.

Lemma 9 (ex ante truthful communication) *Assume that there exists a signal s^* that is a perfect predictor of $\tilde{c}$ — that is, for all $x \in \mathbb{R}$, $D_{s^*}(x) \in \{0,1\}$ — and that Player 2 can inform Player 1 of this signal at a time $t = 0$, before the game begins. Then the Pareto frontier of the game with private information and ex ante communication is the same as the Pareto frontier of the game with full information.*

In other terms, if Player 2 can costlessly inform Player 1 of which signal is a perfect predictor of c, Player 2 can be induced to reveal her type without constraining continuation values. Lemma 9 assumes that signals are describable. When this is not the case and the precision of signals has to be learned from play, limited empathy will generate inefficiencies. As in the game with ambiguity, the pervasiveness of these inefficiencies depends on how fast misbehavior from Player 2 can be detected.

Definition 3 (slow learning) *For all $n \in \mathbb{N}$, consider $\mathbf{s}^n = (s_1, \ldots, s_n)$ a vector of signals of S drawn using the joint distribution $f_S \times f_{S\setminus\{s_1\}} \times \cdots \times f_{S\setminus\{s_1,\ldots,s_{n-1}\}}$. The family of signals $\mathcal{S}$ exhibits slow learning whenever,*

$$\begin{aligned} \forall n \in \mathbb{N}, \textit{ with proba } 1, \quad & Prob(\tilde{c}_1 = c_H \mid \mathbf{s}^n) > 0 \quad \textit{and} \\ & Prob(\tilde{c}_1 = c_L \mid \mathbf{s}^n) > 0. \end{aligned} \tag{15}$$

When slow learning holds, no matter how much information Player 1 has access to, both values of $\tilde{c}$ are always possible given the signals she observes.

Definition 4 (fast learning) *The family of signals $\mathcal{S}$ exhibits fast learning if and only if*

1. when $\tilde{c}_1 = c_H$, there exists $n \in \mathbb{N}$ such that with strictly positive probability, $Prob(\tilde{c}_1 = c_L \mid \mathbf{s}^n) = 0$

2. *when $\tilde{c}_1 = c_L$, there exists $n \in \mathbb{N}$ such that with strictly positive probability, $Prob(\tilde{c}_1 = c_H \mid \mathbf{s}^n) = 0$.*

Lemma 10 *Whenever S exhibits fast learning and Player 1 can observe the past history of the signals she pays attention to, there exists $\overline{\beta} < 1$ such that for all $\beta \geq \overline{\beta}$, Player 2 can be induced to cooperate fully at no efficiency cost.*

Lemma 11 (necessary exit) *Whenever S exhibits slow learning, then, on any equilibrium of game Γ_S, exit will happen on the equilibrium path with strictly positive probability.*

In view of Lemma 8, this implies that in games with slow learning, limited empathy *always* generates inefficiencies. The intuition for this result is straightforward: when learning is slow, it is possible – though unlikely – that Player 2's true cost $\tilde{c}$ is equal to c_H for arbitrarily long periods of time, independently of what the signal observed by Player 1 indicates. If Player 1 did not exit no matter the length of these sequences, Player 2 would be tempted to misrepresent her cost and claim her cost is c_H independently of what the true cost is. Hence exit must happen on the equilibrium path with some positive probability.

5 Conclusion.

This paper explored settings in which players agree on the principle of cooperation, but the details of how cooperation should be implemented are ambiguous. It shows that although there is no uncertainty about the sustainability of cooperation under full information, ambiguity about the contingencies in which cooperation should happen can cause inefficient termination on the equilibrium path. These inefficiencies are closely linked to the indescribability of ambiguous states. In particular, if ambiguous states were describable in the players' common language, Pareto-efficient equilibria would never involve termination on the equilibrium path.

Because ambiguous states are observable, the framework naturally authorizes learning from the other player's actions. When learning takes place, cooperation exhibits interesting dynamics: while learning occurs, the partnership is sensitive to adverse economic shocks, and terminates with positive probability if an ambiguous state with a high cost of cooperation occurs. Once learning is over, however, the players can sustain greater cooperation and the partnership becomes resilient to negative economic shocks. Because inducing information revelation has an efficiency cost, resolving ambiguity can be regarded as an investment in relational capital that comes at the short run cost of potential early termination but yields the future benefit of improved cooperation. When revelation cost are high, it may be optimal for the uninformed player to remain uninformed and have the informed player either under- or over -cooperate.

Finally, the paper shows that resolving ambiguity can be seen as a particular case of an empathy building problem in which players learn how to predict each others' cost. In this richer setup, the basic result that indescribability and lack of empathy hinder cooperation and generate inefficient exit on the equilibrium path still holds. A full-fledged analysis of optimal learning dynamics in this richer setup is beyond the scope of this paper but is an interesting topic for future research.

Appendix A: Extensions

A.1 Repeated game

This section examines the simplifying assumption made in Section 2 that Player 2's decision is an exit decision. The setup used in this section is a variation on that of Section 2: now Player 1's decision to stay or exit is not irreversible any longer: each period, Player 1 decides to participate (P) or not (NP). Player 1's cost of participation is $e > 0$. This section shows that Proposition 3 extends in this repeated game setup. The assumption that $c_H = +\infty$ is maintained. The timing of each period is the following:

1. Player 1 decides to participate or not.
2. An i.i.d. state of the world $w \in \{\underline{w}, \overline{w}, w_a^1, w_a^2\}$ is drawn with probabilities $(\underline{p}, \overline{p}, p_a, p_a)$.
3. Player 2 observes her cost of cooperation $c(w)$. Player 1 observes only w.
4. Player 2 decides whether to cooperate or defect. Final payoffs, are given by

	C	D
P	$(b-e, \pi - c(w))$	$(-e, \pi)$
NP	$(b, -c(w))$	$(0, 0)$.

As in Section 2, states w_a^1 and w_a^2 are ambiguous from the perspective of Player 1.

Consider the following conditions,

$$\underline{p}b < e \,; \quad \frac{\beta}{1-\beta}(\pi - \underline{p}c_L) > c_L \quad \text{and} \quad \frac{\beta}{1-\beta}(\pi - (p_a + \underline{p})c_L) < c_L \tag{16}$$

Under Assumption 16, it is necessary for Player 1 to cooperate at some frequency at an ambiguous state for Player 1 to be induced to stay. Moreover, Player 2 can be induced to cooperate at an ambiguous state associated with cost c_L in addition to cooperating in state $\underline{w}$, but not with probability 1.

Lemma 12 (full participation) *Under Assumption 16 equilibria on the Pareto frontier of the repeated game with full information are such that once the Player 2 has cooperated at any state, Player 1 always participates following an action that is possible on the equilibrium path.*

Proposition 7 (sufficient conditions for costly revelation) *Define* $\underline{V}_{P_2}^{Liar} = \frac{1}{1-(\overline{p}+p_a)\beta}\pi$. *Under Assumption 16, whenever*

$$\beta(V_{P_2}^{Max} - \underline{V}_{,P_2}^{Liar}) < c_L \tag{17}$$

then the Pareto frontier of the game with ambiguity lies strictly below the Pareto frontier of the game with full information.

Note that there is a non-empty set of parameter values such that Assumption 16 and inequality (17) are satisfied together.

Proposition 7 shows that ambiguity can generate inefficiencies even though Player 1's action is not an irreversible exit decision. Inefficiencies will occur whenever under efficient strategies, Player 2 can hope to ride the partnership for a long enough time before her deviation is detected.

A.2 Three-states ambiguity

Section 3 considered forms of ambiguity in which only two states were ambiguous. By studying an example with three states ambiguity, this section makes the point that in all generality, there can be different degrees of ambiguity revelation. As a consequence, it is shown that on the Pareto frontier, the players may end up using cooperative agreements of various precisions depending on the particular path of play.

Consider a model with the same payoff structure as that of section 2, but such that there are no unambiguous states and three ambiguous states $\{w_a^1, w_a^2, w_a^3\}$. Theses three ambiguous states have the same likelihood p_a. Player 1 believes that $(c(w_a^1), c(w_a^2), c(w_a^3))$ is uniformly distributed over $\{(c_L, c_L, c_H), (c_L, c_H, c_L), (c_H, c_L, c_L)\}$. We assume that $c_H = +\infty$.

Since $c_H = +\infty$ and all states are ambiguous, some initial revelation must happen on any equilibrium path for which there is some cooperation. As information is being revealed, there are two levels of ambiguity that can be reached: if the state with cost c_H is first revealed, then there is no more ambiguity about the game; however, if a state with cost c_L is revealed, the two remaining states are ambiguous, and we are exactly in the case studied in Section 3.

In Section 3, we showed that it may be that under ambiguity, the entire Pareto frontier involves under-cooperation on the part of Player 2. In such a case, on the Pareto frontier, depending on the amount of ambiguity that is resolved initially, players will end up using different long run cooperative agreements.

Appendix B: Proofs

Lemma 13 (No Exit) *For any specification of the costs $c(w_a^1)$ and $c(w_a^2)$, under full information, the Pareto frontier of Γ_{FI} is such that:*

(i) either it is reduced to a unique equilibrium for which Player 1 exits with certainty in period $t = 1$

(ii) or, Player 1 always chooses to stay following an action of Player 2 that is possible on the equilibrium path.

Proof: First, if there exist an equilibrium with some cooperation, it must Pareto dominate immediate exit since both players can always guarantee the payoffs of exit.

Now, consider a subgame perfect equilibrium (s_{P_1}, s_{P_2}), involving some cooperation, and such that there exists a history h, attainable on the equilibrium path, such that $s_{P_1}(h) = E$.

Denote $\sqcup$ the concatenation operator for histories, where $h \sqcup h'$ means history h followed by history h'. We say that h includes h' – denoted $h' \subset h$ – if and only if there exists h'' such that $h = h' \sqcup h''$.

Since $s_W(h) = E$, no history of the form $h' = h \sqcup h''$ where $h'' \neq \{E\}$ is reached under (s_{P_1}, s_{P_2}). Hence, without breaking incentive compatibility conditions or changing payoffs on the equilibrium path, we can assume that at any history of the form $h' = h \sqcup h''$ where Player 1 makes a decision $s_{P_1}(h') = E$. From here, the proof is by construction: define the pair of strategies $\tilde{s}_{P_1}$ and $\tilde{s}_{P_2}$ as follows:

1. whenever $h' \in \mathcal{H}$ is of the form $h' = h \sqcup h''$, then $\tilde{s}_{P_2}(h') \equiv s_{P_2}(h'')$ and $\tilde{s}_{P_1}(h') \equiv s_W(h'')$.
2. otherwise, when h' isn't of the form $h \sqcup h''$, then $\tilde{s}_{P_2}(h') \equiv s_{P_2}(h')$ and $\tilde{s}_{P_1}(h') \equiv s_{P_1}(h')$.

Let us first show that when they play according to $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ then at any history h' the players have greater value than if they used (s_{P_2}, s_{P_1}). This is obvious at any history of the form $h' = h \sqcup h''$, since under (s_{P_1}, s_{P_2}) both the worker and the firm would get 0 utility, while under $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$, by construction the firm and the worker are playing a Nash equilibrium which must give them values weakly greater than 0. For any history h' such that $h' \not\subseteq h$ and $h \not\subseteq h'$, the two pairs of strategies (s_{P_1}, s_{P_2}) and $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ prescribe the same behavior at all future histories, and hence they imply the same continuation values. Finally, for any history h' for which there exists $h'' \in \mathcal{H}$ such that $h' \sqcup h'' = h$, the players play the same actions for both pairs of strategies until h is reached and, we have just shown, obtain greater continuation values with $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ once h is reached. It follows that at h' the players must have greater continuation values under $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$.

We now show that the pair $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ forms a subgame perfect equilibrium. At any history of the form $h'' = h \sqcup h'$, by construction, the firm and the worker are playing a Nash equilibrium. The same holds at any history h'' such that $h'' \not\subseteq h$ and $h \not\subseteq h''$. Now consider a history h'' such that $h'' \sqcup h' = h$, with $h' \neq \emptyset$. Note that at h'', (s_{P_1}, s_{P_2}) and $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ prescribe the same actions. Should the players deviate from their prescribed move, then the resulting history cannot be connected to h and at any history following the deviation, s and $\tilde{s}$ prescribe the same behavior. Hence, should they deviate when playing according to $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$, the two players would get the same values they would have had under (s_{P_1}, s_{P_2}). Should they follow their prescribed move, we know that the players get greater continuation value under $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ than under (s_{P_2}, s_{P_1}). Since following their prescribed action was incentive compatible under (s_{P_2}, s_{P_1}), it must be incentive compatible under $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$. It follows that $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ is indeed a subgame perfect equilibrium.

Since at any history where Player 1 stays, the expected value of Player 2 is weakly greater than $\pi > 0$, equilibrium $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ strictly dominates (s_{P_2}, s_{P_1}). It follows that on the Pareto frontier, Player 1 must always stay following actions on the equilibrium path, otherwise it is possible to construct a strictly dominating equilibrium. ■

Proof of Proposition 1: Consider the game with full information Γ_{FI} where by convention $c(w_a^1) = c_L$. The Pareto frontier of Γ_{FI} is entirely spanned by equilibria in which Player 1 exits with probability 1 following any action of player 2 that is not on the equilibrium path. Consider such an equilibrium (s_{P_1}, s_{P_2}) of Γ_{FI}.

Consider now the game with ambiguity and ex ante communication at $t = 0$. Player 2 can be induced to send a message of the form "State w_a^1 [is/isn't] the low cost state" by having Player 1 exit if Player 2 does not send a message at time $t = 0$. A message sent by Player 1 defines a permutation σ over $\{w_a^1, w_a^2\}$ such that $\sigma(w_a^1)$ is the state with cost c_L according to Player 2's claim. Consider the strategy $\tilde{s}_{P_1}$ of Player 1 defined[3] by $\tilde{s}_{P_2} = s_{P_2} \circ \sigma$. When Player 2 tells the truth in period $t = 0$, then $s_{P_1} \circ \sigma$ is a best-reply to $s_{P_2} \circ \sigma$ since (s_{P_1}, s_{P_2}) is an equilibrium of the full-information game Γ_{FI}. Let us now show that when Player 2 uses strategy $\tilde{s}_{P_2}$ in the subgame starting at time $t = 1$, it is incentive compatible for Player 1 to tell the truth at time $t = 0$.

We use some intermediary results. Let us define $z_t = \{w_1, x_1, \cdots, w_t, x_t\}$. For any pair of strategies (s_i, s_{-i}), we will denote by $s_i(z_t)$ the action prescribed by s_i when the state of the world is z_t and players have played according to their strategies in all previous periods. Because (s_{P_1}, s_{P_2}) is on the Pareto frontier of Γ_{FI}, it must be that for any realization of z_t, on the equilibrium path,

$$Prob(a_{t+1} = coop. \,\&\, c(w_{t+1}) = c_H \mid z_t) \leq Prob(a_{t+1} = coop. \,\&\, c(w_{t+1}) = c_L \mid z_t). \qquad (18)$$

Indeed, otherwise, it is possible to improve players' values by shifting cooperation from the high cost state to the low cost state.

Denote by σ^* the permutation defined by truth-telling, $\sigma^\neg$ the permutation defined by lying and $\sigma^\leftrightarrow = \sigma^* \circ \sigma^\neg$. We know that $s_{P_2} \circ \sigma^*$ is a best reply to $s_{P_1} \circ \sigma^*$. Let us now study Player 2's best reply $s^\neg$ to $s_{P_1} \circ \sigma^\neg$. Under s_{P_1} Player 1 exits whenever Player 2 does not play according to s_{P_2}. This implies that under $s^\neg$, for a sequence z_t of states of the world, either $s^\neg$ coincides with $s_{P_2} \circ \sigma^\neg$ at all preceding periods or the game has ended because Player 1 has exited. Hence whenever $s^\neg$ deviates from $s_{P_2} \circ \sigma^\neg$, Player 2 gets a value equal to 0. This is weakly less than any value Player 2 obtains when players use $(s_{P_1} \circ \sigma^*, s_{P_2} \circ \sigma^*)$. Let us now compare the utility obtained by Player 2 in states where $s^\neg$ coincides with $s_{P_2} \circ \sigma^\neg$. Define $Z^\neg \equiv \{z_t \mid \forall t' \leq t, s^\neg(z_{t'}) = s_{P_2} \circ \sigma^\neg(z_{t'})\}$. Let us show that there exists a set Z^* of states of the world z_t such that there is an injective mapping $m : Z^\neg \to Z^*$ and for all $z_t \in Z^\neg$,

$$u_{P_2}(s^\neg, s_{P_1} \circ \sigma^\neg, z_t) \leq u_{P_2}(s_{P_2} \circ \sigma^*, s_{P_1} \circ \sigma^*, m(z_t)) \quad \text{and} \quad Prob(z_t) = Prob \circ m(z_t)$$

where u_{P_2} denotes Player 2's flow payoffs. The existence of such a mapping m implies that Player 2 obtains greater utility under $(s_{P_1} \circ \sigma^*, s_{P_2} \circ \sigma^*)$ than under $(s_{P_1} \circ \sigma^\neg, s_{P_2} \circ \sigma^\neg)$: at states $z_t \in Z^\neg$ states of equal mass can be found that provide greater flow payoffs; at states $z_t \notin Z^\neg$, Player 2 gets value zero under $(s_{P_1} \circ \sigma^\neg, s_{P_2} \circ \sigma^\neg)$ and value weakly greater than zero under $(s_{P_1} \circ \sigma^*, s_{P_2} \circ \sigma^*)$.

We now show that m does exist. At all these states, Pareto efficiency of (s_{P_1}, s_{P_2}) and Lemma13 imply that Player 1 will stay. The only difference is then the cost of cooperation.

[3]Where σ also permutates ambiguous states within histories.

For all states $z_t \in Z^{\neg}$ such that $s^{\neg}(z_t) = D$, then m is defined by $m(z_t) = \sigma^{\leftrightarrow}(z_t)$. Indeed, if $s_{P_2} \circ \sigma^{\neg}(z_t) = D$ then by construction $s_{P_2} \circ \sigma^* \circ \sigma^{\leftrightarrow}(z_t) = D$. When $s^{\neg}(z_t) = C$ and $c(w_t) = c_L$ then equation 18 implies that there exists an injective mapping $(x_t, w_t) \mapsto (x'_t, w'_t)$ such that $c(w'_t) = c_L$ and $s_{P_2}(\{z_{t-1}, x'_t, w'_t\}) = C$.. At such a z_t, m is defined by $m(z_t) = \{z_{t-1}, x'_t, w'_t\}$. Finally for any state z_t such that $s^{\neg}(z_t) = C$ and $c(w_t) = c_H$, any completion of m that makes it a a bijection between the remaining states that maintains probability works. ■

Proof of Proposition 2: This is a direct application of Lemma 13. ■

Proof of Lemma 1: We know from Lemma 2 that on the Pareto frontier Player 1 always stays following equilibrium actions. Furthermore, it is clear that Player 2 never cooperates at states with cost $c_H = +\infty$. Hence, transfer of utility on the Pareto frontier is done entirely by having the Player 2 be cooperating more or less frequently in states with cost c_L. Because players have the same discount rate the timing of these transfers does not affect the efficiency of the equilibrium. The timing of transfers however will affect incentive compatibility conditions and we look for the timing that minimizes the worst case temptation to defect.

Consider a history h such that Player 2 has never cooperated when her cost was c_L before. Behavior at h does not affect any preceding or future incentive constraints. Therefore, as long as cooperation at h is incentive compatible, the rate of cooperation at such a history can be freely chosen. Note that at any history following a history at which Player 2 has cooperated when her cost was c_L, Player 2's behavior will affect past incentive compatibility constraints. In this sense, there is something special about the first time Player 2 cooperates with cost c_L: no past promises need to be upheld.

For any Pareto efficient equilibrium (s_{P_1}, s_{P_2}), denote $V^*_{P_2}$ the continuation value of the Player 2 at any history where she has to cooperate at a state with cost c_L for the first time. Note that in all generality, $V^*_{P_2}$ is a random variable, however, it must satisfy $V^*_F \geq c_L/\beta$. We denote $\mathbf{E}V^*_F$ its expectation, it must satisfy $\mathbf{E}V^*_F \geq c_L/\beta$. We consider stationary strategies in which Player 2 cooperates with probability[4] r^2_L every time her cost is c_L and Player 1 enforces this by exiting when Player 2 does not comply. Denote $V^{r^2_L}_{P_2}$ the continuation value of the Player 2 at any of her decision point when she plays such a strategy. Since such strategies continuously span behavior going from "cooperating always" to "cooperating never", there exists r^2_L such that $V^{r^2_L}_{P_2} = \mathbf{E}V^*_F \geq c_L/\beta$. By construction, this r^2_L makes cooperation incentive compatible and the stationary strategy of with cooperation rate r^2_L is sustainable in equilibrium.

Given this r^2_L, for all possible $r^1_L \in [0, 1]$, consider strategies in which Player 2 cooperates with rate r^1_L at states of cost c_L while she has never cooperated and starts cooperating at rate r^2_L once she has cooperated. Denote $V^{r^1_L, r^2_L}_{P_2}$ the value Player 2 expects at time $t = 0$ under such a strategy. Let $V_{P_2}(0)$ denote the value expected by Player 2 at $t = 0$ under (s_{P_1}, s_{P_2}). Clearly there exists r^1_L such that $V^{r^1_L, r^2_L}_{P_2} = V_{P_2}(0)$. By construction, such a strategy $s^{r^1_L, r^2_L}_{P_2}$

[4] Players make use of the public randomization.

can be sustained in equilibrium. To conclude the proof, note that if $r_L^1 \leq r_L^2$ then one can increase r_L^1 and decrease r_L^2 while keeping the initial value of Player 2 constant and weakly relaxing all incentive compatibility constraints. ■

Proof of Proposition 3: Consider an equilibrium (s_{P_1}, s_{P_2}) on the Pareto frontier of Γ_A. Assume that this equilibrium yields values that are on the Pareto efficient frontier of Γ_{FI}. Then, Lemma 13 implies that under (s_{P_1}, s_{P_2}), Player 1 never exits following behavior that is consistent with equilibrium behavior under full information for some cost structure c. At a revelation stage, not-cooperating is consistent with both possible cost structures. Hence Player 1 should not exit if Player 2 does not cooperate at the state w_a^i of a revelation stage h_t. For (s_{P_1}, s_{P_2}) to yield values that are one the Pareto frontier of Γ_{FI} the only moments when Player 2 may exit following a revelation stage h_t are at a state w_a^{-i} or $\underline{w}$ where Player 2 does not cooperate. Hence, the value Player 2 can obtain when she does not cooperate at w_a^i even though $c(w_a^i) = c_L$ is minimized when Player 1 demands cooperation the first time w_a^{-i} occurs after h_t and whenever $w_t = \underline{w}$ until w_a^{-i} occurs. Hence, the minimum value $V_{P_2}^{Liar}$ that Player 2 can guarantee when she misrepresents her cost is

$$V_{P_2}^{Liar} \equiv \max \left\{ \frac{1}{1 - (\overline{p} + p_a)\beta} \pi \; ; \; \frac{1}{1 - (\underline{p} + \overline{p} + p_a)\beta} (\pi - \underline{p} c_L) \right\}$$

where the value depends on whether it is optimal for Player 2 to cooperate or not when $w_t = \underline{w}$.

The fact that revelation is incentive compatible under (s_{P_1}, s_{P_2}) implies that $\beta(V_{P_2}^{max} - V_{P_2}^{Liar}) \geq c_L$. Hence, if $\beta(V_{P_2}^{max} - V_{P_2}^{Liar}) < c_L$, then (s_{P_1}, s_{P_2}) cannot yield values that are on the Pareto frontier of the game Γ_{FI}. ■

Proof of Lemma 2: The only other circumstance where revelation can occur in equilibrium is when $w_t = \underline{w}$. By cooperating or not cooperating at this state Player 2 can potentially reveal information about ambiguous states. The incentive problem however is exactly the same as at an ambiguous state. Any continuation equilibrium that induces revelation at $w_t = \underline{w}$ would induce revelation at $w_t \in \{w_a^1, w_a^2\}$ and vice-versa. Player 2's temptation is the same in both cases: by misrepresenting her cost structure, Player 2 can save a cost c_L. If revelation is potentially costly, it is weakly Pareto dominant to have revelation occur as late as possible. Hence revelation may as well be delayed to ambiguous states. ■

Proof of Lemma 3: The role of exit is to reduce the continuation value of Player 2 when she misrepresents her cost. If she has misrepresented her cost, Player 2 cannot cooperate at an ambiguous state without revealing she misbehaved. Since it occurs out-of-equilibrium, it is clearly efficient for Player 1 to exit whenever past misbehavior is revealed: it reduces the continuation value of Player 2 when she misbehaves without changing behavior on the equilibrium path.

In these conditions, if she has misrepresented her cost, Player 2 will never choose to cooperate at an ambiguous state. Therefore, behavior that occurs after the Player 2 cooperates at an ambiguous has no impact on the continuation value of a misbehaving Player 2, and hence behavior after Player 2 cooperates at an ambiguous state should be on the Pareto frontier of the game with full information Γ_{FI}. ■

Proof of Lemma 4: Consider an equilibrium (s_{P_1}, s_{P_2}) of game Γ_A that exhibits exit on the equilibrium path following some inconclusive revelation stage with state w_a^i. The point of Lemma 4 is that forcing Player 2 to cooperate is a more efficient way to reduce Player 2's incentive to misrepresent her cost structure than exit. This will bias values following an inconclusive revelation stage in favor of Player 1 but does not restrict ex-ante values since revelation itself can be delayed in order to transfer utility from Player 1 to Player 2.

Let us first consider the case where $r_{max} < 1$. Denote by h^0 the revelation stage under (s_{P_1}, s_{P_2}) and h^1 the first history following h^0 such that the state w is either $\underline{w}$ or w_a^{-i}. Consider the strategies $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ such that at h^1, Player 2 cooperates with probability 1 and at all following states $w \in \{\underline{w}, w_a^{-i}\}$ Player 2 cooperates at a rate r_{max}. At h^1, the continuation value of Player 2 when she has misrepresented her cost is weakly less than her continuation value when her cost was truthfully revealed. Under $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ a truthful Player 2 is weakly indifferent between cooperating or not. Hence it is an equilibrium of the continuation game for a misrepresenting Player 2 not to cooperate at h^1 and hence trigger exit. Hence there need not be any exit on the equilibrium path after h^1. Using $(\tilde{s}_{P_1}, \tilde{s}_{P_2})$ reduces the need for inefficient exit but also transfers utility from Player 2 to Player 1. This transfer can be undone by randomizing the initial revelation stage: with some constant probability $d > 0$ the revelation stage is delayed to the next period where the same ambiguous state occurs.

When $r_{max} = 1$, even when she has misrepresented, Player 2 might want to cooperate at state $\underline{w}$ to keep the partnership going. In those cases we cannot yet rule out exit occurring on the equilibrium path after h^1. However, some exit is unnecessary if Player 2 does not cooperate with probability one at every state $w \in \{\underline{w}, w_a^i\}$. Indeed, if Player 2 does not cooperate at such a history h, then one can demand Player 2 to cooperate at h. If cooperation is not IC for Player 2 under truthful revelation, one can reduce future exit so that cooperation becomes incentive compatible for the truthful type. This weakly reduces the utility of a misrepresenting Player 2 since, she benefits weakly less from the possible reduction in exit. As in the case where $r_{max} = 1$ this induces a transfer of utility from Player 2 to Player 1 that can be undone by randomizing the initial revelation stage. This concludes the proof. ■

Proof of Proposition 4: This proof builds on the proof of Lemma 4. We can restrict our attention to the class of Pareto efficient equilibria described in Lemma 4 where Player 2 cooperates at the maximum possible rate following inconclusive revelation. Let us first consider the case where $r_{max} < 1$. Denote by h_t the inconclusive revelation stage. We first show that having potential exit occur over h_{t+1} and h_{t+2} does not improve upon exiting only in period h_{t+1}.

Since $r_{max} < 1$, we know there is no need for exit after cooperation occurs. Denote by

$\mathbf{q} = (q_1, q_2^{\overline{w}}, q_2^{w_a^i})$ the staying rates at h_{t+1} and h_{t+2}, depending on what state is realized at h_{t+1}. Given that Player 2 cooperates as much as is incentive compatible, the players' values at h_t are,

$$
\begin{aligned}
V_{P_2}^{Liar}(\mathbf{q}) &= q_1\pi + q_1(q_2^{\overline{w}}\overline{p} + q_2^{w_a^i}p_a)\frac{\beta(\overline{p}+p_a)}{1-\beta(\overline{p}+p_a)}\pi && (19)\\
V_{P_2}^{Truth}(\mathbf{q}) &= q_1\left(\pi + (p_a+\underline{p})(-c_L + \beta V_{P_2}^{r_{max}})\right) && (20)\\
&+ q_1(q_2^{\overline{w}}\overline{p} + q_2^{w_a^i}p_a)\frac{\beta(\overline{p}+p_a)}{1-\beta(\overline{p}+p_a)}(\pi + (p_a+\underline{p})(-c_L+\beta V_{P_2}^{r_{max}})) \\
V_{P_1}(\mathbf{q}) &= q_1\left(-e + (p_a+\underline{p})(b + V_{P_1}^{r_{max}})\right) && (21)\\
&+ q_1(q_2^{\overline{w}}\overline{p} + q_2^{w_a^i}p_a)\frac{\beta(\overline{p}+p_a)}{1-\beta(\overline{p}+p_a)}(-e + (p_a+\underline{p})(b+\beta V_{P_1}^{r_{max}}))
\end{aligned}
$$

Any change in $\mathbf{q}$ that keeps $V_{P_2}^{Liar}(\mathbf{q})$ constant keeps $V_{P_1}(\mathbf{q})$ and $V_{P_2}^{Truth}(\mathbf{q})$ constant. Hence delaying exit by one period does not provide any efficiency gains.

This implies that there is also no efficiency gain from delaying exit over $T+1$ periods rather than T and by induction no efficiency gain from delaying exit over $T+1$ periods rather than 1. Given that delaying exit over an infinite number of periods can be arbitrarily approximated by strategies in which exit can happen over long but finite horizons, delaying exit over an infinite horizon does not provide efficiency gains either. A similar proof holds for the case where $r_{max} = 1$. ■

Proof of Lemma 5: We know that exit need only happen in the period following inconclusive revelation. Let us denote q the probability that Player 1 stays after an inconclusive revelation. When she misrepresents her cost, Player 2 gets a value $V_{P_2}^{Liar}(q) = qV_{P_2}^{Liar}$, where $V_{P_2}^{Liar}$ is defined in Proposition 3. Truthful revelation is incentive compatible if and only if

$$V_{P_2}^{Coop} - V_{P_2}^{Liar}(q) \geq \frac{c_L}{\beta}$$

which yields $q \leq \min\left\{1, \frac{\beta V_{P_2}^{Coop} - c_L}{\beta V_{P_2}^{Liar}}\right\}$. Pareto efficient equilibria will use the greatest such q, which yields the result. ■

Proof of Proposition 5: Consider a Pareto efficient equilibrium with revelation and denote $V_{P_1}^{Rev}$ the value expected by Player 1 at some revelation stage with ambiguous state w_a^{-i}. The maximum value Player 1 can get in an under-cooperating equilibrium is $V_{P_1}^{u.\ acc} = \frac{\beta}{1-\beta}(-e + \underline{p}b)$. Under-cooperation will be optimal whenever $V_{P_1}^{Rev} \leq \frac{\beta}{1-\beta}(-e+\underline{p}b)$, under-cooperation will be optimal. Using Proposition 4 and Lemma 5, we obtain that

$$V_{P_1}^{Rev} = \frac{1}{2}(b + \beta V_{P_1}^{Coop}) + \frac{1}{2}\beta q(V_{P_2}^{Coop})V_{P_1}^{1,r_{max}}$$

where $V_{P_1}^{Coop}$ is the value Player 1 would get should Player 2 cooperate, $q(V_{P_2}^{Coop})$ is equal to $\frac{\beta V_{P_2}^{Coop}-c_L}{\beta V_{P_2}^{Liar}}$, and $V_{P_1}^{1,r_{max}}$ is the greatest value Player 1 can obtain. Because $V_{P_1}^{Coop}$ is linear in $V_{P_2}^{Coop}$, $V_{P_1}^{Rev}$ is also linear in $V_{P_2}^{Coop}$. Hence, $V_{P_1}^{Rev}$ is maximized either for $V_{P_2}^{Coop} = V_{P_2}^{r_{min}}$ or $V_{P_2}^{r_{max}}$. We have

$$\begin{aligned} V_{P_1}^{Rev}(V_{P_2}^{r_{max}}) &\leq \frac{1}{2}(b+\frac{\beta}{1-\beta}(-e+b(p_a+\underline{p})) \qquad (22)\\ V_{P_1}^{Rev}(V_{P_2}^{r_{min}}) &\leq \frac{\beta}{2}(b+\frac{\beta}{1-\beta}(-e+b(p_a+\underline{p})). \qquad (23)\end{aligned}$$

Inequalities (22), (23) and simple algebra yield that whenever $b+\frac{\beta}{1-\beta}p_a b \leq \frac{\beta}{1-\beta}(-e+\underline{p}b)$, under-cooperation is optimal. ■

Proof of Lemma 6: This is a direct application of Lemma 13. ■

Proof of Lemma 7: Because at $\overline{w}$, $c(\overline{w}) = +\infty$, Player 2 never cooperates at state $\overline{w}$ on the Pareto frontier of Γ_{FI}. Hence if some revealing equilibrium of Γ_A is on the Pareto frontier of Γ_{FI}, then Player 2 must never cooperate at $\overline{w}$. Hence under such an equilibrium, Player 2 can guarantee a value greater than $\underline{V}_{P_2}^{Liar}$ even when she misrepresents her cost. The incentive compatibility of revelation implies that

$$\beta(V_{P_2}^{max} - \underline{V}_{P_2}^{Liar}) \geq c_L.$$

Hence whenever $\beta(V_{P_2}^{max} - \underline{V}_{P_2}^{Liar}) < c_L$, revelation must imply some inefficiency. ■

Proof of Proposition 6: Consider a possible revelation stage and fix Player 1's conditional value V_{P_1}. There exists some non-revealing strategy of Player 2 that delivers V_{P_1} and gives Player 2 a value of the form $V_{P_2}^{No\ Rev} = \frac{1}{1-\beta}(\pi-\underline{p}-p_a r(c_L+c_H))$, with $r \leq 1$. The question is whether there exists an equilibrium with revelation that gives Player 2 a greater value $V_{P_2}^{Rev}$ while keeping Player 1's value constant.

When $\beta(V_{P_2}^{max} - \underline{V}_{P_2}^{Liar}) < c_L$ there must be inefficiency on the equilibrium path to induce revelation. Whether this takes the form of exit or cooperation at what Player 1 presumes is the high cost state. inefficient behavior is more costly to Player 2 when she has behaved truthfully than when she has misrepresented her cost. Hence, if $V_{P_2}^{FI}$ is the value that Player 2 could expect under full information, we have

$$V_{P_2}^{Rev} \leq V_{P_2}^{FI} - (\frac{c_L}{\beta} + \underline{V}_{P_2}^{Liar} - V_{P_2}^{max}). \qquad (24)$$

Since

$$V_{P_2}^{No\ Rev} \geq V_{P_2}^{FI} - \frac{1}{1-\beta}p_a(c_H - c_L) \qquad (25)$$

we get by combining inequalities (24) and (25) that whenever $\frac{\beta}{1-\beta}p_a(c_H - c_L) < c_L - \beta(V_{P_2}^{max} - \underline{V}_{P_2}^{Liar})$, over-cooperation dominates revelation. ■

Proof of Lemma 8: The proof is identical to that of Lemma 13. ■

Proof of Lemma 9: Given that signals are unbiased, proof similar to that of Proposition 1 holds. Player 1 simply takes the information given by Player 2 seriously. From an ex ante perspective, Player 2 does not benefit from Player 1's confusion. ■

Proof of Lemma 10: Under fast learning when Player 2 does not cooperate at a state where her cost is c_L, PLayer 1 may learn it with certainty in finite time by sampling n consecutive signals. Hence, when β is large enough, there is an equilibrium in which Player 1 entirely trusts the declarations of Player 2 but exits if she learns that Player 2 has misbehaved. ■

Proof of Lemma 11: Under slow learning, it is possible that $\tilde{c} = c_H$ for arbitrarily long sequences independently of the signals that Player 1 observes. If there was no exit on the equilibrium path, then Player 1 would tolerate arbitrarily long sequences in which Player 2 does not cooperate. In that setting, it would not be incentive compatible for Player 2 to cooperate. ■

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