

# Dynamic Pricing in a Competitive Environment

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**Abstract**—We present a dynamic optimization approach for perishable products in a competitive and dynamically changing market. We build a general optimization framework that ties together the competitive and the dynamic nature of pricing. This approach also allows differential pricing for large customers as well as demand learning for the seller. We analyze special cases of the model and illustrate the policies numerically.

**Keywords**—Pricing, Competition, Dynamic programming

## I. INTRODUCTION

Over the years, researchers from a variety of fields have studied pricing theory extensively. These fields include economics (see for example, [34]), marketing (see for example, [23]), revenue management (see for example, [25]) and telecommunications (see for example, [19], [20], [27], [30], [31]) among others. In particular, in recent years, the popularity of the Internet as a marketplace for a wide variety of products, has accelerated the development of new pricing theories and rekindled the interest in dynamic pricing. E-commerce has had a very influential role in the development of pricing and revenue management.

The overall goal of this paper is to present and study a model for a firm which is pricing a product in a dynamic and competitive environment as well as discuss the pricing policies the model generates. Some key features of our model are that the relationship of demand with price is not known *a priori*, but is “learnt” over time. In particular, our model considers the case of two competitive firms. Each of these firms have a known inventory (a fixed total capacity each) of the product which they can sell in the market over the entire time horizon. They compete with each other in a common market where the pricing policies for the product are dynamic over the finite time horizon. As a result our research (contrasted with traditional revenue management literature) takes capacity as given and considers pricing decisions.

### Problem Characteristics

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The pricing problem in this paper has the following features:

(a) We consider a *perishable* product. That is, there is a finite time horizon within which the firm has to sell its stock of products. After this period any unsold inventory or unused capacity is lost. Moreover, *the marginal cost* of an extra unit of demand is relatively *small*. For this reason, our model ignores the cost component in the decision-making process and refers to revenue maximization rather than profit maximization.

(b) We explicitly allow competition in an *oligopolistic* market; that is, a market characterized by a few firms on the supply side and a large number of buyers on the demand side. The theory of oligopoly dates back to the work of Augustin Cournot [10], [11], [12]. A key feature of such a market (in contrast to a monopoly) is that the profit one firm receives depends not just on the prices it sets, but also on the prices set by the competing firms. There is no perfect competition in an oligopolistic market since decisions made by all the firms in the market impact the profits received by each firm. One can consider a cooperative oligopoly (where firms collude) or a non-cooperative oligopoly. In this paper we focus on the latter.

(c) Products are priced *dynamically* over a finite time horizon. This is an important aspect since the demand and the data of the problem evolve dynamically. Most research in the pricing arena does not consider the dynamic and the competitive aspects of the pricing problem jointly. An exception to this involves some work that applies differential game theory (see [1], [2], [7]).

(d) The model we consider allows firms to exercise price discrimination for their larger customers (e.g. offer bulk discounts) by allowing pricing to depend on the order size.

(e) We consider a model where the demand response to the price is not known *a priori*. We assume that this relationship is learnt over time. This part of the model deals with demand learning as the firm acquires more information over time. That is, we exploit the fact that over time firms are able to acquire knowledge regarding demand behavior that can be utilized to improve profitability. Much of the cur-

rent research does not consider this aspect but rather considers demand to be an exogenous stochastic process following a certain distribution. See [5], [6], [8], [9], [14], [15], [17], [27].

### *Application Areas*

There are many markets where the framework we consider in this paper applies. Examples include airline ticket pricing. In this market the products the consumers demand, are the origin-destination (O-D) pairs during a particular time window. The resources are the flight legs (more appropriately seats on a particular flight leg) that have limited capacity. There is a finite horizon to sell the products, after which any unused capacity is lost (perishable products). The airlines compete with one another for the product demand that is stochastic in nature. Other industries sharing the same features include the service industry (for example, hotels, car rentals, and cruise-lines), the retail industry (for example, department stores) and finally, pricing in an e-commerce industry, where products range from commercial bandwidth in a communications network to production capacity at a plant. All these industries attempt to intelligently match capacity with demand via *revenue management*. A review of the huge literature in revenue management can be found in [25], [32] and [33].

### *Problem Statement*

Consider a market for a single perishable product. There are  $n$  firms, indexed by  $i = 1, \dots, n$  with capacity  $C_i$  respectively, for the product. We assume there is no factor distinguishing the product of one firm versus another, other than price. That is, customer preferences for the firm depend only on the price offered. The total time horizon till the entire capacity of the product perishes is  $T$ . Finally, due to the perishability of the product, we assume that at the end of the time horizon, the salvage value of any unsold product is zero.

The total demand of the product in the market (that is, the total arrival of customers) is determined by the prices  $p_i$  that each firm  $i$  sets. That is, total demand is based on an exogenous rate  $\lambda$  which is modulated by the price settings in a manner dictated by a function  $f(\cdot)$ . In general, our model easily extends to allow this exogenous rate to be time dependent (i.e.  $\lambda(t)$ ). The function  $f(\cdot)$  is a non-increasing function of the prices. It starts from the value of one when the prices are zero and reduces to zero as prices are increased.

In this work we assume that the size of the order need not be one unit, but rather follows a distribution. This distribution could be any general discrete distribution (e.g. Geometric) and ideally (though not necessarily) expected to be decreasing as order size increases. The resulting overall demand for the product is then split among the individual demands for each firm depending on the relative pricing dictated by the customer sensitivity to price. We represent this splitting through function  $g(\cdot)$ .

Examples of functions  $g$  include the constant elasticity of substitution function (CES) used commonly by economists to mimic customer behavior. In this paper we use this function to illustrate our results. CES functions are traditionally used as production functions in economics. They map the utility of production factors (like capital and labor) to calculate the production quantity. CES functions were originally introduced by Arrow, Chenery, Minhas and Solow in 1961. As a result, they are also known as ACMS functions. They were generalized to the  $n$ -factor case by Uzawa and McFadden in 1963. A CES function as the name indicates, displays a constant elasticity throughout. This translates to price sensitivity of the customers in our problem context. That is, in the case of two competing firms,  $g(p_1, p_2) = \frac{p_2^\alpha}{p_1^\alpha + p_2^\alpha}$ . This is the fraction of the market that is captured by Firm 1 with prices  $p_2$  for Firm 2. The price sensitivity of the customers is modeled using  $\alpha$  - a higher value indicates higher sensitivity. Besides the value of  $\alpha$ , the value of  $\lambda$  and the demand size with respect to the price are inputs to the model. These parameters could be determined through a learning procedure that can easily be linked to the DP algorithm we will introduce below.

The pricing policy a firm sets at each point of time, does not only change as the deadline draws nearer, but also changes due to the prices set by the competitors. Moreover, the pricing policy a firm sets, changes depending on the unsold capacity of the product at that point of time. Finally, the firm need not in general, have a fixed price rate for orders of all sizes, but could offer *bulk discounts* for larger orders.

### *Problem Formulation*

More formally, the problem we study in this research can be stated as follows.

There are  $n$  firms, indexed by  $i = 1, \dots, n$ , competing for a product in a common market. Each firm  $i$  has total capacity  $C_i$ , and total time horizon  $T$  to sell the product. Notice that after this time the entire

stock of the product perishes, as it becomes obsolete. Let  $D_{kx}$  denote the total number of orders of size  $x$  in period  $k = T, \dots, 1$ . These orders are divided amongst the firms ( $d_{kx}^i$  for firm  $i$ ) according to some splitting function  $g(\cdot)$  such that the vector  $[d_{kx}^i]_i = g([p_{kx}^i]_i)$ . As an example, in the case of two competitive firms

$$[d_{kx}^1, d_{kx}^2] = \left[ \frac{(p_{kx}^2)^\alpha}{(p_{kx}^1)^\alpha + (p_{kx}^2)^\alpha}, \frac{(p_{kx}^1)^\alpha}{(p_{kx}^1)^\alpha + (p_{kx}^2)^\alpha} \right].$$

We denote by  $p_{kx}^i$  the price set by firm  $i$  in period  $k$  for an order of  $x$  units. The total market size is also determined by the prices set by the firms (that is, function  $f(\cdot)$ ). The pricing policy followed by the competitors is  $p_{kx}^i, i \neq 1$ , while the control for Firm 1 is its own pricing policy,  $p_{kx}^1$ .

$$\begin{aligned} & \max_{p_{kx}^1} \quad \sum_{k=1}^T \sum_x d_{kx}^1 \cdot p_{kx}^1 \\ \text{such that} \quad & \sum_{i=1}^n d_{kx}^i = D_{kx} \quad \forall k, x \\ & D_{kx} \sim P(\cdot | f(p_{kx}^i)) \quad \forall k, x \\ & [d_{kx}^i]_i = g([p_{kx}^i]_i) \quad \forall k, x \\ & \sum_{k=1}^T \sum_x d_{kx}^i \cdot x \leq C_i \quad \forall i \\ & p_{kx}^1 \geq 0 \quad \forall k, x \end{aligned}$$

#### A First Dynamic Programming Formulation

Our goal for the remainder of this paper is to study the previous model. To achieve this, we will take a dynamic programming approach. In particular, we will focus on the case of two competing firms ( $n = 2$ ). As a result, in the remainder of the paper, we refer to Firm 2 as the *competitor*. For the sake of simplicity, we assume that the form of distribution for the demand function is known. Nevertheless, we could also incorporate a learning procedure to the model in order to learn its parameters. There is an exogenous rate of arrival for customers denoted by  $\lambda$  with order sizes which have a known probability distribution (denoted by  $h(x) = \text{probability of the order size being } x$ ). We denote by  $p_i(t)$  the price set by Firm  $i = 1, 2$  when there are time  $t$  units from the end of the time horizon. In this problem,  $p_2(t)$  is the pricing policy of the competitor (assumed known) and  $p_1(t)$  is the control. The current capacity of Firm  $i$  with time  $t$  remaining is  $C_i^t$ .

As a result, the expected revenue earned by Firm 1 before the deadline expires is expressed through function  $J(C_1^t, C_2^t, t)$ . Notice that this function depends not only on the pricing policy of Firm 1 but also on that of the competitor firm.

In what follows, we consider a dynamic programming (DP) algorithm. In particular, for a small time interval  $(t, t - \delta)$ , the DP algorithm is described as follows:

#### A Two Dimensional DP Algorithm

The cost-to-go function  $J(C_1^t, C_2^t, t)$  becomes

$$\begin{aligned} J(C_1^t, C_2^t, t) &= \lambda \delta \sum_x h(x) f(p_1, p_2) [g(p_1, p_2) \\ &\quad \cdot (p_1(C_1^t, C_2^t, x, t) \\ &\quad + J(C_1^{t-\delta} - x, C_2^{t-\delta}, t - \delta)) \\ &\quad + (1 - g(p_1, p_2)) \\ &\quad \cdot J(C_1^{t-\delta}, C_2^{t-\delta} - x, t - \delta)] \\ &\quad + (1 - \lambda \delta \sum_x h(x) f(p_1, p_2)) \\ &\quad J(C_1^{t-\delta}, C_2^{t-\delta}, t - \delta) \end{aligned} \quad (2)$$

Alternatively, we can rewrite this dynamic program as follows

$$\begin{aligned} J(C_1^t, C_2^t, t) &= \lambda \delta \sum_x h(x) f [g(p_1(x) \\ &\quad + J(C_1^{t-\delta} - x, C_2^{t-\delta}, t - \delta)) \\ &\quad + (1 - g) J(C_1^{t-\delta}, C_2^{t-\delta} - x, t - \delta)] \\ &\quad + (1 - \lambda \delta \sum_x h(x) f) \\ &\quad \cdot J(C_1^{t-\delta}, C_2^{t-\delta}, t - \delta). \end{aligned} \quad (3)$$

The previous equation leads us to the following differential equation for the cost-to-go function  $J(\cdot)$ .

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \frac{J(C_1^t, C_2^t, t) - J(C_1^{t-\delta}, C_2^{t-\delta}, t - \delta)}{\delta} \\ &= \lambda f \sum_x h(x) [g(p_1(x) + J(C_1^t - x, C_2^t, t)) \\ &\quad + (1 - g) J(C_1^t, C_2^t - x, t) - J(C_1^t, C_2^t, t)]. \end{aligned} \quad (4)$$

This leads us to the following partial differential equation,

$$\begin{aligned} \frac{\partial J(C_1^t, C_2^t, t)}{\partial t} &= \lambda \sum_x h(x) f [g(p_1(x) \\ &\quad + J(C_1^t - x, C_2^t, t)) + \\ &\quad (1 - g) J(C_1^t, C_2^t - x, t) \\ &\quad - J(C_1^t, C_2^t, t)], \end{aligned} \quad (5)$$

with boundary conditions,

$$J(C_1^0, C_2^0, 0) = 0 \quad (6)$$

$$J(0, C_2^t, t) = 0 \quad (7)$$

This leads us in turn to

$$\begin{aligned}
J_{C_1^t, C_2^t}(t) &= e^{-\lambda t} \sum_x h(x) f \\
&\cdot \int_0^t e^{\lambda \tau} \lambda \sum_x h(x) f [g(p_1(x) \\
&+ J_{C_1^t-x, C_2^t}(\tau)) \\
&+ (1-g) J_{C_1^t, C_2^t-x}(\tau)] d\tau \quad (8)
\end{aligned}$$

That is,

$$\begin{aligned}
J_{C_1^t, C_2^t}(t) &= e^{-\lambda t} \sum_x h(x) f(p_1(x), p_2(x)) \\
&\cdot \int_0^t e^{\lambda \tau} \lambda \sum_x h(x) f(p_1(x), p_2(x)) \\
&[g(p_1(x), p_2(x)) (p_{1, C_1^t-x, C_2^t}(x, \tau) \\
&+ J_{C_1^t-x, C_2^t}(\tau)) \\
&+ (1-g(p_1, p_2)) J_{C_1^t, C_2^t-x}(\tau)] d\tau, \quad (9)
\end{aligned}$$

where  $p_i(x)$ ,  $i = 1, 2$  are the prices for order sizes  $x$  for Firm  $i$ ,  $i = 1, 2$  and vary with time.

To gain insight, we have solved this differential equation in closed form for several special cases. For the general case, we have solved this equation numerically. We have taken two approaches in the numerical solution of the problem. The first approach assumes that the competitor's pricing policy is given. This for example may be the case, if the competitor is a follower and implements our previous period prices. A second and more sophisticated approach, is to apply the DP algorithm "iteratively" in order to arrive at an optimal policy for each firm. These iterations consist of alternating in solving the corresponding DP for each firm and using at each iteration an updated pricing policy for the competitor found from the previous iteration.

### Numerical results

To illustrate our results, in what follows we consider a particular numerical example. In particular, we consider a problem with the following parameters.

$$\begin{aligned}
C_1 &= 25 \\
C_2 &= 25 \\
\lambda &= 1 \text{ per day} \\
T &= 5 \text{ days} \\
X &= \{1, 2, 3, 4, 5\} \text{ order sizes} \\
h(x) &= \frac{6-x}{15}, \forall x = 1, \dots, 5
\end{aligned}$$

(Probability distribution of order sizes)

$$\begin{aligned}
\delta &= 0.01 \\
g(p_1, p_2) &= \frac{p_2^\alpha}{p_1^\alpha + p_2^\alpha} \\
\alpha &= 2 \\
f(p_1, p_2) &= \max(0, 1 - \frac{p_1^\beta + p_2^\beta}{2p_{\text{cutoff}}^\beta}) \\
\beta &= 2 \\
p_{\text{cutoff}} &= 5
\end{aligned}$$

The function  $g(\cdot)$  is the CES function discussed earlier. The function  $f(\cdot)$  is a simple approximation to a market where the demand monotonically falls to zero as the average prices are pushed to a level given by  $p_{\text{cutoff}}$ . Using this data, we solved the DP using simulation. The solution took approximately 1.5 hours. This is primarily attributed to the dimensionality of the DP formulation. The results of the simulation gave rise to the following (cost-to-go) function for the optimal expected revenue earned by Firm 1. This is illustrated in the next figure, when the remaining time horizon is  $t = 5$ . In general, we obtain similar results for choices of  $t = 1, \dots, 5$ .

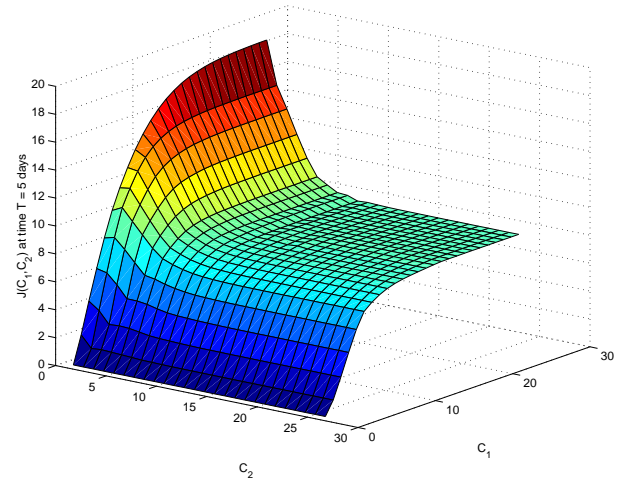


Fig. 1. The form of the cost-to-go function  $J(\cdot)$  for Firm 1 at time = 5 days

This example seems to suggest that the optimal value of the expected return flattens out as the remaining capacities  $C_1$  and  $C_2$  for the two firms increase beyond the average demand. This is the case when the supply exceeds the demand and there would be unsold quantity at the end of the deadline. The figure also shows clearly that the profits of Firm 1 are absorbed by Firm 2 as its capacity  $C_2$  is increased.

Fig. 2 illustrates how Firm 1's optimal pricing policy changes with capacity for various level of order

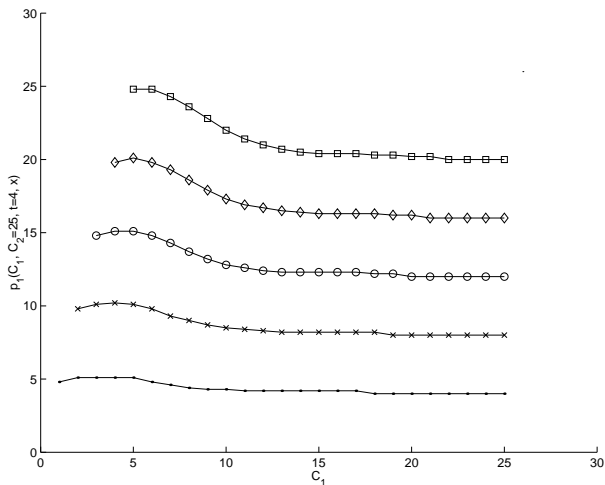


Fig. 2. Firm 1's pricing policy  $p_1(\cdot)$  as it varies with capacity (keeping the competitor's capacity  $C_2$  fixed)

size  $x = 1, \dots, 5$ . The optimal price flattens out to an asymptotic value as Firm 1 becomes over capacitated and the market becomes a buyer's market.

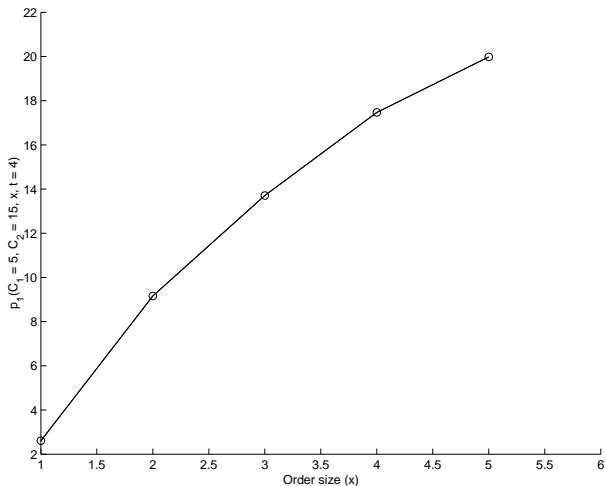


Fig. 3. Observation of bulk discounting by Firm 1 for fixed  $C_1, C_2$  at  $t = 4$  days

Fig. 3 illustrates how the price per unit changes with order size when the capacity for each firm is fixed. We observe that the price rate per unit decreases for bulk orders. This is true because each firm would benefit from large sales and would like to lure bigger customers. Note however, that immediately after a big order is executed the capacity is reduced so the firm can hike up prices immediately.

In the previous model, if we consider the special case where the order size is equal to one, we obtain similar results as before. In particular, the cost-to-go function for Firm 1 as well as the optimal pricing

policy for Firm 1, have a similar behavior as in Figure 1 and Figure 2 respectively. Nevertheless, in this case we cannot consider bulk discounting as in Figure 3.

In summary, our numerical results lead us to conclude the following:

- The optimal cost-to-go (that is, the total expected revenue) for Firm 1 flattens out after some level of capacity.
- The optimal pricing policy for Firm 1 stabilizes after some high enough capacity level.
- Bulk discounting affects the optimal pricing policy.

### Contributions

This paper introduced a general framework for pricing a product in a dynamic as well as competitive environment. Our model allows differential pricing for large buyers in order to incorporate bulk discounts. In addition our model allows demand learning for the firm. We derive closed form solutions for some special cases. Moreover, we analyze the model and determine pricing policies for the general case numerically. Our numerical results match the closed form solution of the special cases. The framework we introduce is flexible and allows us to incorporate a large variety of real world problems.

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