

**A Model of Efficiency and Trading Opportunities
in Financial Markets**

by

Chunyan Jennifer Huang

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

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Abstract

This paper considers an economy with heterogeneous investors and an incomplete securities market. Investors have non-traded income and private information about security payoffs. They trade in the market to allocate the risk from their non-traded income and to speculate on security payoffs. We use the model to examine the equilibrium allocation of risk and the behavior of security prices under different market structures, as defined by the characteristics of all traded securities in the market. In particular, we examine how the addition of derivatives securities affects the trading and prices of existing securities and both the allocational and informational efficiencies of the market. We show that the introduction of derivatives can decrease informational efficiency of the market on security payoffs, increase risk premium and price volatility in the market.

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1. Introduction.

Securities market plays two important roles: to allocate risks and to communicate information among investors [see, e.g., Arrow (1964), Debreu (1959), Hayek (1945)]. How efficient the market performs these two roles crucially depends on the market structure, as defined by a complete characterization of traded securities. The continuous emergence of new securities in the market and the trading volume they can generate provide clear evidence to this dependence. In the literature, how changes in market structure (such as introducing new securities) may affect its allocational and informational roles are mostly studied in separation. For example, in analyzing the informational impact of opening derivative trading, the allocational trading in the market is often specified exogenously (as “noise”) [see, e.g., Grossman (1977)]. As argued eloquently by Grossman (1995), the informational efficiency of the securities market and its allocational function are fundamentally related. The introduction of futures contracts on S&P 500 stock index, say, not only allows investors to communicate their information on future distributions of the index through futures prices, but also *changes* future distributions of the index itself since now investors can use futures, in addition to existing securities, to achieve different allocations of the index risk. The interaction between the two roles of the market, allocational and informational, is important in determining the impact of changes in market structure. It is this interaction and its effect on the impact of derivative trading that we want to study in this paper.

We consider an economy in which investors have non-traded income and private information about security payoffs. The securities market consists of a risk-free security (bond) and a risky security of non-zero supply (stock) as primary securities, and possibly a futures-type derivative security on the stock. Investors trade in the market to allocate the risk from their non-traded income and to speculate on security prices. We solve the equilibrium allocations and security prices under two different market structures, consisting of, respectively, only the primary securities, and the primary securities as well as the derivative security. We examine how the addition of derivative securities changes the trading and prices of existing securities, and how it affects the allocational and informational efficiencies of the market.

In the absence of asymmetric information, the market structure only affects the risk allocations in the market. With the securities market being incomplete, investors are often unable to optimally share the aggregate risk and to perfectly hedge the individual risks. Security prices not only depend on the aggregate risk, but also depend on investors’ individual risks. The introduction of derivative securities then allows investors to better allocate the risks. It tends to decrease the premium on equity and its price volatility.

When there exists information asymmetry among investors, the market also plays the role of transmitting information. The introduction of derivative securities not only affects the efficiency of the market in allocating risk, but also its efficiency in aggregating and revealing private information [see, e.g., Grossman (1977)]. On the one hand, additional prices of the new securities provide more endogenous signals investors can learn from about other investors' private information. On the other hand, the expanded trading opportunities allow investors to better allocate risks, and increase the amount of allocational trading in the market. The increase in allocational trading can generate additional price movements in traded securities, hence make them less informative about investors' private information on security payoffs. In fact, opening derivative trading can worsen the informational efficiency of the market. Contrast to the case of symmetric information, opening derivative trading can increase stock premium and price volatility.

The economy we consider has heterogeneous investors, incomplete securities market with non-traded income and asymmetric information. Many authors have considered the effect of market incompleteness and non-traded income on investors' optimal investment behavior.¹ Market incompleteness and the presence of non-traded income often make investors' optimization problem difficult to solve, and the results on optimal policies are quite limited. Analyzing market equilibrium with non-traded income then becomes more difficult and is mostly done by numerical methods.² The existence of asymmetric information makes the problem even more formidable.³ Our approach here is to impose specific restrictions on investors' preferences and shock distributions, which allows us to obtain closed form solutions of the equilibrium under asymmetric information and different market structures. We sacrifice on generality for the benefit of being able to analyze in more detail investors' portfolio policies, equilibrium security prices and allocational and informational efficiencies under different market structures. The intuition obtained from the model can be helpful in understanding more general models.

The literature on the informational role of derivatives market includes the first formal discussion of Grossman (1977), using a single-period model, and more recently the work of Grossman (1988), Back (1993) and Brennan and Cao (1995), using multiperiod models. All these papers

¹For example, He and Pearson (1991) and Karatzas, Lehoczky, Shreve and Xu (1991) examine the existence and characterization of optimal consumption and investment policies (with finite horizon) under incomplete market. Merton (1971), He and Pagès (1993), Duffie, Fleming and Zariphopoulou (1993), Svensson and Werner (1993), Koo (1994a,b), Cuoco (1995), among others, consider the problem when investors also have non-traded income.

²For equilibrium pricing models with non-traded income, see, e.g., Scheinkman and Weiss (1986), Lucas (1990), Marcet and Singleton (1990), Telmer (1993), Heaton and Lucas (1995), and Detemple (1995).

³Wang (1993) and Detemple (1994) solve multiperiod pricing models under asymmetric information with specific assumptions on preferences and shock distributions. Judd and Bernardo (1994) consider numerical solutions to equilibrium models under asymmetric information.

use the noisy rational expectations framework, where the allocational trading is introduced exogenously. We, however, use a fully rational expectations framework and explicitly model both the allocational and informational trading in the market. For example, investors' demand in the derivatives securities after their introduction is derived endogenously from their optimal consumption and investment policies. This allows us to analyze the allocational and informational efficiency of the securities market, and their interaction in a unified framework.

The paper is organized as follows. Section 2 defines the model. Section 3 considers a benchmark case in which the securities market is complete. The equilibrium of the economy is analyzed under symmetric and asymmetric information in section 4 and 5, respectively, with incomplete markets. Section 6 concludes. Proofs are given in the appendix.

2. The model.

We consider an economy of a single good (taken also as the numeraire) defined on a continuous time-horizon $[0, \infty)$. The economy consists of two classes of investors, denoted by $i = 1, 2$, with population weight ω and $1 - \omega$, respectively. Investors within the same class are identical. For convenience, we will also refer to any investor in class- i as investor i , $i = 1, 2$.

Let (Ω, \mathcal{F}, P) be a complete probability space, where Ω is the set of states of nature describing the exogenous environment of the economy on $[0, \infty)$, \mathcal{F} the σ -algebra of distinguishable events, and P the probability measure on (Ω, \mathcal{F}) . The uncertainty of the economy is generated by an n -dimensional standard Wiener process defined on (Ω, \mathcal{F}, P) , denoted by w . The exogenous information flow is given by the augmented filtration $\{\mathcal{F}_t : t \in [0, \infty)\}$, which are σ -algebras of \mathcal{F} generated by w .

There is a competitive securities market with $m + 1$ traded securities, indexed by $k = 0, 1, \dots, m$. A complete specification of all the traded securities defines the structure of the securities market. In this paper, security 0 is assumed to be a risk-free security (bond) with the price $B_t = e^{rt}$ where $t \geq 0$ and r is a positive constant. The remaining securities are risky. Let $C_{k,t}$ be the cumulative cash flow on security k ($k \geq 1$), $P_{k,t}$ its market price, C_t and P_t , respectively, the cumulative cash flow and price vector of all risky securities. In particular, security 1 is a risky security (stock) with a total number of shares outstanding being one (per capita), and pays a flow of dividends. Let the cumulative dividend process D_t on the stock be given by

$$D_t = \int_0^t G_s ds + b_D dw_s \quad (1a)$$

$$G_t = G_0 + \int_0^t e^{a\sigma(t-s)} b_G dw_s \quad (1b)$$

where $t \in [0, \infty)$, $a_G < 0$ is a negative constant, b_D and b_G are constant matrices of proper order. Thus, the dividend paid on the stock from t to $t + dt$ is $dD_t = G_t dt + b_D dw_t$, where G_t gives the persistent component in the dividend and $b_D dw_t$ the idiosyncratic component. We will use S_t to denote the stock price at t . The remaining securities, i.e., for $k > 1$, are assumed to be contingent claims of zero net supply. In particular, as security 2 we introduce a derivative security which pays a cash flow at the rate equal to the current stock price S_t . This security is similar to a collar contract in interest rate swaps, hence will be called the collar from now on. Let H_t denote the collar price at t .⁴ For any claim traded in the market, being enforceable requires that its payoff can only be contingent on states of the economy observable to all investors. When all investors have perfect information about the underlying state of the economy, the payoff of a contingent claim can in general be made dependent on realizations of the underlying state. When investors do not all observe the underlying state, the payoff of a traded security should only depend on publicly observable states. It is for this reason that we consider the addition of derivative securities such as the collar in changing the market structure since its payoff depends only on market prices of other traded securities.

In addition to the traded securities, each class of investors are also endowed with non-traded income. Let $N_{i,t}$ denote the cumulative non-traded income of investor i , $i = 1, 2$. It is given by

$$N_{i,t} = \int_0^t (\beta_{i,Y} Y_s + \beta_{i,Z} Z_s) dN_s \quad (2a)$$

$$Y_t = Y_0 + \int_0^t e^{a_Y(t-s)} b_Y dw_s \quad (2b)$$

$$Z_t = Z_0 + \int_0^t e^{a_Z(t-s)} b_Z dw_s \quad (2c)$$

$$N_t = b_N w_t \quad (2d)$$

where $i = 1, 2$, a_Y, a_Z are negative constants, b_N, b_Y, b_Z are constant matrices of proper order. For simplicity, we have assumed that all the non-traded income are risky (i.e., there is no drift for the non-traded income process), and dN_t characterizes shocks to the non-traded income. $(\beta_{i,Y} Y_t + \beta_{i,Z} Z_t)$ then determines investor i 's total exposure to the risk of non-traded income. (Extending the current model to allow a linear drift is straightforward.)⁵

Investors may have different information about the economy. They observe the dividend

⁴The collar contract defined here represents a series of bets on future stock prices. For positive stock prices, $S_t > 0$, the long side of the contract receives payments at rate S_t , while for negative stock prices, the short side receives payments at rate $-S_t$. Note that payments here are in the form of continuous flows instead discrete lumps.

⁵Here we have assumed that investors' non-traded risks (all determined by N_t) are perfectly correlated. Introducing additional independent shocks is straightforward and does not change any qualitative nature of our results.

payments and market prices of all traded securities, their own exposure to the non-traded risk and realizations of their non-traded income. However, they may have different information concerning the rate of dividend growth G_t and other investors' exposure to the non-traded risk. Let $\mathcal{F}_{i,t}$ denote the filtration generated by the information set of investor i at $t \in [0, \infty)$ and $\mathcal{F}_t^{\{C, P\}}$ denote the filtration generated by the path of prices and dividends $\{C_t, P_t\}$, which is the public information set. Clearly, $\mathcal{F}_t^{\{C, P\}} \subseteq \mathcal{F}_{i,t} \subseteq \mathcal{F}_t$, $i = 1, 2$. The investors' information sets will be specified in detail in the different situations to be considered.

For investor i ($i = 1, 2$), let $\{c_{i,t} : t \in [0, \infty)\}$ be his consumption policy and $\{\theta_{i,t} : t \in \infty\}$ his trading policy where $\theta_{i,t}$ denotes his holdings in the risky securities at t . His policy is adapted to $\mathcal{F}_{i,t}$. Consumption policies are restricted to integrable processes, and trading policies are restricted to predictable, square-integrable processes with respect to the gain processes of traded securities [see, e.g., Harrison and Pliska (1981) for a discussion on the requirement of square-integrability]. (Here, the integrability is defined over any finite horizon $[0, T]$).

In order to obtain solutions of the equilibrium, we assume that all investors maximize the expected utility of the following form:

$$\mathbb{E} \left[- \int_t^\infty e^{-\rho(s-t) - \gamma c_{i,s}} ds \middle| \mathcal{F}_{i,t} \right], \quad i = 1, 2 \quad (3)$$

where ρ and γ (both positive) are the time discount coefficient and the relative risk-aversion coefficient, respectively.

The economy as defined above exhibits the following features. The securities market is in general incomplete when $m < n$, i.e., the number of risky securities is less than the dimension of uncertainty [see, e.g., Harrison and Kreps (1979) for a formal definition of market completeness]. The existence of non-traded income and its correlation with returns on traded securities will generate trading in the market for allocational reasons, i.e., investors will use the securities market to allocate the risk of their non-traded income. The existence of asymmetric information in the market about the stock's future payoffs gives rise to the informational trading among investors. They speculate in the market based on their private information and expect to earn excess returns. Although the prices of all risky securities will be determined by market equilibrium, the prices of the risk-free security has been exogenously specified for tractability. The assumption of constant absolute risk-aversion implies that investors' holdings of the risky securities may be independent of their wealth. This assumes away any income effect on the investors' trading policies (for the risky securities) and consequently the equilibrium prices.

In this paper, we consider two possible structures of the securities market. The first market structure is when the stock and the bond are the only traded securities. The second market structure is when the collar is also traded in the market, in addition to the stock and the bond.

We will denote these two market structures as I and II, respectively.

In what follows, we use upper case letters to denote random variables and lower case letters to denote deterministic variables (e.g., constants) with only a few exceptions. We use conventional notation c and θ for investors' choice variables, consumption and security holdings, respectively, and w for standard Wiener processes. For a set of elements (of proper order), let $\text{diag}\{a_1, a_2, \dots, a_k\}$, (a_1, a_2, \dots, a_k) and $\text{stack}\{a_1, a_2, \dots, a_k\}$ denote, respectively, the diagonal matrix, row matrix and column matrix with elements a_1, a_2, \dots, a_k . Also let ι denote the identity matrix of any order and $\text{trace}(\cdot)$ denote the trace of a matrix. Furthermore, we introduce the index matrix $1_{jj'}^{(k,k')}$ defined as a $(k \times k')$ matrix with its (j, j') element being 1 and all other elements being zero. Also, for any two random variables $X_{1,t}$ and $X_{2,t}$, where $dX_{j,t} = a_{j,t}dt + b_{j,t}dw_t$, $j = 1, 2$, let $\sigma_{jj'} \equiv b_j b_{j'}$ denote the instantaneous cross-variation between X_j and $X_{j'}$ and $\sigma_j^2 \equiv \sigma_{jj}$, $j, j' = 1, 2$ [see, e.g., Karatzas and Shreve (1988) for a discussion of cross-variation processes]. Also define $\kappa_{jj'} \equiv \sigma_{jj}^{-1/2} \sigma_{jj'} \sigma_{j'j'}^{-1/2}$ as the instantaneous cross correlation.

In order to be more specific, we further assume that w has the following decomposition $w_t = \text{stack}\{w_{D,t}, w_{G,t}, w_{Y,t}, w_{Z,t}, w_{N,t}\}$, where all the components are standard Wiener processes and mutually independent. ($w_{D,t}, w_{G,t}$, etc., need not be one-dimensional, although they can be assumed so in most of future discussions.) Furthermore, we assume

$$\begin{aligned} b_D &= \sigma_D(\iota, 0, 0, 0, 0), & b_G &= \sigma_G(0, \iota, 0, 0, 0) \\ b_Y &= \sigma_Y(0, 0, \iota, 0, 0), & b_Z &= \sigma_Y(0, 0, 0, \iota, 0), & b_N &= \sigma_N \left(\kappa_{DN}\iota, 0, 0, 0, \sqrt{1 - \kappa_{DN}^2} \right). \end{aligned}$$

This specification about the underlying shocks to the economy has simple interpretations. In particular, $w_{D,t}$ and $w_{G,t}$ characterize shocks to the stock's dividend flow, $w_{Y,t}$ and $w_{Z,t}$ characterize shocks to investors' exposures to the non-traded risk. All the shocks are mutually independent. The above assumptions about the b 's impose specific structure on the correlation among the shocks. In particular, stock dividends and investors' non-traded income are correlated when $\kappa_{DN} \neq 0$. To fix ideas, we maintain the assumption that $\kappa_{DN} > 0$ in our future discussions. The specific correlation structure assumed here will simplify our analysis without great loss of generality on the points we want to make.

For future convenience, define $\Psi_t \equiv \text{stack}\{1, D_t, N_t, G_t, Y_t, Z_t\}$ as the vector of underlying state variables, and $X_t \equiv \text{stack}\{1, G_t, Y_t, Z_t\}$, $\tilde{X}_t \equiv \text{stack}\{1, Y_t, Z_t\}$ as two sub-state vectors. (Here, a constant is included as the first component of the state vector to simplify presentation.) Both X_t and \tilde{X}_t follow Gaussian Markov processes:

$$\begin{aligned} dX_t &= a_X X_t dt + b_X dw_t \\ d\tilde{X}_t &= a_{\tilde{X}} \tilde{X}_t dt + b_{\tilde{X}} dw_t \end{aligned}$$

where $a_x = \text{diag}\{0, a_G, a_Y, a_Z\}$, $b_x = \text{stack}\{0, b_G, b_Y, b_Z\}$, $a_{\bar{x}} = \text{diag}\{0, a_Y, a_Z\}$, and $b_{\bar{x}} = \text{stack}\{0, b_Y, b_Z\}$. Let $\beta_i \equiv \text{stack}\{0, \beta_{i,Y}, \beta_{i,Z}\}$, $i = 1, 2$. (Here, we use 0 to denote matrices of zeros without specifying their order, which can be inferred from the context.) Investor i 's non-traded income can then be expressed as

$$N_{i,t} = \int_0^t \tilde{X}'_s \beta_i b_N dw_s, \quad i = 1, 2.$$

Also define

$$F_t \equiv \mathbb{E} \left[\int_0^\infty e^{-r(s-t)} dC_s \middle| \mathcal{F}_t \right]$$

which gives the expected values of future cash flows on all risky securities, discounted at the risk-free rate. In particular, for the stock, $F_t^S = \frac{1}{r-a_G} G_t$, $b_F^S = \frac{1}{r-a_G} b_D$ and $\sigma_F^2 = b_F^S b_F^{S'}$. Furthermore, let

$$dQ_t \equiv dC_t + dP_t - rP_t dt$$

denote the vector of excess share returns of all risky securities. Its first component $dQ_t^S = dD_t + dS_t - rS_t dt$ gives the dollar return on one share of stock financed by borrowing at the risk-free rate. Similarly, its second component $dQ_t^H = S_t dt + dH_t - rH_t dt$ gives the excess share return on the collar.

Prices of all risky securities are determined by the equilibrium of the economy. The equilibrium notion here is the standard one of rational expectations [see, e.g., Radner (1972)]. It is defined as the price process $\{P_t\}$ such that investors adopt consumption and trading policies that maximize their expected utility

$$\begin{aligned} J_{i,t} &\equiv \sup_{c_i, \theta_i} \mathbb{E} \left[- \int_t^\infty e^{-\rho(s-t) - \gamma c_{i,s}} ds \middle| \mathcal{F}_{i,t} \right] \\ \text{s.t. } dW_{i,t} &= (rW_{i,t} - c_{i,t}) dt + \theta'_{i,t} dQ_t + dN_{i,t} \end{aligned} \quad (4)$$

where $i = 1, 2$, and the market clears

$$\omega \theta_{1,t} + (1-\omega) \theta_{2,t} = 1_{11}^{(m,1)}. \quad (5)$$

The transversality condition of the Merton type (1971) will be imposed on investors' control problem: $\lim_{s \rightarrow \infty} \mathbb{E} [J_{i,s} | \mathcal{F}_{i,t}] = 0$.⁶ We only consider stationary equilibrium of the economy. Furthermore, we maintain the following conditions on the parameters throughout the paper:

$$\sigma_N \sigma_Y < \frac{r-2a_Y}{2\sqrt{2}r\gamma}, \quad \sigma_N \sigma_Z < \frac{r-2a_Z}{2\sqrt{2}r\gamma}. \quad (6)$$

⁶For the infinite horizon control problem to have well posed solutions, appropriate boundary conditions are needed. Imposing the above transversality condition is equivalent to the following procedure: first solve the control problem with finite horizon and a bequest function of the terminal wealth in the same form of the utility function, and then let the terminal date goes to infinity.

in order to guarantee the existence of linear equilibrium. (6) requires that the variability in investors' non-traded income cannot be too large.

3. The benchmark case of complete market.

In this paper, the securities market is incomplete in general. Before analyzing the general cases, we first consider the case in which the market is complete as a benchmark case. It provides some basic understanding about the economy, which is useful in analyzing more complicated cases.

With complete securities market, prices fully reveal investors' private information (or provide an efficient aggregation).⁷

In equilibrium, investors are equally informed. Or equivalently, we can start by assuming the same information set for all investors. For simplicity we assume $\mathcal{F}_{i,t} = \mathcal{F}_t$, $i = 1, 2$. Without loss of generality, also let $\omega\beta_{1,Y} + (1-\omega)\beta_{2,Y} = 1$ and $\beta_Z = \omega\beta_{1,Z} + (1-\omega)\beta_{2,Z} \leq 1$ in this section.

Since the market is complete, investors can completely eliminate the idiosyncratic risks in their non-traded income. Consequently, the equilibrium allocation and security prices only depend on the aggregate risk, including the risk in the stock's future payoffs and aggregate exposure to non-traded risk. Since the equilibrium allocation and security prices does not depend on the market structure imposed as long as it satisfies the spanning property [see, e.g., Duffie and Huang (1985)], we omit detailed specifications of the market structure. We have the following theorem:

Theorem 1 *When $\mathcal{F}_{i,t} = \mathcal{F}_t \forall t$ and the market is complete, the economy has a unique linear, stationary equilibrium. In particular, investors' value function is $J_{i,t} = -e^{-\rho t - r\gamma W_{i,t} + \frac{1}{2}\tilde{X}'_t v \tilde{X}_t}$, their optimal consumption policy is*

$$c_{i,t} = rW_{i,t} - \frac{1}{2\gamma}\tilde{X}'_t v \tilde{X}_t - \frac{1}{\gamma} \ln r$$

the stock price is

$$S_t = \frac{1}{r - a_G} G_t + \tilde{\lambda}^S \tilde{X}_t$$

where

$$\tilde{\lambda}_Y^S = \frac{-r\gamma\sigma_{DN} [v_{YZ}\sigma_Z^2 + (r - a_Z - \sigma_Z^2 v_{ZZ})]}{(r - a_Z - \sigma_Z^2 v_{ZZ})(r - a_Y - \sigma_Y^2 v_{YY}) - v_{YZ}^2 \sigma_Y^2 \sigma_Z^2}$$

⁷When investors have asymmetric information, appropriate enforcement mechanism is needed to introduce securities with payoffs contingent on state variables that are not publicly observable. Otherwise, it would be infeasible to include these securities in the market. This may be one of the reason the actual market is incomplete. If, however, the contractual terms of traded securities can be enforced by a court with superior information on the true state variables, then any state-contingent claims can be traded in the market.

$$\lambda_Z^S = \frac{-r\gamma\sigma_{DN} [v_{YZ}\sigma_Y^2 + (r-a_Y - \sigma_Y^2 v_{YY})]}{(r-a_Z - \sigma_Z^2 v_{ZZ})(r-a_Y - \sigma_Y^2 v_{YY}) - v_{YZ}^2 \sigma_Y^2 \sigma_Z^2}$$

$$\lambda_0^S = -\gamma \left[\sigma_D^2 + \frac{1}{(r-a_G)^2} \sigma_G^2 + \lambda_Y^2 \sigma_Y^2 + \lambda_Z^2 \sigma_Z^2 \right]$$

and v is a (3×3) constant matrix given in appendix A.2. For any other traded security $k > 1$ with cumulative payoff $C_t = \int_0^t f(X_t, t) dt + b_C dw_s$, its price at t is $P(X_t, t)$. If $P(X_t, t)$ is twice differentiable with respect to X_t and once differentiable with respect to t , it then satisfies the following equation:

$$rP = \partial_t P + f + X' a'_x \partial_x P + \frac{1}{2} \text{tr} \left(\sigma_{xx} \partial_x^2 P \right) - [r\gamma \lambda^S b_x + (r\gamma \beta b_N - v b_x) X_t] (b'_x \partial_x P + b'_C) \quad (7)$$

where $\beta = (0, 1, \beta_Z)$, $\partial_x P$ denotes the vector of first order derivatives of P with respect to elements of X , $\partial_x^2 P$ the matrix of second order derivatives, and $\partial_t P$ its derivative with respect to t .⁸

No statements about investors' security holdings are made since no specification of the traded securities is given (other than the bond and the stock).

In order to better understand the results, we consider the simple case when $Z_t = 0 \forall t$ and Y_t fully characterizes the aggregate exposure to non-traded risk. In this case, we can write $v = \text{diag}\{v_{00}, v_{YY}, 0\}$,

$$v_{YY} = \left(2\sigma_Y^2\right)^{-1} \left[(r-2a_Y) - \sqrt{(r-2a_Y)^2 - 4(r\gamma)^2 \sigma_N^2 \sigma_Y^2} \right] > 0$$

$$v_{00} = \frac{1}{r} \sigma_Y^2 v_{YY} + r\gamma \lambda_0^S + \frac{2}{r} (r - \rho - r \ln r)$$

and $\lambda_Y^S = -r\gamma\sigma_{DN}/(r-a_Y - \sigma_Y^2 v_{YY})$ and $\lambda_0^S = -\gamma[\sigma_F^2 + \sigma_D^2 + (\lambda_Y^S \sigma_Y)^2]$. Furthermore, v_{YY} , $|\lambda_0^S|$ and $|\lambda_Y^S|$ increase with σ_Y . It then follows that investors' optimal consumption decreases with their exposure to non-traded risk (i.e., $c_{i,t}$ decreases with Y_t holding wealth constant). This reflects the investors' precautionary saving. When investors face higher risk in their future non-traded income, their marginal utility for future consumption increases. Given the interest rate, they want decrease current consumption and save more for future consumption.

The stock price is a simple linear function of the underlying state variables, $S_t = \frac{1}{r-a_G} G_t + \lambda_0^S + \lambda_Y^S Y_t$. The first term $\frac{1}{r-a_G} G_t$ gives the expected value of the stock's future cash flow discounted at the risk-free rate. $\lambda_0^S + \lambda_Y^S Y_t$ then gives the risk premium on the stock. The constant component of the risk premium λ_0^S is simply proportional to the investors' risk aversion and the instantaneous variance of the stock price. This is because that investors' consumption co-varies

⁸For $X_t = (1, G_t, Y_t, Z_t)'$, its variable elements are G_t, Y_t, Z_t . Let $X_i, i = 1, \dots, n$, be the variable elements of X , then $\partial_x P = \text{stack} \left\{ \frac{\partial P}{\partial X_1}, \dots, \frac{\partial P}{\partial X_n} \right\}$ and $\partial_x^2 P = \left\{ \frac{\partial^2 P}{\partial X_i \partial X_j} \right\}$.

linearly with the stock price (since their wealth does). The covariance between consumption changes and stock returns, which determines the premium, then increases linearly with the variance of stock returns.

The time-varying component of the risk premium is linear in Y_t with proportionality coefficient λ_Y^S . It is easy to show that $r - a_Y - \sigma_Y^2 v_{YY} > 0$. Hence, λ_Y^S has the opposite sign of σ_{DN} . Note that σ_{DN} being positive implies positive correlation between shocks to investors' non-traded income and shocks to the stock's payoff. Investors want to use the stock to hedge the risk in their non-traded income. When $Y_t > 0$, investors have a positive exposure to the non-traded risk. Hence, they would like to short the stock in order to reduce the overall risk of their future income. In equilibrium, the stock price has to decrease. Thus λ_Y^S is negative when $\sigma_{DN} > 0$. Also, $|\lambda_Y^S|$ increases with σ_Y , the variability in the aggregate exposure to non-traded risk. As σ_Y increases, investors' expected utility becomes more sensitive to changes in the exposure to non-traded risk (i.e., v_{YY} increases). Consequently, they use the stock (as well as other traded securities) to hedge more actively against these changes. The stock price then becomes more sensitive to Y_t , hence $|\lambda_Y^S|$ increases.

The price of any derivative securities must satisfy the pricing equation (7). Given the appropriate boundary conditions of a security, its price can be obtained by solving (7). As an example, we solve for the collar price. It pays a dividend at a rate equal to current stock price. H_t should be a function of X_t only, independent of the calendar time t , i.e., $H_t = H(X_t)$. Given that X_t follows a Gaussian Markov process, it can be shown that $H(\cdot)$ is linear. Thus,

$$H_t = H(X_t) = \lambda^H X_t \quad (8)$$

where $\lambda^H = (\lambda_0^H, \lambda_G^H, \lambda_Y^H, \lambda_Z^H)$ is a constant matrix. Substitute this into equation (7), we obtain

$$\lambda^H = \lambda^S \left[r\iota - a_X + (r\gamma)\sigma_{XX}\lambda^{S'}1_{11}^{(1,4)} + (r\gamma\sigma_{XN} - \sigma_{XY}v_{YY})1_{13}^{(1,4)} \right]^{-1}.$$

which fully specifies the equilibrium collar price. (The matrix in the square bracket is full ranked.)

4. The case of symmetric information.

We now consider the case of incomplete market, but with symmetric information, and examine how market incompleteness affects the allocation of risk among investors and security prices. Again, we assume $\mathcal{F}_{i,t} = \mathcal{F}_t$, $i = 1, 2$. We start with some general discussions about the equilibrium of the economy, and then examine the equilibrium under market structures I and II, respectively.

4.1. General discussions on equilibrium.

For the risky securities we allow in market structure I and II (stock and collar), their payoffs are governed by Gaussian processes with drifts linear the underlying state vector Ψ_t , which follows a Gaussian Markov process itself. In particular, we can write

$$C_t = \int_0^t \lambda^C X_s ds + b_C dw_s$$

Given the constant risk-aversion preferences of the investors, the equilibrium can exhibit certain linearity [see, e.g., Campbell and Kyle (1992) and Wang (1993)]. Thus, we restrict our analysis to the linear equilibrium of the economy which requires

$$P_t = \lambda^P X_t$$

for prices of all traded securities. Here, $\lambda^C = (\lambda_0^C, \lambda_G^C, \lambda_Y^C, \lambda_Z^C)$, $\lambda^P = (\lambda_0^P, \lambda_G^P, \lambda_Y^P, \lambda_Z^P)$ and b_C are $(m \times 4)$ constant matrices. The process of excess share returns is then

$$dQ_t = a_Q X_t dt + b_Q dw_t$$

where $a_Q = \lambda^P(a_X - r\iota) + \lambda^C$ and $b_Q = \lambda^P b_X + b_C$. Thus, X_t fully characterizes the expected excess share returns on all risky securities considered here. Also, let $\tilde{\lambda}^C = (\lambda_0^C, \lambda_Y^C, \lambda_Z^C)$ and $\tilde{\lambda}^P = (\lambda_0^P, \lambda_Y^P, \lambda_Z^P)$ denote the sub-vectors of λ^C and λ^P that correspond to \tilde{X}_t , respectively.

We have the following result concerning the properties of the linear, stationary equilibrium of the economy:

Proposition 1 *Suppose that $\mathcal{F}_{i,t} = \mathcal{F}_t \forall t$, $i = 1, 2$. In a linear, stationary equilibrium of the economy, the price vector has the form*

$$P_t = \lambda^P X_t, \tag{9}$$

investor i 's optimal policies ($i = 1, 2$) are

$$c_{i,t} = rW_{i,t} - \frac{1}{2\gamma} \tilde{X}_t' v_i \tilde{X}_t - \frac{1}{\gamma} \ln r, \quad \theta_{i,t} = h_{i,t} \tilde{X}_t \tag{10}$$

and his value function $J_{i,t} = -e^{-\rho t - r\gamma W_{i,t} + \frac{1}{2} \tilde{X}_t' v_i \tilde{X}_t}$, where $i = 1, 2$,

$$h_{i,t} = \frac{1}{r\gamma} \sigma_{QQ}^{-1} (a_Q + \sigma_{Q\tilde{X}} v_i - r\gamma \sigma_{QN} \beta_i'),$$

$a_Q = \lambda^P(a_{\tilde{X}} - r\iota) + \tilde{\lambda}^C$, $b_Q = \lambda^P b_X + b_C$, $\sigma_{QQ} = b_Q b_Q'$, and λ^P , v_i satisfy

$$0 = (r\gamma)^2 h_i' \sigma_{QQ} h_i - (r\gamma \beta_i b_N - v_i b_X) (r\gamma \beta_i b_N - v_i b_X)' + r v_i - (v_i a_X + a_X' v_i) - \bar{v}_i 1_{11}^{(4,4)} \tag{11a}$$

$$1_{11}^{(m,4)} = \omega h_1 + (1-\omega) h_2 \tag{11b}$$

with $\bar{v}_i = 2(r - \rho - r \ln r) + \text{tr} \{b_X' v_i b_X\}$, and λ^S , v_i are constant matrices.

Note that the expected excess share returns on traded securities depend only on \tilde{X}_t , the vector of state variables that affects investors' exposure to non-traded risk, and so do the investors' investment-consumption policies and their value functions. In particular, the expected excess share return on the stock is independent of expectations about its payoffs, which is determined by G_t .

Following Merton (1971, 1989), we can interpret investors' security holdings as consists of three sets of portfolios. The first portfolio is the growth portfolio, $\sigma_{QQ}^{-1}a_Q\tilde{X}_t$. The second set of portfolios, defined by each column of $\sigma_{QQ}^{-1}\sigma_{Q\tilde{X}}$, are portfolios used to hedge changes in the state variables. The third portfolio, defined by $\sigma_{QN}^{-1}\sigma_{QN}$, is the portfolio used to hedge the risk in non-traded income. Investor i 's holding in these three sets of portfolios are $(r\gamma)^{-1}$, $(r\gamma)^{-1}v_i\tilde{X}$ and $\beta_i'\tilde{X}$, respectively.

In the remainder of this section, we let $\omega = 1/2$, $\beta_{i,Y} = 1$ and $\beta_{i,Z} = (-1)^{i-1}$, $i = 1, 2$. In this case, Y_t characterizes the aggregate exposure to non-traded risk and Z_t characterizes the idiosyncratic exposure.

4.2. Under market structure I.

We now consider the case when the stock and the bond are the only traded securities. The market is incomplete in this case since there are no securities that allow investors to perfectly hedge their non-traded risk (as characterized by dN_t) as well as changes in their exposures to non-traded risk [as characterized by $(\beta_{i,Y}Y_t + \beta_{i,Z}Z_t)$]. We have the following result on the equilibrium of the economy:

Theorem 2 *When $\mathcal{F}_{i,t} = \mathcal{F}_t \forall t$, $\omega = 1/2$, $\beta_{i,Y} = 1$, $\beta_{i,Z} = (-1)^{i-1}$, $i = 1, 2$, and under market structure I, the economy has a linear, stationary equilibrium of the form in proposition 1 for small values of σ_Z . In particular, the stock price and investors' stock holdings are*

$$S_t = \frac{1}{r - a_G} G_t + \tilde{\lambda}^S \tilde{X}_t$$

$$\theta_{i,t} = 1 + \sigma_{DN} \left(\sigma_{QN} \beta_{i,Z} - \lambda_Y^S \sigma_Y^2 v_{i,YZ} \right) Z_t$$

where $\tilde{\lambda}^S$ and v_i are determined by equation (11) with $a_Q = \tilde{\lambda}^S (a_{\tilde{X}} - r\iota)$ and $b_Q = \lambda^S b_X + b_D$.

Given the symmetry between the two classes of investors, the aggregate exposure is evenly allocated among themselves. Changes in aggregate exposure would not generate trading among investors since it affects everyone the same way. Investors' stock holdings only depend on their idiosyncratic exposures to the non-traded risk.

Since the market is incomplete now, the equilibrium stock price and allocation in general depend on investors' idiosyncratic exposure to non-traded risk. In particular, investors have to use the stock to hedge their non-traded risks. This leads to inefficiencies in both allocating aggregate risk and mutually insuring individual risks. In order to develop some intuition about the equilibrium allocations and the stock price, we first consider special cases in which the aggregate and idiosyncratic exposures to the non-traded risk do not coexist and then discuss the more general case in which they do.

4.2.1. With only aggregate non-traded risk.

One special situation is when there is no idiosyncratic exposure in investors' non-traded risk, i.e., when $Z_t = 0 \forall t$. In this case, all investors are identical and the market is effectively complete [see, e.g., Lucas (1978)]. The equilibrium allocation and the stock price are the same as in the case of complete market, which are given in theorem 1.

4.2.2. With only idiosyncratic non-traded risk.

Another special situation is when there is no aggregate exposure in investors' non-traded risk, i.e., when $\beta_{i,Y} = 0$. Investors, however, do face idiosyncratic non-traded risk, which cannot be eliminated. The equilibrium then has the form in proposition 1 with $\tilde{\lambda}^S = (\lambda_0^S, 0, 0)$ and $h_i = (1, 0, h_{i,Z})$ where $\lambda_0^S = -\gamma(\sigma_F^2 + \sigma_D^2)$, $h_{i,Z} = \beta_{i,Z}\sigma_{DN}/(\sigma_F^2 + \sigma_D^2)$ and

$$v_{i,ZZ} = \frac{1}{2\sigma_Z^2} \left[(r - 2a_Z) - \sqrt{(r - 2a_Z)^2 - 4(r\gamma)^2\sigma_Z^2(\sigma_N^2 - \sigma_{QQ}^{-1}\sigma_{DN}^2)} \right]$$

$$v_{i,0Z} = \frac{(r\gamma)^2\sigma_{DN}}{r - a_Z - \sigma_Z^2 v_{i,ZZ}} \beta_{i,Z}, \quad v_{i,00} = -r\gamma^2(\sigma_F^2 + \sigma_D^2) + \frac{1}{r}\sigma_Z^2(v_{i,0Z} + v_{i,ZZ}) + \frac{2}{r}(r - \rho - r \ln r).$$

In this case, the distribution of investors' exposure to non-traded risk, as determined by a_Z and σ_Z , does not affect the equilibrium stock price. As a matter of fact, the stock price is the same here as in the case of complete market (without non-traded risk at the aggregate level). Given any idiosyncratic exposure of the individual investors to the non-traded risk, some investors sell the stock while others buy to hedge their non-traded risk. With the perfect symmetry among investors in this case, the net buying/selling will be zero. All the trades can be accommodated without moving the price. Investors' utility, however, does depend on the distribution of their exposure to non-traded risk since they cannot perfectly insure their individual through the securities market. The stock only provides an imperfect hedge. When an investor uses the stock to hedge, he incurs the basis risk from price movements as G_t changes. Increasing σ_G increases the basis risk and decreases $|h_{i,Z}|$, hence decreases investors' hedging activities.

4.2.3. With both aggregate and idiosyncratic non-traded risk.

When there are both aggregate and idiosyncratic components in investors' exposures to the non-traded risk, the interaction between them will affect the stock price and investors' trading policies. In this case, we no longer have closed form solutions to the equilibrium. Thus we rely on numerical solutions to equation (11) in analyzing the equilibrium.

Since our future discussions will be based on numerical illustrations, a few comments are in order. Given the large number of parameters in the model, only the results for a small range of parameter values are presented for brevity. The parameter values are chosen to be compatible to the estimated price process in Campbell and Kyle (1993), which has the linear form similar to ours. The remaining degrees of freedom are used to fix a particular set of parameter values which can generate simultaneously all the results in the paper. As a cost, some of the effects may seem small for this particular set of parameter values even though they can be larger for other parameter values. When a particular result under consideration changes qualitatively with certain parameters, we try to show the changes by varying the relevant parameters in the numerical illustrations or to discuss them verbally. However, our exploration of the parameter space and the results presented in the paper are by no means exhaustive.

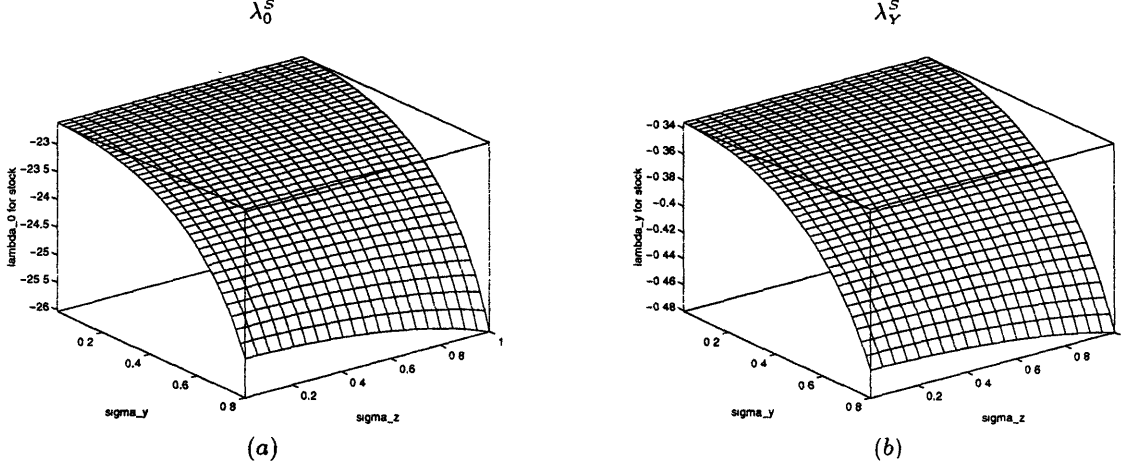


Figure 1: Stock price coefficients plotted against σ_Y and σ_Z under symmetric information and market structure I. The parameters are set at the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.05$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.2$, $\sigma_N = 0.2$, $\kappa_{DN} = 0.6$.

Figure 1 shows how λ_0^S and λ_Y^S change with σ_Y and σ_Z , measuring the variability in aggregate and individual exposures to the non-traded risk, respectively. Same as in the case of complete market, λ_0^S and λ_Y^S are both negative and their absolute values increase with σ_Y . Furthermore, as we increase the dispersion between the investors in non-traded risk σ_Z , both $|\lambda_0^S|$ and $|\lambda_Y^S|$ increase. This is different from the results in the absence of the aggregate non-traded risk. When

investors face more idiosyncratic risk, which cannot be completely hedged away due to market incompleteness, the aggregate risk becomes even more undesirable since it is correlated with the idiosyncratic risks. This implies that the absolute value of λ_Y^S increases and so does the price volatility. Consequently, the stock's risk premium as measure by $|\lambda_0^S|$ increases.

We now examine investor 1's trading strategies. Given that his stock holding is a linear function of the underlying state variables, $\theta_{1,t}^S = h_{1,0}^S + h_{1,Y}^S Y_t + h_{1,Z}^S Z_t$, we can simply look at the coefficients, $h_{1,0}$, $h_{1,Y}$ and $h_{1,Z}$, which characterizes the intensity of his trading in response to the shocks. Given the symmetry between the two classes of investors in the current case, $h_{1,0}^S = 1$ and $h_{1,Y}^S = 0$, implying that on average each investor holds his market share of the stock and bears equal amount of the aggregate non-traded risk. However, $h_{1,Z}^S$ is not zero, reflecting the trading between the two classes of investors to mutually share their individual risks. Figure 2 illustrates how $h_{1,Z}$ changes with σ_Y and σ_Z . Note the $h_{2,Z} = -h_{1,Z}$ as the market clearing condition requires. $h_{1,Z}^S$ is negative (for $\sigma_{DN} > 0$) since investor 1 wants to short the stock when he has a positive exposure to non-traded risk.

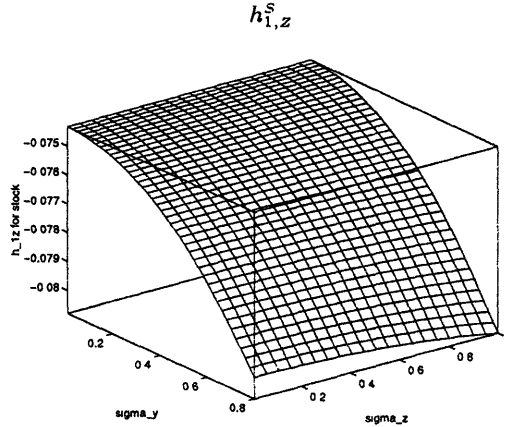


Figure 2: Investor 1's trading policies under symmetric information and market structure I plotted against σ_Y and σ_Z . The parameters are set at the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.05$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.2$, $\sigma_N = 0.2$, $\kappa_{DN} = 0.6$.

As σ_Y and σ_Z change, two factors affect the intensity of the hedging activity, measured by $|h_{1,Z}|$. First, when σ_Y or σ_Z increases, the stock price becomes more volatile (since λ_Y^S increases). Taking any stock positions becomes riskier and investors hence reduce their trading in hedging individual risks. Second, increasing σ_Y or σ_Z makes investors' expected utility more sensitive to changes in their exposures to non-traded risk, hence increases their trading in hedging against these changes. The net effect of an increase in σ_Y or σ_Z on $|h_{1,Z}|$ depends on the trade off between these two factors. For the parameters chosen in figure 2, $|h_{1,Z}|$ increases with σ_Y or σ_Z .

4.3. Under market structure II.

We now consider the case when the collar is added to the market. With both the stock and the collar being traded, investors can synthesize a larger set of possible payoffs. In particular, they can construct trading strategies to better hedge their individual risks without bearing too much additional risk. The new trading opportunities created by the introduction of derivative security certainly affects the stock price and the equilibrium allocation. We have the following proposition about the equilibrium:

Theorem 3 *When $\mathcal{F}_{i,t} = \mathcal{F}_t \forall t$, $\omega = 1/2$, $\beta_{i,Y} = 1$, $\beta_{i,Z} = (-1)^{i-1}$ where $i = 1, 2$, and under market structure II, the economy has a linear, stationary equilibrium of the form in proposition 1 for small values of σ_Y and σ_Z . In particular, the security prices and holdings are*

$$S_t = \frac{1}{(r-a_G)}G_t + \tilde{\lambda}^S \tilde{X}_t, \quad H_t = \frac{1}{(r-a_G)^2}G_t + \tilde{\lambda}^H \tilde{X}_t$$

$$\theta_{i,t} = h_{i,0} - \sigma_{QQ}^{-1} \left(\sigma_{DN} \beta_{i,Z} - \sigma_Y^2 \lambda_Y v_{i,YZ} \right) Z_t$$

where $\tilde{\lambda} \equiv \text{stack}\{\tilde{\lambda}^S, \tilde{\lambda}^H\}$, $\tilde{\lambda}^S = (\lambda_0^S, \lambda_Y^S, 0)$, $\tilde{\lambda}^H = (\lambda_0^H, \lambda_Y^H, 0)$, and v_i are determined by equation (11) with $a_Q \equiv \tilde{\lambda}(a_{\tilde{X}} - r) + \tilde{\lambda}^S 1_{11}^{(2,1)}$ and $b_Q \equiv \tilde{\lambda} b_X + \text{stack}\{b_D, 0\}$.

The equilibrium looks formally similar to that under market structure I. Figure 3 shows how the price coefficients of the stock and collar change with σ_Y and σ_Z . Qualitatively, the stock price coefficients behave the same way as under market structure I.

In order to see more clearly how the introduction of collar changes investors' hedging strategies, let us first consider the special case when $\sigma_Y = 0$, i.e., there is no aggregate exposure to non-traded risk. When $\tilde{Z}_t > 0$, investor 1's non-traded income is positively correlated with the stock's dividends. He then wants to short the stock to hedge the non-traded risk. The stock, however, is an imperfect hedging vehicle since its price can change as G_t changes, which gives rise to a basis risk. When the collar is introduced, its price only depends on G_t . Consequently, investors can use it to completely eliminate the basis risk. In particular, investor 1 can short the stock to hedge the non-traded risk and long the collar to offset the basis risk incurred from his stock position. The net position of the stock and the collar provides a perfect hedge against the hedgable part of the non-traded risk.⁹ Investors now can hedge more aggressively using the off-setting positions in the stock and the collar. Consequently, the level of trading in the stock increases significantly as the collar is introduced. The results carry over to the general case when

⁹The non-traded risk also has a non-hedgable part, associated with $w_{N,t}$, which is independent of the returns on all traded securities (i.e., the stock and the collar).

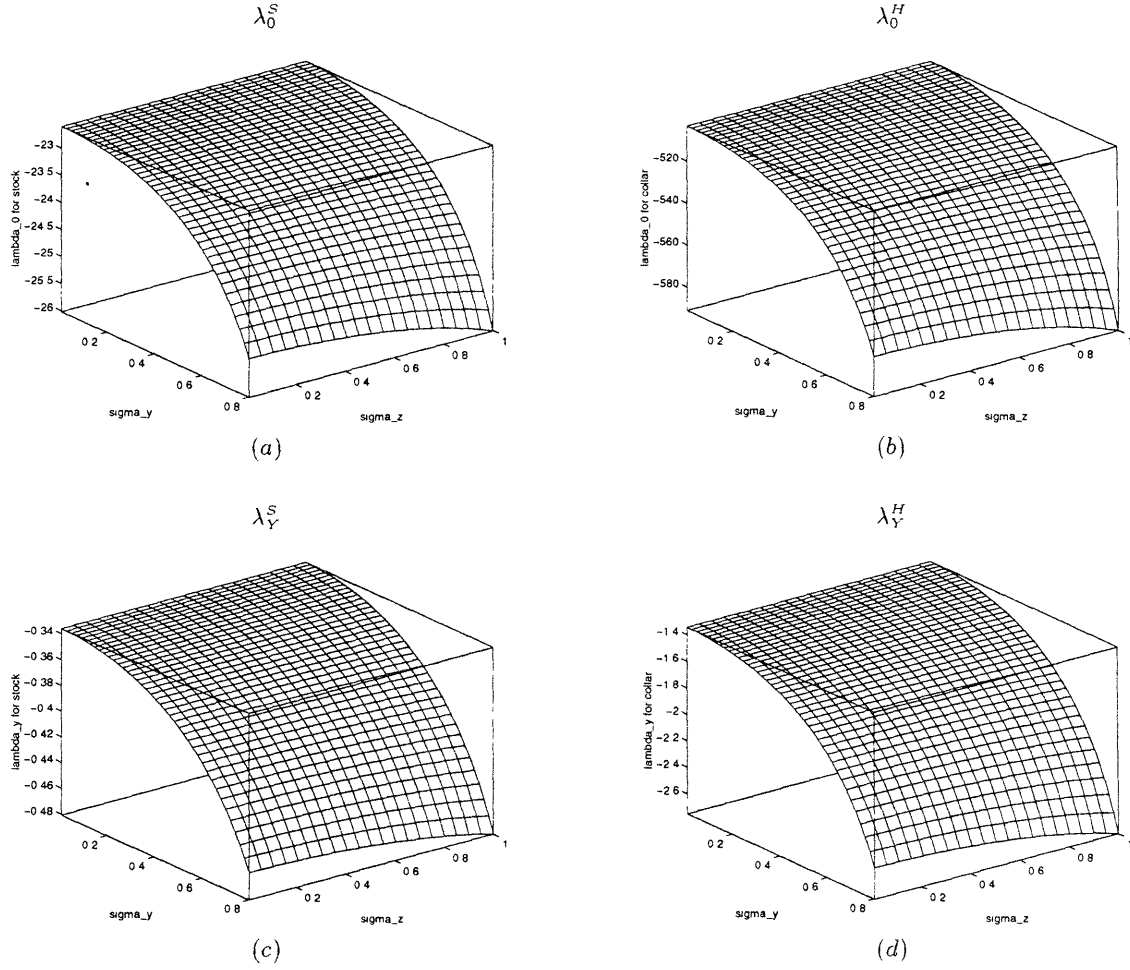


Figure 3: Price coefficients of the stock and collar plotted against σ_Y and σ_Z under symmetric information and market structure II. The parameters are set at the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.05$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.2$, $\sigma_N = 0.2$, $\kappa_{DN} = 0.6$.

there is aggregate exposure to non-traded risk, except that now the collar can only be used to partially reduce the basis risk of stock.

Figure 4 illustrates the different components in investor 1's trading strategies in both markets. Only $h_{1,z}^S$ and $h_{1,z}^H$ are shown there since the other components are the same as under market structure I. As discussed earlier, $h_{1,z}^S < 0$ since investor 1 uses the stock to hedge his non-traded risk, and $h_{1,z}^H > 0$ since he uses the collar to reduce the basis risk. Comparing figure 4 with figure 2, we observe that the absolute value of $h_{1,z}^S$ is much larger under market structure II than under market structure I, reflecting the increase in stock trading when the collar is introduced.

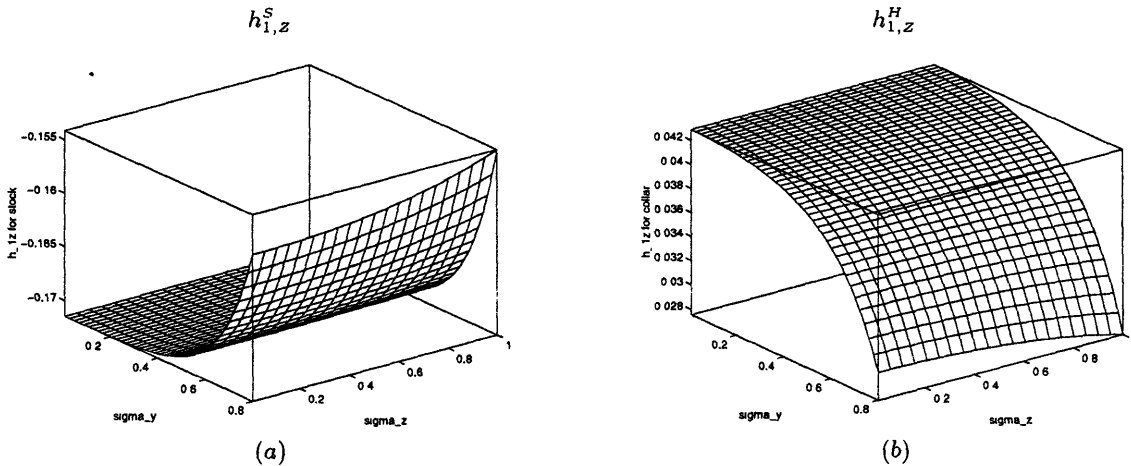


Figure 4: Investor 1's trading policies, plotted against σ_Y and σ_Z under symmetric information and market structure II. The parameters are set at the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.05$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.2$, $\sigma_N = 0.2$, $\kappa_{DN} = 0.6$.

In order to see more clearly the impact of derivative trading on the equilibrium, we illustrate in figure 5 the differences in stock risk premium and price volatility between market structure I and II. From the above discussion, opening trading in the collar allows investors to better hedge their idiosyncratic non-traded risk. This tends to reduce the individual risk they are left with. Consequently, the stock price becomes less sensitive to changes in the aggregate exposure to non-traded risk. Thus, the risk premium and price volatility of the stock decrease. Figure 5(a) shows that the risk premium on the stock, as measured by $|\lambda_0^S|$, decreases as the derivatives market opens, and figure 5(b) shows that the stock price volatility also decreases.

5. The case of asymmetric information.

We now study the equilibrium of the economy under asymmetric information. We will examine how market incompleteness affects its information efficiency, the resulting equilibrium allocation and security prices.

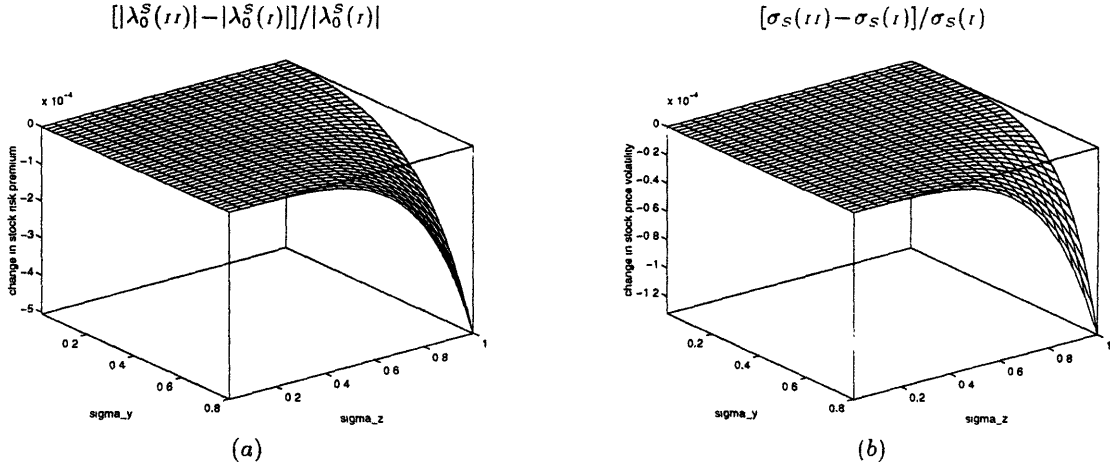


Figure 5: Percentage change in stock risk premium $|\lambda_0^S|$ and price volatility σ_S from market structure I to II under symmetric information, plotted against σ_Y and σ_Z . The parameters are set at the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.05$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.2$, $\sigma_N = 0.2$, $\kappa_{DN} = 0.6$.

For simplicity, we let $\beta_{1,Y} = \beta_{1,Z} = 1$ and $\beta_{2,Y} = \beta_{2,Z} = 0$ in this section. Hence only class-1 investors face non-traded risk. The non-traded income of investor 1 now has the form $dN_{1,t} = (Y_t + Z_t)b_N dw_t$. We will assume that $|a_Y| < |a_Z|$. Thus Y_t and Z_t correspond, respectively, to the relatively more persistent and more transitory components of investor 1's exposure to non-traded risk. We assume that class-1 investors observe the actual dividend growth G_t , their exposure to non-traded risk, Y_t and Z_t , as well as realizations of their non-trade income N_t . Class-2 investors, however, only observe publicly available information, which include dividends paid on all traded securities and their market prices. Thus, $\mathcal{F}_{1,t} = \mathcal{F}_t$ and $\mathcal{F}_{2,t} = \mathcal{F}_t^{\{C, P\}} \subseteq \mathcal{F}_{1,t}$. The actual form of $\mathcal{F}_{2,t}$ crucially depends on the market structure.

In what follows, we first make some general statements about the equilibrium under asymmetric information in the current setting. The actual equilibrium is then examined under market structure I and II.

5.1. General discussions on equilibrium.

Under asymmetric information, class-2 investors do not observe all the underlying state variables, in particular the actual dividend growth and class-1 investors' exposure to the non-traded risk. Their trading policies then depend on their expectations of the unobserved variables. Consequently, their expectations will affect the market prices of traded securities and the equilibrium allocations. Thus the state of the economy under asymmetric information depend not only on the underlying state of the economy, but also on the less informed investors' expectations. Let $\hat{\cdot} \equiv E[\cdot | \mathcal{F}_{2,t}]$ denote class-2 investors' conditional expectation of variable \cdot . In the economy de-

defined here, $\Theta_t \equiv \text{stack}\{G_t, Y_t, Z_t\}$ is the vector of state variables that class-2 investors do not directly observe. $\hat{\Theta}_t \equiv E[\hat{\Theta}|\mathcal{F}_{2,t}]$ then gives their conditional expectation and $\Delta_t \equiv \hat{\Theta}_t - \Theta_t$ their estimation error. Note that $E[\Delta_t|\mathcal{F}_{2,t}] = 0$. Replacing X_t as a state vector, we now need $X_{1,t} \equiv \text{stack}\{1, G_t, Y_t, Z_t, \Delta_t\} = \text{stack}\{X_t, \Delta_t\}$. More generally, higher moments of class-2 investors' conditional expectation can also come in as part of the state vector. However, in a linear equilibrium we consider in this paper, the first moments will be sufficient. For future convenience, also define $X_{2,t} \equiv \text{stack}\{1, \hat{\Theta}_t\} = E[X_t|\mathcal{F}_{2,t}] = \hat{X}_t$. For convenience in exposition, let $\Delta G_t \equiv \hat{G}_t - G_t$ and $\Delta Y_t \equiv \hat{Y}_t - Y_t$.¹⁰

In a linear equilibrium, the expected cash flows and market prices of all traded securities are linear functions of the state variables. The linearity requires that

$$C_t = \int_0^t (\lambda^C X_{1,s} ds + b_C dw_s) \quad (12a)$$

$$P_t = \lambda^P X_{1,t}. \quad (12b)$$

Let $\lambda^P \equiv (\lambda_0^P, \lambda_\Theta^P, \lambda_\Delta^P)$ and $\lambda^C \equiv (\lambda_0^C, \lambda_\Theta^C, \lambda_\Delta^C)$. Since $C_t, P_t \subseteq \mathcal{F}_{2,t}$, we must have $(\lambda_\Theta^P - \lambda_\Delta^P)\Theta_t = (\lambda_\Theta^P - \lambda_\Delta^P)\hat{\Theta}_t$. If the rank of λ_Δ^P is greater than or equal to three, which is the number of state variables unobservable to investor 2, the equilibrium will be fully revealing. Thus in the remaining discussions, we assume that the rank of $(\lambda_\Theta^P - \lambda_\Delta^P)$ is less than 3 and the equilibrium is not fully revealing. Furthermore, we can write $(\lambda_\Theta^P - \lambda_\Delta^P)\Delta_t = 0$. Let $m = 3 - \text{rank}\{\lambda_\Theta^P - \lambda_\Delta^P\}$. It follows that the dimension of investor 2's estimation errors is $3 - m$. Let $\tilde{\Delta}_t$ be the subvector of Δ_t that is not mutually dependent. We then have $\Delta_t = \alpha \tilde{\Delta}_t$, where α is full-ranked matrix of order $3 \times (3 - m)$. For future convenience, we define the reduced form sub-state vectors, $\tilde{X}_{1,t} \equiv (1, Y_t, Z_t, \tilde{\Delta}_t)$ and $\tilde{X}_{2,t} \equiv (1, \hat{Y}_t, \hat{Z}_t)$.

Note that observing P_t is equivalent to observing $(\lambda_\Theta^P - \lambda_\Delta^P)\Theta_t$. Define $\Phi_t \equiv \text{stack}\{C_t, (\lambda_\Theta^P - \lambda_\Delta^P)\Theta_t\}$. Φ_t gives the information of class-2 investors, i.e., $\mathcal{F}_{2,t} = \mathcal{F}_t^{\{\Phi\}}$. We now can derive class-2 investors' expectation of Θ_t given $\mathcal{F}_{2,t}$. First,

$$d\Theta_t = a_\Theta \Theta_t + b_\Theta dw_t$$

where $a_\Theta = \text{diag}\{a_G, a_Y, a_Z\}$ and $b_\Theta = \text{stack}\{b_G, b_Y, b_Z\}$. Next note that

$$d\Phi_t = \left(a_0^\Phi + a_\Theta^\Phi \Theta_t + a_\Theta^\Phi \hat{\Theta}_t \right) dt + b_\Phi dw_t$$

where $a_0^\Phi \equiv \text{stack}\{\lambda_0^C, 0\}$, $a_\Theta^\Phi \equiv \text{stack}\{(\lambda_\Theta^C - \lambda_\Delta^C), (\lambda_\Theta^P - \lambda_\Delta^P)a_\Theta\}$, $a_\Theta^\Phi \equiv \text{stack}\{\lambda_\Delta^C, 0\}$, and $b_\Phi \equiv \text{stack}\{b_C, (\lambda_\Theta^P - \lambda_\Delta^P)b_\Theta\}$. Let $o \equiv E[\Delta_t|\mathcal{F}_{2,t}]$ denote the conditional covariance matrix of the

¹⁰For a more detailed discussion on the nature of equilibrium under asymmetric information, see Wang (1993).

unobserved state variables by class-2 investors. It then follows that o is deterministic and

$$d\hat{\Theta}_t = a_\Theta \hat{\Theta}_t dt + k (d\Phi_t - E[d\Phi_t | \mathcal{F}_{2,t}]) \quad (13a)$$

$$\dot{o} = (a_\Theta o + oa'_\Theta) + b_\Theta b'_\Theta - k (b_\Phi b'_\Phi) k' \quad (13b)$$

where $k = (oa'_\Phi + b_\Theta b'_\Phi)(b_\Phi b'_\Phi)^{-1}$ [see, e.g. Lipster and Shriyayev (1979)]. Also,

$$d\tilde{\Delta}_t = a_{\tilde{\Delta}} \tilde{\Delta}_t dt + b_{\tilde{\Delta}} dw_t$$

where $a_{\tilde{\Delta}}$ and $b_{\tilde{\Delta}}$ are the sub-matrices of the first $(3-m)$ rows of matrices $(a_\Theta - ka_\Phi)\alpha$ and $kb_\Phi - b_\Theta$, respectively. Furthermore,

$$d\tilde{X}_{i,t} = a_{i,\tilde{X}} \tilde{X}_{i,t} dt + b_{i,\tilde{X}} dw_{i,t}, \quad i = 1, 2$$

where $a_{1,\tilde{X}} = \text{diag}\{0, a_\Theta, a_{\tilde{\Delta}}\}$, $b_{1,\tilde{X}} = \text{stack}\{0, b_\Theta, b_{\tilde{\Delta}}\}$, $a_{2,\tilde{X}} = \text{diag}\{0, a_\Theta\}$, $b_{2,\tilde{X}} = \text{stack}\{0, kb_\Phi\}$, $dw_{1,t} = dw_t$ and $dw_{2,t} = (b_\Phi b'_\Phi)^{-1} b_\Phi (d\Phi_t - E[d\Phi_t | \mathcal{F}_{2,t}])$.

Given the dynamics of investor 2's conditional expectations, the excess share returns on the risky securities can be expressed as

$$dQ_t = dC_t + dP_t - rP_t dt = a_{1,Q} X_{1,t} dt + b_Q dw_t$$

where $a_{1,Q} = \lambda^P (a_{1,\tilde{X}} - rI) + \lambda^C$ and $b_Q = \lambda^P b_{1,\tilde{X}} + b_C$. The expected excess share returns then are

$$E[dQ_t | \mathcal{F}_{i,t}] = a_{i,Q} X_{i,t} dt, \quad i = 1, 2$$

where $a_{2,Q} = [\lambda_{\tilde{X}}^C + \lambda_{\tilde{X}}^P (a_{2,\tilde{X}} - rI)]$. Since $X_{i,t}$ follows a Gaussian Markov process under the filtration $\mathcal{F}_{i,t}$, it fully characterizes the investment opportunities for investor i .

Applying the results in appendix A.1, we obtain the following proposition about the equilibrium under asymmetric information:

Proposition 2 *Suppose that $\mathcal{F}_{1,t} = \mathcal{F}_t$, $\mathcal{F}_{2,t} = \mathcal{F}_t^{C,P}$, $\beta_{i,Y} = \beta_{i,Z} = \frac{1}{2}[1 + (-1)^{i-1}]$, and $i = 1, 2$. In a linear, stationary equilibrium of the form (12), the investors' optimal policies and value functions are*

$$\begin{aligned} c_{i,t} &= rW_{i,t} - \frac{1}{2\gamma} \tilde{X}'_{i,t} v_i \tilde{X}_{i,t} - \frac{1}{\gamma} \ln r, \quad \theta_{i,t} = h_i \tilde{X}_{i,t} \\ J_{i,t} &= -e^{-\rho t - r\gamma W_{i,t} + \frac{1}{2} \tilde{X}'_{i,t} v_i \tilde{X}_{i,t}} \end{aligned}$$

where $i = 1, 2$, $h_i = (r\gamma b_Q b'_Q)^{-1} (a_{i,Q} + b_Q b'_{i,X} v_i - r\gamma b_Q b'_N \beta'_i)$. Here, v_i and λ satisfy the following equations:

$$0 = (r\gamma)^2 h'_i (b_Q b'_Q) h_i - (r\gamma \beta_i b_N + v_i b_{i,X}) (r\gamma \beta_i b_N + v_i b_{i,X})' + r v_i - (a'_{i,X} v_i + v_i a_{i,X})$$

$$- \bar{v}_i 1_{11}^{(d_i, d_i)} \quad (14a)$$

$$1_{11}^{(m,1)} = \omega h_1 + (1-\omega) h_2 \tau \quad (14b)$$

where $\bar{v}_i = 2(r - \rho - r \ln r) + \text{tr}(b'_{i,x} v_i b_{i,x})$, $d_i = 5 + 2(-1)^{i-1}$ and $\tau = (\iota, \tilde{\alpha})$.

Similar to proposition 1, the above proposition only provides a characterization of a linear, stationary equilibrium of the economy under asymmetric information if it exists. In the rest of this section, we will examine the existence and the properties of equilibrium under market structure I and II.

5.2. Equilibrium under market structure I.

The economy under asymmetric information and market structure I is similar in spirit to the model considered in Wang (1993). Instead of exogenously introduce noise into the market, here we explicitly model investors' allocational trading which endogenously introduces noise into the market and prevents market prices from revealing too much information. We have the following result on the equilibrium under market structure I.

Theorem 4 *When $\mathcal{F}_{1,t} = \mathcal{F}_t$, $\mathcal{F}_{2,t} = \mathcal{F}_t^{(C,P)}$, $\beta_{i,Y} = \beta_{i,Z} = \frac{1}{2}[1 + (-1)^{i-1}]$, where $i = 1, 2$, and under market structure I, there exists a linear stationary equilibrium of the form in theorem 2 for ω close to 1. In particular, $\tilde{X}_t = \tilde{X}_{1,t} = (1, Y_t, Z_t, \hat{G}_t - G_t, \hat{Y}_t - Y_t)'$, $\tilde{X}_{2,t} = (1, \hat{Y}_t, \hat{Z}_t)$,*

$$\begin{aligned} S_t &= \tilde{\lambda}^s \tilde{X}_t \\ \theta_{i,t} &= h_i \tilde{X}_{i,t} \end{aligned}$$

where $\tilde{\lambda}^s = (\lambda_0^s, \lambda_Y^s, \lambda_Z^s, \lambda_{\Delta Y}^s, \lambda_{\Delta Z}^s)$, $h_1 = (h_{1,0}, h_{1,Y}, h_{1,Z}, h_{1,\Delta G}, h_{1,\Delta Y})$ and $h_2 = (h_{2,0}, h_{2,Y}, h_{2,Z})$ are determined by (14).

The state vector now includes class-2 investors' estimation errors of G_t and Y_t . Given that they observe the stock price, which reveals the following combination of the unobserved state variables $(\frac{1}{r-a_G} - \lambda_G^s) G_t + (\lambda_Y^s - \lambda_{\Delta Y}^s) Y_t + (\lambda_Z^s - \lambda_{\Delta Z}^s) Z_t$, the estimation error in Z_t can be expressed as a linear combination of the estimation errors of G_t and Y_t .

The above existence result is based on a continuation argument (see appendix B). Knowing that a unique equilibrium exists at $\omega = 1$, we can show the generic existence of an equilibrium for ω close to one by appealing to the continuity of equation (14). The proof, however, does not yield explicitly the range of ω where the continuation can be extended to. In our numerical solutions, we always start from the equilibrium at $\omega = 1$ and then reduce ω gradually to get to the particular value we want.

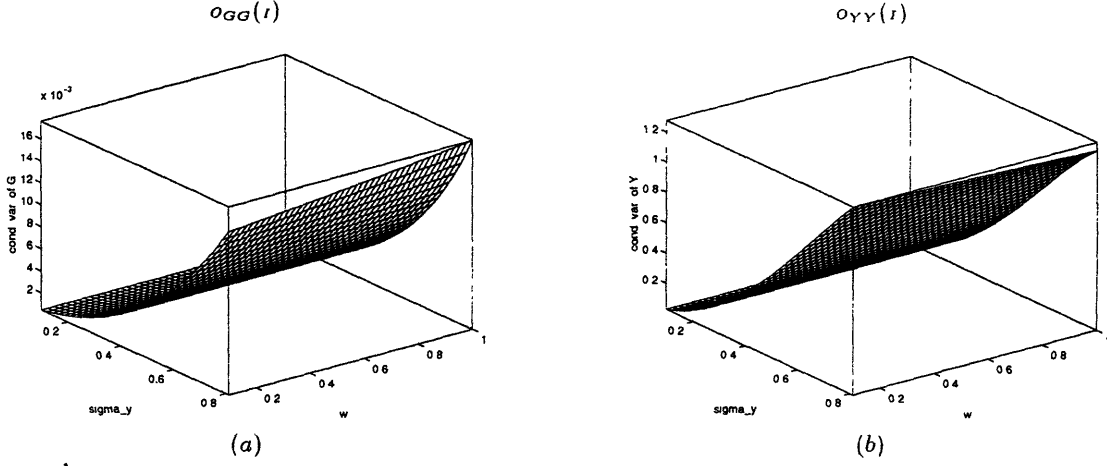


Figure 6: Conditional variance of G_t and Y_t for class-2 investors under market structure I and asymmetric information. The parameters are set at the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.05$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.2$, $\sigma_N = 0.2$, $\kappa_{DN} = 0.6$.

An important element in characterizing the equilibrium is the information asymmetry between the two classes of investors. It can be measured by investor 2's conditional variance of the unobserved state variables, $o_{GG} \equiv E[(\hat{G}_t - G_t)^2 | \mathcal{F}_{2,t}]$ and $o_{YY} \equiv E[(\hat{Y}_t - Y_t)^2 | \mathcal{F}_{2,t}]$. o_{GG} measures the information asymmetry concerning future cash flows of the stock, and o_{YY} measures the information asymmetry concerning class-1 investors' hedging needs. The conditional variance of Z_t , o_{ZZ} , is omitted since it can be expressed as linear function of o_{GG} and o_{YY} (see the discussions in section 4.1). Figure 6 plots o_{GG} and o_{YY} against σ_Y and ω . The effects of σ_Z on o_{GG} and o_{YY} are similar to those of σ_Y .

As σ_Y increases, class-1 investors trade more in the stock to hedge their non-traded risk. The increase in allocational trading introduces additional movements in the stock price and reduces its informativeness about future cash flows. Thus o_{GG} increases with σ_Y . Increasing ω , however, has two off-setting effects. On the one hand, as the fraction of informed investors increases, more information is impounded into the prices through their speculative trading. On the other hand, the amount of allocational trading also increases when more informed investors are hedging their non-traded risk. The net change in o_{GG} as ω increases depends on which of the two effects dominates. In the case shown in figure 6(a), o_{GG} increases with ω . The behavior of o_{YY} can be analyzed similarly. In particular, increasing σ_Y increase the variability in Y_t , hence can increase o_{YY} . But it also increases the allocational trading and stock price sensitivity to changes in Y_t , hence makes the price more informative about Y_t .

Figure 7 illustrates how the stock risk premium and price volatility change with the uncertainty in class-1 investors' exposure to non-traded risk σ_Y and their population weight ω ,

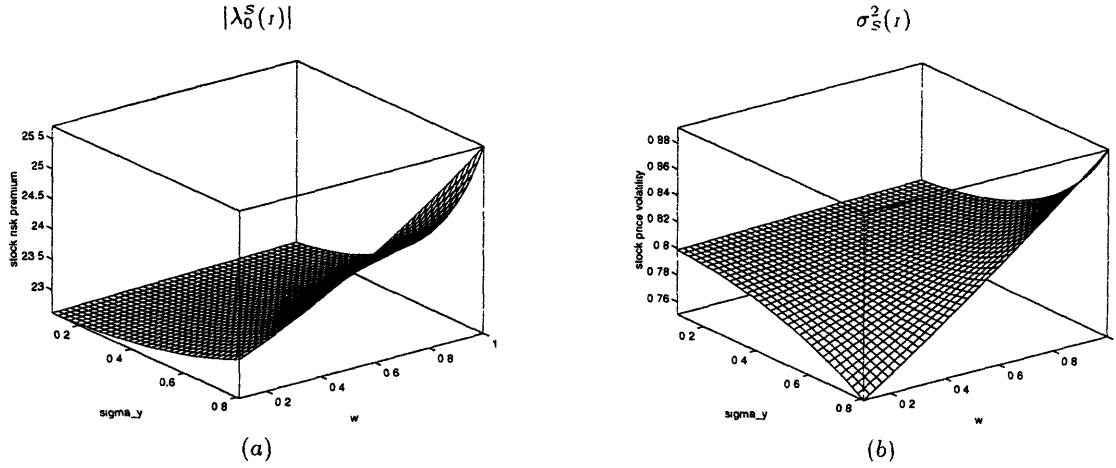


Figure 7: Stock risk premium and price volatility under market structure I and asymmetric information. The parameters are set at the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.05$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.2$, $\sigma_Z = 0.4$, $\sigma_N = 0.2$, $\kappa_{DN} = 0.6$.

respectively. For the given set of parameter values, the risk premium $|\lambda_0^S|$ increases with σ_Y and decreases with ω . This pattern mirrors the pattern in σ_{GG} , i.e., the risk premium increases (decreases) when the information asymmetry between the two classes of investors increases (decreases). Clearly, as class-2 investors become less informed, they will demand higher premium to compensate the increase in their perceived uncertainty about future stock dividends.

The stock price volatility increases with σ_Y when ω is close to one, i.e., when there is a large population of class-1 investors. This result is quite intuitive since higher σ_Y gives higher volatility in class-1 investors' hedging demands, which determines the price volatility when ω is close to one. For ω close to zero, however, the stock price volatility first decreases with σ_Y and then starts increasing. In this case, the stock price varies for two reasons: changes in class-1 investors' hedging needs and changes in class-2 investors' expectation of future dividends. On the one hand, increasing σ_Y increases the volatility of class-1 investors' hedging needs, hence tends to increase price volatility. On the other hand, increasing σ_Y reduces the amount of information class-2 investors extract from prices about future dividends. Less information tends to reduce the variability in their expectation of G_t and the price volatility. This second effect tends to dominate when the population of class-2 investors is large.

5.3. Equilibrium under market structure II

The introduction of the collar changes the risk allocations in the market as discussed in the case of symmetric information. It also has significant impact on the informational efficiency of the market when there is asymmetric information. Introducing a derivative security has the direct

impact of providing additional sources of information to the less informed investors through its market prices. In addition, it has the indirect impact of changing the information content in the prices of existing securities by changing the patterns of investors' allocational trading as well as their informational trading. The interaction between these two can give rise interesting changes in equilibrium as the collar market opens.

The formal results about the equilibrium is summarized in the following theorem.

Theorem 5 When $\mathcal{F}_{1,t} = \mathcal{F}_t$, $\mathcal{F}_{2,t} = \mathcal{F}_t^{(C,P)}$, $\beta_{i,Y} = \beta_{i,Z} = \frac{1}{2}[1 + (-1)^{i-1}]$, where $i = 1, 2$, and under market structure II, the economy has a linear, stationary equilibrium of the form in theorem 2 for ω close to 1. In particular, $\tilde{X}_t = \tilde{X}_{1,t} = (1, Y_t, Z_t, \hat{G}_t - G_t)'$, $\tilde{X}_{2,t} = (1, \hat{Y}_t, \hat{Z}_t)$,

$$\begin{aligned} S_t &= \tilde{\lambda}^S \tilde{X}_t, & H_t &= \tilde{\lambda}^H \tilde{X}_t \\ \theta_{1,t} &= h_1 \tilde{X}_{1,t}, & \theta_{2,t} &= h_2 \tilde{X}_{2,t} \end{aligned}$$

where $\tilde{\lambda}^S = (1, \lambda_Y^S, \lambda_Z^S, \lambda_{\Delta G}^S)$, $\tilde{\lambda}^H = (1, \lambda_Y^H, \lambda_Z^H, \lambda_{\Delta G}^H)$, $h_1 = (h_{1,0}, h_{1,Y}, h_{1,Z}, h_{1,\Delta G})$ and $h_2 = (h_{2,0}, h_{2,Y}, h_{2,Z})$ are determined by (14).

For class-2 investors, there are three unobservable state variables, G_t , Y_t and Z_t . The prices of the stock and the collar, however, provide two endogenous signals as linear functions of the unobserved variables. Thus there is only one degree of uncertainty remaining. The estimating errors of the three variables are then perfectly correlated.

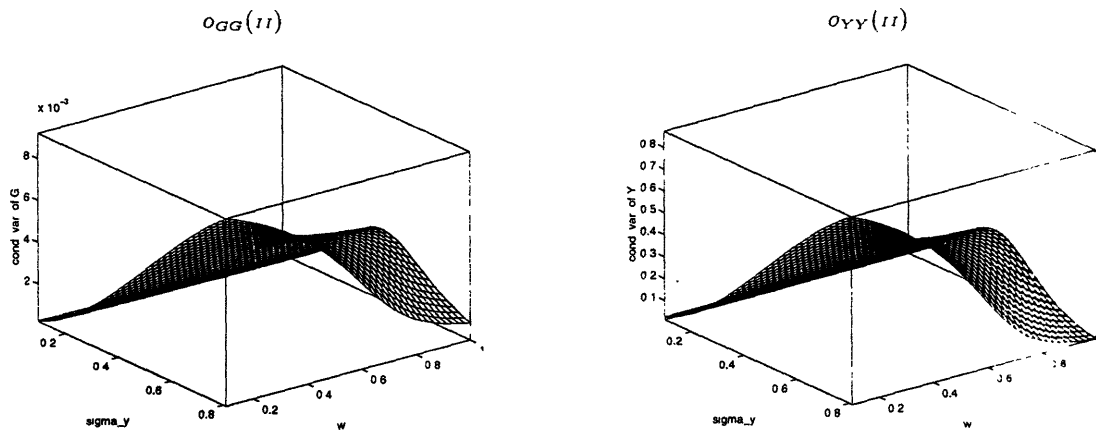


Figure 8: Conditional variance of G_t and Y_t for investor 2 under market structure I and II under asymmetric information. The parameters are set at the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.05$, $a_{i,t} = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.2$, $\sigma_Z = 0.4$, $\sigma_N = 0.2$, $\kappa_{DN} = 0.6$.

Again, we first examine how the information asymmetry between the two classes of investors changes with σ_Y and ω . Figure 8 shows o_{GG} and o_{YY} for different values of σ_Y and ω . As in the case of market structure I, o_{GG} can either increase or decrease with ω , the fraction of class-1 (i.e.,

the informed) investors. As σ_Y increases, σ_{YY} increases when the population of class-1 investors ω is small, also similar to the result under market structure I. However, when the population of class-1 investors is large (i.e., close to one), σ_{GG} increases with σ_Y initially but then starts decreasing.

Figure 9 plots the risk premium and price volatility of the stock for different values of σ_Y and ω . They behave in the similar way as under market structure I.

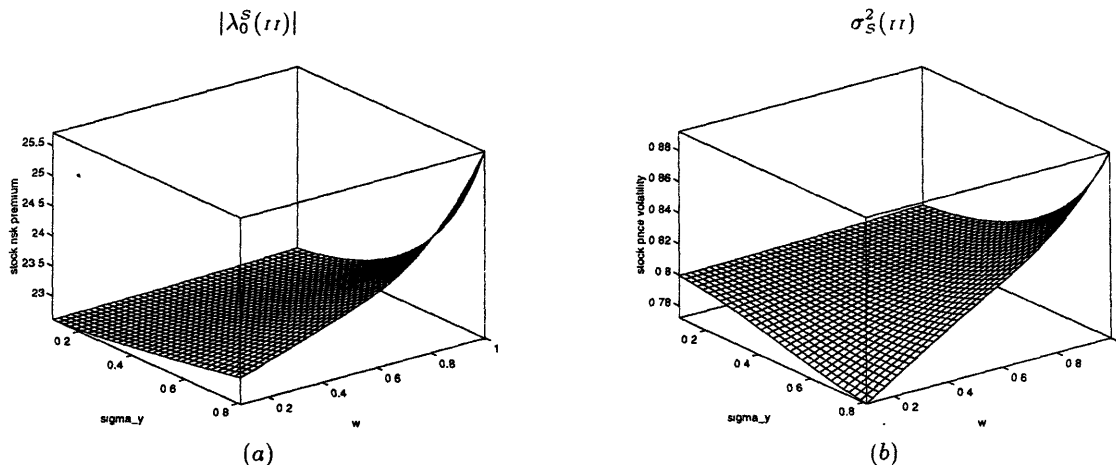


Figure 9: Stock risk premium and price volatility under market structure II and asymmetric information. The parameters are set at the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.05$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.2$, $\sigma_Z = 0.4$, $\sigma_N = 0.2$, $\kappa_{DN} = 0.6$.

5.4. Comparison between market structure I and II.

In order to analyze the impact of derivatives trading on the informational efficiency of the securities market, we now compare the equilibrium of the economy under market structure I and II.

We first consider the change in σ_{GG} from market structure I to II, which is plotted in figure 10 for different values of σ_Y and ω . For most values of σ_Y and ω under consideration, σ_{GG} decreases from market structure I to II, indicating that the information asymmetry between the two classes of investors about the stock's future cash flow decreases after the introduction of collar. However, for certain values of ω and σ_Y , especially when ω is small, the introduction of collar trading can increase the information asymmetry in the market about the stock's future cash flow. In general, σ_{YY} decreases from market structure I to market structure II. This is not surprising since the addition of collar allows class-1 investors to more actively hedge their non-traded risks. The increase in their hedging activities reveals more information through the stock and collar prices about their hedging needs.

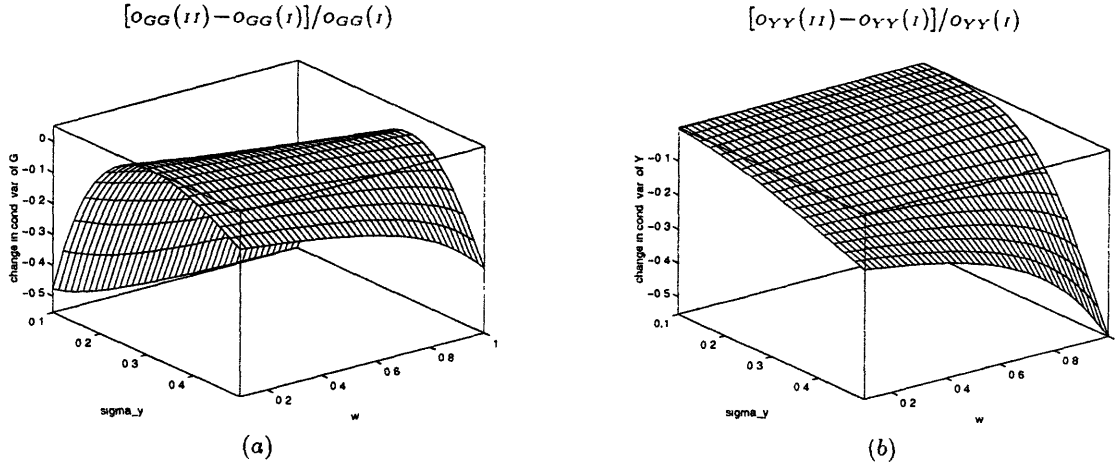


Figure 10: Relative change in information asymmetry, as measured by class-2 (uninformed) investors' conditional variance of future dividends and class-1 (informed) investors' exposure of non-traded risk, from market structure I to II, for different values of ω and σ_Y . The figure in the right panel plots the intersection at $\sigma_Y = 0.3$. Other parameters are set at the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.05$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.2$, $\sigma_Z = 0.4$, $\sigma_N = 0.2$, $\kappa_{DN} = 0.6$.

Using a static, noisy rational expectations setup, Grossman (1977) argues that adding securities to the market should reveal more private information through a larger set of prices and improve its informational efficiency. His results depend on the assumption that the allocational trading (or noise trading) is exogenously specified and unchanged as new securities are introduced. But allocational trading cannot be exogenously specified, especially when we consider changes in the market structure. As we discussed in section 3, when the collar is introduced, investors change their stock trading significantly. An increase in allocational trading can actually make stock prices less informative about its payoffs. In the case that the loss of information contained in the stock price exceeds the gain of information from the collar price, the information asymmetry in the market increases as the collar is introduced. Thus, opening the derivatives market can actually reduce the informational efficiency of the market.

We now can examine the differences in equilibrium stock price between market structures I and II. Figure 11(a) demonstrates changes in stock risk premium when the market structure changes from I to II for different values of ω and σ_Y . At $\omega = 1$, the economy is only populated with class-1 investors, hence becomes a case of identical investors. Changing market structure does not change the equilibrium and security prices since the market is effectively complete in this case. For most values of $\omega < 1$, opening the collar trading decreases the stock risk premium since it reveals more information to class-2 (uninformed) investors about future stock dividends. However, for certain values of σ_Y when ω is small, the stock risk premium can increase with the opening of the collar trading. Not surprisingly, the values of σ_Y and ω for which the premium

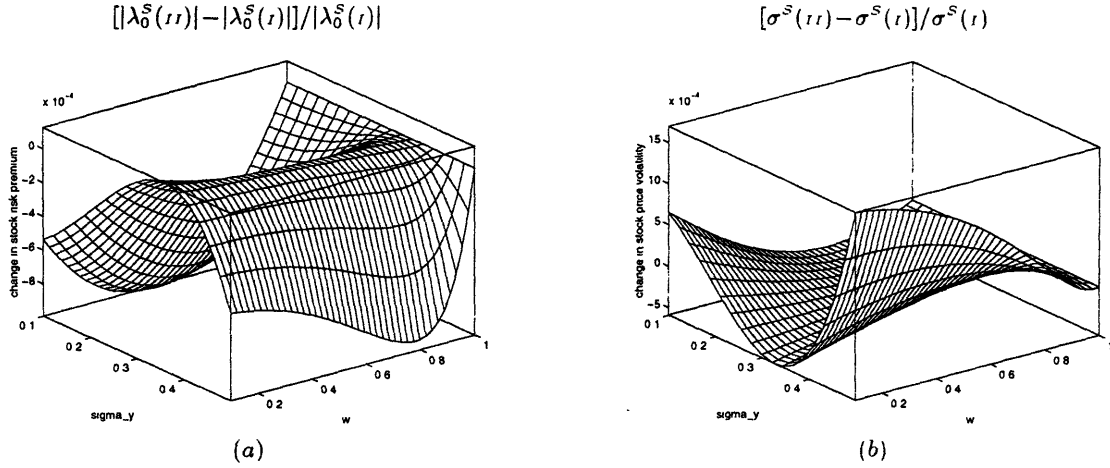


Figure 11: Relative change in stock risk premium and price volatility from market structure I to II under asymmetric information, for different values of ω and σ_Y . The figure in the right panel plots the intersection at $\sigma_Y = 0.3$. Other parameters are set at the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.05$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.2$, $\sigma_Z = 0.4$, $\sigma_N = 0.2$, $\kappa_{DN} = 0.6$.

increases correspond to the same values for which the information asymmetry among investors increases from market structure I to II.

Figure 11(b) shows changes in stock price volatility when the market structure changes from I to II for different values of ω and σ_Y . The impact on stock price volatility by introducing the collar into the market is ambiguous. It can either decrease or increase the stock price volatility, depending on the parameter values. When the economy is mostly populated with class-1 investors (but not completely), collar trading tends to reduce price volatility, which is similar to the case under symmetric information. When the economy is mostly populated with class-2 investors, the effect of information asymmetry is important. Here, for certain ranges of σ_Y , the stock price volatility increases when the collar trading opens.

6. Conclusion

In this paper, we analyze the impact of derivative trading on the allocational and informational efficiencies of the securities market in a fully rational expectations framework. We show that endogenous changes in investors' allocational trading can make the market informationally less efficient when new derivatives are introduced. We also show that introducing derivatives can increase the stock risk premium and price volatility.

Appendix

A.1. Solution for investors' optimization problem.

This appendix considers the solution to investors' control problem when the vector of expected excess share returns on all risky securities follows a continuous Gaussian Markov process. This result will be used repeatedly in further deriving the equilibrium of the economy in different situations.

Suppose that given the information of an investor i , \mathcal{F}_t , the vector of excess share returns is governed by the process

$$\begin{aligned} dQ_t &= a_Q X_t dt + b_Q dw_t \\ dX_t &= a_X X_t dt + b_X dw_t \end{aligned}$$

where w_t denotes the innovations under \mathcal{F}_t . Here, we have omitted the index i for \mathcal{F}_t and w_t for simplicity in notation. Also suppose his non-traded income follows the process $dN_t = (\beta X_t) b_N dw_t$. Further assume that a_Q, a_X, b_Q, b_X, b_N and β are constant over time. The investor's optimization problem can then be stated as

$$\begin{aligned} J_t &\equiv \sup_{c, \theta} \mathbb{E} \left[- \int_t^\infty e^{-\rho(s-t) - \gamma c_s} ds \middle| \mathcal{F}_t \right] \\ \text{s.t. } dW_t &= (rW_t - c_t) dt + \theta'_t dQ_t + (\beta X_t) b_N dw_t \end{aligned}$$

where $\{c, \theta\}$ belong to the feasible policy set. We solve this problem by conjecturing that his value function has the form $J_t = e^{-\rho t - r\gamma W_t + \frac{1}{2} X'_t v X_t}$. It is easy to verify that J_t satisfies the Bellman equation when v solves the following equation:

$$0 = (r\gamma)^2 h' \sigma_{QQ} h - (r\gamma \beta b_N - v b_X) (r\gamma \beta b_N - v b_X)' + r v - (v a_X + a'_X v) - \bar{v} 1_{11}^{(n,n)} \quad (\text{A.1})$$

where

$$h \equiv \frac{1}{r\gamma} \sigma_{QQ}^{-1} (a_Q + b_Q b'_X v - r\gamma b_Q b'_N \beta').$$

Also, J_t satisfies the specified transversality condition. In this case, J_t is indeed the value function and the optimal policies are

$$c_t = rW_t - \frac{1}{2} X_t v X_t - \frac{1}{\gamma} \ln r, \quad \theta_t = h X_t.$$

Thus, the solution to the investors' optimization problem reduces to solving the algebraic matrix equation (A.1).

We now consider the solution to (A.1). Define three matrices m_2 , m_1 and m_0 by

$$\begin{aligned} m_2 m_2' &= b_X \left[\iota - b_Q' (b_Q b_Q')^{-1} b_Q \right] b_X' \\ m_1 &= b_{\bar{X}} b_Q' \sigma_{QQ}^{-1} (a_Q - r \gamma b_Q b_N' \beta') + \left(\frac{r}{2} \iota - a_{\bar{X}} \right) + r \gamma \beta b_N b_X' \\ m_0 m_0' &= (a_Q - r \gamma b_Q b_N' \beta')' \sigma_{QQ}^{-1} (a_Q - r \gamma b_Q b_N' \beta') - (r \gamma \sigma_N)^2 \beta \beta' - \bar{v} 1_{11}^{(n,n)}. \end{aligned} \quad (\text{A.2})$$

(A.1) can then be expressed as

$$m_0 m_0' + m_1' v + v m_1 - v m_2 m_2' v = 0. \quad (\text{A.3})$$

(A.3) is called the algebraic Riccati equation. The algebraic Riccati equation in general have multiple solutions. We need the “smallest” solution for the value function. For two square matrices m and n , we define $m \geq n$ if $m - n$ is positive semi-definite. The strict inequality applies when $m - n$ is positive definite.

There exists the following result on the algebraic Riccati equation [see, e.g., Willems and Callier (1991)]:

Lemma 1 *If m_2 and m_0 are full ranked, the algebraic Riccati equation (A.3) has a unique largest (smallest) solution, which is symmetric and positive (negative) definite.*

Thus, we only need m_2 and m_0 to be full ranked for a solution to (A.1). Let $v = \text{stack}\{(v_{00}, \tilde{v}_0), (\tilde{v}_0', \tilde{v})\}$ where \tilde{v} is the sub-matrix of v that corresponds to the variable part of in X_t (v_{00} corresponds to the constant part and \tilde{v}_0 the cross part). Also let $\tilde{m}_0, \tilde{m}_1, \tilde{m}_2$ denote the sub-matrices of m_0, m_1 and m_2 that correspond to \tilde{v} . We then have the following equation for \tilde{v} :

$$\tilde{m}_0 \tilde{m}_0' + \tilde{m}_1' \tilde{v} + \tilde{v} \tilde{m}_1 - \tilde{v} \tilde{m}_2 \tilde{m}_2' \tilde{v} = 0. \quad (\text{A.4})$$

We only need to solve for \tilde{v} since given \tilde{v} , solving \tilde{v}_0 and v_{00} is trivial. It is obvious that $\tilde{m}_2 \tilde{m}_2'$ is positive definite, since it is the conditional variance of X given Q . Hence \tilde{m}_2 is full ranked. The existence of desired solution of \tilde{v} only requires \tilde{m}_0 to be full ranked. The following lemma is immediate:

Lemma 2 *In the absence of non-traded income, the investor’s control problem given by (A.1) has a unique solution.*

Proof. Given linear price, $m_0 m_0' = a_Q' \sigma_{QQ}^{-1} a_Q$ if there is no non-traded income. Here $a_Q = \bar{\lambda}^P (a_{\bar{X}} - r \iota) + \bar{\lambda}^C$. Since both σ_{QQ}^{-1} and $-(a_{\bar{X}} - r \iota)$ are symmetric positive definite, $m = (a_{\bar{X}} - r \iota)' \sigma_{QQ}^{-1} (a_{\bar{X}} - r \iota)$ is also symmetric and positive definite. Thus, $m_0 = (\lambda^P m^{\frac{1}{2}}, \bar{\lambda}^C \sigma_{QQ}^{-\frac{1}{2}})$ has

full rank, and so does \tilde{m}_0 . By lemma 1, there exists unique, smallest solution to investor's control problem. \blacksquare

If investors have non-traded income, to ensure the existence of the solution, we need to impose conditions on the parameters so that (m_0, m_1) is observable.

A.2. Equilibrium Under Complete Market.

Proof of theorem 1. In a complete market, investors can perfectly hedge the idiosyncratic risk in their non-traded income. Therefore, the equilibrium with two heterogeneous investors is the same as the case with one representative investor who has total exposure to non-traded risk $\omega(\beta_{1,Y}Y_t + \beta_{1,Z}Z_t) + (1 - \omega)(\beta_{2,Y}Y_t + \beta_{2,Z}Z_t) = Y_t + \beta_Z Z_t$. We conjecture a linear equilibrium with price process $P_t = \lambda^P X_t$. The representative agent solves his control problem as defined in appendix A.1 with market clearing condition $h = 1_{1,1}^{m,4}$. Define $\tilde{v} = \{\{v_{YY}, v_{YZ}\}; \{v_{YZ}, v_{ZZ}\}\}$. Then \tilde{v} satisfies the algebraic Riccati equation (A.3). The coefficients are $m_2 = \text{diag}\{\sigma_Y, \sigma_Z\}$, $m_1 = \text{diag}\{\frac{r}{2} - a_Y, \frac{r}{2} - a_Z\}$, $m_0 = r\gamma\sigma_N(1, \beta_Z)$.

Let $d_0 = r^2\gamma^2\sigma_N^2$, $d_1 = (r - 2a_Y)^2/4 - d_0\sigma_Y^2$, $d_2 = (r - 2a_Z)^2/4 - d_0\sigma_Z^2\beta_Z^2$, $d_3 = -d_0\beta_Z$, and $d_4 = \sqrt{d_1d_2 - \sigma_Y^2\sigma_Z^2d_3^2}$. The condition (6) ensures that $d_1d_2 - \sigma_Y^2\sigma_Z^2d_3^2 \geq 0$. Solution \tilde{v} has the following form:

$$\begin{aligned} v_{YZ} &= \pm \sqrt{\frac{d_1 + d_2 \pm 2d_4}{[(d_1 - d_2)/d_3]^2 + 4\sigma_Y^2\sigma_Z^2}} \\ v_{YY} &= \frac{1}{2\sigma_Y^2} \left[(r - 2a_Y) + d_3v_{YZ}^{-1} + [(d_1 - d_2)/d_3]v_{YZ} \right] \\ v_{ZZ} &= \frac{1}{2\sigma_Z^2} \left[(r - 2a_Z) + d_3v_{YZ}^{-1} - [(d_1 - d_2)/d_3]v_{YZ} \right] \\ v_{0Y} = v_{0Z} &= 0, \quad v_{00} = r\gamma\lambda_0^s + \frac{1}{r}(\sigma_Y^2v_{YY} + \sigma_Z^2v_{ZZ}) + \frac{2}{r}(r - \rho - r \ln r) \end{aligned}$$

Note we have four different roots for \tilde{v} corresponding to the four choices of v_{YZ} . We denote \tilde{v}^- as the one corresponding to

$$v_{YZ} = \sqrt{\frac{d_1 + d_2 - 2d_4}{[(d_1 - d_2)/d_3]^2 + 4\sigma_Y^2\sigma_Z^2}}$$

It is easy to check that the difference matrix between \tilde{v}^- and the other \tilde{v} 's is always negative definite. Therefore, \tilde{v}^- maximizes investors' value function $J_t = -e^{-\rho t - r\gamma W_t + \frac{1}{2}X_t'vX_t}$. Having solved \tilde{v} , we can easily solve λ^P from the market clearing condition and consequently $v = \text{diag}\{v_{00}, \tilde{v}\}$. The equilibrium stock price is

$$S_t = \frac{1}{r - a_G} G_t + \tilde{\lambda}^S \tilde{X}_t$$

where

$$\begin{aligned}\lambda_Y^s &= \frac{-r\gamma\sigma_{DN} [v_{YZ}\sigma_Z^2 + (r-a_Z-\sigma_Z^2v_{ZZ})]}{(r-a_Z-\sigma_Z^2v_{ZZ})(r-a_Y-\sigma_Y^2v_{YY}) - v_{YZ}^2\sigma_Y^2\sigma_Z^2} \\ \lambda_Z^s &= \frac{-r\gamma\sigma_{DN} [v_{YZ}\sigma_Y^2 + (r-a_Y-\sigma_Y^2v_{YY})]}{(r-a_Z-\sigma_Z^2v_{ZZ})(r-a_Y-\sigma_Y^2v_{YY}) - v_{YZ}^2\sigma_Y^2\sigma_Z^2} \\ \lambda_0^s &= -\gamma \left[\sigma_D^2 + \frac{1}{(r-a_G)^2}\sigma_G^2 + \lambda_Y^2\sigma_Y^2 + \lambda_Z^2\sigma_Z^2 \right]\end{aligned}$$

The optimal consumption policy is

$$c_t = rW_t - \frac{1}{2\gamma}\tilde{X}'_t v \tilde{X}_t - \frac{1}{\gamma}\ln r$$

Using optimal consumption policy, we can solve the price for any traded security using

$$rP_t dt = E_t \left[\frac{u'(c_{t+dt}, t+dt)}{u'(c_t, t)} (dC_t + dP_t) \right]$$

where u' denotes investor's marginal utility, and $C_t = \int_0^t f(X_t, t)dt + b_C dw_s$ is the cumulative payoff for the security. Using Ito's lemma, we can get the following differential equation for any price process P .

$$rP = \partial_t P + f + X' a'_x \partial_x P + \frac{1}{2} \text{tr} \left(\sigma_{xx} \partial_x^2 P \right) - [r\gamma\lambda^s b_x + (r\gamma\beta b_N - v b_x) X_t] (b'_x \partial_x P + b'_C) \quad (\text{A.5})$$

where $\beta = (0, 1, \beta_Z)$. ■

A.3. Proof of proposition 1 and 2.

Proof of proposition 1. Given linear stock price $P_t = \lambda^P X_t$, the expected returns on the securities follow continuous Gaussian Markov process. Applying the result in appendix A.1, we conclude that v satisfying (11a) solves investors' control problem. To show that the economy is in equilibrium, we only need to check that under this linear price, investors' total demand clears the market. The market clearing condition is described by equation (11b). Therefore, solving the equilibrium under linear price is equivalent to solving the system (11a)-(11b). ■

Proof of proposition 2. It is similar to the symmetric case. As shown in section 5.1, under the linear price, the uninformed investors' conditional expectation of the unobserved state variables solves the Riccati equation in (13b). Also, the expected returns on the securities follow continuous Gaussian Markov process. From appendix A.1, we know that v satisfying (14a) solves both investors' control problems. The market clearing condition should be

$$1_{11}^{(m,1)} = \omega h_1 X_{1,t} + (1-\omega) h_2 X_{2,t}$$

As defined in section 5.1,

$$X_{2,t} = \text{stack}\{1, \hat{\Theta}_t\} = \text{stack}\{1, \Delta_t + \Theta_t\} = \text{stack}\{1, \alpha \tilde{\Delta}_t + \Theta_t\} = (\iota, \alpha) X_{1,t}$$

Thus, we have (14b) as market clearing condition. Solving the equilibrium under linear price is then equivalent to solving the system (13b), (14a) and (14b). \blacksquare

A.4. Existence proofs.

In this section, we prove the existence of a solution to the systems described in theorem 2 and 3 using Implicit Function Theorem[see, Protter and Morrey (1991)]. The proof sketches as follows: First, we show that for a fixed parameter, the system has a solution. Then, we show that at the solution, Jacobian matrix is *generically non-degenerate*.

Definition 1 *Let \mathcal{D} be an open set in \mathbb{R}^n . A function $f : \mathcal{D} \rightarrow \mathbb{R}^m$ is called generically non-degenerate if the n -dimensional Lebesgue measure of its zero set $\{x : f(x) = 0\}$ equal to 0.*

Therefore, by Implicit Function Theorem, the system has a solution in a neighborhood of the initial parameter.

Lemma 3 (Implicit Function Theorem) *Let \mathcal{D} be an open set in \mathbb{R}^{m+n} containing the point (x_0, y_0) . Suppose that $F : \mathcal{D} \rightarrow \mathbb{R}^m$ is continuous and has continuous first partial derivatives in \mathcal{D} . Suppose that*

$$F(x_0, y_0) = 0 \quad \text{and} \quad \det(\nabla_x F(x_0, y_0)) \neq 0. \quad (\text{A.6})$$

Then positive numbers ϵ_x and ϵ_y can be chosen so that:

1. *the direct product of the closed balls $\overline{B_m(x_0, \epsilon_x)}$ and $\overline{B_n(y_0, \epsilon_y)}$ with centers at x_0, y_0 and radii ϵ_x and ϵ_y , respectively, is in \mathcal{D} ;*
2. *ϵ_x and ϵ_y are such that for each $y \in B_n(y_0, \epsilon_y)$ there is a unique $x \in B_m(x_0, \epsilon_x)$ satisfying $F(x, y) = 0$. If f is the function from $B_n(y_0, \epsilon_y)$ to $B_m(x_0, \epsilon_x)$ defined by these ordered pairs (x, y) , then $F(f(y), y) = 0$; furthermore, f and all its partial derivatives are continuous on $B_n(y_0, \epsilon_y)$.*

Lemma 4 *Let $f : \mathcal{D} \rightarrow \mathbb{R}$ be a real analytic function, where $\mathcal{D} = D_1 \otimes \dots \otimes D_n$ is an open subset of \mathbb{R}^n . Let $\mathcal{N} \equiv \{x \in \mathcal{D} : f(x) = 0\}$ be its zero set. Then either $\mathcal{N} = \mathcal{D}$ or $\mu_n(\mathcal{N}) = 0$ where μ_n is the n -dimensional Lebesgue measure.*

Proof. We will prove by induction. First note that \mathcal{N} is closed and therefore measurable. For $n = 1$, \mathcal{N} is either finite, or has an accumulation point. In the latter case we have from [see, Ahlfors (1979)] that the function f is identically zero on \mathcal{D} . Noting that any finite set has zero Lebesgue measure concludes this part of proof.

Let us assume the conclusion of the lemma holds for certain $k \geq 1$ and prove it for $n = k + 1$. Denoting f as a function of two variables $f(t, x)$ on $D_1 \otimes D_2$, where $D_2 \equiv D_2 \otimes \cdots \otimes D_{k+1}$. We see that f is a real analytic function in both t and x separately as well. Consider the set $S = \{t \in D_1 : \forall x \in D_2, f(t, x) = 0\}$. For $t \notin S$ we have $\int_{D_2} f(t, x) dx = 0$ by the inductive assumption. If set S is finite, it is of zero Lebesgue measure in D_1 . Therefore $\mu_n(N) = \int_{D_1} \int_{D_2} 1_{f(t,x)=0} dx dt = 0$ by Fubini theorem [see, e.g., Doob(1991)]. If, on the other hand, S is not finite, then it has an accumulation point. From the result of $n = 1$, we see that for any fixed $x \in D_2$, $f(t, x)$ is identically zero in D_1 , and therefore identically zero on $\mathcal{D} = D_1 \otimes D_2$ which closes the proof. ■

Lemma 5 *Under symmetric information, system (11a)-(11b) has a solution for $\sigma_z = 0$.*

Proof. When $\sigma_z = 0$, the two investors are facing the same non-traded risk. Therefore, the market is essentially complete in the sense that no trading will occur. Applying result in appendix (A.2), we conclude that there exists a solution. ■

Proof of theorem 2. To conform to the notation of the Implicit Function Theorem, we reformulate our system as $F(x, y) = 0$, where $F = (F_1, F_2, F_3)$, with F_1, F_2 corresponding to equation (11a) for $i = 1, 2$, respectively, and F_3 corresponding to equation (11b). Define $x \equiv [[v_1], [v_2], \lambda^P]$, where $[v_1], [v_2]$ are the coefficients in investors' value function, and λ^P is the coefficient of the price process. (Here we use $[v]$ to denote the vector of all non-identical entries in matrix v .) Define $y \equiv [r, \gamma, \rho, \omega, a_G, a_Y, a_Z, \sigma_D, \sigma_G, \sigma_Y, \sigma_Z, \sigma_N, [\kappa]]$ ($[\kappa]$ is the covariance coefficients among all shocks) as the vector of all parameters for system (11a)-(11b). We denote $y = (\bar{y}, \tilde{y})$, where $\bar{y} = \sigma_z$ is the variable and \tilde{y} is rest of the parameters.

From lemma 5, $\exists x_0$, such that $F(x_0, \bar{y}_0, \tilde{y}) = 0$ for $\bar{y}_0 = 0$. By lemma 6, the Jacobian of F at point $(x_0, \bar{y}_0, \tilde{y})$ is generically non-degenerate. Therefore, by Implicit Function Theorem, there exists a solution to system (11a)-(11b) for σ_z close to 0. ■

Lemma 6 *$\det(\nabla_x F(x_0, \bar{y}_0, \tilde{y}))$ is generically non-degenerate.*

Proof. Define $f(\tilde{y}) \equiv \det(\nabla_x F(x_0, \bar{y}_0, \tilde{y}))$. It is clear that $f(\tilde{y})$ is analytic. Also, for $\tilde{y}_0 = [1, 1, 1, .5, 0, 0, 0, 1, 1, 1, 0, 0]$, $f(\tilde{y}_0) \neq 0$. Applying the result of lemma 4, we conclude that $\det(\nabla_x F(x_0, \bar{y}_0, \tilde{y}))$ is generically non-degenerate. ■

Proof of theorem 3. The proof is similar to that of theorem 2. ■

Lemma 7 *Under asymmetric information, if the parameters satisfy assumption (6), then system (13b), (14a)- (14b) has a solution for $\omega = 1$.*

Proof. When $\omega = 1$, the equilibrium price should be the same as the representative investor case with all investor being informed. This is because the uniformed investor has no impact on the equilibrium prices. Therefore, in equilibrium, the informed investor solves his control problem without any consideration to the uninformed investor. The uninformed then takes prices as given and form his rational expectation about the unobserved state variable. Consequently, he solves his control problem based on his new information set.

From appendix A.2 we know that there exists a solution to the informed investor's control problem and the stock price process.

The uninformed investor's expectation of the unobservable state variable satisfies equation (13b). Let α be the $3 \times (3-m)$ matrix that maps Δ_t to $\tilde{\Delta}_t$ in section 5.1, then $o = \alpha \rightarrow \alpha'$, where \rightarrow is full rank sub matrix of o . Simple algebra shows that equation (13b) reduces to the algebraic Riccati equation (A.3) with $m_2 m_2' = \alpha' a_\phi' (b_\phi b_\phi')^{-1} a_\phi \alpha$, $m_1 = [\alpha' (a_\phi' (b_\phi b_\phi')^{-1} (b_\phi b_\phi')' + a_\phi')]$ $|_{(3-m) \times (3-m)}$ and $m_0 m_0' = [(b_\phi b_\phi') (b_\phi b_\phi')^{-1} (b_\phi b_\phi')' + b_\phi b_\phi']$ $|_{(3-m) \times (3-m)}$. Here $|_{(3-m) \times (3-m)}$ means taking the first $(3-m) \times (3-m)$ sub matrix. Using the same argument as in appendix A.1, we can see that $m_2 m_2' = \alpha' \lambda m (\alpha' \lambda)'$ where $m = a_\phi' \sigma_\phi^{-1} a_\phi$ is symmetric and positive definite. Therefore, m_2 is full rank. $m_0 m_0'$ is the sum of two covariance matrix and is therefore both symmetric and positive definite. So, m_0 has full rank. We conclude that (m_1, m_2) is controllable and (m_0, m_1) is observable. By lemma 1, the conditional expectation has unique positive definite solution.

The uninformed investor has no non-traded income, and he takes prices as given. Given his rational expectation about the unobserved state variables, he solves his control problem as defined in appendix A.1. By lemma (2), his control problem has a solution. Thus, for $\omega = 1$, system (13b), (14a)-(14b) has a solution. ■

Proof of theorem 4. To conform to the notation of the Implicit Function Theorem, we reformulate our system as $F(x, y) = 0$, where $F = (F_1, F_2, F_3, F_4)$ with F_1, F_2 corresponding to (14a) $i = 1, 2$, respectively, F_3 corresponding to (14b) and F_4 to (13b).

Similar to the proof of theorem 1, we define $x \equiv [[v_1], [v_2], \lambda^P, [o]]$ where o is uninformed investor's conditional variance of the unobserved state variables. Define y to be the same as in theorem 1. Denote $y = (\bar{y}, \tilde{y})$, where $\bar{y} = w$ is the variable and \tilde{y} is rest of the parameters.

From lemma 7, $\exists x_0$, such that $F(x_0, \bar{y}_0, \tilde{y}) = 0$ for $\bar{y}_0 = 1$. By lemma 8, the Jacobian of F is generically non-degenerate at $(x_0, \bar{y}_0, \tilde{y})$. Therefore, by Implicit Function Theorem, there exists a solution to system (13b), (14a)-(14b) for ω close to 1. ■

Lemma 8 $\det(\nabla_x F(x_0, \bar{y}_0, \tilde{y}))$ is generically non-degenerate.

Proof. The proof is similar to that of lemma 6. Note that $f(\tilde{y})$ is analytic, and $f(\tilde{y}_0) \neq 0$ at $\tilde{y}_0 = [1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0]$. By lemma 4, we see that $\det(\nabla_x F(x_0, \bar{y}_0, \tilde{y}))$ is generically non-degenerate. ■

Proof of theorem 5. The proof is similar to that of theorem 4. ■

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