KINETIC THEORY ANALYSIS OF PROBLEMS IN THE TRANSITION RANGE OF KNUDSEN NUMBERS

by

George K. Bienkowski
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Certified by _____
Thesis Supervisor

Accepted by _____
Chairman, Departmental Committee on Graduate Students

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ABSTRACT

The problem of analysis of gas dynamic problems in the transition region between free molecular and continuum conditions is treated through an integral equation formulation of the Boltzmann equation. The troublesome collision terms are approximated by a collision model. It is shown in a formal manner that this model can be made to give quantitatively correct answers up to any order in the Chapman-Enskog expansion. A preliminary solution of the problem of an expanding sphere of an initially pressurized gas is obtained through a recursion relation in time derived from the integral equation. The structure of a plane steady normal shock is determined by iteration from the zero shock thickness solution with a simple Krook's collision model used in the integral equation. The problem of heat transfer between parallel plates at Knudsen numbers close to unity is solved by iteration from Lester Lees' moment method solution and the use of a modified asymmetric collision model. Preliminary experimental data determining the heat transfer between parallel plates in helium was obtained for the whole range between free molecular and continuum conditions.

Detail results for the expanding sphere case are obtained for helium expanding into air with an initial pressure ratio of 18. The shock structure for Mach number of 1.5 and $\gamma = 1.667$ is calculated up to the second iteration. The temperature profile between two plates at a temperature ratio of 4 and Knudsen number of 8/15 is calculated in detail for the first iteration. The experimental data is not of sufficient accuracy to allow comparison between different theories but it does indicate that the Lester Lees' moment method probably underestimates the heat flux in the transition range. All the theoretical results obtained are reasonable and give cause to have confidence in the theory. A great deal of further numerical work remains to be done before all the implications of this theoretical approach are assessed and a definitive judgement can be made as to its validity and usefulness.

Thesis Supervisor: M. Finston

Title: Associate Professor of Aeronautics and Astronautics

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OBJECT

The behavior of non-uniform gases in the range between free molecular and continuum conditions is investigated. Theoretical analysis is carried out by means of an integral equation formulation of kinetic theory. The three different types of problems investigated theoretically are initial value, macroscopic discontinuity and boundary value problems. An example of a boundary value problem, heat transfer between parallel plates, is also investigated experimentally.

I. INTRODUCTION

Until recent years, most if not all aerodynamic problems with aeronautical application could be treated on the continuum basis. The Navier-Stokes relations could be assumed as valid, and the transport coefficients considered as quantities determined by interpreting experimental information through the application of these same Navier-Stokes equations. There is now, however, a whole store of aerodynamic problems with aeronautical applications where the Navier-Stokes relations do not hold, and the continuum approach is insufficient to explain the physical phenomena.

In trying to treat these problems analytically it is necessary to determine the limitations of the Navier-Stokes equations and then look for an alternative treatment outside the range of their validity. Since matter is composed of molecules one way the range of validity of the Navier-Stokes relations can be determined is through a theory on a molecular scale. In continuum theory the linear relation between the stress and rate of strain tensors, or the linear relation between the heat flux and the temperature gradient is often "justified" by saying that these are only the first terms of the Taylor's expansions for the stress tensor and heat flux vector respectively. The use of these first terms alone carries the implication that the successive terms have to be small. Continuum theory, however, gives no easy reference to determine what is a small gradient. Even the crudest kinetic theory gives the mean free path as the proper scale by which to judge variations, and with very little effort gives, at least qualitatively, the right behavior for the "coefficient" of this "first term" of the Taylor series expansion. With a great deal of more effort quantitative results can be obtained that agree well with experimental data. It is apparent, therefore, that to treat problems where the validity of the Navier-Stokes relations is in doubt, one should turn to

kinetic theory provided, of course, the assumptions inherent in this theory can be satisfied. This is, in general, true in neutral gases away from the condensation point.

In a gas of sufficient rarefaction the molecules can be assumed to be interacting with each other only over a small part of their time and trajectory. In this case the concept of a collision, i.e., interaction between particles in a finite time can be defined. Furthermore, because of the sufficient rarefaction of the gas the collisions of more than two particles at a time can be neglected because of their relatively lower probability. Under these conditions the basic equation of kinetic theory is the Boltzmann equation, and the fundamental unknown is the distribution function. The distribution function is a probability density of the number of molecules in the six-dimensional space of position and velocity coordinates. The Boltzmann equation can be considered as just an equation of continuity for this probability density, where the right hand side is the effect of collisions.

Kinetic theory is, of course, an old subject and the Boltzmann equation has been investigated over many years. All the initial attempts, however, were directed toward explaining and justifying the continuum equations. The interest was mainly in showing that the kinetic theory formalism will explain, on a molecular scale, all the well-known phenomena of gas dynamics. This effort was culminated by the well-known book of Chapman and Cowling[1] in which formulas were obtained for the transport coefficients in terms of molecular parameters. Since agreement with experimental results was surprisingly good, to many aerodynamicists it appeared that kinetic theory had fulfilled its purpose and was now a completed body of knowledge.

The Chapman-Enskog solution, expounded in this book, is basically an expansion for the distribution function in inverse powers of the collision frequency. Appropriate separation into terms of the same order allows successive solution (at least in principle) of the next higher order term in terms of the lower order ones. Though the lack of a direct proof of convergence makes the theory mathematically incomplete, the immediate success in applying its results to calculating transport properties

to first order, gained it wide acceptance. The ability to obtain the Navier-Stokes relations with quantitatively correct transport coefficients by means of this molecular theory is, of course, no small achievement. It is unfortunate, however, that this success actually discouraged people for many years from investigating many facets of kinetic theory, that one now finds are not at all tractable by the Chapman-Enskog method.

Although the limitations of the Navier-Stokes relations are directly apparent in the Chapman-Enskog solution, until recent years not much was done in studying the cases when these limitations are exceeded. The advent in recent years of great interest in rarefied gas flows has prompted a much greater activity in this field. Of course the range that has been investigated the most because of its relative simplicity has been the other extreme, free molecular flow. This region is defined as the region of gas phenomena where the collisions of molecules between themselves in the vicinity of a solid boundary are so infrequent compared to collisions with this boundary that they can be entirely neglected. Though problems of this kind can be treated quite effectively on an intuitive basis by actually keeping track of an "average" particle, they also fit very well into the kinetic theory formalism. In this formalism the free molecular range consists of solutions of the Boltzmann equation where the troublesome right hand collision terms can be neglected. The equation then has a single family of characteristics and solution is, in principle, at least straight forward. There have been many particular problems solved in this range by many different techniques, but the only really serious difficulty encountered has been the inability to adequately describe the interaction at a solid boundary.

In an effort to bridge the gap between the Navier-Stokes region and the free molecular region attempts have been made at expanding from either side. Expansions from the continuum side are, of course, obtainable within the Chapman-Enskog formalism. These second order effects were actually computed by Chapman and Cowling in a particular case and found to be small for normal pressures. The equations of conservation that result when "second order" terms are included are

commonly known as the Burnett Equations[2]. They have recently been applied to certain problems such as dispersion of sound at wavelengths near a mean free path[3, 4] and found to be of dubious validity in spite of tremendous mathematical complexity. In the case of the problem of shock structure results obtained from these Burnett equations[5] actually are further away from available experimental results for low Mach numbers[6] than the solutions obtained from the Navier-Stokes relations.

Expansion from free molecular flow was originally carried out in a straight forward manner by expanding in inverse powers of the Knudsen number. That this is inappropriate has since been shown by Willis[7] who by an integral iteration method indicated that the near free molecular dependence on Knudsen number is as logarithm of the Knudsen number over the Knudsen number. Another method devised for near free molecular flow was carried out by Lunc and Lubonski[8]. This is based on entirely physical arguments and extends the range from free molecular assumption by taking into account not only the particles that strike the boundary from infinity but, also, the first collisions between particles coming away from the boundary and those approaching it. This method requires a large amount of "bookkeeping" to keep track of all the particles and does not offer much hope of extension beyond the first collision model.

The most successful method of expansion from the free molecular was carried out by Willis[7]. He transformed the Boltzmann equation into an integral equation and then iterated by starting with the free molecular solution. This method, though possessing its own problems, has at least a certain amount of generality as well as possibilities of extension beyond the first term. It has shown the inadequacy of expansions in inverse powers of Knudsen number and has also been used to calculate orifice mass flows that compare favourably with experiments[9, 10, 11].

A recent attempt at solution of problems in the transition range that does not fall in any of the above categories has been carried out at Massachusetts Institute of Technology by Haviland[39] and Lavin[40]. This attempt consists of an application of Monte Carlo methods to following the history of an individual molecule through a "field" of target molecules

whose distribution is previously assumed. All the possible "states" of this probe particle are interpreted as a new distribution function, and the process can again be repeated with this as the new distribution of target particles. This method therefore consists of using a computer as a piece of test equipment in simulation of actual physical conditions. The method has an advantage of being able to accurately represent the "collision" process, but suffers from the necessity of calculating an extremely large number of events. The results thus far obtained in Refs. 39 and 40 suffer from certain unexplained oscillations and certainly cannot yet be interpreted as an exact solution to which all others have to be referred, but the method does offer promise as a means of checking particular cases of more approximate solutions if its great computational difficulties can be overcome.

All the methods thus far discussed, aside from certain simplifications, have been in a sense exact. Exact, that is, in the fact that any expansion is exact provided all the terms are at least, in principle, calculable. They have all in one way or another, at least indirectly, solved for the distribution function. In recent years, however, there has been a lot of activity in trying to solve kinetic theory problems by the approximate moment methods. These moment methods consist of taking velocity moments of the Boltzmann equation and integrating over velocity space. The resulting moments are considered as state variables; and the equations are called the moment equations. The first three are the usual conservation equations but with heat flux vector and stress tensor as independent variables. The next two moment equations can be considered as the equations for the stress tensor and heat flux vector but will, of course, also involve higher moments. The crux of the moment method consists of cutting off this series of equations by assuming a distribution function with certain undetermined parameters that are functions of space and time. By taking moments of this distribution function and taking a sufficient number of the moment equations to solve for all the undetermined parameters, one obtains the distribution function to the best approximation within the original assumptions of its form. One cannot expect to have great accuracy for the

distribution function itself, but its moments, the physically meaningful quantities, can be accurate as they, at the very least, satisfy the conservation equations.

There are many variants of these moment methods but the differences consist only of different choices for the form of the assumed distribution function. Probably the best known moment method is Grad's thirteen moment method[12] which consists of constructing the trial function by multiplying a Maxwellian distribution by a series of Hermite polynomials and leaving thirteen arbitrary parameters. It gives results that couple the stress tensor and heat flux vector but reduce to the proper values in the Navier-Stokes limit. Many other variants of the moment methods have been applied to particular problems by Gross, Ziering and Jackson[13, 14, 15]. Krook[16] gives a relatively general account of a whole class of moment methods and discusses both the full and half moment methods. A most recent variant of the moment methods has been suggested and applied to several problems by Lees[17, 18]. This differs in an important way from the others by the fact that the form of distribution function makes the macroscopic variables converge automatically to the proper limits at both the continuum and free molecular regions, with only a single assumed trial function.

Moment methods do, in general, have certain problems associated with them. The arbitrariness of the choice of the trial function and the lack of a really good test of accuracy leave some doubt as to the validity of the results. The great complexity in taking into account more than the first few moments also generally limits the accuracy of the detail properties of the solution. Also, the necessity of either changing the trial function or switching from full-moment to half-moment methods to find solutions for the whole range of Knudsen numbers, in all but Lees' method, limit their usefulness in studying the "transition" regime.

Another development in kinetic theory in recent years has been an attempt to find models for the collision terms that do not require detail analysis of the collision process. The analysis of an approach to equilibrium by approximating the collision terms by a term proportional to the difference between the distribution function and its equilibrium

value has been known for a long time. A more recent refinement of this to take into account the fact that the equilibrium distribution function may itself be varying, has been carried out by Bhatnager, Gross and Krook[19] and carried on by Krook[16, 20]. This model, known as Krook's model, assumes that particles scattered at any point are scattered spherically with a locally Maxwellian distribution. For the most used case, where the collision frequency is assumed independent of velocity, one uses the actual local density, temperature and velocity values for the Maxwellian distribution; while for cases where the collision frequency does depend on velocity, one has to use what one can call the collision density, temperature and velocity defined by applying the conservation conditions.

The purpose of this thesis is to find a technique for solving fluid dynamic problems right in the transition regime, i.e., for the whole range of Knudsen numbers, in a way that will allow for increasing the accuracy without excessive complication. To do this the Boltzmann equation (or one with a model for the collsion term) is first transformed into a purely integral equation. This integral equation can then, in principle, be solved exactly for problems where the distribution function is known exactly at some time t, for all times later than t. In general, however, this has to be done numerically and use of the full collision terms precludes an easy solution. Furthermore, in steady state problems the distribution function is not known at any time t and thus a direct solution is impossible. This can be circumvented by iterating from an assumed solution until the answer converges.

To test the power of the integral equation approach, three types of problems were solved by using it. An initial value problem of an initially dense helium sphere expanding into much lower density air has been solved numerically using the simple Krook's model for collisions. An infinite space problem with large gradients of a normal shock structure has been solved by iteration (to the second iteration), using Krook's model. If the initial distribution function is assumed discontinuous, the first iteration has a discontinuity in the first derivative, the second iteration in the second derivative, etc. A boundary value problem of heat transfer between parallel plates is solved by iteration (to the first iteration) using a modified asymmetric Krook's model. Here to ensure convergence to the proper limits the iteration

is started from a solution obtained by Lees' moment method.

The lack of any precise standard with which to compare precludes the possibility of conclusively analysing the accuracy of the results. Also the lack of a mathematical convergence proof for the general case does not allow us to claim that exactness can be achieved even with many iterations. The reasonable behavior and relatively quick convergence in some of the cases carried out indicates that the methods can be useful for analysing fluid phenomena in the transition regime. Much further work needs to be done, however, before any statements can be made about the exactness and power of the method. The use of Krook's model, justified here because only the gross behavior of solutions was desired, adds an additional indeterminate factor that has to be considered in any accuracy analysis. It can be stated, however, on the basis of the results obtained in this paper that the integral equation formulation of kinetic theory can be applied with reasonable success to problems in the transition regime of Knudsen numbers, as well as be very useful in aiding physical intuition in analysis of problems that are not presently tractable quantitatively.

In addition to the theoretical work mentioned above an attempt was made at measuring experimentally the heat transfer between parallel plates over a wide range of Knudsen numbers. Because of many technical difficulties with the experimental apparatus only preliminary results are available. These consist of the measured heat flux between parallel plates in helium as a function of pressure. The experiments cover the whole range between free molecular and continuum while retaining a reasonably small ratio of radiation to conduction heat transfer. The accuracy of the results is, however, not sufficient to determine if the iterated solution is any improvement over the Lees' moment method result, thus no attempt at comparison was made. The experimental results are compared to the Lees' moment method solution in which the accomodation coefficient determined from the experiment is used. The two results are generally in agreement though the small range over which either differs from the free molecular or continuum value almost guarantees that any solution that goes to the correct limits is also reasonably correct in between.

II. INTEGRAL EQUATION FORMULATION OF THE KINETIC THEORY EQUATION

One of the big difficulties in trying to solve the Boltzmann equation is the integro-differential nature of the equation. The right hand side is a complicated non-linear integral of the unknown appearing on the left hand, differential side. Even when models are substituted for the integral term, the model parameters have to retain, at least implicitly, some of their properties as functions dependent on the unknown distribution. Since these parameters are, in general, velocity space integrals, one still has an integro-differential equation in practice. Of course the established methods discussed in the first chapter can be applied to these equations with quite satisfactory results at the two extremes, continuum and free molecular. Behavior between these two extremes, however, cannot, even in principle, be obtained without carrying excessively large numbers of terms in the expansions necessary.

It is therefore suggested that another possible method of solution in the transition region, around Knudsen number of unity, be investigated. This method consists of transforming the kinetic theory equation into a purely integral form and then using some form of iteration starting with a solution that goes to the proper limits at both extremes. The integral equation formulation itself is only a formal transformation and was first suggested by Jaffe[21] long ago. It has recently been applied by Willis[7, 11, 22] primarily as iteration from free molecular flow, and in the particular case of linearized couette flow[23] has been solved numerically for all Knudsen numbers using an approximate collision model. The investigation in this thesis is concerned with the analysis of general classes of fluid problems in the transition range, i.e., in between free molecular and continuum conditions.

The integral equation formulation is chosen because of its explicit indication of the boundary conditions, as well as its expected better convergence properties characteristic of integral iterations.

Because of the lack of a general derivation of the integral formulation in the literature, it is given here as a transformation of the general Boltzmann type differential equation and then interpreted physically and specialized to particular cases. The Boltzmann type equation is considered where the right hand side is identified only as the net density of particles scattered into the phase space volume per unit time as a result of collisions. The equation can be written as:

$$\frac{\partial f}{\partial t} + \vec{v}\cdot(\nabla_r f)_{\vec{v}} + \vec{a}\cdot(\nabla_v f)_r = \mathcal{P}(\vec{r},\vec{v},t) - \nu(\vec{r},\vec{v},t) f(\vec{r},\vec{v},t) \qquad (II-1)$$

This is now a general kinetic theory equation valid whenever the concepts of discrete collisions are valid. $\mathcal{P}(\vec{r},\vec{v},t)$ is the number of molecules scattered into the phase space volume $d\vec{r}$ $d\vec{v}$ per unit time, and $\nu(\vec{r},\vec{v},t)$ is the collision frequency of particles of velocity $\vec{v}$ at the point $\vec{r}$. The Boltzmann equation can be obtained by substituting

$$\mathcal{P}(\vec{r},\vec{v},t) = \iint \sigma(\Omega)\, g\, f'(\vec{r},\vec{v}',t)\, f'(\vec{r},\vec{w}',t)\, d\Omega\, d^3\vec{w} \qquad (II-2)$$

and

$$\nu(\vec{r},\vec{v},t) = \iint \sigma(\Omega)\, g\, f(\vec{r},\vec{w},t)\, d\Omega\, d^3\vec{w} \qquad (II-3)$$

where $\sigma(\Omega)$ is the differential collision cross section

$g = |\vec{v}-\vec{w}| = |\vec{v}'-\vec{w}'|$ the magnitude of the relative velocity

$'$ designates conditions before collision

$\vec{v},\vec{w}$ and $\vec{v}',\vec{w}'$ are the four velocities involved in a description of a binary collision.

Any other model for the collision process can, however, be incorporated into the formalism by identifying $\mathcal{P}(\vec{r},\vec{v},t)$ and $\nu(\vec{r},\vec{v},t)$ through their dependence either on $f(\vec{r},\vec{v},t)$ or the macroscopic variables

incorporated into an assumed function of $\vec{v}$. Mathematically the assumption underlying this formalism is that $\mathcal{P}$ and ν are both finite. It is well known that $\mathcal{P}$ and ν as defined in the Boltzmann equation case are actually infinite for Maxwellian particles while $\mathcal{P}-\nu f$ is finite. This can, of course, be fixed by limiting the integration to collisions that alter the momentum beyond a certain minimum. Since both for physical reasons and for mathematical consistency, required to retain the definition of a "collision", $\mathcal{P}$ and ν should always be finite, the splitting of the collision term should be possible in any practical problem.

For brevity let us call the differential part of the equation:

$$\frac{\mathcal{D}f}{\mathcal{D}t} = \frac{\partial f}{\partial t} + \vec{v}\cdot(\nabla_r f)_{\vec{v}} + \vec{a}\cdot(\nabla_v f)_{\vec{r}} \qquad (II\text{-}4)$$

Here it is desirable to give a word of caution about the meaning of the symbols. By $\nabla_r f$ one means in this notation $(\nabla_r f)_{\vec{v}}$ i.e., the gradient of f in $\vec{r}$ space while holding $\vec{v}$ fixed in magnitude and direction. As long as in the description of both $\vec{v}$ and $\vec{r}$ a fixed direction in space is used as a reference, this definition corresponds directly to the usual gradient operator ∇f. If, however, as is often desirable $\vec{v}$ is defined by essentially giving its direction relative to $\vec{r}$ and $\vec{r}$ space has a curvature, than $(\nabla_r)_{\vec{v}}$ does not necessarily correspond to the usual gradient. The usual gradient has the physical meaning of taking the derivative in $\vec{r}$ space while keeping $\vec{v}$ constant as defined relative to $\vec{r}$, while $(\nabla_r)_{\vec{v}}$ is the derivative in $\vec{r}$ space while keeping $\vec{v}$ constant in absolute space. It is obvious that there will be a difference between the two equal to the change in the function due to the change in $\vec{v}$ relative to $\vec{r}$. Thus $(\nabla_r)_{\vec{v}}$ can always be interpreted as $\nabla_r - (\nabla_r\vec{v})\cdot\nabla_v$. Thus when $\vec{v}$ is defined relative to $\vec{r}$, then $\nabla_r\vec{v}$ exists when there is coordinate curvature.

It is possible to take the equation

$$\frac{\mathcal{D}f}{\mathcal{D}t} = \mathcal{P} - \nu f \qquad (II\text{-}5)$$

and directly integrate, but integration over $\mathcal{P}-\nu f$ which remains finite for all non-uniform conditions gives integrals that diverge as the volume in physical space is increased. A better approach is to use an integrating factor and thus effectively transfer νf to the left hand side.

Let

$$g(\vec{r},\vec{v},t) = f(\vec{r},\vec{v},t)\exp\left(\Omega(\vec{r},\vec{v},t)\right) \qquad (II-6)$$

then

$$\frac{\mathcal{D}f}{\mathcal{D}t} = \frac{\mathcal{D}g}{\mathcal{D}t}\exp(-\Omega) - f\frac{\mathcal{D}\Omega}{\mathcal{D}t} = \mathcal{P}-\nu f \qquad (II-7)$$

This can be satisfied by the two equations

$$\frac{\mathcal{D}\Omega}{\mathcal{D}t} = \nu \qquad \text{and} \qquad \frac{\mathcal{D}g}{\mathcal{D}t} = \mathcal{P}\exp(\Omega) \qquad (II-8)$$

separately.

Thus, to transform the equation into a purely integral form only, the inversion of the $\mathcal{D}/\mathcal{D}t$ is required, i.e., the Green's function for the operator $\mathcal{D}/\mathcal{D}t$. For the case of no external forces i.e., $\vec{a}=0$ this can be done in a straight forward manner.

Consider the equations

$$\frac{\partial\phi}{\partial t} + \vec{v}\cdot\nabla_r\phi = W(\vec{r},\vec{v},t) \qquad (II-9)$$

and

$$\frac{\partial\tilde{G}(\vec{r},\vec{r}_o)}{\partial t} + \vec{v}\cdot\nabla_r\tilde{G}(\vec{r},\vec{r}_o) = -\delta^{(3)}(\vec{r}-\vec{r}_o)\,\delta(t-t_o) \qquad (II-10)$$

The minus sign in the equation for the Green's function is a result of the fact that the operator is not self-adjoint, but rather the negative of the operator is its adjoint.

Consider now solving the equation

$$\frac{\partial \phi}{\partial t} + \vec{v} \cdot \nabla_r \phi = W \qquad (II-11)$$

in terms of $\tilde{G}(\vec{r}, \vec{r}_o)$ inside the volume V bounded by the surface S and with initial conditions at time $t=0$.

By multiplying the first equation by $\tilde{G}$ and the second by ϕ, integrating over V and time t from 0 to t_o^+ and interchanging the meaning of $\vec{r}_o$ and $\vec{r}$, and t_o and t, one gets

$$\phi(\vec{r}, \vec{v}, t) = \int_0^{t^+} \iiint_V W(\vec{r}_o, \vec{v}, t_o)\, \tilde{G}(\vec{r}_o, \vec{r})\, dV_o\, dt_o$$

$$+ \int_0^{t^+} \iint_S \vec{n} \cdot \vec{v}\, \phi(\vec{r}_{os}, \vec{v}, t_o)\, \tilde{G}(\vec{r}_{os}, \vec{r})\, dS_o\, dt_o$$

$$+ \iiint_V \phi(\vec{r}_o, \vec{v}, 0)\, \tilde{G}(\vec{r}_o, \vec{r})\, dV_o \qquad (II-12)$$

where $\vec{n}$ is the normal vector along the surface pointing into the volume V.

Since $\tilde{G}(\vec{r}_o, \vec{r}) = G(\vec{r}, \vec{r}_o)$ one has to only solve the equation.

$$\frac{\partial G(r, r_o)}{\partial t} + \vec{v} \cdot \nabla_r G(\vec{r}, \vec{r}_o) = \delta^{(3)}(\vec{r} - \vec{r}_o)\, \delta(t - t_o) \qquad (II-13)$$

To do this one takes the Fourier transform in physical space

$$G = \frac{1}{(2\pi)^3} \iiint \gamma(\vec{p}, t-t_0) \exp\left[i\vec{p}\cdot(\vec{r}-\vec{r}_0)\right] dV_{\vec{p}} \qquad (II-14)$$

$$\delta^{(3)}(\vec{r}-\vec{r}_0) = \frac{1}{(2\pi)^3} \iiint \exp\left[i\vec{p}\cdot(\vec{r}-\vec{r}_0)\right] dV_{\vec{p}} \qquad (II-15)$$

The resulting equation becomes

$$\frac{\partial \gamma}{\partial(t-t_0)} + i\vec{v}\cdot\vec{p}\,\gamma = \delta^{(1)}(t-t_0) \qquad (II-16)$$

which gives the solution

$$\gamma(\vec{p}, t-t_0) = \exp\left[-i\vec{v}\cdot\vec{p}(t-t_0)\right] u(t-t_0) \qquad (II-17)$$

where

$$u(\tau) = \begin{cases} 0 & \tau \le 0 \\ 1 & \tau > 0 \end{cases} \qquad (II-18)$$

Thus

$$G(\vec{r}, \vec{r}_0; t, t_0) = \delta^{(3)}(\vec{r}-\vec{r}_0-\vec{v}(t-t_0))\, u(t-t_0) \qquad (II-19)$$

Substituting this result into the equation for ϕ one gets

$$\phi(\vec{r},\vec{v},t) = \int_{0^+}^{t^+}\iiint_V W(\vec{r}_0,\vec{v},t_0)\,\delta^{(3)}(\vec{r}-\vec{r}_0-\vec{v}(t-t_0))\,dV_{\vec{r}_0}\,dt_0$$

$$+\ \vec{n}\cdot\vec{v}\int_{0^+}^{t^+}\iint_S \phi(\vec{r}_{0s},\vec{v},t_0)\,\delta^{(3)}(\vec{r}-\vec{r}_{0s}-\vec{v}(t-t_0))\,dS_{\vec{r}_0}\,dt_0$$

$$+\ \iiint_V \phi(\vec{r}_0,\vec{v},0)\,\delta^{(3)}(\vec{r}-\vec{r}_0-\vec{v}t)\,dV_{\vec{r}_0} \qquad (II\text{-}20)$$

The first term can be immediately integrated over the volume giving the result

$$\int_0^t W(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)\,dt_0$$

The other two terms, however, never contribute simultaneously. It is shown in Appendix A that when $\vec{v}$ and t are such that $\vec{r}-\vec{v}t$ lies within the volume V then $\vec{r}-\vec{v}(t-t_0)-\vec{r}_{0s}$ never lies in the effective volume $dS_{\vec{r}_0}\ \vec{n}\cdot\vec{v}\,dt_0$ and vice versa. Thus, only one of these terms will contribute in any particular problem. The contribution of the last term is quite obviously

$$\phi(\vec{r}-\vec{v}t,\vec{v},0)$$

While if the second term contributes it takes some manipulation to show that the result is

$$\phi(\vec{r}-\vec{v}(t-t_s),\vec{v},t_s)$$

where t_s is the time at which a particle traveling with velocity $\vec{v}$ must have left the surface to arrive at $\vec{r}$ at time t without any collisions. The derivation of this result is carried out in Appendix A.

Substitution of these results into the expressions for $\Omega(\vec{r},\vec{v},t)$ and $f(\vec{r},\vec{v},t)$ yields

$$\Omega(\vec{r},\vec{v},t) = \int_{t_s u(t_s)}^{t} \nu(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)\,dt_0$$

$$+ \Omega(\vec{r}-\vec{v}t,\vec{v},0)\,u(-t_s) + \Omega(\vec{r}-\vec{v}(t-t_s),\vec{v},t_s)\,u(t_s) \qquad \text{(II-21)}$$

and

$$f(\vec{r},\vec{v},t) = \int_{t_s u(t_s)}^{t} \mathcal{P}(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)\exp\left[-(\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0))\right]dt_0$$

$$+ f(\vec{r}-\vec{v}(t-t_s),\vec{v},t_s)\exp\left[-(\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}-\vec{v}(t-t_s),\vec{v},t_s))\right]u(t_s)$$

$$+ f(\vec{r}-\vec{v}t,\vec{v},0)\exp\left[-(\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}-\vec{v}t,\vec{v},0))\right]u(-t_s)$$

$$\text{(II-22)}$$

where t_s is a solution of the equation

$$\vec{r}-\vec{v}(t-t_s)=\vec{r}_s \quad \text{(i.e., } \vec{r} \text{ on the surface)}$$

First of all it is immediately apparent that the boundary conditions of $\Omega(\vec{r},\vec{v},t)$ are unimportant as only $\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}',\vec{v},t')$ appears in the equation for $f(\vec{r},\vec{v},t)$. This means that $\Omega(\vec{r},\vec{v},t)$

can be defined as

$$\Omega(\vec{r},\vec{v},t) = \int_{t_s,u(t_s)}^{t} \nu(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)\,dt_0 \qquad (II-23)$$

with the physical meaning of the number of collisions a stream of particles traveling with velocity $\vec{v}$ would suffer in the distance from the boundary surface to the point $\vec{r}$, or in the time t as the case may be.

Several things become clear from these considerations. Most of all it is apparent that the expression derived for $f(\vec{r},\vec{v},t)$ can be physically interpreted and shown to be identical to that derived on purely physical grounds. Secondly, it can be seen that boundary conditions whether in space or time enter in such a way as to make clear the fact that three or four collisions away is equal to infinity for all practical purposes. This limits the sphere of influence on $f(\vec{r},\vec{v},t)$ to conditions within three or four mean free paths or mean free times away which, of course, is also known to be true from physical considerations. The appearance of the equations of straight lines along velocities $\vec{v}$ or "characteristics" of the differential operator $\frac{\partial f}{\partial t} + \vec{v}\cdot(\nabla_r f)_{\vec{v}}$ further points the way to finding the more general Green's functions when $\vec{a}$ is not identically zero.

The expression for $f(\vec{r},\vec{v},t)$ can easily be seen to represent just an accounting of the density of the particles at the position $\vec{r}$ and with the velocity $\vec{v}$. This density of particles in phase space is obtained by integrating backwards along the direction $\vec{v}$ and counting all the particles that are scattered into the stream with velocity $\vec{v}$ and then reducing them by the number that are scattered out before they reach position $\vec{r}$. Thus, as one integrates along this line of $\vec{v}$ back away from $\vec{r}$ and, of course, backwards in time, one integrates over $\mathcal{P}(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)$ which is the scattering of particles into the stream, multiplied by $\exp\left[-(\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0))\right]$ which corrects to give the number

of those scattered that will reach point $\vec{r}$. Thus, one includes all the particles that could possibly have the velocity $\vec{v}$ at position $\vec{r}$ and time t. One carries this integration out only until time equal to zero at which time one presumably includes the initial conditions also decayed by the number that have been scattered out of the stream in time t, or until one reaches a boundary where one includes the particles scattering from the boundary in the direction $\vec{v}$.

The fact that the integration is carried out along straight lines is a direct result of the assumption of no external forces. As in this case lines along $\vec{v}$ are the characteristics of the differential equation along which conditions remain unchanged except for the "driving function" effect which is precisely what is being integrated. This, however, suggests a more general approach which will give the Green's function for the general case including external forces.

From the above results one can generalize to show that the characteristics of the differential operator are the equations of motion of a particle under the action of the external forces and having the velocity $\vec{v}$ at position $\vec{r}$ at time t. Thus, the characteristic going away from this point is $\vec{r}_c = \vec{r} - \int_{t_1}^{t} \vec{w}(\tau)\, d\tau$ where $\vec{w}$ is the velocity that a particle would have to have at time τ to have the velocity $\vec{v}$ at time t and position $\vec{r}$. This velocity in turn can be related to the integral of the acceleration

$$\vec{w}(\tau) = \vec{v} - \int_{t_1}^{\tau} \vec{a}(\vec{r}_c(\tau'), \tau')\, d\tau' \qquad (II-24)$$

Thus, the characteristic can be expressed as

$$\vec{r}_c = \vec{r} - \vec{v}(t-t_0) + \int_{t_1}^{t}\int_{t_1}^{\tau} \vec{a}(\vec{r}_c(\tau'), \tau')\, d\tau'\, d\tau \qquad (II-25)$$

In general, this is really an integral equation for $\vec{r}_c$ as the acceleration $\vec{a}$ is itself a function of the location of the characteristic.

Fortunately in most practical problems the external force does not vary appreciably over a mean free path. Thus, for the range over which the characteristic $\vec{r}_c$ is useful, i.e., several mean free paths, the acceleration $\vec{a}$ can be assumed constant. This greatly simplifies the equation to

$$\vec{r}_c = \vec{r} - \vec{v}(t-t_0) + \vec{a}\frac{(t-t_0)^2}{2} \qquad (II-26)$$

for the steady case and

$$\vec{r}_c = \vec{r} - \vec{v}(t-t_0) + \int_{t_0}^{t} a(\tau)(\tau-t_0)\,d\tau \qquad (II-27)$$

for a purely time varying case.

This simplification should be valid for a wide class of problems, as the acceleration $\vec{a}$ is usually a result of such "field type" external forces as gravity or electromagnetic fields. Except in cosmological problems the variation of the gravitational potential within a mean free path is certainly negligible. In problems where a magnetic or electric field is applied there is, however, a whole class of problems where the field can vary appreciably over a mean free path. Any problem involving electromagnetic wave phenomena where the wavelength is of the same order as the mean free path on the surface at least results in just such a condition. Fortunately in most cases of electromagnetic wave interaction with a gas the field force effects on the trajectories are negligibly small and can be altogether neglected. Furthermore, it can be shown that for periodic fields the $\vec{r}$ dependence of $\vec{a}$ usually need not be considered. When the frequency of the field is of the same order as the collision frequency, then the ratio of the mean free path to the wavelength is of the order of $\frac{\bar{v}}{c}$ i.e., very small for normal non-relativistic conditions. And thus, the variation of $\vec{a}$ along $\vec{r}_c$ can be neglected and only the effect of the time variation, which is known, included. On the other hand, it is apparent that when the mean free path and wavelength are of the same order the period of the wave is $\bar{v}/c$ of the mean free time and thus under integration only the D.C. level is important.

In D.C. problems with strong interactions the geometry is usually such that again there is no great variation of the field in a single mean free path. Thus, for this great variety of problems the characteristics are directly available to within a good accuracy. Thus, it should be possible to use these characteristics to integrate the kinetic theory equation even for a large class of problems involving external forces.

The very definition of a characteristic implies that

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_{\vec{r}_c, \vec{w}} \qquad (II\text{-}28)$$

This can, however, be easily checked by carrying out the following coordinate transformation

$$\vec{r} = \vec{r}_c + \vec{w}\tau + \int_0^{\tau}\int_0^{\tau} \vec{a}(\tau'')\, d\tau'' d\tau' \qquad (II\text{-}29)$$

$$\vec{v} = \vec{w} + \int_0^{\tau} \vec{a}(\tau')\, d\tau' \qquad (II\text{-}30)$$

$$t = \tau + t_0 \qquad (II\text{-}31)$$

and evaluating $\left(\partial/\partial\tau\right)_{\vec{r}_c, \vec{w}}$.

By the definition of the differential and the use of the chain rule one gets

$$\left(\frac{\partial}{\partial\tau}\right)_{\vec{r}_c, \vec{w}} = \left(\frac{\partial t}{\partial\tau}\right)_{\vec{r}_c, \vec{w}} \frac{\partial}{\partial t} + \left(\frac{\partial \vec{r}}{\partial\tau}\right)_{\vec{r}_c, \vec{w}} \cdot \vec{\nabla}_r + \left(\frac{\partial \vec{v}}{\partial\tau}\right)_{\vec{r}_c, \vec{w}} \cdot \vec{\nabla}_v \qquad (II\text{-}32)$$

Since

$$\left(\frac{\partial t}{\partial \tau}\right)_{\vec{r}_c,\vec{w}} = 1 \qquad (II\text{-}33)$$

$$\left(\frac{\partial \vec{r}}{\partial \tau}\right)_{\vec{r}_c,\vec{w}} = \vec{v} \qquad (II\text{-}34)$$

$$\left(\frac{\partial \vec{v}}{\partial \tau}\right)_{\vec{r}_c,\vec{w}} = \vec{a}(\vec{r},t) \qquad (II\text{-}35)$$

$$\frac{\mathcal{D}}{\mathcal{D}t} = \left(\frac{\partial}{\partial \tau}\right)_{\vec{r}_c,\vec{w}} = \frac{\partial}{\partial t} + \vec{v}\cdot(\nabla_r)_v + \vec{a}(\vec{r},t)\cdot(\nabla_v)_{\vec{r}} \qquad (II\text{-}36)$$

Thus, the equations

$$\frac{\mathcal{D}\,\Omega(\vec{r},\vec{v},t)}{\mathcal{D}\,t} = \mathcal{V}(\vec{r},\vec{v},t) \qquad (II\text{-}37)$$

and

$$\frac{\mathcal{D}}{\mathcal{D}t}\left(f(\vec{r},\vec{v},t)\exp[\Omega(\vec{r},\vec{v},t)]\right) = \mathcal{P}(\vec{r},\vec{v},t)\exp[\Omega(\vec{r},\vec{v},t)] \qquad (II\text{-}38)$$

can be transformed into

$$\left(\frac{\partial\,\Omega(\vec{r},\vec{v},t)}{\partial \tau}\right)_{\vec{r}_c,\vec{w}} = \mathcal{V}(\vec{r}_c,\vec{w},\tau) \qquad (II\text{-}39)$$

and

$$\left(\frac{\partial}{\partial\tau}\left(f(\vec{r}_c,\vec{w},\tau)\exp[\Omega(\vec{r}_c,\vec{w},\tau)]\right)\right)_{\vec{r}_c,\vec{w}} = \mathcal{P}(\vec{r}_c,\vec{w},\tau)\exp[\Omega(\vec{r}_c,\vec{w},\tau)] \qquad (II-40)$$

Since these are now simple integrable first order equations in one variable, integration yields the simple results

$$\Omega(\vec{r}_c,\vec{w},\tau_2) - \Omega(\vec{r}_c,\vec{w},\tau_1) = \int_{\tau_1}^{\tau_2} \nu(\vec{r}_c,\vec{w},\tau)\,d\tau \qquad (II-41)$$

and

$$f(\vec{r}_c,\vec{w},\tau_2)\exp[\Omega(\vec{r}_c,\vec{w},\tau_2)] - f(\vec{r}_c,\vec{w},\tau_1)\exp[\Omega(\vec{r}_c,\vec{w},\tau_1)]$$

$$= \int_{\tau_1}^{\tau_2} \mathcal{P}(\vec{r}_c,\vec{w},\tau)\exp[\Omega(\vec{r}_c,\vec{w},\tau)]\,d\tau \qquad (II-42)$$

Now it is possible to transform back to the original coordinates where $\vec{r}$, and $\vec{v}$ are physically meaningful. By properly defining $t_0 = t_1$ it is possible to set $\vec{r}_2(\tau_2) = \vec{r}_c$ and $\vec{v}_2(\tau_2) = \vec{w}$. This results in the following equations for $\Omega(\vec{r}_2,\vec{v}_2,t_2)$ and $f(\vec{r}_2,\vec{v}_2,t_2)$ in terms of

$\Omega(\vec{r}_1, \vec{v}_1, t_1)$ and $f(\vec{r}_1, \vec{v}_1, t_1)$.

$$\Omega(\vec{r}_2, \vec{v}_2, t_2) = \Omega\left(\vec{r}_2 - \int_{t_1}^{t_2} \vec{w}\, d\tau, \vec{v}_2 - \int_{t_1}^{t_2} \vec{a}\, d\tau, t_1\right)$$

$$+ \int_{t_1}^{t_2} \nu\left(\vec{r}_2 - \int_{\tau}^{t_2} \vec{w}\, d\tau', \vec{v}_2 - \int_{\tau}^{t_2} \vec{a}\, d\tau', \tau\right) d\tau \qquad \text{(II-43)}$$

$$f(\vec{r}_2, \vec{v}_2, t_2) = f\left(\vec{r}_2 - \int_{t_1}^{t_2} \vec{w}\, d\tau, \vec{v}_2 - \int_{t_1}^{t_2} \vec{a}\, d\tau, t_1\right) \cdot$$

$$\exp\left[-\left(\Omega(\vec{r}_2, \vec{v}_2, t_2) - \Omega\left(\vec{r}_2 - \int_{t_1}^{t_2} \vec{w}\, d\tau, \vec{v}_2 - \int_{t_1}^{t_2} \vec{a}\, d\tau, t_1\right)\right)\right]$$

$$+ \int_{t_1}^{t_2} \mathcal{P}\left(\vec{r}_2 - \int_{\tau}^{t_2} \vec{w}\, d\tau', \vec{v}_2 - \int_{\tau}^{t_2} \vec{a}\, d\tau', \tau\right) \exp\left[-\left(\Omega(\vec{r}_2, \vec{v}_2, t_2) - \Omega\left(\vec{r}_2 - \int_{\tau}^{t_2} \vec{w}\, d\tau', \vec{v}_2 - \int_{\tau}^{t_2} \vec{a}\, d\tau', \tau\right)\right)\right] d\tau$$

(II-44)

One must not be deceived into thinking that this purely formal solution is actually the answer. In one way or another $\mathcal{P}$ and ν depend on f and thus "the solution" is really a terribly complicated and non-linear integral equation. It is apparent, however, that provided $\mathcal{P}$ and ν are not excessively complicated functions of f a possibility of iteration exists for steady state problems, while direct step by step integration appears possible for time dependent problems, as f at t_2 only depends on f at t_1 and ν and $\mathcal{P}$ between t_1 and t_2 . Therefore by choosing $\Delta t = t_2 - t_1$, small enough so that properties can be expanded around t_1 one can obtain a simple recursion relation for f .

Before proceeding to investigate the application of this equation, either to steady state or initial value problems, it is important to investigate the particular expressions for $\mathcal{P}$ and ν that can be used. Of course, the Boltzmann definition of $\mathcal{P}$ and ν is exact, but it has the great disadvantage of being related to f through the detail expressions in a collision process. To avoid these complexities it is possible to approximate the averaged results of the collision processes by some scattering model that gives $\mathcal{P}$ and ν as a function of velocity and the macroscopic properties. The various possibilities in choosing these are discussed in the next chapter.

NOMENCLATURE

$\vec{a}$	Acceleration per particle due to external forces
$f(\vec{r},\vec{v},t)$	The distribution function in six dimensional space of $\vec{r}$ and $\vec{v}$
g	The magnitude of relative velocity in a two body collision
$g(\vec{r},\vec{v},t)$	$f(\vec{r},\vec{v},t)\exp[-\Omega(\vec{r},\vec{v},t)]$ Modified distribution function
$G(\vec{r},\vec{r}_0;t,t_0)$	Green's function for differential operator in Boltzmann equation
$\tilde{G}(\vec{r},\vec{r}_0;t,t_0)$	The adjoint of the above Green's function
$\vec{n}$	Vector normal to the boundary S and pointing into the volume V
$\vec{p}$	Fourier transform space vector
$\mathcal{P}(\vec{r},\vec{v},t)$	The scattering function specifying the density of particles scattered into the phase space at $\vec{r}$ and $\vec{v}$, at time t
$\vec{r}$	Position vector
$\vec{r}_c$	Vector in the transformed space designating the characteristic
$\vec{r}_s$	Position vector locating the surface S
S	The surface that bounds the volume V
t	Time
t_s	The time at which a particle with velocity $\vec{v}$ must leave the surface S to be at $\vec{r}$ at time t
$u(\tau)$	Step function (unity for positive argument and zero everywhere else)
$\vec{v}$	The velocity vector

V	The volume in which solution is desired
$\vec{w}$	Velocity along characteristic which gives the velocity $\vec{v}$ at $\vec{r}$ at time t
$W(\vec{r},\vec{v},t)$	Arbitrary weighting function in the generalized Boltzmann type differential equation
$\mathcal{D}/\mathcal{D}t$	$\partial/\partial t + \vec{v}\cdot(\nabla_r)_{\vec{v}} + \vec{a}\cdot(\nabla_v)_{\vec{r}}$ the differential operator appearing in Boltzmann equation
$\gamma(\vec{p},t-t_0)$	Fourier transform of the Green's function
$\delta(t-t_0)$	The delta function in time (is zero everywhere except when argument is zero and has unit area)
$\delta^{(3)}(\vec{r}-\vec{r}_0)$	Generalized three dimensional delta function
$\nu(\vec{r},\vec{v},t)$	The collision frequency of particles with velocity $\vec{v}$ at position $\vec{r}$ and time t .
$\sigma(\Omega)$	Differential collision cross section as a function of solid angle Ω
τ	Time along a characteristic
$\phi(\vec{r},\vec{v},t)$	Generalized function in Boltzmann type differential equation
$\Omega(\vec{r},\vec{v},t)$	Integrating factor appearing in the substitution of

$$g(\vec{r},\vec{v},t) = f(\vec{r},\vec{v},t)\exp\left[\Omega(\vec{r},\vec{v},t)\right]$$

III. COLLISION MODELS

In trying to solve the Boltzmann equation by any means most of the difficulty comes from the collision term which requires detailed knowledge of two-body collisions. The necessary detail analysis of the collision process is not only tedious but the resulting integrals cannot, in general, be evaluated. The fact that only the averaged results of collisions enter into the equation suggest looking for approximate scattering models that represent the averaged results of collisions to some desired accuracy. Although one knows that the only exact representation of binary scattering is the Boltzmann collision integral, one can hope to achieve a reasonable approximation by forcing the model to satisfy some of the same overall conditions as does the collision integral. Thus, any model, as a minimum requirement, has to satisfy the conservation laws. Beyond this one can, of course, add conditions improving the accuracy while increasing the complexity with each additional one.

The value of using models to get a quick order of magnitude estimate was already known years ago. The approach of a gas to equilibrium can be treated for small disturbances from equilibrium by simply substituting for the collision term the expression

$$\frac{(f_e - f)}{\tau}$$

where τ is the mean freetime, f_e the equilibrium value of the distribution, and f the unknown distribution function. The equilibrium distribution and τ are treated as constants and yield the exponential approach to equilibrium with the mean free time as the time constant. Further trivial extension of this model can serve to give correct parametric dependence of the transport properties and even correct order of magnitude quantitative results. When the above mentioned model is plugged into the Boltzmann

equation and the distribution function is expanded into

$$f = f_e + f_1 \qquad (\text{III}-1)$$

where

$$f_1 << f_e$$

the solution for f_1 becomes

$$f_1 = -\frac{D}{Dt} f_e \qquad (\text{III}-2)$$

The evaluation of transport properties resulting from this distribution gives answers which depend on the mean free path in the right way and are only incorrect in the numerical coefficients.

In recent years, the above approach to representing the collision term has been extended and formalized by Bhatnager, Gross and Krook[19] and further studied by Krook[20, 16]. In a general way Krook discusses the class of models that can be represented as

$$\frac{D f}{D t} = \nu(\vec{r},\vec{v},t)\left(\mathcal{E}(\vec{r},\vec{v},t) - f(\vec{r},\vec{v},t)\right) \qquad (\text{III}-3)$$

where

$$\mathcal{E}(\vec{r},\vec{v},t) = \frac{n_c}{(2\pi R T_c)^{3/2}} \exp\left[-(\vec{v}-\vec{u}_c)^2/2RT_c\right] \qquad (\text{III}-4)$$

with n_c, T_c and u_c defined as collisional number density, temperature and velocity respectively. In general, these "collisional" properties do not have to correspond to the actual macroscopic variables at the point in question. They are defined by guaranteeing the conservation of the

collision invariants:

$$\iiint \nu(\vec{r},\vec{v},t)\,\mathcal{E}(\vec{r},\vec{v},t)\,\psi_i(\vec{v})\,d^3\vec{v} = \iiint \nu(\vec{r},\vec{v},t)\,f(\vec{r},\vec{v},t)\,\psi_i(\vec{v})\,d^3\vec{v} \qquad \text{(III - 5)}$$

where

$$\psi_i(\vec{v}) = 1,\ \vec{v},\ v^2 \qquad \text{(III - 6)}$$

In the special case of $\nu(\vec{r},\vec{v},t) = \bar{\nu}(\vec{r},t)$ (independent of $\vec{v}$), the $\mathcal{E}(\vec{r},\vec{v},t)$ is just the standard Maxwellian distribution for the local macroscopic variables. This, of course, makes the model determinable from the macroscopic properties alone. This particular case is usually referred to as Krook's model and has been applied to a number of problems in rarefied flows. It converges to the continuum limit in a qualitatively correct way but does not give the correct quantitative results for the transport coefficients or the right ratios between them. The latter inaccuracy is a direct result of the assumption of constant collision frequency $\bar{\nu}(\vec{r},t)$ in velocity space. This prevents the proper weighting of the different velocity moments, but rather weights them all in the same way. The former inaccuracy is a result of the assumption of spherical symmetry inherent in this class of models. This assumption approximates the scattering of particles at any point in a spherically symmetric fashion, which is known to be incorrect on physical grounds. It is obvious that in a non-uniform flow particles colliding at any point on the average come from distances of the order of a mean free path. Now, if conditions are different at this average distance in different directions, then the scattering in different directions will also have to be different.

The possibility of correcting Krook's model by adding asymmetric components has been mentioned by the original authors and others but not carried out in detail. The integral equation formulation of the kinetic theory equation is particularly useful in recognizing the implications of

Krook's model and finding the proper means of improving its accuracy. Therefore, a detail investigation is carried out in this chapter, and modifications are suggested in the Krook's model that can, in principle, make it give quantitatively correct results to any order in the Chapman-Enskog expansion.

Before proceeding with the derivation it is best, however, to examine the reasons for devising a collision model and the qualities desired of it. The reason for wanting a collision model is simplicity in solving the Boltzmann equation in the region where few solutions now exist i.e., between free molecular and continuum conditions. Since in this region no straight forward expansion for the distribution function resulting in a possible linearization exists, the simplicity of the collision model is almost a necessity to solve the Boltzmann equation as a boundary value problem. Collisions increase in their importance to the solution of the problem from having no effect in free molecular flow to being the dominating phenomenon in the continuum range. It is therefore apparent that to check the accuracy of any collision model one has to turn to the continuum limit. Thus, the qualities that one desires in a model are; that it will conserve number, momentum and energy, and that it will give quantitatively correct results when the continuum condition is approached.

In order to illustrate the method without excessive mathematics the analysis is carried out here for a one-component, steady state, one-dimensional case. A more general three-dimensional case is carried out in vector notation in Appendix B. The general integral equation

$$f(\vec{r},\vec{v},t) = \int_{t_s u(t_s)}^{t} \wp(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)\, \exp\left[-\left(\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)\right)\right] dt_0$$

$$+ f(\vec{r}-\vec{v}(t-t_s),\vec{v},t_s)\, \exp\left[-\left(\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}-\vec{v}(t-t_s),\vec{v},t_s)\right)\right] u(t_s)$$

$$+ f(\vec{r}-\vec{v}t,\vec{v},0)\, \exp\left[-\left(\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}-\vec{v}t,\vec{v},0)\right)\right] u(-t_s) \qquad (III-7)$$

reduces for the one-dimensional steady state case to

$$f(x,\vec{v}) = \int_0^{\tau_s} \mathcal{P}(x-v_1\tau,\vec{v}) \exp\left[-(\Omega(x,\vec{v})-\Omega(x-v_1\tau,\vec{v})\right] d\tau$$

$$+ f(x-v_1\tau_s,\vec{v}) \exp\left[-(\Omega(x,\vec{v})-\Omega(x-v_1\tau_s,\vec{v}))\right]$$

where τ_s is the time a particle travels at velocity $\vec{v}_1$ from the boundary to get to position x and v_1 is the x component of velocity.

After the substitution of $x' = x - v_1\tau$ and definition of x_1 and x_2 as the boundaries of the problem, the equation reduces to

$$f_+(x,v) = \int_{x_1}^{x} \mathcal{P}(x',\vec{v}) \exp\left[-(\Omega(x,\vec{v})-\Omega(x',v))\right] dx'/v_1$$

$$+ f(x_1,\vec{v}) \exp\left[-(\Omega(x,\vec{v})-\Omega(x_1,\vec{v}))\right] \qquad v_1>0 \qquad \text{(III-8)}$$

$$f_-(x,\vec{v}) = \int_{x}^{x_2} \mathcal{P}(x',\vec{v}) \exp\left[-(\Omega(x,\vec{v})-\Omega(x',\vec{v}))\right] dx'/|v_1|$$

$$+ f(x_2,\vec{v}) \exp\left[-(\Omega(x,\vec{v})-\Omega(x_2,\vec{v}))\right] \qquad v_1<0 \qquad \text{(III-9)}$$

where $$\Omega(x,\vec{v}) = \int_{x_1}^{x} \nu(x',\vec{v})\, dx'/v_1$$

When Krook's model is substituted for the scattering terms, the equation becomes

$$f_+(x,\vec{v}) = \int_{x_1}^{x} \varepsilon(x',\vec{v}) \exp\left[-(\Omega(x,\vec{v})-\Omega(x',\vec{v}))\right] \nu(x',\vec{v})\, dx'/v_1$$

$$+ f(x_1,\vec{v}) \exp\left[-(\Omega(x,\vec{v})-\Omega(x_1,\vec{v}))\right] \qquad v_1>0 \qquad \text{(III-10)}$$

$$f_-(x,\vec{v}) = \int_{x}^{x_2} \varepsilon(x',\vec{v}) \exp\left[-(\Omega(x,\vec{v})-\Omega(x',\vec{v}))\right] \nu(x',\vec{v})\, dx'/|v_1|$$

$$+ f(x_2,\vec{v}) \exp\left[-(\Omega(x,\vec{v})-\Omega(x_2,\vec{v}))\right] \qquad v_1<0 \qquad \text{(III-11)}$$

Since our interest is now directed toward the behavior of the model as the continuum limit is approached, the boundary condition can be neglected by assuming that the boundaries are infinitely far away. In this case the equation becomes

$$f_+(x,\vec{v}) = \int_{-\infty}^{x} \varepsilon(x',\vec{v}) \exp\left[-(\Omega(x,\vec{v}) - \Omega(x',\vec{v}))\right] \nu(x',\vec{v})\, dx'/v_1$$

$$\text{for} \qquad v_1 > 0 \qquad \text{(III-12)}$$

$$f_-(x,\vec{v}) = \int_{x}^{\infty} \varepsilon(x',\vec{v}) \exp\left[-(\Omega(x,\vec{v}) - \Omega(x',\vec{v}))\right] \nu(x',\vec{v})\, dx'/|v_1|$$

$$\text{for} \qquad v_1 < 0 \qquad \text{(III-13)}$$

By noticing that

$$\nu(x',\vec{v})\frac{dx'}{v_1} = -d\left(\Omega(x,\vec{v}) - \Omega(x',\vec{v})\right) \qquad \text{(III-14)}$$

one can immediately integrate this equation by parts as many times as one desires. The following change of variables, however, simplifies the partial integrations immensely. Let $K(x,x') = \Omega(x,\vec{v}) - \Omega(x',\vec{v})$, then the equation simply becomes

$$f(x,\vec{v}) = \int_{0}^{K(x,\pm\infty)} \varepsilon(x',\vec{v}) \exp\left(-K(x,x')\right) d\,K(x,x') \qquad \text{(III-15)}$$

where $K(x,-\infty)$ is used for $v_1 > 0$ and $K(x,+\infty)$ for $v_1 < 0$.

From the definition of

$$K(x,x') = \int_{x'}^{x} \nu(x'')\frac{dx''}{v_1} \qquad \text{(III-16)}$$

it is apparent that $K(x,-\infty) \Rightarrow \infty$ when $v_1 > 0$ and $\nu(x'')$ finite and also $K(x,+\infty) \Rightarrow \infty$ for $v_1 < 0$ and $\nu(x'')$ finite. The equation for f becomes

$$f(x,\vec{v}) = \int_0^\infty \mathcal{E}(x',\vec{v}) \exp\left[-K(x,x')\right] dK(x,x') \qquad \text{(III-17)}$$

for all v_1. This is reasonable, as when the boundaries do not enter the problem there should be no difference in representation of $f(x,\vec{v})$ for different directions of the velocity.

This equation can be immediately integrated to give

$$f(x,\vec{v}) = \mathcal{E}(x,\vec{v}) + \int_0^\infty \frac{d\mathcal{E}}{dK} \exp\left[-K(x,x')\right] dK \qquad \text{(III-18)}$$

and further

$$f(x,\vec{v}) = \mathcal{E}(x,\vec{v}) + \left(\frac{d\mathcal{E}}{dK}\right)_{K=0}$$

$$+ \int_0^\infty \frac{d^2\mathcal{E}}{dK^2} \exp\left(-K(x,x')\right) dK(x,x') \qquad \text{(III-19)}$$

$$f(x,\vec{v}) = \mathcal{E}(x,\vec{v}) + \left(\frac{d\mathcal{E}}{dK}\right)_{K=0}$$

$$+ \left(\frac{d^2\mathcal{E}}{dK^2}\right)_{K=0} + \left(\frac{d^3\mathcal{E}}{dK^3}\right)_{K=0} + \cdots\cdots\cdots \qquad \text{(III-20)}$$

When this is transformed back to the meaningful physical variables

$$f(x,\vec{v}) = \mathcal{E}(x,\vec{v}) + \left(\frac{d\mathcal{E}}{dx'}\Big/\frac{dK}{dx'}\right)_{x'=x}$$

$$+\left[\left(\frac{d}{dx'}\left(\frac{d\mathcal{E}}{dx'}\Big/\frac{dK}{dx'}\right)\right)\Big/\frac{dK}{dx'}\right]_{x'=x} + \cdots\cdots \qquad \text{(III-21)}$$

or

$$f(x,\vec{v}) = \mathcal{E}(x,\vec{v}) - \frac{v_1}{\nu(x,\vec{v})}\frac{d\mathcal{E}}{dx}(x,\vec{v})$$

$$+ \frac{v_1}{\nu(x,\vec{v})}\frac{d}{dx}\left(\frac{v_1}{\nu(x,\vec{v})}\frac{d\mathcal{E}}{dx}(x,\vec{v})\right) + \cdots\cdots \qquad \text{(III-22)}$$

This is obviously an expansion for f in terms of inverse collision frequency and might be correct at least very near the infinite ν condition. This expansion therefore can be easily used to evaluate exactly the expression for $\mathcal{E}(x,\vec{v})$ that one desires in order to make this model give the correct results at continuum and near it. All that is necessary is to choose a model that has enough free parameters to satisfy the conservation laws as well as yield correct transport coefficients evaluated from $f(x,\vec{v})$ at continuum.

To fix ideas let us start with the usual Krook's model, i.e., $\mathcal{E}(x,\vec{v}) = f_0(x,\vec{v}) = (n(x)/(2\pi RT(x))^{3/2})\exp\left[-\frac{(v-u(x))^2}{2RT(x)}\right]$ and $\nu(x)$ equal to the averaged collision frequency independent of $\vec{v}$. In this case we know this model satisfies the conservation equations and all that remains is to evaluate the transport coefficients to see this model's behavior at continuum. The definition of continuum implied that only terms up to order $(v_1/\nu, \alpha/dx)$ are to be kept. Thus, the equation for f becomes

$$f(x,\vec{v}) = f_0(x,\vec{v})\left(1 - \frac{v_1}{\nu(x)}\frac{d\ln f_0}{dx}\right) \qquad \text{(III-23)}$$

Since $f_o(x,\vec{v})$ depends on x only through the parameters $n(x)$, $T(x)$ and $u(x)$ the derivative $d\ln f_o/dx$ can be immediately carried out in terms of gradients of these macroscopic quantities.

$$\frac{d\ln f_o}{dx} = \frac{1}{n(x)}\frac{dn(x)}{dx} - \frac{3}{2T(x)}\frac{dT(x)}{dx}$$

$$+ \frac{(v_1-u)^2+v_2^2+v_3^2}{2RT(x)}\frac{1}{T(x)}\frac{dT}{dx} + 2\frac{(v_1-u)}{2RT(x)}\frac{du}{dx}(x) \qquad \text{(III-24)}$$

It can immediately be seen that for the case of no macroscopic velocity and no pressure gradient

$$\frac{d\ln f_o}{dx} = \left(\frac{v_1^2+v_2^2+v_3^2}{2RT(x)} - \frac{5}{2}\right)\frac{d\ln T}{dx}(x) \qquad \text{(III-25)}$$

and the resulting heat flux in the x direction becomes

$$q_x = \iiint f(x,\vec{v})\frac{mv^2}{2}v_1\,dv_1\,dv_2\,dv_3$$

$$= -\iiint f_o(x,\vec{v})\,v_1^2\frac{mv^2}{2}\left(\frac{v^2}{2RT}-\frac{5}{2}\right)dv_1\,dv_2\,dv_3\,\frac{1}{\nu(x)}\frac{d\ln T}{dx}$$

$$= -\frac{5}{32}\frac{m(2RT)^{3/2}\sqrt{\pi}}{S\,T}\frac{dT}{dx} \qquad \text{(III-26)}$$

where $\nu = \sqrt{2}\,n\,S \times 2/\sqrt{\pi}\sqrt{2RT}$ and S is the collision cross section.

This makes the heat flux proportional to the temperature gradient as it should be for continuum, but it further gives the right parametric dependence of the coefficient on the properties like temperature and collision cross section. The numerical coefficient is, however, wrong as the correct answer is [34]

$$q_x = -\frac{75}{128}(1+\alpha)\frac{m(2RT)^{3/2}\sqrt{\pi/2}}{sT}\frac{dT}{dx} \qquad (III-27)$$

where $0 < \alpha < .026$ depending on the repulsion law. Thus, the answer is wrong by the ratio 8/15 or roughly 45 percent.

The more general case for both temperature and velocity gradients can be calculated by using the conservation equations to properly separate the terms contributing to the heat flux and the viscous stress. Here it is important to remember that the term that is being evaluated $d\ln f_0/dx$ is already multiplied by the inverse of the collision frequency. Thus, it is only necessary to evaluate it to zeroth order in inverse collision frequency to retain accuracy to first order. The proper conservation equations therefore are

$$\frac{dnu_1}{dx} = 0 \qquad (III-28)$$

$$\frac{dp}{dx} = -\rho u_1 \frac{du_1}{dx} \qquad (III-29)$$

and

$$u_1\frac{dT}{dx} = -\frac{2}{3}\frac{p}{kn}\frac{du_1}{dx} \qquad (III-30)$$

and the state equation is

$$p = nkT \qquad \text{(III-31)}$$

The expression can be written as follows

$$v_1 \frac{d\ln f_0}{dx} = (v_1 - u_1) \frac{d\ln f_0}{dx}$$

$$+ u_1 \frac{d\ln f_0}{dx} \qquad \text{(III-32)}$$

The first term gives

$$(v_1-u_1)\left(\frac{d\ln n}{dx} - \frac{3}{2}\frac{d\ln T}{dx} + \frac{(\vec{v}-\vec{u})^2}{2RT}\frac{d\ln T}{dx}\right.$$

$$\left. + \frac{v_1-u_1}{RT}\frac{du_1}{dx}\right) = (v_1-u_1)\left(\frac{(\vec{v}-\vec{u})^2}{2RT} - \frac{5}{2}\right)\frac{d\ln T}{dx}$$

$$+ \frac{(v_1-u_1)}{RT}(v_1-u_1-u_1)\frac{du_1}{dx} \qquad \text{(III-33)}$$

The second term can be manipulated to give a result purely proportional to $\frac{du_1}{dx}$

$$u_1 \frac{d\ln f_0}{dx} = u_1\left(\frac{d\ln n}{dx} + \left(\frac{(\vec{v}-\vec{u})^2}{2RT} - \frac{3}{2}\right)\frac{d\ln T}{dx}\right.$$

$$+ u_1 \frac{(v_1-u_1)}{RT}\frac{du_1}{dx} \qquad \text{(III-34)}$$

By the equation of continuity and the energy equation this gives

$$u_1(v_1-u_1)\frac{du_1}{dx} - \frac{1}{3}\frac{(\vec{v}-\vec{u})^2}{RT}\frac{du_1}{dx}$$

Combination of the terms gives

$$v_1 \frac{d \ln f_o}{dx} = (v_1 - u_1)\left(\frac{(\vec{v}-\vec{u})^2}{2RT} - \frac{5}{2}\right)\frac{d \ln T}{dx}$$

$$+ \frac{(v_1-u_1)^2 - \frac{1}{3}(\vec{v}-\vec{u})^2}{RT}\frac{du_1}{dx} \qquad (III-35)$$

Substitution into integrals of the collision invariants over velocity space yields the desired results that

$$\iiint v_1 \frac{d \ln f_o}{dx} \psi_i(\vec{v})\, d^3\vec{v} \equiv 0 \qquad (III-36)$$

where

$$\psi_i = 1, \vec{v}, v^2$$

Furthermore, from the form of the expression for $v_1 \, d \ln f_o/dx$ it is apparent that the heat flux will be proportional to the temperature gradient alone, while the stress will be proportional to the velocity gradient alone. It is not necessary to carry out the calculation in detail for the stress as it is apparent that analogously to the heat flux it will give the right parametric dependence but the wrong numerical coefficient.

The important result that can be gained from this analysis is an immediate correction for the two inaccuracies in the Krook's model. Both the incorrect numerical coefficients and their incorrect ratio can be immediately corrected by multiplying the usual Krook's model by

$$1 - \frac{c_1}{\nu}(v_1-u_1)\left(\frac{(\vec{v}-\vec{u})^2}{2RT} - \frac{5}{2}\right)\frac{d \ln T}{dx} - \frac{c_2}{\nu}\frac{(v_1-u_1)^2 - \frac{1}{3}(\vec{v}-\vec{u})^2}{RT}\frac{du_1}{dx}$$

where c_1 and c_2 are chosen in such a way as to make the transport coefficients come out correctly in the continuum range. Since only terms of first order in inverse collision frequency are retained the expression $\nu_1/\nu \; d\ln\varepsilon/dx$ is still just $\nu_1/\nu \; d\ln f_0/dx$ The expression for f, however, is now multiplied by the additional asymmetric term above, which to first order in $1/\nu$ gives

$$f(x,\vec{v}) = f_0(x,\vec{v}) \left[1 + \frac{c_1+1}{\nu}(u_1-v_1)\left(\frac{(\vec{v}-\vec{w})^2}{2RT} - \frac{5}{2}\right)\frac{d\ln T}{dx} \right.$$

$$\left. - \frac{c_2+1}{\nu} \frac{(v_1-u_1)^2 - \frac{1}{3}(\vec{v}-\vec{w})^2}{RT} \frac{du_1}{dx} \right] \qquad (III-37)$$

The value of c_1 can be easily obtained by referring back to the discrepancy between Krook's model value and the correct one for the thermal conductivity.

$$\frac{5}{16}(c_1+1) = \frac{75}{128}(1+\alpha)$$

$$c_1 = \frac{7}{8} + \frac{5}{8}\alpha \qquad (III-38)$$

In a similar way the value of c_2 can be found to be.

This method is not limited to correcting Krook's model to first order in $1/\nu$ but can be easily extended to any higher order by just adding higher order asymmetry into $\mathcal{E}(x,\vec{v}$ and retaining more terms. Since the analysis basically follows the same pattern as the Chapman-Enskog expansion it is not necessary to carry out the details just to illustrate the point.

The usefulness of any higher order corrections to Krook's model is dubious anyway because of the lack of good verification that the inclusion of the higher order terms in the Chapman-Enskog series actually does extend the range of validity.

It is important to notice that the correct quantitative limits were obtained without having to resort to using a collision frequency dependent on velocity. This avoids the unmanageable integrations resulting from any realistic representation of $\nu(\vec{r},\vec{v})$ while retaining the proper ratios between the transport coefficients. One cannot, of course, predict the accuracy of this model representation in the transition regime until some exact solutions are available for comparison. One can, however, suggest that since collisions between particles become of less and less importance as the free molecular limit is approached, any inaccuracy in the model is offset, at least partially, by the decreasing importance of the model to the solution of the problem. It is important to notice that though the model is obtained by expansion in inverse powers of the collision frequency the usefulness of the model, even to first order, is not necessarily as limited as the Navier-Stokes results. When the model is actually plugged into the integral equation it need no longer be solved by partial integrations. Thus, though the asymmetry in the model is proportional to the gradients this asymmetry is integrated over, making the answer not necessarily limited to the range of validity of the expansion.

NOMENCLATURE

c_1	Constant in correction of Krook's model to give correct heat flux at continuum
c_2	Constant in correction of Krook's model to give correct pressure tensor at continuum
f	The distribution function
f_e	The equilibrium distribution function
f_o	The local Maxwellian distribution function
f_1	The first order correction for the distribution function
k	The Boltzmann constant
n	The number density of particles in physical space
p	The pressure of the gas
R	The gas constant of the gas
S	Collision cross section
T	The temperature of the gas
$\vec{u}$	The macroscopic velocity vector
$u(x)$	The macroscopic velocity at position x in the x direction
$\vec{v}$	The velocity vector in phase space, v_1, v_2, v_3 are the x, y and z components of $\vec{v}$
x	The component of $\vec{r}$ for which one dimensional problems are analysed
D/Dt	The Boltzmann differential operator
α	Constant appearing in theoretical expression for thermal conductivity

$\mathcal{E}(\vec{r},\vec{v},t)$	The distribution of particles scattered at position $\vec{r}$ and having the velocity $\vec{v}$ at time t, used in the approximate collision model
$K(x,x')$	The number of collision a stream of particles traveling with velocity $\vec{v}_i$, suffers between x' and x $\left(\int_{x'}^{x}\vec{\nu}(x'')\,dx''\right)$
$\nu(\vec{r},\vec{v},t)$	The collision frequency at $\vec{r}$ and t for particles with the velocity $\vec{v}$
$\nu(\vec{r},t)$	The average collision frequency at $\vec{r}$ and t
τ_f	The mean free time between collisions
ψ_i	Any collision invariant

Subscripts

c	Designates "collisional" properties
$+$	The part of velocity space where $v_i > 0$
$-$	The part of velocity space where $v_i < 0$

IV. TRANSITION REGIME PROBLEMS

We have obtained a purely integral formulation of the Boltzmann equation and have discussed the approximate models that can be substituted for the exact collision terms in order to simplify solution in the transition regime. Now it is necessary to discuss the types of problems to which this formulation might be most applicable and indicate, in major outline, the methods of attack for solving them.

The types of problems that arise in the "transition" regime can be divided into subgroups along many lines. It certainly can be done according to the basic geometry or according to the basic physical phenomenon, such as diffusion, conduction, etc. No method of subdivision can purport to include all possible subclasses and thus determine all the problems to be encountered. Thus, though in no way claiming completeness, we divide the transition regime into three categories, initial value, macroscopic discontinuity and boundary value problems. An initial value problem in kinetic theory starts with completely determined distribution function in some region V at a time t. The desired solution is the distribution function within V for all times later than t. Since in kinetic theory time is essentially scaled by the mean free time between collisions, initial value problems consist of solutions for times less than and of the same order as the mean free time. In a general case, the region V may be bounded or not bounded by a solid surface S. We shall limit ourselves to discussion of initial value problems without solid boundaries. The more general case can be included by combining the initial value with a boundary value type problem.

A macroscopic discontinuity type problem consists of a sharp variation of some property over distances of the order of a mean free path, which makes it appear as a discontinuity on the macroscopic scale. This could include such phenomena as a shock, a shear layer,

or an interface between two gases. Here no solid boundaries enter the problem and the distribution function is not known at any particular time t. The only information available is usually in the form of asymptotic boundary conditions on the distribution function.

The third type, a boundary value problem, consists of the steady state solution of the distribution function in a region V bounded by a surface S on which certain thermodynamic variables are known. The characteristic that identifies this type of problem is the necessity of evaluating the distribution function of particles coming from the boundaries. Here the actual knowledge of what takes place is very limited and the available experimental information quite meager and imperfect. In the class of boundary value problems treated here, the usual accomodation type of treatment for the surface interactions is assumed valid.

The integral equation for the initial value case with no solid boundaries and no external forces becomes

$$f(\vec{r},\vec{v},t) = \int_0^t \mathcal{P}(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)\exp\left[-(\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0))\right]dt_0$$

$$+ f(\vec{r}-\vec{v}t,\vec{v},0)\exp\left[-(\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}-\vec{v}t,\vec{v},0))\right] \qquad (\text{IV}-1)$$

$$\text{where} \qquad \Omega(\vec{r},\vec{v},t) = \int_0^t \nu(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)\,dt_0 \qquad (\text{IV}-2)$$

In order to make the substitution of the Krook's type model easier one can immediately divide up $\mathcal{P}(\vec{r},\vec{v},t)$ into $\nu(\vec{r},\vec{v},t)\times\mathcal{E}(\vec{r},\vec{v},t)/\nu(\vec{r},\vec{v},t)$, and call the latter term $\mathcal{E}(\vec{r},\vec{v},t)$. Furthermore, the lower limit need not be zero but can be taken as any time t_1. With this change and the

new notation the equation for f becomes

$$f(\vec{r},\vec{v},t) = \int_{t_1}^{t} \mathcal{E}(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)\exp\left[-(\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0))\right]\times$$

$$\nu(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)\,dt_0 + f(\vec{r}-\vec{v}(t-t_1),\vec{v},t_1)\exp\left[-(\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}-\vec{v}(t-t_1),\vec{v},t_0))\right] \qquad (IV-3)$$

It is immediately apparent that the distribution function at time t depends only on the distribution function at time t_1 and the scattering terms from the time t_1, to the time t. Thus, the equation is similar to the parabolic type and might be tractable, at least numerically, in small increments of time.

Letting $t-t_1=\Delta t$ and defining a new variable $t-t_0=\tau$ the expression for f becomes

$$f(\vec{r},\vec{v},t) = \int_{0}^{\Delta t} \mathcal{E}(\vec{r}-\vec{v}\tau,\vec{v},t-\tau)\exp\left[-\int_{0}^{\tau}\nu(\vec{r}-\vec{v}\tau',\vec{v},t-\tau')\,d\tau'\right]\times$$

$$\nu(\vec{r}-\vec{v}\tau,\vec{v},t-\tau)\,d\tau$$

$$+ f(\vec{r}-\vec{v}\Delta t,\vec{v},t-\Delta t)\exp\left[-\int_{0}^{\Delta t}\nu(\vec{r}-\vec{v}\tau',\vec{v},t-\tau')\,d\tau'\right] \qquad (IV-4)$$

Now if Δt is made to approach a small number the integral can be

evaluated by Taylor series expansion about the point t_1 , where all conditions are known.

$$\mathcal{E}(\vec{r}-\vec{v}\tau, \vec{v}, t-\tau) = \mathcal{E}(\vec{r}-\vec{v}\Delta t, \vec{v}, t-\Delta t)$$

$$+ (\Delta t-\tau)\left(\frac{d\mathcal{E}}{d\tau}(\vec{r}-\vec{v}\tau, \vec{v}, t-\tau)\right)_{\tau=\Delta t}$$

$$+ \frac{(\Delta t-\tau)^2}{2}\left(\frac{d^2\mathcal{E}}{d\tau^2}(\vec{r}-\vec{v}\tau, \vec{v}, t-\tau)\right)_{\tau=\Delta t} + \cdots\cdots \qquad (\text{IV}-5)$$

and

$$\nu(\vec{r}-\vec{v}\tau', \vec{v}, t-\tau) = \nu(\vec{r}-\vec{v}\Delta t, \vec{v}, t-\Delta t)$$

$$+ (\Delta t-\tau')\left(\frac{d\nu}{d\tau'}(\vec{r}-\vec{v}\tau', \vec{v}, t-\tau')\right)_{\tau'=\Delta t}$$

$$+ \frac{(\Delta t-\tau')^2}{2}\left(\frac{d^2\nu}{d\tau'^2}(\vec{r}-\vec{v}\tau', \vec{v}, t-\tau')\right)_{\tau'=\Delta t} + \cdots\cdots \qquad (\text{IV}-6)$$

Plugging this into the expression for f yields the result

$$f(\vec{r},\vec{v},t) = \int_0^{\Delta t} \Big(\mathcal{E}(\vec{r}-\vec{v}\Delta t,\vec{v},t_1) + (\Delta t-\tau)\left(\frac{d\mathcal{E}}{d\tau}\right)_{\tau=\Delta t} + \dots\Big) \times$$

$$\exp\left[-\nu(\vec{r}-\vec{v}\Delta t,\vec{v},t_1)\tau - \left(\frac{d\nu}{d\tau}\right)_{\tau=\Delta t} \times \left(\Delta t\,\tau - \frac{\tau^2}{2}\right) + \dots\right] \times$$

$$\left(\nu(\vec{r}-\vec{v}\Delta t,\vec{v},t_1) + (\Delta t-\tau)\left(\frac{d\nu}{d\tau}\right)_{\tau=\Delta t} + \dots\right) d\tau$$

$$+ f(\vec{r}-\vec{v}\Delta t,\vec{v},t_1)\exp\left[-\nu(\vec{r}-\vec{v}\Delta t,\vec{v},t_1)\Delta t\right.$$

$$\left. - \left(\frac{d\nu}{d\tau}\right)_{\tau=\Delta t} \times \frac{(\Delta t)^2}{2} + \dots\right] \qquad \text{IV-7}$$

Integration to second order in Δt yields

$$f(\vec{r},\vec{v},t) = f(\vec{r}-\vec{v}\Delta t,\vec{v},t_1)$$

$$+ \left(\mathcal{E}(\vec{r}-\vec{v}\Delta t,\vec{v},t_1) - f(\vec{r}-\vec{v}\Delta t,\vec{v},t_1)\right)\left(\nu(\vec{r}-\vec{v}\Delta t,\vec{v},t_1)\Delta t\right)$$

$$- \left[\left(\mathcal{E}(\vec{r}-\vec{v}\Delta t,\vec{v},t_1) - f(\vec{r}-\vec{v}\Delta t,\vec{v},t_1)\right)\left(1 - \left(\frac{1}{\nu^2}\frac{d\nu}{d\tau}\right)_{\tau=\Delta t}\right)\right.$$

$$\left. - \left(\frac{1}{\nu}\frac{d\mathcal{E}}{d\tau}\right)_{\tau=\Delta t}\right] \times \frac{(\nu(\vec{r}-\vec{v}\Delta t,\vec{v},t_1)\Delta t)^2}{2} \qquad \text{IV-8}$$

This equation can now be solved, in principle, in a step by step numerical process when f, ν, and $\mathcal{E}$ are known at sometime t_1, for anytime t later than t_1. The result to first order in Δt is the same as that obtained by transforming the original differential equation into a difference equation to first order in $\nu\Delta t$. The higher order terms differ, however, because in the integration of the equation the νf on the right hand side has been transposed into the left hand differential operator. Thus, whereas in the direct change to a difference equation terms like $(df/d\tau)_{\tau=\Delta t}$ have to appear and thus result in a three term recussion relation for f, in the integrated form only the "driving function" appears in the derivatives. It is important to remark here that $(d/d\tau)_{\tau=\Delta t}$ is actually equal to $+\left(\partial/\partial t + \vec{v}\cdot\nabla_r\right)_{t=t_1}$ but it can always be reduced to terms proportional to the gradients of the macroscopic properties by use of the conservation equations and the dependence of the collision model on the macroscopic variables.

Treatment of a macroscopic discontinuity problem can be started from the same basic integral equation, but with the initial value and boundary terms omitted. Further, since effects in kinetic theory are only felt several mean free paths away, the limits of integration can be made to go to infinity once it is known no boundaries exist within a few mean free paths away from the point in question. Thus, the basic equation becomes

$$f(\vec{r},\vec{v}) = \int_0^\infty \mathcal{E}(\vec{r}-\vec{v}\tau,\vec{v}) \exp\left[-\left(\Omega(\vec{r},\vec{v}) - \Omega(\vec{r}-\vec{v}\tau,\vec{v})\right)\right] \times$$

$$\nu(\vec{r}-\vec{v}\tau,\vec{v})\, d\tau \qquad \text{(IV-9)}$$

with $\mathcal{P}$ again replaced by $\nu\mathcal{E}$ to make the transition to Krook's model simpler. One notices that this equation can be integrated by parts as

$$d\left(\Omega(\vec{r},\vec{v}) - \Omega(\vec{r}-\vec{v}\tau,\vec{v})\right) = -\nu(\vec{r}-\vec{v}\tau,\vec{v})\, d\tau \qquad \text{(IV-10)}$$

This, however, yields results exactly as those obtained in the analysis of the modifications of Krook's model in Chapter III, i.e., an expansion in inverse powers of the collision frequency. Thus, in this case, where the operator $(1/\nu \; d/d\tau)$ cannot be assumed small because of the large variations in a mean free path, this expansion is of very dubious value. As a matter of fact, the very purpose in turning to kinetic theory in studies of structures of macroscopic discontinuities is to avoid this assumption which is inherent in the Navier-Stokes relations.

Since the integral equation is not of the type that can be solved by standard methods, the only alternative is to solve it approximately by some form of iteration. The lack of standard means of solution also precludes, in the general case, the application of the standard tests of convergence. Thus, for the present, iteration is suggested as a possibility with actual convergence in each case being made plausible by the approach of two successive iterations to the same value.

For the method to be powerful and have practical application it should not be too sensitive to the initial choice of the trial function provided the proper boundary conditions are satisfied. One should be able to obtain a reasonable approximation to the structure of a discontinuity by starting with no information about the structure but only the necessary asymptotic values on each side of the discontinuity. If this is the case, then the method can have great power in analysing problems about which there is little or no detail information, provided of course, only a few iterations are needed for convergence.

In general, the results of Euler's equation for a macroscopic discontinuity will give discontinuous $\mathcal{E}$ and ν which can be expressed in the following form

$$\mathcal{E}(\vec{r},\vec{v}) = \mathcal{E}_1(\vec{v}) \quad \text{within region } V_1 \qquad (IV\text{-}11)$$

$$\mathcal{E}(\vec{r},\vec{v}) = \mathcal{E}_2(\vec{v}) \quad \text{within region } V_2 \qquad (IV\text{-}12)$$

and

$$\nu(\vec{r},\vec{v}) = \nu_1(\vec{v}) \qquad \text{within region } V_1 \qquad (IV\text{-}13)$$

$$\nu(\vec{r},\vec{v}) = \nu_2(\vec{v}) \qquad \text{within region } V_2 \qquad (IV\text{-}14)$$

The equation

$$\vec{r} - \vec{r}_s = 0$$

giving the shape of boundary locating the discontinuity and thus the boundary between regions V_1 and V_2 is determined also from solution of Euler's equations. Substitution of this into the equation for f yields

$$f(\vec{r},\vec{v}) = \int_0^{\tau_s(\vec{r},\vec{v})} \mathcal{E}_1(\vec{v}) \exp[-\nu_1 \tau]\, \nu_1\, d\tau$$

$$+ \int_{\tau_s(\vec{r},\vec{v})}^{\infty} \mathcal{E}_2(\vec{v}) \exp[-\nu_2 \tau]\, \nu_2\, d\tau \qquad \text{within } V_1 \qquad (IV\text{-}15)$$

$$f(\vec{r},\vec{v}) = \int_0^{\tau_s(\vec{r},\vec{v})} \mathcal{E}_2 \exp[-\nu_2 \tau]\, \nu_2\, d\tau$$

$$+ \int_{\tau_s(\vec{r},\vec{v})}^{\infty} \mathcal{E}_1 \exp[-\nu_1 \tau]\, \nu_1\, d\tau \qquad \text{within } V_2 \qquad (IV\text{-}16)$$

where τ_s is the value at which $\vec{r} - \vec{v}\tau$ crosses the boundary between V_1 and V_2. Now if $f(\vec{r}_s,\vec{v})$ is evaluated it is apparent that $\tau_s(\vec{r},\vec{v})$ is such that depending on the direction of velocity it is either zero or infinity. Thus, if the direction of velocity is from V_1 to V_2 the

expression for f becomes

$$f(\vec{r}_s,\vec{v}) = \int_0^\infty \varepsilon_1 \exp[-\nu_1\tau]\nu_1\, d\tau \qquad (IV-17)$$

for $\vec{v}$ from V_1 to V_2

regardless of whether the formula in V_1 or V_2 is used. Likewise, when direction of velocity $\vec{v}$ is from V_2 to V_1, f becomes

$$f(\vec{r}_s,\vec{v}) = \int_0^\infty \varepsilon_2 \exp[-\nu_2\tau]\nu_2\, d\tau \qquad (IV-18)$$

for $\vec{v}$ from V_2 to V_1

regardless of which formula is used.

This result is quite important for it shows that though $f(\vec{r}_s,\vec{v})$ has now become discontinuous in velocity space, it is continuous in physical space. Therefore, all macroscopic properties, being integrals over velocity space, become continuous across the original discontinuity. The discontinuity in velocity space is only a spurious result of the wide initial trial function and will probably also disappear with successive iterations. As a matter of fact, the great crudeness of the initial iterate means that iteration from this trial function is an extreme test of the method's usefulness and validity, and thus can not be taken as an absolute test of the method.

The steady state boundary value problem can be analysed by starting with the basic equation

$$f(\vec{r},\vec{v},t) = \int_0^{t_s} \mathcal{E}(\vec{r}-\vec{v}\tau,\vec{v},t-\tau)\exp\left[-\left(\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}-\vec{v}\tau,\vec{v},t-\tau)\right)\right]\times$$

$$\times\,\nu(\vec{r}-\vec{v}\tau,\vec{v},t-\tau)\,d\tau$$

$$+ f(\vec{r}-\vec{v}t_s,\vec{v},t-t_s)\exp\left[-\left(\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}-\vec{v}t_s,\vec{v},t-t_s)\right)\right] \qquad (\text{IV-19})$$

where τ_s is the time for a particle traveling at velocity $\vec{v}$ to come from the surface S to the point $\vec{r}$. To obtain the steady state solution, i.e., $t \Rightarrow \infty$ one needs only to drop the time dependence

$$f(\vec{r},\vec{v}) = \int_0^{t_s} \mathcal{E}(\vec{r}-\vec{v}\tau,\vec{v},t-\tau)\exp\left[-\left(\Omega(\vec{r},\vec{v})-\Omega(\vec{r}-\vec{v}\tau,\vec{v})\right)\right]$$

$$\times\,\nu(\vec{r}-\vec{v}\tau,\vec{v})\,d\tau$$

$$+ f(\vec{r}-\vec{v}t_s,\vec{v})\exp\left[-\left(\Omega(\vec{r},\vec{v})-\Omega(\vec{r}-\vec{v}t_s,\vec{v})\right)\right] \qquad (\text{IV-20})$$

Here again it is apparent that because of the non-linearity no standard form of solving integral equations will work. Integration by parts is again undesirable because of the resultant expansion in inverse powers of collision frequency. Iteration is again a possibility, but two new problems arise that did not appear in the macroscopic discontinuity problem. First of all the zeroth iterate, i.e., the trial function is not immediately apparent. It is already known that by starting with the free

molecular solution iteration yields an expansion in inverse powers of Knudsen number (with a logarithmic term included). This is unsatisfactory if the whole "transition regime" is to be covered. Secondly, necessity of knowing $f(\vec{r}-\vec{v}t_s, \vec{v})$, i.e., the distribution of particles emitted from a solid boundary brings up the whole problem of what really happens to particles as they strike the boundary and are re-emitted. This problem is the subject of much study at present, but no really conclusive results are yet available.

The choice of the initial trial function can be narrowed down by looking at behavior of the integral equation at the two extremes. The free molecular solution is immediately available from this equation in exact form

$$f(\vec{r},\vec{v}) = f(\vec{r}-\vec{v}t_s,\vec{v})\exp\left[+(\Omega(\vec{r}-\vec{v}t_s)-\Omega(\vec{r},\vec{v}))\right] \qquad (IV-21)$$

and

$$\Omega(\vec{r},\vec{v})-\Omega(\vec{r}-\vec{v}t_s) = \int_0^{t_s} \nu(\vec{r}-\vec{v}\tau,\vec{v})\,d\tau \Rightarrow 0 \qquad (IV-22)$$

Since $\int_0^{t_s} \nu\, d\tau$ near the surface has to be small to satisfy the free molecular requirement the solution becomes simply

$$f(\vec{r},\vec{v}) = f(\vec{r}-\vec{v}t_s,\vec{v}) \qquad (IV-23)$$

Thus, regardless of the initial choice of the trial function the solution of the integral equation will always converge to the proper limit at the free molecular extreme.

At the continuum extreme, where the boundaries can be neglected, we have seen that the solution converges to the right limit by partial integration, when the right model is used. In the iterative scheme this implies that the model used have the macroscopic properties in it

converge to the right variations at the continuum end. If the exact form of the collision integral is used, it is necessary that the trial distribution function converge to the proper value to first order in the Chapman-Enskog expansion to obtain the proper limit at continuum; and that the parameters in it be solutions of the Navier Stokes relations at this limit.

It is apparent, therefore, that a possible choice of trial function is one that converges to the correct limits at the two extremes. In such a case the integral equation has the best chance of giving a good approximation to the behavior in between the two extremes. When a Krook's type model is used in the equation no initial assumption has to be made about the form of the distribution function. Only the behavior of the macroscopic variables has to be known approximately over the whole transition regime.

From the above considerations it seems apparent that the simpler moment methods might supply the best choice of the initial trial function as these will, at least, guarantee the conservation laws. Furthermore, if a model is used in the iteration, only the macroscopic properties and not the distribution function resulting from the moment method needs to be used. This is desirable as the moment methods do not purport to solve for the distribution function but tend to treat it more as just a convenient weighting function to satisfy the conservation equations in some average way.

For the iterative scheme to be a powerful and practical tool the zeroth iterate cannot demand too much labor to obtain. Also, the accuracy requirement on this zeroth iterate cannot be too stringent. Lees' moment method, therefore, seems the ideal choice for the initial trial function (zeroth iterate). This method, though certainly not the most accurate of the moment methods, requires the least amount of labor to obtain solutions and always guarantees that they converge to the proper limits in the macroscopic variables at both extremes. The undesirable discontinuity in velocity space of the distribution function offers no problem if it is considered as a convenient weighting function alone, while the

macroscopic properties are considered as the solution.

The problem of applying the boundary conditions is a serious one and proper analysis of it will undoubtedly have to wait for better and more complete data. The presently accepted accomodation type of treatment is actually quite well suited to the integral equation formulation. In this treatment the distribution function of particles in the half of velocity space with velocity direction pointing away from the boundary is Maxwellian with the energy, density and velocity parameters related to the wall conditions and to the same parameters of the distribution function in the other half of velocity space. Thus, the boundary conditions can, at least in principle, be applied after each iteration. Integrating of the equation with the appropriate velocity moments over each half of velocity space at the boundary and then relating the resulting parameters through the accomodation coefficients and the continuity equation yields a sufficient number of relations to determine the parameters in the distribution of particles leaving the boundary.

The possibility of applying the integral equation formulation, in principle, to the three types of "transition" regime problems discussed here, though not a conclusive proof of its practicability and usefulness, shows that the method should yield results for a wide class of problems. The three particular cases of, an initial expansion of a pressurized sphere of gas expanding into another gas, the structure of a steady state normal shock and heat transfer between parallel plates, carried out in the next three chapters, illustrate the problems encountered in the practical applications of the methods discussed in this chapter.

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NOMENCLATURE

$f(\vec{r},\vec{v},t)$	The distribution function
$\mathcal{P}(\vec{r},\vec{v},t)$	The scattering function specifying the density in phase space of particles scattered at $\vec{r}$ and t and having velocity $\vec{v}$
$\vec{r}$	Position vector
$\vec{r}_s$	Position vector locating boundary between V_1 and V_2 in the macroscopic discontinuity problem
S	Surface bounding volume
t	Time
t_s	Time required for particle traveling at velocity $\vec{v}$ to go from surface S to position $\vec{r}$
$\vec{v}$	Velocity of particle
V	Volume in which solution is desired
V_1, V_2	Volumes in the macroscopic discontinuity problem
Δt	Time increment in initial value problem
$\mathcal{E}(\vec{r},\vec{v},t)$	Density of particles with velocity $\vec{v}$ that have just suffered a collision at $\vec{r}$ and t
$\mathcal{E}_1, \mathcal{E}_2$	The values $\mathcal{E}$ in the two regions V_1 and V_2 in the macroscopic discontinuity problem
$\nu(\vec{r},\vec{v},t)$	Collision frequency for particles of velocity $\vec{v}$ at position $\vec{r}$ and time t
ν_1, ν_2	The values of ν in V_1 and V_2 in the macroscopic discontinuity problem
τ	Dummy variable appearing in integrations over time
$\Omega(\vec{r},\vec{v},t)$	Integrating factor in the integral form of the kinetic theory equation

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V. EXPANDING SPHERE PROBLEM

An example of an initial value problem that has been of some interest for a long time and has been relatively well analysed for the inviscid case[24, 25, 26] is the problem of a high pressure sphere of gas being suddenly released into a surrounding medium at a much lower pressure. The similarity solution first obtained by Taylor[24, 25] gives the strength and speed of outgoing shock resulting from the sudden expansion. The numerical solutions by Brode[26] determine the outgoing shock, as well as other secondary shocks, and the density, velocity and temperature profiles in the whole region. No continuum type solution (especially the inviscid one) can hope, however, to solve for the initial behavior before the shocks are fully formed and, thus, all continuum solutions are long time solutions in comparison to the mean free time between collisions.

For all practical purposes at normal temperatures and pressures and macroscopic size initial spheres the "long time" solution is the only one of importance. The initial expansion during a time of the order of a mean free time is a negligible portion of the total time history of the problem. The inviscid solution is also sufficient because the effects of transport properties are limited to the actual regions of the shocks and to the contact surface between the two media. These are macroscopic discontinuities and can be, at least in principle, left for separate later solution when details are wanted. The fact that the contact surface thickness is a diffusion effect increasing as $\sqrt{t}$ while the shock propagates in the main proportionally to t further justifies initially neglecting the contact surface thickness in solving for the shock pattern.

Now, if instead of looking at this problem in the case where the original sphere diameter is many times the mean free path, as in the case at normal conditions, we look at the identical problem but in the case where the original sphere diameter is of the same order of

magnitude as the mean free path of the outside gas. For a one meter sphere in air this would occur at outside pressures just below a micron on mercury. In this case an appreciable portion of the problem takes place during a time less than and equal to the mean free time in the outside gas. Thus, no continuum theory can hope to give reasonable results. The problem can, however, be treated directly by means of the formalism developed in the last chapter for treating initial value problems. One can hope to obtain from this theory complete maps of the macroscopic properties for times of the order of a mean free time in steps small compared to the mean free time. From this one should be able to tell exactly how much diffusion has taken place at the contact surface and exactly how long it has taken to form a macroscopic discontinuity that can be identified as a shock.

The particular problem carried out is one of a helium sphere initially at a pressure ratio of 18 to the outside gas expanding into air. This particular case, under inviscid conditions has been treated theoretically by Brode in Ref. 26 and investigated experimentally by Boyer in Ref. 27. The only difference introduced into our problem has been the fact that the overall pressure level has been scaled to the point where the initial sphere diameter is only equal to about one mean free path in the air outside.

The problem can be stated in the following way. For times less than zero the region of space for radii up to $r = r_0$ consists of quiescent helium of density ρ_{h_0}, temperature T_{h_0} and pressure p_{h_0}. The region of space for radii greater than r_0 consists of quiescent air of density ρ_{a_0}, temperature T_{a_0} and pressure p_{a_0}. At time equal to zero the membrane separating the two media and allowing them to maintain separate equilibrium levels is broken. The problem is to determine for $t > 0$, $\rho_h(r,t)$, $T_h(r,t)$, $p_h(r,t)$, $\rho_a(r,t)$, $T_a(r,t)$, $p_a(r,t)$ as well as the velocities $u_a(r,t)$ and $u_h(r,t)$. Also in the region where the gases are mixed one would like to determine the mixture properties like density $\rho_{ah}(r,t)$, temperature $T_{ah}(r,t)$ pressure $p_{ah}(r,t)$ and the overall mass velocity $u_{ah}(r,t)$. In kinetic theory, it is only necessary to determine $f_h(r,\vec{v},t)$ and $f_a(r,\vec{v},t)$

and all the macroscopic properties will be known. The initial conditions in kinetic theory therefore become

$$f_A(r,\vec{v},0) = n_{A0}\left(\frac{1}{2\pi R_A T_{A0}}\right)^{3/2} \exp\left[-\frac{v^2}{2R_A T_{A0}}\right] \qquad r < r_0$$

$$f_A(r,\vec{v},0) = 0 \qquad r > r_0 \qquad (V-1)$$

and

$$f_a(r,\vec{v},0) = n_{a0}\left(\frac{1}{2\pi R_a T_{a0}}\right)^{3/2} \exp\left[-\frac{v^2}{2R_a T_{a0}}\right] \qquad r > r_0$$

$$f_a(r,\vec{v},0) = 0 \qquad r < r_0 \qquad (V-2)$$

The relationship between f at t and f at $t-\Delta t$ derived in the last chapter should be sufficient to determine f_A and f_a for all times greater than zero. The steps Δt, however, have to be small compared to the shortest mean free time in the problem and the solution cannot be carried for excessively long times because of the accumulated error.

The equation relating $f(\vec{r},\vec{v},t)$ to $f(\vec{r}-\vec{v}\Delta t,\vec{v},t_1)$ derived in the last chapter when reduced to the spherically symmetric case becomes

$$f(r,v,\cos\theta,t) = f(r',v,\cos\theta,t-\Delta t) + \left[\mathcal{E}(r',v,\cos\theta,t-\Delta t) - f(r',v,\cos\theta,t-\Delta t)\right]\nu(r',v,\cos\theta,t-\Delta t)\Delta t - \left[\left(\mathcal{E}(r',v,\cos\theta,t-\Delta t) - f(r',v,\cos\theta,t-\Delta t)\right)\left(1-\frac{1}{\nu^2}\frac{d\nu}{dr}\right) - \frac{1}{\nu}\frac{d\mathcal{E}}{dr}\right]\frac{(\nu(r',v,\cos\theta,t-\Delta t)\Delta t)^2}{2} \cdots \qquad (V-3)$$

where θ is the angle between the velocity vector and the $\vec{r}$ vector, and

$$r' = \sqrt{r^2 - 2rv\cos\theta\,\Delta t + (v\Delta t)^2} \qquad (\overline{V}-4)$$

and

$$\cos\theta' = \frac{r\cos\theta - v\Delta t}{r'} \qquad (\overline{V}-5)$$

Here no assumptions have yet been made as to the form of ν and ε. In principle, these could be calculated exactly as $f(r', v, \cos\theta', t-\Delta t)$ is a known function. Unfortunately the collision integrals that would have to be evaluated to carry this out are not easily done, even on an electronic computer, because of the necessity of carrying out at least quadruple numerical integration. A collision model is, therefore, ideal in this case because ε and ν become simply algebraically related to the macroscopic variables which have to be evaluated anyway. In the problem calculated the simple Krook's model was chosen, but extension to a more sophisticated model is straight forward.

The solution is carried out in the following way. All variables

are non-dimensionalized by initial properties within the sphere

$$\bar{\rho}_\lambda = \rho_\lambda/\rho_{\lambda 0} \quad , \quad \bar{T}_\lambda = \frac{T_\lambda}{T_{\lambda 0}} \quad , \quad \bar{P}_\lambda = P_\lambda/P_{\lambda 0} \quad , \quad \bar{u}_\lambda = u_\lambda/\sqrt{2R_\lambda T_{\lambda 0}}$$

$$\bar{\rho}_a = \rho_a/\rho_{\lambda 0} \quad , \quad \bar{T}_a = T_a/T_{\lambda 0} \quad , \quad \bar{P}_a = P_a/P_{\lambda 0} \quad , \quad \bar{u}_a = u_a/\sqrt{2R_a T_{\lambda 0}}$$

$$\bar{\rho}_{a\lambda} = \bar{\rho}_\lambda + \bar{\rho}_a \qquad \bar{T}_{a\lambda} = \frac{\bar{\rho}_\lambda \bar{T}_\lambda + \bar{\rho}_a R_a/R_\lambda \bar{T}_a}{\bar{\rho}_\lambda + R_a/R_\lambda \bar{\rho}_a}$$

$$\bar{u}_{a\lambda} = \left(\bar{\rho}_\lambda \bar{u}_\lambda + \bar{\rho}_a \sqrt{R_a/R_\lambda}\, \bar{u}_a\right)/\bar{\rho}_{a\lambda}$$

$$\bar{\tau} = \frac{\tau \nu_{\lambda 0}}{\sqrt{2R_\lambda T_{\lambda 0}}} \qquad \bar{V}_\lambda = \frac{V}{\sqrt{2R_\lambda T_{\lambda 0}}} \qquad \bar{V}_a = \frac{V}{\sqrt{2R_a T_{\lambda 0}}}$$

$$\bar{t} = \nu_{\lambda 0} t \qquad \Delta\bar{t} = \nu_{\lambda 0} \Delta t \qquad x = \cos\theta \qquad \bar{\nu} = \frac{\nu}{\nu_{\lambda 0}}$$

$$(\bar{V}-6)$$

Normalizing factors are also put into the distribution functions to make the initial conditions appear as follows

$$\bar{f}_\lambda(\bar{\tau}, \bar{v}_\lambda, x, 0) = \begin{cases} \exp[-\bar{v}_\lambda^2] & \bar{\tau} < \bar{\tau}_0 \\ 0 & \bar{\tau} > \bar{\tau}_0 \end{cases}$$

and

$$\bar{f}_a(\bar{\tau}, \bar{v}_a, x, 0) = \begin{cases} 0 & \bar{\tau} < \bar{\tau}_0 \\ \bar{\rho}_{a0}\left(\frac{1}{\bar{T}_{a0}}\right)^{3/2} \exp[-\bar{v}_a^2] & \bar{\tau} > \bar{\tau}_0 \end{cases} \qquad (\bar{V}-7)$$

and the macroscopic variables to be obtained from the distribution function by the following integrations

$$\bar{\rho}_h = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \int_{-1}^{+1} \bar{f}_h \, \bar{v}_h^2 \, dx \, d\bar{v}_h \qquad (\bar{V}-8)$$

$$\bar{\rho}_a = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \int_{-1}^{+1} \bar{f}_a \, \bar{v}_a^2 \, dx \, d\bar{v}_a \qquad (\bar{V}-9)$$

$$\bar{u}_h = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \int_{-1}^{+1} \bar{f}_h \, \bar{v}_h^3 \, x \, dx \, d\bar{v}_h \Big/ \bar{\rho}_h \qquad (\bar{V}-10)$$

$$\bar{u}_a = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \int_{-1}^{+1} \bar{f}_a \, \bar{v}_a^3 \, x \, dx \, d\bar{v}_a \Big/ \bar{\rho}_a \qquad (\bar{V}-11)$$

$$\bar{\rho}_{ah} = \bar{\rho}_a + \bar{\rho}_h \qquad (\bar{V}-12)$$

$$\bar{u}_{ah} = \left(\bar{u}_h \bar{\rho}_h + \bar{u}_a \bar{\rho}_a \sqrt{\frac{R_a}{R_h}} \right) \Big/ \bar{\rho}_{ah} \qquad (\bar{V}-13)$$

$$\bar{T}_h = \frac{2}{3} \left(\frac{2}{\sqrt{\pi}} \int_0^{\infty} \int_{-1}^{+1} \bar{v}_h^4 \bar{f}_h \, dx \, d\bar{v}_h \Big/ \bar{\rho}_h - \bar{u}_{ah} (2\bar{u}_h - \bar{u}_{ah}) \right) \qquad (\bar{V}-14)$$

$$\bar{T}_a = \frac{2}{3} \left(\frac{2}{\sqrt{\pi}} \int_0^{\infty} \int_{-1}^{+1} \bar{v}_a^4 \bar{f}_a \, dx \, d\bar{v}_a \Big/ \bar{\rho}_a - \bar{u}_{ah} \sqrt{\frac{R_h}{R_a}} \left(2\bar{u}_a - \bar{u}_{ah} \sqrt{\frac{R_h}{R_a}} \right) \right) \qquad (\bar{V}-15)$$

$$\bar{T}_{ah} = \left(\bar{\rho}_h \bar{T}_h + \bar{\rho}_a T_a \frac{R_a}{R_h} \right) \Big/ \left(\bar{\rho}_h + \frac{R_a}{R_h} \bar{\rho}_a \right) \qquad (\bar{V}-16)$$

The expressions for $\mathcal{E}$ and ν have to be evaluated in terms of these non-dimensional variables. The scattering function $\mathcal{E}$ takes the form of a Maxwellian in Krook's model and ν is independent of velocity. A further restriction fixes the parameters in $\mathcal{E}$ by requiring the conservation laws to be satisfied. Since momentum and energy are conserved in a mixture only for all the components together, and not individually, the macroscopic velocity and temperature in the Krook's model for either species should be the total mixture, velocity and temperature respectively. If the individual velocity and temperature is used in Krook's model the model becomes more restrictive than the exact collision integral. With this in mind the scattering terms $\bar{\mathcal{E}}_A$ and $\bar{\mathcal{E}}_a$ are determined as

$$\bar{\mathcal{E}}_A = \bar{\rho}_A / \bar{T}_{a_A}^{3/2} \exp\left[- \frac{(\bar{v}_A - \bar{u}_{aA})^2}{\bar{T}_{aA}} \right] \qquad (V\text{-}17)$$

and

$$\bar{\mathcal{E}}_a = \bar{\rho}_a / \bar{T}_{a_A}^{3/2} \exp\left[- \frac{(\bar{v}_a - \bar{u}_{aA}\sqrt{R_A/R_a})^2}{\bar{T}_{aA}} \right] \qquad (V\text{-}18)$$

The collision frequency ν in Krook's model is just the average collision frequency independent of velocity. It is, however, composed of two terms: one due to collisions with like particles and the other due to collisions with the other species. In this non-dimensional notation the collision frequencies become

$$\bar{\nu}_A = \frac{\nu_A}{\nu_{A_0}} = \bar{\rho}_A \sqrt{\bar{T}_A} + \frac{R_a}{R_A} \bar{\rho}_a \frac{S_{AA}}{S_{aA}} \sqrt{\frac{\bar{T}_A + R_a/R_A \bar{T}_a}{2}} \qquad (V\text{-}19)$$

and

$$\bar{\nu}_a = \frac{\nu_a}{\nu_{ao}} = \sqrt{\bar{T}_a}\,\bar{\rho}_a\left(\frac{R_a}{R_\lambda}\frac{S_{\lambda\lambda}}{S_{aa}}\right)^{3/4} + \bar{\rho}_\lambda \frac{S_{\lambda\lambda}}{S_{a\lambda}}\sqrt{\frac{\bar{T}_\lambda + \frac{R_a}{R_\lambda}\bar{T}_a}{2}} \qquad (V-20)$$

where $S_{\lambda\lambda}$, $S_{a\lambda}$ and S_{aa} are the collision cross sections for helium-helium, helium-air and air-air collisions respectively. They are, in general, functions of the temperatures of the species but here they are assumed constant to correspond to a hard sphere collision model. This may be inconsistent with the assumption of velocity independent collision frequency assumed in the Krook's model, as velocity independent collision frequency is a result of fifth-power repulsion law. If this is carried to the extreme, however, one should also make the collision frequency infinite as this also is a result of the fifth-power repulsion. We, therefore, choose to consider the velocity independent collision frequency in the Krook's model as just an averaged frequency substituted for the actual one in order to simplify the model. With such an interpretation the choice of the repulsion law is still left arbitrary.

The basic equations, under the application of all these assumptions and non-dimensionalization, become

$$\bar{f}_\lambda(\bar{r},\bar{v}_\lambda,x,\bar{t}) = \bar{f}_\lambda(\bar{r}',\bar{v}_\lambda,x',\bar{t}-\Delta\bar{t})$$

$$+\left[\bar{\rho}_\lambda(\bar{r};\bar{t}-\Delta\bar{t})/(\bar{T}_{a\lambda}(\bar{r};\bar{t}-\Delta\bar{t}))^{3/2}\exp\left(-\frac{(\bar{v}_\lambda-\bar{u}_{a\lambda}(\bar{r};\bar{t}-\Delta\bar{t}))^2}{\bar{T}_{a\lambda}(\bar{r}',\bar{t}-\Delta t)}\right)\right.$$

$$\left.-\bar{f}_\lambda(\bar{r}';\bar{v}_\lambda,x';\bar{t}-\Delta\bar{t})\right]\bar{\nu}_\lambda(\bar{r}';\bar{t}-\Delta\bar{t})\,\Delta\bar{t} + \text{higher order terms} \qquad (V-21)$$

and

$$f_a(\bar{r},\bar{v}_a,x,\bar{t}) = f_a(\bar{r}',\bar{v}_a,x',\bar{t}-\Delta\bar{t})$$

$$+\left[\left(\rho_a(\bar{r}')/\left(T_{ah}(\bar{r}',\bar{t}-\Delta\bar{t})\right)^{3/2}\right)\exp\left(-\frac{(\bar{v}_a-\sqrt{\frac{R_4}{R_a}}\,\bar{u}_{ah}(\bar{r}',\bar{t}-\Delta\bar{t}))^2}{T_{ah}(\bar{r}',\bar{t}-\Delta\bar{t})}\right)\right.$$

$$\left.-f(\bar{r}',\bar{v}_a,x',\bar{t}-\Delta t)\right]\bar{\nu}_a(\bar{r}',\bar{t}-\Delta\bar{t})\,\Delta\bar{t} + \text{h.o.t.} \qquad (\bar{V}\text{-}22)$$

Thus, although a large amount of bookkeeping is involved, solution of the problem relies entirely on evaluating $\bar{r}'$ and x' for each $\bar{v}$ and x and determining $\bar{f}(\bar{r},\bar{v},x,\bar{t})$ from $\bar{f}(\bar{r}',\bar{v},x',\bar{t}-\Delta t)$ and the macroscopic variables at $(\bar{r}',\bar{t}-\Delta\bar{t})$. From this $\bar{f}(\bar{r},\bar{v},x,\bar{t})$ the new macroscopic variables can be evaluated for all $\bar{r}$ at time $\bar{t}$. From this, a new $\bar{f}$ can be evaluated for time $\bar{t}+\Delta\bar{t}$; this process continuing, in principle, up to any time $\bar{t}$, that one desires.

The large amount of bookkeeping and the necessary numerical evaluation of the macroscopic variables suggest machine solution of the problem. Since a digital computer works only in a numerical fashion, all functions, thus far discussed, become matrices. The fact that the relationships between conditions at time $\bar{t}$ and $\bar{t}-\Delta\bar{t}$ involve a shift of position in physical space and direction in velocity space through $\bar{r}'$ and x' respectively, necessitates interpolation as it is obvious that $\bar{r}'$ and x' will not, in general, fall on points at which conditions have been evaluated. The integrations to obtain the macroscopic variables become matrix multiplications of the $\bar{f}$ matrices with certain weighting matrices.

The actual machine solution was carried out with the relationship to first order in $\Delta\bar{t}$, as computers are more efficient at

carrying out simpler manipulations more often. At time $\bar{t}=0$ the macroscopic variables are given for a certain mesh size in physical space. Two new matrices representing the right hand sides of the recursion relations are defined as follows:

$$\bar{E}_\lambda(\bar{r}_i,\bar{v}_{\lambda j},x_k,\bar{t}) = \bar{f}_\lambda(\bar{r}_i,\bar{v}_{\lambda j},x_k,\bar{t})$$

$$+\left[\bar{E}_\lambda(\bar{r}_i,\bar{v}_{\lambda j},x_k,\bar{t})\nu - \bar{f}_\lambda(\bar{r}_i,\bar{v}_{\lambda j},\bar{x}_k,\bar{t})\right]\bar{\nu}_\lambda(\bar{r}_i,\bar{t})\Delta\bar{t} \qquad (V\text{-}23)$$

and

$$\bar{E}_a(\bar{r}_i,\bar{v}_{aj},x_k,\bar{t}) = \bar{f}_a(\bar{r}_i,\bar{v}_{aj},x_k,\bar{t})$$

$$+\left[\bar{E}_a(\bar{r}_i,\bar{v}_{aj},x_k,\bar{t}) - f_a(\bar{r}_i,\bar{v}_{aj},x_k,\bar{t})\right]\bar{\nu}_a(\bar{r}_i,\bar{t})\Delta\bar{t} \qquad (V\text{-}24)$$

Thus, to evaluate the distribution function matrix at $\bar{t}+\Delta t$ it is only necessary to determine $\bar{r}'$ and x' and then interpolate on the $\bar{E}_\lambda$ and $\bar{E}_a$ matrices. To evaluate the macroscopic properties it is only necessary to multiply the $\bar{f}_\lambda$ and $\bar{f}_a$ matrices by the proper weighting matrices. To carry the solution to the next time one needs only to evaluate the $\bar{E}_\lambda$ and $\bar{E}_a$ matrices again, find the new $\bar{r}'$ and x' for each $\bar{r}_i$, $\bar{v}_j$ and x_k, and calculate the new $\bar{f}_\lambda$ and $\bar{f}_a$ matrices by interpolation.

In the actual case calculated, the physical space mesh size was not held fixed but was adjusted from a very fine mesh size of .05 initial mean free paths to larger sizes as the storage was filled up.

The double integration was carried out by quadratures in both

directions. The integral was first transformed into

$$\int_{0}^{\infty}\int_{-1}^{+1} d\bar{v}\,dx = \int_{-\infty}^{\infty}\int_{0}^{1} d\bar{v}\,dx \qquad (\bar{V}\text{-}25)$$

which can be seen to be equivalent from geometric considerations. The velocity integration was carried out by a Hermite-Gauss quadrature to take advantage of the fact that the integrand always behaves as $exp(-\bar{v}^2)$ in the limits of $\bar{v} \Rightarrow \pm\infty$. Ten points and weighting functions were chosen, as it was found that this gave less than a one percent error for such common integrands as polynomials or trigonometric functions. The x integration was carried out by Legendre-Gauss quadrature as this technique was found most useful for cases where no a priori knowledge was available about the integrand. A six point evaluation was used because of lack of enough memory storage locations on the computer. This evaluation, however, is equivalent to an eleventh order Legendre polynomial representation of the function which should certainly suffice for most reasonable angular variations of the integrand.

The initial mesh size was chosen as .05 in the non-dimensionalized units. This mesh was retained until the memory was filled up for either matrix $\bar{f}_2$ or $\bar{f}_a$. This occurred when the disturbance on $\bar{f}_2$ reached 80 mesh size units out or $\bar{f}_a$ 120 mesh size units out. At such time the program automatically readjusted the mesh size by doubling it and, thus, freeing half the original storage available. This process was, of course, repeated each time the storage was again filled. This doubling of the mesh size was accompanied by the doubling of $\Delta\bar{t}$ for consistency in the expected accuracy. The reason this was possible is a result of the fact that though, in the beginning, the initial discontinuity requires a small mesh size and a small increment of time to obtain reasonable accuracy, the gradual lessening of the gradients on the molecular scale allows a

a larger mesh size and a larger increment in time to retain the same accuracy.

It was also found that the accumulated numerical errors in calculating the macroscopic velocities by integrations of the distribution function were sufficient to give results that did not conserve the total number of particles for large times. This was circumvented, however, by calculating the macroscopic velocity through the direct integration of the continuity equation.

$$\bar{u}(\bar{r}) = \frac{1}{\bar{r}^2 \bar{\rho}(\bar{r})} \left(\bar{r}_1^2 \bar{\rho}(\bar{r}_1) \bar{u}(\bar{r}_1) - \frac{\partial}{\partial t} \int_{\bar{r}_1}^{\bar{r}} s^2 \bar{\rho}(s) ds \right) \qquad (V-26)$$

The integration was always started from the two asymptotic regions where the velocity was known to be zero and carried on towards the middle. This allowed calculating the velocity at any point by merely knowing the velocity at a point one mesh size away and the density change per unit time in between the two points. The calculations are carried out from the two extremes toward the center to minimize accumulated errors by placing the greatest accumulation of error at a point where the velocity is likely to be the highest.

Analysis of the accuracy of the results obtained is quite difficult. The large number of variables and the appearance of nonlinear integral terms in the recursion relation make the standard technique for choosing optimum mesh sizes inapplicable. All that we can do is quote the sources of error in the computation and guess at their magnitude. First of all there are interpolation errors on both $\bar{r}$ and x. These are probably of the order of one percent; except in the initial step the functions don't vary more than a few percent from point to point. The next source of error is, of course, from the numerical integrations. This is impossible to predict analytically when the integrand is itself a numerical function. The orders of the quadratures were chosen principally because

of the limits of storage locations on the computer. These quadratures are, however, very accurate for well behaved functions. Thus, it is hoped that they are also sufficient for the actual numerical functions being integrated.

Beside these computational errors there is, of course, some uncertainty of the effect of the Krook's model on the accuracy of the results. It is known that at the continuum limit Krook's model gives all the correct physical behavior though quantitatively incorrect by 25 to 50 percent. The accuracy of the model itself gets worse when the distribution function is further away from local equilibrium, as is the case for times much less than a mean free time near the initial discontinuity. The effect of the model, however, is less as the initial effects are almost entirely from the collision - less motion of the particles. No conclusive statement can, therefore, be made as to the models effect on the results, though we have every reason to believe that the general trends in the physical behavior are correct.

The results for the helium sphere expanding into air at an initial pressure ratio of 18 and initial diameter of $\pi/2$ times the outside air mean free path are shown in Figs. V-1 through V-9. If we look at the mixture density profiles shown in Fig. V-1 through V-4 we can see the building up of a bump in the density of the air and the propagation of this bump outwards into the gas.

Comparison with the inviscid results for the same initial pressure ratio immediately indicates, however, that any identification of this disturbance as a shock is impossible. This becomes reasonable when one realizes that the large "shock" thickness to radius of curvature ratio precludes the possibility of identification of upstream and downstream equilibrium conditions which are necessary for a shock. The fact that the distrubance doesn't even appear to be tending towards the inviscid results as well as the lack of any secondary density disturbances is a direct result of the fact that the volume effects and the results of just simple collisionless "kinematic diffusion" tend to drive the whole profile to a uniform condition in a time much shorter than the time for collisions to create a

"shock". This is evident from a simple comparison of the results to those of the collisionless motion calculated from results given by Kornowski[41] in Figs. V-1 to V-3. Figure V-4 is just a superposition of the present results for three times to make apparent the outward propagation. Figures V-5 through V-7 show the densities of the individual components and thus indicate the large effects of diffusion in the problem. Figure V-8 indicates the change in the mixture velocity profile and Fig. V-9 shows the mixture temperature profiles for the three representative times.

The overall results though not justifiable as correct because numerical inaccuracies and incomplete understanding of the effects of the approximate collision model, are certainly reasonable in nature and clearly indicate the piling up effect of collisions. An important physical result of the calculations is the determination that for initial diameters of the order of a mean free path and moderate pressure ratios the pressure overshoots are much smaller and become negligible much sooner than the inviscid calculation indicate. The relative similarity of the present calculations to the collisionless results also indicate that probably the single most important effect is the simple kinematic diffusion of the two species and the collisions only alter the profiles slightly from this.

NOMENCLATURE

f	The distribution function
n	Number density of particles in physical space
p	The pressure of the gas
R	The gas constant
$\vec{r}$	Position vector
r	Magnitude of position vector (radius from origin)
r'	$\sqrt{r^2 - 2rvx\Delta t + (v\Delta t)^2}$ modified radius
r_0	Initial radius of pressurized sphere
S_{aa}	Collision cross section for air-air collisions
S_{ah}	Collision cross section for air-helium collisions
S_{hh}	Collision cross section for helium-helium collisions
T	The temperature of the gas
t	Time
u	Radial macroscopic velocity of gas
$\vec{v}$	Velocity vector of particle
v	Velocity magnitude of particle
$x = \cos\theta$	The cosine of the angle between position and velocity vectors
$x' = \cos\theta'$	Modified cosine of θ $x' = \frac{xr - v\Delta t}{r'}$
Δt	Time increment in step by step solution
$\mathcal{E}(\vec{r},\vec{v},t)$	Density of particles that have velocity $\vec{v}$ and have just suffered a collision at $\vec{r}$ and t

θ	Angle between position and velocity vector
θ'	Modified angle defined by equation $\cos\theta' = \frac{r\cos\theta - v\Delta t}{r'}$
$\nu(\vec{r}, \vec{v}, t)$	Collision frequency of particles with velocity $\vec{v}$ at $\vec{r}$ and t
ρ	The mass density of the gas

Subscripts

a	Designates air
ah	Designates mixture of air and helium
h	Designates helium
o	Designates initial conditions
i, j, k	Are subscripts used to designate discrete points used in numerical computations

Superscript

$\bar{}$	Designates normalized, non-dimensional quantity (Normalizations are listed in text on pages 63 through 64).

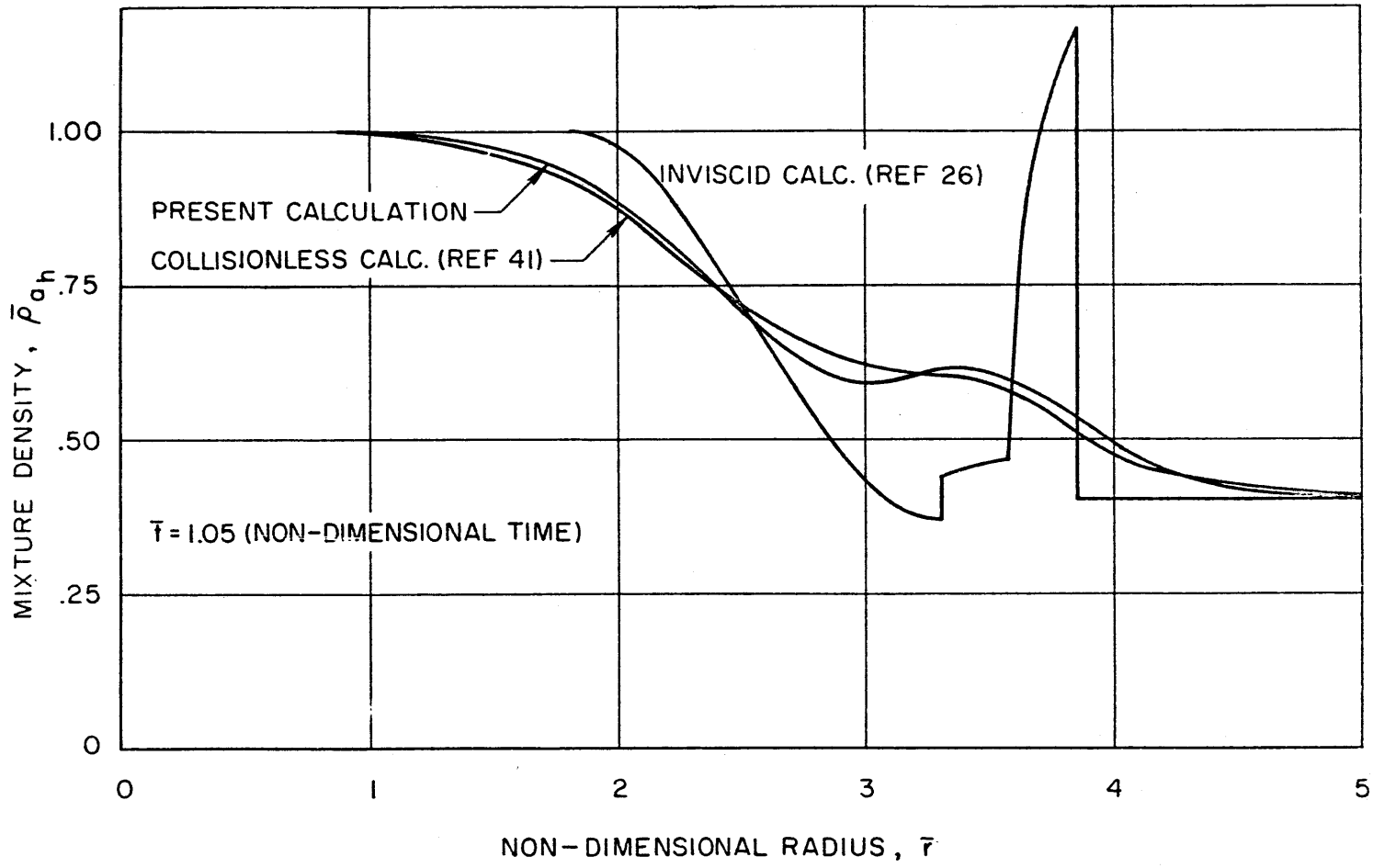

Figure V-1. Mixture density ($\bar{t}$ = 1.05)

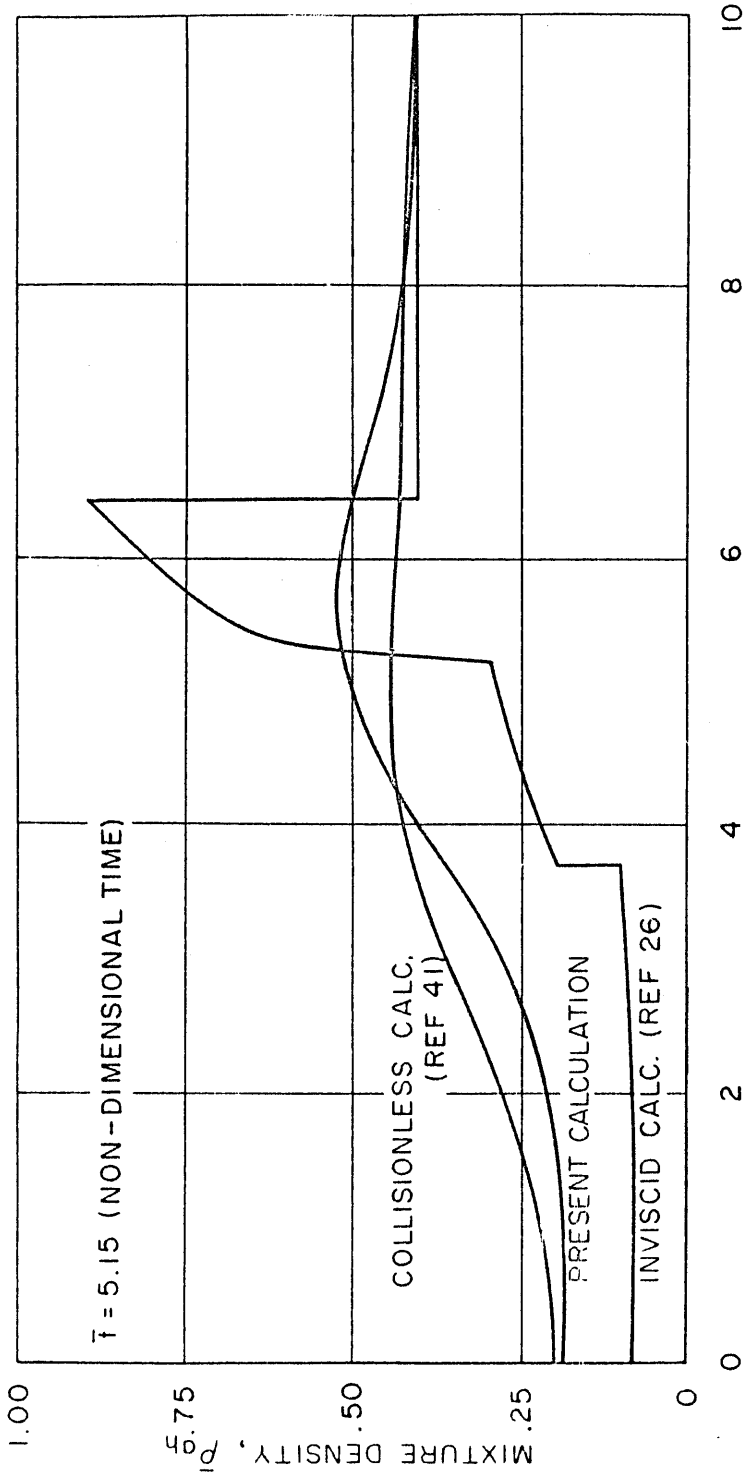

Figure V-2. Mixture density ($\bar{t}$ = 5.15)

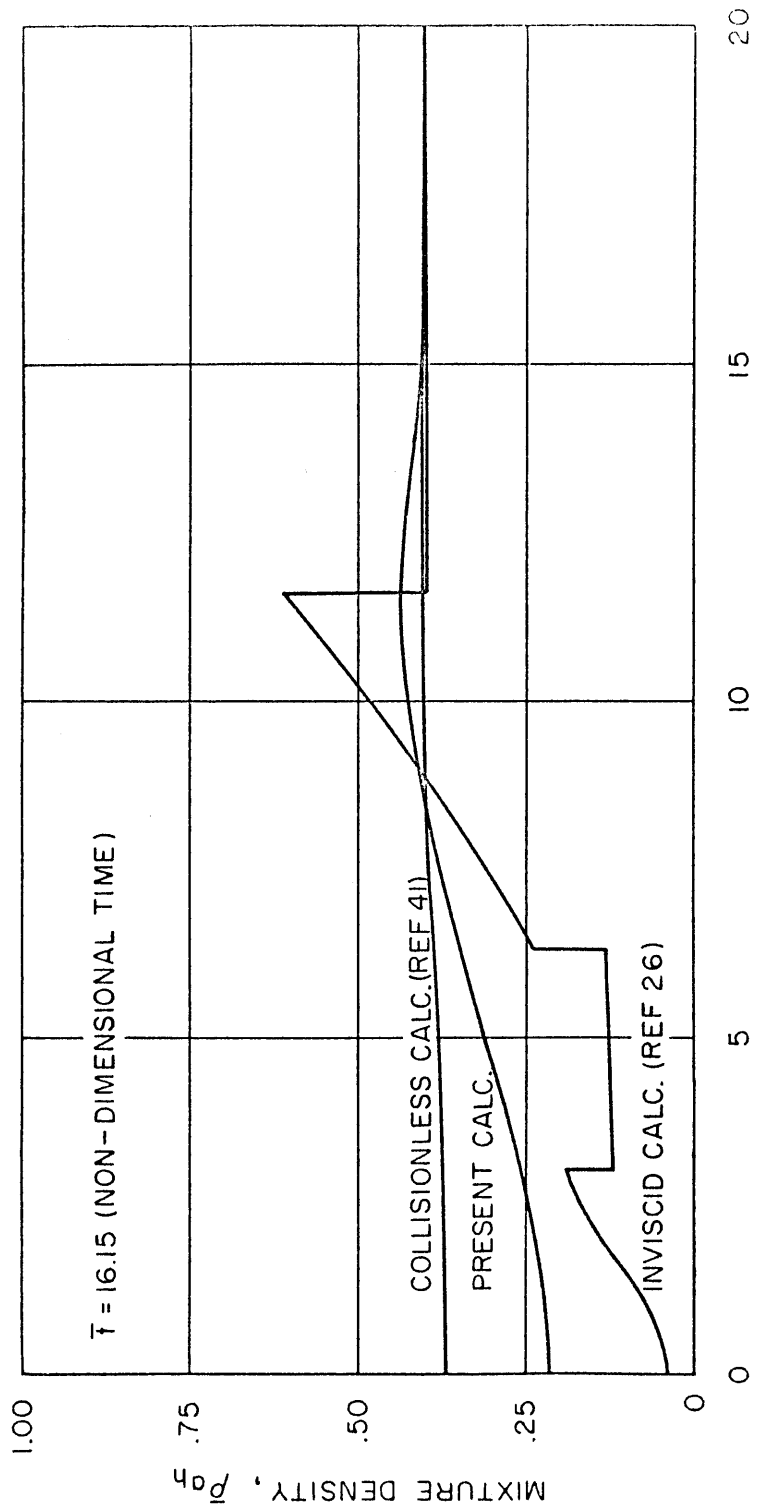

Figure V-3. Mixture density ($\bar{t}$ = 16.15)

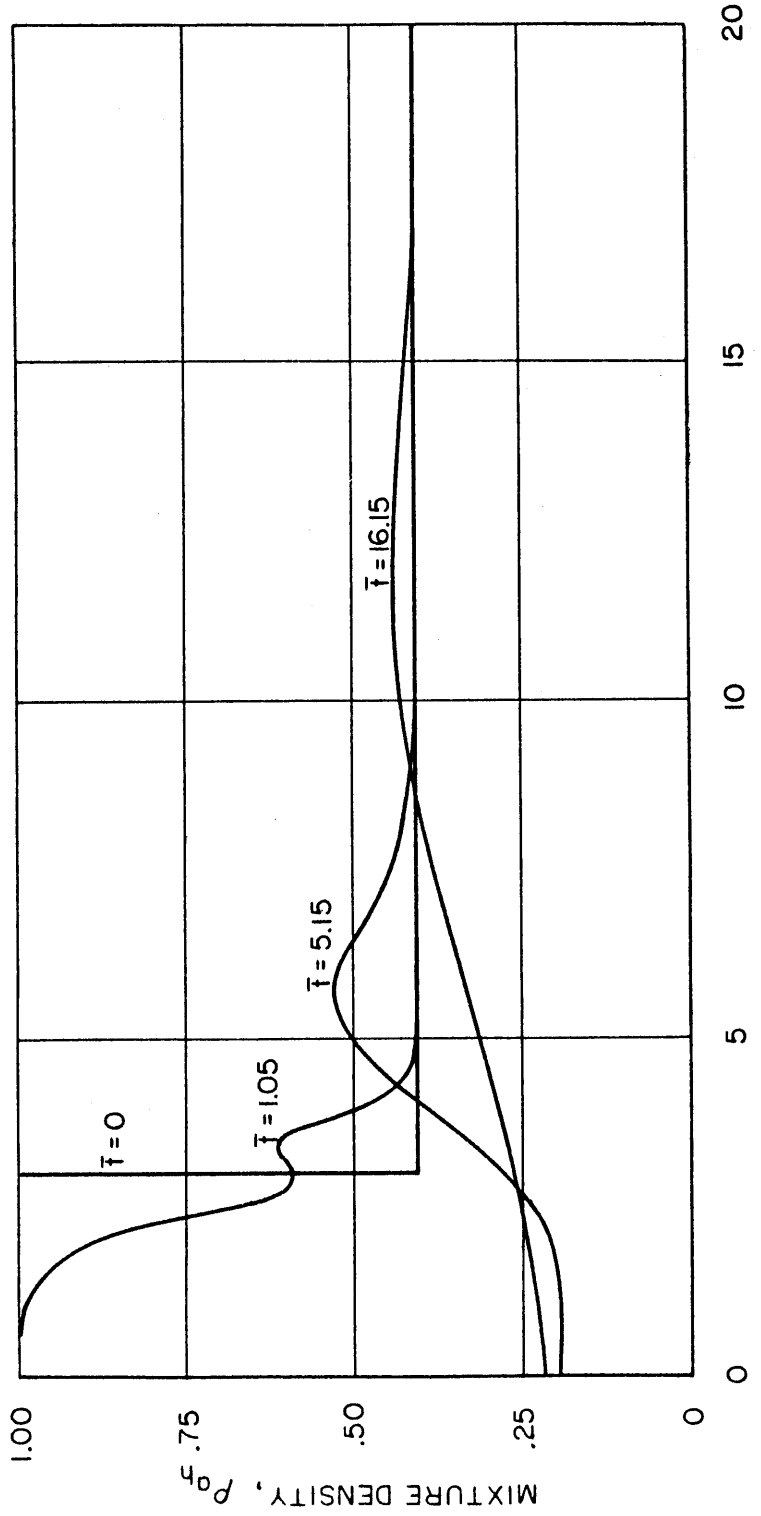

Figure V-4. Density disturbance propagation

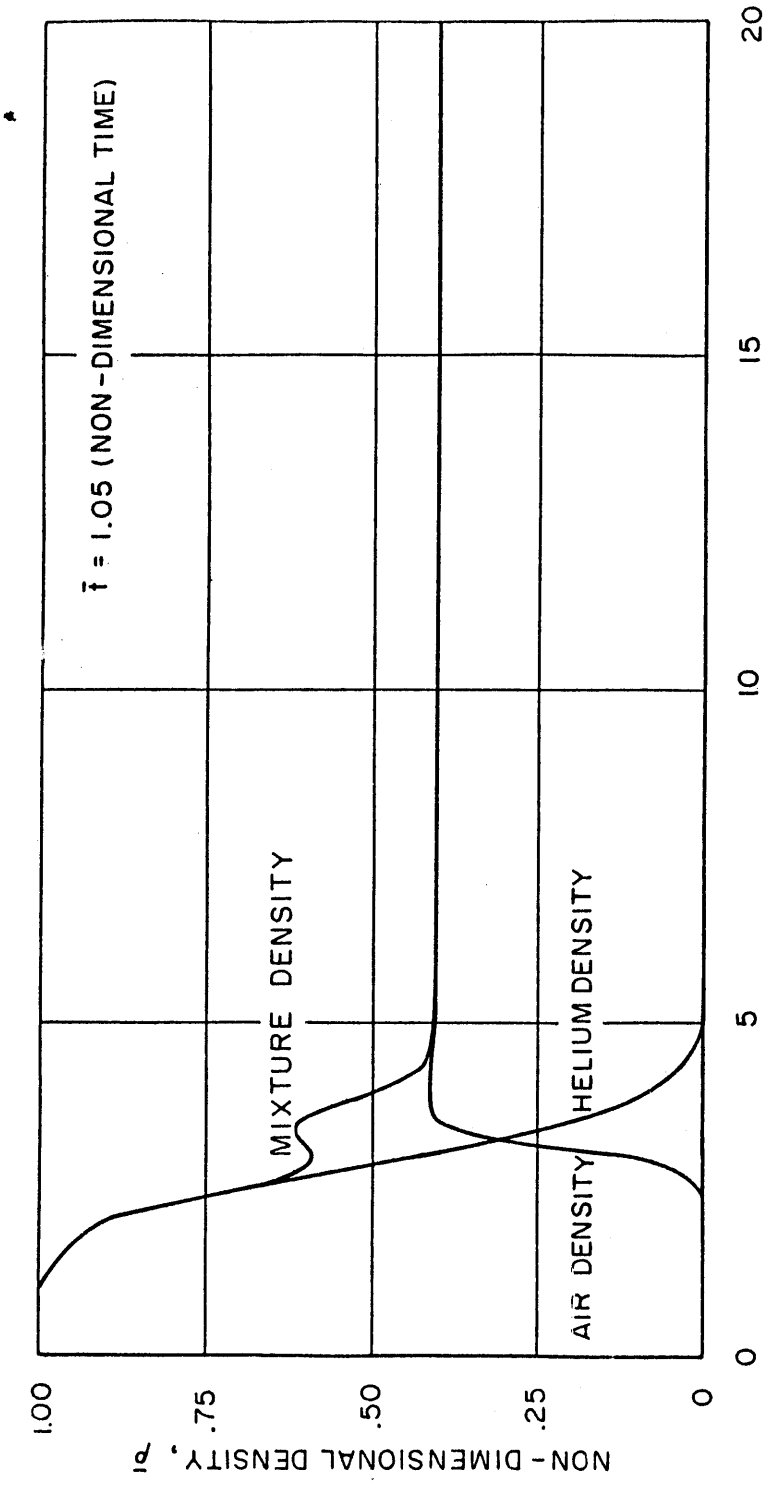

Figure V-5. Density profiles at $\bar{t}$ = 1.05

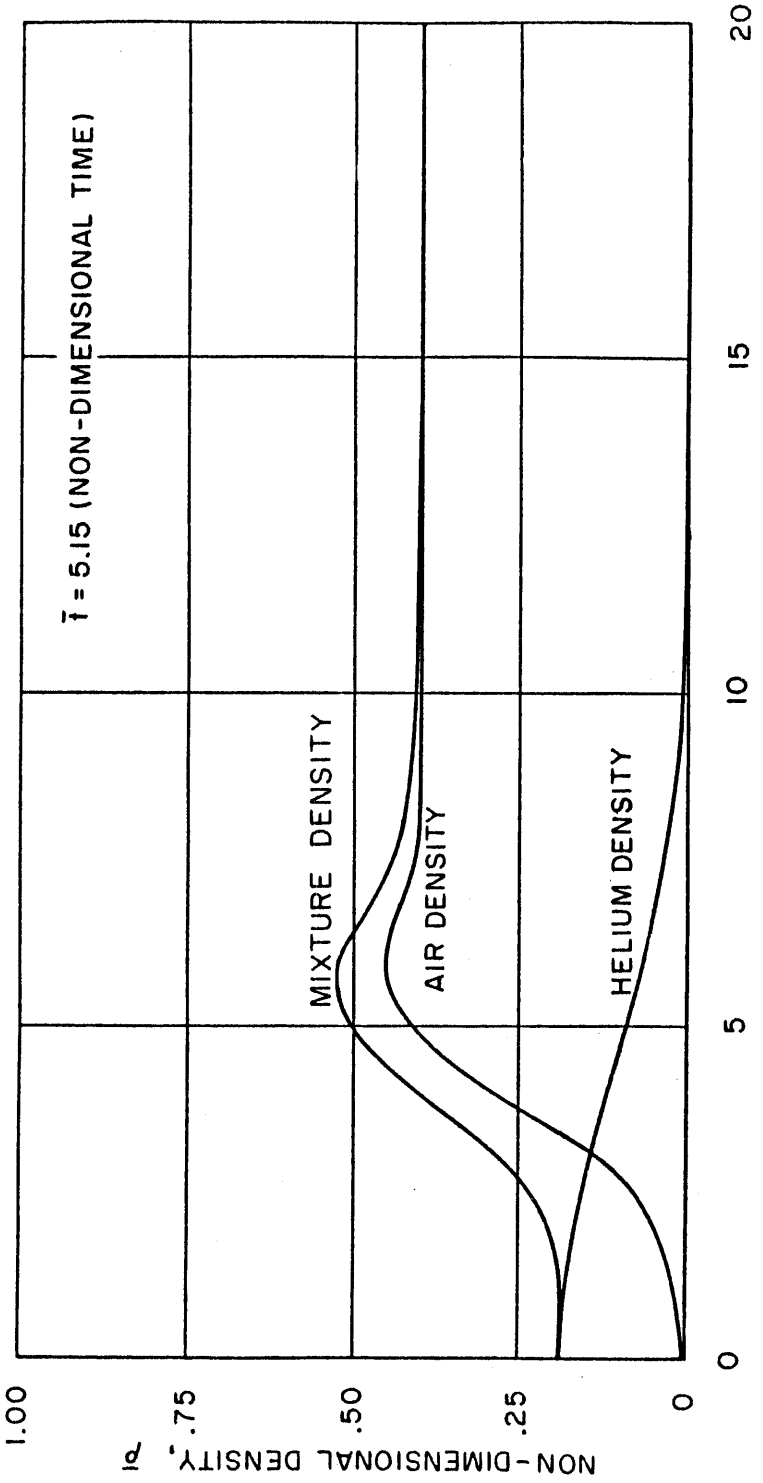

Figure V-6. Density profiles at $\bar{t}$ = 5.15

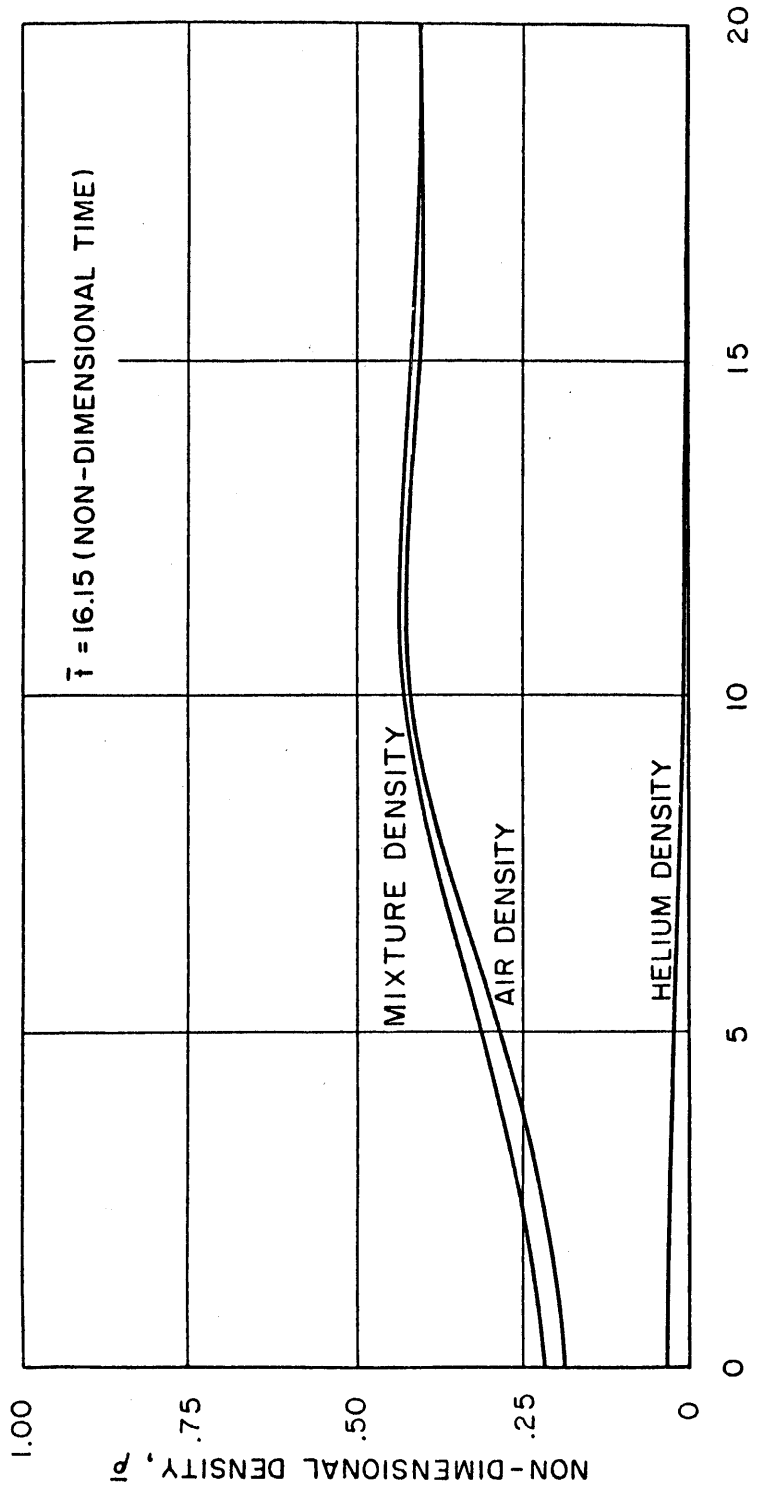

Figure V-7. Density profiles at $\bar{t}$ = 16.15

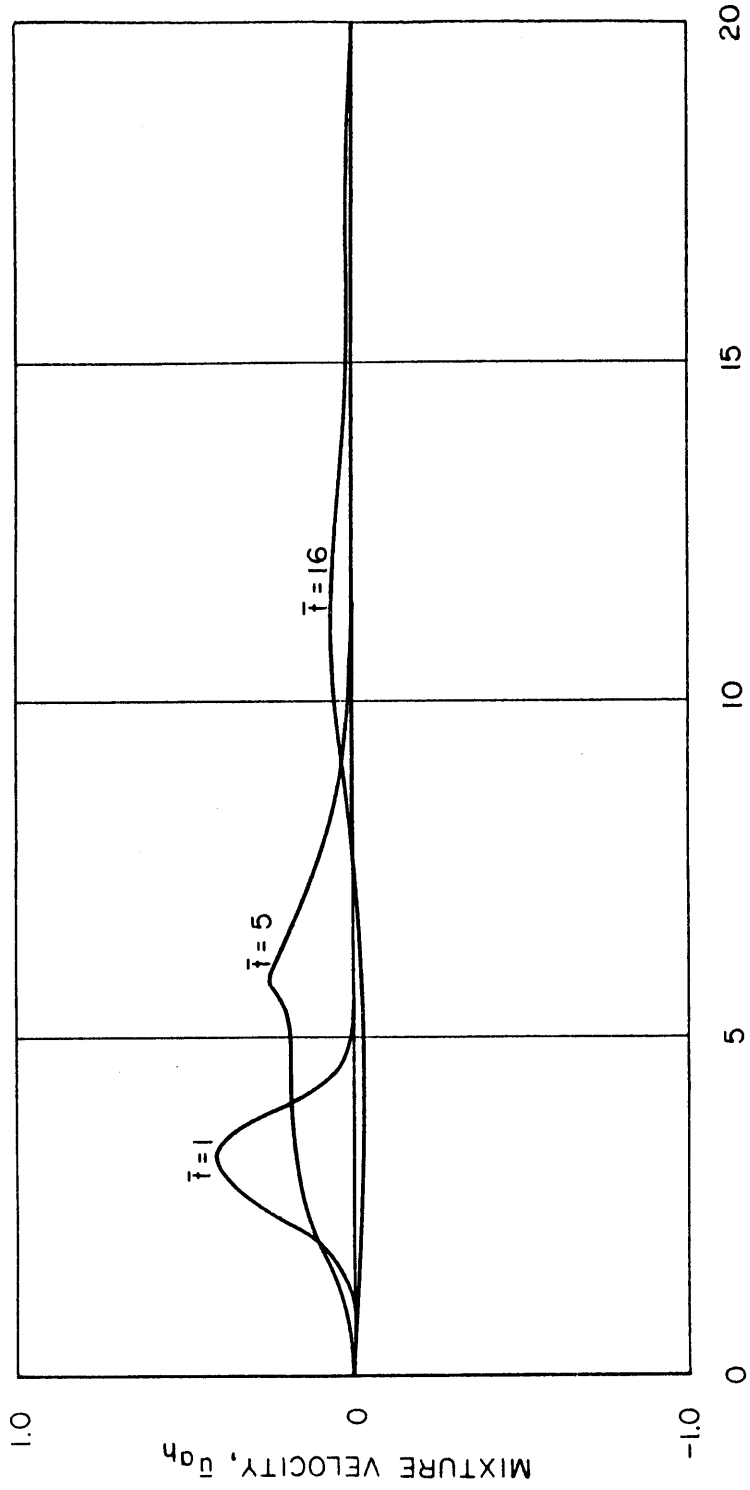

Figure V-8. Velocity profiles

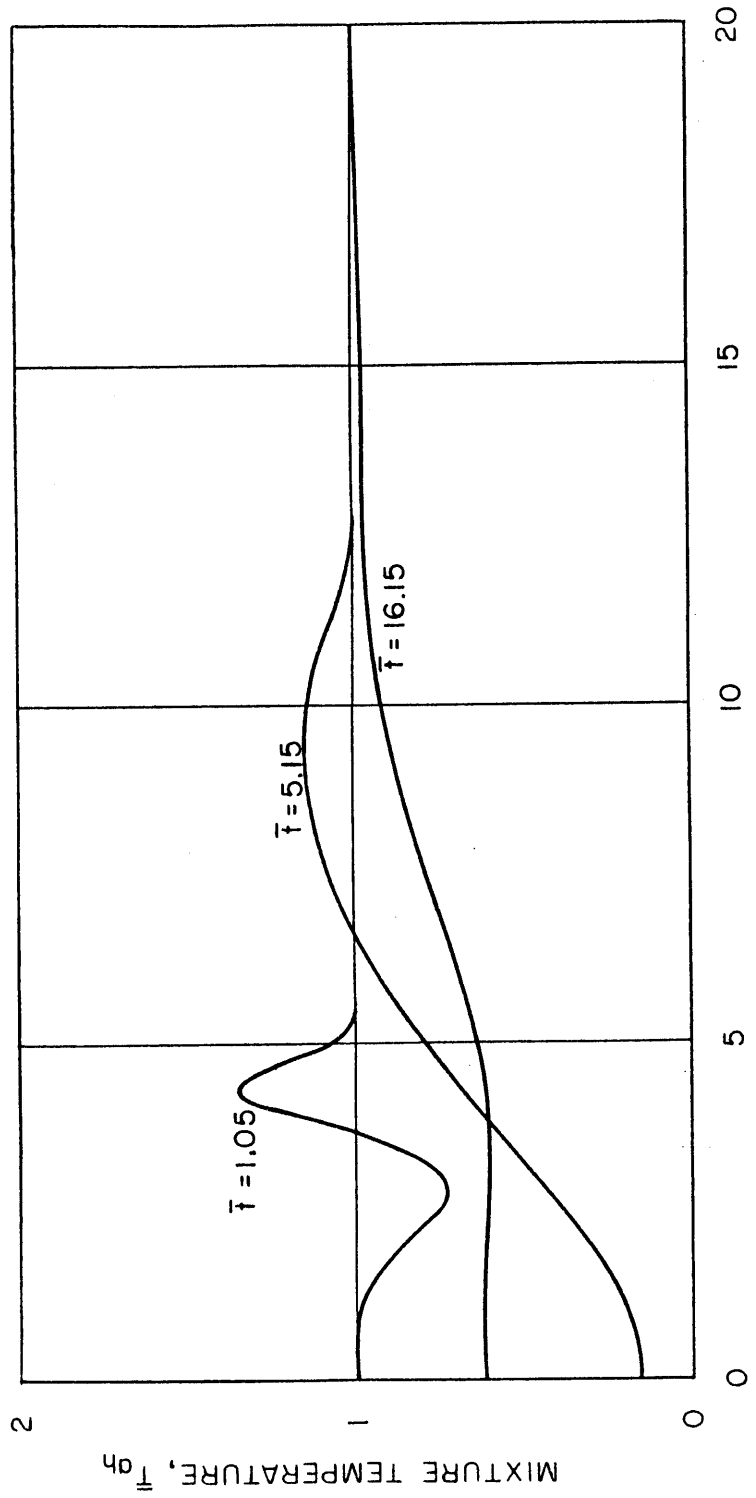

Figure V-9. Temperature profiles

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VI. NORMAL SHOCK STRUCTURE

One macroscopic discontinuity that has been of interest to aerodynamicists for years is the normal shock. A solution of the one dimensional Euler equations gives a possibility of a discontinuous jump in the density, temperature, pressure and velocity.

The second law of thermodynamics further limits the direction of the property jumps to correspond to an entropy rise. This solution is, of course, the simplest representation of a shock, but for most purposes is sufficient if only the overall flow field is desired. The actual shock cannot be an exact discontinuity but must rather have some thickness through which the properties change from the upstream to the downstream values.

The logical next step would, of course, be to solve for the shock structure by going to the Navier-Stokes equation. This has been done[28] but the solution should really be rejected on theoretical grounds because it gives gradients which are large per mean free path in contradiction to the theory which relies on an expansion based on small gradients per mean free path. Qualitatively, one can believe the order of magnitude of thickness obtained. It is, however, still an unresolved question why the theory, though theoretically unjustified, gives answers that have at least for low Mach numbers been relatively well verified by the few experimental results available[29, 6].

Regardless of this, however, analysis of shock structure should, in principle, be carried out by means of kinetic theory. Various approximate means have been used in recent years to solve for the shock structure directly by kinetic theory. Probably the best known of these is the Mott-Smith method[30]. There have been others done by different moment methods[31, 32]. Comparison to determine which is the best method is very difficult because of the great scarcity of accurate experimental

results. All the methods, however, give shock thicknesses of the order of several mean free paths.

Since the integral equation formulation is particularly well suited for problems involving a scale of a mean free path, it would be a useful result to obtain the normal shock structure by means of integral iteration. It is obvious that to test the power of the method it is desirable to start with the crudest possible zeroth iterate (trial function). The Rankine-Hugoniot shock relations give the necessary boundary conditions, but every estimate of the shock thickness itself requires a lot of labor. The zeroth iterate was, therefore, chosen as a discontinuous set of macroscopic parameters satisfying the Rankine-Hugoniot conditions across the discontinuity. This corresponds to the zero shock thickness assumption and will thus test the convergence of the iteration most severely. It is apparent that if within several iterations the solution converged and gave a reasonable shock structure the method would be quite powerful. The basic integral equation when reduced to the steady one dimensional case without boundaries becomes

$$f_+(x,\vec{v}) = \int_{-\infty}^{x} \varepsilon(x',\vec{v}) \exp\left[-\frac{\Lambda(x,\vec{v})-\Lambda(x',\vec{v})}{v_1}\right] \nu(x',\vec{v}) \frac{dx'}{v_1}$$

$$\text{for} \quad v_1 > 0 \qquad \text{(VI-1)}$$

$$f_-(x,\vec{v}) = \int_{x}^{\infty} \varepsilon(x',\vec{v}) \exp\left[-\frac{\Lambda(x',\vec{v})-\Lambda(x,\vec{v})}{|v_1|}\right] \nu(x',\vec{v}) \frac{dx'}{|v_1|}$$

$$\text{for} \quad v_1 < 0 \qquad \text{(VI-2)}$$

and

$$\Lambda(x,\vec{v}) = \int^{x} \nu(x')\, dx'$$

The distribution function is represented differently in each half of velocity space because of the obvious physical fact that the distribution of particles at position x with velocity components positive in the x direction will be determined by all the particles scattered in the physical space for all $x' < x$ while the distribution function for particles with $v_1 < 0$ will be determined by conditions in the region of physical space for $x' > x$.

The initial trial function can be represented in the kinetic theory notation as

$$f^{(0)}(x,\vec{v}) = f_1$$

$$= \frac{n_1}{(2\pi RT_1)^{3/2}} \exp\left[\frac{-(v_1-u_1)^2+v_2^2+v_3^2}{2RT_1}\right] \qquad x<0 \qquad (VI-3)$$

$$f^{0}(x,\vec{v}) = f_2$$

$$= \frac{n_2}{(2\pi RT_2)^{3/2}} \exp\left[\frac{-(v_1-u_2)^2+v_2^2+v_3^2}{2RT_2}\right] \qquad x>0 \qquad (VI-4)$$

where n_2/n_1, T_2/T_1 and u_2/u_1 satisfy the normal shock relations. The first iteration can be immediately obtained since the integrations over physical space can be carried out exactly.

The lower limit on the integration in the expression for $\Lambda(x,\vec{v})$ is taken as zero

$$\Lambda(x,\vec{v}) = \nu_1(\vec{v})\,x \qquad x<0$$

$$\Lambda(x,\vec{v}) = \nu_2(\vec{v})\,x \qquad x>0 \qquad (VI-5)$$

and ν_1 and ν_2 are defined as the usual collision frequency

$$\nu(\vec{v}) = \iiint\iint \sigma(\Omega)\,|\vec{v}-\vec{w}|\,f(\vec{w})\,d\Omega\,d^3\vec{w} \qquad (VI-6)$$

evaluated for $X<0$ and $X>0$ respectively. Substitution of the above into the integral equation yields for the distribution function.

For $X>0$

$$f^{(1)}(x,\vec{v}) = \begin{cases} \mathcal{E}_2 + (\mathcal{E}_1-\mathcal{E}_2)\exp\left(-\frac{\nu_2 x}{v_1}\right) & v_1>0 \\ \mathcal{E}_2 & v_1<0 \end{cases} \qquad (VI-7)$$

For $X<0$

$$f^{(1)}(x,\vec{v}) = \begin{cases} \mathcal{E}_1 & v_1>0 \\ \mathcal{E}_2 + (\mathcal{E}_2-\mathcal{E}_1)\exp\left(\frac{\nu_1 x}{|v_1|}\right) & v_1<0 \end{cases} \qquad (VI-8)$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are known exactly for this first iteration since the zeroth iterate distributions are Maxwellian on either side of the discontinuity. The detail balance at equilibrium guarantees that $\mathcal{E}_1$

and $\mathcal{E}_2$, therefore, correspond to the distributions themselves and .

$$\mathcal{E}_1 = \frac{n_1}{(2\pi R T_1)^{3/2}} \exp\left[-\frac{(v_1-u_1)^2+v_2^2+v_3^2}{2RT_1}\right] \qquad (VI-9)$$

$$\mathcal{E}_2 = \frac{n_2}{(2\pi R T_2)^{3/2}} \exp\left[-\frac{(v_1-u_2)^2+v_2^2+v_3^2}{2RT_2}\right] \qquad (VI-10)$$

To obtain the first iterate on macroscopic variables it is only necessary to integrate $f^{(1)}$ multiplied by the various velocity moments. The general expression for $\nu(\vec{v})$ based on the original Maxwellian distributions is extremely complicated and contains error functions even in the hard sphere case. It is, therefore, desired for simplicity to take $\nu(\vec{v}) = \bar{\nu}$ independent of $\vec{v}$. This is essentially the same assumption as was made in the Krook's model except that in this case the form of $\mathcal{E}$ is exactly correct. It can also be considered as an assumption of the inverse fifth power repulsion law which gives collision frequency independent of velocity.

Before proceeding to try to evaluate the macroscopic variables it is advisable here to indicate the behavior of the solution at the original discontinuity. If we evaluate

$$f^{(1)}(0,\vec{v}) = \begin{cases} \mathcal{E}_1 & v_1 > 0 \\ \mathcal{E}_2 & v_1 < 0 \end{cases} \qquad (VI-11)$$

we notice that it converges to the same limit from either direction.

The first derivative, however, does not.

For $X < 0$

$$\left(\frac{\partial f^{(1)}}{\partial x}\right)_{x=0^-} = \frac{\nu_1}{|v_1|}(\varepsilon_2 - \varepsilon_1) \qquad \text{for } v_1 < 0$$

(VI-12)

$$\left(\frac{\partial f^{(1)}}{\partial x}\right)_{x=0^-} = 0 \qquad \text{for } v_1 > 0$$

While for $X > 0$

$$\left(\frac{\partial f^{(1)}}{\partial x}\right)_{x=0^+} = 0 \qquad \text{for } v_1 < 0$$

(VI-13)

$$\left(\frac{\partial f^{(1)}}{\partial x}\right)_{x=0^+} = \frac{\nu_2}{v_1}(\varepsilon_2 - \varepsilon_1) \qquad \text{for } v_1 > 0$$

Thus the first iteration eliminated the discontinuity but placed a discontinuity in the first derivative.

To evaluate macroscopic variables it is, in general, necessary (even with $\nu = \overline{\nu}$) to evaluate functions that have not been tabulated. These have the form

$$\Gamma_n(X,u) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left[-(v-u)^2 - \frac{X}{v}\right] v^n \, dv \qquad (VI-14)$$

Even though certain expansions are possible for evaluating these functions in certain ranges of K and u it was found that it was quickest to obtain them to within a few percent by numerical integration. The method used and the results obtained are shown in Appendix C. It is important here only to remark that these functions $\Gamma_n(K,u)$ obey a recursion relation which makes it necessary to compute the function for only three successive values of n as all the other $\Gamma_n(K,u)$'s can be obtained from these.

The first iterates of the macroscopic variables can be expressed entirely in terms of a sum of the above functions, while successive iterations, in general, can be expressed as integrals of these functions over X where $\Gamma_n(K,u)$ is a function of X through $K(x)$ and $u(x)$ obtained in the previous iteration.

When the parameters in the problem are all non-dimensionalized as follows

$$\bar{n}(\bar{x}) = \frac{n(x)}{n_1} \quad , \quad \bar{T}(x) = \frac{T(x)}{T_1} \quad , \quad \bar{u}(x) = \frac{u(x)}{\sqrt{2RT_1}}$$

$$\bar{\nu}(x) = \frac{\nu(x)}{\nu_1} \quad , \quad \bar{x} = \frac{x\nu_1}{\sqrt{2RT_1}} \quad \bar{V}_1 = \frac{V_1}{\sqrt{2RT_1}} \qquad \text{(VI-15)}$$

$$\bar{V}_2 = \frac{V_2}{\sqrt{2RT_1}} \quad , \bar{V}_3 = \frac{V_3}{\sqrt{2RT_1}} \quad , M_1 = \frac{u_1}{\sqrt{\gamma RT_1}} \quad , M_2 = \frac{u_2}{\sqrt{\gamma RT_2}}$$

the first iterate for the density in the hard sphere case becomes

$X > 0$

$$\bar{n}(\bar{x}) = \bar{n}_2 - \left(\bar{n}_2 \Gamma_0\left(\bar{n}_2|\bar{x}|, \sqrt{\tfrac{\gamma}{2}} M_2\right) - \Gamma_0\left(\bar{n}_2\sqrt{\bar{T}_2}|\bar{x}|, \sqrt{\tfrac{\gamma}{2}} M_1\right)\right)$$

$X < 0$ (VI-16)

$$\bar{n}(\bar{x}) = 1 + \left(\bar{n}_2 \Gamma_0\left(\frac{|\bar{x}|}{\sqrt{T_2}}, -\sqrt{\tfrac{\gamma}{2}} M_2\right) - \Gamma_0\left(|\bar{x}|, -\sqrt{\tfrac{\gamma}{2}} M_1\right)\right)$$

The velocity at any point can be evaluated from the continuity equation $\bar{n}\bar{u} = \bar{u}_1$

$$\bar{u}(x) = \sqrt{\frac{\gamma}{2}} M_1 / \bar{n}(x) \qquad \text{(VI-17)}$$

The temperature in turn can be evaluated in terms of $\Gamma_0(k,u)$, $\Gamma_2(k,u)$ and $\bar{u}(x)$.

$$\bar{T}(x) = \frac{\bar{n}_2 \bar{T}_2}{\bar{n}(x)} \left[1 - \frac{2}{3}\left(\Gamma_2(\bar{n}_2 \bar{x}, \sqrt{\frac{\gamma}{2}} M_2) + \Gamma_0(\bar{n}_2 \bar{x}, \sqrt{\frac{\gamma}{2}} M_2)\right)\right]$$

$$+ \frac{2}{3\bar{n}(x)} \left[\Gamma_2\left(\bar{n}_2 \sqrt{\bar{T}_2}\, \bar{x}, \sqrt{\frac{\gamma}{2}} M_1\right) + \Gamma_0\left(\bar{n}_2 \sqrt{\bar{T}_2}\, \bar{x}, \sqrt{\frac{\gamma}{2}} M_1\right)\right]$$

$$- \frac{2}{3}\left(\bar{u}^2 - \frac{\bar{n}_2}{\bar{n}} \frac{\gamma}{2} M_2^2\right) \qquad x > 0 \qquad \text{(VI-18)}$$

$$\bar{T}(x) = \frac{1}{\bar{n}(x)} \left[1 - \frac{2}{3}\left(\Gamma_2(|\bar{x}|, -\sqrt{\frac{\gamma}{2}} M_1) + \Gamma_0(|\bar{x}|, -\sqrt{\frac{\gamma}{2}} M_1)\right)\right]$$

$$+ \frac{2}{3} \frac{\bar{n}_2 \bar{T}_2}{\bar{n}(x)} \left[\Gamma_2\left(\frac{|\bar{x}|}{\sqrt{\bar{T}_2}}, -\sqrt{\frac{\gamma}{2}} M_2\right) + \Gamma_0\left(\frac{|\bar{x}|}{\sqrt{\bar{T}_2}}, -\sqrt{\frac{\gamma}{2}} M_2\right)\right]$$

$$- \frac{2}{3}\left(\bar{u}^2 - \frac{1}{\bar{n}(x)} \frac{\gamma}{2} M_1^2\right) \qquad \text{(VI-19)}$$

Other properties in the shock such as the heat flux and the stress can be obtained in terms of $\Gamma_0(k,u)$, $\Gamma_1(k,u)$ and $\Gamma_2(k,u)$ by direct integration over velocity space and the use of the recursion relation derived in Appendix C and appearing as Eq. (C-3). These same properties can, however, also be evaluated in terms of the properties already calculated through the application of the momentum and energy equations, which give the following results in this one

dimensional case.

$$\left(\frac{P_{xx}-P}{P_1}\right) = 1+\gamma M_1^2\left(1-\frac{1}{\bar{\kappa}(x)}\right)-\bar{\kappa}(x)\bar{T}(x) \qquad (VI-20)$$

$$q_x/p_1\sqrt{\gamma RT_1} = M_1\left[\left(1+\gamma M_1^2\right)\left(1-\frac{1}{\bar{\kappa}(x)}\right)+\frac{3}{2}(1-\bar{T})-\frac{\gamma M_1^2}{2}\left(1-\frac{1}{\bar{\kappa}^2}\right)\right] \qquad (VI-21)$$

Comparison of the results for these transport quantities obtained by the two different methods can serve as a test of the accuracy of the results. The difference in the results between two successive iterations is, of course, another way of testing the accuracy of the method.

In attempting to evaluate the second and higher order iterations it becomes immediately apparent that if this is done directly functions like $\Gamma_{-1}(\kappa,\mu)$ are going to appear because of the $1/v_1$ term in the integral. If the work could be continued analytically this would be no problem as the logarithmic singularity that this function possesses at $\kappa = 0$ is integrable. Since, however, successive iterations have to be done numerically it is advisable to try to avoid dealing with singular functions. This can be easily done by integrating the equation by parts once.

$$f_+(x,\vec{v}) = \mathcal{E}(x,\vec{v})$$

$$-\int_{-\infty}^{x}\frac{d\mathcal{E}}{dx'}(x',\vec{v})\exp\left[-\kappa(x,x')/v_1\right]dx' \qquad v_1>0$$

$$f_-(x,\vec{v}) = \mathcal{E}(x,\vec{v}) \qquad (VI-22)$$

$$+\int_{x}^{\infty}\frac{d\mathcal{E}}{dx'}(x',\vec{v})\exp\left[-\kappa(x',x)/|v_1|\right]dx' \qquad v_1<0$$

It is apparent from the above equation that the undesirable $1/v_1$ giving the $\Gamma_{-1}(k,u)$ function has been eliminated. The general solution for any macroscopic property can, therefore, be represented in terms of the previous value given by integrating $\mathcal{E}(\bar{x},\vec{v})$ plus a correction term composed of integrals over x' of the $\Gamma_n(k,u)$ functions for positive n multiplied by the property gradients resulting from the previous iteration. The property gradients come from the dependence of $\mathcal{E}(x,\vec{v})$ on the macroscopic properties. For the case of the simple Krook's model in the one dimensional case

$$\mathcal{E}(x,\vec{v}) = \frac{n(x)}{(2\pi RT(x))^{3/2}} \exp\left[-\frac{(v_1-u)^2+v_2^2+v_3^2}{2RT(x)}\right] \qquad (\text{VI}-23)$$

the gradient becomes

$$\frac{d\mathcal{E}}{dx}(x,\vec{v}) = \mathcal{E}(x,\vec{v})\left[\frac{d\ln n}{dx}(x) + \left(\frac{(v_1-u)^2+v_2^2+v_3^2}{2RT(x)} - \frac{3}{2}\right)\frac{d\ln T}{dx}(x)\right.$$

$$\left. + \frac{(v_1-u)}{RT(x)}\frac{du}{dx}\right] \qquad (\text{VI}-24)$$

The resulting expression for the density becomes

$$\bar{n}^{(\ell)}(\bar{x}) = \bar{n}^{(\ell-1)}(\bar{x}) - \int_{-\infty}^{\bar{x}} \left\{\Gamma_0\left(\bar{k}(\bar{x},\bar{x}'), \frac{\bar{u}^{\ell-1}}{\sqrt{\bar{T}^{\ell-1}}}\right)\times\right.$$

$$\left.\left[\frac{d\bar{n}^{\ell-1}}{d\bar{x}'} + \left(\frac{\bar{u}^{(\ell-1)2}}{\bar{T}^{\ell-1}} - \frac{1}{2}\right)\bar{n}^{\ell-1}\frac{d\ln\bar{T}^{\ell-1}}{d\bar{x}'} - 2\frac{\bar{u}^{\ell-1}\bar{n}^{\ell-1}}{\bar{T}^{\ell-1}}\frac{d\bar{u}^{\ell-1}}{d\bar{x}'}\right]\right.$$

$$+\Gamma_1\left(\bar{K}(\bar{x},\bar{x}'),\frac{\bar{u}^{\ell-1}}{\sqrt{\bar{T}^{\ell-1}}}\right)\left[\frac{2\bar{x}^{\ell-1}}{\sqrt{\bar{T}^{\ell-1}}}\frac{d\bar{u}^{\ell-1}}{d\bar{x}'}-\frac{2\bar{u}\bar{x}^{\ell-1}}{\sqrt{\bar{T}^{\ell-1}}}\frac{d\ln T^{\ell-1}}{d\bar{x}'}\right]$$

$$+\Gamma_2\left(\bar{K}(\bar{x},\bar{x}'),\frac{\bar{u}^{\ell-1}}{\sqrt{\bar{T}^{\ell-1}}}\right)\frac{d\ln T^{\ell-1}}{d\bar{x}'}\bar{n}^{\ell-1}\Big\}\,d\bar{x}'$$

$$+\int_{\bar{x}}^{\infty}\Big\{\Gamma_0\left(\bar{K}(\bar{x}',\bar{x}),-\frac{\bar{u}^{\ell-1}}{\sqrt{\bar{T}^{\ell-1}}}\right)\times$$

$$\left[\frac{d\bar{x}^{\ell-1}}{d\bar{x}'}+\left(\frac{\bar{u}^{(\ell-1)^2}}{\bar{T}^{\ell-1}}-\frac{1}{2}\right)\frac{d\ln\bar{T}^{\ell-1}}{d\bar{x}'}\bar{x}^{\ell-1}-\frac{2\bar{u}^{\ell-1}\bar{x}^{\ell-1}}{\sqrt{\bar{T}^{\ell-1}}}\frac{d\bar{u}^{\ell-1}}{d\bar{x}'}\right]$$

$$+\Gamma_1\left(\bar{K}(\bar{x}',\bar{x}),-\frac{\bar{u}^{\ell-1}}{\sqrt{\bar{T}^{\ell-1}}}\right)\left[\frac{2\bar{u}^{\ell-1}\bar{x}^{\ell-1}}{\sqrt{\bar{T}^{\ell-1}}}\frac{d\ln\bar{T}^{\ell-1}}{d\bar{x}'}-\frac{2\bar{x}^{\ell-1}}{\sqrt{\bar{T}^{\ell-1}}}\frac{d\bar{u}^{\ell-1}}{d\bar{x}'}\right]$$

$$+\Gamma_2\left(\bar{K}(\bar{x}',\bar{x}),-\frac{\bar{u}^{\ell-1}}{\sqrt{\bar{T}^{\ell-1}}}\right)\frac{d\ln\bar{T}^{\ell-1}}{d\bar{x}'}\bar{n}^{\ell-1}\Big\}\,d\bar{x}' \qquad \text{(VI-25)}$$

where

$$\bar{K}(\bar{x},\bar{x}')=\int_{\bar{x}'}^{\bar{x}}\bar{\nu}(\bar{x}'')\,d\bar{x}''\Big/\sqrt{\bar{T}(x')} \qquad \text{(VI-26)}$$

and

$$\bar{K}(\bar{x}';\bar{x}) = \int_{\bar{x}}^{\bar{x}'} \bar{\sigma}(\bar{x}'')\, d\bar{x}'' / \sqrt{\bar{T}(\bar{x}')} \qquad (VI-27)$$

A similary complicated result for the temperature can be obtained but, in principle, no new complications are added and the results can still be expressed in terms of integrals of the previously calculated properties and their gradients and the tabulated $\Gamma_n(K, u)$ functions.

As an example of the iterate solution the density profile for the case of $\gamma = 1.667$ and $M_1 = 1.5$ is plotted in Fig. VI-1. The discontinuous derivative at $X = 0$ in the first iterate is, of course, a spurious result of the initial choice of trial function and does not have a physical interpretation. This first iterate, however, is a crude representation of the shock, and the second iteration gives only a slight improvement. The difficulty lies near the point where the function was originally discontinuous. This can be interpreted to mean that a more realistic choice of initial trial function should be made as the discontinuous one places too stringent a demand on the integral iteration process.

The results thus far obtained are thus insufficient to determine the feasibility of integral iteration in calculating shock structure though the tendency in each iteration is in the right direction. Thus new efforts should be made by iterating from less unrealistic initial assumptions and thus improving the chances of convergence of the method within only a few iterations. Only in this way will the method be given a fair trial.

NOMENCLATURE

$f(x,\vec{v})$	The distribution function
M	The Mach number
n	The number density of particles in physical space
p	The pressure of the gas
p_{xx}	The x x component of the pressure tensor
q_x	The heat flux in x direction
R	The gas constant
T	The temperature of the gas
u	The macroscopic velocity of the gas
$\vec{v}$	The velocity vector of a particle
v_1, v_2, v_3	The x , y , z components of $\vec{v}$
x	Coordinate in direction normal to the shock
$\Gamma_n(k,u)$	Special function tabulated in Appendix C
γ	The specific heat ratio
$\mathcal{E}(x,\vec{v})$	The scattering function specifying the density of particles with velocity $\vec{v}$ that have suffered a collision at x
$\mathcal{L}(x,\vec{v})$	Integral over the collision frequency defined on
$\nu(x,\vec{v})$	The collision frequency of particles with velocity $\vec{v}$ at position x
$\bar{\nu}(x)$	The average collision frequency at x
$\sigma(\Omega)$	The differential collision cross section

Subscripts

$+$	Designates the half of velocity space where $v_1 > 0$
$-$	Designates the half of velocity space where $v_1 < 0$
1	Designates asymptotic conditions on the far upstream side of the shock
2	Designates asymptotic conditions on the far downstream side of the shock

Superscripts

$\overline{}$	Designates normalization with respect to upstream conditions (Defined on pages 91 through 92)
l	Designates the l'th iteration solution

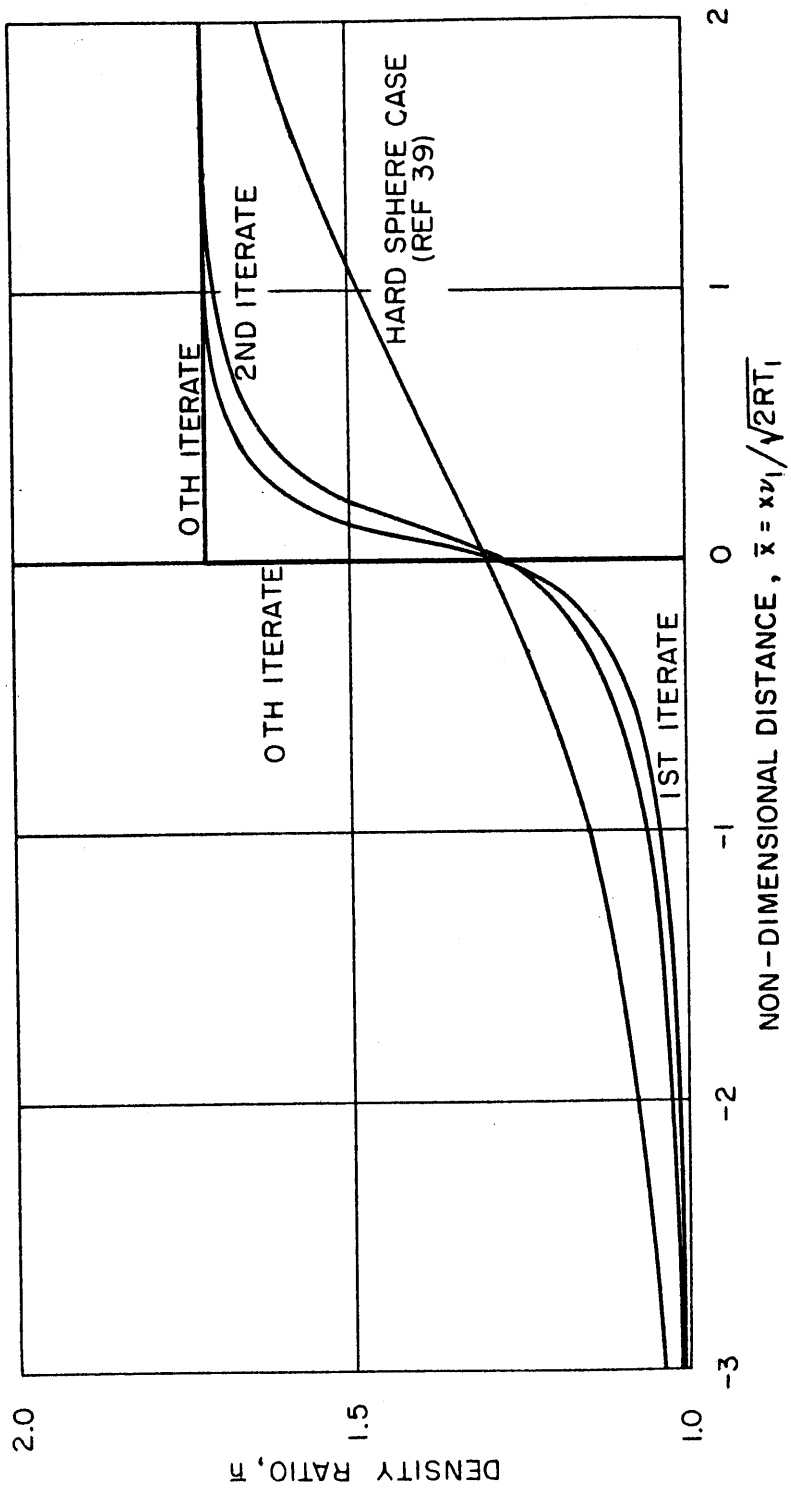


Figure VI-1. Density profile of normal shock at M = 1.5, γ = 1.67

VII. HEAT TRANSFER BETWEEN PARALLEL PLATES

A relatively simple example of a boundary value problem in the transition regime is the problem of heat transfer between parallel plates at pressures low enough to make the mean free path of the same order as the plate separation. The problem has all the basic features of the general boundary value problem in the transition regime, but the geometry is very simple, and the description of the problem is limited to a single variable, the temperature profile. There is also the additional advantage in this problem in the fact that the continuum and free molecular solutions are both well known and simple. We have, therefore, chosen this problem as an illustration of the application of the integral iteration method to solution of transition boundary value problems.

The problem can be stated as follows; two plates infinite in extant are separated by a distance b and are maintained at temperatures T_{w_1} and T_{w_2} respectively. The solution desired consists of the temperature profile of the gas in the gap between the two plates, and the overall heat flux, as a function of the pressure of the gas. The solutions for pressures high enough to make the gap be many times the mean free path are of course well known. The boundary conditions are such as to force the gas temperature to the wall temperatures at the walls, and the profile is either a straight line or varies as y^n with n depending on the assumed temperature dependence of the thermal conductivity. The heat flux is directly proportional to the temperature difference and inversely proportional to the plate separation while being independent of the pressure. The low pressure or free molecular solution is also well known. The temperature is a constant with a jump existing between the wall temperature and the fluid temperature at the wall. The heat flux is still proportional to the temperature difference between the two walls, but is independent of the plate separation, while being proportional to the pressure.

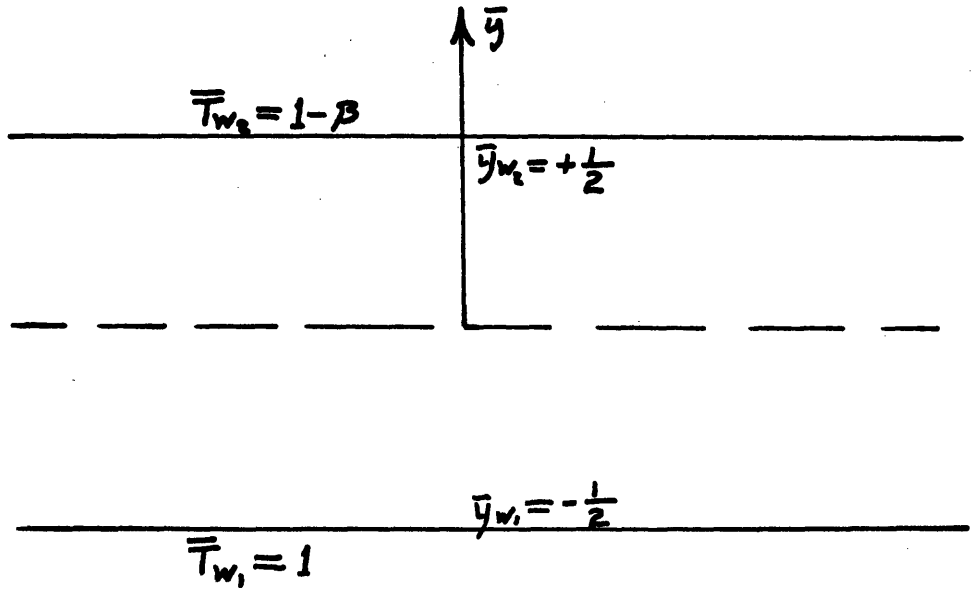
It is apparent, therefore, that the change in the mode of heat transfer takes place someplace between the two extremes. It is our goal to try to determine the temperature profile and heat flux for the whole range of pressures between the continuum and the free-molecular condition. Since it is apparent that the heat flux cannot really be a function of the absolute value of the pressure, but rather some non-dimensional quantity, it must depend on the pressure through the Knudsen number, which in this case is the ratio of the mean free path to the plate separation.

When the variables are non-dimensionalized and the temperature difference introduced as a parameter, the problem takes the following form

$$\bar{T} = T/T_{w_1} \qquad \bar{y} = y/b$$

VII - 1

$$\beta = \frac{\Delta T_w}{T_{w_1}} = \frac{T_{w_1} - T_{w_2}}{T_{w_1}}$$



The continuum solution is obtained from the energy equation which reduces to

$$\frac{d}{d\bar{y}} k(\bar{T}) \frac{d\bar{T}}{d\bar{y}} = 0 \qquad (VII-2)$$

where k is the thermal conductivity, and the boundary conditions are, $\bar{T}(-\frac{1}{2}) = 1$ and $\bar{T}(+\frac{1}{2}) = 1-\beta$. The heat flux is $q_{y_0} = k(\bar{T}) T_{M_1}/b \; d\bar{T}/d\bar{y}$ and is constant. Solution can be easily obtained for simple relations between k and T.

The free molecular solution is likewise easily obtainable by considering all the particles with positive y velocities to have come from the bottom plate, and all those with negative y velocities to have come from the top plate. The condition of no net mass transfer and the accommodation coefficients at the walls completely determine the solution, which is simply

$$q_y = \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2 - \alpha_1 \alpha_2} \frac{\rho}{\sqrt{\bar{T}}} \frac{\sqrt{2RT_{M_1}}}{\sqrt{\pi}} \beta \qquad (VII-3)$$

where α_1 and α_2 are the accomodation coefficients

and $$\bar{T} = 4\left(\frac{\bar{T}_1 \bar{T}_2}{\bar{T}_2 + \bar{T}_1 + 2\sqrt{\bar{T}_1 \bar{T}_2}}\right) \qquad (VII-4)$$

where $$\bar{T}_1 = \frac{\alpha_1 + \alpha_2(1-\alpha_1)(1-\beta)}{\alpha_1 + \alpha_2 - \alpha_1 \alpha_2}$$

$$\bar{T}_2 = \frac{\alpha_2(1-\beta) + \alpha_1(1-\alpha_2)}{\alpha_1 + \alpha_2 - \alpha_1 \alpha_2}$$

We, however, want to find the solution for all Knudson numbers in between. As discussed in Section IV, the application of the integral equation formulation to a boundary value problem requires iteration. Since the choice of the initial trial function will either greatly enhance or hinder the iteration process, it is very important to choose a good one. As was already mentioned, the integral iteration process will in general quickly converge to the right limit in and near the free molecular end. It is, therefore, most important that the initial function chosen approach the continuum limit in a correct way as there convergence is poorest. At the same time a choice of a particulary complicated function will require a large amount of labor in the iteration and will defeat its own purpose. The best compromise choice for the zeroth iterate, therefore, seems to be a solution obtained by Lester Lees' moment method. Though it is certainly true that this solution does not approach the free molecular limit in the proper fashion and that the detail profile cannot be correct, the gross behaviour is correct at both limits and the trend in between is represented by a relatively simple smooth function. We, therefore, propose to solve the heat transfer between parallel plates problem by Lees' moment method and then iterate from this in the integral equation. The solution by Lees' moment method is carried out in Appendix D so as not to complicate the train of thought here by too many details.

The results for any of the several cases carried out in the appendix always give a temperature profile that is identical with the continuum profile in shape (for the same k(T) dependence) but has a temperature jump at each wall. The heat flux is always equal to the heat flux at continuum times some function of Knudsen number that reduces to $\frac{1}{K}$ at the free molecular limit. Keeping this in mind, we can analyze the integral iteration procedure without fixing our attention on any particular k(T) dependence or equivalently any particular collision cross section dependence.

The integral equation for the steady state one dimensional

case with boundaries becomes

$$f_+(y,\vec{v}) = \int_{y_1}^{y} \varepsilon(y';\vec{v}) \exp\left[-\frac{\Lambda(y,\vec{v})-\Lambda(y';\vec{v})}{v_1}\right] \nu(y';\vec{v}) \frac{dy'}{v_1}$$

$$+ f_+(y_1,\vec{v}) \exp\left[-\frac{\Lambda(y)}{v_1}\right] \qquad v_1>0$$

(VII-5)

$$f_-(y,\vec{v}) = \int_{y}^{y_2} \varepsilon(y';\vec{v}) \exp\left[-\frac{\Lambda(y';\vec{v})-\Lambda(y,\vec{v})}{|v_1|}\right] \nu(y';\vec{v}) \frac{dy'}{|v_1|}$$

$$+ f_-(y_2,\vec{v}) \exp\left[-\frac{\Lambda(y_2)-\Lambda(y)}{|v_1|}\right] \qquad v_1<0$$

where

$$\Lambda(y,\vec{v}) = \int_{y_1}^{y} \nu(y';\vec{v})\, dy'$$

Iteration follows the same lines as that carried out for the normal shock case but with a different zeroth iterate and the additional complication of $f_+(y_1,\vec{v})$ and $f_-(y_2,\vec{v})$ entering into the problem.

We again have a choice of using an exact evaluation of $\nu(y,\vec{v})$ and $\varepsilon(y,\vec{v})$ based on the zeroth iterate or using a model based on the macroscopic properties alone from the zeroth iterate. Here, the choice is quite obvious. Since our zeroth iterate is a moment method solution, it is apparent that a model for ε and ν is actually a better as well as a more convenient choice. The moment method solution does not really purport to solve for the distribution function, but rather uses certain free parameters in an assumed distribution function to satisfy the conservation laws as well as some higher moment equations. The macroscopic variables can then be treated as the solution while the distribution function can be considered as only a weighing function used in the solution. Especially in the case of Lees' method which assumes a distribution

function discontinuous in velocity space the use of the distribution function as a solution is undesirable.

We, therefore, choose for the model $\mathcal{E}$ and ν a modified version of the Krook's model. The collision frequency ν is still the velocity independent average collision frequency. We choose an $\mathcal{E}(y, \vec{v})$, however, that has an asymmetry proportional to the temperature gradient as derived in Chapter III. In this case of no net macroscopic velocity and for no pressure gradient (Lees' moment method solution) this becomes

$$\mathcal{E}(y,\vec{v}) = \frac{n(y)}{(2\pi RT(y))^{3/2}} \exp\left[-\frac{v_1^2+v_2^2+v_3^2}{2RT(y)}\right] \times$$

$$\left[1 - \frac{c_1 v_1}{\nu(y)}\left(\frac{v_1^2+v_2^2+v_3^2}{2RT(y)} - \frac{5}{2}\right)\frac{d\ln T}{dy}\right] \qquad \text{(VII-6)}$$

When this type of model is substituted into the integral equation and one tries to evaluate the macroscopic variables again an untabulated kind of function appears of importance.

$$I_n(K) = \int_0^\infty \exp\left[-\left(v^2+\frac{K}{v}\right)\right] v^n dv \qquad \text{(VII-7)}$$

Even though these functions are only special cases of the $\Gamma_n(K,u)$ functions they are evaluated and tabulated separately in Appendix E. They are evaluated to a much greater accuracy than the $\Gamma_n(K,u)$ functions because they can be evaluated by expansion for small K and by method of steepest descents for large K. Again there is a recursion relation between different orders n and some integral relations as well. The method of evaluation and the tabulated results are presented in Appendix E.

With these results available we can now proceed to evaluate the first iterate for our problem of heat transfer between parallel plates. Lees' moment method is based on the assumption of the inverse fifth power repulsion between particles, thus, for consistancy one should use the thermal conductivity proportional to T the temperature, and a collision frequency ν proportional to density alone. With these assumptions our zeroth iterate becomes (from Appendix D)

$$\psi = q_y/q_{y_0} = \frac{\alpha}{\beta^2 K^2}\left[\frac{1+K\left(\frac{2-\alpha}{\alpha}\right)\left(1+(1-\beta)^{3/2}\right)}{4\frac{1-\alpha}{\alpha}+\alpha\left(\frac{2-\alpha}{\alpha}\right)^2}\right]\times$$

$$\times\left[1-\sqrt{1-\frac{2\beta^2K^2\left[4\frac{1-\alpha}{\alpha}+\alpha\left(\frac{2-\alpha}{\alpha}\right)^2\right]\left(1-\frac{\beta}{2}\right)}{\alpha\left[1+\frac{2-\alpha}{\alpha}K\left(1+(1-\beta)^{3/2}\right)\right]^2}}\right] \qquad \text{(VII-8)}$$

$$\mp = \sqrt{\phi^2 - 2\psi\beta\bar{y}}$$

where $$K = \frac{k_{w_1}}{pb}\frac{T_{w_1}}{\frac{2}{\pi}\sqrt{2RT_{w_1}}} = \frac{15}{8}\frac{\lambda_{w_1}}{b}$$

Now for a first try we can take the case of perfect accomodation which reduces these equations to

$$\psi = \frac{1+K\left(1+(1-\beta)^{3/2}\right)}{[K\beta]^2}\left[1-\sqrt{1-(2-\beta)\left[\frac{\beta K}{1+K\left(1+(1-\beta)^{3/2}\right)}\right]^2}\right] \qquad \text{(VII-9)}$$

$$\mp = \sqrt{\phi^2 - 2\psi\beta\bar{y}} \qquad \text{(VII-10)}$$

where

$$\phi^2 = 1-\beta\left[1+\psi K\left(1-\sqrt{1-\beta}\right)-\frac{\beta}{2}\left(1+(2-\beta)(\psi K)^2\right)\right] \qquad \text{(VII-11)}$$

In principle it should be possible to substitute this $\bar{y}$ dependence for $\bar{T}$ and the corresponding $\bar{y}$ dependence of $\bar{n}$ into the expressions for $\mathcal{E}(\bar{y},\bar{v})$ and $\nu(\bar{y},\bar{v})$ and evaluate directly the first iterate in terms of the $I_n(K)$ functions. The $1/v_1$ appearing in the integral equation necessitates evaluating the solution in terms of integrals over the $I_{-1}(k)$ functions which have a logarithmic singularity. (Appendix E shows that $I_0(K)$ behaves as $(a_0 + b_1 K \ln K)$ for small K and the differential relation $I_n = dI_{n+1}/dK$ shows that $I_{-1}(K)$ has a logarithmic singularity at $K=0$.) The fact that this singularity is integrable does not cause any problems in an analytic solution if such were possible. If, however, the iteration is going to be carried out numerically the singularity will cause problems in the integrations. It is, therefore, advisable to avoid this singularity by integrating the integral equation by parts once and thus, eliminating the $1/v_1$ term. This as in the normal shock case also has the advantage in the fact that it gives the ℓ'th iterate of any macroscopic property directly as the $(\ell-1)$'th iterate plus some terms that are evaluated in terms of the gradients of the macroscopic properties evaluated by the $(\ell-1)$ iterate.

The integral equation for f in terms of the normalized coordinates becomes

$$f_+(\bar{y},\bar{v}) = b\int_{-\frac{1}{2}}^{\bar{y}} \mathcal{E}(\bar{y}',\bar{v})\exp\left[-\frac{\Lambda(\bar{y})-\Lambda(\bar{y}')}{v_1}\right]\nu(\bar{y}')\frac{d\bar{y}'}{v_1}$$

$$+ f_+(-\tfrac{1}{2},\bar{v})\exp\left[-\frac{\Lambda(\bar{y})}{v_1}\right] \qquad v_1 > 0$$

$$f_-(\bar{y},\bar{v}) = b\int_{\bar{y}}^{\frac{1}{2}} \mathcal{E}(\bar{y}',\bar{v})\exp\left[-\frac{\Lambda(\bar{y})-\Lambda(\bar{y})}{|v_1|}\right]\nu(\bar{y}')\frac{d\bar{y}'}{|v_1|}$$

$$+ f_-(+\tfrac{1}{2},\bar{v})\exp\left[-\frac{\Lambda(\tfrac{1}{2})-\Lambda(\bar{y})}{|v_1|}\right] \qquad v_1 < 0$$

where $$\Lambda(\bar{y}) = b\int_{-\frac{1}{2}}^{\bar{y}} \nu(\bar{y}'')\,d\bar{y}'' \qquad (\text{VII-12})$$

When this is integrated by parts once it becomes

$$f_+(\bar{y},\vec{v}) = \mathcal{E}(\bar{y},\vec{v}) + \left(f_+(-\tfrac{1}{2},\vec{v}) - \mathcal{E}(-\tfrac{1}{2},\vec{v})\right)\exp\left[-\frac{\Lambda(\bar{y})}{v_1}\right]$$

$$-b\int_{-\frac{1}{2}}^{\bar{y}} \frac{d\mathcal{E}}{d\bar{y}'}\exp\left[-\frac{\Lambda(\bar{y})-\Lambda(\bar{y}')}{v_1}\right]d\bar{y}'$$

$$\text{for } v_1 > 0 \qquad \text{(VII-13)}$$

$$f_-(\bar{y},\vec{v}) = \mathcal{E}(\bar{y},\vec{v}) + \left(f_-(+\tfrac{1}{2},\vec{v}) - \mathcal{E}(+\tfrac{1}{2},\vec{v})\right)\exp\left[-\frac{\Lambda(\tfrac{1}{2})-\Lambda(\bar{y})}{|v_1|}\right]$$

$$+b\int_{\bar{y}}^{\frac{1}{2}} \frac{d\mathcal{E}}{d\bar{y}'}\exp\left[-\frac{\Lambda(\bar{y}')-\Lambda(\bar{y})}{|v_1|}\right]d\bar{y}'$$

$$\text{for} \quad v_1 < 0 \qquad \text{(VII-14)}$$

When any velocity moment is calculated to determine the macroscopic properties the integration of the first term contributes the previous iterate automatically because of the choice of $\mathcal{E}(\bar{y},\vec{v})$. The correction to this term is supplied by a boundary term from each side and an integral of the property gradients times the $\Pi_n(K)$ (i.e. $I_n(K)/I_n(0)$) functions.

Thus, the first iterate on the density becomes

$$\bar{n}'(\bar{y}) = \frac{1}{\bar{T}^{\circ}(\bar{y})}$$

$$+ \frac{\Pi_0}{2}\left(\frac{\Lambda^{\circ}(\bar{y})}{\sqrt{2RT_{w_1}}}\right)$$

$$- \frac{1}{2\bar{T}^{\circ}(-\frac{1}{2})}\Pi_0\left(\frac{\Lambda^{\circ}(\bar{y})}{\sqrt{2R\bar{T}^{\circ}(-\frac{1}{2})}}\right)$$

$$+ \left(\frac{1}{1-\beta}\right)\frac{\Pi_0}{2}\left(\frac{\Lambda^{\circ}(\frac{1}{2})-\Lambda^{\circ}(\bar{y})}{\sqrt{2RT_{w_2}}}\right)$$

$$- \frac{1}{2\bar{T}^{\circ}(+\frac{1}{2})}\Pi_0\left(\frac{\Lambda^{\circ}(\frac{1}{2})-\Lambda^{\circ}(\bar{y})}{\sqrt{2RT^{\circ}(\frac{1}{2})}}\right)$$

$$- \frac{(1+c_1)}{4}\int_{-\frac{1}{2}}^{\bar{y}}\left(\frac{1}{\bar{T}^{\circ}(\bar{y})}\frac{d\ln\bar{T}^{\circ}}{d\bar{y}'}\right)\left\{\Pi_2\left(\frac{\Lambda^{\circ}(\bar{y})-\Lambda^{\circ}(\bar{y}')}{\sqrt{2RT^{\circ}(\bar{y}')}}\right)\right.$$

$$\left. - 3\Pi_0\left(\frac{\Lambda^{\circ}(\bar{y})-\Lambda^{\circ}(\bar{y}')}{\sqrt{2RT^{\circ}(\bar{y})}}\right)\right\} d\bar{y}'$$

$$+ \frac{(1+c_1)}{4}\int_{\bar{y}}^{\frac{1}{2}}\left(\frac{1}{\bar{T}^{\circ}(\bar{y}')}\frac{d\ln T^{\circ}}{d\bar{y}'}\right)\left\{\Pi_2\left(\frac{\Lambda^{\circ}(\bar{y}')-\Lambda^{\circ}(\bar{y})}{\sqrt{2RT^{\circ}(\bar{y}')}}\right)\right.$$

$$\left. - 3\Pi_0\left(\frac{\Lambda^{\circ}(\bar{y}')-\Lambda^{\circ}(\bar{y})}{\sqrt{2RT^{\circ}(\bar{y}')}}\right)\right\} d\bar{y}' \qquad (VII-15)$$

The first iterate on the temperature can be obtained also by a similar procedure and can be expressed as follows.

$$\frac{3}{2} n m R T(\bar{y}) = \frac{3}{2} m n^{o}(\bar{y}) R T^{o}(\bar{y})$$

$$+ \frac{n_{w_1} m R T_{w_1}}{4} \left(\Pi_2 \left(\frac{\Lambda(\bar{y})}{\sqrt{2RT_{w_1}}} \right) + 2\Pi_0 \left(\frac{\Lambda(\bar{y})}{\sqrt{2RT_{w_1}}} \right) \right)$$

$$- \frac{n^{o}(-\frac{1}{2}) m R T^{o}(-\frac{1}{2})}{4} \left(\Pi_2 \left(\frac{\Lambda(\bar{y})}{\sqrt{2RT^{o}(-\frac{1}{2})}} \right) + 2\Pi_0 \left(\frac{\Lambda(\bar{y})}{\sqrt{2RT^{o}(-\frac{1}{2})}} \right) \right)$$

$$+ \frac{n_{w_2} m R T_{w_2}}{4} \left(\Pi_2 \left(\frac{\Lambda(\frac{1}{2}) - \Lambda(\bar{y})}{\sqrt{2RT_{w_2}}} \right) + 2\Pi_0 \left(\frac{\Lambda(\frac{1}{2}) - \Lambda(\bar{y})}{\sqrt{2RT_{w_2}}} \right) \right)$$

$$- \frac{n^{o}(\frac{1}{2}) m R T^{o}(\frac{1}{2})}{4} \left(\Pi_2 \left(\frac{\Lambda(\frac{1}{2}) - \Lambda(\bar{y})}{\sqrt{2RT^{o}(\frac{1}{2})}} \right) + 2\Pi_0 \left(\frac{\Lambda(\frac{1}{2}) - \Lambda(\bar{y})}{\sqrt{2RT^{o}(\frac{1}{2})}} \right) \right)$$

$$- (1+c_1) \int_{-\frac{1}{2}}^{\bar{y}} n^{o}(\bar{y}') m R T_{o}^{o}(\bar{y}') \frac{d \ln T^{o}(\bar{y}')}{d\bar{y}'} \times$$

$$\left\{ \frac{3}{8} \Pi_4 \left(\frac{\Lambda(\bar{y}) - \Lambda(\bar{y}')}{\sqrt{2RT^{o}(\bar{y}')}} \right) - \frac{3}{4} \Pi_2 \left(\frac{\Lambda(\bar{y}) - \Lambda(\bar{y}')}{\sqrt{2RT^{o}(\bar{y}')}} \right) \right.$$

$$\left. - \frac{1}{4} \Pi_0 \left(\frac{\Lambda(\bar{y}) - \Lambda(\bar{y}')}{\sqrt{2RT^{o}(\bar{y}')}} \right) \right\} d\bar{y}'$$

$$+ (1+c_1) \int_{-\frac{1}{2}}^{\bar{y}} n^{o}(\bar{y}') m R T_{o}^{o}(\bar{y}') \frac{d \ln T^{o}(\bar{y}')}{d\bar{y}'} \times$$

$$\left\{ \frac{3}{8} \Pi_4 \left(\frac{\Lambda(\bar{y}') - \Lambda(\bar{y})}{\sqrt{2RT^{o}(\bar{y}')}} \right) - \frac{3}{4} \Pi_2 \left(\frac{\Lambda(\bar{y}') - \Lambda(\bar{y})}{\sqrt{2RT^{o}(\bar{y}')}} \right) \right.$$

$$\left. - \frac{1}{4} \Pi_0 \left(\frac{\Lambda(\bar{y}') - \Lambda(\bar{y})}{\sqrt{2RT^{o}(\bar{y}')}} \right) \right\} d\bar{y}' \qquad \text{(VII-16)}$$

The heat flux can be evaluated directly from the original integral equation because the ν_1 term in the numerator of the weighing function cancels out the ν_1, in the denominator of the equation. The expression for the heat flux, therefore, becomes

$$q_y^{(1)} = p^\circ \frac{2}{\sqrt{\pi}} \left[\sqrt{2RT_{w_1}} \left(\frac{\Pi_3}{4} \left(\frac{\Lambda^\circ(\bar{y})}{\sqrt{2RT_{w_1}}} \right) + \frac{\Pi_1}{4} \left(\frac{\Lambda^\circ(\bar{y})}{\sqrt{2RT_{w_1}}} \right) \right) \right.$$

$$\left. - \sqrt{2RT_{w_2}} \left(\frac{\Pi_3}{4} \left(\frac{\Lambda^\circ(\frac{1}{2}) - \Lambda^\circ(\bar{y})}{\sqrt{2RT_{w_2}}} \right) + \frac{\Pi_1}{4} \left(\frac{\Lambda^\circ(\frac{1}{2}) - \Lambda^\circ(\bar{y})}{\sqrt{2RT_{w_2}}} \right) \right) \right]$$

$$+ p^\circ b \left\{ \int_{-\frac{1}{2}}^{\bar{y}} \nu(\bar{y}') \left(\frac{\Pi_2}{4} \left(\frac{\Lambda^\circ(\bar{y}) - \Lambda^\circ(\bar{y}')}{\sqrt{2RT^\circ(\bar{y}')}} \right) + \frac{\Pi_0}{2} \left(\frac{\Lambda^\circ(\bar{y}) - \Lambda^\circ(\bar{y}')}{\sqrt{2RT^\circ(\bar{y}')}} \right) \right) d\bar{y}' \right.$$

$$\left. - \int_{\bar{y}}^{\frac{1}{2}} \nu(\bar{y}') \left(\frac{\Pi_2}{4} \left(\frac{\Lambda^\circ(\bar{y}') - \Lambda^\circ(\bar{y})}{\sqrt{2RT^\circ(\bar{y}')}} \right) + \frac{\Pi_0}{2} \left(\frac{\Lambda^\circ(\bar{y}') - \Lambda^\circ(\bar{y})}{\sqrt{2RT^\circ(\bar{y}')}} \right) \right) d\bar{y}' \right\}$$

$$- c_1 p^\circ \frac{2}{\sqrt{\pi}} \left\{ \int_{-\frac{1}{2}}^{\bar{y}} \sqrt{2RT^\circ(\bar{y}')} \frac{d \ln T^\circ}{d\bar{y}'} \left[\frac{\Pi_5}{2} \left(\frac{\Lambda^\circ(\bar{y}) - \Lambda^\circ(\bar{y}')}{\sqrt{2RT^\circ(\bar{y}')}} \right) \right. \right.$$

$$\left. - \frac{3}{8} \Pi_3 \left(\frac{\Lambda^\circ(\bar{y}) - \Lambda^\circ(\bar{y}')}{\sqrt{2RT^\circ(\bar{y}')}} \right) - \frac{1}{8} \Pi_1 \left(\frac{\Lambda^\circ(\bar{y}) - \Lambda^\circ(\bar{y}')}{\sqrt{2RT^\circ(\bar{y}')}} \right) \right] d\bar{y}'$$

$$+ \int_{\bar{y}}^{\frac{1}{2}} \sqrt{2RT^\circ(\bar{y}')} \frac{d \ln T^\circ}{d\bar{y}'} \left[\frac{\Pi_5}{2} \left(\frac{\Lambda^\circ(\bar{y}') - \Lambda^\circ(\bar{y})}{\sqrt{2RT^\circ(\bar{y}')}} \right) \right.$$

$$\left. \left. - \frac{3}{8} \Pi_3 \left(\frac{\Lambda^\circ(\bar{y}') - \Lambda^\circ(\bar{y})}{\sqrt{2RT^\circ(\bar{y}')}} \right) - \frac{1}{8} \Pi_1 \left(\frac{\Lambda^\circ(\bar{y}') - \Lambda^\circ(\bar{y})}{\sqrt{2RT^\circ(\bar{y}')}} \right) \right] d\bar{y}' \right\} \qquad (\text{VII}-17)$$

The $\Lambda(\bar{y})$ appearing in the above expressions can be evaluated analytically for this first iterate. $\Lambda(\bar{y}')$ is defined as

$$\Lambda(\bar{y}') = b \int_{-\frac{1}{2}}^{\bar{y}'} \nu(\bar{y}'') \, d\bar{y}'' \qquad (\text{VII}-18)$$

and $\nu(\bar{y}'')$ can be taken as $\nu_0 n^0(\bar{y}'')$ where $n^0(\bar{y})$ is a constant and $n^0(\bar{y}'')$ is the zeroth iterate solution for the density. The fact that the Lester Lees' moment method solution gives a constant pressure allows us to evaluate this immediately in terms of the temperature. This gives the equation

$$\Lambda(\bar{y}') = b \int_{-\frac{1}{2}}^{\bar{y}'} \frac{\nu_0 p}{Rm} \, d\bar{y}''/T(\bar{y}'') \qquad \text{(VII-19)}$$

This in turn can be easily integrated for the case of Maxwellian particles as

$$T(\bar{y}'') = T_{w_1} \sqrt{\phi^2 - 2\psi_B \bar{y}}$$

and

$$\frac{d\bar{y}''}{T(\bar{y}'')} = - dT(\bar{y}'')/(T_{w_1}^2 \psi_B)$$

thus

$$\Lambda(\bar{y}') = \frac{b\nu_0 p}{Rm T_{w_1} \psi_B} \int_{\bar{T}(\bar{y}')}^{\bar{T}(-\frac{1}{2})} d\bar{T}$$

or

$$\Lambda(\bar{y}') = b\nu_{w_1}/\psi_B \left[\bar{T}(-\tfrac{1}{2}) - \bar{T}(\bar{y}')\right] \qquad \text{(VII-20)}$$

With the above relations, it is now possible to evaluate the first iterate by numerical integrations. Successive iterations can be continued by use of the same basic integral relation, but the fact that the pressure will not in general be a constant will cause some modification in the evaluation of the asymmetric term in the collision model, and of course the $\Lambda(\bar{y}')$ will not in general be available analytically.

As an example of the effect of the first iteration on Lees' moment method solution the temperature profile was calculated for the case of Knudsen number equal to 8/15. The results are plotted in Fig. VII-1. It can be immediately seen that the first iteration introduces a boundary "layer type" effect near the boundaries and reduces the actual temperature jump at the walls. Both these effects are in the right direction as on physical grounds it is easy to see that near the walls the effect of the walls is going to produce a kind of boundary layer resulting from the $\exp\left[-\int d\bar{y}/\lambda(\bar{y})\right]$ scattering of the particles that came originally from the wall. Likewise, it is known that Lees' moment method overestimates the slip at a wall in the linearized Couette flow problem[23] and thus, it is reasonable to assume that it also overestimates the temperature jump in the heat transfer problem. The results in Fig. VII-1 therefore, show that the iteration process is at least tending towards results that are physically more realistic than the original trial function. A real test of the convergence of the method in this case, however, will require further iteration as well as calculations in other Knudsen number ranges. The preliminary results indicated here, certainly allows us to have high hopes for the quick convergence of the results and their physical validity.

NOMENCLATURE

b	The plate separation
c_1	Constant in correction to Krook's model to give proper continuum limit
$f(y,\vec{v})$	The distribution function
$I_n(K)$	Function defined and evaluated in Appendix E
K	Parameter (which is 15/8 times the Knudson number) appearing in Lester Lees' moment method solution.
$k(\bar{T})$	Thermal conductivity
p	The pressure of the gas
q_y	The heat flux between the plates
R	The gas constant
T	The temperature of the gas
$\bar{T}$	Temperature normalized by the hot wall temperature (T/T_{w_1})
$\vec{v}$	Velocity vector of a particle
v_1, v_2, v_3	The y, x and z components of $\vec{v}$ respectively
y	Coordinate normal to the plates and having its origin halfway between them
$\bar{y}$	Normalized coordinate (y/b)
α	The accomodation coefficient
β	Normalized temperature difference
$\mathcal{E}(y,\vec{v})$	The scattering function designating the density of particles with velocity $\vec{v}$ that have just suffered a collision at y

$\Lambda(y,\vec{v})$	Integral of collision frequency over y (defined on page)
λ	The mean free path
$\nu(y,\vec{v})$	The collision frequency of particles with velocity $\vec{v}$ at position y
$\Pi_n(K)$	A function defined as $I_n(K)/I_n(0)$ and tabulated in Appendix E
ϕ	Constant appearing in Lester Lees' moment method solution (defines the normalized temperature at $y = 0$)
$\psi = q_y/q_{y0}$	Heat flux divided by the value at continuum conditions

Subscripts

w_1	Designates conditions based on the hot wall temperature
w_2	Designates conditions based on the cold wall temperature
0	Designates the continuum asymptote
$+$	Designates the half of velocity space where the y component $\vec{v}$ is positive
$-$	Designates the half of velocity space where the y component of $\vec{v}$ is negative

Superscripts

0	Designates the zeroth iterate, i.e., Lester Lees' moment method solution
1	Designates the first iteration solution

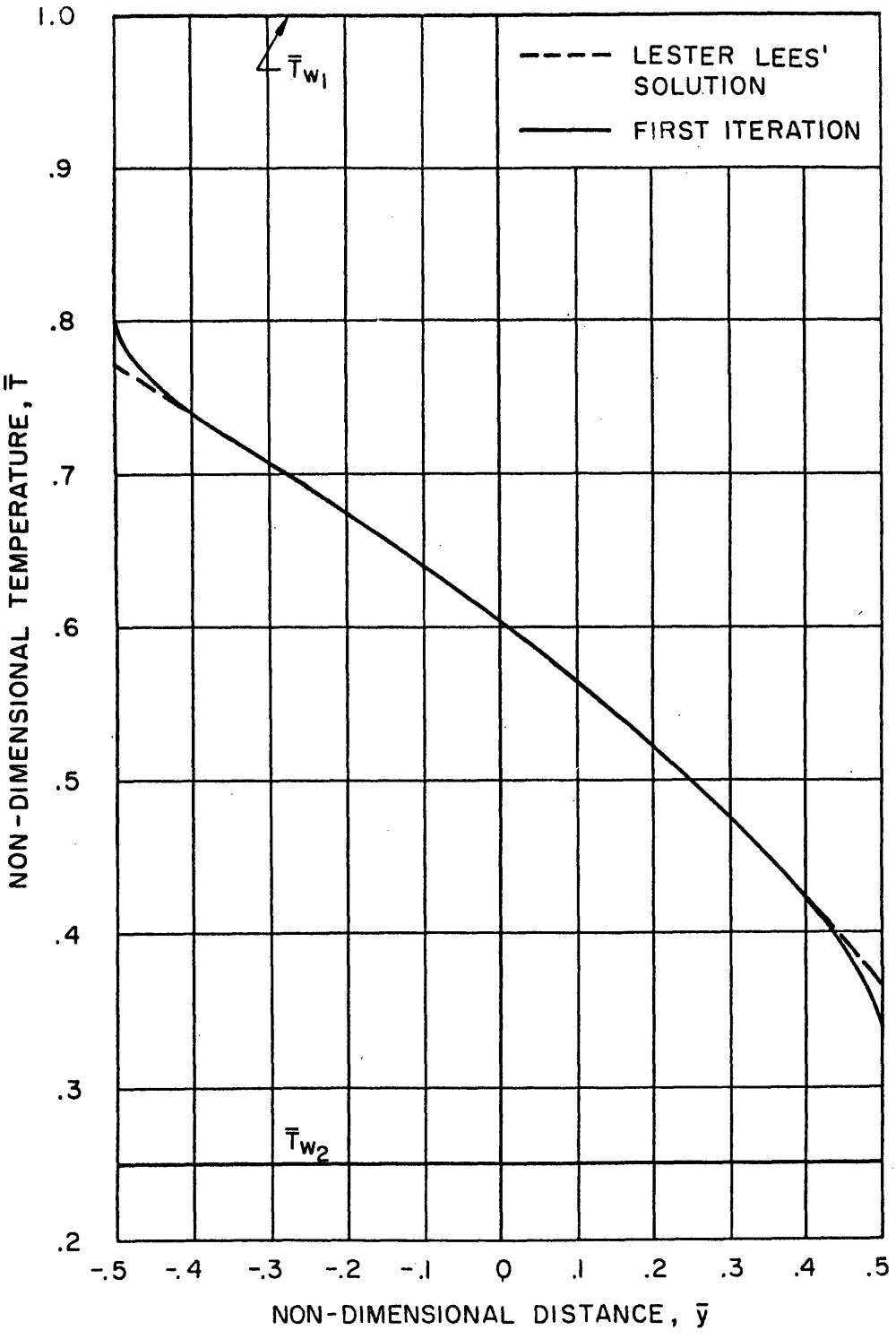


Figure VII-1. Temperature profile between parallel plates $\Delta T/T = 3/4$ and $\lambda_{w_1}/b = 8/15$

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VIII. HEAT TRANSFER EXPERIMENT

Because of the great scarcity of available experimental information in the transition range of Knudsen numbers even very simple experiments that only determine gross effects in this range are of some value. One experiment that, at least in principle, can be performed with very simple apparatus in the measurement of heat flux between two large parallel plates, separated by a gap containing a gas, as a function of the pressure of the gas. Though this will not answer such important questions as what is the temperature jump at the walls or how does the temperature profile change with the pressure, it will give an indication of the variation of the heat flux as a function of Knudsen number and thus serve, at least, as a partial reference for comparison of different theoretical methods.

The experiment as such is, of course, not new as it has been done as long ago as 1898 by Von Smoluchowski[36] and more recently with a slightly modified geometry by Wiedman and Trumpler[37] in connection with determination of accomodation coefficients. The results up to the present, however, are by no means conclusive, and certainly more work remains to be done.

Of course, very similar measurements can be obtained with the different geometry of a wire inside a cylinder. This geometry allows great simplicity in the apparatus as the central wire acts as the heater and the measuring instrument as well as one of the surfaces between which the heat flux is taking place. Unfortunately, there are several disadvantages to this experiment. First of all there are really two Knudsen numbers in this problem based on the radii of the inner and outer cylinder. This complicates the interpretation of the results as the mode of the heat flux may actually be both free molecular near the central wire and continuum near the outer cylinder at a single level of the pressure. Secondly, besides the radiation transfer and fringe effects which will appear in any heat

transfer problem, this geometry includes the possibility of convective effects which unlike the other two is pressure dependent and thus more difficult to isolate. The parallel plate geometry introduces neither of these difficulties. Convection can be entirely eliminated in the central portion between the plates by having the hotter plate on top and thus inducing a very stable density gradient. The fact that there is only one physical dimension of importance, provided the aspect ratio (plate dimension divided by the plate separation) is high, means that only one Knudsen number is necessary to describe the physical situation. This, in turn, aids in the interpretation of results by guaranteeing that for any one Knudsen number the mode of heat flux is the same everywhere between the two plates.

The experimental apparatus consists of two polished and gold plated copper plates each 3/4 inches thick and 10 inches in diameter separated by a gap of .034 inches. The separation is maintained by 3 quartz tubes of 2mm I.D. and 4mm O.D. and about 1/2 long. These are mounted on micrometer screws so as to allow adjustment of plate separation between zero and 1/4 inch. Into the top side of the top copper plate a teflon insulated nichrome wire is wound so as to act as a heating coil. On top of this is a 5/8 inch thick teflon plate with an identical heating coil on the other side and a stainless steel plate on top of that. The teflon plate and second heating coil are included in order to have a means of nulling the thermal gradient across the teflon and thus guaranteeing that all the power going into the lower heating coil is transmitted as heat flux across the gap between the plates. The bottom plate is mounted on 4 copper coils potted together. These coils carry cold methanol and are capable of maintaining the bottom plate at any temperature down to $-90\,^{\circ}F$ with the existing pump and heat exchanger. A sketch of the two plates in assembled position is shown in Fig. VIII-1.

The plates as described above are mounted on a stainless steel plate with electrical feed throughs. A bell jar with an inlet for feeding in helium at the top is placed on top of the stainless steel plate. The vacuum equipment consists of a two inch diffusion pump and a backing

pump, as well as some vacuum tanks, to increase the volume and thus decrease the percentage leakage of air into the system. The instrumentation consists of two thermocouples mounted in each copper plate, one on each side of the teflon plate and one on the top stainless steel plate. The pressure measurements are taken with a McLeod gage, with a tube coming through the bottom plate, for pressures below a millimeter, and with a Haas manometer for pressures above a millimeter. The heat flux is read directly on a wattmeter attached to the line going into the coil on top of the copper plate. A schematic diagram of the apparatus and associated equipment is shown in Fig. VIII-2.

After many months of correcting trivial details and making the vacuum equipment tight, some preliminary data was obtained with helium. It was found that taking the bottom plate to temperatures lower than about -30°F resulted in leaks around the seals in the stainless steel base and thus operations were restricted to temperatures above this level. A series of 14 data points were taken at operating conditions of about 100°F temperature difference and pressures from .02 microns to 100 millimeters. The raw data is shown in Table VIII-1 and the heat flux normalized to a 100°F temperature difference is plotted in Fig. VIII-3 as a function of pressure. The reason only 14 points were obtained and the fact that the temperature difference was not always exactly maintained at the same level is a result of the very long time constant associated with taking of each data point.

It was found that the heat flux for pressures below about a micron is constant and independent of the pressure. This value must, therefore, correspond to the sum of the radiation heat flux and the end losses through the quartz tubes. To get the conduction heat flux it is only necessary to subtract this from the total heat flux at any pressure. The fact that the temperature difference was not always the same means that a different amount should be subtracted at each point to normalize properly. Since no information is available as to the relative ratio of radiation and quartz tube conduction in this asymptotic heat flux, it was normalized linearly with the temperature difference to facillitate calculation. This is reasonable

as the radiation heat flux can be written as proportional to

$$\left(T_{w_1}^2 + T_{w_2}^2\right)\left(T_{w_1} + T_{w_2}\right)\left(T_{w_1} - T_{w_2}\right)$$

and in this experiment the variation in the quantity

$$\left(T_{w_1}^2 + T_{w_2}^2\right)\left(T_{w_1} + T_{w_2}\right)$$

was well below the experimental accuracy of the pressure measurments.

When this asymptotic heat flux is subtracted the remaining conduction heat flux varies linearly with pressure for pressures below 100 microns. This means that in this range the heat flux is definitely free molecular. Since the formula for free molecular heat flux is well known as a function of accomodation coefficient, we can determine an average accomodation coefficient for the two walls by merely plugging our experimentally determined slope of dq/dp into the theoretical formula and determining the average accomodation coefficient. The resulting value of accomodation coefficient $\alpha = .455$ is at the high end of experimental scatter of other investigators[38]. Though this does not give any information as to its correctness, it probably does mean that the equipment used in this experiment was relatively dirty. It also means that for this particular apparatus the accomodation coefficient is probably correct.

Once the accomodation coefficient is available it is possible to compare the experimental results to some theoretical results for the same accomodation coefficient. A plot of heat flux normalized by the continuum value versus Knudsen number is shown in Fig. VIII-4. The theoretical results are those obtained by Lees' moment method with four moment equations. The results shown are for the constant property case as for this value of $\Delta T_w/T_{w_1} \cong .19$ there is no appreciable difference between this

and the Maxwellian molecule case. The experimental results are also shown on this graph. The Knudsen number for the theoretical results is calculated by making sure that in the free molecular limit the experimental points agree with theory. This is necessary as the inaccuracies in the measurement of plate separation, and the lack of clarity as to what properties the Knudsen number should be based on are such as to make comparison between theory and experiment unfeasible unless some reference point is fixed.

As could be expected, the experiments generally follow the basic pattern of this simple theory. The accuracy of the experiments is certainly not sufficient to compare different theoretical results purely on the basis of available data. The experiment does, however, give some information as to the accomodation coefficient and also the general behavior between free molecular and continuum conditions. It verifies, quite conclusively, the predictions of almost any theory that deviation from the asymptotes of free molecular and continuum occurs over a relatively narrow range of only two orders of magnitude of Knudsen number. The level where this transition occurs is, of course, strongly dependent on the accomodation coefficient. This, in turn, means that probably the single most important piece of information necessary for proper prediction of results in the transition regime is the knowledge of the accomodation coefficient, which is certainly imperfect at present. Of course, more basic data that would allow prediction of accomodation coefficients theoretically is even more desirable and hopefully will some day be available.

The data, so far obtained, is certainly neither extensive enough nor accurate enough to draw any far reaching conclusions. It does indicate, however, that the apparatus can be used for measurements of heat flux over a range wide enough to include both limiting conditions on Knudsen number without excessively high radiation and end effects.

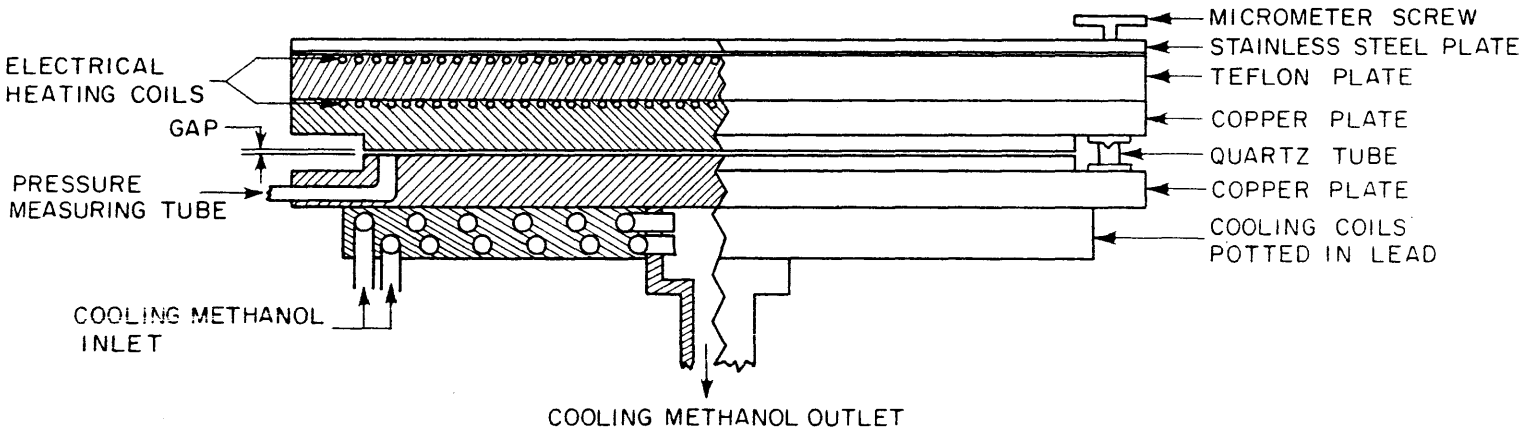

Figure VIII-1. Basic configuration of heat transfer plates

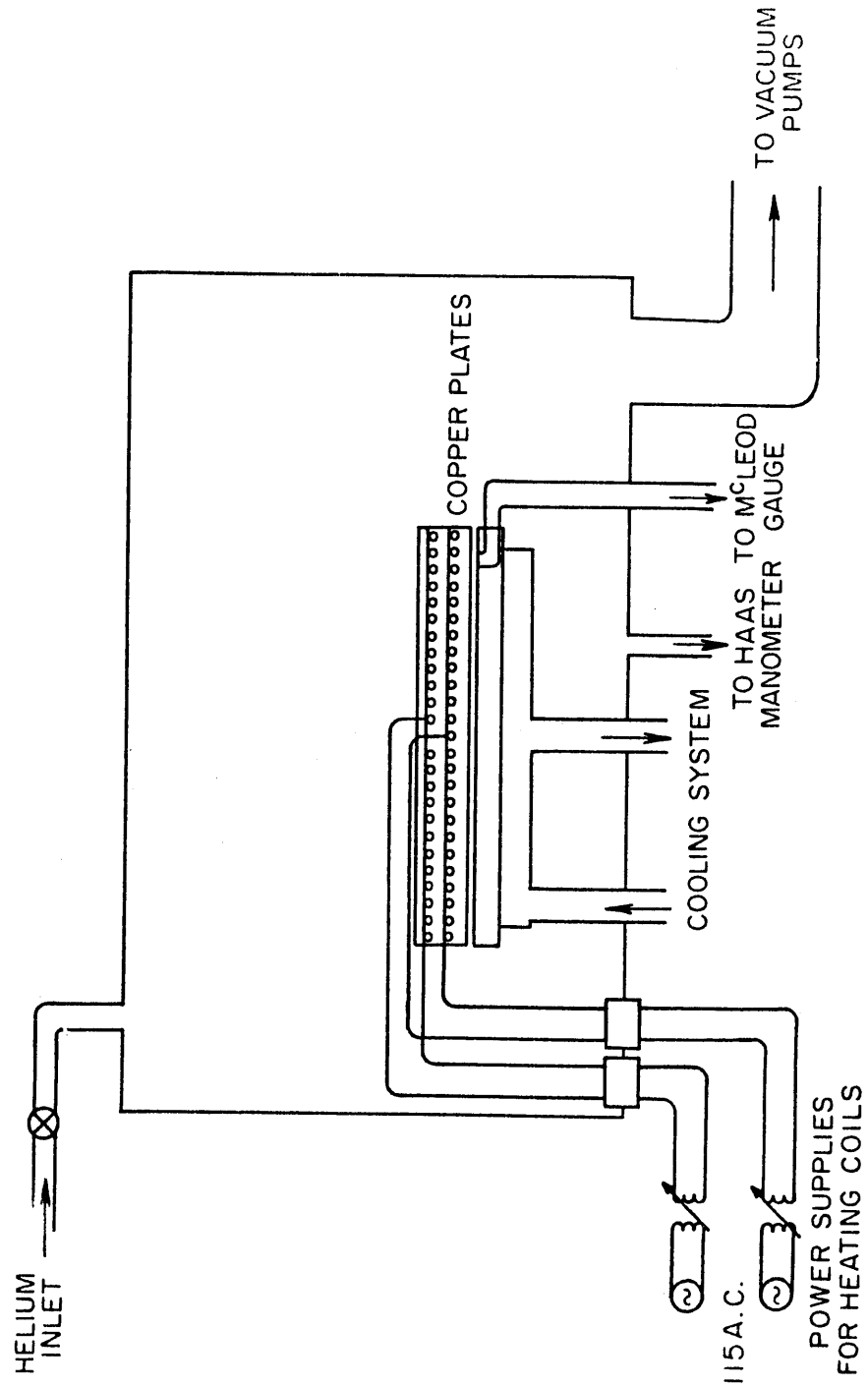

Figure VIII-2. Schematic of experimental system

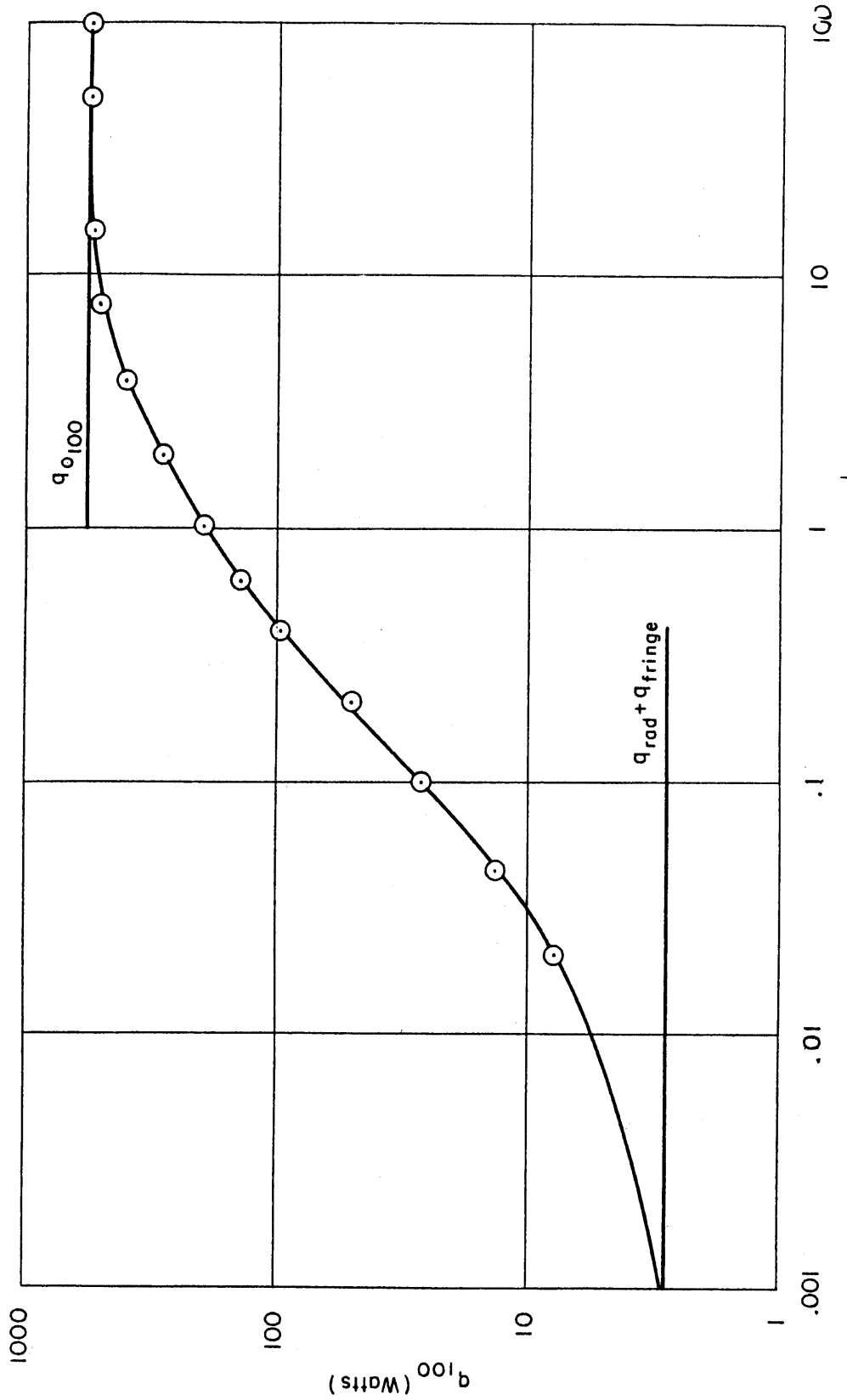

Figure VIII-3. Heat flux versus pressure

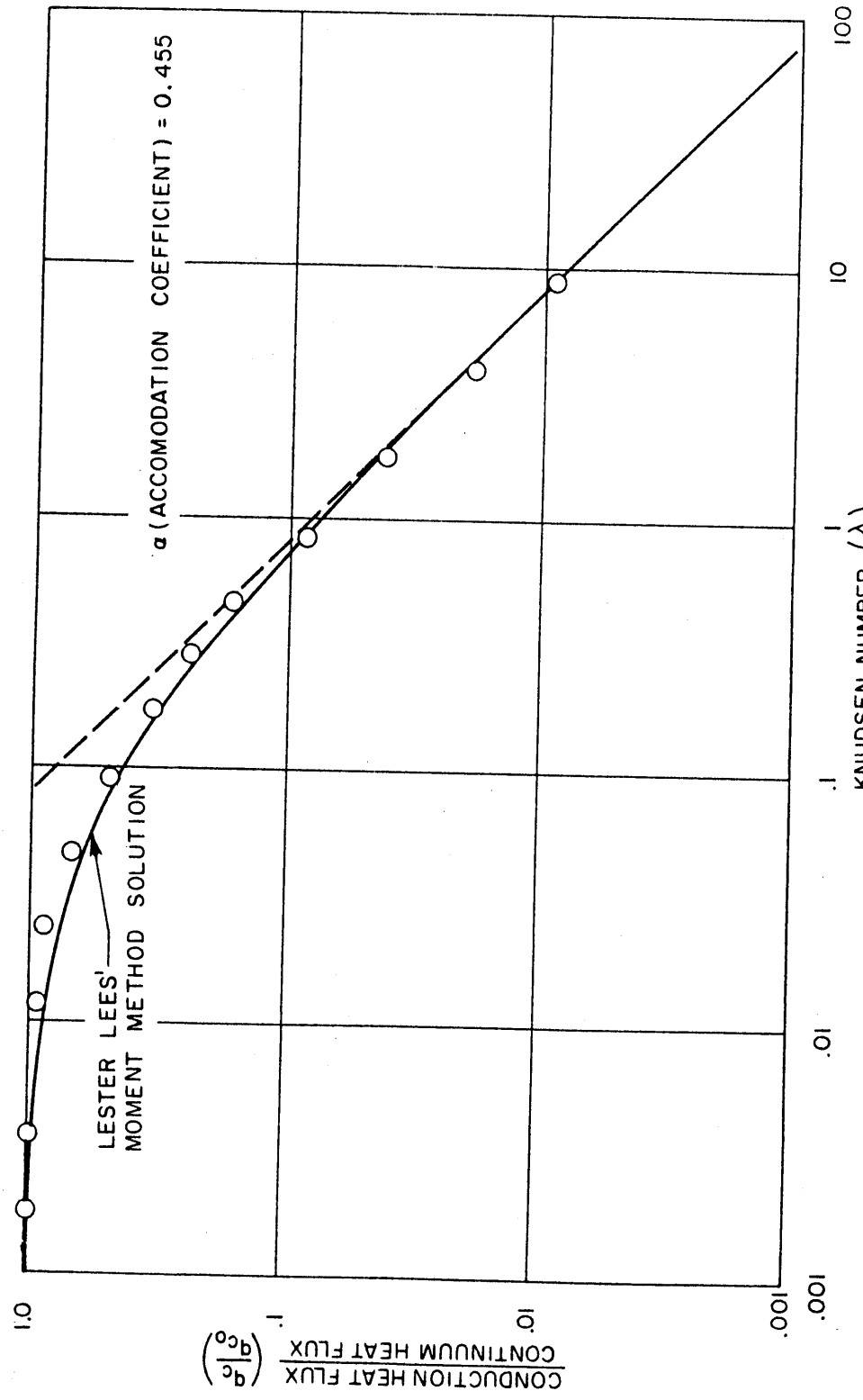

Figure VIII-4. Normalized conduction heat flux versus Knudsen number

TABLE I

Heat Transfer Data

P Millimeters	q_y Watts	T_{w_2} °F	T_{w_1} °F	T_{ss} °F	ΔT_w °F
2×10^{-5}	2.5	-23.0	66.0	65.0	89.0
.02075	7.0	-24.9	63.8	63.2	88.7
.04500	13.0	-23.9	74.0	74.0	97.9
.1000	22.5	-23.6	62.0	62.5	85.6
.2050	50.0	-22.7	77.5	77.0	100.2
.3870	83.5	-24.4	61.1	60.8	85.5
.6150	145.0	-22.6	79.8	80.0	102.4
1.024	177.0	-23.9	65.9	65.5	89.8
1.940	265.0	-23.3	69.8	69.7	93.1
3.780	390.0	-21.0	76.5	76.5	97.5
7.470	464.0	-19.5	72.0	72.0	91.5
14.85	500.0	-21.9	72.2	72.1	94.1
50.10	565.0	-23.7	77.2	77.5	100.9
100.0	520.0	-23.0	70.0	70.5	93.0

p	Pressure in gap in millimeters of mercury
q_y	Total heat flux in watts
T_{w_2}	Lower plate temperature in °F
T_{w_1}	Upper plate temperature in °F
T_{ss}	Top stainless steel cover temperature in °F
ΔT_w	Temperature difference in °F

IX. CONCLUSIONS

The stated objective of this thesis is the analysis and description of physical phenomena in the transition between free molecular and continuum conditions. The approach used has been to transform the Boltzmann equation into a purely integral form and substitute simplified models for the collision term. This description has the advantage of showing the effects of boundaries explicitly and is amenable to numerical iteration procedures.

The method has been applied to three different types of problems. Only relatively preliminary results have been obtained in all three as the excessive complication in the numerical computations coupled with a limited availability of computer time prevented extensive study of any individual problem. Furthermore, it was felt more desirable to carry the method far enough in several problems to allow numerical computation, but to avoid getting involved in the many technical refinements of numerical techniques that are necessary in an accurate numerical solution. These preliminary results, however, are certainly sufficient to indicate that the integral equation formulation of kinetic theory can be of some use in the transition regime, both quantitatively and as an aid to intuition.

The results shown at the end of Section V for the expanding sphere problem cannot be considered final as the accuracy of the numerical techniques has not been sufficiently well analysed. Though numerical refinements could undoubtedly improve the accuracy and make all the macroscopic quantities well behaved, the available results are sufficient to show a reasonable behavior of the density disturbance propagation. This is immediately apparent from a glance at the density profiles shown in Figs. V-1 through V-4. These figures also show that for the moderate pressure ratio chosen and an initial diameter of the order of the outside mean free path the solution lies much closer to the collisionless case

than to the inviscid results. Though there is an apparent tendency for the solution to get closer to the inviscid results, the effects of just simple collisionless "kinematic diffusion" overpower this effect and drive the gas toward uniform properties in a time much shorter than the time required to approach the inviscid profile. Figures V-5 through V-7 indicate the fact that diffusion of the two species far from being a small effect is the dominant one for the conditions chosen. The much higher mean speed of the helium particles directly accounts for the much quicker diffusion of helium into the air than vice versa. This further suggests that extreme care should be used in interpreting results as collisionless motion of two components with different "diffusion" rates represented by their different mean speeds can have results that at first glance appear similar to collision dominated effects. This in interpreting the results shown in Figs. V-1 through V-3 one realizes that only part of the density overshoot is a result of collisions while a portion of it is only the effect of the different mean speeds of the two species. The results of the present calculation are, however, sufficient to indicate the effects of collisions and thus show usefulness of the method for solving problems between the collisionless and collision-dominated regimes.

The results for the normal shock structure shown in Fig. 6-1 are not complete because they were carried out by hand computation. The fact that the $\Gamma_n(k,\mu)$'s needed in the calculations are calculated with doubtful accuracy would make any more extensive calculations of dubious value. The results available are certainly sufficient, however, to show some power of the method in tending toward reasonable answers in two iterations from the very unrealistic zero shock thickness assumption. Further iteration poses no new problems and with electronic computers should be easily obtainable provided an accurate technique for calculating the $\Gamma_n(k,\mu)$'s is devised. Though a better alternative seems to be to iterate from a more reasonable initial trial function and thus relax the convergence requirements of the method.

The dependence of solutions on integrals over the distance normalized by the local mean free path further justifies making general statements

as to the asymmetry of the shock structure as well as suggesting a different definition of shock thickness. Because of the actual asymmetric shape of the shock structure the usual definition based on the maximum slope has a dubious meaning. A more meaningful definition would seem to be based on the integral

$$\int \frac{dx}{\lambda(x)}$$

carried out between limits where the properties are a certain fixed percentage away from the asymptotes.

The results obtained for the heat transfer between parallel plates by a single iteration are sufficient to justify confidence in the method though certainly additional results are needed before any conclusive statements can be made. The deviation from Lees' method occurs principally near the walls where that method is known to be least accurate. The resulting reduction in the actual temperature jump at the wall is physically justifiable because of the boundary layer type influence of the particles coming from the walls, but the inability of ever actually measuring the temperature of the gas at a wall casts doubt as to whether the temperature jump obtained by extrapolation of the temperature gradient to the wall is not the more easily verifiable quantity. Whatever the case, however, we can certainly say that the integral equation solution appears capable of adequately describing the physical situation.

The $I_n(K)$ functions calculated to a relatively good accuracy in Appendix E will certainly arise in other problems and are thus certainly of general usefulness. The fact that the accuracy can be improved to any order desired by just numerical computations from the quoted results, makes this a feasible method for more extensive tabulations. The availability of analytic expressions for the behavior of the functions throughout the range allows making certain statements that have physical significance. The $\exp\left[-3\left(\frac{K}{2}\right)^{2/3}\right]$ behavior for large K indicates that the influence of the wall or any distrubance on a macroscopic property is felt within a distance where $3\left(\frac{K}{2}\right)^{2/3}$ is roughly 4 or 5. The fact that K can be related to

an integral of

$$\int \frac{dx}{\lambda(x)}$$

where $\lambda(x)$ is the local mean free path gives us a sort of "sphere of influence" criterion for kinetic theory phenomena. Furthermore, the ratio between I_2 and I_0 which is proportional to $\left(\frac{x}{2}\right)^{2/3}$ indicates again that energy disturbances have a larger "sphere of influence" than density disturbances.

The experimental results obtained certainly cannot be used as a conclusive proof of validity of any single theory, but in this field conclusive results are not yet available for any problem. The fact that the accomodation coefficient measured with the apparatus lies at the high end of the experimental scatter of other investigators is not too surprising, as no special care was taken to chemically clean or treat the surfaces. The resultant possibility of adsorbed material on the surfaces would certainly make the accomodation coefficient tend to be higher than for clean surfaces. As a matter of fact, the value of .455 obtained from the experiment is close to the value of .500 predicted by the Baule[34] theory of accomodation of helium with adsorbed helium at the wall. Though this theory is not, in general, too accurate it does indicate the fact that walls with adsorbed layers of the same material as is being accomodated tend to increase the accomodation coefficient by as much as an order of magnitude over the "clean" wall result.

The fact that all the experimental points in the transition regime tend to lie above the Lees' moment method solution, suggests that this solution may underestimate the heat flux in this region. The fact that the difference is only of the order of 10 percent, which is also the order of accuracy of the measurements, precludes making a definitive judgement on this or on the relative merits of any other theory.

The theoretical results obtained, at least tentatively, indicate that the integral equation formulation of kinetic theory can have great power in the analysis and solution of problems in the transition range of Knudsen numbers. The experimental results obtained are certainly of some value in defining the behavior of heat transfer between free molecular and

continuum conditions and also give the helium accomodation coefficient obtained by a method different from that usually used. A large amount of additional work, however, remains in both the theoretical and experimental parts before the implications of the work started are fully determined.

Aside from the general improvement in the numerical techniques that should be done to better utilize the integral equation formulation, and the more detail calculation of the $\Gamma_n(K, w)$ functions to obtain their analytic properties, the imperfect conservation of mass, momentum and energy that occurs in all the problems should be investigated. In the initial value problem the accuracy can be increased relatively simply by not only solving the problem through the step by step recursion relation, but upon this initial solution iterating in the actual integral equation until the successive iterations are the same. In the steady state problems the fact that the model uses the previous iterate values of the macroscopic variables and thus does not conserve the collision invariants exactly, could cause problems in the convergence of the results and should be investigated more fully. The fact, however, that in the limit when the successive iterations are the same the conservation laws are satisfied exactly, indicates that probably when the initial trial function is not too incorrect the inaccuracies in the conservation laws should be no more serious than the other numerical computation problems.

The experimental apparatus designed and built for this thesis should certainly be used for additional measurements on helium as well as other gases. Some modifications in the apparatus should also allow increasing the accuracy of the results as well as possibly taking some temperature profile data. The apparatus as such can also be used for more extensive measurements of accomodation coefficients by a technique different from the usual "hot wire in a cylinder" method. The relatively low level of radiation heat flux and the ability to adjust the gap size should allow more accurate measurements than with the standard method. The fact that a whole series of points can be taken in the free molecular region further allows computing the accomodation coefficient by taking the slope of the heat flux with respect to pressure and thus avoiding the inherently inaccurate procedure in subtracting the radiation heat flux from the total.

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APPENDIX A

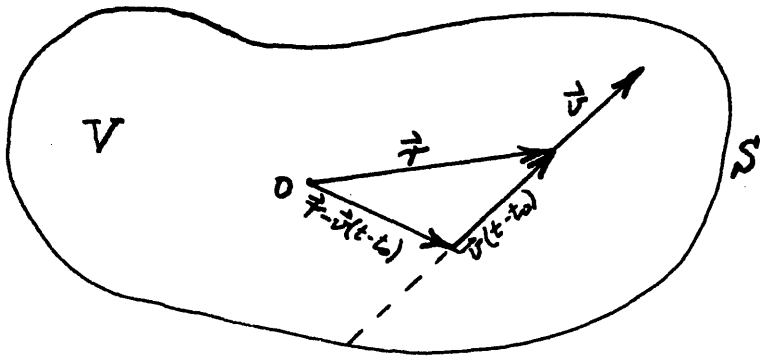
BOUNDARY CONDITIONS IN INTEGRAL EQUATION

The boundary conditions in the integral equation consist of two terms

$$\vec{n}\cdot\vec{v}\int_0^t\iint_S \phi(\vec{r}_{0s},\vec{v},t_0)\,\delta^{(3)}(\vec{r}-\vec{r}_{0s}-\vec{v}(t-t_0))\,dS_0'\,dt_0$$

$$+\iiint_V \phi(\vec{r}_0,\vec{v},0)\,\delta^{(3)}(\vec{r}-\vec{r}_0-\vec{v}t)\,dV_0$$

The solution desired is for the volume V bounded by the surface S. With a little geometry it is easy to show that both terms do not contribute simultaneously. Putting the coordinate system at an arbitrary point O within V we get the following geometry



Now we notice that both boundary terms contain delta functions which contribute only when the argument is zero. Thus, the second term can contribute only if $\vec{r}_0 = \vec{r} - \vec{v}t$ lies within the volume V. When this is true, however, $\vec{r}_0 = \vec{r} - \vec{v}(t-t_0)$ can only lie on a line between $\vec{r}$ and $\vec{r} - \vec{v}t$ for positive values of t_0 less than t. Thus, it cannot be on the boundary and the argument of the delta function for any $\vec{r}_{0s}$ on the boundary and time t_0 between zero and t cannot be zero. The converse can easily be seen to be true, for when the argument of the delta function is zero for a positive value of t_0, than $\vec{r} - \vec{v}t$ always lies outside the volume V. The case of $\vec{r} - \vec{v}t$ lying on the surface creates problems because both delta functions have their arguments zero at the edges of volumes of integration. In this case either term can be taken, as physically there is no difference between them. Mathematically this can be accomplished by considering t to go from 0^+ to t, thus excluding the $t=0$ case from the first term. This was actually assumed in the original derivation anyway as the Green's function contained a term $u(t_0)$ which is non-zero only for $t_0 > 0$.

It is immediately obvious that the second term contribution, when it exists, will be simply

$$\phi(\vec{r} - \vec{v}t, \vec{v}, 0)$$

This is purely a result of integrating the delta function over V.

To first term contribution can be evaluated by considering a coordinate system of which two coordinates lie within the surface S and the third coordinate is $\vec{n} \cdot \vec{v}\, t_0$. This means that this term will only contribute for the case when $t_0 = t_s$ where t_s is determined by the equation.

$$\vec{r}_{0s} = \vec{r} - \vec{v}(t - t_s) \qquad (A-1)$$

where $\vec{r}_{0s}$ is the intersection of the line from $\vec{r}$ in the negative $\vec{v}$ direction with the surface S. The result becomes

$$\phi(\vec{r} - \vec{v}(t-t_s), \vec{v}, t_s)$$

APPENDIX B

COLLISION MODEL

We discuss here a class of collision models that can be represented as

$$\bar{\nu}(\vec{r},t)\left(\mathcal{E}(\vec{r},\vec{v},t)-f(\vec{r},\vec{v},t)\right)$$

where $\bar{\nu}$ is the average collision frequency and $\mathcal{E}(\vec{r},\vec{v},t)$ is determined in such a way as to satisfy the conservation equations, as well as give quantitatively correct transport coefficients in the continuum limit. The $\mathcal{E}$ chosen is, therefore, a local Maxwellian multiplied by a series

$$\mathcal{E} = f_0\left(1+\frac{1}{\nu}\phi_1(\vec{r},\vec{v},t)+\frac{1}{\nu^2}\phi_2(\vec{r},\vec{v},t)+\cdots\cdots\right) \qquad (B-1)$$

where all the ϕ's are such as to give zero contribution to the density, momentum and energy integrals.

$$\iiint \psi_i(\vec{v}) f_0(\vec{r},\vec{v},t)\,\phi_n(\vec{r},\vec{v},t)\,d^3\vec{v} = 0 \qquad (B-2)$$

where

$$\psi_i(\vec{v}) = 1,\ \vec{v},\ v^2 \qquad (B-3)$$

Furthermore, ϕ_1 alone guarantees proper transport properties to first order in the Chapman-Enskog expansion in the limit while ϕ_1 plus ϕ_2 give proper limits to second order, etc.

The method of determining the functions ϕ_n consists of substituting this form into the general integral equation and then integrating by parts, identifying the various terms in the resulting expansion in inverse collision frequency with the respective terms in the Chapman-Enskog expansion, and choosing the ϕ's to give the same answer to that order in $1/\nu$. The general integral equation can be used with the boundary terms omitted as the expansion from continuum is only valid at locations a sufficient distance away from the boundaries to make their effect negligible, i.e., several mean free paths.

The equation, therefore, becomes

$$f(\vec{r},\vec{v},t) = \int_0^t \varepsilon(\vec{r}',\vec{v},t_0)\,\nu(\vec{r}',t_0)\times$$

$$\exp\left[-\left(\Omega(\vec{r},\vec{v},t)-\Omega(\vec{r}',\vec{v},t_0)\right)\right] dt_0 \qquad (B-4)$$

where

$$\Omega(\vec{s},\vec{v},t') = \int_0^{t'} \nu(\vec{s}-\vec{v}(t'-t''),t'')\,dt'' \qquad (B-5)$$

and

$$\vec{r}' = \vec{r} - \vec{v}(t-t_0) \qquad (B-6)$$

The contribution from $t_0 = t$ is left out because the Chapman-Enskog expansion also assumes that time variations are small for times of the order of a mean free time between collisions. This means only solutions for large times compared to the mean free time are desired. This makes the initial value give zero contribution because of the exponential

$$\exp\left[-\left(\Omega(\vec{r},\vec{v},t) - \Omega(\vec{r}-\vec{v}t,\vec{v},0)\right)\right] \Rightarrow 0$$

Before trying to evaluate the integral it is advisable to define the term

$$K(t,t_0;\vec{v}) = \Omega(\vec{r},\vec{v},t) - \Omega(\vec{r}',\vec{v},t_0) \tag{B-7}$$

This term can, in turn, be related to the collision frequency through the definitions of $\Omega(\vec{r},\vec{v},t')$

$$\Omega(\vec{r},\vec{v},t) = \int_0^t \nu(\vec{r}-\vec{v}(t-t_0'),t_0')\,dt_0' \tag{B-8}$$

and

$$\Omega(\vec{r}',\vec{v},t_0) = \int_0^{t_0} \nu(\vec{r}-\vec{v}(t-t_0)-\vec{v}(t_0-t_0'),t_0')\,dt_0' \tag{B-9}$$

This gives the expression for $K(t,t_0;\vec{v})$

$$K(t,t_0;\vec{v}) = \int_{t_0}^{t} \nu(\vec{r}-\vec{v}(t-t_0'),t_0')\,dt_0' \tag{B-10}$$

From this it is immediately apparent that

$$dK(t,t_0;\vec{v}) = -\nu(\vec{r}-\vec{v}(t-t_0),t_0)\,dt_0$$

and the limits $t_0=0$ and $t_0=t$ become $K(t,t_0;\vec{v})=\infty$ and $K(t,t_0;\vec{v})=0$ respectively. This makes the equation for $f(\vec{r},\vec{v},t)$ take the following form

$$f(\vec{r},\vec{v},t) = \int_0^{\infty} \mathcal{E}(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)\times$$

$$\exp\left[-K(t,t_0;\vec{v})\right]\,dK(t,t_0;\vec{v}) \qquad (B-11)$$

This equation can be immediately evaluated by successive integrations by parts.

$$f(\vec{r},\vec{v},t) = \mathcal{E}(\vec{r},\vec{v},t) + \left(\frac{d\mathcal{E}}{dK}\right)_{K=0}$$

$$+ \left(\frac{d}{dK}\,\frac{d\mathcal{E}}{dK}\right)_{K=0} + \left(\frac{d^3\mathcal{E}}{dK^3}\right)_{K=0} + \cdots\cdots \qquad (B-12)$$

This can now be transformed back to the physical variables $\vec{r}, \vec{v}, t$.

$$f(\vec{r},\vec{v},t) = \mathcal{E}(\vec{r},\vec{v},t)$$

$$+\left[\frac{d\mathcal{E}(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)}{dt_0} \Big/ \frac{dK(t,t_0;\vec{v})}{dt_0}\right]_{t_0=t}$$

$$+\left[\left(\frac{d}{dt_0}\left(\frac{d\mathcal{E}(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)}{dt_0} \Big/ \frac{dK(t,t_0;\vec{v})}{dt_0}\right)\right) \Big/ \frac{dK(t,t_0;\vec{v})}{dt_0}\right]_{t_0=t} \qquad (B-13)$$

and with the substitution $\frac{dK}{dt_0}(t,t_0;\vec{v}) = -\nu(\vec{r}-\vec{v}(t-t_0),t_0)$

$$f(\vec{r},\vec{v},t) = \mathcal{E}(\vec{r},\vec{v},t)$$

$$-\frac{1}{\nu(\vec{r},t)}\left(\frac{d\mathcal{E}(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)}{dt_0}\right)_{t_0=t}$$

$$+\frac{1}{\nu(\vec{r},t)}\times\left(\frac{d}{dt_0}\frac{1}{\nu(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)}\frac{d\mathcal{E}(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)}{dt_0}\right)_{t_0=t} \qquad (B-14)$$

The meaning of $\left(\frac{d\mathcal{E}(\vec{r}-\vec{v}(t-t_0),\vec{v},t_0)}{dt_0}\right)_{t_0=t}$ here is actually

$$\frac{\partial\mathcal{E}}{\partial t}(\vec{r},\vec{v},t) + \vec{v}\cdot\vec{\nabla}_r\mathcal{E}(\vec{r},\vec{v},t)$$

as it is the total derivative with respect to t_0 .

It is now possible to determine the ϕ_n functions from this equation up to any order n . This is done in a step by step process. First of all the $\mathcal{E}$ up to the first order in inverse collision frequency is substituted into the equation and only terms up to this order are retained. This gives

$$f(\vec{r},\vec{v},t) = f_0(\vec{r},\vec{v},t)\left[1 + \frac{\phi_1(\vec{r},\vec{v},t)}{\nu(\vec{r},t)}\right.$$

$$\left. - \frac{1}{\nu(\vec{r},t)}\left(\frac{\partial \ln f_0(\vec{r},\vec{v},t)}{\partial t} + \vec{v}\cdot\vec{\nabla}_r \ln f_0(\vec{r},\vec{v},t)\right)\right] \qquad (B\text{-}15)$$

the

$$\frac{\partial \ln f_0(\vec{r},\vec{v},t)}{\partial t} + \vec{v}\cdot\vec{\nabla}_r \ln f_0(\vec{r},\vec{v},t)$$

corresponds to the $\mathcal{D}^{(1)}$ operator in the Chapman-Enskog expansion[1]. Thus, with the help of the conservation equations (to zeroth order in $1/\nu$)

$$\frac{\partial n}{\partial t} + \nabla_r \cdot n\vec{u} = 0 \qquad (B\text{-}16)$$

$$nm\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla}_r)\vec{u}\right) = -\vec{\nabla}_r p \qquad (B\text{-}17)$$

$$\frac{\partial T}{\partial t} + \vec{u}\cdot\vec{\nabla}_r T = -\frac{2}{3kn}\left\{p\nabla\cdot\vec{u} + \nabla\cdot\vec{q}\right\} \qquad (B\text{-}18)$$

where

ρ is the density
$\vec{u}$ is the macroscopic velocity
p is the pressure
T is the temperature
$\vec{q}$ is the heat flux
k is the Boltzmann constant
m is the molecular mass

and the state equation

$$p = nkT$$

this term can be reduced to

$$\mathcal{D}' \ln f_0 = \left\{ \left(\frac{m(\vec{v}-\vec{u})^2}{2kT} - \frac{5}{2} \right) (\vec{v}-\vec{u}) \cdot \vec{\nabla}_r \ln T \right.$$

$$+ \frac{m}{kT} \left[\left[(\vec{v}-\vec{u})(\vec{v}-\vec{u}) - \frac{1}{3}(\vec{v}-\vec{u})^2 \overleftrightarrow{I} \right] : \right.$$

$$\left. \left. \vec{\nabla}_r \vec{u} \right] \right\} \qquad (B\text{-}19)$$

where $\overleftrightarrow{I}$ is the unit tensor and the double dot product designates $A_{ij} B_{ij}$ summed over all i and j in the summation notation.

These terms are identical to those appearing on the right hand side of the Chapman-Enskog equation for f_1, the first order solution. Thus, they satisfy all the same requirements, i.e., their velocity moments over mass, momentum and energy are zero and the $\vec{\nabla}_r \ln T$ contributes only to the heat flux while $\vec{\nabla}_r \vec{u}$ contributes only to the pressure tensor. The evaluation of these quantities using our expression

for $f(\vec{r},\vec{v},t)$ with $\phi_1(\vec{r},\vec{v},t)$ assumed zero give the resulting transport coefficients with a correct parametric dependence but the wrong numerical factor. From this it is obvious that $\phi_1(\vec{r},\vec{v},t)$ should be chosen in such a way that the correct numerical coefficients are obtained for the thermal conductivity and viscosity independently. This condition is satisfied when $\phi_1(\vec{r},\vec{v},t)$ takes the form

$$\phi_1(\vec{r},\vec{v},t) = -\left\{ c_1\left(\frac{m(\vec{v}-\vec{u})^2}{2kT} - \frac{5}{2}\right)(\vec{v}-\vec{u})\cdot\nabla_r \ln T \right.$$

$$\left. + \frac{c_2 m}{kT}\left[\left\{(\vec{v}-\vec{u})(\vec{v}-\vec{u}) - \frac{1}{3}(\vec{v}-\vec{u})^2\vec{\vec{I}}\right\} : \nabla_r \vec{u}\right]\right\} \qquad (B\text{-}20)$$

where c_1 and c_2 are determined by insisting that the transport coefficients are correct when evaluated by using the distribution function evaluated to first order in ν .

This determines the coefficients c_1 and c_2 as

$$c_1 = \frac{7}{8} + \frac{15}{8}\alpha \qquad (B\text{-}21)$$

and

$$c_2 = \frac{1}{4} + \frac{5}{4}\varepsilon \qquad (B\text{-}22)$$

where α varies from zero to .026 and ε varies from zero to .016 depending on the collision model.

The complete scattering function in the collision model takes the form

$$\mathcal{E}(\vec{r},\vec{v},t) = \frac{n(\vec{r},t)}{[2\pi RT(\vec{r},t)]^{3/2}} \exp\left[-\frac{(\vec{v}-\vec{u}(\vec{r},t))^2}{2RT(\vec{r},t)}\right] \times$$

$$\left\{1 - \left(\frac{7+15\alpha}{8\nu(\vec{r},t)}\right)\left(\frac{(\vec{v}-\vec{u}(\vec{r},t))^2}{2RT(\vec{r},t)} - \frac{5}{2}\right) \times\right.$$

$$(\vec{v}-\vec{u}(\vec{r},t)) \cdot \nabla_{\vec{r}} \ln T(\vec{r},t) - \left(\frac{1+5\varepsilon}{4\nu(\vec{r},t)}\right) \times$$

$$\left.\left[(\vec{v}-\vec{u}(\vec{r},t))(\vec{v}-\vec{u}(\vec{r},t) - \frac{1}{3}(\vec{v}-\vec{u}(\vec{r},t))^2 \overleftrightarrow{1}\right] : \nabla_{\vec{r}}\vec{u}(\vec{r},t)\right\} \qquad (B-23)$$

where the constants α and ε and the variable $\nu(\vec{r},t)$ are determined for the collision repulsion law used. For Maxwellian (inverse fifth power repulsion) molecules α and ε are both zero and $\nu(\vec{r},t)$ is proportional to the density alone, while for hard sphere molecules α and ε take on their maximum values of .026 and .016 respectively while the collision frequency $\nu(\vec{r},t)$ is proportional to the density times the square root of the temperature.

The next order function can be obtained by straight forward extension of the method. Terms of order $1/\nu^2$ are retained in the relation between $f(\vec{r},\vec{v},t)$ and $\mathcal{E}(\vec{r},\vec{v},t)$. The conservation equations can now be used up to first order in $1/\nu$ which can be considered the solution for f containing terms up to ϕ_1. Of course, a whole series of terms will arise just like in the Chapman-Enskog expansion but ϕ_2 can, in principle, be evaluated by insisting that the coefficients in front of the

terms in the heat flux vector and pressure tensor are correct. The extension is not, however, carried out because there is doubt as to the theoretical validity of the Chapman-Enskog expansion beyond the first term, as well as complete lack of experimental means of determining any of the coefficients in the higher order terms.

APPENDIX C

EVALUATION OF $\Gamma_n(k, u)$ FUNCTIONS IN SOLVING KINETIC THEORY PROBLEMS

In solving kinetic theory problems by integral iteration a particular form of an integral repeatedly appears. It is, therefore, advisable to define and tabulate this function as well as discover some of its properties. The general function is defined as

$$\Gamma_n(K,u)=\frac{1}{\sqrt{\pi}}\int_0^\infty \exp\left[-(v-u)^2-\frac{K}{v}\right]v^n\,dv \qquad (C\text{-}1)$$

where

$$0\le K<\infty \quad \text{and} \quad -\infty<u<\infty$$

For the special case $K=0$ this function can be evaluated in terms of error functions, also in the special case of $u=0$ it can be expanded for small K and integrated by the method of steepest descents for large K. This is done in Appendix E in connection with the heat transfer problem.

Before proceeding to determine the functions $\Gamma_n(K,u)$, however, we can save ourselves a lot of work by determining analytically certain properties and relations that these functions satisfy. By integrating by

parts in velocity space and juggling terms around we can get the relation

$$\Gamma_n(K,u) = -\frac{e^{xp}}{\sqrt{\pi}}\left[-(v-u)^2-\frac{K}{v}\right]\frac{v^{n-1}}{2}\Bigg|_0^\infty$$

$$+\frac{n-1}{2\sqrt{\pi}}\int_0^\infty exp\left[-(v-u)^2-\frac{K}{v}\right]v^{n-2}dv$$

$$+\frac{K}{2}\frac{1}{\sqrt{\pi}}\int_0^\infty exp\left[-(v-u)^2-\frac{K}{v}\right]v^{n-3}dv$$

$$+\frac{K}{\sqrt{\pi}}\int_0^\infty exp\left[-(v-u)^2-\frac{K}{v}\right]v^{n-1}dv \qquad (C-2)$$

which gives us the third order recursion relation for different orders n.

$$\Gamma_n(K,u) = u\,\Gamma_{n-1}(K,u)$$

$$+\frac{n-1}{2}\Gamma_{n-2}(K,u)+\frac{K}{2}\Gamma_{n-3}(K,u) \qquad (C-3)$$

This means that once the Γ_n functions are known for three orders n, all others can be determined from an algebraic expression.

If we try to integrate the Γ_n function over K we quickly realize that certain integral relations exist between Γ_n's for different n's.

Integration over K just puts another ν into the numerator raising the value of n by one. The following integral relations are, therefore, true

$$\Gamma_n(K,u) = \int_K^\infty \Gamma_{n-1}(K',u)\,dK' \qquad (C\text{-}4)$$

and

$$\Gamma_n(K,u) = \Gamma_n(0,u) - \int_0^K \Gamma_{n-1}(K',u)\,dK' \qquad (C\text{-}5)$$

Also the inverse differential relation is satisfied

$$\frac{d\Gamma_n}{dK}(K,u) = -\Gamma_{n-1}(K,u)$$

The actual evaluation of the functions can be carried out by expansions in powers of K for small K and by method of steepest decent for large K. The expansion, however, still requires evaluating error function type integrals and is very messy. Also, the region of convergence is very much dependent on the value of u. The method of steepest descents requires solving a cubic equation and is again only valid for certain regions of u for anything but extremely large K. To get a first order estimate of the behavior of the functions it was, therefore, decided to evaluate them by numerical integrations.

The numerical integrations were done on a small slow computer and were thus carried out by only a sixth order Legendre-Gauss quadrature. The results were analysed for accuracy by comparing the

functions obtained with the exact values for the case of $K=0$. In all cases the error was less than a percent. An additional check was carried out for the case $u=0$ by comparing the functions with those obtained by the more exact method of Appendix E. Here again the error was under one percent.

The numerical integration scheme consisted of replacing the infinite integration by the integral

$$\Gamma_n(K,u) = \frac{1}{\pi}\int_0^{u+3.2} \exp\left[-(v-u)^2 - \frac{K}{v}\right] v^n \, dv$$

$$= \frac{1}{\pi}\int_0^{u+3.2} F(v)\, dv \qquad (C-6)$$

This allowed evaluation by the Legendre-Gauss quadrature

$$\Gamma_n(K,u) = \frac{u+3.2}{2\pi} \sum_{j=1}^{3} W_j \left[F\left(\frac{u+3.2}{2}(1+X_j)\right)\right.$$

$$\left. + F\left(\frac{u+3.2}{2}(1-X_j)\right)\right] \qquad (C-7)$$

where X_j and W_j are the ordinates and weighting factors for the sixth order Legendre-Gauss quadrature[33].

The results for $\Gamma_0(K,u)$, $\Gamma_1(K,u)$ and $\Gamma_2(K,u)$ are shown in tables 1 through 10. The normalized forms $\Gamma_0(K,u)/\Gamma_0(0,u)$, $\Gamma_1(K,u)/\Gamma_1(0,u)$ and $\Gamma_2(K,u)/\Gamma_2(0,u)$ are plotted in figures C-1 through C-3, so as to allow determination of the effect of u on the behavior of the functions. The fact that the decay to zero is much quicker for negative u's than for positive ones is quite evident, and is certainly consistent with the physical fact that effects are felt farther downstream than upstream.

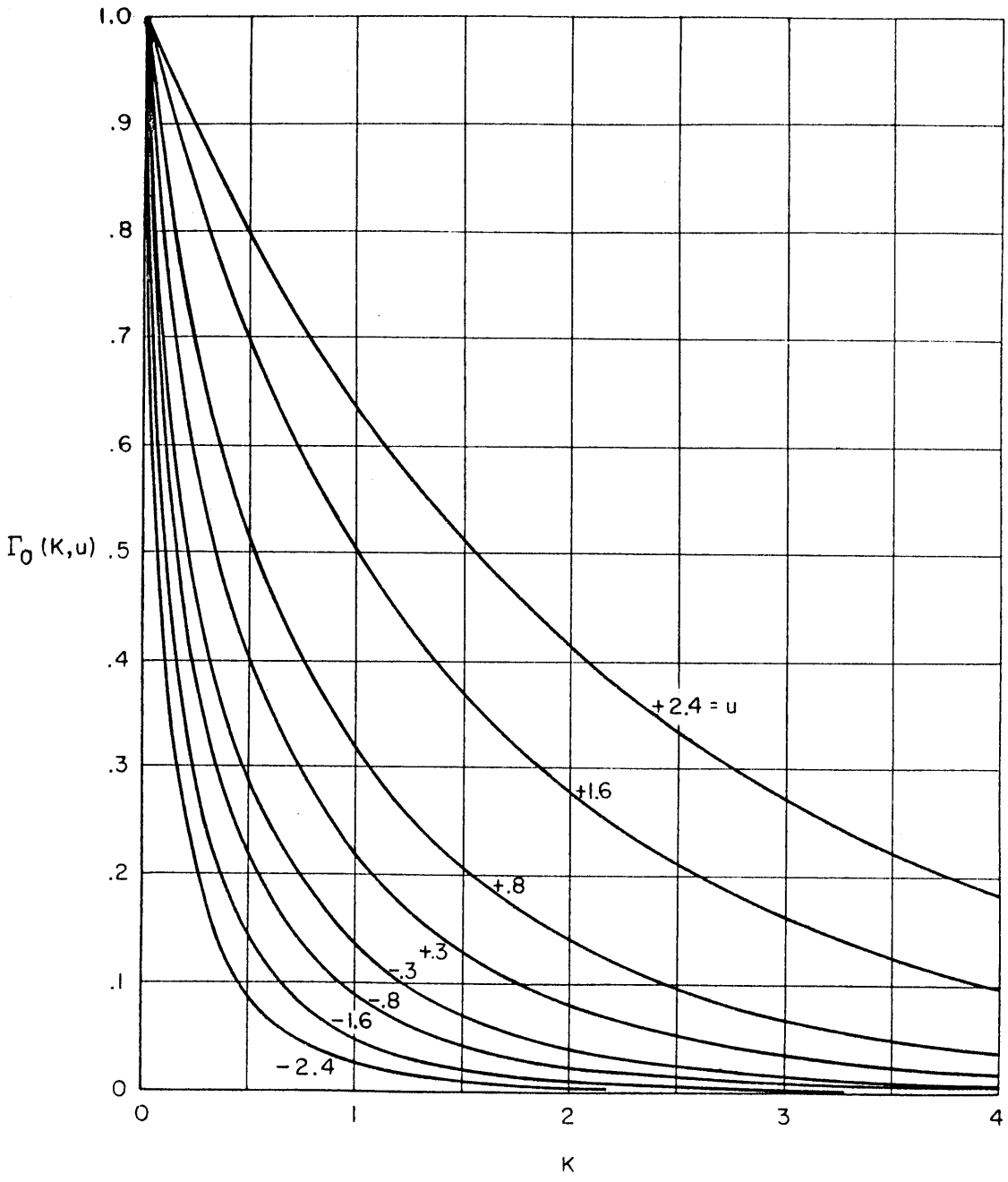


Figure C-1. Γ_0 function

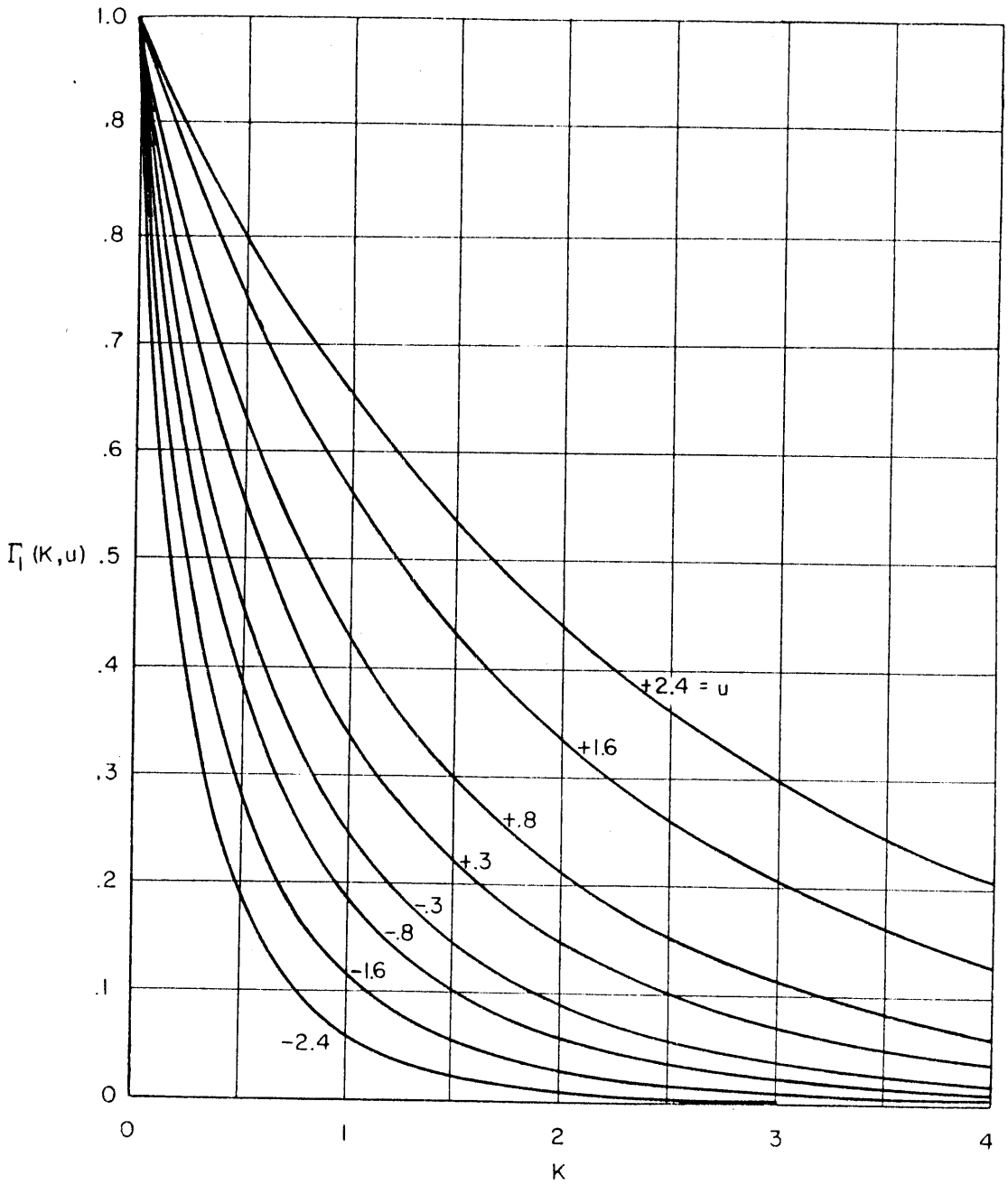


Figure C-2. Γ_1 function

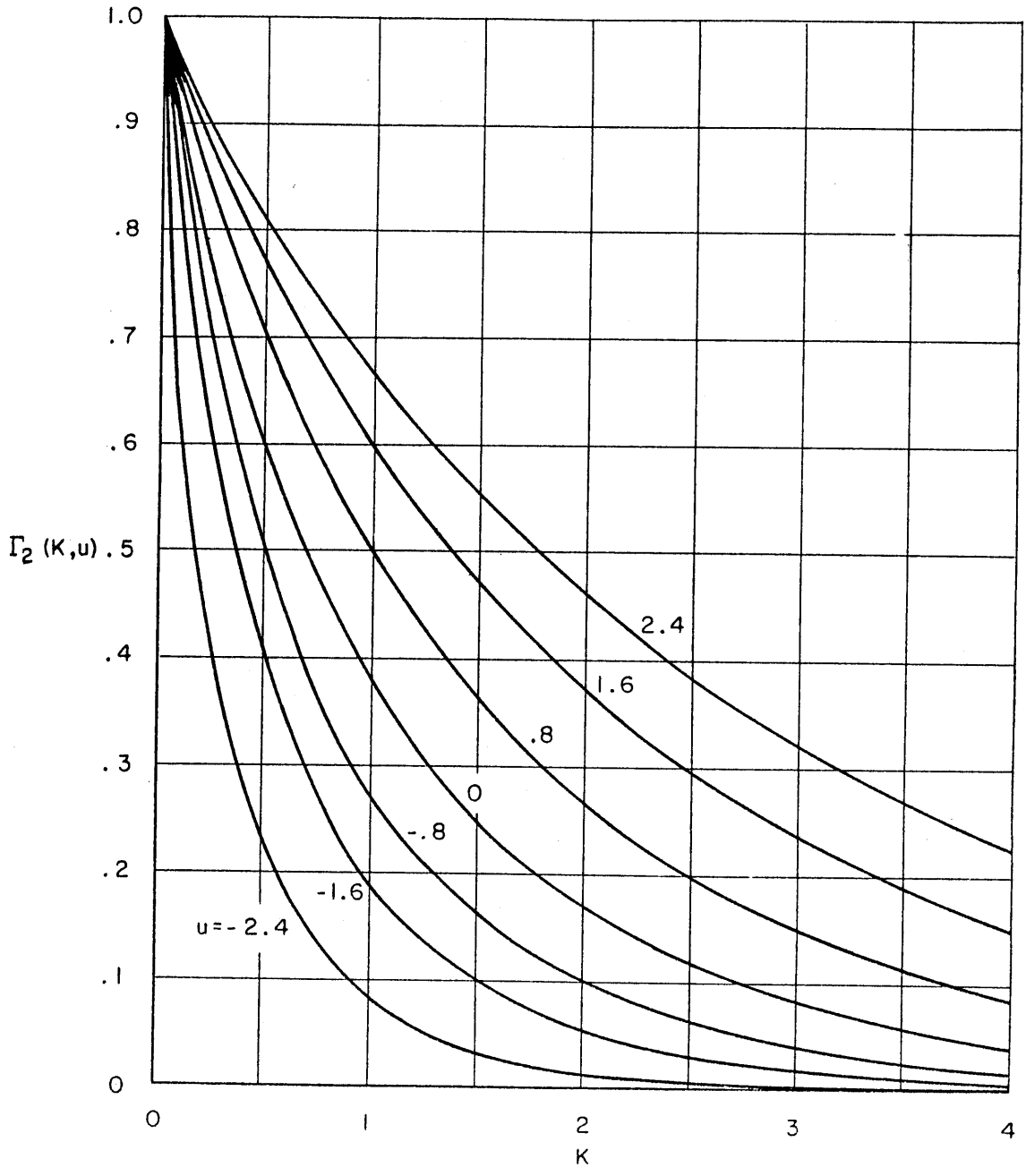


Figure C-3. Γ_2 function

Table C-I

$\Gamma_0(\kappa, u)$ Function

κ=	u= -2.4	-2.3	-2.2	-2.1	-2.0	-1.9	-1.8	-1.7	-1.6	-1.5	-1.4
0	$.345^{-3}$	$.570^{-3}$	$.930^{-3}$	$.149^{-2}$	$.234^{-2}$	$.361^{-2}$	$.546^{-2}$	$.811^{-2}$	$.118^{-1}$	$.169^{-1}$	$.239^{-1}$
.1	$.143^{-3}$	$.243^{-3}$	$.404^{-3}$	$.658^{-3}$	$.105^{-2}$	$.166^{-2}$	$.256^{-2}$	$.389^{-2}$	$.581^{-2}$	$.852^{-2}$	$.123^{-1}$
.2	$.890^{-4}$	$.154^{-3}$	$.259^{-3}$	$.426^{-3}$	$.687^{-3}$	$.109^{-2}$	$.169^{-2}$	$.258^{-2}$	$.387^{-2}$	$.570^{-2}$	$.827^{-2}$
.3	$.592^{-4}$	$.104^{-3}$	$.178^{-3}$	$.297^{-2}$	$.485^{-3}$	$.777^{-3}$	$.122^{-2}$	$.188^{-2}$	$.284^{-2}$	$.422^{-2}$	$.616^{-2}$
.4	$.412^{-4}$	$.731^{-4}$	$.126^{-3}$	$.214^{-3}$	$.353^{-3}$	$.571^{-3}$	$.907^{-3}$	$.141^{-2}$	$.216^{-2}$	$.323^{-2}$	$.476^{-2}$
.5	$.298^{-4}$	$.532^{-4}$	$.927^{-4}$	$.158^{-3}$	$.263^{-3}$	$.429^{-3}$	$.687^{-3}$	$.108^{-2}$	$.166^{-2}$	$.251^{-2}$	$.373^{-2}$
.6	$.221^{-4}$	$.398^{-4}$	$.696^{-4}$	$.119^{-3}$	$.200^{-3}$	$.328^{-3}$	$.529^{-3}$	$.836^{-3}$	$.130^{-2}$	$.198^{-2}$	$.296^{-2}$
.7	$.167^{-4}$	$.303^{-4}$	$.534^{-4}$	$.919^{-4}$	$.155^{-3}$	$.255^{-3}$	$.413^{-3}$	$.658^{-3}$	$.103^{-2}$	$.157^{-2}$	$.237^{-2}$
.8	$.129^{-4}$	$.235^{-4}$	$.417^{-4}$	$.720^{-4}$	$.122^{-3}$	$.202^{-3}$	$.328^{-3}$	$.524^{-3}$	$.822^{-3}$	$.127^{-2}$	$.192^{-2}$
.9	$.100^{-4}$	$.185^{-4}$	$.330^{-4}$	$.573^{-4}$	$.971^{-4}$	$.161^{-3}$	$.263^{-3}$	$.422^{-3}$	$.665^{-3}$	$.103^{-2}$	$.156^{-2}$
1.0	$.788^{-5}$	$.147^{-4}$	$.264^{-4}$	$.460^{-4}$	$.784^{-4}$	$.131^{-3}$	$.214^{-3}$	$.344^{-3}$	$.543^{-3}$	$.844^{-3}$	$.129^{-2}$
1.2	$.499^{-5}$	$.951^{-5}$	$.173^{-4}$	$.306^{-4}$	$.525^{-4}$	$.882^{-4}$	$.145^{-3}$	$.235^{-3}$	$.373^{-3}$	$.583^{-3}$	$.895^{-3}$
1.4	$.324^{-5}$	$.630^{-5}$	$.117^{-4}$	$.209^{-4}$	$.362^{-4}$	$.613^{-4}$	$.102^{-3}$	$.165^{-3}$	$.264^{-3}$	$.414^{-3}$	$.640^{-3}$
1.6	$.214^{-5}$	$.425^{-5}$	$.801^{-5}$	$.145^{-4}$	$.255^{-4}$	$.435^{-4}$	$.727^{-4}$	$.119^{-3}$	$.191^{-3}$	$.301^{-3}$	$.468^{-3}$
1.8	$.144^{-5}$	$.292^{-5}$	$.558^{-5}$	$.102^{-4}$	$.182^{-4}$	$.314^{-4}$	$.528^{-4}$	$.871^{-4}$	$.141^{-3}$	$.223^{-3}$	$.348^{-3}$
2.0	$.983^{-6}$	$.203^{-5}$	$.394^{-5}$	$.732^{-5}$	$.131^{-4}$	$.229^{-4}$	$.389^{-4}$	$.646^{-4}$	$.105^{-3}$	$.168^{-3}$	$.263^{-3}$
2.2	$.679^{-6}$	$.143^{-5}$	$.282^{-5}$	$.530^{-5}$	$.961^{-5}$	$.169^{-4}$	$.289^{-4}$	$.485^{-4}$	$.794^{-4}$	$.128^{-3}$	$.201^{-3}$
2.4	$.474^{-6}$	$.102^{-5}$	$.204^{-5}$	$.387^{-5}$	$.709^{-5}$	$.126^{-4}$	$.217^{-4}$	$.367^{-4}$	$.606^{-4}$	$.909^{-4}$	$.155^{-3}$
2.6	$.333^{-6}$	$.731^{-6}$	$.148^{-5}$	$.286^{-5}$	$.528^{-5}$	$.945^{-5}$	$.164^{-4}$	$.280^{-4}$	$.465^{-4}$	$.757^{-4}$	$.121^{-3}$
2.8	$.236^{-6}$	$.529^{-6}$	$.109^{-5}$	$.213^{-5}$	$.397^{-5}$	$.715^{-5}$	$.125^{-4}$	$.215^{-4}$	$.359^{-4}$	$.588^{-4}$	$.946^{-4}$
3.0	$.169^{-6}$	$.386^{-6}$	$.809^{-6}$	$.159^{-5}$	$.300^{-5}$	$.545^{-5}$	$.963^{-4}$	$.166^{-4}$	$.279^{-4}$	$.460^{-4}$	$.744^{-4}$
3.2	$.121^{-6}$	$.283^{-6}$	$.603^{-6}$	$.120^{-5}$	$.229^{-5}$	$.419^{-5}$	$.744^{-4}$	$.129^{-4}$	$.218^{-4}$	$.362^{-4}$	$.589^{-4}$
3.4	$.872^{-7}$	$.209^{-6}$	$.452^{-6}$	$.914^{-6}$	$.175^{-5}$	$.323^{-5}$	$.578^{-4}$	$.101^{-4}$	$.172^{-4}$	$.286^{-4}$	$.468^{-4}$
3.6	$.631^{-7}$	$.155^{-6}$	$.341^{-6}$	$.698^{-6}$	$.135^{-5}$	$.251^{-5}$	$.452^{-4}$	$.792^{-4}$	$.136^{-4}$	$.227^{-4}$	$.373^{-4}$
3.8	$.459^{-7}$	$.115^{-6}$	$.258^{-6}$	$.535^{-6}$	$.105^{-5}$	$.196^{-5}$	$.355^{-4}$	$.626^{-4}$	$.108^{-4}$	$.181^{-4}$	$.299^{-4}$
4.0	$.335^{-7}$	$.860^{-7}$	$.196^{-6}$	$.413^{-6}$	$.815^{-6}$	$.154^{-5}$	$.281^{-4}$	$.497^{-5}$	$.859^{-5}$	$.145^{-4}$	$.241^{-4}$

Table C-1 (continued)

x=	u= -1.3	-1.2	-1.1	-1.0	-.9	-.8	-.7	-.6	-.5	-.4	-.3
0	$.330^{-1}$	$.448^{-1}$	$.599^{-1}$	$.787^{-1}$	.102	.129	.161	.198	.240	.286	.336
.1	$.174^{-1}$	$.242^{-1}$	$.332^{-1}$	$.446^{-1}$	$.591^{-1}$	$.768^{-1}$	$.983^{-1}$	.124	.153	.187	.225
.2	$.118^{-1}$	$.165^{-1}$	$.228^{-1}$	$.310^{-1}$	$.414^{-1}$	$.554^{-1}$	$.704^{-1}$	$.898^{-1}$	.113	.139	.169
.3	$.885^{-2}$	$.125^{-1}$	$.173^{-1}$	$.237^{-1}$	$.318^{-1}$	$.421^{-1}$	$.549^{-1}$	$.704^{-1}$	$.890^{-1}$	.111	.136
.4	$.688^{-2}$	$.979^{-2}$	$.137^{-1}$	$.188^{-1}$	$.255^{-1}$	$.339^{-1}$	$.445^{-1}$	$.574^{-1}$	$.730^{-1}$	$.915^{-1}$	.113
.5	$.544^{-2}$	$.780^{-2}$	$.110^{-1}$	$.152^{-1}$	$.207^{-1}$	$.278^{-1}$	$.367^{-1}$	$.476^{-1}$	$.609^{-1}$	$.768^{-1}$	$.954^{-1}$
.6	$.435^{-2}$	$.628^{-1}$	$.891^{-2}$	$.124^{-1}$	$.170^{-1}$	$.230^{-1}$	$.305^{-1}$	$.399^{-1}$	$.513^{-1}$	$.651^{-1}$	$.813^{-1}$
.7	$.351^{-2}$	$.510^{-2}$	$.728^{-2}$	$.102^{-1}$	$.141^{-1}$	$.191^{-1}$	$.256^{-1}$	$.336^{-1}$	$.435^{-1}$	$.555^{-1}$	$.697^{-1}$
.8	$.285^{-2}$	$.417^{-2}$	$.599^{-2}$	$.846^{-2}$	$.117^{-1}$	$.160^{-1}$	$.215^{-1}$	$.285^{-1}$	$.371^{-1}$	$.475^{-1}$	$.600^{-1}$
.9	$.234^{-2}$	$.344^{-2}$	$.496^{-2}$	$.704^{-2}$	$.983^{-2}$	$.135^{-1}$	$.182^{-1}$	$.242^{-1}$	$.317^{-1}$	$.409^{-1}$	$.519^{-1}$
1.0	$.193^{-2}$	$.285^{-2}$	$.414^{-2}$	$.590^{-2}$	$.828^{-2}$	$.114^{-1}$	$.155^{-1}$	$.207^{-1}$	$.272^{-1}$	$.353^{-1}$	$.450^{-1}$
1.2	$.135^{-2}$	$.201^{-2}$	$.294^{-2}$	$.422^{-2}$	$.597^{-2}$	$.831^{-2}$	$.114^{-1}$	$.153^{-1}$	$.204^{-1}$	$.266^{-1}$	$.342^{-1}$
1.4	$.971^{-3}$	$.145^{-2}$	$.213^{-2}$	$.304^{-2}$	$.440^{-2}$	$.617^{-2}$	$.851^{-2}$	$.116^{-1}$	$.155^{-1}$	$.204^{-1}$	$.264^{-1}$
1.6	$.714^{-3}$	$.107^{-2}$	$.158^{-2}$	$.231^{-2}$	$.330^{-2}$	$.466^{-2}$	$.647^{-2}$	$.884^{-2}$	$.119^{-1}$	$.158^{-1}$	$.206^{-1}$
1.8	$.534^{-3}$	$.806^{-3}$	$.120^{-2}$	$.175^{-2}$	$.252^{-2}$	$.357^{-2}$	$.499^{-2}$	$.686^{-2}$	$.929^{-2}$	$.124^{-1}$	$.163^{-1}$
2.0	$.406^{-3}$	$.616^{-3}$	$.919^{-2}$	$.135^{-2}$	$.195^{-2}$	$.278^{-2}$	$.390^{-2}$	$.539^{-2}$	$.735^{-2}$	$.986^{-2}$	$.130^{-1}$
2.2	$.312^{-3}$	$.476^{-3}$	$.713^{-3}$	$.105^{-2}$	$.153^{-2}$	$.219^{-2}$	$.309^{-2}$	$.429^{-2}$	$.587^{-2}$	$.792^{-2}$	$.105^{-1}$
2.4	$.242^{-3}$	$.371^{-3}$	$.559^{-3}$	$.830^{-3}$	$.121^{-2}$	$.174^{-2}$	$.247^{-2}$	$.344^{-2}$	$.473^{-2}$	$.642^{-2}$	$.857^{-2}$
2.6	$.189^{-3}$	$.292^{-3}$	$.442^{-3}$	$.659^{-3}$	$.967^{-3}$	$.140^{-2}$	$.199^{-2}$	$.278^{-2}$	$.385^{-2}$	$.524^{-2}$	$.703^{-2}$
2.8	$.149^{-3}$	$.231^{-3}$	$.352^{-3}$	$.527^{-3}$	$.777^{-3}$	$.113^{-2}$	$.161^{-2}$	$.227^{-2}$	$.315^{-2}$	$.431^{-2}$	$.581^{-2}$
3.0	$.118^{-3}$	$.184^{-3}$	$.282^{-3}$	$.424^{-3}$	$.628^{-3}$	$.916^{-3}$	$.131^{-2}$	$.186^{-2}$	$.259^{-2}$	$.356^{-2}$	$.482^{-2}$
3.2	$.939^{-4}$	$.147^{-3}$	$.226^{-3}$	$.343^{-3}$	$.510^{-3}$	$.747^{-3}$	$.108^{-2}$	$.153^{-2}$	$.214^{-2}$	$.296^{-2}$	$.402^{-2}$
3.4	$.750^{-4}$	$.118^{-3}$	$.183^{-3}$	$.278^{-3}$	$.416^{-3}$	$.612^{-3}$	$.887^{-3}$	$.127^{-2}$	$.178^{-2}$	$.247^{-2}$	$.337^{-2}$
3.6	$.602^{-4}$	$.953^{-4}$	$.148^{-3}$	$.227^{-3}$	$.341^{-3}$	$.504^{-3}$	$.733^{-3}$	$.105^{-2}$	$.148^{-2}$	$.206^{-2}$	$.283^{-2}$
3.8	$.485^{-4}$	$.772^{-4}$	$.121^{-3}$	$.185^{-3}$	$.280^{-3}$	$.416^{-3}$	$.608^{-3}$	$.875^{-3}$	$.124^{-2}$	$.173^{-2}$	$.239^{-2}$
4.0	$.392^{-4}$	$.627^{-4}$	$.984^{-4}$	$.152^{-3}$	$.230^{-3}$	$.344^{-3}$	$.505^{-3}$	$.730^{-3}$	$.104^{-2}$	$.146^{-2}$	$.202^{-2}$

Table C-1 (continued)

M=	M= .3	.4	.5	.6	.7	.8	.9	1.0	1.1	1.2	1.3
0	.664	.714	.760	.802	.839	.871	.898	.921	.940	.955	.967
.1	.508	.557	.604	.649	.691	.730	.764	.795	.823	.847	.867
.2	.414	.459	.505	.549	.591	.631	.669	.703	.734	.762	.787
.3	.350	.392	.434	.476	.517	.557	.594	.630	.662	.692	.719
.4	.302	.341	.380	.420	.459	.497	.534	.569	.602	.633	.661
.5	.264	.299	.336	.373	.410	.447	.483	.517	.550	.581	.610
.6	.232	.265	.299	.334	.369	.404	.438	.471	.504	.534	.564
.7	.205	.236	.267	.299	.332	.366	.399	.431	.463	.493	.522
.8	.182	.210	.239	.270	.301	.332	.364	.395	.426	.455	.484
.9	.162	.188	.215	.243	.273	.302	.333	.363	.392	.421	.450
1.0	.145	.168	.194	.220	.248	.276	.305	.333	.362	.391	.418
1.2	.116	.136	.158	.181	.205	.231	.257	.283	.310	.337	.363
1.4	$.941^{-1}$	.111	.130	.150	.172	.194	.218	.242	.266	.291	.316
1.6	$.769^{-1}$	$.915^{-1}$	.108	.125	.144	.164	.186	.208	.230	.253	.276
1.8	$.633^{-1}$	$.758^{-1}$	$.898^{-1}$	.105	.122	.140	.159	.179	.200	.221	.242
2.0	$.525^{-1}$	$.633^{-1}$	$.755^{-1}$	$.889^{-1}$	.104	.120	.137	.155	.174	.193	.213
2.2	$.438^{-1}$	$.532^{-1}$	$.637^{-1}$	$.756^{-1}$	$.886^{-1}$	.103	.118	.134	.152	.170	.188
2.4	$.368^{-1}$	$.449^{-1}$	$.541^{-1}$	$.645^{-1}$	$.761^{-1}$	$.887^{-1}$	.103	.117	.133	.150	.166
2.6	$.311^{-1}$	$.381^{-1}$	$.462^{-1}$	$.553^{-1}$	$.656^{-1}$	$.768^{-1}$	$.892^{-1}$	.102	.117	.132	.147
2.8	$.264^{-1}$	$.325^{-1}$	$.396^{-1}$	$.476^{-1}$	$.567^{-1}$	$.668^{-1}$	$.778^{-1}$	$.899^{-1}$	.103	.116	.131
3.0	$.225^{-1}$	$.278^{-1}$	$.340^{-1}$	$.411^{-1}$	$.492^{-1}$	$.582^{-1}$	$.681^{-1}$	$.790^{-1}$	$.907^{-1}$	.103	.116
3.2	$.193^{-1}$	$.239^{-1}$	$.294^{-1}$	$.357^{-1}$	$.428^{-1}$	$.508^{-1}$	$.597^{-1}$	$.696^{-1}$	$.802^{-1}$	$.916^{-1}$	.104
3.4	$.165^{-1}$	$.206^{-1}$	$.254^{-1}$	$.310^{-1}$	$.373^{-1}$	$.445^{-1}$	$.525^{-1}$	$.614^{-1}$	$.710^{-1}$	$.814^{-1}$	$.925^{-1}$
3.6	$.142^{-1}$	$.178^{-1}$	$.221^{-1}$	$.270^{-1}$	$.326^{-1}$	$.391^{-1}$	$.463^{-1}$	$.543^{-1}$	$.630^{-1}$	$.725^{-1}$	$.826^{-1}$
3.8	$.123^{-1}$	$.154^{-1}$	$.192^{-1}$	$.236^{-1}$	$.286^{-1}$	$.343^{-1}$	$.408^{-1}$	$.481^{-1}$	$.560^{-1}$	$.646^{-1}$	$.739^{-1}$
4.0	$.106^{-1}$	$.134^{-1}$	$.167^{-1}$	$.206^{-1}$	$.251^{-1}$	$.303^{-1}$	$.361^{-1}$	$.426^{-1}$	$.498^{-1}$	$.577^{-1}$	$.662^{-1}$

Table C-1 (concluded)

κ=	u= 1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4
0	.976	.983	.988	.992	.995	.996	.998	.999	.999	.999	$.100^{1}$
.1	.885	.900	.913	.923	.932	.939	.944	.949	.952	.953	.953
.2	.809	.828	.845	.860	.872	.883	.891	.898	.904	.907	.910
.3	.744	.766	.786	.803	.818	.831	.842	.851	.859	.864	.869
.4	.687	.711	.732	.751	.769	.784	.796	.808	.817	.824	.830
.5	.636	.661	.684	.705	.723	.739	.754	.766	.777	.785	.793
.6	.591	.616	.640	.661	.681	.699	.714	.728	.740	.750	.758
.7	.549	.575	.599	.621	.642	.660	.677	.692	.704	.715	.725
.8	.512	.537	.562	.584	.605	.625	.642	.657	.671	.683	.693
.9	.477	.503	.527	.550	.571	.591	.609	.625	.639	.652	.663
1.0	.445	.471	.495	.518	.540	.560	.578	.595	.610	.623	.634
1.2	.389	.414	.438	.461	.482	.503	.521	.539	.554	.569	.581
1.4	.341	.365	.388	.410	.432	.452	.471	.489	.505	.520	.533
1.6	.299	.322	.345	.366	.387	.407	.426	.444	.460	.475	.489
1.8	.264	.285	.307	.328	.348	.368	.386	.404	.420	.435	.449
2.0	.233	.254	.274	.294	.313	.332	.350	.367	.383	.399	.413
2.2	.207	.226	.245	.264	.282	.301	.318	.334	.350	.365	.379
2.4	.184	.201	.219	.237	.255	.272	.289	.305	.321	.335	.349
2.6	.163	.180	.197	.214	.230	.247	.263	.278	.293	.308	.321
2.8	.146	.161	.177	.193	.208	.224	.240	.254	.269	.283	.296
3.0	.130	.144	.159	.174	.186	.203	.218	.232	.246	.260	.273
3.2	.116	.130	.143	.157	.171	.185	.199	.212	.226	.239	.251
3.4	.104	.116	.129	.142	.155	.168	.181	.194	.207	.220	.232
3.6	$.934^{-1}$	.105	.116	.128	.142	.153	.166	.178	.190	.202	.214
3.8	$.838^{-1}$	$.942^{-1}$	.105	.116	.128	.139	.151	.163	.175	.186	.198
4.0	$.753^{-1}$	$.849^{-1}$	$.950^{-1}$	.105	.116	.127	.138	.149	.161	.172	.182

Table C-II

Γ_1 (κ, u) Function

κ=	u= -2.4	-2.3	-2.2	-2.1	-2.0	-1.9	-1.8	-1.7	-1.6	-1.5	-1.4
0	$.620^{-4}$	$.110^{-3}$	$.184^{-3}$	$.300^{-3}$	$.490^{-3}$	$.780^{-3}$	$.123^{-2}$	$.190^{-2}$	$.289^{-2}$	$.431^{-2}$	$.635^{-2}$
.1	$.388^{-4}$	$.689^{-4}$	$.119^{-3}$	$.201^{-3}$	$.333^{-3}$	$.540^{-3}$	$.859^{-3}$	$.134^{-2}$	$.207^{-2}$	$.312^{-2}$	$.464^{-2}$
.2	$.275^{-4}$	$.496^{-4}$	$.868^{-4}$	$.148^{-3}$	$.248^{-3}$	$.406^{-3}$	$.652^{-3}$	$.103^{-2}$	$.150^{-2}$	$.243^{-2}$	$.364^{-2}$
.3	$.202^{-4}$	$.369^{-4}$	$.653^{-4}$	$.113^{-3}$	$.190^{-3}$	$.314^{-3}$	$,509^{-3}$	$.809^{-3}$	$.126^{-2}$	$.194^{-2}$	$.293^{-2}$
.4	$.153^{-4}$	$.282^{-4}$	$.503^{-4}$	$.875^{-4}$	$.149^{-3}$	$.247^{-3}$	$.403^{-3}$	$.646^{-3}$	$.102^{-2}$	$.157^{-2}$	$.238^{-2}$
.5	$.117^{-4}$	$.219^{-4}$	$.395^{-4}$	$.691^{-4}$	$.118^{-3}$	$.198^{-3}$	$.324^{-3}$	$.522^{-3}$	$.826^{-3}$	$.128^{-2}$	$.196^{-2}$
.6	$.917^{-5}$	$.173^{-4}$	$.314^{-4}$	$.554^{-4}$	$.952^{-4}$	$.160^{-3}$	$.264^{-3}$	$.427^{-3}$	$.679^{-3}$	$.106^{-2}$	$.163^{-2}$
.7	$.725^{-5}$	$.138^{-4}$	$.253^{-4}$	$.449^{-4}$	$.776^{-4}$	$.131^{-3}$	$.217^{-3}$	$.353^{-3}$	$.564^{-3}$	$.884^{-3}$	$.136^{-2}$
.8	$.578^{-5}$	$.111^{-4}$	$.205^{-4}$	$.368^{-4}$	$.639^{-4}$	$.108^{-3}$	$.180^{-3}$	$.294^{-3}$	$.472^{-3}$	$.743^{-3}$	$.115^{-2}$
.9	$.464^{-5}$	$.905^{-5}$	$.169^{-4}$	$.303^{-4}$	$.530^{-4}$	$.904^{-4}$	$.151^{-3}$	$.247^{-3}$	$.398^{-3}$	$.628^{-3}$	$,976^{-3}$
1.0	$.375^{-5}$	$.740^{-5}$	$.139^{-4}$	$.252^{-4}$	$.443^{-4}$	$.759^{-4}$	$.127^{-3}$	$.209^{-3}$	$.338^{-3}$	$.535^{-3}$	$.834^{-3}$
1.2	$.249^{-5}$	$.502^{-5}$	$.961^{-5}$	$.176^{-4}$	$.314^{-4}$	$.543^{-4}$	$.918^{-4}$	$.152^{-3}$	$.247^{-3}$	$.395^{-3}$	$.619^{-3}$
1.4	$.168^{-5}$	$.347^{-5}$	$.675^{-5}$	$.126^{-4}$	$.226^{-4}$	$.395^{-4}$	$.675^{-4}$	$.113^{-3}$	$.184^{-3}$	$.296^{-3}$	$.467^{-3}$
1.6	$.115^{-5}$	$.243^{-5}$	$.481^{-5}$	$.908^{-5}$	$.165^{-4}$	$.291^{-4}$	$.502^{-4}$	$.845^{-4}$	$.139^{-3}$	$.225^{-3}$	$.358^{-3}$
1.8	$.798^{-6}$	$.172^{-5}$	$.346^{-5}$	$.663^{-5}$	$.122^{-4}$	$.218^{-4}$	$.378^{-4}$	$.641^{-4}$	$.106^{-3}$	$.173^{-3}$	$.277^{-3}$
2.0	$.559^{-6}$	$.123^{-5}$	$.252^{-5}$	$.489^{-5}$	$.910^{-5}$	$.164^{-4}$	$.287^{-4}$	$.490^{-4}$	$.819^{-4}$	$.134^{-3}$	$.216^{-3}$
2.2	$.395^{-6}$	$.890^{-6}$	$.185^{-5}$	$.364^{-5}$	$.685^{-5}$	$.124^{-4}$	$.220^{-4}$	$.378^{-4}$	$.636^{-4}$	$.105^{-3}$	$.170^{-3}$
2.4	$.281^{-6}$	$.648^{-6}$	$.137^{-5}$	$.273^{-5}$	$.519^{-5}$	$.951^{-5}$	$.169^{-4}$	$.293^{-4}$	$.497^{-4}$	$.825^{-4}$	$.134^{-3}$
2.6	$.201^{-6}$	$.475^{-6}$	$.102^{-5}$	$.206^{-5}$	$.396^{-5}$	$.733^{-5}$	$.131^{-4}$	$.229^{-4}$	$.391^{-4}$	$.652^{-4}$	$.107^{-3}$
2.8	$.145^{-6}$	$.350^{-6}$	$.767^{-6}$	$.157^{-5}$	$.305^{-5}$	$.568^{-5}$	$.103^{-4}$	$.180^{-4}$	$.309^{-4}$	$.519^{-4}$	$.854^{-4}$
3.0	$.105^{-6}$	$.259^{-6}$	$.579^{-6}$	$.120^{-5}$	$.235^{-5}$	$.443^{-5}$	$.805^{-5}$	$.142^{-4}$	$.246^{-4}$	$.415^{-4}$	$.686^{-4}$
3.2	$.761^{-7}$	$.193^{-6}$	$.438^{-6}$	$.922^{-6}$	$.183^{-5}$	$.347^{-5}$	$.635^{-5}$	$.113^{-4}$	$.196^{-4}$	$.333^{-4}$	$.554^{-4}$
3.4	$.555^{-7}$	$.144^{-6}$	$.334^{-6}$	$.711^{-6}$	$.143^{-5}$	$.273^{-5}$	$.504^{-5}$	$.901^{-5}$	$.157^{-4}$	$.268^{-4}$	$.448^{-4}$
3.6	$.406^{-7}$	$.108^{-6}$	$.255^{-6}$	$.551^{-6}$	$.112^{-5}$	$.216^{-5}$	$.401^{-5}$	$.722^{-5}$	$.127^{-4}$	$.217^{-4}$	$.365^{-4}$
3.8	$.298^{-7}$	$.812^{-7}$	$.195^{-6}$	$.429^{-6}$	$.880^{-6}$	$.171^{-5}$	$.321^{-5}$	$.581^{-5}$	$.103^{-4}$	$.177^{-4}$	$.298^{-4}$
4.0	$.219^{-7}$	$.613^{-7}$	$.150^{-6}$	$.335^{-6}$	$.695^{-6}$	$.137^{-5}$	$.258^{-5}$	$.470^{-5}$	$.832^{-5}$	$.144^{-4}$	$.244^{-4}$

Table C-II (continued)

κ=	u= -1.3	-1.2	-1.1	-1.0	-.9	-.8	-.7	-.6	-.5	-.4	-.2
0	$.916^{-2}$	$.130^{-1}$	$.182^{-1}$	$.251^{-1}$	$.341^{-1}$	$.456^{-1}$	$.600^{-1}$	$.780^{-1}$	$.998^{-1}$	.126	.157
.1	$.678^{-2}$	$.975^{-2}$	$.138^{-1}$	$.192^{-1}$	$.264^{-1}$	$.357^{-1}$	$.475^{-1}$	$.624^{-1}$	$.807^{-1}$	.103	.130
.2	$.536^{-2}$	$.777^{-2}$	$.111^{-1}$	$.155^{-1}$	$.215^{-1}$	$.293^{-1}$	$.393^{-1}$	$.519^{-1}$	$.676^{-1}$	$.870^{-1}$	.110
.3	$.434^{-2}$	$.633^{-2}$	$.909^{-2}$	$.128^{-1}$	$.179^{-1}$	$.245^{-1}$	$.331^{-1}$	$.440^{-1}$	$.577^{-1}$	$.746^{-1}$	$.951^{-1}$
.4	$.356^{-2}$	$.523^{-2}$	$.755^{-2}$	$.107^{-1}$	$.150^{-1}$	$.207^{-1}$	$.281^{-1}$	$.376^{-1}$	$.496^{-1}$	$.645^{-1}$	$.827^{-1}$
.5	$.295^{-2}$	$.435^{-2}$	$.632^{-2}$	$.904^{-2}$	$.127^{-1}$	$.176^{-1}$	$.241^{-1}$	$.324^{-1}$	$.429^{-1}$	$.561^{-1}$	$.723^{-1}$
.6	$.246^{-2}$	$.365^{-2}$	$.533^{-2}$	$.766^{-2}$	$.108^{-1}$	$.151^{-1}$	$.207^{-1}$	$.280^{-1}$	$.373^{-1}$	$.490^{-1}$	$.635^{-1}$
.7	$.207^{-2}$	$.309^{-2}$	$.453^{-2}$	$.654^{-2}$	$.929^{-2}$	$.130^{-1}$	$.179^{-1}$	$.244^{-1}$	$.326^{-1}$	$.430^{-1}$	$.560^{-1}$
.8	$.175^{-2}$	$.262^{-2}$	$.386^{-2}$	$.561^{-2}$	$.800^{-2}$	$.113^{-1}$	$.156^{-1}$	$.213^{-1}$	$.286^{-1}$	$.379^{-1}$	$.495^{-1}$
.9	$.149^{-2}$	$.224^{-2}$	$.332^{-2}$	$.483^{-2}$	$.693^{-2}$	$.978^{-2}$	$.136^{-1}$	$.186^{-1}$	$.251^{-1}$	$.335^{-1}$	$.439^{-1}$
1.0	$.128^{-2}$	$.193^{-2}$	$.287^{-2}$	$.419^{-2}$	$.603^{-2}$	$.854^{-2}$	$.119^{-1}$	$.163^{-1}$	$.222^{-1}$	$.297^{-1}$	$.391^{-1}$
1.2	$.955^{-3}$	$.145^{-2}$	$.217^{-2}$	$.319^{-2}$	$.462^{-2}$	$.658^{-2}$	$.925^{-2}$	$.128^{-1}$	$.175^{-1}$	$.235^{-1}$	$.312^{-1}$
1.4	$.725^{-3}$	$.111^{-2}$	$.166^{-2}$	$.246^{-2}$	$.359^{-2}$	$.515^{-2}$	$.728^{-2}$	$.101^{-1}$	$.139^{-1}$	$.189^{-1}$	$.252^{-1}$
1.6	$.558^{-3}$	$.857^{-3}$	$.130^{-2}$	$.193^{-2}$	$.282^{-2}$	$.408^{-2}$	$.580^{-2}$	$.812^{-2}$	$.112^{-1}$	$.153^{-1}$	$.205^{-1}$
1.8	$.434^{-3}$	$.671^{-3}$	$.102^{-2}$	$.152^{-2}$	$.225^{-2}$	$.326^{-2}$	$.466^{-2}$	$.656^{-2}$	$.911^{-2}$	$.125^{-1}$	$.169^{-1}$
2.0	$.341^{-3}$	$.530^{-3}$	$.809^{-3}$	$.122^{-2}$	$.180^{-2}$	$.263^{-2}$	$.377^{-2}$	$.534^{-2}$	$.746^{-2}$	$.103^{-1}$	$.140^{-1}$
2.2	$.270^{-3}$	$.421^{-3}$	$.647^{-3}$	$.978^{-3}$	$.146^{-2}$	$.213^{-2}$	$.308^{-2}$	$.438^{-2}$	$.615^{-2}$	$.850^{-2}$	$.116^{-1}$
2.4	$.215^{-3}$	$.337^{-3}$	$.521^{-3}$	$.791^{-3}$	$.118^{-2}$	$.174^{-2}$	$.253^{-2}$	$.361^{-2}$	$.509^{-2}$	$.707^{-2}$	$.970^{-2}$
2.6	$.172^{-3}$	$.271^{-3}$	$.421^{-3}$	$.643^{-3}$	$.966^{-3}$	$.143^{-2}$	$.208^{-2}$	$.299^{-2}$	$.424^{-2}$	$.591^{-2}$	$.814^{-2}$
2.8	$.138^{-3}$	$.219^{-3}$	$.342^{-3}$	$.525^{-3}$	$.792^{-3}$	$.118^{-2}$	$.173^{-2}$	$.249^{-2}$	$.354^{-2}$	$.496^{-2}$	$.686^{-2}$
3.0	$.111^{-3}$	$.178^{-3}$	$.279^{-3}$	$.430^{-3}$	$.652^{-3}$	$.975^{-3}$	$.143^{-2}$	$.208^{-2}$	$.297^{-2}$	$.418^{-2}$	$.580^{-2}$
3.2	$.904^{-4}$	$.145^{-3}$	$.228^{-3}$	$.354^{-3}$	$.539^{-3}$	$.809^{-3}$	$.120^{-2}$	$.174^{-2}$	$.250^{-2}$	$.353^{-2}$	$.492^{-2}$
3.4	$.735^{-4}$	$.118^{-3}$	$.187^{-3}$	$.292^{-3}$	$.447^{-3}$	$.673^{-3}$	$.999^{-3}$	$.146^{-2}$	$.210^{-2}$	$.299^{-2}$	$.418^{-2}$
3.6	$.601^{-4}$	$.972^{-4}$	$.154^{-3}$	$.242^{-3}$	$.371^{-3}$	$.562^{-3}$	$.838^{-3}$	$.123^{-2}$	$.178^{-2}$	$.254^{-2}$	$.356^{-2}$
3.8	$.493^{-4}$	$.800^{-4}$	$.128^{-3}$	$.200^{-3}$	$.310^{-3}$	$.471^{-3}$	$.704^{-3}$	$.104^{-2}$	$.151^{-2}$	$.216^{-2}$	$.304^{-2}$
4.0	$.405^{-4}$	$.661^{-4}$	$.106^{-3}$	$.167^{-3}$	$.259^{-3}$	$.395^{-3}$	$.593^{-3}$	$.878^{-3}$	$.128^{-2}$	$.184^{-2}$	$.260^{-2}$

Table C-II (continued)

κ=	u=.3	.4	.5	.6	.7	.8	.9	1.0	1.1	1.2	1.3
0	.457	.526	.600	.678	.760	.846	.934	$.103^{1}$	$.112^{1}$	$.121^{1}$	$.132^{1}$
.1	.399	.463	.532	.606	.684	.767	.853	.942	$.103^{1}$	$.113^{1}$	$.122^{1}$
.2	.353	.413	.477	.546	.621	.699	.781	.867	.956	$.105^{1}$	$.114^{1}$
.3	.315	.370	.430	.495	.565	.640	.718	.800	.886	.974	$.106^{1}$
.4	.283	.334	.390	.451	.517	.587	.662	.740	.823	.908	.996
.5	.255	.302	.354	.411	.473	.540	.611	.686	.765	.847	.932
.6	.230	.274	.322	.376	.434	.497	.565	.637	.713	.792	.874
.7	.208	.249	.294	.344	.399	.459	.523	.592	.664	.740	.820
.8	.189	.226	.269	.316	.368	.424	.485	.551	.620	.693	.769
.9	.171	.206	.246	.290	.339	.392	.450	.513	.579	.649	.723
1.0	.156	.187	.226	.267	.313	.364	.419	.478	.541	.609	.679
1.2	.130	.158	.191	.227	.268	.313	.363	.416	.474	.536	.601
1.4	.109	.134	.162	.194	.230	.270	.315	.364	.417	.473	.533
1.6	$.922^{-1}$	.114	.138	.167	.199	.235	.275	.319	.367	.419	.474
1.8	$.782^{-1}$	$.968^{-1}$	.118	.144	.172	.205	.241	.281	.324	.372	.422
2.0	$.667^{-1}$	$.830^{-1}$	.102	.124	.150	.179	.211	.247	.287	.330	.377
2.2	$.571^{-1}$	$.714^{-1}$	$.882^{-1}$	.108	.131	.157	.186	.218	.255	.294	.337
2.4	$.490^{-1}$	$.616^{-1}$	$.765^{-1}$	$.939^{-1}$	.114	.137	.164	.193	.226	.262	.302
2.6	$.423^{-1}$	$.533^{-1}$	$.665^{-1}$	$.820^{-1}$	.100	.121	.145	.171	.201	.234	.270
2.8	$.365^{-1}$	$.462^{-1}$	$.579^{-1}$	$.717^{-1}$	$.879^{-1}$	.107	.128	.152	.179	.209	.243
3.0	$.316^{-1}$	$.402^{-1}$	$.506^{-1}$	$.628^{-1}$	$.773^{-1}$	$.941^{-1}$	.113	.135	.160	.188	.218
3.2	$.275^{-1}$	$.351^{-1}$	$.442^{-1}$	$.552^{-1}$	$.681^{-1}$	$.832^{-1}$	.101	.121	.143	.168	.196
3.4	$.239^{-1}$	$.306^{-1}$	$.388^{-1}$	$.485^{-1}$	$.601^{-1}$	$.737^{-1}$	$.894^{-1}$	.107	.128	.151	.176
3.6	$.208^{-1}$	$.268^{-1}$	$.340^{-1}$	$.427^{-1}$	$.531^{-1}$	$.653^{-1}$	$.795^{-1}$	$.959^{-1}$	.114	.135	.159
3.8	$.182^{-1}$	$.235^{-1}$	$.299^{-1}$	$.377^{-1}$	$.470^{-1}$	$.580^{-1}$	$.708^{-1}$	$.857^{-1}$	.103	.122	.143
4.0	$.159^{-1}$	$.206^{-1}$	$.263^{-1}$	$.333^{-1}$	$.417^{-1}$	$.516^{-1}$	$.632^{-1}$	$.766^{-1}$	$.920^{-1}$	.110	.129

Table C-II (concluded)

κ=	u=1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4
0	.141[1]	.150[1]	.160[1]	.170[1]	.180[1]	.190[1]	.200[1]	.210[1]	.220[1]	.230[1]	.240[1]
.1	.132[1]	.142[1]	.152[1]	.161[1]	.171[1]	.181[1]	.190[1]	.200[1]	.209[1]	.218[1]	.229[1]
.2	.124[1]	.133[1]	.143[1]	.152[1]	.162[1]	.172[1]	.181[1]	.191[1]	.200[1]	.209[1]	.218[1]
.3	.116[1]	.125[1]	.135[1]	.144[1]	.154[1]	.163[1]	.173[1]	.182[1]	.191[1]	.200[1]	.209[1]
.4	.109[1]	.118[1]	.127[1]	.136[1]	.146[1]	.155[1]	.164[1]	.174[1]	.183[1]	.192[1]	.200[1]
.5	.102[1]	.111[1]	.120[1]	.129[1]	.138[1]	.148[1]	.157[1]	.166[1]	.175[1]	.184[1]	.192[1]
.6	.958	.105[1]	.113[1]	.122[1]	.131[1]	.140[1]	.149[1]	.158[1]	.167[1]	.176[1]	.185[1]
.7	.901	.986	.107[1]	.116[1]	.125[1]	.134[1]	.142[1]	.151[1]	.160[1]	.169[1]	.177[1]
.8	.848	.930	.101[1]	.110[1]	.118[1]	.127[1]	.136[1]	.144[1]	.153[1]	.162[1]	.170[1]
.9	.799	.878	.959	.104[1]	.113[1]	.121[1]	.130[1]	.138[1]	.147[1]	.155[1]	.163[1]
1.0	.753	.829	.908	.989	.107[1]	.115[1]	.124[1]	.132[1]	.140[1]	.149[1]	.157[1]
1.2	.670	.741	.815	.891	.968	.105[1]	.113[1]	.121[1]	.129[1]	.137[1]	.145[1]
1.4	.597	.663	.732	.804	.877	.952	.103[1]	.110[1]	.118[1]	.126[1]	.134[1]
1.6	.553	.595	.659	.726	.795	.866	.938	.101[1]	.108[1]	.116[1]	.123[1]
1.8	.477	.534	.594	.657	.722	.788	.856	.926	.996	.107[1]	.114[1]
2.0	.427	.480	.536	.595	.656	.718	.783	.849	.916	.984	.105[1]
2.2	.383	.432	.484	.539	.596	.655	.716	.779	.842	.908	.974
2.4	.344	.390	.438	.489	.542	.598	.655	.715	.775	.838	.901
2.6	.310	.352	.397	.444	.494	.546	.600	.656	.714	.773	.834
2.8	.279	.318	.359	.403	.451	.499	.550	.603	.658	.714	.772
3.0	.251	.287	.326	.367	.411	.456	.505	.555	.606	.660	.716
3.2	.226	.260	.296	.334	.375	.418	.463	.510	.559	.610	.663
3.4	.204	.235	.268	.304	.342	.382	.425	.469	.516	.564	.615
3.6	.185	.213	.244	.277	.313	.350	.390	.432	.476	.522	.570
3.8	.167	.193	.212	.253	.286	.321	.359	.398	.440	.484	.530
4.0	.151	.175	.202	.230	.261	.294	.330	.367	.406	.448	.491

Table C-III

$\Gamma_2(\kappa, u)$ Function

κ=	u= -2.4	-2.2	-2.0	-1.8	-1.6	-1.4	-1.2	-1.0	-.8	-.6	-.4	-.2
0	$.237^{-4}$	$.600^{-4}$	$.190^{-3}$	$.514^{-3}$	$.129^{-2}$	$.304^{-2}$	$.680^{-2}$	$.142^{-1}$	$.280^{-1}$	$.523^{-1}$	$.925^{-1}$	.156
.1	$.141^{-4}$	$.482^{-4}$	$.146^{-3}$	$.405^{-3}$	$.104^{-2}$	$.251^{-2}$	$.566^{-2}$	$.120^{-1}$	$.240^{-1}$	$.453^{-1}$	$.811^{-1}$	.138
.2	$.109^{-4}$	$.380^{-4}$	$.117^{-3}$	$.330^{-3}$	$.862^{-3}$	$.210^{-2}$	$.479^{-2}$	$.103^{-1}$	$.207^{-1}$	$.396^{-1}$	$.716^{-1}$	.123
.3	$.851^{-5}$	$.305^{-4}$	$.956^{-4}$	$.273^{-3}$	$.720^{-3}$	$.177^{-2}$	$.408^{-2}$	$.885^{-2}$	$.181^{-1}$	$.348^{-1}$	$.635^{-1}$	.110
.4	$.675^{-5}$	$.247^{-4}$	$.787^{-4}$	$.227^{-3}$	$.607^{-3}$	$.151^{-2}$	$.351^{-2}$	$.767^{-2}$	$.158^{-1}$	$.307^{-1}$	$.566^{-1}$	$.990^{-1}$
.5	$.541^{-5}$	$.203^{-4}$	$.655^{-4}$	$.191^{-3}$	$.515^{-3}$	$.129^{-2}$	$.303^{-2}$	$.669^{-2}$	$.139^{-1}$	$.272^{-1}$	$.506^{-1}$	$.892^{-1}$
.6	$.437^{-5}$	$.167^{-4}$	$.549^{-4}$	$.162^{-3}$	$.440^{-3}$	$.111^{-2}$	$.263^{-2}$	$.585^{-2}$	$.123^{-1}$	$.242^{-1}$	$.453^{-1}$	$.806^{-1}$
.7	$.355^{-5}$	$.139^{-4}$	$.463^{-4}$	$.136^{-3}$	$.378^{-3}$	$.964^{-3}$	$.230^{-2}$	$.515^{-2}$	$.109^{-1}$	$.216^{-1}$	$.408^{-1}$	$.729^{-1}$
.8	$.291^{-5}$	$.116^{-4}$	$.392^{-4}$	$.118^{-3}$	$.326^{-3}$	$.838^{-3}$	$.201^{-2}$	$.454^{-2}$	$.964^{-2}$	$.193^{-1}$	$.367^{-1}$	$.661^{-1}$
.9	$.239^{-5}$	$.976^{-5}$	$.334^{-4}$	$.102^{-3}$	$.283^{-3}$	$.732^{-3}$	$.177^{-2}$	$.402^{-2}$	$.860^{-2}$	$.174^{-1}$	$.332^{-1}$	$.601^{-1}$
1.0	$.197^{-5}$	$.823^{-5}$	$.285^{-4}$	$.878^{-4}$	$.246^{-3}$	$.642^{-3}$	$.156^{-2}$	$.357^{-2}$	$.768^{-2}$	$.156^{-1}$	$.300^{-1}$	$.547^{-1}$
1.2	$.135^{-5}$	$.591^{-5}$	$.211^{-4}$	$.661^{-4}$	$.189^{-3}$	$.498^{-3}$	$.123^{-2}$	$.284^{-2}$	$.618^{-2}$	$.127^{-1}$	$.247^{-1}$	$.456^{-1}$
1.4	$.942^{-6}$	$.429^{-5}$	$.157^{-4}$	$.503^{-4}$	$.146^{-3}$	$.390^{-3}$	$.973^{-3}$	$.228^{-2}$	$.501^{-2}$	$.104^{-1}$	$.205^{-1}$	$.382^{-1}$
1.6	$.662^{-6}$	$.315^{-5}$	$.119^{-4}$	$.386^{-4}$	$.114^{-3}$	$.308^{-3}$	$.778^{-3}$	$.184^{-2}$	$.410^{-2}$	$.860^{-2}$	$.171^{-1}$	$.322^{-1}$
1.8	$.470^{-6}$	$.233^{-5}$	$.900^{-5}$	$.299^{-4}$	$.893^{-4}$	$.245^{-3}$	$.626^{-3}$	$.150^{-2}$	$.337^{-2}$	$.714^{-2}$	$.143^{-1}$	$.272^{-1}$
2.0	$.335^{-6}$	$.174^{-5}$	$.689^{-5}$	$.233^{-4}$	$.706^{-4}$	$.196^{-3}$	$.507^{-3}$	$.122^{-2}$	$.278^{-2}$	$.595^{-2}$	$.120^{-1}$	$.231^{-1}$
2.2	$.241^{-6}$	$.130^{-5}$	$.531^{-5}$	$.183^{-4}$	$.561^{-4}$	$.158^{-3}$	$.412^{-3}$	$.101^{-2}$	$.231^{-2}$	$.498^{-2}$	$.102^{-1}$	$.197^{-1}$
2.4	$.174^{-6}$	$.983^{-6}$	$.411^{-5}$	$.144^{-4}$	$.448^{-4}$	$.128^{-3}$	$.337^{-3}$	$.830^{-3}$	$.192^{-2}$	$.419^{-2}$	$.863^{-2}$	$.168^{-1}$
2.6	$.126^{-6}$	$.745^{-6}$	$.320^{-5}$	$.114^{-4}$	$.361^{-4}$	$.104^{-3}$	$.276^{-3}$	$.687^{-3}$	$.160^{-2}$	$.353^{-2}$	$.733^{-2}$	$.144^{-1}$
2.8	$.920^{-7}$	$.568^{-6}$	$.250^{-5}$	$.909^{-5}$	$.290^{-4}$	$.845^{-4}$	$.227^{-3}$	$.570^{-3}$	$.134^{-2}$	$.298^{-2}$	$.625^{-2}$	$.124^{-1}$
3.0	$.673^{-7}$	$.434^{-6}$	$.197^{-5}$	$.727^{-5}$	$.235^{-4}$	$.691^{-4}$	$.188^{-3}$	$.475^{-3}$	$.113^{-2}$	$.253^{-2}$	$.533^{-2}$	$.107^{-1}$
3.2	$.494^{-7}$	$.333^{-6}$	$.155^{-5}$	$.584^{-5}$	$.191^{-4}$	$.568^{-4}$	$.156^{-3}$	$.397^{-3}$	$.951^{-3}$	$.215^{-2}$	$.457^{-2}$	$.920^{-2}$
3.4	$.363^{-7}$	$.256^{-6}$	$.123^{-5}$	$.470^{-5}$	$.156^{-4}$	$.468^{-4}$	$.129^{-3}$	$.333^{-3}$	$.804^{-3}$	$.183^{-2}$	$.392^{-2}$	$.795^{-2}$
3.6	$.268^{-7}$	$.198^{-6}$	$.975^{-6}$	$.380^{-5}$	$.128^{-4}$	$.387^{-4}$	$.108^{-3}$	$.280^{-3}$	$.680^{-3}$	$.156^{-2}$	$.337^{-2}$	$.688^{-2}$
3.8	$.198^{-7}$	$.153^{-6}$	$.777^{-6}$	$.309^{-5}$	$.105^{-4}$	$.321^{-4}$	$.901^{-4}$	$.236^{-3}$	$.577^{-3}$	$.133^{-2}$	$.290^{-2}$	$.596^{-2}$
4.0	$.147^{-7}$	$.119^{-6}$	$.620^{-6}$	$.251^{-5}$	$.865^{-5}$	$.267^{-4}$	$.756^{-4}$	$.199^{-3}$	$.491^{-3}$	$.114^{-2}$	$.250^{-2}$	$.518^{-2}$

Table C-III (concluded)

κ=	u=0	.2	.4	.6	.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4
0	.250	.384	.568	.808	.111[1]	.149[1]	.193[1]	.246[1]	.306[1]	.374[1]	.450[1]	.534[1]	.625[1]
.1	.224	.348	.519	.745	.103[1]	.139[1]	.182[1]	.233[1]	.290[1]	.355[1]	.426[1]	.504[1]	.588[1]
.2	.201	.316	.475	.688	.961	.130[1]	.171[1]	.220[1]	.276[1]	.338[1]	.408[1]	.483[1]	.565[1]
.3	.182	.288	.436	.635	.894	.122[1]	.161[1]	.208[1]	.262[1]	.323[1]	.390[1]	.464[1]	.544[1]
.4	.165	.263	.401	.588	.833	.114[1]	.152[1]	.197[1]	.249[1]	.308[1]	.373[1]	.445[1]	.524[1]
.5	.150	.240	.369	.545	.777	.107[1]	.143[1]	.186[1]	.236[1]	.293[1]	.357[1]	.427[1]	.504[1]
.6	.136	.220	.340	.506	.725	.100[1]	.135[1]	.176[1]	.225[1]	.280[1]	.342[1]	.410[1]	.485[1]
.7	.124	.202	.314	.470	.677	.943	.127[1]	.167[1]	.214[1]	.267[1]	.327[1]	.394[1]	.467[1]
.8	.113	.185	.290	.437	.633	.886	.120[1]	.158[1]	.203[1]	.255[1]	.313[1]	.378[1]	.450[1]
.9	.104	.171	.269	.407	.592	.833	.113[1]	.150[1]	.193[1]	.243[1]	.300[1]	.363[1]	.433[1]
1.0	.950[-1]	.157	.249	.379	.554	.783	.107[1]	.142[1]	.184[1]	.232[1]	.287[1]	.349[1]	.417[1]
1.2	.800[-1]	.134	.215	.329	.487	.694	.958	.128[1]	.167[1]	.212[1]	.264[1]	.322[1]	.387[1]
1.4	.678[-1]	.115	.185	.287	.429	.616	.857	.116[1]	.151[1]	.194[1]	.242[1]	.297[1]	.359[1]
1.6	.577[-1]	.985[-1]	.161	.251	.378	.548	.768	.104[1]	.138[1]	.177[1]	.223[1]	.275[1]	.333[1]
1.8	.492[-1]	.849[-1]	.140	.221	.334	.488	.689	.942	.125[1]	.162[1]	.205[1]	.254[1]	.310[1]
2.0	.422[-1]	.733[-1]	.122	.194	.296	.435	.619	.851	.114[1]	.148[1]	.188[1]	.235[1]	.288[1]
2.2	.363[-1]	.636[-1]	.106	.171	.263	.389	.556	.770	.104[1]	.134[1]	.173[1]	.217[1]	.268[1]
2.4	.312[-1]	.552[-1]	.931[-1]	.150	.233	.348	.501	.698	.943	.124[1]	.160[1]	.201[1]	.249[1]
2.6	.270[-1]	.481[-1]	.817[-1]	.133	.207	.311	.451	.632	.860	.114[1]	.147[1]	.186[1]	.231[1]
2.8	.234[-1]	.419[-1]	.717[-1]	.118	.185	.279	.407	.574	.784	.104[1]	.136[1]	.172[1]	.215[1]
3.0	.203[-1]	.366[-1]	.631[-1]	.104	.165	.250	.367	.521	.716	.958	.125[1]	.160[1]	.201[1]
3.2	.176[-1]	.320[-1]	.556[-1]	.923[-1]	.147	.225	.332	.473	.654	.879	.115[1]	.148[1]	.187[1]
3.4	.153[-1]	.281[-1]	.490[-1]	.819[-1]	.131	.202	.300	.430	.598	.808	.106[1]	.137[1]	.174[1]
3.6	.133[-1]	.246[-1]	.433[-1]	.728[-1]	.117	.182	.271	.391	.546	.742	.983	.127[1]	.162[1]
3.8	.116[-1]	.216[-1]	.383[-1]	.648[-1]	.105	.164	.246	.356	.500	.682	.908	.118[1]	.151[1]
4.0	.102[-1]	.190[-1]	.339[-1]	.577[-1]	.941[-1]	.147	.222	.324	.458	.628	.840	.110[1]	.141[1]

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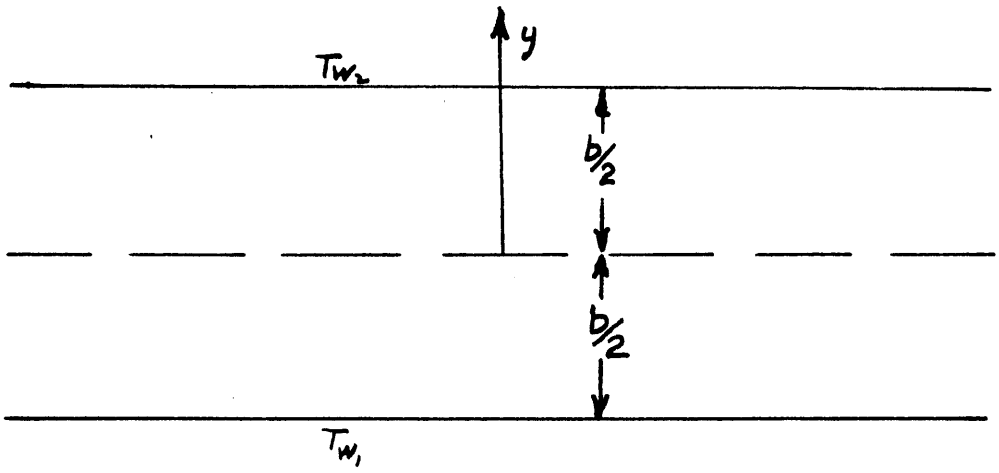
APPENDIX D

SOLUTION OF THE HEAT TRANSFER BETWEEN PARALLEL PLATES PROBLEM BY LESTER LEES' MOMENT METHOD

The geometry of the problem consists of



where T_{w_1} and T_{w_2} are the wall temperatures and b is the plate separation.

Lester Lees' moment method suggests using an assumed distribution function of the form

$$f(\bar{y},\vec{v}) = \begin{cases} \dfrac{n_1(\bar{y})}{(2\pi RT_1(y))^{3/2}} \exp\left[-\dfrac{v^2}{2RT_1(y)}\right] & v_y > 0 \\ \dfrac{n_2(y)}{(2\pi RT_2(y))^{3/2}} \exp\left[-\dfrac{v^2}{2RT_2(y)}\right] & v_y < 0 \end{cases} \qquad (D-1)$$

From this distribution function it is possible to evaluate the relationship between the physical properties which are integrals over all velocity space and the assumed parameters n_1, n_2, T_1, T_2 .

$n(y)$ = number density

$$= \int_{-\infty}^{\infty}\int\int\int f(y,\vec{v})\, dv_y\, dv_x\, dv_z$$

$$= \frac{n_1(y) + n_2(y)}{2} \qquad (D-2)$$

$T(y)$ = Temperature

$$= \frac{1}{\frac{3}{2} n(y) R} \int_{-\infty}^{\infty}\int\int\left(\frac{v_y^2+v_x^2+v_z^2}{2}\right) f(y,\vec{v})\, dv_y\, dv_x\, dv_z$$

$$= \frac{n_1(y)\,T_1(y) + n_2(y)\,T_2(y)}{n_1(y) + n_2(y)} \qquad (D-3)$$

$p(y)$ = pressure

$$= p\,R\,T = n\,m\,R\,T$$

$$= \frac{m\,R}{2}\left(n_1(y)\,T_1(y) + n_2(y)\,T_2(y)\right) \qquad (D-4)$$

$\overline{v_y}$ = macroscopic velocity in y direction $= \frac{2}{\pi}\sqrt{2R}\left(n_1\sqrt{T_1} - n_2\sqrt{T_2}\right)/n(y)$

The physical equations for this geometry and restraints are

$$\overline{v}_y = 0$$

$$P_{yy} = \text{constant}$$

$$\frac{dq}{dy} = 0 \text{ where } q \text{ is the heat flux} \qquad (D\text{-}5)$$

$$\frac{dQ_{yy}}{dy} = -\frac{2}{3}\frac{P}{\mu} q \qquad Q_{yy} \text{ is the transport of the heat flux}$$

The first two equations become

$$n_1\sqrt{T_1} = n_2\sqrt{T_2} \qquad (D\text{-}6)$$

$$n_1 T_1 + n_2 T_2 = \frac{2P}{mR} \qquad (D\text{-}7)$$

These can be used to eliminate n_1 and n_2

$$n_1 = \frac{2P}{mR}\Big/\left(\sqrt{T_1}\left(\sqrt{T_1}+\sqrt{T_2}\right)\right) \qquad (D\text{-}8)$$

$$n_2 = \frac{2P}{mR}\Big/\left(\sqrt{T_2}\left(\sqrt{T_1}+\sqrt{T_2}\right)\right) \qquad (D\text{-}9)$$

The second two equations become the following when the form of $f(x,v)$ assumed is used

$$n_1 T_1^{3/2} - n_2 T_2^{3/2} = \sqrt{\frac{\pi}{2R}}\,\frac{q}{Rm} \qquad (D\text{-}10)$$

$$\frac{5}{4} R^2 m \frac{d}{dy}\left(n_1 T_1^2 + n_2 T_2^2\right) = -\frac{2}{3}\frac{P}{\mu} q \qquad (D\text{-}11)$$

Substitution of expressions for n_1 and n_2 yields

$$\sqrt{T_1} - \sqrt{T_2} = \sqrt{\frac{\pi}{2R}} \frac{q}{2p} \qquad (D-12)$$

$$\frac{d}{dy}\left[\frac{T_1^{3/2} + T_2^{3/2}}{\sqrt{T_1} + \sqrt{T_2}}\right] = -\frac{4}{15}\frac{q}{\mu R} = -\frac{q}{k} \qquad (D-13)$$

where k is the conductivity $k = 15/4\,\mu R$. Equation (D-13) can be reduced by juggling and using Eq. (D-12) to this form

$$\frac{d\sqrt{T_1 T_2}}{dy} = -q/k \qquad (D-14)$$

Substitution for n_1 and n_2 in the expression for T also shows that

$$T(y) = \sqrt{T_1 T_2} \qquad (D-15)$$

The two pertinent equations can now be solved. The difference from the continuum equations appears only in the boundary conditions as now these are imposed on T_1 and T_2 not just on T. Before solving

these equations it is desirable to non-dimensionalize

$$\bar{y} = y/b \qquad \theta_1 = \sqrt{T_1/T_{w_1}} \qquad \theta_2 = \sqrt{T_2/T_{w_1}}$$

$$\beta = \frac{T_{w_1} - T_{w_2}}{T_{w_1}} \qquad \psi = \frac{q\,b}{k_{w_1}(T_{w_1} - T_{w_2})}$$

$$K = \frac{15}{16}\frac{\mu}{\rho b}\sqrt{2\pi R T_{w_1}} \tag{D-16}$$

$$= \frac{k_{w_1} T_{w_1}}{b p}\frac{1}{\frac{2}{\sqrt{\pi}}\sqrt{2RT_{w_1}}} = \frac{15}{8}\frac{\lambda_{w_1}}{b}$$

where λ_{w_1} is the mean free path for Maxwell particles at wall 1. In this notation the equations become

$$\theta_1 - \theta_2 = \frac{q}{p\,\frac{2}{\sqrt{\pi}}\sqrt{2RT_{w_1}}}$$

$$= \left(\frac{q\,b}{k_{w_1}(T_{w_1} - T_{w_2})}\right)\left(\frac{T_{w_1} - T_{w_2}}{T_{w_1}}\right)\left(\frac{T_{w_1} k_{w_1}}{p\,b\,\frac{2}{\sqrt{\pi}}\sqrt{2RT_{w_1}}}\right)$$

$$= \psi\,\beta\,K \tag{D-17}$$

$$\frac{d}{d\bar{y}}\theta_1\,\theta_2 = -\frac{q\,b}{k(T_{w_1} - T_{w_2})}\left(\frac{T_{w_1} - T_{w_2}}{T_{w_1}}\right)$$

$$= -\frac{k_{w_1}}{k}\psi\,\beta \tag{D-18}$$

For Maxwellian particles $k = k_{w_1}\theta_1\theta_2$ but keeping the problem more general even if not rigorously consistent let us first assume that $k = k_{w_1}(\theta_1\theta_2)^n$ where n can be made to represent the real variation over a certain range of temperatures. For this assumption the general solution looks as follows

$$[\theta_1\theta_2]^{n+1} = \phi^{n+1} - (n+1)\psi\beta\bar{y} \qquad (D\text{-}19)$$

$$\theta_1 - \theta_2 = \psi\beta K \qquad (D\text{-}20)$$

Thus

$$[\theta_1(\theta_1 - \psi\beta K)]^{n+1} = \phi^{n+1} - (n+1)\psi\beta\bar{y} \qquad (D\text{-}21)$$

$$[\theta_2(\theta_2 + \psi\beta K)]^{n+1} = \phi^{n+1} - (n+1)\psi\beta\bar{y} \qquad (D\text{-}22)$$

These can be solved for θ_1 and θ_2 quite easily, but the application of boundary conditions to determine the undetermined constants cannot, in general, be carried out for both arbitrary n and arbitrary accomodation coefficient. These constants are ϕ and ψ and it is easily seen that a solution for arbitrary n may involve solving higher order equations.

As a first attempt one can assume that the accomodation coefficient is unity. This allows putting the boundary conditions in directly:

$$\theta_1(-\tfrac{1}{2}) = 1$$

and

$$\theta_2(+\tfrac{1}{2}) = \sqrt{1-\beta} \qquad (D\text{-}23)$$

Thus

$$\left[1-\psi\beta K\right]^{n+1} = \phi^{n+1} + \frac{n+1}{2}\psi\beta$$

$$\left[1-\beta+\sqrt{1-\beta}\,\psi\beta K\right]^{n+1} = \phi^{n+1} - \frac{n+1}{2}\psi\beta \tag{D-24}$$

The three cases that can be considered analytically are $n = 0,\ 1/2,\ 1$. These correspond to the assumptions of constant property case, hard sphere case, and Maxwellian particle case respectively. The $n = 0$ solution is very easy

$$1-\psi\beta K = \phi + \frac{1}{2}\psi\beta$$

$$1-\beta+\sqrt{1-\beta}\,\psi\beta K = \phi - \frac{1}{2}\psi\beta$$

$$\psi = \frac{1}{1+K\left(1+\sqrt{1-\beta}\right)} \tag{D-25}$$

$$\phi = 1 - \beta/2 - \frac{\psi\beta K}{2}\left(1-\sqrt{1-\beta}\right)$$

$$\phi = 1 - \beta/2\left[1 + \frac{K\left(1-\sqrt{1-\beta}\right)}{1+K\left(1+\sqrt{1-\beta}\right)}\right] \tag{D-26}$$

Transforming this back to the physical quantities

$$q = \frac{k}{b}(T_{w_1} - T_{w_2}) \Big/ \left[1 + \frac{15}{8}\frac{\lambda_{w_1}}{b}\left(1+\gamma\sqrt{\frac{T_{w_2}}{T_{w_1}}}\right)\right] \qquad (D\text{-}27)$$

$$\left(\frac{T_{w_1} - T}{T_{w_1} - T_{w_2}}\right) = \frac{1}{2}\left[1 + \frac{\frac{2y}{b} + \frac{15}{8}\frac{\lambda_{w_1}}{b}\left(1-\sqrt{T_{w_2}/T_{w_1}}\right)}{1 + \frac{15}{8}\frac{\lambda_{w_1}}{b}\left(1+\sqrt{T_{w_2}/T_{w_1}}\right)}\right] \qquad (D\text{-}28)$$

The next case we can consider is the case $\eta = 1$ and accomodation coefficient of unity. This is the case that is consistant with the equations used in Lester Lees' moment method and represents Maxwellian particles. The equations for the boundary conditions are

$$\left[1 - \psi\beta K\right]^2 = \phi^2 + \psi\beta$$

$$\left[1-\beta + \sqrt{1-\beta}\,\psi\beta K\right]^2 = \phi^2 - \psi\beta$$

$$2\beta - \beta^2 - 2\psi\beta K\left(1+(1-\beta)^{3/2}\right) + \beta(\psi\beta K)^2 = 2\psi\beta$$

$$(\psi\beta K)^2 - 2(\psi\beta K)\left(\frac{\left(1+(1-\beta)^{3/2}\right)K+1}{K\beta}\right) + 2-\beta = 0$$

$$\psi\beta K = \frac{1+K\left(1+(1-\beta)^{3/2}\right)}{K\beta} \pm \sqrt{\beta - 2 + \left(\frac{1+K\left(1+(1-\beta)^{3/2}\right)}{K\beta}\right)^2}$$

$$\psi = \frac{1 + K(1+(1-\beta)^{3/2})}{(K\beta)^2}\left[1 \pm \sqrt{1 - (2-\beta)\left[\frac{K\beta}{1+K(1+(1-\beta)^{3/2})}\right]^2}\right] \qquad (D\text{-}29)$$

This appears as a very complicated mess and also has an ambiguity in sign. It can be shown, however, that for $\beta << 1$ this reduces to a form similar to that for $n = 0$ and also the ambiguity is dissolved as only the minus sign can have physical significance

$$\psi = \frac{1 + K(2 - 3/2\,\beta)}{(K\beta)^2}\left[1 \pm \left(1 - \frac{2-\beta}{2}\frac{(K\beta)^2}{[1+K(2-3/2\,\beta)]^2}\right)\right]$$

$$\psi = \frac{1 - \beta/2}{1 + K(2 - 3/2\,\beta)} \qquad (D\text{-}30)$$

This can be seen to approach the solution for $n = 0$ as $\beta \Rightarrow 0$ where the fact that using the average conductivity brings in the term $1 - \beta/2$ and using the average Knudson number brings in the term $K(2 - 3/2\,\beta)$ instead of $K(1+\sqrt{1-\beta})$.

ϕ^2 can also be found to be

$$\phi^2 = 1 - \beta\left[1 + \psi K(1 - \sqrt{1-\beta}) - \frac{\beta}{2}\left(1 + (2-\beta)(\psi K)^2\right)\right] \qquad (D\text{-}31)$$

If one realizes that β is always less than one and thus the parameter

$$\frac{2-\beta}{2}\left[\frac{K\beta}{1+K(1+(1-\beta)^{1/2})}\right]$$

is also less than unity one can carry the expansion further.

Defining

$$\varepsilon = \frac{K\beta}{1+K(1+(1-\beta)^{1/2})} \qquad (D-32)$$

the expression takes the form

$$\psi = \frac{1-\beta/2}{1+K(1+(1-\beta)^{1/2})}\left[1+\frac{1}{2}\left((1-\beta/2)\varepsilon^2+(1-\beta/2)^2\varepsilon^4\right.\right.$$

$$\left.\left.+\frac{5}{4}(1-\beta/2)^3\varepsilon^6+\cdots\right)\right] \qquad (D-33)$$

For $5/4(1-\beta/2)^3\varepsilon^6 << 1$ this can be put in closed form to that order of accuracy

$$\psi = \frac{1}{2}\frac{1-\beta/2}{1+K(1+(1-\beta)^{1/2})} \times$$

$$\left[1+\frac{1}{1-(1-\beta/2)\left[\frac{K\beta}{1+K(1+(1-\beta)^{1/2})}\right]^2}\right] \qquad (D-34)$$

The important results needed for future use are, however, contained in the expression that

$$T = \sqrt{\phi^2 - 2\psi\beta\bar{y}}\quad T_{W_1} \qquad (D\text{-}35)$$

The solution for $n=\frac{1}{2}$ can be seen to be even more complicated as the resulting equation for the boundary conditions is a cubic, with all that this implies in complication. This solution is, therefore, not carried out in detail, though if need be this can be done in a straightforward manner.

A more interesting phenomenon to investigate is the influence of thermal accomodation at the walls. This is expressible in the following way

$$\frac{T_1(-\frac{1}{2}) - T_2(-\frac{1}{2})}{T_{W_1} - T_2(-\frac{1}{2})} = \alpha$$

$$\frac{T_2(\frac{1}{2}) - T_1(\frac{1}{2})}{T_{W_2} - T_1(\frac{1}{2})} = \alpha \qquad (D\text{-}36)$$

In the non-dimensionalized coordinates used so far, these become:

$$\frac{\theta_1^2(-\frac{1}{2}) - \theta_2^2(-\frac{1}{2})}{1 - \theta_2^2(-\frac{1}{2})} = \alpha$$

$$\frac{\theta_2^2(\frac{1}{2}) - \theta_1^2(\frac{1}{2})}{1-\beta - \theta_1^2(\frac{1}{2})} = \alpha \qquad (D\text{-}37)$$

Calling $\theta_1(-\frac{1}{2})$, $\theta_2(-\frac{1}{2})$, θ_{11} and θ_{21} respectively and

$\theta_1(\frac{1}{2})$ and $\theta_2(\frac{1}{2})$, θ_{12} and θ_{22} respectively one obtains the relations.

$$\theta_{11}^2 - \theta_{21}^2(1-\alpha) = \alpha$$

$$\theta_{22}^2 - \theta_{12}^2(1-\alpha) = (1-\beta)\alpha \qquad (D-38)$$

Using the relation $\theta_1 - \theta_2 = \psi\beta K$ at wall (1) and (2) one gets

$$\theta_{11}^2 - (1-\alpha)(\theta_{11} - \psi\beta K)^2 = \alpha$$

$$\theta_{22}^2 - (1-\alpha)(\theta_{22} + \psi\beta K)^2 = (1-\beta)\alpha \qquad (D-39)$$

The solutions of these equations are

$$\theta_{11} = \sqrt{1 + \left(\frac{\psi\beta K}{\alpha}\right)^2(1-\alpha)} - (1-\alpha)\frac{\psi\beta K}{\alpha}$$

$$\theta_{22} = \sqrt{1-\beta + \left(\frac{\psi\beta K}{\alpha}\right)^2(1-\alpha)} + (1-\alpha)\frac{\psi\beta K}{\alpha} \qquad (D-40)$$

and the previous equations become

$$\left[\theta_{11}(\theta_{11} - \psi\beta K)\right]^{n+1} = \phi^{n+1} + \psi\beta\left(\frac{n+1}{2}\right)$$

$$\left[\theta_{22}(\theta_{22} + \psi\beta K)\right]^{n+1} = \phi^{n+1} - \frac{n+1}{2}\psi\beta \qquad (D-41)$$

It can be seen that the solution is going to be difficult even for the $n=0$ case unless some assumption can be made to eliminate the square root term. The general equations for Ψ and ϕ look as follows:

$$\left[1+2\left(\frac{\Psi\beta K}{\alpha}\right)^2(1-\alpha)-\frac{\Psi\beta K}{\alpha}(2-\alpha)\sqrt{1+\left(\frac{\Psi\beta K}{\alpha}\right)^2(1-\alpha)}\right]^{n+1}$$

$$=\phi^{n+1}+\frac{n+1}{2}\Psi\beta \qquad (D\text{-}42)$$

$$\left[1-\beta+2\left(\frac{\Psi\beta K}{\alpha}\right)^2(1-\alpha)+\left(\frac{\Psi\beta K}{\alpha}\right)(2-\alpha)\sqrt{1-\beta+\left(\frac{\Psi\beta K}{\alpha}\right)^2(1-\alpha)}\right]^{n+1}$$

$$=\phi^{n+1}-\frac{n+1}{2}\Psi\beta \qquad (D\text{-}43)$$

for $n=0$ the equation for Ψ becomes

$$\Psi = 1-\Psi K\left(\frac{2-\alpha}{\alpha}\right)\times$$

$$\left[\sqrt{1+\left(\frac{\Psi\beta K}{\alpha}\right)^2(1-\alpha)}+\sqrt{1-\beta+\left(\frac{\Psi\beta K}{\alpha}\right)^2(1-\alpha)}\right] \qquad (D\text{-}44)$$

This cannot be solved directly unless one assumes that

$$\left(\frac{\Psi\beta K}{\alpha}\right)^2(1-\alpha)\left(\frac{1+\sqrt{1-\beta}}{2\sqrt{1-\beta}}\right) << 1$$

In this case

$$\Psi = \frac{1}{1+\frac{2-\alpha}{\alpha}K(1+\sqrt{1-\beta})} \qquad (D\text{-}45)$$

Thus

$$\left(\frac{\Psi\beta K}{\alpha}\right)^2(1-\alpha)\left(\frac{1+\sqrt{1-\beta}}{2\sqrt{1-\beta}}\right)=\left[\frac{\beta K}{\alpha+(2-\alpha)K(1+\sqrt{1-\beta})}\right]^2\left(\frac{1+\sqrt{1-\beta}}{2\sqrt{1-\beta}}\right)(1-\alpha) \qquad (D\text{-}46)$$

Since the contribution of the term becomes important only for K large, let us investigate the magnitude of this term for $K \Rightarrow \infty$

$$\frac{1}{2}\frac{1-\alpha}{(2-\alpha)^2}\frac{\beta^2}{1-\beta+\sqrt{1-\beta}}$$

Thus, for reasonable values of $\beta = \frac{\Delta T_w}{T_{w_1}} \simeq \frac{1}{2}$ for which the $n=0$ case can be expected to be valid

$$\frac{1-\alpha}{2(2-\alpha)^2} << 4.828$$

Thus, even for $\alpha << 1$

$$1-\alpha^2 << 38.6$$

For the same β but $K=1$ (right on the transition range)

$$1-\alpha << 154$$

One can thus conclude that for $\beta < \frac{1}{2}$ (which is necessary to allow assumption of $n=0$) the approximation holds for all values of α and thus

$$\psi = \frac{1}{1+\frac{2-\alpha}{\alpha}K(1+\sqrt{1-\beta})} \qquad (D-47)$$

The situation for $n = 1$ is not quite as simple, but under a similar assumption of

$$\left(\frac{\Psi\beta K}{\alpha}\right)^2 \frac{(1-\alpha)}{\sqrt{1-\beta}} \ll 1$$

The equation for Ψ becomes:

$$\Psi = \left(1 + 2\left(\frac{\Psi\beta K}{\alpha}\right)^2\left[(1-\alpha) + \left(\frac{2-\alpha}{2}\right)^2\right] + \frac{2-\alpha}{\alpha}\Psi\beta K\sqrt{1-\beta}\right)$$

$$- \beta/2 - \Psi K \frac{(2-\alpha)}{\alpha}\left(1+\sqrt{1-\beta}\right) \qquad (D-48)$$

The solution of this equation is

$$\Psi = \frac{\alpha}{(\beta K)^2}\left[\frac{1 + K\left(\frac{2-\alpha}{\alpha}\right)\left(1+(1-\beta)^{3/2}\right)}{4\frac{1-\alpha}{\alpha} + \alpha\left(\frac{2-\alpha}{\alpha}\right)^2}\right] x$$

$$\left[1 - \sqrt{1 - \frac{2\beta^2K^2\left(4\frac{1-\alpha}{\alpha} + \alpha\left(\frac{2-\alpha}{\alpha}\right)^2\right)(1-\beta/2)}{\alpha\left[1 + \frac{2-\alpha}{\alpha}K\left(1+(1-\beta)^{3/2}\right)\right]^2}}\right] \qquad (D-49)$$

Which for $\beta <<$ becomes

$$\Psi = \frac{1 - \beta/2}{1 + \frac{2-\alpha}{\alpha}K\left(2 - \frac{3}{2}\beta\right)} \qquad (D-50)$$

Thus, again, converging to the answer for $n=0$, provided one realizes that the $1-\beta/2$ term only normalizes ψ by multiplying by $\bar{k}/k_{w_i}$ and the $2-3/2\beta$ term only redefines the Knudson number in terms of averaged conditions based on $k \sim T$.

Carrying the expansion further just as in the $n=0$ case one gets the following

$$\psi = \frac{1}{1+\frac{3-\alpha}{\alpha}K(1+(1-\beta)^{3/2})}\frac{1}{f(\alpha)\varepsilon^2}\left[\frac{1}{2}\left(f(\alpha)(2-\beta)\varepsilon^2\right)\right.$$

$$\left. + \frac{1}{2}f(\alpha)^2(1-\beta/2)^2\varepsilon^4 + \frac{1}{2}f(\alpha)^3(1-\beta/2)^3\varepsilon^6 + \frac{5}{8}f(\alpha)^4(1-\beta/2)^4\varepsilon^8 \ldots\right] \quad (D-51)$$

where

$$\varepsilon^2 = \left[\frac{K\beta}{1+\frac{3-\alpha}{\alpha}K(1+(1-\beta)^{3/2})}\right]^2$$

and

$$f(\alpha) = \left(\frac{2-\alpha}{\alpha}\right)^2 + 4\frac{1-\alpha}{\alpha^2}$$

This again reduces to a closed form solution

$$\psi = \frac{1-\beta/2}{1+\frac{3-\alpha}{\alpha}K(1+(1-\beta)^{3/2})} \times \frac{1}{2}\left[1+\frac{1}{1-(1-\beta/2)\left(\left(\frac{2-\alpha}{\alpha}\right)^2+4\frac{1-\alpha}{\alpha^2}\right)\left[\frac{K\beta}{1+\frac{3-\alpha}{\alpha}K(1+(1-\beta)^{3/2})}\right]^2}\right] \quad (D-52)$$

for $\frac{1}{4}(1-\beta)^3\varepsilon^6 << 1$

This can be seen to converge to the proper answers for $\beta << 1$ and also for $\alpha = 1$.

The case for $n = \frac{1}{2}$ will, of course, be more complicated because the equation to be solved will be a cubic. It can be shown, however, that a solution can be found for the case of

$$\left(\frac{\psi\beta K}{\alpha}\right)^2 \frac{1-\alpha}{\sqrt{1-\beta}} << 1$$

just as in the $n = 0$, and $n = 1$ cases.

A plot of heat flux versus Knudson number for both the constant property and Maxwellian case is shown in Fig. D-1. It can be seen that all answers converge to the same point (because of the normalization) in the continuum while their slopes always become the same in the free molecular limit. It can further be seen that only between a Knudson number of .05 and 10 is there any measurable deviation from the asymptotic solutions.

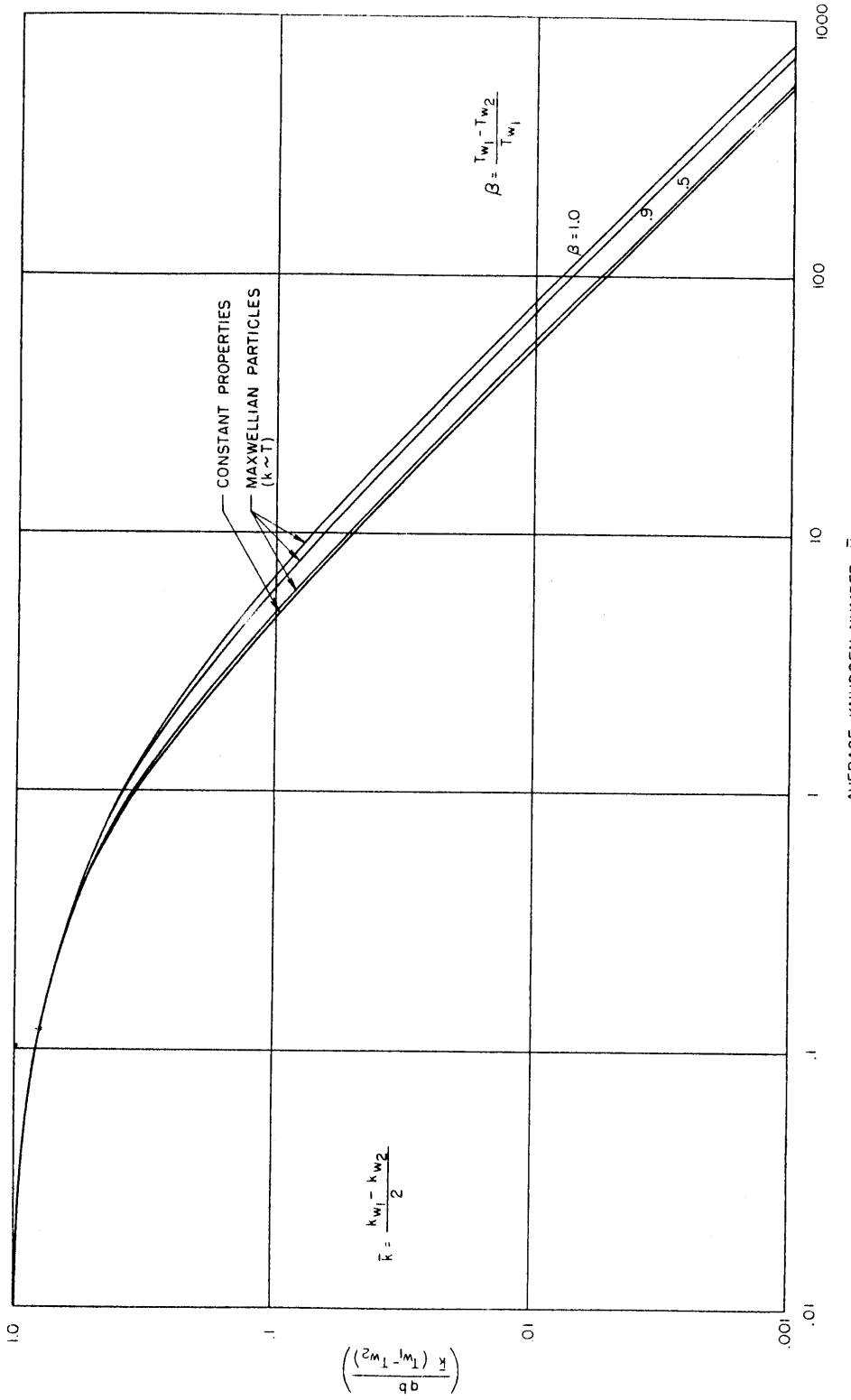


Figure D-1. Heat transfer versus Knudsen number (α = 1)

APPENDIX E

EVALUATION OF THE $I_n(k)$ FUNCTIONS

In the application of the integral-iteration scheme to the solution of transition regime heat transfer problems a certain type of integral continually reappears. It is, therefore, advisable to analyse the functions defined by this type of integral and if they are not representable by known functions tabulate them. The integral is of the form

$$I_n(K) = \int_0^\infty \exp\left[-\left(v^2+\frac{K}{v}\right)\right] v^n dv \qquad (E\text{-}1)$$

The goal is to determine as many analytic properties of the set of functions $I_n(K)$ and then find a method of evaluating them for all K and n.

Integration by parts in velocity space yields the immediate result

$$I_n(K) = -\frac{v^{n-1}}{2}\exp\left[-v^2-\frac{K}{v}\right]\Bigg|_0^\infty + \frac{n-1}{2}\int_0^\infty \exp\left[-v^2-\frac{K}{v}\right] v^{n-2}dv + \frac{K}{2}\int_0^\infty \exp\left[-v^2-\frac{K}{v}\right] v^{n-3}dv \qquad (E\text{-}2)$$

which give the recursion relation

$$I_n(K) = \frac{n-1}{2} I_{n-2}(K) + \frac{K}{2} I_{n-3}(K) \qquad (E-3)$$

thus, requiring the determination of only three I_n's in order to determine all the others through algebraic relations.

Integrations over K yield the useful results

$$I_n(K) = \int_K^\infty I_{n-1}(K)\, dK$$

and

$$I_n(K) = I_n(0) - \int_0^K I_{n-1}(K)\, dK$$

$I_n(0)$ can be easily determined as these definite integrals are well known for positive n

$$I_n(0) = \int_0^\infty \exp[-v^2] v^n dv$$

$$= \left(\frac{n-1}{2}\right)!/2 \qquad \text{for } n \text{ odd}$$

$$= \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2^{\frac{n}{2}+1}} \sqrt{\pi} \qquad \text{for } n \text{ even} \qquad (E-4)$$

Differentiation with respect to K yields the differential relation

$$dI_n(K)/dK = I_{n-1}(K) \qquad (E\text{-}5)$$

Since only three $I_n(K)$'s are needed to determine all others through algebraic relations we choose to determine $I_0(K)$, $I_1(K)$ and $I_2(K)$. The function $I_{-1}(K)$ is not directly determined even though it comes in directly in the integral iteration scheme because it has a logarithmic singularity at $K=0$. The first object, therefore, is to evaluate

$$I_0(K) = \int_0^\infty \exp\left[-v^2 - \frac{K}{v}\right] dv$$

For small values of K this can be done by expanding the integrand in powers of K. It cannot, however, be done blindly as K/v is not necessarily small even for small K when v approaches zero. It is possible to first split up the integrations

$$I_0(K) = \int_0^1 \exp\left[-\frac{K}{v}\right]\left(1 - v^2 + \frac{v^4}{2!} - \frac{v^6}{3!} \cdots\right) dv$$

$$+ \int_1^\infty \exp\left[-v^2\right]\left(1 - \frac{K}{v} + \frac{K^2}{2v^2} - \frac{1}{3!}\left(\frac{K}{v}\right)^3 \cdots\right) dv \qquad (E\text{-}6)$$

By proper changing of variables these can be reduced to integrals that can be evaluated term by term in terms of polynomials and error functions

and Ei functions. (Ei functions are defined as

$$Ei(x) = \int_1^\infty \exp[-xt]\,\frac{dt}{t} \qquad (E-7)$$

Further expansion of these functions in powers of K, where possible, yields the results that $I_0(K)$ can, in general, be expressed as

$$I_0(K) = \sum_{m=0}^{\infty} \left(a_m^\circ + b_m^\circ \ln K\right) K^m$$

The coefficients can, in turn, be determined from the expansion mentioned. They are actually themselves determined in terms of infinite series which we can define as follows

$$\alpha_0 = \left(\frac{1!}{3!\times 1!} - \frac{3!}{5!\times 2!} + \frac{5!}{7!\times 3!} - \frac{7!}{9!\times 4!} + \frac{9!}{11!\times 5!} - \frac{11!}{13!\times 6!} \cdots\right)$$

$$\alpha_1 = \left(\frac{0!}{3!\times 1!} - \frac{2!}{5!\times 2!} + \frac{4!}{7!\times 3!} - \frac{6!}{9!\,4!} + \frac{8!}{11!\times 5!} - \frac{10!}{13!\times 6!} \cdots\right)$$

$$\alpha_2 = \left(-\frac{1!}{5!\times 2!} + \frac{3!}{7!\times 3!} - \frac{5!}{9!\times 4!} + \frac{7!}{11!\times 5!} - \frac{9!}{13!\times 6!} + \frac{11!}{15!\times 7!} \cdots\right) \qquad (E-8)$$

etc.

The general term is

$$\alpha_n = (-1)^{n/2} \sum_{l=0}^{\infty} \frac{(-1)^l\,(2l+1)!}{[n+1+2(l+1)]!\times(l+1+n/2)!}$$

for n even (E-9)

and

$$\alpha_n = (-1)^{\frac{n-1}{2}} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\,(2\ell)!}{[n+2(\ell+1)]!\,(\ell+1+\frac{n-1}{2})!}$$

for n odd (E-10)

From this we can determine new constants

$$c_n = \frac{1}{n!}\left(\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(1)\right) - \sum_{\ell=0}^{n-1} \frac{\alpha_\ell}{(n-1-\ell)!}$$

$$- \sum_{\ell=0}^{\frac{n}{2}-1} \frac{(-1)^{\ell}}{(n-1-2\ell)(n-1-2\ell)!\,(1+2\ell)!\,\ell!}$$

for n even (E-11)

$$c_n = \frac{\gamma\,(-1)^{\frac{n-1}{2}}}{n!\,(\frac{n-1}{2})!} - \frac{1}{n!}\left(\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(1)\right)$$

$$+ \sum_{\ell=0}^{n-1} \frac{\alpha_\ell}{(n-1-\ell)!} + \sum_{\ell=0}^{\frac{n-3}{2}} \frac{(-1)^{\ell}}{(n-1-2\ell)(n-1-2\ell)!\,(1+2\ell)!\,\ell!}$$

for n odd (E-12)

where

$$\gamma = \lim_{t \Rightarrow \infty} \left[1 + \frac{1}{2} + \frac{1}{3} \cdots\cdots \frac{1}{t} - \log t\right] = 0.5772157$$

and where all sums are assumed zero when the upper limit is lower in value than the lower limit. In addition, we need to calculate the additional constants e_n which are

$$e_0 = \frac{\sqrt{\pi}}{2}\left(1 - erf(1)\right)$$

$$e_1 = \frac{1}{2} Ei(1)$$

$$e_2 = \frac{1}{2!}\left[\exp(-1) - 2e_0\right] = \frac{1}{2!} B_2$$

$$e_3 = \frac{-1}{2}\frac{1}{3!}\left[\exp(-1) + 2e_1\right] = -\frac{1}{3!} B_3$$

and

$$e_n = \frac{\left[(n-3)(n-5)\cdots 1\right]\exp(-1) - B_{n-2}}{n!\left[(n-1)(n-3)\cdots 1\right]} = \frac{B_n}{n!\left[(n-1)(n-3)\cdots 1\right]} \qquad (E\text{-}13)$$

for n even and > 2

and

$$e_n = \frac{-1}{(n-1)n!}\left[\exp(-1) + \frac{2 B_{n-2}}{n-3}\right] = -\frac{1}{(n-1)!\,n!} B_n \qquad (E\text{-}14)$$

for n odd and > 3

With these constants it is now possible to evaluate a_n^o which is

$$a_n^o = c_n + e_n \qquad \text{(E-15)}$$

The value of b_n^o is much more easily obtained from the expansion and is

$$b_n^o = \frac{(-1)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!\, n!} \qquad \text{for } n \text{ odd} \qquad \text{(E-16)}$$

and $b_n^o \equiv 0$ for n even

When the calculated values of a_n^o and b_n^o are substituted into the expression for $I_o(K)$ it becomes

$$I_o(K) = .8862269 - (.1341766 - \ln K)K$$

$$- .8862269K^2 + (.2445846 - .1666667\ln K)K^3$$

$$+ .04923503K^4 - (9.031340\times10^{-3} - 4.166667\times10^{-3}\ln K)K^5$$

$$- 6.569535\times10^{-4}K^6 + (8.742174\times10^{-5} - 3.306878\times10^{-5}\ln K)K^7$$

$$+ 3.372035\times10^{-6}K^8 - (1.509185\times10^{-7} - 1.148222\times10^{-7}\ln K)K^9 \qquad \text{(E-17)}$$

If we examine the next term we see that it is about 1×10^{-8}. Thus, even for $K = 2.5$ the above expansion neglects only terms of order 10^{-4} and below. This makes the above expansion up to the ninth term accurate to .01 percent up to $K = 2.5$.

It is now necessary to find a means of evaluating $I_0(K)$ for large K. This can be done most easily by obtaining the asympotic series by the method of steepest descents. The generalized method given in Ref. 35 consists of making the substitution

$$f(t) = f(t_0) - w^2$$

where t_0 is the point at which $\partial f/\partial t = 0$ and where the desired integral is

$$\int_C \exp\left[z f(t)\right] dt$$

The substitution results in the integral

$$\exp\left[z f(t_0)\right] \int_C \exp\left[-z w^2\right] \frac{dt}{dw}\, dw$$

where dt/dw can be evaluated in terms of a power series

$$\frac{dt}{dw} = \sum_{n=0}^{\infty} a_n w^n$$

The answer is expressible in terms of the coefficients a_n as follows

$$\exp\left[z f(t_0)\right] \sqrt{\frac{\pi a_0^2}{z}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})} \left(\frac{1}{z}\right)^n \frac{a_{2n}}{a_0}$$

and the coefficients a_{2n} are in turn expressible in terms the derivatives of $f(t)$ at $t=t_0$. When this method is applied to evaluating $I_0(K)$ we choose $z=1$, $f(t)=-\left(t^2+\frac{K}{t}\right)$. The solution of $f'(t_0)=0$ becomes

$$2t_0-\frac{K}{t_0}=0 \qquad \text{or} \qquad t_0=\left(\frac{K}{2}\right)^{1/3} \tag{E-18}$$

The values of the coefficients a_{2n} become

$$a_0=\frac{1}{\sqrt{3}}$$

$$a_2/a_0=-\frac{1}{18}\left(\frac{2}{K}\right)^{2/3}$$

$$a_4/a_0=+.01285\left(\frac{2}{K}\right)^{4/3}$$

$$a_6/a_0=-.002333\left(\frac{2}{K}\right)^{2}$$

and the value of $I_0(K)$ becomes

$$I_0(K)=\exp\left[-3\left(\frac{K}{2}\right)^{2/3}\right]\sqrt{\frac{\pi}{3}}\left[1-.0277778\left(\frac{2}{K}\right)^{2/3}\right.$$

$$\left.+.0096450\left(\frac{2}{K}\right)^{4/3}-.00437499\left(\frac{2}{K}\right)^{2}\cdots\right] \tag{E-19}$$

It is important to realize that this is an asympotic series and that one should always retain only the terms up to the one where the series begins to diverge. Fortunately, in our case for values of $K > 2.5$ the last term is always the smallest. Furthermore, we can increase the accuracy of the answer by averaging over the series with and without the last term. This makes the expected maximum error of the order of 1/2 the last term which, in our case, becomes

$$.0021873\left(\frac{2}{K}\right)^2$$

or roughly .1 percent. This appears not as accurate at this point as the series expansion from $K = 0$ but we compared the term in that expansion to the initial value of the function whereas here we are comparing it to the actual value of the function at $K = 2.5$. At this point the value of the junction is only about 4 percent of the initial value at $K = 0$. Thus, based on the same scale the error becomes of the order of .01 percent. We have shown that in order to have the complete set of $I_n(K)$'s available it is only necessary to evaluate $I_0(K)$, $I_1(K)$ and $I_2(K)$ and obtain the rest through the recursion relation to the same accuracy. We have, however, also the integral relations which can be used to evaluate $I_1(K)$ and $I_2(K)$ from $I_0(K)$. For the expansion from $K = 0$ it is only necessary to integrate the expression and evaluate the zero order term to obtain $I_1(K)$. A second integration will, in turn, yield $I_2(K)$. In general, the following recursion relation is satisfied by the coefficients in the expansion of $I_n(K)$ for positive n.

$$a_0^n = \frac{1\cdot 3\cdot 5 \cdots K-1}{2^{\frac{K}{2}+1}}\sqrt{\pi} \qquad n \text{ even}$$

$$a_0^n = \left(\frac{n-1}{2}\right)!/2 \qquad n \text{ odd}$$

$$b_0^n = 0$$

$$a_\ell^n = -\left(\frac{a_{\ell-1}^{n-1}}{\ell} - b_{\ell-1}^{n-1}/\ell^2\right) \qquad \ell \geq 1$$

$$b_\ell^n = -b_{\ell-1}^{n-1}/\ell \qquad \ell \geq 1$$

The reason the I_n's for negative values of n cannot be obtained by this process is the fact that though series can be integrated term by term they cannot, in general, be differentiated term by term. These functions can, however, be obtained by use of the three term recursion relation on the functions themselves.

The I_n's for positive n's and large K can also be obtained by direct integration but the recursion relations are different for even and odd orders of n. Furthermore, it becomes quickly apparent that the accuracy is decreased as n increases. We, therefore, evaluate here only the $I_1(K)$ and $I_2(K)$ by this method. If we define the variable

$$x = \sqrt{3}\left(\frac{K}{2}\right)^{1/3}$$

The expression for $I_0(K)$ can be written as

$$I_0(x) = \sqrt{\frac{\pi}{3}}\, e^{-x^2}\left[1 + d_1^0/x^2 + \frac{d_2^0}{x^4} + \frac{d_3^0}{x^6} \cdots\right] \qquad (E-20)$$

The expressions for I_1 and I_2 in this notation become

$$I_1(x) = \sqrt{\frac{\pi}{3}}\left(\frac{x}{\sqrt{3}}\right)\exp(-x^2)\left[1 + \frac{d_1'}{x^2} + \frac{d_2'}{x^4} + \frac{d_3'}{x^6} \cdots\right] \qquad (E\text{-}21)$$

where

$$d_1' = \frac{1 + 2d_1^\circ}{2}$$

$$d_2' = -\frac{1 + 2d_1^\circ + 4d_2^\circ}{4}$$

$$d_3' = 3\left(\frac{1 + 2d_1^\circ - 4d_2^\circ + 8d_3^\circ}{8}\right)$$

and

$$I_2(x) = \sqrt{\frac{\pi}{3}}\left(\frac{x^2}{3}\right)\exp(-x^2)\left[1 + \frac{d_1^2}{x^2} + \frac{d_2^2}{x^4} + \frac{d_3^2}{x^6} \cdots\right] \qquad (E\text{-}22)$$

where

$$d_1^2 = 1 + d_1'$$

$$d_2^2 = d_2'$$

$$d_3^2 = d_3' - d_2'$$

The functions $I_0(K)$, $I_1(K)$, $I_2(K)$ were calculated by the above derived formulas as follows: for K from zero to 2.5 they were calculated by the expansion up to 9th order in K while for values of $K > 2.5$ they were calculated from the asymptotic expansions, up to the fourth term. The results are tabulated in a form normalized by the value for $K = 0$. These normalized functions called $\Pi_n(K)$ are shown in Table E-1 and are also plotted in Fig. E-1 to facillitate comparison.

It is interesting to notice here that the higher order functions decay more gradually at large values of K than do the lower order ones. Whereas the $\Pi_0(K)$ function is down to one percent at $K = 3.95$ the $\Pi_1(K)$ function does not come down to one percent until $K = 5.23$ and $\Pi_2(K)$ not until $K = 6.23$. This behavior is, of course, expected as the higher moments represented by the higher values of n should have their influence extend further because of the heavier weighting on the higher velocities. Physically, this will mean that the influence of the temperature at a point is felt further away than the influence of the density.

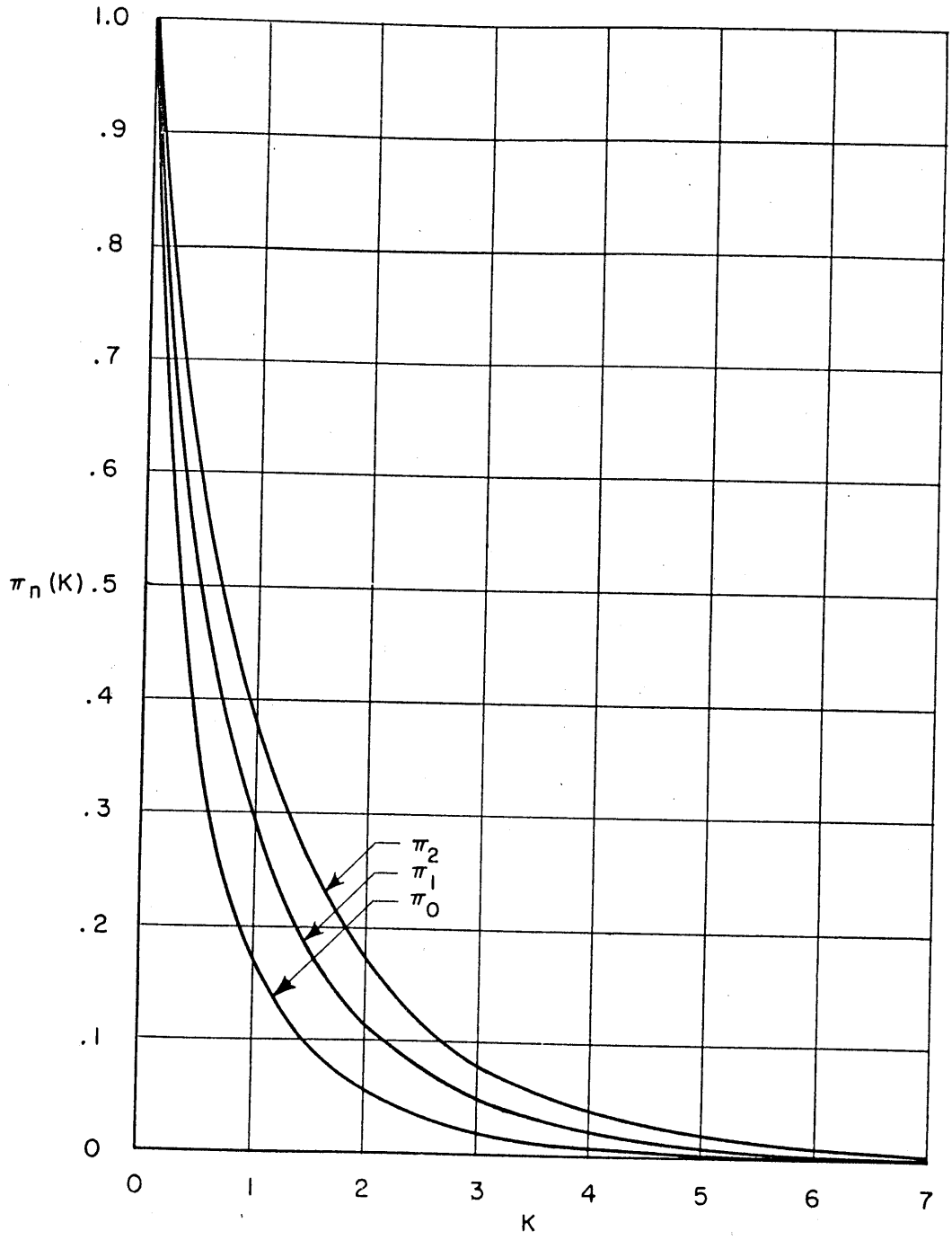


Figure E-1. π functions

Table E-1

$\pi_n(\kappa)$ Function

κ	π_0	π_1	π_2
0	1	1	1
.1	.71575	.85268	.88812
.2	.57123	.73953	.80632
.3	.47099	.64764	.72823
.4	.39579	.57111	.65960
.5	.33709	.50634	.59891
.6	.28988	.45093	.54498
.7	.25120	.40311	.49685
.8	.21918	.36147	.45378
.9	.19215	.32514	.41506
1.0	.16998	.29313	.38023
1.1	.14976	.26495	.34878
1.2	.13229	.23988	.32033
1.3	.11842	.21764	.29454
1.4	.10574	.19820	.27113
1.5	.09469	.18004	.24984
1.6	.08541	.16459	.23044
1.7	.07672	.14974	.21275
1.8	.06915	.13675	.19659
1.9	.06250	.12545	.18180
2.0	.05632	.11480	.16824
2.5	.03488	.07534	.11538
3.0	.01937	.05050	.08039
3.5	.01434	.03446	.05677
4.0	.00973	.02386	.04053
5.0	.00454	.01187	.02124
6.0	.00222	.00554	.01039
7.0	.00112	.00326	.00636
8.0	.00060	.00178	.00360
9.0	.00032	.00100	.00199
10.0	.00018	.00037	.00122

BIOGRAPHICAL NOTE

The author was born in Bialystok, Poland on March 25, 1937. After many moves resulting from World War II he came to the United States from England in October 1948. He graduated from the Utica Free Academy in Utica, New York in June 1954. He enrolled the same year at the Massachusetts Institute of Technology and graduated in June 1959 with the degrees of Bachelor of Science and Master of Science in Aeronautical Engineering. He embarked upon the doctoral program at the Massachusetts Institute of Technology in the department of Aeronautics and Astronautics in September of the same year. For the past year he has worked as a Research Assistant at the Aerophysics Laboratory, M.I.T. He is married and has one daughter.