GENERALIZED STRAIGHTENING LAWS FOR PRODUCTS OF DETERMINANTS

by

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Abstract

The (semi)standard Young tableau have been known since Hodge and Littlewood to naturally index a basis for the multihomogeneous coordinate rings of flag varieties under the Plücker embedding. In representation theory, the irreducible representations of $GL_n(\mathbf{C})$ arise as the multihomogeneous components of these rings.

I introduce a new class of *straight* tableau by slightly weakening the requirements for standard tableaux. The straight tableau are defined for the more general class of *row-convex* shapes. I show for any row-convex shape that these tableaux index a basis for the associated representation. I provide an explicit straightening algorithm for expanding elements of the representation into this basis. For skew shapes, this algorithm specializes to the classical straightening law. I define the *anti-straight* tableaux, which provide a similar basis and straightening law for the kernels of the projection maps in certain James-Peel complexes.

The above results allow me to provide degree 2 Groebner bases for the homogeneous coordinate rings of certain of the Magyar "configuration varieties." Further, I provide canonical subalgebra (or SAGBI) bases for these rings.

I establish a quantum version of the basis results and straightening algorithms, using a new notion of SAGBI bases suitable for this non-commutative setting. A benefit of this technique is a new (and strengthened) proof of the Huang-Zhang standard basis theorem for quantum bitableaux.

Rota and his collaborators have generalized the commutative constructions for $GL(V)$ representations to the setting of superalgebras and representations of the general linear Lie superalgebras. The notions of straight and anti-straight tableaux are defined in this general context, thus allowing the case of Weyl and Schur modules to be handled concurrently. I show in this setting how the basis indexed by the straight tableaux is naturally related to a basis for the dual representation given by a supersymmetric version of the Reiner-Shimozono decomposable tableaux.

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Mathematics is of course comprised of people as well as ideas. I owe much to the support of more mathematicians than I can readily list–thank you all. I owe especial thanks to Richard Ehrenborg, Gabor Hetyei, Lauren Rose and Marguerite Eisenstein-Taylor. The last has listened to me rehearse, and then ruthlessly improved, each of my talks. I would be remiss in not thanking Steve Maurer, my mentor at Swarthmore, for first showing me that mathematics was beautiful, that algorithms were part and parcel of it, and that one could be a good parent whilst being a mathematician.

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To my family; Marguerite, Rebecca, Dad, Mom

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0.3 Introduction

The focus of this thesis is the study of straightening laws and bases for the representations of GL_n (and its generalizations) that are defined by "row-convex" shapes. These representations are constructed by a process (originating with Deruyts and described in general by Akin, Buchsbaum, and Weyman) of "antisymmetrizing along rows" and then "symmetrizing along columns." It has long been known in the case the the shapes involved consist of the Ferrer's diagrams associated to partitions, the representations created are precisely the (characteristic zero) irreducible representations. Study of the slightly more general class of skew shapes dates to the work of Littlewood. It is only in recent years however that the representations associated to more general shapes have been actively considered, most notably by Magyar, Reiner, and Shimozono. As a result of their work, we now know how to construct a basis, indexed by tableaux of any fixed "%-avoiding" shape, for the representation associated to that shape. The study of this basis relies on the use of the "row-reading word" of these tableaux–a sequence formed by reading off the entries of the rows of a tableau. Since the shape of the tableaux is known, these row-reading word of a tableau determines the tableaux.

Simultaneously with this work, there has some study of tableaux in terms of their column-reading words. For partition and skew shapes, the symmetry of the standard tableaux for makes this approach combinatorially equivalent to the use of row-reading words. However, one achieves significant algebraic gains from this approach, namely a SAGBI basis algorithm presented by Sturmfels and a deformation theory for coordinate rings of flag varieties developed by Rippel and (using different machinery) by Sturmfels. Nevertheless, to generalize these results to more general shapes, one need replace the notion of a "column word" with some "modified column word," which lists the entries in each column successively but with the entries from a given column appearing in sorted order. The general philosophy of this thesis then is that one should look for bases indexed by tableaux which are determined by their modified column words. Rephrasing, in any basis, there should exist at most one tableaux with the content of it columns given. A first, and very beautiful, step in this direction are the results of Woodcock in which he shows that for "almost-skew" shapes, such bases can be found where the rows of the indexing tableaux still strictly increase. One virtue inherent in the study of tableaux by modified column words is that it becomes trivial to determine a basis for the "flagged" B_n-representation associated to a given shape; one discards precisely the basis elements in which an element of the ith column exceeds i.

David Buchsbaum first suggested to this author that the results of Woodcock might be strengthened and generalized. In particular, Woodcock's result is tantalizingly incomplete–to produce one of his bases, one needs make numerous arbitrary choices. In Chapter 1, I introduce a combinatorially natural way to make those choices. In the process, I produce a "straight" basis for any module defined by a row-convex shape. Furthermore, I produce a two-rowed straightening algorithm for expressing any element of the representation in terms of this new "straight" basis. Straightening algorithms are unknown for Magyar and Lakshmibai's %-avoiding bases, and even in the case of the column-convex representations studied by Reiner and Shimozono, no two-rowed straightening algorithm is known, although theoretical results of Magyar guarantee the relations among the module elements are generated by two-row relations. I further show how to produce an "anti-straight" basis, again determined by modified column words but in which the shape is permitted to vary. Anti-straight tableaux are then used to index modules determined by collections of row-convex shapes. All of these constructions are characteristic-free and valid for super-Schur modules, the superalgebra generalization of the Weyl and Schur modules.

In Chapter 2, I show how the techniques developed in Chapter 1 produces Groebner and Subalgebra Analogues to Groebner Bases for Ideals bases for commutative (and non-commutative) rings generated by the super-Schur modules. In the process, I develop the notion–of independent interest–of a non-commutative SAGBI basis. As an application, I use the SAGBI bases produced in this chapter to prove the Cohen-Macaulayness of certain rings generated by products of determinants.

Chapter 3 is devoted to the development of quantum analogues of the straight and anti-straight bases and their straightening algorithms. It relies heavily on the term orders developed for non-commutative SAGBI bases in Chapter 2. This technique allows us to obtain a quicker proof (in the negative letter case) of the Huang-Zhang standard basis theorem for quantum bitableaux. This new proof strengthens their result to show that quantum bitableaux expand into standard bitableaux with shapes that are longer in the *dominance* order. I close with some short exact sequences for quantized Schur modules. In particular, I obtain quantum versions of the Akin-Buchsbaum and some of the Klucznik short-exact sequences used in the construction of characteristic free resolutions for Schur modules Indeed, the use of anti-straight bases for modules determined by collections of shapes allows me to generate short-exact sequences new even in the unquantized case.

Finally, in Chapter 4, I collect a number of results concerning the initial terms of super-Schur modules. Although, remarkably, the straight tableaux (still of row-convex shape) are distinct from the tableaux obtained by Magyar and by Reiner and Shimozono, there is a simple bijection between them and the "decomposable" column-convex tableaux of Reiner and Shimozono. I prove an algebraic version of this correspondence that applies to any basis which satisfies the generalized Woodcock property of corresponding to row-standard tableaux with distinct modified column words. I further show how the straight tableaux give canonical decompositions of Reiner and Shimozono's recording tableaux and give another proof of a corner-cell recurrence due to Reiner. I close with some conjectures on the initial terms showing up in any %-avoiding diagram motivated by some combinatorial results on initial terms of row-convex and two-rowed tableaux.

Chapter 1

Straightening row-convex tableaux

1.1 Introduction

The irreducible representations of S_n and GL_n (in characteristic 0) were constructed by Deruyts as vector spaces spanned by the products of determinants associated to certain sets of minors of a generic matrix. This chapter begins with a compact exposition of the *letter-place superalgebra* of Grosshans-Rota-Stein [GRS87]. Within this context we define a supersymmetric generalization of the Deruyts construction. In particular, to any generalized diagram D (that is any subset of $\mathbf{Z}^+ \times \mathbf{Z}^+$) we associate a representation $\mathcal{S}^D$ of a general linear Lie superalgebra. This definition is the natural generalization of the Schur and Weyl modules defined in Akin-Buchsbaum-Weyman [ABW82]. In recent years considerable progress studying the GL_n representations corresponding to generalized diagrams has been made by Victor Reiner and Mark Shimozono and by Peter Magyar and V. Lakshmibai.

In this chapter I consider row convex diagrams D–that is diagrams where if two cells are present in row l of D, then all intervening cells are present in D. I define a class of "straight tableau" recognizable by local combinatorial criteria and present a straightening algorithm for expanding row convex tableaux of shape D into linear combinations of straight tableaux of shape D. This algorithm generalizes the usual straightening law for expanding skew (or partition-shaped) tableaux into linear combinations of standard skew (or partition) tableaux. The straight basis theorem solves a problem posed by David Buchsbaum of producing explicitly one of the bases for "almost-skew" tableaux whose existence was proved by David Woodcock [W94].

Finally, given multiple row-convex shapes $D_1, \ldots, D_k$, I find a related basis of "anti-straight tableaux" for the span of $\mathcal{S}^{D_1}, \ldots, \mathcal{S}^{D_k}$. The anti-straight tableaux are notable for no longer being restricted to be in the set $\{D_i\}$.

1.2 Notation

In this chapter, vectors such as $(c_1, c_2, \ldots, c_k)$ are abbreviated as $\underline{c}$.

1.3 Representations, shapes, and determinants

To begin with I will sketch the Deruyts construction and its direct generalization to arbitrarily shaped tableaux. In the interest of accessibility, no use is made in this section of either supersymmetry or letterplace notation.

We will construct a large class of $GL_n(\mathbf{k})$ representations as subspaces of $\mathbf{k}[x_{i,j}]$ where $\mathbf{k}$ is an arbitrary field and $i = 1 \ldots n$, $j = 1 \ldots m$. We can picture $\mathbf{k}[x_{i,j}]$ as the ring of coordinate functions on the space $\mathbf{k}^{n\times m}$ of $n \times m$ matrices with entries in $\mathbf{k}$. We could consider GL_n acting on this space by left multiplication. In this case $g \in GL_n$ acts contragrediently on any function $f \in \mathbf{k}[x_{i,j}]$ by $g \circ f(A) = f(g^{-1}A)$ for all $A \in \mathbf{k}^{m\times n}$. This is a rational representation of GL_n on the function space; i.e. g sends any polynomial in $\mathbf{k}[x_{i,j}]$ to another polynomial whose coefficients are rational functions in the entries of g. One of our ongoing concerns will be to produce constructions and results applicable when $\mathbf{k}$ is any commutative ring. For this purpose, a rational action will not serve.

Instead, cause $g \in GL_n(\mathbf{k})$ to act on $\mathbf{k}^{m\times n}$ as left multiplication by ${g^{-1}}^{\mathrm{T}}$, the transpose of its inverse. Now if $g = (g_{i,j})$, then

$$g(x_{r,s}) = \sum_i g_{i,r} x_{i,s}.$$

This action extends to a homomorphism on all of $\mathbf{k}[x_{i,j}]$. The resulting representation is $Sym(\bigoplus_{i=1}^m \mathbf{k}^n)$. Not only is this a polynomial representation, but it has the distinct advantage that in our constructions we no longer need the space $\mathbf{k}^{n\times m}$ for anything but motivation. Taking this to its logical conclusion, we will derive results about super-representations and (in Chapter 3) quantum representations where the algebra $\mathbf{k}[x_{i,j}]$ of functions on the space $\mathbf{k}^{n\times m}$ is replaced by a suitably deformed polynomial ring. The first step in this process is to replace the algebra $\mathbf{k}[x_{i,j}]$ with $\mathbf{Z}[x_{i,j}]$. We will produce basis results for certain $\mathbf{Z}$-submodules of $\mathbf{Z}[x_{i,j}]$ which, after tensoring with $\mathbf{k}$ will be $GL_n(\mathbf{k})$-representations.

The two simplest examples to consider are $S^k(\mathbf{k}^n)$ and $\Lambda^k(\mathbf{k}^n)$. These can be realized as the subrepresentations of $\mathbf{k}[x_{i,j}]$ spanned respectively by the sets

$$\{x_{i_1,1}x_{i_2,1}\cdots x_{i_k,1}\} \qquad \text{and} \qquad \left\{ \begin{vmatrix} x_{i_1,1} & x_{i_2,2} & \cdots & x_{i_k,k} \\ x_{i_2,1} & x_{i_2,2} & \cdots & x_{i_2,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i_k,1} & x_{i_k,2} & \cdots & x_{i_k,k} \end{vmatrix} \right\},$$

where $1 \le i_1, \ldots, i_k \le n$. We can think of the elements of the former set as certain products of 1×1-minors of the matrix $(x_{i,j})$. We will roughly follow [AB85] in constructing various representations of GL_n as the span of certain products of minors, a method dating back to Deruyts.

The first item to have at hand is a useful notation for the determinants of the minors of $(x_{i,j})$. Suppose that a minor A of $(x_{i,j})$ has row indices $i_1, \cdots, i_k$ and ordered column indices $j_1 < j_2 < \cdots < j_k$. The submatrix A may also be indexed by writing a sequence $\underline{i}$ of row indices so that i_l is in position j_l. This requires that we allow "empty" positions in the sequence. For example, the two pairs of index sequences $(5,9,8)$, $(1,2,4)$ and (i_1,i_2,i_3), $(2,3,5)$ correspond respectively to the sequences

$$\begin{array}{ll} \text{Column \#:} & 1\,2\,3\,4 \\ & 5\,9\;\;\,8 \end{array} \qquad \text{and} \qquad \begin{array}{ll} \text{Column \#:} & 1\;2\;3\;4\;5 \\ & \;\;\;i_1\,i_2\;\;\;i_3 \end{array}$$

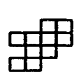
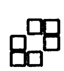
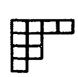
Given a product $\prod_{i=1}^{k}|A_i|$ of determinants of minors A_i, we can index this product by a *tableau* formed by writing k rows of such sequences where the lth row consists of the sequence indexing A_l. If T is a tableau, let $[T]$ denote the product of determinants indexed by T. Thus $Sym^k(\mathbf{k}^n)$ and $\Lambda^k(\mathbf{k}^n)$ are spanned respectively by all polynomials of the form

$$\begin{bmatrix} i_1 \\ i_2 \\ \cdots \\ i_k \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} i_1 & i_2 & \ldots & i_k \end{bmatrix},$$

where $1 \le i_1, \ldots, i_k \le n$. These are the representations associated to the *shapes*

$$\left.\begin{array}{c}\square\\[-4pt]\square\\ \vdots \\ \square\end{array}\right\} k \text{ cells} \qquad \text{and} \qquad \underbrace{\square\square\square\ldots\square}_{k \text{ cells}}.$$

Following Akin-Buchsbaum-Weyman [ABW82], we will associate representations to more general shapes like

 or  or  or .

We want to think of a shape as a finite subset of the integer plane $\mathbf{Z}\times\mathbf{Z}$. Since we have adopted the notations of Buchsbaum and of Rota, we coordinatize the cells in a diagram following the convention of matrices. That is, the coordinates of cells in the third diagram could be written (top row to bottom row) as $\{(1,3),(2,1),(3,2)\}$ or perhaps $\{(5,3),(6,1),(7,2)\}$. For the theory we develop, translating a diagram in the plane will have little effect on either the representations or the combinatorics associated to the diagram, so in practice we omit precise coordinates, assuming that cells in the top row have first coordinate 1 and cells in the leftmost column have second coordinate 1.

Example 1.3.1 Let λ be a partition, i.e. a finite decreasing sequence of positive integers, A shape is a *partition-shape* of shape λ (also called the diagram of λ) if it consists of all cells (i, j) for $1 \preceq j \leq \lambda_i$. Much of the combinatorics literature assumes that a shape is automatically a partition shape.

Suppose λ and μ are both partitions and μ is componentwise weakly less than λ. Then the skew-shape λ/μ is the set $D_\lambda \backslash D_\mu$ where D_λ, D_μ are the diagrams of λ and μ.

As a purely combinatorial definition, if D is a generalized shape, define a *tableau* T of shape D with entries in a set $\mathcal{L}$ to be a function from D to $\mathcal{L}$. Given $(i, j) \in D$ we write $T_{i,j}$ for the value of this function on (i, j) and think of $T_{i,j}$ as written in the (i, j) cell of D. The span of all polynomials associated to tableaux of a fixed shape is easily seen to be closed under GL_n action.

Definition 1.3.1 *Let D be a diagram. Define*

$$\mathcal{S}^D(\mathbf{k}^n) = \operatorname{span}_T[T]$$

where T ranges over all tableaux of shape D whose entries are chosen from $\{1, \ldots, n\}$.

Example 1.3.2 The Schur module of shape  over $\mathbf{k}^4$ is spanned by the 16 polynomials $[T]$ indexed by all tableaux T filled with elements of $\{1, \ldots, 4\}$, whose rows strictly increase. Eliminating the polynomial $\left[\begin{smallmatrix} 1 & 3 & 4 \\ & 2 & \end{smallmatrix}\right]$ makes this set a basis.

The action of S_n on tableaux defined by making $\sigma \in S_n$ act on each cell by replacing its content (say i) with $\sigma(i)$ descends immediately to make the Schur modules (and thus their multihomogenous submodules) in S_n-representations.

Let $\phi : \mathbf{k}[x_{i,j}] \twoheadrightarrow \mathbf{k}[x_{i,j} : i \leq j]$ be the canonical projection sending $x_{i,j}$ to $x_{i,j}$ for all $i \leq j$ and sending $x_{i,j}$ to 0 for $i > j$. The kernel of ϕ is closed under the action of the Borel subgroup B_n of lower triangular matrices. This leads us to the following.

Definition 1.3.2 *For any diagram D, define*

$$\mathcal{S}_f^D(\mathbf{k}^n) = \phi(\mathcal{S}^D(\mathbf{k}^n)).$$

Since the above remarks say that B_n acts on $\mathbf{k}[x_{i,j}]/\ker(\phi)$, the space $\mathcal{S}_f^D(\mathbf{k}^n)$ has the structure of a B_n-representation.

Being a GL_n-representation, $\mathcal{S}^D(\mathbf{C}^4)$ is a gl_n and a $U(gl_n)$ representation. The following "characteristic free" version of this fact is extended in the next section.

The $\mathbf{Z}$-module $\mathcal{S}^D(\mathbf{Z}^n)$ is a representation of $U(gl_n(\mathbf{k}))$, the $\mathbf{Z}$-subalgebra of the universal enveloping algebra of gl_n generated by $E_{i,j}^s/s!$ for all $1 \leq i, j \leq n$ and all $s \geq 0$. Here the $E_{i,j}$ are the generators for the universal enveloping algebra modulo the relations $E_{i,j}E_{k,l} - E_{k,l}E_{i,j} = \delta_{j,k}E_{i,l} - \delta_{i,l}E_{k,j}$ where δ is the Kronecker delta. The generator $E_{i,j}$ acts on $\mathbf{k}[x_{i,j}]$ as a derivation, i.e. $E_{i,j}$ satisfies the Leibniz rule $E_{i,j}(fg) = E_{i,j}(f)g + fE_{i,j}(g)$ on products. The action is completely specified by requiring that $E_{i,j}x_{k,l} = \delta_{j,k}x_{i,l}$.

1.4 Superalgebra constructions

Various results involving duality are most naturally stated in terms of superalgebras, however most of the results in this thesis pertaining to commutative algebra and to representations of S_n and GL_n can be read with little knowledge of superalgebras. To facilitate this process, I have provided in Table 1.1 at the end of Section 1.5 a "dictionary" for translating between selected superalgebra constructions and their more familiar counterparts. Indeed, the casual reader is highly advised to briefly skim the present section and refer back to it when necessary.

1.4.1 Polynomial superalgebras

The prefix "super" in the term superalgebra indicates that the algebra in question is $\mathbf{Z}_2$-graded. We will be dealing with *polynomial* superalgebras, i.e. the super-analogue of a polynomial algebra. Furthermore, since we will be dealing with $\mathbf{Z}$-algebras or with $\mathbf{k}$-algebras over an arbitrary commutative ring $\mathbf{k}$, we will need to distinguish between symmetric algebras and algebras of divided powers.

The symmetric and exterior $\mathbf{Z}$-algebras associated to a set $\mathcal{L}$ are written $\mathcal{S}ym(\mathcal{L})$ and $\Lambda(\mathcal{L})$. We write the symmetric and exterior $\mathbf{Q}$-algebras associated to $\mathcal{L}$ as $\mathcal{S}ym_{\mathbf{Q}}(\mathcal{L})$ and $\Lambda_{\mathbf{Q}}(\mathcal{L})$. The algebra $\mathcal{D}iv(x)$ of divided powers of a variable x is the commutative $\mathbf{Z}$-algebra generated by all symbols $x^{(i)}$ and satisfying the relations $x^{(i)}x^{(j)} = \binom{i+j}{i}x^{(i+j)}$. This is isomorphic to the $\mathbf{Z}$-subalgebra of $\mathbf{Q}[x]$ generated by $x^i/i!$, hence $x^{(i)}$ is referred to as the ith divided power of x. The divided powers algebra of a set is the tensor product of the divided powers algebras of its elements.

A *signed set* $\mathcal{L}$ is a set together with a function $|\ | : \mathcal{L} \mapsto \mathbf{Z}_2$. An element $s \in \mathcal{L}$ is *negative* if $|s| = 1$ and *positive* if $|s| = 0$. Define $\mathcal{L}^-$ to be the subset of all negative elements of $\mathcal{L}$ and likewise let $\mathcal{L}^+$ to be all positive elements of $\mathcal{L}$. An *alphabet* is a linearly ordered signed set.

Definition 1.4.1 *For any signed set $\mathcal{L}$, define the associated* polynomial superalgebra *to be the $\mathbf{Z}$-algebra*

$$\mathcal{S}uper(\mathcal{L}) = \Lambda(\mathcal{L}^-) \otimes \mathcal{D}iv(\mathcal{L}^+)$$

and

$$\mathcal{S}uper_{\mathbf{Q}}(\mathcal{L}) = \Lambda_{\mathbf{Q}}(\mathcal{L}^-) \otimes \mathcal{S}ym_{Q}(\mathcal{L}^+).$$

It will often be convenient to regard $\mathcal{S}uper(\mathcal{L})$ as the subalgebra of $\mathcal{S}uper_{\mathbf{Q}}(\mathcal{L})$ generated by all $l \in \mathcal{L}$ and by $\frac{a^k}{k!}$ for all $a \in \mathcal{L}^+$.

1.4.2 Lie Superalgebras

A free **k**-module F is *signed* when it has two distinguished free submodules F_0 and F_1 whose direct sum is F. Elements of F_0 and F_1 are called homogeneous and $|x| = i$ for $x \in F_i$.

Following Scheunert [Sc79], we call a free signed $\mathbf{Z}$-module a *Lie superalgebra* when it is endowed with a *superbracket* [,] satisfying the commutativity relation,

$$[x, y] = -(-1)^{|x||y|}[y, x]$$

for homogeneous elements x, y and the super-analogue of the Jacobi identity

$$(-1)^{|a||c|}[a, [b, c]] + (-1)^{|a||b|}[b, [c, a]] + (-1)^{|b||c|}[c, [a, b]] = 0$$

for homogeneous elements a, b, c.

For any signed alphabet, $\mathcal{L}$, the general linear Lie superalgebra $pl_{\mathcal{L}}$, as used in [BT91], is the vector space (over $\mathbf{Q}$) with basis $E_{a,b}$ for $a, b \in \mathcal{L}$ and bracket

$$[E_{a,b}, E_{c,d}] = \delta_{b,c}E_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{d,a}E_{c,b}.$$

The *enveloping algebra* $U_{\mathbf{Q}}(pl_{\mathcal{L}})$ over $\mathbf{Q}$ of $pl_{\mathcal{L}}$ is a $\mathbf{Z}$-graded superalgebra generated in degree 1 by variables $E_{a,b}$ subject to the relations

$$E_{a,b}E_{c,d} - (-1)^{(|a|+|b|)(|c|+|d|)}E_{c,d}E_{a,b} = \delta_{b,c}E_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{d,a}E_{c,b}.$$

The sign of a variable $E_{a,b}$ is $|a| + |b|$. The vector space spanned by degree 1 elements of $U_{\mathbf{Q}}(pl_{\mathcal{L}})$ is the Lie superalgebra $pl_{\mathcal{L}}$ when the supercommutator $[A, B] = AB - (-1)^{|A||B|}BA$ is used as the bracket.

In obtaining characteristic free results, I state and prove most results in terms of $\mathbf{Z}$-modules. To this end I will define $U(pl_{\mathcal{L}})$ to be the $\mathbf{Z}$-subalgebra of $U_{\mathbf{Q}}(pl_{\mathcal{L}})$

generated by all $E_{a,b}$ and by $\frac{E^i_{a,b}}{i!}$ for all a, b of the same sign and all $i \in \mathbf{N}$. We shall work in the context of $U(pl_{\mathcal{L}})$-representations. First we establish some technical lemmas.

A (left) *superderivation* D on a $\mathbf{k}$-superalgebra S is a $\mathbf{k}$-linear endomorphism of S such that for any two homogeneous elements p, q of S, the super-analogue $D(pq) = (Dp)q + (-1)^{\epsilon|p|}p(Dq)$ of the Leibniz rule holds for some fixed $\epsilon \in \mathbf{Z}_2$. This ϵ is the *sign* of D. A *right superderivation* R is defined similarly except that the Leibniz rule now generalizes to $(pq)R = (-1)^{|R||q|}(pR)q + p(qR)$.

In general, a function f on a superalgebra is $\mathbf{Z}_2$-graded if either it preserves the sign of all homogeneous elements or it reverses the sign of all homogeneous elements. In the former case we write $|f| = 0$ and in the latter, $|f| = 1$.

Lemma 1.4.1 *Any $\mathbf{Z}_2$-graded function $f : \mathcal{A} \to Super_{\mathbf{Q}}(\mathcal{A})$ lifts uniquely to a left superderivation of $Super_{\mathbf{Q}}(\mathcal{A})$. Similarly, f lifts uniquely to a right superderivation.*

If additionally, $f(\mathcal{A}) \subset Super(\mathcal{A}) \subset Super_{\mathbf{Q}}(\mathcal{A})$, then f lifts to a superderivation on $Super(\mathcal{A})$.

Proof. (Sketch) The property of being a superderivation completely specifies f. Indeed f satisfies the generalized Leibniz rule on monomials in $\mathcal{A}$ iff it is a superderivation.

If $f(\mathcal{A}) \subset Super(\mathcal{A})$ it suffices to check that $f(\frac{a^i}{i!}) \in Super(\mathcal{A})$ for all $i \in \mathbf{N}$ and all positively signed $a \in \mathcal{A}$. But since f is a superderivation, $f(\frac{a^i}{i!}) = f(a)\frac{a^{i-1}}{i-1!}$. □

Lemma 1.4.2 *Any sign-preserving function $f : \mathcal{A} \to Super(\mathcal{B}) \subset Super_{\mathbf{Q}}(\mathcal{B})$ defines a superalgebra homomorphism $f : Super_{\mathbf{Q}}(\mathcal{A}) \to Super_{\mathbf{Q}}(\mathcal{B})$ which lifts to a superalgebra homomorphism $f : Super(\mathcal{A}) \to Super(\mathcal{B})$ under the usual identification of $a^{(i)}$ with $\frac{a^i}{i!}$.*

Proof.(sketch) It suffices to show that for any homogeneous element $x \in \mathcal{A}$ with $|x| = 0$, $f(\frac{x^i}{i!})$ is in $Super(\mathcal{B})$. Write $f(x)$ as a sum of monomials in $Super(\mathcal{B})$. Use the formula $(y+z)^k/k! = \sum_{i=0}^k \frac{y^i}{i!}\frac{z^{k-i}}{k-i!}$ to induct on the number of summands in $f(x)$. For the base case it actually suffices to assume that $f(x)$ is the kth divided power of a positively signed variable b. But $\frac{b^{ik}}{i!(k!)^i}$ is an integer multiple of $\frac{b^{ik}}{(ik)!} \in Super(\mathcal{B})$, since $\frac{(ik)!}{i!(k!)^i}$ is the number of ways to partition an ik-set into i k-sets. □

Note that a sign preserving function $f : \mathcal{A} \to Super([\mathcal{L} \mid \mathcal{P}])$ does not necessarily lift to a superalgebra homomorphism. Suppose for example that $\mathcal{A} = \{x\}$, $\mathcal{L} = \mathcal{L}^- = \{l\}$, and $\mathcal{P} = \mathcal{P}^- = \{p\}$. The map $f(x) = (l|p)$ lifts to a homomorphism from $\mathbf{Q}[x] \to \mathbf{Q}[x_{l,p}]$ but not to a homomorphism from $\mathcal{D}iv(x) \to \mathbf{Z}[x_{l,p}]$.

Definition 1.4.2 *Given an algebra $\mathcal{L}$, define a superderivation $D_{a,b} : Super(\mathcal{L}) \to Super(\mathcal{L})$ for all $a, b \in \mathcal{L}$ by defining $D_{a,b}c = \delta_{b,c}a$.*

Similarly define a right superderivation ${}_{a,b}\mathrm{R} : Super(\mathcal{L}) \to Super(\mathcal{L})$ for all $a, b \in \mathcal{L}$ by defining $c\ {}_{a,b}\mathrm{R} = \delta_{c,a}b$.

For a, b positive $\frac{D^j_{b,a}}{j!}\frac{a^i}{i!} = \frac{b^j}{j!}\frac{a^{i-j}}{i-j!}$, hence the divided powers of these derivations (which following convention are written $D^{(j)}_{b,a}$) are also well-defined on $Super(\mathcal{L})$.

Proposition 1.4.3 *The maps $E_{a,b} \mapsto D_{a,b}$ and $E_{a,b} \mapsto {}_{a,b}\mathrm{R}$ extend respectively to a representation and a right representation of $U(pl_{\mathcal{L}})$ and $U(pl_{\mathcal{P}})$ on $Super(\mathcal{L})$ and $Super(\mathcal{P})$ respectively.*

Proof. By symmetry it suffices to consider the letter polarizations $D_{a,b}$. We need to prove that the polarizations satisfy the bracket identity of $U(pl_{\mathcal{L}})$, namely that

$$D_{a,b}D_{c,d} - (-1)^{|D_{c,d}||D_{a,b}|}D_{c,d}D_{a,b} = \delta_{b,c}D_{a,d} - (-1)^{|D_{c,d}||D_{a,b}|}\delta_{a,d}D_{c,b}.$$

It suffices to do the checking on monomials in $Super_{\mathbf{Q}}(\mathcal{L}) \supseteq Super(\mathcal{L})$. Thus we can use the fact that the polarization $D_{a,b}(x_1x_2\cdots x_k)$ is a signed sum

$$\sum_{i=1}^k \pm x_1x_2\cdots D_{a,b}(x_i)\cdots x_k.$$

We first check the subcase where $b \neq c$ and $a \neq d$. It is enough to establish a sign-preserving bijection between the monomials appearing in $D_{a,b}D_{c,d}(x_1 \cdots x_k)$ and those in $(-1)^{|D_{c,d}||D_{a,b}|}D_{c,d}D_{a,b}$. It is easy to check that the bijection pairing the two monomials in which $D_{a,b}$ acts on the ith term and $D_{c,d}$ acts on the jth term works.

If $i \neq j$ then without restriction on b, c, a, d, the sign-preserving bijection described above holds. However, when $b = c$ or $a = d$, terms where $i = j$ occur in, respectively, the action of $D_{a,b}D_{c,d}(x_1 \cdots x_k)$ and $(-1)^{|D_{c,d}||D_{a,b}|}D_{c,d}D_{a,b}$. These leftover terms are are accounted for respectively by the terms $\delta_{b,c}D_{a,d}$ and $-(-1)^{|D_{c,d}||D_{a,b}|}\delta_{a,d}D_{c,b}$ on the right-hand side. □

1.4.3 Biproducts

This section consists of a short exposition of *biproducts* as introduced in [GRS87]. Biproducts are bilinear maps from $\mathcal{S}uper(L) \times \mathcal{S}uper(P)$ to a "letter-place" algebra $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ analogous to the natural map from $\Lambda(\mathbf{Q}^m) \otimes \Lambda(\mathbf{Q}^{m*}) \to \mathbf{Q}$ defined by

$$v_1 \wedge \cdots \wedge v_k \times w_1^* \wedge \cdots \wedge w_k^* \mapsto \det(w_j^*(v_i)).$$

We start by introducing the algebra $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$. For any pair $\mathcal{L}$ and $\mathcal{P}$ of alphabets, we can define $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ to be

$$\Lambda(\mathcal{L}^- \times \mathcal{P}^+ \uplus \mathcal{L}^- \times \mathcal{P}^+) \otimes \mathcal{S}ym(\mathcal{L}^- \times \mathcal{P}^-) \otimes \mathcal{D}iv(\mathcal{L}^+ \times \mathcal{P}^+).$$

We write the generators of $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ as signed variables $(l|p)$ with the obvious convention that $|(l|p)| = |l| + |p|$. The superalgebra is then defined by the relations

$$(a|b)(c|d) = (-1)^{|(a|b)||(c|d)|}(c|d)(a|b)$$

for all $a, c \in \mathcal{L}$ and $b, d \in \mathcal{P}$; and

$$i!(a|b)^{(i)} = (a|b)^i \qquad \text{and} \qquad (a|b)^{(i)}(a|b)^{(j)} = \binom{i+j}{i}(a|b)^{(i+j)}$$

when a and b are each positively signed. This algebra is naturally isomorphic to a $\mathbf{Z}$-subalgebra of $Super_{\mathbf{Q}}(\{x_{a,d}\}_{a\in\mathcal{L},\, d\in\mathcal{P}})$ where $|x_{a,d}| = |a| + |d|$. In particular, as a $\mathbf{Z}$-subalgebra, $Super([\mathcal{L} \mid \mathcal{P}])$ is generated by all $x_{a,d}$ and by all $\frac{x^i_{a,d}}{i!}$ where a, b are positive and $i \in \mathbf{N}$.

We define the *letter polarization* $D_{a,b} : Super_{\mathbf{Q}}([\mathcal{L} \mid \mathcal{P}]) \to Super_{\mathbf{Q}}([\mathcal{L} \mid \mathcal{P}])$ to be the superderivation such that $D_{a,b}(c|p) = \delta_{b,c}(a|p)$. Similarly, let the *place polarization* ${}_{a,b}\mathrm{R} : Super_{\mathbf{Q}}([\mathcal{L} \mid \mathcal{P}]) \to Super_{\mathbf{Q}}([\mathcal{L} \mid \mathcal{P}])$ be the right superderivation such that $(l|c)\ {}_{a,b}\mathrm{R} = \delta_{c,a}(l|b)$. These lift to functions $D_{a,b}, {}_{a,b}\mathrm{R} : Super([\mathcal{L} \mid \mathcal{P}]) \to Super([\mathcal{L} \mid \mathcal{P}])$.

The proof of the following proposition is essentially identical to that of Proposition 1.4.3.

Proposition 1.4.4 *The maps $E_{a,b} \mapsto D_{a,b}$ and $E_{a,b} \mapsto {}_{a,b}\mathrm{R}$ extend respectively to a representation and a right representation of $U_{\mathbf{Q}}(pl_L)$ and $U_{\mathbf{Q}}(pl_P)$ acting on $Super_{\mathbf{Q}}([\mathcal{L} \mid \mathcal{P}])$.* $\square$

Furthermore, I claim for $a, b \in \mathcal{L}$ (respectively $a, b \in \mathcal{P}$), the polarization $\frac{D^k_{b,a}}{k!}$ $(\frac{{}_{a,b}\mathrm{R}^k}{k!})$ of $Super_{\mathbf{Q}}([\mathcal{L} \mid \mathcal{P}])$ lifts to a polarization on the subalgebra $Super([\mathcal{L} \mid \mathcal{P}])$. To check this it suffices to show that $Super([\mathcal{L} \mid \mathcal{P}])$, considered as a subalgebra of $Super_{\mathbf{Q}}([\mathcal{L} \mid \mathcal{P}])$, is closed under the action of the polarizations. In fact, since for $a, b \in \mathcal{L}^+$,

$$\frac{D^j_{b,a}}{j!}\frac{(a|x)^i}{i!} = \frac{(b|x)^j}{j!}\frac{(a|x)^{i-j}}{(i-j)!} \in Super([\mathcal{L} \mid \mathcal{P}])$$

and for all a, b

$$D^j_{b,a}(a|x)^i \in Super([\mathcal{L} \mid \mathcal{P}])$$

we have the following.

Proposition 1.4.5 *The maps $E_{a,b} \mapsto D_{a,b}$ and $E_{a,b} \mapsto {}_{a,b}\mathrm{R}$ provide a representation and a right representation of $U(pl_L)$ and $U(pl_P)$ acting on $Super([\mathcal{L} \mid \mathcal{P}])$.* □

We are now in a position to define the biproduct introduced in Grosshans-Rota-Stein.

Definition 1.4.3 *Given sequences $l_1, l_2, \ldots, l_k \in \mathcal{L}$ and $p_1, p_2, \ldots, p_k \in \mathcal{P}$, define*

$$(l_1, l_2, \ldots, l_k | p_1, p_2, \ldots, p_k) = \sum_{\sigma \in S_k} (-1)^{n_\sigma} (l_{\sigma(1)} | p_1)(l_{\sigma(2)} | p_2) \cdots (l_{\sigma(k)} | p_k)$$

where

$$\begin{aligned} n_\sigma = \#\{(i,j) : i < j,\ \sigma^{-1}(i) > \sigma^{-1}(j),\ l_i, l_j \text{ are negative}\} \\ +\#\{(i,j) : i > j \text{ and } l_{\sigma(i)}, p_j \text{ are negative}\}. \end{aligned}$$

The sign convention becomes more intuitive if we think of picking up a negative sign every time two negative letters cross when reordering (by exchanging letters) the sequence $l_1, l_2, \ldots, l_k$ to the sequence $l_{\sigma(1)}, l_{\sigma(2)}, \cdots, l_{\sigma(k)}$. We also picture a negative sign being acquired every time a negative place crosses a negative letter when the places are brought over to pair with the letters.

This complicated sign rule is justified by the following easily verified result.

Proposition 1.4.6 *Let $l_1, \ldots, l_k \in \mathcal{L}$ and $p_1, \ldots, p_k \in \mathcal{P}$ and suppose $a \in \mathcal{L}^+$ and $b \in \mathcal{P}^+$ are distinct from the l_i's and p_j's. The expression*

$$D_{l_1,a} D_{l_2,a} \cdots D_{l_k,a} \frac{(a|b)^k}{k!}\ {}_{b,p_1}\mathrm{R}\ {}_{b,p_2}\mathrm{R} \cdots {}_{b,p_k}\mathrm{R} = D_{l_1,a} D_{l_2,a} \cdots D_{l_k,a} (a|p_1)(a|p_2) \cdots (a|p_k).$$

evaluates to $(l_1, l_2, \ldots, l_k | p_1, p_2, \ldots, p_k)$. □

Since $D_{l_i,a}, D_{l_j,a}$ satisfy the same commutation rules as l_i, l_j, and likewise for ${}_{b,p_i}\mathrm{R}$, ${}_{b,p_j}\mathrm{R}$ and p_i, p_j, the biproduct is well defined on pairs of monomials u, v in

$Super_{\mathbf{Q}}(\mathcal{L}) \times Super_{\mathbf{Q}}(\mathcal{P})$. Denote this biproduct $(u|v)$ and extend it by bilinearity to all of $Super_{\mathbf{Q}}(\mathcal{L}) \times Super_{\mathbf{Q}}(\mathcal{P})$.

Checking that the biproduct extends to a map from $Super(\mathcal{L}) \times Super(\mathcal{P})$ to $Super([\mathcal{L} \mid \mathcal{P}])$, amounts to checking that, for pairs of monomials u, v in $Super(\mathcal{L}) \times Super(\mathcal{P})$, $(u|v) \in Super([\mathcal{L} \mid \mathcal{P}])$. But this is immediate since for $a, l \in \mathcal{L}^+$ and $b, p \in \mathcal{P}^+$, $\frac{D^k_{b,a}}{k!}$ and $\frac{{}_{a,b}\mathrm{R}^k}{k!}$ are well defined operations on $Super([\mathcal{L} \mid \mathcal{P}])$.

If w is a monomial in $Super_{\mathbf{Q}}(\mathcal{A})$ for some signed set A, we define $\mathbf{c}\,(w)!$ to be $\prod_{i\in\mathcal{A}}(\#\text{times } i \text{ appears in } w)!$. $Super(\mathcal{A})$ is a free $\mathbf{Z}$-module; considered as a submodule of $Super_{\mathbf{Q}}(\mathcal{A})$ it has basis $\{\frac{1}{\mathbf{c}(w)!}w\}$ for all nonzero monomials $w \in Super_{\mathbf{Q}}(\mathcal{A})$. The nonzero terms $\{\frac{1}{\mathbf{c}(w)!}w\}$ are called *divided powers monomials.*

Suppose $w \in Super_{\mathbf{Q}}([\mathcal{L} \mid \mathcal{P}])$. Let $\mathbf{c}\,(w)! = \prod_{i\in\mathcal{L}^+\, j\in\mathcal{P}^+}(\#\text{times } (i|j) \text{ appears in } w)!$. Similarly, $Super([\mathcal{L} \mid \mathcal{P}])$ is a free $\mathbf{Z}$-module with basis consisting of the *divided powers monomials* $\{\frac{1}{\mathbf{c}(w)!}w\}$ for all nonzero monomials $w \in Super_{\mathbf{Q}}([\mathcal{L} \mid \mathcal{P}])$.

We will occasionally have cause to write $\prod_{i=1}^k (w_i|v_i)$ as $\left(\begin{array}{c|c} w_1 & v_1 \\ \vdots & \vdots \\ w_k & v_k \end{array}\right)$ for $w_i \in Super(L)$ and $v_i \in Super(P)$. Note that his differs by a sign from the "vertical notation" of [GRS87].

Finally, we introduce a new version $\mathrm{Tab}(w_1, w_2, \ldots, w_k|v_1, v_2, \ldots, v_k) \in Super([\mathcal{L} \mid \mathcal{P}])$ of the biproduct. This is no longer defined on elements of $Super(\mathcal{L}) \times Super(\mathcal{P})$ but simply on a pair of k-tuples of letters in $\mathcal{L}$ and $\mathcal{P}$. It agrees with $(w_1, w_2, \ldots, w_k|v_1, v_2, \ldots, v_k)$ up to a nonzero scalar multiple and is scaled such that if $w_1 <\!\!+ w_2 <\!\!+ \cdots <\!\!+ w_k$ and $v_1 <\!\!+ v_2 <\!\!+ \cdots <\!\!+ v_k$. then the basis element $\frac{1}{\mathbf{c}(\prod_i (w_i|v_i))!}\prod_i (w_i|v_i)$ appears with coefficient 1. Accordingly, define

$$\mathrm{Tab}(w_1, \ldots, w_k|v_1, \ldots, w_k) = \frac{1}{\mathbf{c}\,(w)!\mathbf{c}\,(v)!}(-1)^{\#\ \{(i,j): i>j \text{ and } w_i, v_j \text{ are negative}\}}(w|v).$$

Further, suppose S and T are tableaux of the same shape. Let S_i be the ith row of S and let T_i be the ith row of T. Define $[S|T] = \prod_i \mathrm{Tab}(S_i|T_i)$.

This notation is justified by the following proposition.

Proposition 1.4.7 *Let S and T be two tableaux of shape D. Let $S_{i,j}$ be the entry in the i,jth position of S and similarly for T. Suppose S and T $(<_+)$-increase across rows. The coefficient of* $\frac{1}{c\left(\prod_{i,j}(S_{i,j}|T_{i,j})\right)!}\prod_{i,j}(S_{i,j}|T_{i,j})$ *in $[S|T]$ is either 0 or ± 1.* $\square$

An easy calculation checking the action of E_{l_1,l_2} and E_{p_1,p_2} on monomials in $Super([\mathcal{L} \mid \mathcal{P}])$ verifies the following.

Proposition 1.4.8 *The actions of $U(pl_{\mathcal{L}})$ and $U(pl_{\mathcal{P}})$ on $Super([\mathcal{L} \mid \mathcal{P}])$ commute.* $\square$

Proposition 1.4.9 *For $E \in U(pl_{\mathcal{L}})$ and $F \in U(pl_{\mathcal{P}})$ we have $E(v|w) = (Ev|w)$ and $(v|w)F = (v|wF)$.*

Proof. It suffices to check the statement for $E_{y,x} \in U(pl_{\mathcal{L}})$. This is immediate from the claim

$$E_{y,x}E_{v_1,a}\cdots E_{v_k,a}(a^{(k)}|b^{(k)}) = \sum_{i=1}^{k}\delta_{v_i,x}(-1)^{|E_{y,x}|(|E_{v_1,a}\cdots E_{v_{i-1},a}|)}E_{v_1,a}\cdots E_{v_{i-1},a}E_{y,a}\cdots E_{v_k,a}(a^{(k)}|b^{(k)}),$$

where a does not appear in $v_1,\ldots,v_k$. The proof of the claim is a routine induction on k using the fact that

$$E_{y,x}E_{v_1,a} = \delta_{v_1,x}E_{y,a} + (-1)^{|E_{y,x}||E_{v_1,a}|}E_{v_1,a}E_{y,x}.$$

Alternately, the claim follows from the fact that if u, v, w are homogeneous elements of the enveloping algebra of a Lie superalgebra, then ad_u (defined by $ad_u(v) = uv - (-1)^{|u||v|}vu$) is a superderivation and $uv = ad_u(v) + (-1)^{|u||v|}vu$. Thus

$$E_{y,x}E_{v_1,a}\cdots E_{v_k,a}(a^{(k)}|b^{(k)}) = ad_{E_{y,x}}\left(E_{v_1,a}\cdots E_{v_k,a}\right)(a^{(k)}|b^{(k)}).$$

$\square$

1.5 Schur modules, Weyl modules, generalizations

Remarkably, the various modules (over S_n, GL_n, B_n, a Schur algebra, etc...) associated to a generalized shape can often be effectively studied with little recourse to representation theory. We will be guided by the philosophy that the "right" way to ask representation-theoretic questions about these modules is to turn them into combinatorial questions about some submodule or subalgebra of letterplace superalgebra.

Following this philosophy, we will define our primary object of study, the super-Schur module as a $\mathbf{Z}$-submodule of a letterplace algebra. We then show how the Schur and Weyl modules arise as special cases, in particular proving the equivalence of our construction with the Akin-Buchsbaum construction of [ABW82].

A tableau of shape D is termed Deruyts if it is obtained by filling each cell in the diagram with the cell's column index viewed as a negative variable. We denote such a tableau by $Der^-(D)$. Similarly, if each cell is filled with a positive variable indexing that cell's row, then we have defined the tableaux $Der^+(D)$.

Example 1.5.1

$$Der^-\left(\begin{smallmatrix} & & \square & \square \\ \square & & \square & \square \\ & & \square & \\ \square & & & \end{smallmatrix}\right) = \begin{array}{cccc} & & 3^- & 4^- \\ 1^- & & 3^- & 4^- \\ & & 3^- & \\ 1^- & & & \end{array} \qquad Der^+\left(\begin{smallmatrix} & & \square & \square \\ \square & & \square & \square \\ & & \square & \\ \square & & & \end{smallmatrix}\right) = \begin{array}{cccc} & & 1^+ & 1^+ \\ 2^+ & & 2^+ & 2^+ \\ & & 3^+ & \\ 4^+ & & & \end{array}$$

Definition 1.5.1 *Suppose that T is a tableau of shape D. Suppose that $\mathcal{L}$ contains the set of letters present in T and that $\mathcal{P}^-$ contains the indices for all columns present in D. Define an element $[T] \in Super([\mathcal{L} \mid \mathcal{P}])$ indexed by T by,*

$$e_T = Tab\big[T|Der^-(D)\big].$$

Example 1.5.2 Let $\mathcal{L} = \mathcal{L}^- = \{a, b, c, d, e, f\}$ and let $\mathcal{P} = \mathcal{P}^- = \{1, 2, 3\}$. Let $T = \begin{array}{ccc} & a & d \\ b & c & e \\ & f & \end{array}$. Then

$$[T] = \left[\begin{array}{ccc} & a & d \\ b & c & e \\ & f & \end{array} \,\middle|\, \begin{array}{ccc} & 2 & 3 \\ 1 & 2 & 3 \\ & 2 & \end{array}\right].$$

In other words, $[T] = (ad|23)(bce|123)(f|2)$ or,

$$e_T = \det\begin{pmatrix} (d|2) & (d|3) \\ (a|2) & (a|3) \end{pmatrix} \det\begin{pmatrix} (e|1) & (e|2) & (e|3) \\ (c|1) & (c|2) & (c|3) \\ (b|1) & (b|2) & (b|3) \end{pmatrix} (f|2) \quad .$$

Definition 1.5.2 *Suppose that $\mathcal{D}$ is a collection of shapes. Define the* super-Schur module

$$\mathcal{S}^{\mathcal{D}}(\mathcal{L}) = \mathrm{span}_{\mathbf{Z}}\Big\{[T] : \mathrm{shape}(T) \in \mathcal{D} \text{ and } T \text{ is filled with letters from } \mathcal{L}\Big\}.$$

In the case that $\mathcal{D} = \{D\}$ and $\mathcal{L}$ is negative (respectively positive) then $\mathcal{S}^{\mathcal{D}}(\mathcal{L})$ is called the Schur (respectively Weyl) module associated with the diagram D. These terms are justified by the following result.

Proposition 1.5.1 *Let R be a commutative ring. Let F be a free R-module of rank n. Let α be the $0/1$-matrix having 1's precisely where D has cells.*

If $\mathcal{L} = \mathcal{L}^-$ has cardinality n, then $R \otimes_{\mathbf{Z}} \mathcal{S}^D(\mathcal{L}^-) = L_\alpha(F)$, where $L_\alpha(F)$ is the Akin-Buchsbaum-Weyman "Schur functor" associated to the generalized shape matrix α.

If $\mathcal{L} = \mathcal{L}^+$ has cardinality n, then $R \otimes_{\mathbf{Z}} \mathcal{S}^D(\mathcal{L}^+) = K_\alpha(F)$, where $K_\alpha(F)$ is the Akin-Buchsbaum-Weyman "coSchur functor."

Proof. I will assume familiarity with the Akin-Buchsbaum-Weyman constructions and show how to interpret $\mathcal{S}^D(\mathcal{L}^-)$ and $\mathcal{S}^D(\mathcal{L}^+)$ as instances of these constructions. The key step will be to write $[T]$ as the image of a composition of two maps. These maps will be, up to sign, the maps of Akin-Buchsbaum-Weyman. First we need another definition.

Definition 1.5.3 *Let $F(D)$ be the tableau of shape D formed by writing a 1 in the northmost cell in the first column, a 2 in the next highest cell, etc., continuing with the second column, then the third, etc.*

Suppose that the shape D has l cells. Let $\underline{c} = (c_1, c_2, \ldots, c_l')$ be the composition of l where c_i is the number of cells in the ith column of D. Similarly, let $\underline{r} = (r_1, r_2, \ldots, r_{l''})$ where r_j is the number of cells in the jth row of D. Let $\mathcal{P} = \{a_1, a_2, \ldots, a_{l''}, 1, 2, \ldots, l'\}$ where all a_i are positive and the remaining "places" are negative. Let $\mathcal{P}' = \{1, 2, \ldots, l\}$; we assume these letters are negative, though for our construction positive letters could have been employed. We define maps $\phi : Super^{\underline{r}}([\mathcal{L} \mid \mathcal{P}^+]) \longrightarrow Super([\mathcal{L} \mid \mathcal{P}'])$ and $\psi : Super([\mathcal{L} \mid \mathcal{P}']) \longrightarrow Super^{\underline{c}}([\mathcal{L} \mid \mathcal{P}^-])$. The map ϕ is defined by its action on the monomials in $Super^{\underline{r}}([\mathcal{L} \mid \mathcal{P}^+])$. In particular, it is defined so if T is an arbitrary tableau of shape D, then

$$\phi\left([T|Der^+(D)]\right) = \epsilon_T[T|F(D)],$$

where $\epsilon_{w_1,w_2,\ldots} = \pm 1$. If $\mathcal{L}$ contains letters of only one sign then ϵ_T does not depend on the choice of tableau T. The map ϕ is is easily defined to be a product of place polarizations and hence commutes with letter polarizations.

The map ψ is even easier tc define. It is the restriction of the algebra homomorphism defined by sending $(x|j)$, to $(x|i)$ for all $x \in \mathcal{L}$ and $j \in \{c_1 + \cdots + c_{i-1} + 1, \ldots, c_1 + \cdots + c_i\}$ This map of modules can also be defined as a product of place polarizations, so it too commutes with letter polarizations.

Staying with the same shape as the preceding example, let $T = \begin{matrix} & a & d \\ b & b & e \\ & f & \end{matrix}$. We can write $[T]$ (up to sign) as the image

$$\left[\begin{matrix} & a & d \\ b & b & e \\ & f & \end{matrix} \;\middle|\; \begin{matrix} & a_1 & a_1 \\ a_2 & a_2 & a_2 \\ & a_3 & \end{matrix}\right] \mapsto \epsilon_T \left[\begin{matrix} & a & d \\ b & b & e \\ & f & \end{matrix} \;\middle|\; \begin{matrix} & 2 & 5 \\ 1 & 3 & 6 \\ & 4 & \end{matrix}\right] \mapsto \epsilon_T \left[\begin{matrix} & a & d \\ b & b & e \\ & f & \end{matrix} \;\middle|\; \begin{matrix} & 2 & 3 \\ 1 & 2 & 3 \\ & 2 & \end{matrix}\right].$$

under $\psi \circ \phi$. Here $\epsilon_T = (-1)^{2(|b|+|b|+|e|+|f|)+3(|f|)}$, and we can express ϕ, ψ on respectively $Super^{\underline{r}}([\mathcal{L} \mid \mathcal{P}^+])$ and $Super([\mathcal{L} \mid \mathcal{P}'])$ by $\phi = {}_{a_1,2}R\ {}_{a_1,5}R\ {}_{a_2,1}R\ {}_{a_2,3}R\ {}_{a_2,6}R\ {}_{a_3,4}R$, and $\phi = {}_{1,1}R\ {}_{2,2}R\ {}_{3,2}R\ {}_{4,2}R\ {}_{5,3}R\ {}_{6,3}R$. If $a, \ldots, f$ are all positive then $[T]$ turns out to be $(-1)^{2+1+1+1}$ times the element of $\Lambda^{\underline{c}}$ associated to T in [ABW82].

Now suppose that $\mathcal{L} = \mathcal{L}^+$. This implies that $Super^{\underline{r}}([\mathcal{L} \mid \mathcal{P}^+])$ is naturally isomorphic to $\mathcal{D}iv^{\underline{r}}$. Recall that if $w \in \mathcal{D}iv^k$ then the map from $\mathcal{D}iv^k$ to $\mathcal{S}ym^{(1,1,\ldots,1)} = (R^n)^{\otimes l}$ defined by $(\cdots w \cdots | a^{(k)}) \mapsto (\cdots w \cdots | 12 \cdots k)$ is defined as the degree-$(1,1,\ldots,1)$ component of the k-fold coproduct Δ^k. Thus up to sign ϕ is $\Delta^{r_1} \otimes \Delta^{r_2} \otimes \cdots \otimes \Delta^{r_{l''}}$.

Similarly we check that ψ agrees (up to sign) with the "twist map" followed by multiplication in $\Lambda(R^n)^{\otimes l'}$ defined in Akin-Buchsbaum-Weyman. But the "twist map" is accomplished (up to a fixed sign) by anti-commuting the letterplaces in any monomial so that $(x|j)$ comes before any $(x'|k)$ where the index of the column containing cell j precedes the index of the column containing cell k. Multiplying elements of $(R^n)^{\otimes l}$ together to form elements of $\Lambda(R^n)^{\otimes l'}$ is precisely what is accomplished by sending $(x|j)$ to $(x|i)$ where i is the index of the column containing the jth cell.

In the example, we had

$$(a|a_1)(d|a_1)(b|a_2)^{(2)}(e|a_2)(f|a_3) \mapsto (a|2)(d|5)(b|1)(b|3)(e|6)(f|4)+$$
$$+(a|2)(d|5)(b|1)(e|3)(b|6)(f|4) + \cdots + (d|2)(a|5)(e|1)(b|3)(b|6)(f|4)$$

This is equal to

$$(-1)^{2+1+1+1}(b|1)(a|2)(b|3)(f|4)(d|5)(e|6) + (b|1)(a|2)(e|3)(f|4)(d|5)(b|6) + \cdots$$
$$\cdots + (e|1)(d|2)(b|3)(f|4)(a|5)(b|6)$$

and is sent by ψ to

$$(-1)^{2+1+1+1}(b|1)(a|2)(b|2)(f|2)(d|3)(e|3) + (b|1)(a|2)(e|2)(f|2)(d|3)(b|3) + \cdots$$
$$\cdots + (e|1)(d|2)(b|2)(f|2)(a|3)(b|3).$$

If $\mathcal{L} = \mathcal{L}^-$ then the analysis of the construction is essentially the same. As modules over $R \otimes U(pl_{\mathcal{L}}) = U(gl(F))$, $\mathcal{S}uper^{\underline{r}}([\mathcal{L} \mid \mathcal{P}^+]) \simeq \Lambda^{\underline{r}}(F)$, $\mathcal{S}uperlpLP' \simeq F^{\otimes l}$ and $\mathcal{S}uper^{\underline{c}}([\mathcal{L} \mid \mathcal{P}^-]) \simeq \mathcal{S}ym^{\underline{c}}(F)$. The image of $\psi \circ \phi$ is $K_\alpha(F)$. □

Example 1.5.3 The Weyl module of shape  on positive letters a, b is a GL_2-representation with basis

$$\left[\begin{smallmatrix} a\,a\,a \\ b \end{smallmatrix}\right], \quad \left[\begin{smallmatrix} a\,a\,b \\ b \end{smallmatrix}\right], \quad \left[\begin{smallmatrix} b\,b\,b \\ a \end{smallmatrix}\right].$$

Specialization when $\mathcal{L} =$ $\mathcal{L}^+$	$\mathcal{L}^-$	Superalgebra construction. (size of $\mathcal{L}$ is n)
$Sym(\mathbf{k}^n) = \mathbf{k}[x_l : l \in \mathcal{L}]$	$\Lambda(\mathbf{k}^n)$	$\mathbf{k} \otimes_{\mathbf{Z}} Super(\mathcal{L})$
$\Lambda(\mathbf{k}^{mn})$	$\mathbf{k}[x_{l,p} : l \in \mathcal{L},\ p \in \mathcal{P}]$	$\mathbf{k} \otimes_{\mathbf{Z}} Super([\mathcal{L} \mid \mathcal{P}])$
$\sum_\sigma \frac{1}{c\,(i_1 \ldots i_k)!} \varepsilon_{i_{\sigma_1},j_1} \cdots \varepsilon_{i_{\sigma_k},j_k}$ the variables $\varepsilon_{l,p}$ anti-commute	$\det \begin{pmatrix} x_{i_1,j_1} & \cdots & x_{i_1,j_k} \\ \vdots & \ddots & \vdots \\ x_{i_k,j_1} & \cdots & x_{i_k,j_k} \end{pmatrix}$	$\mathrm{Tab}(i_1, \ldots, i_k \mid j_1, \ldots, j_k)$
Weyl module, K_D	Schur module, L_D	$\mathrm{span}_T[T \mid Der^-(D)] = S^D$
gl_n	gl_n	$pl_{\mathcal{L}}$
$x_i \frac{\partial}{\partial x_j}$	‡	$D_{i,j}$ acting on $Super(\mathcal{L})$
‡	$\sum_{p \in \mathcal{P}} x_{i,p} \frac{\partial}{\partial x_{j,p}}$	$D_{i,j}$ on $Super([\mathcal{L} \mid \mathcal{P}])$
$\leq$	$<$	$<_+$
$<$	$\leq$	$<_-$

Table 1.1: Specialization of superalgebra constructions to cases where the sign is solely positive or negative. I assume that $\mathcal{P} = \mathcal{P}^-$ and that $\mathcal{P}$ has m elements. All bitableaux are assumed to be filled with letters from $\mathcal{L}$ on the left side and $\mathcal{P}$ on the right side. Single tableau are assumed to be filled with letters from $\mathcal{L}$.
‡ In this case the specialization of the superalgebra construction is not familiar enough to be written down briefly.

Table 1.1 summarizes a few of the correspondences between the preceding superalgebra constructions and their more common unsigned versions.

The Schur and Weyl modules constructed in [ABW82] are GL_n representations since each map is natural with respect to GL_n-action. This can be seen directly from the observation that these maps can be expressed as place polarizations and that the action of GL_n on elements $[T]$ can be expressed as a letter polarization. For example, the action of the permutation (12) on the $\mathbf{Z}$-submodule of $S^D\mathcal{L}$ spanned by $[T]$ where

T has i occurrences of 1 and j occurrences of 2 is given by $D_{2,x}^{(x)}D_{1,2}^{(j)}D_{x,1}^{(i)}$ where $x \neq \mathcal{L}$ and $|x| = 1$.

Exercise 1.5.1 *Express the action of the other generators of GL_n on e_T in terms of letter polarizations.*

Apply the relation $D_{2,x}E = [D_{2,x}, E] + ED_{2,x}$ to show that the extra variable x can be eliminated from the preceding expression of the action of (12).

It will be useful to have a further normalized version of $[T]$. Let T be a tableau and define $\mathrm{Tab}(T) = (-1)^{N_T}[T]$ where

$$N_T = \#\{\{(i,j),(i',j')\} \ : \ i < i',\ j' < j \text{ and } T_{i,j}, T_{i',j'} \text{ are positive}\} + \{\{(i,j),(i',j)\} \ : \ i < i',\ T_{i,j} > T_{i',j}\}.$$

This notation is justified by the following result analogous to Proposition 1.4.7.

Proposition 1.5.2 *Let T be a tableau and define permutations σ_j of the cells in column j of T such that if $r_1 < r_2$ are row indices for cells in column j then $T_{\sigma_j(r_1),j} < T_{\sigma_j(r_2),j}$. The coefficient of*

$$\frac{1}{\mathbf{c}\left(\prod_{i,j}(T_{i,j}|j)\right)!}\prod_j\prod_i(T_{\sigma_j(i),j}|j)$$

in Tab(T) is 1 so long as no two cells in the same column contain the same positive letter. It is 0 otherwise.

1.6 Straight bases

The usual bases for skew Weyl modules consist of the semistandard Young tableaux, namely all tableaux which weakly increase in their rows and strictly increase in their columns. Example 1.5.3 showed that this in not the case for more general shapes. The question that naturally arises is whether for a given shape D a well-behaved set of tableaux can be chosen to index a basis of $\mathcal{S}^D$. The basis of standard Young tableaux for skew Weyl modules has a number of properties one would like to generalize. In particular:

1. The rows of a tableaux weakly increase.

2. Knowing the number of times a letter appears in each column of a standard Young tableaux determines that tableaux.

3. It is combinatorially "obvious" when a tableau is in the indexing set.

4. There is an easy to describe algorithm for rewriting $[T]$ in terms of basis elements.

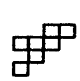
Property 2 may not be familiar–it is the fundamental property used in the SAGBI-basis algorithms of [Stu93] and in [W94] Woodcock shows that there must exist bases satisfying this property when D is "almost-skew."

In this section I define a class of "straight" tableaux that satisfy all of the above properties. The elements $[T]$ where T is straight and of shape D will from a basis for the super-Schur module $\mathcal{S}^D$ for any "row-convex" shape D. As an immediate corollary I obtain an explicit construction as called for by Buchsbaum of a basis for the modules considered by Woodcock.

If one is willing to strengthen Property 2 somewhat then only slightly more complicated shapes,  for instance, fail to possess both properties 1 and 2.

1.6.1 Row-convex diagrams and straight tableau

Shapes appearing in this section are assumed, unless otherwise noted, to have first coordinate 1 in their top rows and second coordinate 1 in their leftmost columns.

Definition 1.6.1 *A* row-convex shape, *such as* , *is a shape with no gaps in any row. I.E., if cells (r,i) and (r,k) are in a shape D, then (r,j) is in D, for all $i < j < k$. Since the constructions of section 1.5 are not sensitive to the order of rows* *in a diagram, we assume that* the rows of a row convex diagram are sorted so that higher rows end at least as far to the right as lower rows.

We generalize the notions of strict and weak inequalities to take the sign of a letter into account. For $a, b \in \mathcal{L}$, write $a <_+ b$ if $a < b$ or $a = b$ and both are positively signed. Similarly, write $a <_- b$ if $a < b$ or $a = b$ and both are negative. These inequalities are both entered in Table 1.1.

Following [GRS87] a tableau T with entries in a signed set is *standard* when it $(<_+)$-increases across rows and $(<_-)$-increases down columns.

I introduce the notion of a *straight tableau* of row-convex shape by slightly relaxing the usual conditions for standardness of a tableau.

Definition 1.6.2 *A row-convex tableau is called* straight *when*

1. *The contents of any row $<_+$-increase from left to right, and*

2. *Given two cells in the same column, say (i, k) and (j, k) for $i < j$, the entry in the top cell, (i, k), may be $(>_+)$-larger than the entry in (j, k) (i.e. the cells form an* inversion*) only if cell $(i, k-1)$ exists and its content is $(>_-)$-larger than the content of (j, k).*

This definition amounts to requiring that the columns are as close as possible to $(<_-)$-increasing, subject to the condition that the rows remain $(<_+)$-increasing. A more precise version of the preceding fact is implicit in the correctness Algorithm Straight-Filling. A tableau satisfying condition 1 is called *row-standard* and an inversion violating condition 2 is called a *flippable* inversion.

Proposition 1.6.1 *A skew tableau, T, is straight iff it is standard.*

Proof. Since a standard tableau has no inversions, it suffices to prove the only-if part. We prove the contrapositive. We can assume that T is row-standard. Suppose that the cells $(i, k), (j, k)$ with $i < j$ are an inversion. Let k_0 be the least (leftmost) column

such that $(i, k_0), (j, k_0)$ is an inversion. If $(i, k_0 - 1)$ exists then by skewness so does $(j, k_0 - 1)$ and thus by assumption $T_{i,k_0-1} <\!\!- T_{j,k_0-1} <\!\!+ T_{j,k_0}$ hence T is not straight.
□

Corollary 1.6.2 *The straight tableaux of skew shape with only positively signed letters are the usual semistandard Young tableaux.* □

Given a tableau T its *column word*, c_T is the word formed by reading the entries of T from bottom to top and left to right. Its *modified column word* is the word w_T formed by writing the entries of the first column in decreasing order followed by the entries of the second column in decreasing order, etc.

Theorem 1.6.3 *If T and T' are straight tableaux of the same shape, then $T \neq T'$ implies $w_T \neq w_{T'}$. More strongly, if there exists a straight tableau T with $w_T = w$ then the algorithm* `Straight-Filling` *in Figure 1-1 produces it.*

Proof. We need the notion of the *reverse column word* w'_T of T formed by writing the entries of the first column of T in increasing order then those of the second column in increasing order etc.

Input: A word w' of length n, and an n-celled shape D.
Output: A straight tableau T with $w'_T = w'$ or "IMPOSSIBLE" if no such tableau exists.

> Let c_j be the column index of the jth cell in D reading column by column from left to right.
> Let T be an empty tableau of shape D
> **for** $k = 1 \ldots n$
> Let i be the smallest (northmost) index such that $(i, c_k) \in D$ is still empty and either there is no cell in position $(i, c_k - 1)$ or $T_{i,c_k-1} <\!\!+ w'_k$.
> **if** there is no such i **then return** "IMPOSSIBLE"
> **else** $T_{i,c_k} \leftarrow w'_k$.

Figure 1-1: **Algorithm** Straight-Filling

A tableau, T, produced by this algorithm must be straight. If in a fixed column, k, the letter y is inserted into row i by the algorithm while $x <\!\!+ y$ was inserted into row $j > i$, then it must be that $T_{i,k-1} \nrightarrow x$ else x would have been inserted into row i.

Now suppose that the algorithm produces a tableau T with reverse column word w'. Let $\underline{c}$ be as in the algorithm. Any tableau with reverse column word w' can be produced by a similar filling process. Define $\underline{i}$ so that reading through w' and inserting w'_k into cell (i_k, c_k) gives the desired tableau. Let us assume that if w'_k appears in multiple cells in column c_k that the northmost appearance is filled by the first w'_k in w', the second northmost by the second, etc. Let $\underline{i}$ be the filling sequence corresponding to T, this is the sequence produced by the Algorithm Straight-Filling. Let $\underline{i}'$ be the filling sequence corresponding to some other tableau T'. Let k_0 be the smallest integer such that $i_{k_0} \neq i'_{k_0}$. So in filling T', we have placed w'_{k_0} into cell (i'_{k_0}, c_{k_0}) when according to Algorithm Straight-Filling, it could have been put into (i_{k_0}, c_{k_0}) where $i_{k_0} < i'_{k_0}$. By necessity, in filling T', something $(\geq)$-larger than w_{k_0} must be placed in (i_{k_0}, c_{k_0}). In fact by our assumptions about repeated letters in the definition of $\underline{i}$, this inequality is strict. But these facts guarantee that the inversion $\{(i_{k_0}, c_{k_0}), (i'_{k_0}, c_{k_0})\}$ of T' violates condition 2 in the definition of straight tableaux.

The above argument says that if we try to create a straight tableau by reading across w' and sequentially filling its letters into a tableau then at each step the choice of where to insert the letters is forced on us. If at any point during execution of the algorithm there is no place to put a letter which preserves row-standardness, then it is in fact not possible to find a straight tableau with the designated column content and shape. This is precisely the circumstance under which "IMPOSSIBLE" is returned.

We conclude that not only does `Straight-Filling` produce a straight tableau, but any other tableau, T' having the same modified (equivalently reverse) column word is not straight. $\square$

Corollary 1.6.4 *The matrix expressing the super-polynomials $[T]$ indexed by straight tableaux as $\mathbf{Z}$-linear combinations of divided powers monomials in the polynomial superalgebra is in echelon form with ± 1 at each pivot. Hence the straight basis elements* *are linearly independent.*

Since echelon form of course requires an ordering on basis elements and monomials we defer the proof until the appropriate orders have been developed. Monomials are ordered according to a "diagonal term order" as in [Stu93] which requires that the smallest monomial in $\det(A)$, where A is a minor of $(x_{i,j})$, be the product of the elements on the diagonal. (This is unfortunately backwards from the convention in commutative algebra which has $\prod_i(x_{i,i})$ be the *largest* monomial in $\det(A)$.) In general, a *diagonal term order* on $Super([\mathcal{L} \mid \mathcal{P}])$ is

1. A total order, $\prec$, on monomials in $Super([\mathcal{L} \mid \mathcal{P}])$ such that for monomials m, m', n, n', the relations $m \prec m'$ and $n \prec n'$ imply that $mn \prec m'n'$ or $mn = 0$ or $m'n' = 0$.

2. The smallest monomial in a nonzero biproduct $(i_1 \ldots, i_k | j_1, \ldots, j_k)$ with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$ is $\prod_l (i_l | j_l)$.

The *default diagonal term order*, $\prec_{\text{diag}}$ that we utilize is characterized as follows. We order letterplaces $(i|j)$ by $(i|j) > (i'|j')$ when $j < j'$ or when $j = j'$ and $i > i'$. Let $M, N \in Super([\mathcal{L} \mid \mathcal{P}])$ be two nonzero monomials. Suppose $(i|j)$ is the largest letterplace appearing to a different power in M and N. Write $M > N$ when M is divisible by a higher power of $(i|j)$ than is N.

Example 1.6.1 Suppose $\mathcal{L} = \{1^-, 2^-\}$ and $\mathcal{P} = \{a^+, x^-\}$, then $(2|a) > (1|a) > (2|x) > (1|x)$. Further, we have

$$(1|x) < (1|x)^6 < (2|a)(1|x)^2(2|x) < (2|a)(2|x)^2(1|x) < (2|a)(1|a)(1|x)^2.$$

The following lemma is immediate.

Lemma 1.6.5 *A normalized monomial $\prod_{l=1}^k (i_l|j_l) \neq 0$ in $Super([\mathcal{L} \mid \mathcal{P}])$ is a monomial written so that $(i_l|j_l) \geq (i_{l+1}|j_{l+1})$ in the default diagonal term order. For two* *normalized monomials,* $M = \prod_{l=1}^{k}(i_l|j_l)$ *and* $N = \prod_{l=1}^{k}(i'_l|j_l)$ *differing only in their letters,* $M < N$ *in the default diagonal term order iff* $i_1, \dots, i_k$ *is lexicographically less than* $i'_1, \dots, i'_k$. □

Let Ψ be the function taking a normalized monomial $\prod_{l=1}^{k}(i_l|j_l) \in \mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ to $i_1, \dots, i_k$.

Definition 1.6.3 *Given* $p \in \mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ *and an order* $\prec$ *on monomials, define the* initial monomial $\mathrm{init}_{\prec}(p)$ *of* p *to be the smallest (divided powers) monomial appearing in* p.

Sometimes the phrase "initial term" will be used when the coefficient of the initial monomial is to be included.

The following result says that in most cases the modified column word of T can be read directly from the smallest monomial appearing in $[T]$.

Proposition 1.6.6 *If* T *is a tableau whose rows* ($<_+$)*-increase and whose columns contain no repeated positive letters, then*

$$w_T = \Psi(\mathrm{init}_{\prec_{\mathbf{diag}}}([T])).$$

Proof. Suppose $[T] = \prod_i \mathrm{Tab}(w_{i,c_{i,1}}, w_{i,c_{i,2}}, \dots, w_{i,c_{i,l_i}} | c_{i,1}, c_{i,2}, \dots, c_{i,l_i})$. The initial term (with coefficient) of the ith multiplicand is $\frac{1}{\mathbf{c}(\underline{w}_i)!}\prod_j (w_{i,c_{i,j}}|c_{i,j})$. and since positive letters never repeat in a column the product of these initial terms is nonzero and hence equals $\mathrm{init}_{\prec_{\mathbf{diag}}}([T])$. □

Note that the initial term

$$\frac{1}{\mathbf{c}\,(\underline{w}_i)!}\prod_j (w_{i,c_{i,j}}|c_{i,j})$$

appearing above is a basis element in the monomial $\mathbf{Z}$-basis for $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ viewed as a submodule of $\mathcal{S}uper_{\mathbf{Q}}([\mathcal{L} \mid \mathcal{P}])$.

Proposition 1.6.7 *If T is straight of shape D, and if c_l is the index of the column of $F(D)$ containing l then*

$$\frac{1}{\mathrm{c}\,(w_T)!}\prod_l ((w'_T)_l|c_l) = \mathrm{init}_{\prec_{\mathrm{diag}}}(Tab(T))$$

where by abuse of notation $\mathrm{init}_{\prec_{\mathrm{diag}}}()$ is here taken to be the initial term–i.e. it includes the coefficient. □

Corollary 1.6.8 *Suppose T is a straight tableau, then $w_T = \Psi(\mathrm{init}_{\prec_{\mathrm{diag}}}([T]))$.* □

We now complete the proof of the independence result.

Proof.(of Corollary 1.6.4.) Since Theorem 1.6.3 says that distinct straight tableau have distinct modified column words, we conclude from Corollary 1.6.8 that if divided powers monomials are ordered by $\prec_{\mathrm{diag}}$ and the polynomials $[T]$ corresponding to straight tableaux are ordered lexicographically by their modified column words, then the matrix expressing the $[T]$ in terms of divided powers monomials is in echelon form with ± 1's as pivots. □

Corollary 1.6.9 *Suppose $p = \sum_i \alpha_i T_i$ is a linear combination of row-standard tableaux such that $p = \sum_j \beta_j[S_j]$ where the S_j are distinct straight tableaux and where all tableaux have the same row-convex shape D. The smallest modified column word of a tableaux in the S_j's is weakly larger (lexicographically) than the smallest modified column word appearing in the T'_i.*

Proof. Let c_l be the column of $F(D)$ containing l. Suppose that $w_{T_{i_0}} \leq w_{T_i}$ for all i and suppose $w_{S_{j_0}} < w_{S_j}$ for all $j \neq j_0$–recall that distinct straight tableaux have distinct modified column words. We want to show $w_{T_{i_0}} \leq w_{S_{j_0}}$. Now because straight tableaux have distinct modified column words $\prod_l (w_{S_{j_0}l}|c_l)$ is the smallest monomial occurring in p. That means the it must appear in $\sum_i \alpha_i T_i$ if that expression is expanded out to a polynomial in $Super([\mathcal{L} \mid \mathcal{P}])$. But if w_{T_i} is always larger than $w_{S_{j_0}}$ then no monomial as small as $\prod_l (w_{S_{j_0 l}}|c_l)$ can appear in $\sum_i \alpha_i T_i$. □

The point of the next section is to show that any $\sum_i \alpha_i [T_i]$ can be rewritten in the above fashion.

1.6.2 A straightening law via Grosshans-Rota-Stein relations

The purpose of this section is to produce a two-rowed straightening law for reducing any tableau to a linear combination of straight tableau. As shown in Chapter 2 when $\mathcal{L} = \mathcal{L}^-$, this amounts to explicitly producing a nonreduced degree 2 Groebner basis for the homogeneous coordinate rings of a certain configuration variety. In more combinatorial language, we produce an algorithm, `straighten-tableau` shown in Figure 1-2 which starts with a tableau T and returns a formal linear combination $\sum_i \alpha_i S_i$ of straight tableau with integer coefficients such that $[T] = \sum_i \alpha_i [S_i]$ and such that each step in the algorithm modifies at most two rows in a tableau.

We provide an example of the straightening law below. In this case, the value c_1 in Algorithm `row-straighten` happens to always index the leftmost column in the lower of the two rows.

Example 1.6.2 We will mark the shuffles indexed by σ in Definition 1.6.4 by $\star$ and mark the shuffles indexed by τ with $\bullet$. Intuitively, symmetrizing the $\star$'d entries yields an expression equal to collecting the $\star$'d entries and splitting the $\bullet$'d entries in all possible ways. elements of w below with $^\bullet$ and the elements of v with *. Observing that the entries 4 and 2 form a flippable inversion, we first have,

$$\left[\begin{array}{cccc} 1 & 3 & \overset{4^\star 5^\star}{5\ 7} & \\ & 3 & \overset{2^\star}{8} & \end{array}\right] = \left[\begin{array}{cccc} 1 & 3 & \overset{2^\star 5^\star}{5\ 7} & \\ & 3 & \overset{4^\star}{8} & \end{array}\right] - \left[\begin{array}{cccc} 1 & 3 & \overset{2^\star 4^\star}{5\ 7} & \\ & 3 & \overset{5^\star}{8} & \end{array}\right].$$

Input: A row-convex tableau T.
Output: $\sum_\iota \alpha_\iota S_\iota$ such that $[T] = \sum_\iota \alpha_\iota [S_\iota]$ where each S_ι is a straight tableau and $\alpha_\iota \in \mathbf{Z}$.

> **if** T is straight **then** output T.
> **else** there exists a flippable inversion in some rows i, j
>> Let $\sum_\kappa \beta_\kappa \cdot \substack{\cdots v_\kappa \cdots \\ \cdots w_\kappa \cdots}$ be the output of **row-straighten**$\left(\substack{\cdots T_i \cdots \\ \cdots T_j \cdots}\right)$; let N_κ be (# pos. letters in w_κ + # pos. letters in T_j) $\cdot$ (# pos letters in $T_{i+1} \cdots T_{j-1}$).
>>
>> $$\text{Output } \sum_\kappa \beta_\kappa \cdot \textbf{straighten-tableau}\begin{pmatrix} \cdots T_1 \cdots \\ \cdots\cdots \\ \cdots T_{i-1} \cdots \\ \cdots v_\kappa \cdots \\ \cdots T_{i+1} \cdots \\ \cdots\cdots \\ \cdots T_{j-1} \cdots \\ \cdots w_\kappa \cdots \\ \cdots T_{j+1} \cdots \\ \cdots\cdots \end{pmatrix}.$$

Figure 1-2: **Algorithm straighten-tableau.** T_i, T_j, etc. are the ith, jth, etc. rows of the tableau T. The summation over tableau has greek letters as indices so as to not overlap notationally with the notion of extracting a specific row of an unindexed tableau.

But the cells in column 3 and rows 2 and 3 of the first tableau on the right hand side now contain a flippable inversion. We straighten as follows,

$$\begin{bmatrix} & 2 & 5 & \\ 1^\bullet & 3^\bullet & 5^\star & 7^\star \\ & & 4^\star & \\ & 3 & 8 & \end{bmatrix} = \begin{bmatrix} & 2 & 5 & \\ 1^\bullet & 3^\bullet & 4^\star & 7^\star \\ & & 5^\star & \\ & 3 & 8 & \end{bmatrix} - \begin{bmatrix} & 2 & 5 & \\ 1^\bullet & 3^\bullet & 4^\star & 5^\star \\ & & 7^\star & \\ & 3 & 8 & \end{bmatrix} + \begin{bmatrix} & 2 & 5 & \\ 1^\bullet & 4^\star & 5^\star & 7^\star \\ & & 3^\bullet & \\ & 3 & 8 & \end{bmatrix} - \begin{bmatrix} & 2 & 5 & \\ 3^\bullet & 4^\star & 5^\star & 7^\star \\ & & 1^\bullet & \\ & 3 & 8 & \end{bmatrix}.$$

Now the first two tableaux above are straight, but the last two are not. We straighten the next to last tableau by,

$$\begin{bmatrix} & 2 & 5 & \\ 1^\bullet & 4^\star & 5^\star & 7^\star \\ & & 3 & \\ & 3^\star & 8 & \end{bmatrix} = \begin{bmatrix} & 2 & 5 & \\ 1^\bullet & 3^\star & 5^\star & 7^\star \\ & & 3 & \\ & 4^\star & 8 & \end{bmatrix} - \begin{bmatrix} & 2 & 5 & \\ 1^\bullet & 3^\star & 4^\star & 7^\star \\ & & 3 & \\ & 5^\star & 8 & \end{bmatrix} + \begin{bmatrix} & 2 & 5 & \\ 1^\bullet & 3^\star & 4^\star & 5^\star \\ & & 3 & \\ & 7^\star & 8 & \end{bmatrix} + \begin{bmatrix} & 2 & 5 & \\ 3^\star & 4^\star & 5^\star & 7^\star \\ & & 3 & \\ & 1^\bullet & 8 & \end{bmatrix}$$

and the last tableau by

$$-\begin{bmatrix} & & 2^\star & 5^\star \\ 3 & 4 & 5 & 7 \\ & & 1^\star & \\ & 3 & 8 & \end{bmatrix} = -\begin{bmatrix} & & 1^\star & 5^\star \\ 3 & 4 & 5 & 7 \\ & & 2^\star & \\ & 3 & 8 & \end{bmatrix} + \begin{bmatrix} & & 1^\star & 2^\star \\ 3 & 4 & 5 & 7 \\ & & 5^\star & \\ & 3 & 8 & \end{bmatrix},$$

Input: A two-rowed row-convex tableau $T = \begin{smallmatrix} v_{m_1} v_{m_1+1} \cdots v_{\lambda_1} \\ w_{m_2} w_{m_2+1} \cdots w_{\lambda_2} \end{smallmatrix}$ which is row-standard but not straight.
Output: $\sum_\kappa \alpha_\kappa \cdot \begin{smallmatrix} \cdots v_\kappa \cdots \\ \cdots w_\kappa \cdots \end{smallmatrix}$ such that
Claim 1: $\left[\begin{smallmatrix} \cdots v \cdots \\ \cdots w \cdots \end{smallmatrix}\right] = \sum_\kappa \alpha_\kappa \left[\begin{smallmatrix} \cdots v_\kappa \cdots \\ \cdots w_\kappa \cdots \end{smallmatrix}\right]$ where $\alpha_k \in \mathbf{Z}$ and
Claim 2: the column word of $\begin{smallmatrix} \cdots v_\kappa \cdots \\ \cdots w_\kappa \cdots \end{smallmatrix}$ is lexicographically larger than the column word of $\begin{smallmatrix} \cdots v \cdots \\ \cdots w \cdots \end{smallmatrix}$.

> Let c_2 be the index of the column containing the leftmost flippable inversion.
> Let c_1 be the smallest column such that $c_1 \geq m_2$ and either $v_{c_1-1} \not< w_{c_1}$ or $c_1 - 1 < m_1$ (i.e. v_{c_1-1} does not exist.)
> Let c_3 be the rightmost column such that $w_{c_2} = w_{c_3}$.
> **if** $c_1 < c_2$ **then**
>   Let $\sum_{\iota \in I} \beta_\iota T_\iota = Syz_{c_2, c_2+1, \cdots, \lambda_1; c_1, c_1+1, \cdots, c_2}(T)$.
>   *Expansion* $\leftarrow 0$.
>   **for** $\iota \in I$
>   **if** $w_{T_\iota} > w_T$ **then** *Expansion* $\leftarrow$ *Expansion* $+ \beta_\iota T_\iota$.
>   **else** *Expansion* $\leftarrow$ *Expansion* $+ \beta_\iota \cdot$ **row-straighten**(T_ι).
>   Output *Expansion.*
> **else** $\triangleright$ Comment: $c_1 = c_2$.
>   Let c_0 be the leftmost column such that $v_{c_0} \not> w_{c_2}$. $\triangleright$ $c_0 < c_2 \Rightarrow v_{c_0} = w_{c_2}$.
>   Let $\sum_{\iota \in I} \beta_\iota T_\iota = Syz_{c_0, c_0+1, \cdots, \lambda_1; c_1, c_1+1, \cdots, c_2}(T)$.
>   *Expansion* $\leftarrow 0$.
>   **for** $\iota \in I$
>   **if** $w_{T_\iota} > w_T$ **then** *Expansion* $\leftarrow$ *Expansion* $+ \beta_\iota T_\iota$.
>   **else** *Expansion* $\leftarrow$ *Expansion* $+ \beta_\iota \cdot$ **row-straighten**(T_ι).
>   Output *Expansion.*

Figure 1-3: Algorithm row-straighten. This algorithm is partly motivated in the process of proving its correctness; see Proposition 1.6.12.
If $\mathcal{L} = \mathcal{L}^-$, then we will always have $w_{T_\iota} > w_T$ so the algorithm will never recurse and instead could have directly output the expressions $Syz(T)$. The expression $Syz(T)$ is defined in Definition 1.6.4.

so

$$\begin{bmatrix} & & 4 & 5 \\ 1 & 3 & 5 & 7 \\ & & 2 & \\ & 3 & 8 & \end{bmatrix} = -\begin{bmatrix} & & 2 & 4 \\ 1 & 3 & 5 & 7 \\ & & 5 & \\ & 3 & 8 & \end{bmatrix} + \begin{bmatrix} & & 2 & 5 \\ 1 & 3 & 4 & 7 \\ & & 5 & \\ & 3 & 8 & \end{bmatrix} - \begin{bmatrix} & & 2 & 5 \\ 1 & 3 & 4 & 5 \\ & & 7 & \\ & 3 & 8 & \end{bmatrix} + \begin{bmatrix} & & 2 & 5 \\ 1 & 3 & 5 & 7 \\ & & 3 & \\ & 4 & 8 & \end{bmatrix} +$$
$$-\begin{bmatrix} & & 2 & 5 \\ 1 & 3 & 4 & 7 \\ & & 3 & \\ & 5 & 8 & \end{bmatrix} + \begin{bmatrix} & & 2 & 5 \\ 1 & 3 & 4 & 5 \\ & & 3 & \\ & 7 & 8 & \end{bmatrix} + \begin{bmatrix} & & 2 & 5 \\ 3 & 4 & 5 & 7 \\ & & 3 & \\ & 1 & 8 & \end{bmatrix} - \begin{bmatrix} & & 1 & 5 \\ 3 & 4 & 5 & 7 \\ & & 2 & \\ & 3 & 8 & \end{bmatrix} + \begin{bmatrix} & & 1 & 2 \\ 3 & 4 & 5 & 7 \\ & & 5 & \\ & 3 & 8 & \end{bmatrix}.$$

The first step in verifying Algorithm `straighten-tableau` is to prove that when T is replaced with $\sum_i \beta_i T_i$ by Algorithm `row-straighten` we have $[T] = \sum_i \beta_i [T_i]$. The second step involves showing that each T_i is somehow closer to being straight than was T. The first of these facts is an immediate consequence of the correctness of Algorithm `row-straighten`. Essentially this comes down to verifying the identities used in the preceding example. The second follows from the correctness of `row-straighten` and the fact that given a tableau T and another tableau T' differing only in two rows i, j, then the column word of the two-rowed subtableaux consisting of rows i, j of T is less than the corresponding column word determined by T' iff $c_T < c_{T'}$.

The proof of Algorithm `straighten-tableau` thus depends solely on the correctness of Algorithm `row-straighten`. We will prove both claimed properties of Algorithm `row-straighten` for each of the two cases appearing in the algorithm. First we will produce the "determinantal" identities used in Algorithm `row-straighten`. These identities are immediate from the more general Theorem ? of [GRS87].

Proposition 1.6.10 *Let a, b, c be positive letters. Let i, j, k, l be nonnegative integers. Let $1, 2, \ldots, i+j+l$ be negative letters. Fix a two-rowed row-convex shape D whose top row contains its bottom row by specifying the starting and ending columns, 1 through $i+j+1$ and m through $m+l+k-1$ of the top and bottom rows respectively. The following identity holds for tableaux of shape D.*

$$\left(\begin{array}{cc|cccccc} a^{(i+l)} & b^{(j)} & 1 & 2 & \ldots & \ldots & \ldots & {\scriptstyle i+j+l} \\ b^{(l)} & c^{(k)} & & m & {\scriptstyle m+1} & \ldots & {\scriptstyle m+l+k-1} & \end{array} \right)$$

$$= (-1)^l \left(\begin{array}{cc|cccc} b^{(j+l)} & a^{(i)} & 1\ 2 & \ldots\ \ldots & \ldots & i+j+l \\ a^{(l)} & c^{(k)} & m & m+1 & \ldots & m+l+k-1 \end{array} \right)$$

Proof. It suffices to check that the monomials arising from the expansion of each expression have the same coefficients. But since each row on the left side of the biproduct contains a divided powers monomial, we find that the letterplace monomials in the expansion of first biproduct are indexed by the set of all tableau of shape D with $i+1$ a's and j b's in the first row, with l b's and k c's in the second row, and not having a or b repeated in a column; these last index the summand 0. Any tableau T satisfying this condition corresponds to a single occurrence with coefficient ± 1 of the normalized monomial $\Psi^{-1}(w_T)$ where Ψ is as defined after Lemma 1.6.5. Suppose two tableaux T, T' have the same modified column word, then the monomial indexed by T differs from the monomial indexed by T' by $(-1)^\epsilon$ where ϵ is the number of transpositions (within columns) needed to make the tableau T into T'.

The tableaux indexing the monomials in the right hand side expansion are formed similarly. It suffices to check that there is a bijection between the tableau indexing each expansion, that this bijection preserves modified column words, and that l transpositions are required to turn one tableau into the tableau it is paired with. In either case, one can first decide where in the bottom row the c's appear. After this one is forced on the left-hand side to put a's above the l b's and on the right-hand side to put b's above the l a's. Finally, in either case there are $\binom{i+j}{k}$ ways to arrange the remaining a's and b's in the top row. Two tableaux formed by making the same choices for both the right and left-hand sides will thus differ only by the transposition within their columns of l pairs of of a's with b's. $\square$

Choosing letters $v_1, \ldots, v_{i+l}$, $w_1, \ldots, w_k$, and $u_1, \ldots, u_{j+l}$. and applying the product, $D_{v_1,a} D_{v_2,a} \cdots D_{v_{i+l},a}\ D_{w_1,c} \cdots D_{w_k,c}\ D_{u_1,b} \cdots D_{u_{j+l},b}$, of polarizations to the identity in Proposition 1.6.10, provides the basic identities used for straightening.

A *shuffle* of a word $W = w_1, \ldots, w_n$ into parts of length k, k' is a pair of words

$W_{(1)}$ and $W_{(2)}$ of W having lengths k and k' respectively, such that $W_{(1)}$ and $W_{(2)}$ can be found as a pair of disjoint subwords of W. Neither $W_{(1)}$ nor $W_{(2)}$ need be contiguous as a subword of W. When $w = 1, \ldots, n$ this amounts to a permutation σ of the index set $1, \ldots, n$ such that $\sigma_1 < \cdots < \sigma_k$ and $\sigma_{k+1} < \cdots < \sigma_{k+k'}$.

Corollary 1.6.11 *Let T be a two-rowed row-convex tableau of shape D as in Proposition 1.6.10. Fix two sequences $1 \le c_1 < \cdots < c_j \le i+j+l$ and $m \le c'_1 < \cdots < c'_l \le k+l$. Call the cells $c_1, \ldots, c_j$ in the top row and cells $c'_1, \ldots, c'_l$ in the bottom row of T* marked cells.

There exist integers α_σ and β_τ indexed by shuffles such that $\alpha_e[T]=$

$$\sum_{\substack{\text{all nontrivial shuffles} \\ u_{\sigma(c_1)}u_{\sigma(c_2)}\cdots u_{\sigma(c_j)};\ u_{\sigma(c'_1)}u_{\sigma(c'_2)}\cdots u_{\sigma(c'_l)} \\ \text{of } v_{\sigma(c_1)}v_{\sigma(c_2)}\cdots v_{\sigma(c_j)}w_{\sigma(c'_1)}w_{\sigma(c'_2)}\cdots w_{\sigma(c'_l)} \\ \text{with } \sigma \in S_{c_1,\ldots,c_j,c'_1,\ldots,c'_l}}} \alpha_\sigma[T'_\sigma] + \sum_{\substack{\text{all shuffles} \\ v_{\tau(d_1)}v_{\tau(d_2)}\cdots v_{\tau(d_i)};\ v_{\tau(d_{i+1})}v_{\tau(d_{i+2})}\cdots v_{\tau(d_{i+l})} \\ \text{of } v_1 v_2 \cdots \hat{v_{c_1}} \cdots \hat{v_{c_j}} \cdots v_{i+j+l} \\ \text{with } \tau \in S_{d_1,d_2,\ldots,d_{i+l}} := S_{1,2,\ldots,\hat{c_1},\ldots,\hat{c_j},\ldots,i+j+l}}} \beta_\tau[T''_\tau],$$

where T'_σ is the tableau resulting from sorting the rows of

$$\begin{array}{c} v_1 \ \ldots \ v_{c_1-1} \ u_{\sigma(c_1)} \ \ldots \ \ldots \ u_{\sigma(c_{j-1})} \ \ldots \ \ldots u_{\sigma(c_j)} \ \ldots \ v_{i+j+l} \\ w_m \ \ldots w_{c'_1-1} \ u_{\sigma(c'_1)} \ \ldots \ u_{\sigma(c'_l)} \ \ldots \ w_{k+l-1} \ w_{k+l} \end{array}$$

and T''_τ is obtained by row-sorting the tableau

$$\begin{array}{c} v_{c_1} \ \ldots \ v_{c_j} \ w_{c'_1} \ \ldots \ w_{c'_l} \ v_{\tau(d_1)} \ v_{\tau(d_2)} \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots \ v_{v_{\tau(d_i)}} \\ v_{\tau(d_{i+1})} \ v_{\tau(d_{i+2})} \ \ldots \ v_{\tau(d_{i+l})} \ w_m \ \ldots \ \hat{w}_{c'_1} \ \ldots \ \hat{w}_{c'_l} \ \ldots \ w_{k+l} \end{array}.$$

If no positive letter appears in both marked and unmarked cells in the top row of T and no positive letter appears in both marked and unmarked cells in the bottom row of T then we can set $\alpha_e = 1$. The coefficients α_σ and β_τ are found by obeying the rules for polarizations. □

Definition 1.6.4 *Preserving the notation of Corollary 1.6.11, define the formal linear* *combination of tableaux, $Syz_{c_1,\ldots,c_j;c'_1,\ldots,c'_l}(T)$, to be*

$$\sum_\sigma \alpha_\sigma/\alpha_e T'_\sigma + \sum_\tau \beta_\tau/\alpha_e T''_\tau,$$

where as in the Corollary, the sum runs over all nontrivial shuffles σ of the word $v_{\sigma(c_1)}v_{\sigma(c_2)}\cdots v_{\sigma(c_j)}w_{\sigma(c'_1)}w_{\sigma(c'_2)}\ldots w_{\sigma(c'_l)}$ and over all shuffles τ of the word

$$v_1v_2\cdots \hat{v_{c_1}}\cdots \hat{v_{c_j}}\cdots v_{i+j+l}.$$

We have just verified that any formal linear combination of tableau with integer coefficients produced by Algorithm `row-straighten` satisfies Claim 1 made in the algorithm specifications; we now go to work on the heart of the proof, namely Claim 2. In the process of proving Claim 2, we will check that a formal linear combination is actually produced.

Proposition 1.6.12 *Given a non-straight two rowed row-convex tableau, T, Algorithm* `row-straighten` *produces a formal linear combination of tableaux each of which has a lexicographically larger column word than c_T.*

Let D be the row-convex shape $(\lambda_1, \lambda_2)/(m_1, m_2)$.

Proof. The proof is by induction on c_T. Suppose that T is the tableau

$$\begin{array}{cccccccc}
\text{Column \#} & m_1 & m_1{+}1 & m_2 & m_2{+}1 & & \lambda_2 & \lambda_1 \\
 & & & & & & & \\
 & v_{m_1} & v_{m_1+1} & & & \cdots & & v_{\lambda_1} \\
 & & & w_{m_2} & w_{m_2+1} & \cdots & w_{\lambda_2} &
\end{array}$$

For the moment we will assume $m_1 \le m_2$. The case $m_1 \ge m_2$ handles the classical straightening algorithm. We will be able to derive it and the fact that it increases column words directly as Porism 1.6.13 from the present result.

If T is straight, then we are done. Otherwise, it becomes necessary to set up a straightening syzygy that expresses T in terms of tableau T_i such that c_T is always lexically smaller than c_{T_i}. So let us examine what T looks like if T has a flippable inversion.

First we find the earliest column q, reading the tableau from left to right, such that if the next column were to contain an inversion, then that inversion would be flippable. Thus

$$q = \begin{cases} m_1 - 1 & \text{if } m_1 \geq m_2 \\ \min_{\substack{m_2 \leq i \leq \lambda_2 \\ v_{i-1} <_+ w_i}} i - 1 & \text{otherwise} \end{cases}$$

The presence of a flippable inversion guarantees that q exists. The value of c_1 in Algorithm `row-straighten` is $q+1$. Continue reading the tableau left to right in order to find the first column after q that actually *has* an inversion. If this column is $r+1$ then we have

$$r = \min_{\substack{q < j \leq \lambda_2 \\ v_j <_+ w_j}} j - 1.$$

This inversion is guaranteed to be flippable, and thus $r+1$ is the location of the leftmost flippable inversion in the tableau; this location is recorded as c_2 in Algorithm `row-straighten`.

Two cases arise in the algorithm namely $q < r$ and $q = r$. The pictures in Figures 1-4 and 1-5 outline these situations. The symbol, "$\bullet$", indicates a cell in the diagram. An arrow from one cell to another indicates that the contents of the first cell are larger than the contents of the second. Arrows can be modified by the addition of a "$-$" or "$+$", indicating that the contents of the cells are permitted to be equal if the contents are respectively negatively or positively signed. Sequences of cells surrounded by parentheses or braces may be omitted. The braced sequence is omitted precisely when $q = m_2 - 1$. Finally, although if $m_1 = m_2$, the cell in column q isn't really present, q will still be well-defined (it equals $m_1 - 1$) and the northwest arrow from column $q+1$ to q should be thought of as present.

If no positive letter appears multiple times in T, then $c_0 = c_1$ in Case II and so Cases I and II can be treated simultaneously. To begin with we handle Case I.

Case I: $q < r$.

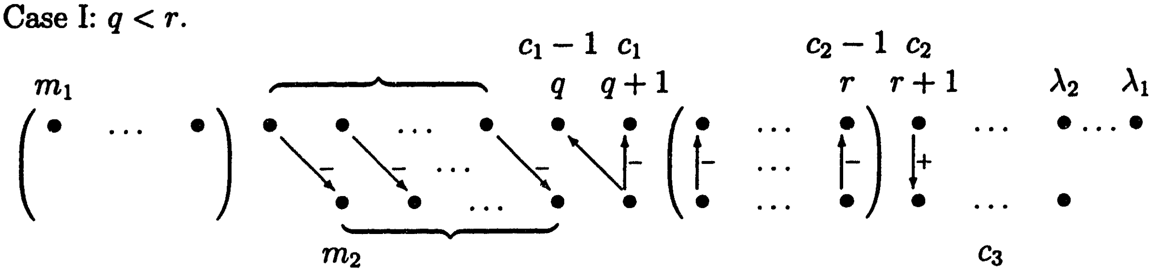


Figure 1-4: **Case I**: $c_1 < c_2$: Relations between entries in a two-row tableau being straightened by Algorithm `row-straighten`.
The column $q = c_1 - 1$. All entries in the bottom row from c_2 through c_3 are equal but distinct from any entry in column $c_3 + 1$ of that row.

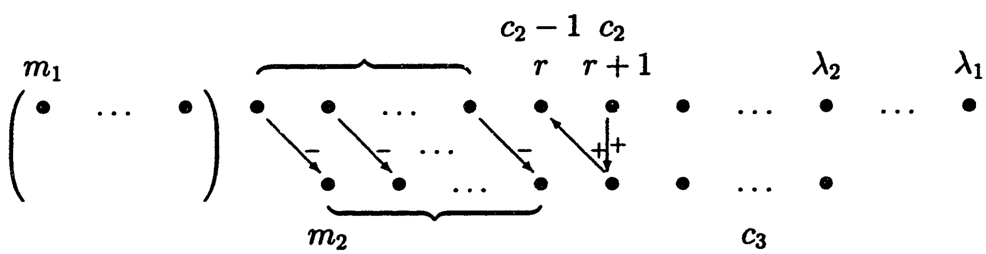


Figure 1-5: **Case II**: $c_1 = c_2$: Relations between entries in a two-row tableau being straightened by Algorithm `row-straighten`.
All entries in the bottom row from c_2 through c_3 are equal but distinct from any entry in column $c_3 + 1$ of that row. If $c_0 < c_2$, then the entries in the top row that equal the bottom row entry in column c_2 must start at c_0 and extend at least as far as $c_2 - 1$.

We apply the syzygy of Definition 1.6.4 to the letters in positions $c_2, \ldots, \lambda_1$ of the top row and positions $c_1, \ldots, c_3$ of the bottom row.

In particular, apply Corollary 1.6.11 to write

$$\left[\begin{array}{l} v_{m_1} \quad \cdots \quad \cdots \quad \cdots \quad \overline{v_{c_2} \quad \cdots \quad v_{\lambda_1}} \\ \qquad w_{m_2} \cdots \underline{w_{c_1} \cdots w_{c_2} \cdots w_{c_3}} \cdots \end{array}\right] = A + B. \tag{1.1}$$

The entries used to define the expression $Syz(T)$ in Definition 1.6.4 are marked by over/underlines. Here A is a signed sum over all shuffles of the $c_2 - m_1$ non-overlined entries in the top row, of the brackets of the tableau formed by leaving $c_2 - 1 - m_1 - c_3 + c_1$ entries in the top row and moving the other $c_3 - c_1 + 1$ entries to the bottom row. *All* of the under/overlined elements are collected into the top row. As usual, the tableaux are sorted so as to be row-standard.

B, in turn, is a sum over all tableau found by non-trivially shuffling the under/overlined elements in place and then row-sorting.

It suffices to show that each tableau appearing in A or B has lexically larger column word than c_T.

I will tackle the tableau in A first. Suppose T' appears in A. To this purpose, let $W = v_{m_1} \ldots v_r$ be the word being shuffled. Then we can write $W_{(1)}$, the first part of the shuffle, as $x_{m_1} \ldots x_{t-1}$, and we can write $W_{(2)} = y_1 \ldots y_{c_2 - t}$ where $t = c_2 + c_1 - c_3 - 1$. I claim that

$$T' = \begin{array}{l} \boxed{x_{m_1} \ \cdots \ x_{t-1} \ w_{q+1} \ \cdots \ w_{c_3}} \ v_{r+1} \ \cdots \ \cdots \ \cdots \ v_{\lambda_1} \\ \qquad \boxed{w_{m_2} \ \cdots \ w_q \ y_1 \ \cdots \ y_{c_2 - t}} \ w_{c_3+1} \ \cdots \ w_{\lambda_2} \end{array},$$

where the boxed entries must be sorted in order to get a row-standard tableau. Denote the entries in the bottom row by $z_{m_2}, \ldots, z_{\lambda_2}$. Having run out of Roman letters, denote the entries in the top row by $\varpi_{m_1}, \ldots, \varpi_{\lambda_1}$.

To verify this claim, it suffices to observe that the rows (as written) are in order. Since the x_i are taken from $v_{m_1}, \ldots, v_r$, checking the top row amounts to observing that $v_{r+1} \not> w_{c_3}$.

For the bottom row, it suffices to note that $w_{c_3+1} \not> v_r$.

Now let

$$k = \min_{\substack{m_1 \le i \le t \\ \varpi_i \ne v_i}} i - 1 \qquad .$$

We check that k in fact exists and for the first time must appeal to the fact that we are in Case I and hence $v_q < w_{q+1}$. Suppose $\varpi_i = v_i$ for $m_1 \le i < t = c_1 - 1 - c_3 + c_2$. Since by construction $v_{t-1} \not< v_q < w_{q+1}$, we have that $x_i = v_i$ for all i as above and thus the boxed elements in the top row are already in order. But then $\varpi_t = w_{q+1} \ne v_t$, so k exists.

Now for some inequalities. Having shown that $k+1$ is the leftmost position in which $\varpi_k \ne v_k$, we can conclude that $\varpi > v$. So if $k+1 < m_2$ we conclude directly that $c_{T'}$ is lexically larger than c_T and in this case we are done.

Suppose that $k+1 \ge m_2$. We show that $v_{k+1} \le y_1$. Suppose to the contrary that $v_{k+1} > y_1$. Since y_1 comes from $v_{m_1}, \ldots, v_r$ this says that $y_1 = v_j$ for some $j \le k$ and $y_1 \ne v_{j'}$ for $j' > k$. Now the upper row of T' still contains $v_1 \ldots v_k$ even though y_1 has been removed to the bottom row. But this implies that y_1 also appears in $w_{q+1} \ldots w_{r+1}$ which is impossible since $w_{q+1} \not> v_q \not> v_{k+1} > y_1$.

Thus since the diagram for Case I shows that $w_{k+1} < v_{k+1}$, we have $w_{k+1} < y_1$. So, after sorting, we find that $z_{m_2} = w_{m_2}$; $z_{m_2+1} = w_{m_2+1}$; $\ldots, z_{k+1} = w_{k+1}$. So in tableaux T, T', the columns $m_1, \ldots, k$ agree as does the bottom entry of column $k+1$. But the top entry in column $k+1$ is larger in T' than in T. Hence $c_{T'}$ is lexically larger than c_T.

At last we deal with tableaux appearing in B in equation 1.1. Recall that tableaux in B arise from nontrivially shuffling the over/underlined entries and then resorting the rows. Let $W = w_{q+1} \cdots w_{c_3} v_{r+1} \cdots v_{\lambda_1}$. Let $W_{(1)}, W_{(2)}$ be a shuffle of W into two parts of size $\lambda_1 - r$ and $c_3 - q$ respectively. Since $w_{q+1} \not> v_q$, such a tableau will look like

$$T' = \begin{array}{lccl} v_{m_1} & \cdots & v_q & \boxed{v_{q+1} \ \cdots \ v_r \qquad \cdots\cdots W_{(1)} \cdots\cdots \qquad} \\ \quad w_{m_2} & \cdots & w_q & \boxed{\cdots\cdots W_{(2)} \cdots\cdots \qquad w_{c_3+1} \ \cdots \ w_{\lambda_2}} \end{array},$$

where, as before, the boxed elements must be sorted so that T' will be row standard. Denote the contents of the top and bottom rows of T' by $\varpi_{m_1}, \ldots, \varpi_{\lambda_1}$ and $z_{m_2}, \ldots, z_{\lambda_2}$ respectively.

Now let k be the last column in which the bottom rows of T and T' agree, i.e.

$$k = \min_{\substack{q+1 \leq i \leq c_3 \\ z_i \neq w_i}} i - 1.$$

I claim that k exists as defined–i.e. that there *is* an $i \leq r+1$ in which T and T' disagree in the bottom row and I further claim that $z_{k+1} > w_{k+1}$. We know by construction that $W_{(2)} \neq w_{q+1} \cdots w_{c_3}$. Hence, writing $W_{(2)} = W_{(2)_{q+1}} \cdots W_{(2)_{c_3}}$, there exists a minimal j such that $W_{(2)_j} \neq w_j$. Because $v_{r+1} \nrightarrow w_{r+1} = w_{c_3}$, this implies that $W_{(2)_j} > w_j$. But since also $w_{c_3+1} > w_{c_3}$, we find $z_{j+q} > w_{j+q}$. and $z_i = w_i$ for all $i < j+q$.

Subcase 1. Any letter appearing in the multiset difference $W_{(1)} - \{\{v_{r+1}, \ldots, v_{\lambda_1}\}\}$ is ($\nrightarrow$)-greater than w_{k+1}. But since $k < r$, the picture of case I shows that $v_{k+1} \nleftarrow w_{k+1}$, this means that on resorting the boxed elements, every element in $W_{(1)}$ stays in column $k+2$ or higher. Thus, $\varpi_{q+1} = v_{q+1}; \ldots; \varpi_k = v_k; \varpi_{k+1} = v_{k+1}$. Hence columns $m_1 \ldots k$ agree in T and T'. But the bottom of column $k+1$ is larger in T' than T. Thus $w_{T'}$ is lexically larger than w_T.

The above argument also generates the fact (unused in this proof, but see the comment after Corollary 1.6.15) that, T, T' agree in the top element of column $k+1$.

Subcase 2: suppose that $k \geq c_2 - 1$. This says that the bottom rows of T, T' agree at least through column $c_2 - 1$. Since $v_{c_2-1} \nleftarrow w_{c_2}$, we have immediately that the top rows of T, T' agree through column $c_2 - 1$. Now either the bottom of column c_2

changes (hence increases) so $c_{T'}$ is lexically larger than c_T and we are done or the the number of positive letters in the bottom row that equal w_{c_2} decreases. In the latter case, not only do the tableaux T, T' agree up to column $c_2 - 1$ but T' still has a flippable inversion in column c_2 since the entry in the top of that column now equals the positive letter w_{c_2} that remains at the bottom.

We repeat the straightening law on T', producing some tableaux with lexicographically larger modified column words and some tableaux that are unchanged in columns smaller than c_2 and unchanged at the bottom of column c_2 but which have fewer copies of w_{c_2} in their bottom rows. Eventually, we must run out of positive letters equal to w_{c_2} in the bottom row and so eventually the modified column word increases.

We now treat Case II. Here $c_1 = c_2$. This case follows essentially the same lines as Case I.

We apply the syzygy of Definition 1.6.4 to the letters in positions $c_0, \ldots, \lambda_1$ of the top row and positions $c_1, \ldots, c_3$ of the bottom row.

In particular, apply Corollary 1.6.11 to write

$$\left[\begin{array}{ccccccccccc} v_{m_1} & & \cdots & \overline{\overline{v_{c_0}}} & \overline{\overline{\cdots}} & \overline{v_{c_2}} & & \overline{\cdots} & & \overline{v_\lambda} \\ & w_{m_2} & \cdots & w_{c_1} & \cdots & \underline{\underline{w_{c_2}}} & \underline{\underline{\cdots}} & \underline{\underline{w_{c_3}}} & \cdots & \end{array}\right] = A + B. \tag{1.2}$$

The entries used to define the expression $Syz(T)$ in Definition 1.6.4 are marked by over/underlines. The entries that have been marked twice are positive letters all equal to each other. Here A is a signed sum over all shuffles of the $c_0 - m_1$ non-overlined entries in the top row, of the tableau formed by leaving $c_0 - 1 - m_1 - c_3 + c_2$ of the non-overlined entries in the top row and moving the other $c_3 - c_2 + 1$ non-overlined entries to the bottom row. *All* of the under/overlined elements are collected into the top row. As usual, the tableaux in A are sorted to be row-standard.

B, in turn, is a sum over all tableau obtained by non-trivially shuffling the under/overlined elements in place and then row-sorting.

Again we show that each tableau appearing in A or B has lexically column word lexically larger than c_T.

Start with the tableau in A. Suppose T' appears in A. To this purpose, let $w = v_{m_1} \dots v_r$ be the word being shuffled. Then we can write $W_{(1)}$, the first part of the shuffle, as $x_{m_1} \dots x_{t-1}$, and we can write $W_{(2)} = y_1 \dots y_{c_0-t}$ where $t = c_0 - c_3 + c_2 - 1$. It is easily verified that

$$T' = \begin{array}{llllllllllll} x_{m_1} & \cdots & x_{t-1} & v_{c_0} = & \cdots & = w_{c_3} & v_{c_2} & \cdots & & & \cdots & v_{\lambda_1} \\ \boxed{w_{m_2} \quad \cdots} & & & \cdots & & w_{c_2-1} & y_1 & \cdots & \boxed{y_{c_0-t}} & w_{c_3+1} & \cdots & w_{\lambda_2} \end{array},$$

where the boxed entries must be sorted in order to get a row-standard tableau. Denote the entries in the bottom row by $z_{m_2}, \dots, z_{\lambda_2}$. Denote the entries in the top row by $\varpi_{m_1}, \dots, \varpi_{\lambda_1}$.

Since $t < c_0$ we have $v_{c_0} \neq v_t$ and thus

$$k = \min_{\substack{m_1 \leq i \leq t \\ \varpi_i \neq v_i}} i - 1$$

is well defined. As in Case I, if $k + 1 < m_2$ we conclude directly that $w_{T'}$ is lexically larger than w_T.

Suppose that $k + 1 \geq m_2$. We show that $v_{k+1} \leq y_1$. Suppose to the contrary that $v_{k+1} > y_1$. Since y_1 comes from $v_{m_1}, \dots, v_{c_0-1}$ this says that $y_1 = v_j$ for some $j \leq k$ and $y_1 \neq v_{j'}$ for $j' > k$. But this says that if the letter y_1 occurs in the $y_1 \cdots y_{c_0-t}$ part of the shuffle, then the $x_{m_1} \cdots x_k$ part cannot start with $v_1 \cdots v_k$–contradiction.

Thus since the diagram for Case II shows that $w_{k+1} < v_{k+1}$, we find $w_{k+1} < y_1$. So, after sorting, we discover that $z_{m_2} = w_{m_2}$; $z_{m_2+1} = w_{m_2+1}$; $\dots, z_{k+1} = w_{k+1}$. So in tableaux T, T', the columns $m_1, \dots, k$ agree as does the bottom entry of column $k+1$. But the top entry in column $k+1$ is larger in T' than in T. Hence $c_{T'}$ is lexically larger than $c_{T'}$.

Now we handle the tableaux appearing in B in equation 1.1. Recall that tableaux in B arise from nontrivially shuffling the over/underlined entries and then resorting the rows. Let $c_4 = \min_{\substack{c_2 \le i \le \lambda_1 \\ w_{c_2} < v_i}} i$ Let $W = v_{c_4} \cdots v_{\lambda_1}$. Let $W_{(1)}, W_{(2)}$ be a shuffle of W into two parts of size $\lambda_1 - c_4 - s + 1$ and $1 \le s \le c_3 - c_2 + 1$ respectively. Since $w_{c_2} \not> v_{c_2-1} = v_{c_0}$, such a tableau will look like

$$T' = \begin{array}{cccccccc} & & & & & & \text{Column: } c_4+s-1 & \\ v_{m_1} & & \cdots \ v_{c_2-1} & w_{c_2} = & \cdots & & = w_{c_2} & \cdots\cdots\cdots W_{(1)} \cdots\cdots\cdots \\ & w_{m_2} & \cdots \ w_{c_2-1} & w_{c_2+s} & \cdots & & w_{c_3} & \boxed{\cdots W_{(2)} \cdots \quad w_{c_3+1} \ \cdots \ w_{\lambda_2}} \end{array},$$

where, as before, the boxed elements must be sorted so that T' will be row standard. Denote the contents of the top and bottom rows of T' by $\varpi_{m_1}, \dots, \varpi_{\lambda_1}$ and $z_{m_2}, \dots, z_{\lambda_2}$ respectively.

The top rows of T, T' agree through column $c_2 - 1$. Again either the bottom of column c_2 increases and we are done or the the number of positive letters in the bottom row that equal w_{c_2} decreases.

And again iterating the straightening law on T' eventually increases the modified column word. □

The preceding proposition only straightened two-rowed tableau whose top row started at least as far left as its bottom row. In fact the proof extends to skew shapes as follows.

Porism 1.6.13 *Algorithm* `row-straighten` *applied to a non-standard skew-tableau terminates and produces tableaux with strictly larger column word.*

Proof. Fill the vacant cells in the the top row with "fake" negative letters–i.e. new negative letters disjoint from and smaller than the letters in $\mathcal{L}$. Straighten this partition shaped tableau. The straightening algorithm will never touch those fake letters. Now remove the fake letters (removing these letters preserves identities in the letter-place algebra because it is essentially the algebra homomorphism that sends $(i|j)$ to $\delta_{i,j}$ where i is the ith fake negative letter. □

We have now established the correctness of Algorithm `row-straighten`.

Theorem 1.6.14 *The straight tableaux of shape D form a $\mathbf{Z}$-basis for $\mathcal{S}^D$ and Algorithm* `straighten-tableau` *expands any generator of $\mathcal{S}^D$ in terms of this basis. Further, given a row-standard tableau T, the expansion of $[T]$ is in terms of tableaux with larger column words than w_T.*

By Corollary 1.6.9 we can extend the preceding result.

Corollary 1.6.15 *Algorithms* `row-straightening` *and* `straighten-tableau` *produce tableaux with weakly larger modified column words than that of the input tableau.* $\square$

A sufficiently careful analysis of the preceding proof will also reveal the above fact.

1.7 Straightening for unions of row-convex tableaux

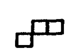
The definitions of section 1.5 say that given shapes D_1 and D_2, the $\mathbf{Z}$-modules $\mathcal{S}^{D_1}(\mathcal{L})$ and $\mathcal{S}^{D_2}(\mathcal{L})$ both live inside $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$. Thus we can ask to extend the "straight basis" results of the previous section to the $\mathbf{Z}$-linear span of $\mathcal{S}^{D_1}(\mathcal{L})$ and $\mathcal{S}^{D_2}(\mathcal{L})$.

Definition 1.7.1 *Let $\mathcal{D}$ be a collection of shapes. Define*

$$\mathcal{S}^{\mathcal{D}}(\mathcal{L}) = \operatorname*{span_{\mathbf{Z}}}_{D \in \mathcal{D}} \mathcal{S}^D(\mathcal{L}).$$

Similarly, let $R^{\mathcal{D}}(\mathcal{L})$ be the $\mathbf{Z}$-subalgebra of $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ generated by $\mathcal{S}^{\mathcal{D}}(\mathcal{L})$.

The naive approach, hoping that the union of the straight bases for the $\mathcal{S}^D(\mathcal{L})$ provides a basis for $\mathcal{S}^{\mathcal{D}}(\mathcal{L})$, fails to work. I will present a modification of the ideas that led to straight tableaux. This will provide a basis for $\mathcal{S}^D(\mathcal{L})$ indexed by "anti-straight" tableaux such that the union of the the anti-straight bases for all $\mathcal{S}^D(\mathcal{L})$ with $D \in \mathcal{D}$ indeed is a basis for $\mathcal{S}^{\mathcal{D}}(\mathcal{L})$.

We want to find that the intersection of the bases for $\mathcal{S}^D$ and $\mathcal{S}^{D'}$ is a basis for the intersection $\mathcal{S}^D \cap \mathcal{S}^{D'}$. For example, suppose $D =$  and $D' =$ . The intersection turns out to be $\mathcal{S}^{D''}$ where $D'' = \square\square\square$. So we need bases for the super-Schur modules corresponding to D, D' which each possess an "obvious" subset that is a basis for $\mathcal{S}^{D''}$.

As a first step, I will formalize the combinatorics that tells us $\mathcal{S}^{D''} \subset \mathcal{S}^{D} \cap \mathcal{S}^{D'}$. We rely on the inclusion portion of a James-Peel complex [JP79]. In particular, start with a shape D_1. Pick two rows $i < j$ in D_1 and define D_2 as follows. First, require that D_2 agrees with D_1 in all rows except i, j. Second, require D_2 to have a cell in column c of row i iff D_1 has a cell in either row i or row j of column c; D_2 has a cell in column c of row j iff D_1 has a cell in that column of both row i and row j. Observe that this operation creates a new tableau that also (modulo empty rows) has its rows sorted by their last cell. The following proposition is due (in the non superalgebra case) to James and Peel [].

Proposition 1.7.1 (James-Peel) *Let $\mathcal{L}_0$ be a signed set of letters. If the preceding definitions of D_1 and D_2 are preserved then $\mathcal{S}^{D_2}(\mathcal{L}_0) \subset \mathcal{S}^{D_1}(\mathcal{L}_0)$.*

Proof. Let $\mathcal{L} = \mathcal{L}_0 \uplus \{a^+, b^+\}$. Let T' be a tableau of shape D_1 such that each cell of row i sharing a column with a cell in row j contains the letter b and the remaining cells in rows i and j contain a's.

Let T'' be tableau of shape D_2 that agrees with T'' except in rows i and j. In row i it contains only a's and in row j only b's.

It is easy to see that $[T'] = \pm[T'']$. Further, any tableau of shape D_2 arises from polarizing the a's and b's into $\mathcal{L}_0$. We conclude that any generator of $\mathcal{S}^{D_2}$ is expressible as an element of $\mathcal{S}^{D_1}$ by suitably polarizing $\pm[T']$. $\square$

Observe that the proof is constructive–this fact guarantees that the proof of Theorem 1.7.7 can be rewritten as a constructive algorithm. The next step is to write down a basis for $\mathcal{S}^D$ consisting of tableaux of various shapes. Tableau of shape different from D will index elements of super-Schur-modules strictly contained in $\mathcal{S}^D$.

Definition 1.7.2 *A row-convex shape D_2 is called* a direct compression *of a row-convex shape D_1 if it is formed from D_1 by the construction preceding Proposi-*

tion 1.7.1.

A shape D' is a compression *of another shape D when it can be formed by a sequence of direct compressions starting with D. By construction, the superSchur module corresponding to a compression of a shape D is a submodule of $\mathcal{S}^D$.*

Example 1.7.1 If

$D =$ , $D' =$ , and $D'' =$ ,

then D' is a direct compression of D. The maximally compressed shape, D'', which is a compression of D is found by pushing all cells in D as far north in their columns as possible.

Roughly speaking, we will try to reduce a pair of rows forming a skew subtableau to a pair of rows forming a non-skew subtableau; given a tableau of shape D we will be straightening it into tableaux of shape D and tableaux having shapes which are compressions of D.

We are still aiming at a two-rowed straightening law in which valid indexing tableaux are determined by their modified column words. I now define the impermissible two-rowed skew subtableaux. Roughly, speaking these will be skew tableau for which there exists a column c such that cutting the tableau after column c, flipping the first half top to bottom and gluing the halves back together results in a row-standard row-convex tableaux. This formalizes as follows.

Definition 1.7.3 *A two-rowed skew tableau, $\begin{smallmatrix} u_{\mu_1}\cdots u_{\lambda_1} \\ v_{\mu_2}\cdots v_{\lambda_1 2} \end{smallmatrix}$, (with u_i, v_i in respectively the top and bottom of column i.) is* flippable *iff it is strictly skew (i.e. not of partition shape) and there exists $\mu_1 \le c < \lambda_2$ such that $u_c \lessdot\!\!+\, v_{c+1}$ and $v_c \lessdot\!\!+\, u_{c+1}$, or $v_{\mu_1 - 1} \lessdot\!\!+\, u_{\mu_1}$, or $v_{\lambda_2} \lessdot\!\!+\, u_{\lambda_2+1}$, or $\lambda_1 = \lambda_2$.*

We also need some condition analogous to requiring that a tableau have no flippable inversions.

Definition 1.7.4 *A* anti-inversion *in a tableau T is a pair of cells (i, c) and (j, c) in the same column of T such that $i < j$, $T_{i,c} <\!\!+ T_{j,c}$. This anti-inversion is* flippable *if there is no intermediate entry in cell $(i, c+1)$ such that $T_{i,c} <\!\!+ T_{i,c+1} <\!\!- T_{j,c}$.*

Putting these conditions together we arrive at the desired definition for an index tableau.

Definition 1.7.5 *A tableau T is* anti-straight *if it contains no flippable skew tableaux and no flippable anti-inversions.*

Theorem 1.7.2 *Let D be a row-convex shape. A basis for $\mathcal{S}^D(\mathcal{L})$ is given by all $[T]$ where T runs over all anti-straight tableaux on $\mathcal{L}$ having shape D' where D' is a compression of D.*

We will prove a stronger version of this result as Theorem 1.7.7.

Dealing with multiple shapes makes it useful to extend the notion of a (modified) column word to record the shape of a tableau as well as its content. To do this we will make the column word of an n-celled tableau T the lower half of a *biword* $\mathbf{c}_T = \binom{\hat{w}_1,\ldots,\hat{w}_n}{\check{w}_1,\ldots,\check{w}_n}$. For the upper word record as $\hat{w}_i$ the index of the column that the ith letter in the lower word came from. Two biwords are *equivalent* if there exists a permutation of the n columns that makes them equal. This motivates the following definitions.

If $\mathbf{w} = (\hat{w}, \check{w}) = {\hat{w} \atop \check{w}}$ is a biword with $\hat{w}$ comprised solely of negative letters then I will call w a *modified standard biword* when its upper word increases and $\hat{w}_i = \hat{w}_{i+1}$ implies $\check{w}_i -\!\!> \check{w}_{i+1}$. Define $\mathbf{w}_T$ to be the modified standard biword formed by permuting $\mathbf{c}_T$. Similarly, a *reverse standard biword* $\mathbf{w}'_T$ has column indices written in increasing order in its upper word and has the reverse column word w'_T as its lower word.

The following proposition claims that all modified column biwords occur as the modified column biword of a unique anti-straight tableau.

Input: A modified standard biword $\mathbf{w}$ of length n.
Output: A compressed anti-straight tableau, T, with $\mathbf{w}_T = \mathbf{w}$.

> Let $\mathbf{u}$ be the reordering of $\mathbf{w}$ such that $\hat{u}$ is increasing
> and $\hat{u}_i = \hat{u}_{i+1}$ implies $\check{u}_i <_- \check{u}_{i+1}$.
> Let T be an empty matrix
> **for** $k \leftarrow n$ **downto** 1
> Let i be the smallest (northmost) index such $(i, \hat{u}_k)$ is empty and either
> there is no cell in position $(i, \hat{u}_k + 1)$ or $\check{u}_k <_+ T_{i,\hat{u}_k+1}$.
> $T_{i,\hat{u}_k} \leftarrow \check{u}_k$.

Figure 1-6: **Algorithm** Anti-Straight-Filling

Proposition 1.7.3 *If $\mathbf{w}$ is a modified column biword then there exists a unique anti-straight tableau, T such that $\mathbf{w}_T = \mathbf{w}$.*

Proof.(existence) I claim that Algorithm `Anti-Straight-Filling` in Figure 1-6 produces the desired tableau. The construction of `Anti-Straight-Filling` guarantees row-standardness. Further, there can never be a flippable anti-inversion because if the larger element of an anti-inversion *could* have been placed in more northerly position while preserving row-standardness, then the algorithm would have done so.

It suffices to show the tableau produced by the algorithm contains no flippable skew tableaux. Consider a pair of rows $i < j$ that form a strictly skew subtableau. Say that k is the first column in which row i has a cell. By construction we may assume that rows i, j have no flippable anti-inversions. But also by construction we have the following situation:

$$\begin{tikzcd}[row sep=small]
\text{Column \#} & k-1 & k & \\
 & & y & \cdots \\
\cdots & x \arrow[ur, "-"] & z \arrow[u] & \cdots
\end{tikzcd}$$

The lack of flippable anti-inversions implies that all pairs of cells in the same column of rows i, j must form anti-inversions Since no anti-inversions are flippable, this guarantees that the pair of rows i, j is not flippable. □

Proof.(uniqueness) Just as in the proof of Proposition 1.6.3 we can assume that any tableau with modified column biword $\mathbf{w}$ can be produced by reading through $\mathbf{u}$ in reverse order and inserting $\check{u}_k$ into cell $(j_k, \hat{u}_k)$. Let $\underline{j}$ be the sequence corresponding to the tableau T produced by `Anti-Straight-Filling`. Suppose in fact that T' is a tableau with the same modified column biword as T. Let $\underline{j}'$ be the filling sequence corresponding to T'. (If multiple copies of some letter x appear in some column then use the filling sequence that puts the first from right occurrence of x into the most northerly cell occupied by x, the second occurrence into the next most northern cell etc.) Let k_0 be the largest integer (i.e. earliest in the filling algorithm) such that $j_{k_0} \neq j'_{k_0}$. If cell $(j_{k_0}, \hat{u}_{k_0})$ of T' is nonempty, this guarantees that some element smaller than $\check{u}_{k_0}$ is placed there. By construction, $j_{k_0} < j'_{k_0}$. Hence the cells $(j_{k_0}, \hat{u}_{k_0})$ and $(j'_{k_0}, \hat{u}_{k_0})$ form a flippable anti-inversion in T'.

On the other hand, if $(j_{k_0}, \hat{u}_{k_0})$ is empty then rows j_{k_0} and j'_{k_0} form a skew subtableau of T'. Since by construction $T'_{j_{k_0}, \hat{u}_{k_0}} \nleftarrow T'_{j_{k_0}, \hat{u}_{k_0}+1}$, this is in fact a flippable skew tableaux. We conclude that T' is not anti-straight. □

Corollary 1.7.4 *The elements, $[T]$, of $Super([\mathcal{L} \mid \mathcal{P}])$ corresponding to distinct anti-straight tableaux T are linearly independent.*

Proof. By Proposition 1.6.7 the initial monomials of $[T]$ and $\mathrm{Tab}(T)$ are all distinct and appear with coefficient ± 1. □

For the next proposition, I will need a partial order on shapes.

Definition 1.7.6 *Two shapes, D, D' are comparable when, viewing D, D' as 0/1-matrices, $(1,1,1,\ldots,1)\cdot D = (1,1,1,\ldots,1)\cdot D'$. Define $D < D'$ when $D\cdot(1,1,1,\ldots,1)^t$ is lexically bigger than $D' \cdot (1,1,1,\ldots,1)$. Roughly speaking, D is small when it has many cells in its top rows.*

This poset is simply a disjoint union of chains and each chain has as minimal element a maximally compressed shape.

Porism 1.7.5 *Let T be the tableau produced by* `Anti-Straight-Filling`. *The shape D of T is minimal among the shapes of all tableau T' with $\mathbf{w}_{T'} = \mathbf{w}_T$.*

Proof. I claim that T' can be compressed to T by combinatorial operations preserving its modified column biword. Preserve the notation from the proof of Proposition 1.7.3.

Suppose the rows j_{k_0}, j'_{k_0} form a strictly skew subdiagram of T'. Flip top to bottom the columns of these two rows up through column $\hat{u}_{k_0}$ to find a tableau of smaller shape with the same modified column biword.

Suppose instead that row j_{k_0} contains row j'_{k_0}. Then there exists a segment of columns in these two rows starting just right of the first non anti-inversion left of column k_0 and ending in the anti-inversion in column k_0. Flipping this segment top to bottom yields a row-standard tableau T'' of the same shape. Now, when read from the right, $c_{T''}$ is lexically bigger than than $c_{T'}$.

This last fact guarantees that if we iterate the above process it will eventually terminate. Termination means that we have reached the unique anti-straight tableau T with given modified column biword. But since this process constructs T as a compression of any tableau T' with $\mathbf{w}_{T'} = \mathbf{w}_T$, we have just shown that T has minimum shape. □

Observe that this proof provides an algorithm for transforming any row-convex tableau with modified column word w into the tableau found by `Anti-Straight-Filling` by a repeated process of flipping the same segment in two rows or flipping initial segments ending at the same point in a two-rowed skew subdiagram.

The immediately preceding analysis proves the following.

Corollary 1.7.6 *If T and T' of shapes D, D' are two tableau such that $w_T = w_{T'}$ then there exists a tableau S with $w_S = w_T$ and whose shape is a (possibly trivial) compression of both D and D'.* □

We are now ready to prove the promised strengthening of Theorem 1.7.2.

Theorem 1.7.7 *Let $\mathcal{B}$ be the set of all anti-straight tableau such that the shape of $T \in \mathcal{B}$ is a compression of some shape in $\mathcal{D}$. Then the elements of the (multi)set $\{[T] \mid T \in \mathcal{B}\} \subset Super([\mathcal{L} \mid \mathcal{P}])$ are a $\mathbf{Z}$-basis for the $\mathbf{Z}$-module $\mathcal{S}^{\mathcal{D}}$.*

Proof. It suffices to prove spanning. The spanning argument is an induction on the modified column words and the shapes of the tableaux involved.

Start with $p \in \mathcal{S}^{\mathcal{D}}$. Without loss of generality, we can assume that any two shape in $\mathcal{D}$ are comparable in the sense of Definition 1.7.6. That is

$$p = \sum_i \alpha_i \mathrm{Tab}(T_i) \qquad , \tag{1.3}$$

with $0 \neq \alpha_i \in \mathbf{Z}$. Let $D_i \in \mathcal{D}$ be shape(T_i). By Theorem 1.6.14 we can assume that each T_i is straight.

Now break p down as $p = q + r$ where $q = \sum_{j \in J} \alpha_j \mathrm{Tab}(T_j)$ and the u_{T_j} for $j \in J$ are the lexically minimal modified column words arising from tableaux in equation 1.3. Since r is written in terms of tableau with lexically bigger modified column words, by induction the elements of r can be written as a linear combination of basis elements indexed by anti-straight tableaux. It suffices to show the same for q.

Let S be the anti-straight tableau with $\mathbf{w}_S = \mathbf{w}_{T_j}$. Let $\alpha = \sum_j \alpha_j$. Then

$$q = \alpha \mathrm{Tab}(S) + \sum_j \alpha_j \left(\mathrm{Tab}(T_j) - \mathrm{Tab}(S)\right).$$

So it suffices to be able to write $[T_j] - [S]$ as a linear combination of compressed straight tableaux. But because the shape of S is a compression of shape (T_j) we can, for each j, write $[S]$ as linear combination of polynomials indexed by tableaux having the same shape as T_j. The proof of Proposition 1.7.1 shows how to accomplish this constructively. So, using Algorithm `straighten-tableau` we can rewrite $[T_j] - [S]$ in terms of straight tableaux of shape T_j which by Corollaries 1.6.4 and 1.6.9 have lexically larger modified column words than does T_j. By induction, these tableaux may be expanded in terms of anti-straight tableaux whose shapes are compressions of shape (T_j). □

Corollary 1.7.8 *If D, D' are two row-convex shapes, then a basis for for $\mathcal{S}^D(\mathcal{L}) \cap \mathcal{S}^{D'}(\mathcal{L})$ is given by all $[T]$ where T runs over all anti-straight tableaux on $\mathcal{L}$ having shape D'' which is a compression of both D and D'.*

Proof. We have already shown independence ant that the elements $[T] = \pm\text{Tab}(T)$ lie in the intersection. The algorithm used in the proof of Theorem 1.7.7 shows that if $\sum \alpha_i[S_i] = \sum \beta_j[T_j]$ with shape$(S_i) = D$ and shape$(T_i) = D'$, then each summation can be rewritten an a $\mathbf{Z}$-linear combination of elements $[T]$ where T is anti-straight having shape as specified above. □

1.8 Flagged super-Schur modules

The *flagged* Schur modules $\mathcal{S}_f^D$ have been the subject of considerable interest (see for instance [LS90, RS96b, LM96]). One advantage of having bases compatible with the diagonal term orders described on page 40 is that they descend to bases of the corresponding flagged module. I will start by formalizing the notion of a flagged superSchur module.

Definition 1.8.1 *Let $\underline{f}$ be a weakly increasing sequence of letters in the alphabet $\mathcal{L}$. Regard this sequence as indexed by elements of $\mathcal{P}$. The* flagged superSchur module *$\mathcal{S}_{\underline{f}}^D(\mathcal{L})$ is the subquotient of $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ equal to the image of the submodule $\mathcal{S}^D(\mathcal{L})$ under the map $\phi_{\underline{f}}$ which quotients $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ by setting $(l|p) = 0$ whenever $l > f_p$.*

A tableau T is flagged *if, in each column i, T has no entry exceeding f_i.*

The flagged tableaux are natural to consider since if T is row-standard (of any shape) and fails to be flagged, then $\phi_f([T]) = 0$– in particular each monomial in the expansion of any row in which the flagging condition is violated has some factor $(l|p)$ with $l > f_p$.

Classical results tell us that if D is a skew-tableau then a basis for $\mathcal{S}^D_{\underline{f}}$ is given by all $[T]$ such that T is standard and flagged. This result carries over to flagged row-convex superSchur modules.

Theorem 1.8.1 *Let D be a row-convex shape. Fix a weakly increasing flag $\underline{f}$. A basis for $\mathcal{S}^D_{\underline{f}}(\mathcal{L})$ is given by the elements $\phi_{\underline{f}}([T])$ where T runs over all flagged, straight tableaux of shape D with entries chosen from $\mathcal{L}$.*

Proof. It suffices to show that the purported basis elements are linearly independent. In particular I claim that their initial terms under any diagonal term order are still distinct. But by Proposition 1.6.6, and the observation immediately following it these initial terms are all distinct (since the tableaux T were straight.) They are all basis elements for $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$, and by the flagging condition, they are all nonzero under ϕ_f. □

This result has the following easy generalization. Let $\underline{f}, \underline{g}$ both be weakly increasing sequences of letters in $\mathcal{L}$ indexed by elements of $\mathcal{P}$ such that $\underline{f} \leq \underline{g}$ compentwise. Define the doubly flagged superSchur module $\mathcal{S}^D_{\underline{f},\underline{g}}(\mathcal{L})$ to be the image of $\mathcal{S}^D(\mathcal{L})$ under the map $\phi_{\underline{f},\underline{g}}$ quotienting $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ by the ideal generated by $\{(l|p) : l \notin f_l, \ldots, g_l\}$. Call a tableau T *doubly flagged* with respect to $\underline{f}, \underline{g}$ if every entry in column i is between f_i and g_i. The same proof as above shows the following.

Theorem 1.8.2 *A basis for $\mathcal{S}^D_{\underline{f},\underline{g}}(\mathcal{L})$ is given by the elements $\phi_{\underline{f}}([T])$ where T runs over all doubly flagged, straight tableaux of shape D with entries chosen from $\mathcal{L}$.* □

1.9 A flagged corner-cell recurrence.

This section shows how the straightening algorithm immediately produces a branching rule expressing a $pl_{\mathcal{L}}$-representation, corresponding to a row-convex shape D, in terms of $pl_{\mathcal{L}\setminus\{a\}}$ representations (for some $a \in \mathcal{L}$) corresponding to subshapes of D. In the case that $\mathcal{L} = \mathcal{L}^-$ and after tensoring with $\mathbf{Q}$, this recovers, by completely different means, the column-convex case of the branching rule in [RS96c]. The branching presented below generalizes to the case of flagged super-Schur modules.

I start with the observation that there are alternatives to Algorithm Straight-Filling for producing straight tableau with specified column content and shape. Column content and shape is of course determined by an equivalence class of biwords. We could sort the biwords by their *lower* word and fill the tableau by starting with an empty tableau of the given shape and adding successive letters as follows (reading left to right through the biword). Suppose $\check{w}_j$ is the next letter. It is supposed to appear in column $\hat{w}_j$. Place it in the northmost available cell (say row i) in column $\hat{w}_j$ such that either $(i, \hat{w}_j - 1)$ is not in the diagram or such that the cell $(i, \hat{w}_j - 1)$ contains a letter x with $x <_{+} \check{w}_j$. If no such cell exists, then the biword does not arise from a straight tableau of the given shape. To check this algorithm, it suffices to show that if we put $\check{w}_j$ into some more southerly row i', then the inversion created in cells $(i, \hat{w}_j)$ and $(i', \hat{w}_j)$ is flippable. This is immediate.

Definition 1.9.1 *Let D be a sorted row-convex shape. Define* a horizontal strip, *E^+, in D to be any subset of the cells of D such that there exists a shape D straight tableau, T, on some alphabet $a^+ < b_1 < b_2 < \cdots$ where the cells in T that contain a^+ are precisely the cells of E. Similarly, define* a vertical strip, *E^- as any set of cells containing all the negative letters a^- appearing in some straight tableau of shape D on some alphabet $a^- < b_1 < b_2 < \cdots$.*

Let $\underline{g}, \underline{f}$ be two weakly increasing sequences of letters indexed by the elements of $\mathcal{P}$. A vertical or horizontal strip is a-flagged (with respect to $\underline{g}, \underline{f}$) if it contains cells only in columns i where $g_i \leq a \leq f_i$.

Note that strips are allowed to be empty!

An immediate consequence of the algorithm which started this section is the fact that a vertical or a horizontal strip is immediately determined if one knows the multiset of column indices appearing in the cells of the strip.

Suppose that D is a row-convex shape and that $\mathcal{L}$ is an alphabet. Let $\underline{g}, \underline{f}$ be two weakly increasing sequences in $\mathcal{L}$ indexed by the elements of $\mathcal{P}$. Define

$$ch^D_{\underline{g},\underline{f}}(\mathcal{L}) = \sum_{T \text{ straight}} \prod_{(i,j)\in D} t_{T_{(i,j)}}$$

where the sum runs over all $\underline{g}, \underline{f}$-doubly flagged straight tableaux of shape D on $\mathcal{L}$ and where $T_{(i,j)}$ is the (i,j)th entry of T. In the case that $\underline{f}$ and $\underline{g}$ are trivial–they contain respectively only the largest and smallest elements of $\mathcal{L}$–and $\mathcal{L}$ contains letters of only one sign, then this amounts to the formal character of the $GL(|\mathcal{L}|)$-representation $\mathcal{S}^D(\mathcal{L})$. If just one of $\underline{f}, \underline{g}$ is trivial, we get the formal character of a representation of a Borel subgroup. The following identity is immediate from the definition of a straight tableau.

Proposition 1.9.1 *Fix two weakly increasing sequences $\underline{g}, \underline{f}$ of letters, and choose $a \in \mathcal{L}$. If D is a sorted row-convex diagram, then*

$$ch^D_{\underline{g},\underline{f}}(\mathcal{L}) = \sum_{\text{strip}E} ch^{D/E}_{\underline{g},\underline{f}}(\mathcal{L}\backslash\{a\})$$

where the sum runs over all a-flagged horizontal (respectively vertical) strips E in D when $a \in \mathcal{L}$ is positive (respectively negative), and where D/E is the diagram formed by removing E from D. □

The next proposition establishes a filtration for $pl_{\mathcal{L}}$-modules $\mathcal{S}^D(\mathcal{L})$ that realizes this identity. We start by ordering all multisets in $\mathcal{P}^- = \{1, 2, \ldots\}$ by dominance order–i.e. $\mathcal{I}_1 < \mathcal{I}_2$ when for all i there are at least as many copies of elements in $\{1, \ldots, i\}$ living in $\mathcal{I}_1$ as in $\mathcal{I}_2$. Let a be the smallest letter in $\mathcal{L}$. Define

$$\mathcal{S}^D(\mathcal{L};\mathcal{I}) = \operatorname*{span}_{T \text{ straight}}{}_{\mathbf{Z}} \{[T]\}$$

where T runs over all shape D straight tableaux on $\mathcal{L}$ in which the multiset $\mathcal{I}_1$ of columns indicating where a appears in T must weakly dominate $\mathcal{I}$.

Theorem 1.9.2 *Let a be a letter in $\mathcal{L}$. (Without loss of generality order $\mathcal{L}$ so that a is minimal.) Choose a multiset $\mathcal{I}$ from $\mathcal{P}$. If a is positive (respectively negative) and E is a, necessarily unique, horizontal (respectively vertical) strip in D occupying as many cells in column i as there are i's in $\mathcal{I}$ then*

$$\mathcal{S}^D(\mathcal{L};\mathcal{I}) \Big/ \sum_{\mathcal{I}_0>\mathcal{I}} \mathcal{S}^D(\mathcal{L};\mathcal{I}_0) \quad \simeq \quad \mathcal{S}^{D/E}(\mathcal{L}\backslash\{a\})$$

as a $pl_{\mathcal{L}\backslash\{a\}}$-module. Here D/E is the shape formed by removing E from D.

Proof. It suffices to observe that given a row-standard tableau T such that the cells occupied by a comprise E, then any tableaux appearing in the straightened form of $[T]$ has the cells occupied by a form a strip E' determined by a multiset $I' \geq I$. This can be seen by directly examining the straightening relations. In particular, any straightening relation which moves the a's produces a row-standard tableau in which the a's form a horizontal (respectively vertical) strip indexed by some $I' > I$. $\square$

A more sophisticated result on the allowable contents of a tableau appearing in the straightening of $[T]$ is proved at the end of Chapter 3, Section 3.6.

The preceding result generalizes to the case in which the super-Schur modules are replaced by their singly flagged versions. In this case the isomorphism is over the subalgebra of $U(pl_L)$ generated by all $E_{b,a}$ for $b \geq a$ with $a, b \in \mathcal{L}$. Finally, when $\mathcal{L} = \mathcal{L}^-$, both the ordinary and flagged results generalize directly to the quantum Schur modules studied in Chapter 3.

Chapter 2

Rings generated by products of determinants

2.1 Introduction

In Chapter 1 I used the term "straightening law" essentially as a synonym for any roughly combinatorial algorithm to reduce an element of a free module to normal form modulo some relations. More specifically, the modules in Chapter I were constructed inside a ring. In this chapter I lift some of the results in Chapter 1 concerning modules $\mathcal{S}^{\mathcal{D}}$ to results concerning the subrings $R^{\mathcal{D}}$ they generate.

While I will not here presume to formally define a straightening law in the general case, there are several extant constructs in ring theory that encompass much of what one wants in a straightening law. Section 2.2 interprets the results of Chapter 1 in terms of generalized Groebner bases for the rings $R^{\mathcal{D}}$.

It should be noticed that in contrast to Algorithm `straighten-tableau`, the algorithm used to prove Theorem 1.7.7 bears little formal resemblance to Groebner basis reduction. It is in fact inspired by the SAGBI (Subalgebra Analogue of a Groebner Basis for Ideals) basis of [RoSw90, KaMa89] which I recall below. The heart of this chapter is the application of the combinatorics of straight and anti-straight tableaux from Chapter I in combination with the noncommutative generalization of SAGBI bases that I introduce in Section 2.3.

2.2 Quadratic Groebner bases for some $R^{\mathcal{D}}$.

I will recall the definition of a *noncommutative Groebner basis* following Bergman [Be78]. A noncommutative Groebner basis for an ideal I in the free noncommutative $\mathbf{k}$-algebra $\mathbf{k}\langle b_1, b_2, \ldots, b_n\rangle$ on variables $b_1, \ldots, b_n$ is defined by a *Noetherian*–all strictly increasing sequences are finite–total order $\prec$ on all monomials in the variables b_i and a finite set of noncommutative polynomials $p_i(b_1, \ldots, b_n)$ which generate I. This order is required to be a *term order*, namely given two monomials $M \preceq M'$ we must have that $NMP \preceq NM'P$ for any other monomials N, P. Given a noncommutative polynomial $p \in \mathbf{k}\langle b_1, b_2, \ldots, b_n\rangle$ denote the *smallest* monomial appearing in p by $\mathrm{init}_{\prec}(p)$. Denote that monomial times its coefficient by $\mathrm{LT}_{\prec}(p)$ and denote $p - \mathrm{LT}_{\prec}(p)$ by $\mathrm{tail}_{\prec}(p)$. Consider the set $\mathcal{M}$ of all monomials not divisible by $\mathrm{LT}(p_i)$ for any i. The set $\{p_i(b_1, \ldots, b_n) : i \in \mathcal{I}\}$ forms a *Groebner basis* with respect to $\prec$ for the ideal I that it generates when the projections of the monomials in $\mathcal{M}$ are linearly independent in $\mathbf{k}\langle b_1, \ldots, b_n\rangle/I$. In Groebner basis theory the monomials in $\mathcal{M}$ are usually called *standard monomials*. Any Groebner basis immediately provides a straightening algorithm for expressing an element $p \in \mathbf{k}\langle b_1, \ldots, b_n\rangle/I$ as a linear combination of standard monomials when p is presented as a polynomial in $\mathbf{k}\langle b_1, \ldots, b_n\rangle$; choose a term in p divisible (in $\mathbf{k}\langle b_1, \ldots, b_n\rangle$) by the leading term of some p_i and replace that multiplicand (choose one arbitrarily if necessary) by $\mathrm{tail}(p_i)$. Repeat until done. The assumption that $\prec$ is a total order can be relaxed so long as each p_i in the Groebner basis has a unique leading monomial. In this case $\mathrm{LT}(p)$ is the sum, with coefficients, of all minimal monomials in p.

The *degree of a Groebner basis* is the maximal degree of the initial monomials $\mathrm{init}(p_i)$ for the p_i in that basis. Unless otherwise stated I will assume the degree of of a generator b_i of the noncommutative polynomial ring is 1. If the ring $\mathbf{k}\langle b_i\rangle$ is replaced by a commutative ring all the above definitions carry over word for word to the commonly used notions in commutative algebra. However, this means that if R is presented as a quotient of the noncommutative polynomial ring by $\mathbf{k}\langle b_i\rangle$ and alternately as a quotient of the commutative polynomial ring $\mathbf{k}[b_i]$ by I_2, then I_1 and I_2 can have Groebner bases (with respect to essentially the same term order) of different degrees. Essentially, a reduction rule (of degree $\deg(M)+2$) in the noncommutative setting that replaces b_iMb_j with $\sum_l b_lMb_l'$ becomes the degree 2 rule $b_ib_j \mapsto \sum_l b_lb_l'$ in the commutative setting. A similar lowering of degree occurs if one fixes quasi-commutation relations between generators, namely $b_ib_j = \alpha_{i,j}b_jb_i$ for all i,j and with $\alpha_{i,j} \in \mathbf{k}$. In general, call a $\mathbf{k}$-algebra, A, *quasi-commutative* if it is generated by elements which quasi-commute. We will adopt this refined notion of degree when working with quasi-commutative rings.

In general, given an ideal I in an algebra generated by quasi-commuting variables, define $\mathrm{LT}\,(I)$ to be the ideal spanned by all $\mathrm{LT}\,(p)$ for all $p \in I$.

Fix an alphabet $\mathcal{L}$ and a shape D. Let S be the quasi-commutative polynomial $\mathbf{Z}$-algebrafreely generated by variables

$$X_T \;:\; T \text{ is a straight tableau of shape } D \text{ filled from } \mathcal{L}.$$

subject to the commutation relation

$$X_TX_{T'} = (-1)^{(\#\text{ of positive letters in } T)\ (\#\text{ of positive letters in } T')}\ X_{T'}X_T.$$

Caveat: S can have $\mathbf{Z}$-torsion. Since $[T]^2 = 0$ if T has an odd number of positive letters, we could have imposed $X_T^2 = 0$ if we had wanted to avoid this torsion.

Grade this algebra in the usual fashion by defining X_T to have degree 1.

A degree d monomial $\prod_{i=1}^d X_{T_i}$ in this ring is indexed by the sequence $T_1,\ldots,T_d$ of d tableau of shape D. We could rewrite this sequence as a single tableau consisting of the first row of T_1 atop the first row of T_2 through the first row of T_d atop the second row of T_1 etc. Denote this new tableau by $T_1 \circ T_2 \circ \cdots \circ T_d$. Denote its shape by $d \circ D$.

Suppose that T, T' are tableaux of shape D. By Theorem 1.6.14, we have $[T][T'] = \sum_i \alpha_i [S_i][S_i']$ where each tableau $S_i \circ S_i'$ is straight of shape $2 \circ D$. If T, T' are straight tableaux, define a syzygy $S_{T,T'} \in \mathbf{Z}[X_T]$ by

$$S_{T,T'} = X_T X_{T'} - \sum_i \alpha_i X_{S_i} X_{S_i'}.$$

The following result claims that the above syzygies actually form a Groebner basis. First we define the appropriate term order. Suppose that M and M' are two monomials in the X_T's. We only require that M, M' to be comparable when they have the same degree d. Suppose that T and T' are the tableaux of shape $d \circ D$ indexing these monomials. Define $M \prec_{\mathrm{cw}} M'$ iff these tableaux's column words satisfy $c_T < c_{T'}$ in lexicographic order.

Theorem 2.2.1 *If D is a row-convex diagram then the relations $S_{T,T'}$ form a degree 2 Groebner basis with respect to the order $\prec_{\mathrm{cw}}$. for the ideal I_D of relations in S between generators of R^D.*

Proof. Let $T_1, \ldots, T_k$ be straight tableaux of shape D. By Theorem 1.6.14 the products $\{\prod_i [T_i] \ : \ T_1 \circ \cdots \circ T_k \text{ is straight}\}$ are linearly independent in $S/(S_{T,T'})$. It suffices to show that reduction (with respect to $\prec$) by the listed syzygies yields a linear combination of straight tableaux. In other words, it suffices to check that if $T_1 \circ \cdots \circ T_k$ fails to be straight then $\prod_i [T_i]$ is non-standard in the Groebner basis sense. That is there must exist T_j, T_k such that $T_j \circ T_k$ fails to be straight. But since straightness can be checked by examining two rows at a time, this is clear. $\square$

Now let $\mathcal{D}$ be a collection of shapes and consider the ring $R^{\mathcal{D}}$ generated by $S^{\mathcal{D}}$. Present this ring as above as the quotient of a quasi-commutative ring with generators X_T by an ideal $I_{\mathcal{D}}$. We say that $R^{\mathcal{D}}$ is multihomogeneous with respect to $\mathcal{D}$ if it is multigraded by $\mathbf{Z}^{|\mathcal{D}|}$ as follows. View elements of $\mathbf{Z}^{|\mathcal{D}|}$ as functions from $\mathcal{D}$ to $\mathbf{Z}$ and require that the generator X_T where shape$(T) = D$ has multidegree g such that $g(D') = \delta_{D,D'}$. The following result is immediate from the techniques of the preceding proof. The operation, $\circ$, of *tableau composition* generalizes as follows. The tableau $T \circ T'$ is a new tableau formed by interleaving the rows of T and T' so that the right hand edge of $T \circ T'$ is still partition-like, and so that if row i of T and row i' of T' end in the same position, then in $T \circ T'$, row i from T precedes row i' from T'. The definition of the syzygy $S_{T,T'}$ generalizes directly.

Porism 2.2.2 *If $\mathcal{D}$ contains only row-convex shapes and $R^{\mathcal{D}}$ is multihomogeneous with respect to D then, for any ordering of $\mathcal{D}$, $I_{\mathcal{D}}$ has a degree 2 Groebner basis with respect to $\prec_{\mathrm{cw}}$ consisting of all relations $S_{T,T'}$ with* shape(T) *and* shape(T') *in $\mathcal{D}$. In particular, the standard monomials are the $\prod_{i=1}^{k} X_{T_i}$ where T_i has shape $D_i \in \mathcal{D}$, the sequence $D_1, \ldots, D_k$ increases with respect to the chosen order on $\mathcal{D}$, and the tableau $T_1 \circ \cdots \circ T_k$ is straight.* □

If $\mathcal{D}$ consists of all one-rowed shapes starting in column 1 and ending in columns $l_1 < l_2 < \ldots < l_k$, then this recovers the usual degree 2 Groebner basis for the multihomogeneous coordinate ring (under the Plücker embedding) of the variety of flags $V_1 \subset \cdots \subset V_k$ where $\dim(V_i) = l_k$. In fact, if the ith shape in the set $\mathcal{D}$ is a single row that starts in column l'_i and ends in $l''_i = l'_i + l_k$ and if $l'_i \geq l'_{i+1}$ and $l''_i \leq l''_{i+1}$ for all i then we discover an unusual degree 2 Groebner basis for this ring. If one "flags" the rings–i.e. consider the ring generated by a flagged super-Schur module from Section 1.8 of Chapter 1–then the various $\underline{l}'$ above produce rings (corresponding when $\mathcal{L} = \mathcal{L}^-$ to certain Schubert varieties) and for these rings we still obtain a degree 2 Groebner basis.

The anti-straight basis theorem can be exploited to provide Groebner bases for rings $R^{\mathcal{D}}$ which are not multihomogeneous with respect to $\mathcal{D}$. First I will need to introduce some concepts related to shapes and their compressions. Suppose that $\mathcal{D}$ is a collection of shapes. If D, D' are two shapes denote by $D \circ D'$ their *composition*, namely the shape formed by interleaving rows of D and D' exactly as for tableaux. Denote by $\mathbf{Z}\{\mathcal{D}\}$ the monoid (under composition) generated by $\mathcal{D}$. The set $\mathbf{Z}\{\mathcal{D}\}$ is *closed under compression* when any compression of any shape in $\mathbf{Z}\{\mathcal{D}\}$ remains in $\mathbf{Z}\{\mathcal{D}\}$.

We start by expressing the $\mathbf{Z}$-algebra $R^{\mathcal{D}}(\mathcal{L})$ generated by $\mathcal{S}^{D}(\mathcal{L})$ as a quotient of the quasi-commutative polynomial ring S' generated by

$$Y_T : Y \text{ is anti-straight with shape in } \mathbf{Z}\{\mathcal{C}\}$$

subject to the usual quasi-commutation relation

$$Y_S Y_T = (-1)^{(\#\text{ of positive letters in } S)\ (\#\text{ of positive letters in } T)}\ Y_T Y_S.$$

In particular, express $R^{\mathcal{D}}$ as the quotient of the $\mathbf{Z}$-algebra S by an ideal $J_{\mathcal{D}}$. since $R^{\mathcal{D}}$ is not graded by degree in Y_T's, we will need to be able to distinguish in our term order between monomials of differing degrees. (If we set the degree of Y_T to be the number of cells in T then we recover a graded structure, but we will still need to distinguish in our term order between degree d tableaux of differing shapes.) If M, M' are indexed by tableaux T, T' of shape $D, D' \in \mathbf{Z}\{\mathcal{D}\}$ then define $M \prec'_{\mathrm{cw}} M'$ if $D > D'$ in the order of definition 1.7.6. Suppose that $D = D'$. then $M \prec'_{\mathrm{cw}} M'$ if $w_T < w_{T'}$ in lexicographic order or if $w_T = w_{T'}$ and $c_T <_r c_{T'}$ where $<_r$ is lexicographic order when c_T, c'_T are read from the right.

We are now ready to define the basic syzygies I will use for constructing the non-homogeneous Groebner bases. Suppose that T, T' are two anti-straight tableaux whose shapes belong to $\mathcal{D}$. By theorem 1.7.7 we can write $[T][T'] = \sum_i \alpha_i [S_i] + \sum_j \beta_j [S'_j][S''_j]$ where each tableau S_i and each tableau $S_j \circ S'_j$ is anti-straight. Further the shape of each S_i is a compression of that of $T \circ T'$ and S'_j, S''_j have the same shapes respectively as T'_j, T''_j. This shows that for each i, $Y_T Y_{T'} < Y_{S_i}$. The following lemma shows that the products $Y_{S'_j} Y_{S''_j}$ are also bigger than $Y_T Y_{T'}$.

Lemma 2.2.3 *If, given a tableaux T, $[T]$ is expanded according to theorem 1.7.7 into a linear of $[S_i]$ where each S_i is anti-straight, then the shape of each S_i is at least as small as the shape of T. Further, for each S_i, either $w_{S_i} > w_T$ or $w_{S_i} = w_T$ and $c_{S_i} >_r c_T$.*

Proof. This fact is implicit in the proof of Theorem 1.7.7. If in the straightening algorithm of that proof we replace the tableaux with smallest modified column word by a more compressed tableau then we have increased in the order $\prec'_{\mathrm{cw}}$.

Initial tableaux that are replaced with a tableaux of the same shape either see their modified column word decrease or, by the nature of algorithm `anti-straight-filling1`, the new column word is larger with respect to $<_r$. □

Reflecting the two-rowed straightening algorithm of Chapter 1 from left to right provides a method for reducing the tableaux from the last step of the above proof which would have been replaced by a tableau of the same shape.

With the notation above define

$$Syz_{T,T'} = Y_T Y_{T'} - \sum_i \alpha_i Y_{S_i} - \sum_j \beta_j Y_{S'_j} Y_{S''_j}.$$

Theorem 2.2.4 *If $\mathcal{S}_1, \mathcal{S}_2$ are two sets of column indices and if $\mathcal{D}$ consists of all rows starting at a column in $\mathcal{S}_1$ and ending at a column in $\mathcal{S}_2$, then the ideals of relations among generators of R^D and R^D_f have degree 2 quasi-commutative Groebner bases given respectively by $Syz_{T,T'}$ and $\phi_{\underline{f}}(Syz_{T,T'})$ with $T, T' \in \mathcal{D}$.*

Proof. To show that the syzygies $Syz_{T,T'}$ form the desired Groebner basis for $R^{\mathcal{D}}$ one need show that given any non-straight tableaux $T_1 \circ \cdots \circ T_k$ where the shape of each T_i lies in $\mathcal{D}$ there exists $j \leq k$ such that $T_j \circ T_k$ fails to be anti-straight. But since anti-straightness like straightness relies only upon comparing pairs of rows, this is clear. □

The above theorems carry over immediately to the rings $R^D_f(\mathcal{L})$ generated by the flagged superSchur modules $S^D_f(\mathcal{L})$ studied in Section 1.8 of Chapter 1. These results likewise generalize to the doubly flagged rings of that section and to the rings generated by the doubly flagged versions of $\mathcal{S}^{\mathcal{D}}(\mathcal{L})$.

By way of contrast, it is worth observing that if $\mathcal{D}$ is an arbitrary collection of shapes such that $\mathbf{Z}\{\mathcal{D}\}$ is closed under compression, $R^{\mathcal{D}}$ has a degree 2 straightening law in the sense that any tableau failing to be anti-straight, can be reduced via identities involving only two rows at a time, to a linear combination of straight tableaux, each expressible as a product tableaux with shape in $\mathcal{D}$. However, it need not have a degree 2 Groebner basis since a given anti-straight tableaux may factor as a product of tableaux with shapes in $\mathcal{D}$ in multiple ways. While the straightening algorithm needn't concern itself with this factorization, the Groebner basis algorithm must.

2.3 Generalized SAGBI bases

This section introduces a noncommutative generalization of the SAGBI bases of [RoSw90, KaMa89]. First we recall the definition for a commutative algebra over a field $\mathbf{k}$. Suppose that $S = \mathbf{k}[x_1, \ldots, x_n]$ is a commutative polynomial ring over $\mathbf{k}$ and that R is a subalgebra of S. Suppose that $\prec$ is a term order on S (carrying the definition of Section 2.2 over word for word and assuming that $\prec$ is a total order.) A finite set $\{p_i : i \in I\}$ of polynomials in S is said to be a SAGBI basis for R if

$$\left\{\alpha \mathrm{LT}_{\prec}\left(\prod_i p_i^{a_i}\right) : \alpha \in \mathbf{k},\ a_i \in \mathbf{N}\right\} = \{\mathrm{LT}_{\prec}(p) : p \in R\}.$$

The existence of a SAGBI basis for a subalgebra immediately implies a *subduction* algorithm testing whether elements in S lie in R and, if so, expressing them in terms of the generators p_i. In particular, given $p \in S$, check to see whether its initial term lies in the monoid generated by the initial terms of the p_i. If so, express $\mathrm{LT}(p)$ as $\mathrm{LT}(\alpha \prod_i p_i^{a_i})$, replace $\mathrm{LT}(p)$ with $\alpha \prod_i p_i^{a_i}$ and perform subduction on the difference $p - \alpha \prod_i p_i^{a_i}$. Observe that while this algorithm is well-defined, it naively requires an exhaustive search in order to tell whether the leading term of p in a valid leading term for an element of R. The fact that we have at hand a combinatorially well-behaved straightening law will enable us to explicitly construct products $\prod_i p_i^{a_i}$ having a specified leading term.

We have constructed $R^{D}_{\underline{f},\underline{g}}(\mathcal{L})$ as a subalgebra of $Super([\mathcal{L} \mid \mathcal{P}])$, so it is reasonable to ask for a natural notion of a SAGBI basis suitable for working with $R^{D}_{\underline{f},\underline{g}}(\mathcal{L})$. The solution I describe in this section is inherently quasi-commutative as made precise in Section 2.4. It is applied in Chapter 3 to quantum determinants and is applicable to subalgebras of the universal enveloping algebra of a Lie algebra.

2.3.1 The ambient ring

The first point to observe is that an algorithm designed to mimic SAGBI basis or Groebner basis methods needs a clear notion of what constitutes an initial monomial. With this in mind I will work with a $\mathbf{k}$-algebra S for some commutative ring $\mathbf{k}$ and a partial order $\prec$ on monomials satisfying the following properties.

1. S is generated as a $\mathbf{k}$-algebra by a finite set $\{x_1, x_2, \ldots, x_n\} \subset S$.

2. Call the set $\mathcal{T} = \left\{x_{i_1}x_{i_2}\cdots x_{i_j} \mid 1 \leq i_1 \leq i_2 \leq \cdots \leq i_j \leq n\right\} - \{0\}$ the set of *straight monomials* in $\{x_1, x_2, \ldots, x_n\}$– the overlap in nomenclature should resolve itself in context. The ring S must be a free $\mathbf{k}$-module with basis $\mathcal{T}$. Two ordered products $x_{i_1}x_{i_2}\cdots x_{i_{j'}}$ and $x_{k_1}x_{k_2}\cdots x_{k_{j''}}$ with $\underline{i} \neq \underline{j}$ are equal iff they are both 0.

3. The order $\prec$ must be Noetherian on $\mathcal{T}$. Given $p \in S$, let $\text{init}\,(p)$ denote the collection of minimial monomials appearing with non-zero coefficient when p is expanded in terms of the basis $\mathcal{T}$. Let $\text{LT}\,(p)$ be the sum of all these minimal monomials each scaled by its coefficient in the expansion of p. Let $\text{tail}\,(p) = p - \text{LT}\,(p)$. For all $m, m', n, n' \in \mathcal{T}$ if $m \succeq m'$ and $n \succeq n'$

$$\text{init}\,(mn) \succeq \text{init}\,(m'n')$$

so long as neither side of the inequality is 0. The fact that init (mn) is a single monomial follows from Property 4 below. If either of the first two inequalities were strict, then the conclusion must be a strict inequality, again modulo the vanishing of either product.

4. For any $1 \leq i, j \leq n$ we have $x_j x_i = \alpha x_i x_j + T$ where $\alpha, \alpha^{-1} \in \mathbf{k}$ and where T is a sum of straight monomials each larger than $x_i x_j$. Further, if $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ and $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_{n'}}$ are monomials in $\mathcal{T}$ and if

$$(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n})(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_{n'}}) = (x_1^{i_1+k_1} x_2^{i_2+k_2} \cdots x_n^{i_n+k_n} + T)$$

then $x_1^{i_1+k_1} x_2^{i_2+k_2} \cdots x_n^{i_n+k_n} = 0$ implies $T = 0$.

A partial order on monomials in $\{x_1, \cdots, x_n\}$ is said to be *compatible* with the ordered monomials of in the x_i (or, by abuse of notation, compatible with the x_i) when it satisfies the preceding properties.

These kinds of properties have been exploited by Weispfenning in the context of Groebner bases and "skew-polynomial" rings, although I'm not aware of previous extensions to exterior algebras.

2.3.2 The initial set and SAGBI bases

Let S and $\prec$ be as described above, and let A be a subalgebra of S. Define $\mathrm{LT}^{\mathrm{monom}}(A)$ of A with respect to $\prec$, to be the set $\{$all terms appearing in $\mathrm{LT}(p) \ \mid p \in A\}$. For example, considering $\mathbf{Z}[2x^3] \subset \mathbf{Z}[x]$, $\mathrm{LT}(\mathbf{Z}[2x^3]) = \{j2^k x^{3k} | j, k \in \mathbf{N}\}$. For $\mathbf{Q}[x+y] \subset \mathbf{Q}[x,y]$ under the partial order in which x, y are incomparable, we find $\mathrm{LT}(\mathbf{Q}[x+y]) = \mathrm{LT}(\mathbf{Q}[x+y])$. If $\mathcal{F} \subset \mathcal{A}$ is a finite set such that $\mathrm{LT}(f)$ is a monomial for all $f \in \mathcal{F}$ and such that

$$\left\{\mathrm{LT}\left(\prod_{f \in R} \mathrm{LT}(f)\right) \mid R \subset \mathcal{F}\right\} = \mathrm{LT}^{\mathrm{monom}}(A) \quad ,$$

then we'll call $\mathcal{F}$ a SAGBI basis (with respect to $\prec$) for A.

The proof that a SAGBI basis for A generates A now follows exactly the lines of the commutative proof.

Proposition 2.3.1 *Suppose $\mathcal{F} = \{f_i\}$ is a* SAGBI *basis for A. Then $\mathcal{F}$ generates A.*

Proof. Suppose $p \in A$. Let m be a term in LT (p). By assumption $m = \mathrm{LT}\,(f_{i_1}, f_{i_2}, \cdots, f_{i_k})$ for some $i_1, i_2, \cdots, i_k$. Thus $p - f_{i_1} f_{i_2} \cdots f_{i_k}$ is in A and it has fewer leading terms not strictly less than m.

Since $\succ$ in Noetherian, p can be expressed as a sum of products of SAGBI basis elements by iterating the above process. □

Porism 2.3.2 *Suppose $\mathcal{F} = \{f_i\}$ is a* SAGBI *basis for A. Suppose $p \in A$. The above algorithm allows us to write $p = \sum_j \alpha_j \underline{f}^{\underline{i}_j}$ where any monomial m in* LT (p) *is weakly smaller than the monomials* init $(\prod_k f_{l_k})$ *for all $\underline{l} = \underline{i}_j$. Further, these monomials are all distinct and satisfy* init $(\prod_k f_{l_k}) =$ init $(\prod_k$ init $(f_{l_k}))$. □

The following lemma is useful in establishing the Noetherianess of a partial order.

Lemma 2.3.3 *Suppose that $\prec$ is compatible with $\{x_1, \ldots, x_n\}$ modulo the Noetherian property. If $1 \prec x_i$ for all x_i then an ascending sequence $m_1 \prec m_2 \prec m_3 \prec \ldots$ of straight monomials must be finite, i.e. $\prec$ in Noetherian and hence compatible with the x_i.*

Proof. The proof is almost a reduction to the commutative case. Consider the sequence of exponent vectors for the monomials. Namely, if $m_i = x_1^{l_{i,1}} x_2^{l_{i,2}} \cdots x_n^{l_{i,n}}$, consider the sequence

$$\{(l_{i,1}, l_{i,2}, \ldots, l_{i,n})\} \quad .$$

By Dickson's Lemma (see [CLO'S]), the submonoid in $\mathbf{Z}^n$ generated by this sequence is in fact finitely generated. Thus if the sequence were infinite, there would exists $j < k$ such that $(l_{j,1}, l_{j,2}, \ldots, l_{j,n}) < (l_{k,1}, l_{k,2}, \ldots, l_{k,n})$ in the product partial order.

Rephrasing this in terms of the algebra S, and recalling that m_k and m_j must be nonzero (since they are straight), we find that

$$m_k + T = m_j \left(x_1^{l_{k,1}-l_{j,1}} x_2^{l_{k,2}-l_{j,2}} \dots x_n^{l_{k,n}-l_{j,n}} \right)$$

where T is a sum of straight monomials each greater than m_k. Thus

$$\begin{aligned} m_k &= \text{init}\left(m_j \left(x_1^{l_{k,1}-l_{j,1}}, x_2^{l_{k,2}-l_{j,2}}, \dots, x_n^{l_{k,n}-l_{j,n}} \right) \right) \\ &\preceq \text{init}\left(m_j \cdot 1 \right) \\ &= m_j \end{aligned}$$

which contradicts the assumption that the sequence ascends. □

2.4 Deformations

In [Ri94] Rippel shows how to produce a flat deformation from a certain $R^{\mathcal{D}}(\mathcal{L}^-)$ to a ring generated by monomials. In particular, for Rippel, $\mathcal{D}$ consists of a finite set of single rows each starting in the first column; in geometric language these are multihomogeneous coordinate rings of flag varieties. In [Stu94, Stu96] Sturmfels generalizes this result to any commutative domain R which as a subalgebra of $\mathbf{Q}[x_1, \dots, x_n]$ possesses a SAGBI basis. In particular, he shows that R can be expressed as S/I where S is a polynomial ring and that there exists a term order (naturally arising from the term order on $\mathbf{Q}[x_i]$ yielding the SAGBI basis) such that LT (I) is a prime ideal generated by binomials. Among other uses, this technique provides a quick proof (modulo some well-known commutative algebra) of the Cohen-Macaulayness of the multihomogeneous coordinate rings of flag varieties under the Plücker embedding. In this section, I generalize of Sturmfels' result to the noncommutative SAGBI bases I introduced in Section 2.3.

For the remainder of this section assume that A is a $\mathbf{k}$-algebra as in Subsection 2.3.1 generated by a finite set of variables t_i for $i \in \mathcal{I}$. Let $\prec$ be a term order compatible with the variables t_i. We define a graded ring associated to A. First, suppose that M is a straight monomial in the t_i's. Define $A_{\succ M}$ to be the subring of A spanned by all straight monomials $M' \succ M$. Likewise, define the subring $A_{\succeq M}$ to be the span of all straight monomials $M' \succeq M$. For convenience, if M is an ordered monomial that fails to be straight, define $A_{\succeq M} = A_{\succ M} = 0$.

Definition 2.4.1 *Define a* $\mathbf{N}^{|\mathcal{I}|}$*-multigraded ring,*

$$G_{\prec}(A) = \bigoplus_{\underline{l} \in \mathbf{N}^{|\mathcal{I}|}} A_{\succeq\left(\prod_i t_i^{l_i}\right)} \Big/ A_{\succ\left(\prod_i t_i^{l_i}\right)}.$$

A term pre-order compatible with variables x_i is a partial pre-order on nonzero ordered monomials in the x_i obeying the same restrictions as a term order compatible with those variables. Any homomorphism $\phi : \mathbf{k}\langle x_1, \ldots, x_n\rangle \to A$ induces a term pre-order $\prec_\phi$ on $\mathbf{k}\langle x_1, \ldots, x_n\rangle$ as follows. Define $prod_l x_{j_l} \prec_\phi \prod_l x_{j'_l}$ when $\operatorname{init}_\prec\left(\prod_l \operatorname{init}_\prec\left(\phi(x_{j_l})\right)\right) \prec \operatorname{init}_\prec\left(\prod_l \operatorname{init}_\prec\left(\phi(x_{j'_l})\right)\right)$; remember that 0 is not a straight monomial in A and hence is certainly incomparable under $\prec$ with any straight monomial.

Theorem 2.4.1 *Let the A be a $\mathbf{k}$-algebra as in Subsection 2.3.1. Let $G_\prec(A)$ be the graded ring associated to A and $\prec$. Suppose that B is a subalgebra of A possessing a* SAGBI *basis $\{f_1, \ldots, f_n\}$ with respect to $\prec$. Let I be the kernel of the map $\phi : \mathbf{k}\langle x_1, \ldots, x_n\rangle \to B$ under which $x_i \mapsto f_i$. Let I' be the kernel of the map $\phi' : \mathbf{k}\langle x_1, \ldots, x_n\rangle \to G_\prec(A)$ under which x_i is sent to the class of $\operatorname{init}_\prec(f_i)$. We find that $I' = \mathrm{LT}_{\prec_\phi}(I)$.*

Proof. I first show that $I \subset I'$. Suppose $p \in I$ Write $p = \operatorname{init}_{\prec_\phi}(p) + \operatorname{tail}_{\prec_\phi}(p)$. Since $\phi(p) = 0$, we must have $\phi(\operatorname{init}(p)) \in A_{\succ \operatorname{init}(p)}$. But elements of $\phi'(\operatorname{init}(p))$ live in $A_{\succeq \operatorname{init}(p)}/A_{\succ \operatorname{init}(p)}$ hence $\phi'(\operatorname{init}(p)) = 0$.

Suppose on the other hand that $q \in I'$. Now any monomial M in the x_i's is either sent by ϕ' to 0 or to a unique multigraded piece of $G_{\prec}(A)$. Since $M = \mathrm{init}\,(M)$ and any monomial M such that $\phi'(M) = 0$ satisfies $\phi(M) = 0$, it suffices to consider the case that each monomial in q is sent to a single graded component of $G_{\prec}(A)$, say the component indexed by $\underline{i} \in \mathbf{N}^{\mathcal{I}}$. But in this case, $\phi'(q) = 0$ implies $\phi(q) \in A_{\succ \underline{t}^{\underline{i}}}$. Thus by Porism 2.3.2 q is expandable as $r(f_1, \ldots, f_k)$ such that each monomial in $r(x_1, \ldots, x_k)$ is $(\prec_\phi)$-larger than $\underline{t}^{\underline{i}}$. Thus $\mathrm{init}_{\prec_\phi}(q(\underline{x})) = \mathrm{init}_{\prec_\phi}(q(\underline{x}) - r(\underline{x}))$ with $\phi(q(\underline{x}) - r(\underline{x})) = 0$. □

Assume A is commutative. Let $S = \mathbf{k}[x_1, \ldots, x_n]$ Hence $G_{\prec}(A) = A$ and close examination precisely recovers Sturmfels' result. In particular, suppose we define a *weight order* on the monomials in a polynomial ring via a function w from the variables x_i to $\mathbf{Z}$: namely $\underline{x}^{\underline{i}} < \underline{x}^{\underline{j}}$ when $\sum_l i_l w(x_l) < \sum_l j_l w(x_l)$. With this notation we have rederived the main theorem of [Stu94].

Corollary 2.4.2 (Sturmfels) *Let A be a polynomial ring over a field. Let $B \subset A$ be a subalgebra having* SAGBI *basis $f_1, \ldots, f_k$ under a weight order $\prec$. Define $\phi : \mathbf{k}[x_1, \ldots, x_k] \to B$ by $phi(x_i) = f_i$. Then $I' = \mathrm{init}_{\prec_\phi}(\ker \phi)$ is a toric ideal and $\mathbf{k}[x_1, \ldots, x_k]/I'$ is a commutative semigroup ring generated by monomials* $\mathrm{init}_{\prec}(f_i)$.
□

This result implies (see [Ei]) the existence of a flat deformation from $B = S/I$ to S/I' and hence various nice properties of A/I' may be lifted to B.

2.5 Super Bracket Algebras

In this section, we show that the letterplace superalgebra with diagonal term order is an appropriate setting for generalized SAGBI bases.

In this case, the ambient algebra S will be the superalgebra $Super([\mathcal{L} \mid \mathcal{P}])$ on the "letters" $\mathcal{L}$ and negative "places" $\mathcal{P} = \mathcal{P}^- = \{1, 2, \ldots, n\}$. Recall that $Super([\mathcal{L} \mid \mathcal{P}])$

is defined as a $\mathbf{Z}$-algebra. We will tensor it with $\mathbf{k}$ when a $\mathbf{k}$-algebra is desired. We might as well let $\mathcal{P}$ be ordered compatibly with the integers.

We verify that $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ with any graded diagonal term order (see Chapter 1, page 40) satisfies the axioms of Subsection 2.3.1. By a graded term order, I mean one which requires that any monomial of degree i is larger than any monomial of degree $i+1$.

1. S is generated by the letterplaces.

2. Since $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ is a multigraded algebra, under the fine multigrading by $\mathbf{Z}^{|\mathcal{L}^-|n} \oplus \mathbf{Z}_2^{|\mathcal{L}^+|n}$ indicating the presence of each letterplace, $\mathcal{T}$ is a vector space basis for S over $\mathbf{Z}$, and two differently written ordered letterplace monomials are equal if and only if they are both 0.

3. For all $m, m', n, n' \in \mathcal{T}$ if $m \succeq m'$ and $n \succeq n'$

$$\text{init}\,(mn) \succeq \text{init}\,(m'n')$$

so long as neither side of the inequality is 0. If either of the first two inequalities were strict, then the conclusion must be a strict inequality. By Lemma 2.3.3, the term order $\succ$ is Noetherian.

4. The final axiom, that the exponent vector for the leading term of a nonzero product of two straight monomials can be found by adding the exponent vectors of the multiplicands is also immediate.

2.6 Straightening using SAGBI bases

The definition of a SAGBI basis even in the commutative case does not guarantee the reduction of an algebra element to a unique polynomial in the SAGBI basis elements. For example, consider the commutative bracket algebra $R^{\sqcup\!\sqcup}(\{1^-, 2^-, 3^-, 4^-\})$ under a diagonal term order. This gives the usual SAGBI basis for the bracket algebra, namely the set of all brackets. However, since init ([13][24]) = init ([14][23]) we can expand

$$(1|1)(2|1)(4|2)(3|2)-(3|1)(2|1)(1|2)(4|2)-(1|1)(4|1)(3|2)(2|2)+(4|1)(3|1)(1|2)(2|2)$$

(better known as [13][24]) by subduction into

$$\begin{aligned}&[14][23]+(1|1)(3|1)(2|2)(4|2)\\&\qquad -(1|1)(4|1)(2|2)(3|2)-(2|1)(3|1)(1|2)(4|2)+(2|1)(4|1)(1|2)(3|2)\end{aligned}$$

which reduces (again by the subduction algorithm) [14][23] + [12][34]. Of course the subduction algorithm of [Stu93] for the bracket algebra fixes this problem by requiring that the product of SAGBI basis elements used in each step of the subduction algorithm should be standard (in the sense of Young tableaux). One could generalize this by requiring that, for each monomial m in the initial algebra (or initial set), one fix a particular product of SAGBI basis elements that has m as its initial term. Requiring that the subduction algorithm always use this product when removing the term m guarantees confluence of the subduction algorithm, since the collection of such products must be linearly independent. I will call a SAGBI basis with such choices made a *confluent* SAGBI basis. We want a confluent SAGBI basis to come equipped with an efficient algorithm which given an initial term will determine the appropriate product realizing that initial term. In the cases I discuss, this algorithm will be provided by the filling algorithms of Chapter 1.

The next several propositions illustrate the application of generalized SAGBI bases to the superalgebra analogues of certain homogeneous coordinate rings of some algebraic varieties. In order to make the presentation more intuitive, I will present successively more general applications.

Recall that $Super([\mathcal{L} \mid \mathcal{P}])$ can be graded by content. Namely the index group is

$\mathbf{Z}^{|\mathcal{L}|} \oplus \mathbf{Z}^n$ with $(l_i|j)$ being in the (e_i, e_j) homogeneous component if l_i is the ith letter in $\mathcal{L}$ and letting e_i, e_j be the ith and jth standard basis vectors. Similarly, $\mathcal{S}uper(L)$ can be graded by $\mathbf{Z}^{|\mathcal{L}|}$ with l_i being in the homogeneous component indexed by e_i. Suppose w is in the homogeneous component of $\mathcal{S}uper(L)$ indexed by the vector $\underline{a}$. Then the *bracket* $[w]$ is in the homogeneous component of $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ indexed by $(\underline{a}, (1, 1, \ldots, 1))$. Define the *bracket algebra* of *step* n, as the subalgebra $R^{\overbrace{\square\square\cdots\square}^{n \text{ cells}}}$ of $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$.

Proposition 2.6.1 *The set of all* $[w_1, \ldots, w_n]$ *with* $w_i \in \mathcal{L}$ *and* $w_i <\!\!+ w_{i+1}$ *forms a SAGBI basis for* $R^{\overbrace{\square\square\cdots\square}^{n \text{ cells}}}(\mathcal{L})$ *under any diagonal term order.*

Suppose an initial term $\prod_{i=1}^n \prod_{j=1}^d (l_{j,i}|i)$ *with* $\sigma_{j,i} <\!- \sigma_{j+1,i}$ *for* $1 \leq j < d$ *is required to be reduced using the product* $Tab(T)$ *of brackets where* $T = (l_{j,i})$. *With this reduction rule the set of brackets form a confluent SAGBI basis.*

Proof. Apply Proposition 1.6.6 to determine that the standard tableaux have distinct initial terms. The fact that the standard tableaux form a basis completes the proof.

□

The problem that this approach runs into that does not appear in the commutative case lies precisely in the fact that a product of leading terms might vanish. In fact, it is not clear how to determine the leading term of a product of brackets in sub-exponential time even in very special cases. The most notorious example is the product $[l_1, l_2, \ldots, l_n]^n$ where all l_i are positive. In this case, the leading term being nonzero is equivalent to Rota's basis conjecture for representable matroids [].

If $\lambda = 1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_k \leq n$ we define the algebra $F(\lambda)$ to be the subalgebra $R^{\mathcal{D}}$ of $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ where $\mathcal{D}$ consists of all one-rowed row-convex shapes starting at 1 and ending at some λ_i. If $\mathcal{L} = \mathcal{L}^-$ and has cardinality m, then this is the homogeneous coordinate ring for the variety of flags $F_0 \subset F_1 \subset \cdots \subset F_k$ with $\dim(F_i) = \lambda_i$ when the flag variety is considered as a subvariety of a product of Grassmannians embedded in projective space by the Plücker embedding.

Proposition 2.6.2 *The set $\mathcal{F}$ of all biproducts $(w|12\cdots\lambda_i)$ with $w \in Super(L)$ form a SAGBI basis for $F(\lambda)$ under any diagonal term order.*

Let $\underline{d} = d_1 \geq d_2 \geq \cdots \geq d_k > 0$ be a partition, and let $\underline{c} = c_1 \geq c_2 \geq \cdots \geq c_{d_1} > 0$ be its conjugate partition. Impose the reduction rule that, for all standard tableaux T with shape $\underline{d}$ and whose (j,i) cell contains $l_{j,i}$, replaces $\prod_{j=1}^{d_i}\prod_{i=1}^{c_j}(l_{j,i}|i)$ with $\prod_j T_j - \mathrm{tail}([T])$. Here T_j is the jth row of T.

With these reduction rules $\mathcal{F}$ is a coherent SAGBI basis.

Proof. We conclude from Proposition 1.6.6, Theorem 1.8.2 and the fact that distinct straight (a forteriori standard) tableaux have distinct modified column words, that any initial monomial must be the initial monomial $[S]$ for some standard tableau S. Thus any initial term is an integer times something of the form

$$\frac{1}{\mathbf{c}\left(\prod(l_{j,i}|i)\right)!}\prod_{j=1}^{d_1}\prod_{i=1}^{c_j}(l_{j,i}|i)$$

where for each j, $c_j \in \{\lambda_1, \lambda_2, \ldots\}$. The proposition follows. □

We now define another superalgebra to use as our ambient space. This construction will rely on the order of $\mathcal{L}$. To start, let $\underline{a}, \underline{b}$ be length k sequences in $\mathcal{L}$. Without loss of generality we will be able to assume $\underline{a}$ and $\underline{b}$ are $(<_+)$-increasing. Let $\underline{a} = a_1, a_2, \ldots, a_k$ and let $\underline{b} = b_1, b_2, \cdots, b_k$ with $b_i \leq a_i$. Define the ideal $I(\underline{a}, \underline{b})$ to be the ideal of $Super([\mathcal{L} \mid \mathcal{P}])$ generated by $(c|j)$ where $c < b_j$ or $c > a_j$. Let ψ be the canonical projection $\psi : Super([\mathcal{L} \mid \mathcal{P}]) \to Super([\mathcal{L} \mid \mathcal{P}])/I(\underline{a}, \underline{b})$. Observe that $Super([\mathcal{L} \mid \mathcal{P}])/I(\underline{a}, \underline{b})$ can also be viewed as a sub-superalgebra of $Super([\mathcal{L} \mid \mathcal{P}])$, choosing for $\mathcal{T}$ the subset of all straight monomials in $Super([\mathcal{L} \mid \mathcal{P}])$ outside $\ker\psi$, this algebra inherits the properties necessary for having the diagonal term orders be compatible with the variables $(l|p)$.

We define the algebra $C(\underline{a}, \underline{b})$ to be the subalgebra of $Super([\mathcal{L} \mid \mathcal{P}])/I(\underline{a}, \underline{b})$ generated by the images of all biproducts $\psi((w|12\cdots\lambda_i))$ where w is a word in $Super(L)$ and $0 \leq i \leq k$.

Suppose $\mathcal{L} = \mathcal{L}^-$ and has cardinality m. If K is algebraically closed, then $K \otimes_{\mathbf{Z}} C(\underline{a}, \underline{b})$ is the homogeneous coordinate ring for the skew Schubert variety of a standard pair of flags parameterized by $\underline{a}$ and $\underline{b}$. (See page 224 of [Sta76].)

Proposition 2.6.3 *Suppose u, v are two $(<\!\!\cdot)$-increasing sequences of length n in $\mathcal{L}$. We write $u \leq v$ if $u = u_1u_2 \ldots u_n$, $v = v_1v_2 \ldots v_n$ and $u_i \leq v_i$. Let $\mathcal{F}$ be the set of length n brackets $[w]$ where w $(<\!\!\cdot)$-increases and w is doubly flagged, i.e. $\underline{b} < w < \underline{a}$.*

The set $\mathcal{F}$ forms a SAGBI basis for $C(\underline{a}, \underline{b})$. Further, suppose we impose the subduction rule

$$\frac{1}{c(\prod_{j,i}(l_{j,i}|i))!} \prod_{j=1}^{d} \prod_{i=1}^{n} (l_{j,i}|i) \mapsto \prod_{j} T_j - \text{tail}\left([T]\right),$$

for all doubly flagged standard tableaux T where the (j, i) entry is denoted $l_{j,i}$ and where T_j is the subtableau consisting of the jth row of T.

Then $\mathcal{F}$ is a coherent SAGBI basis and subduction coincides with straightening.

Before proving the result, I will generalize to superalgebras a well known result for flagged minors.

Lemma 2.6.4 *If w is $(<\!\!\cdot)$-increasing then $\psi_{\underline{a},\underline{b}}([w])$ is nonzero iff w is doubly flagged. Furthermore, if $\psi_{\underline{a},\underline{b}}([w]) \neq 0$, its leading term under any diagonal term order is the image of the leading term of $[w]$ and is a divided powers monomial in the $\mathbf{Z}$-algebra $Super([\mathcal{L} \mid \mathcal{P}])$.*

Proof. Assume w has length n. I will abuse notation and term and identify an element of S with its image in $S/I(\underline{a}, \underline{b})$ under ψ.

Recall that $(w|12 \cdots n)$ equals

$$\frac{1}{c(w)!} \sum_{\sigma \in S_n} \pm \prod_{j=1}^{n} \left(w_j|\sigma(j)\right) \quad . \tag{2.1}$$

Now if $w \notin \mathcal{F}$ then for some i, either $w_i < b_i$ or $w_i > a_i$. We will show that the former implies $(w|12 \cdots n) = 0$. The latter case is symmetric. But $w_i < b_i$ implies that for

$1 \leq j \leq i$, $(w_j|k) = 0$ whenever $k \geq i$. So for (2.1) to be nonzero, $\prod_{j=1}^{i} \left(w_j|\sigma(j)\right)$ cannot be identically zero. But by the pigeonhole principle, for some $1 \leq j \leq i$, $\sigma(j) \geq i$.

Conversely, suppose $w \in \mathcal{F}$ then init $(w) = \prod_{j=1}^{n}(w_j|j)$ so long as that product is nonzero. Again, init (w) is being identified with its image $\psi(\mathrm{LT}\,(w)\,)$. But since for all j $b_j \leq w_j \leq a_j$, we have $(w_j|j) \notin I(\underline{a}, \underline{b})$ and hence the product is nonzero. □

Proof.(of the Proposition) The straightening algorithm for rectangular tableau guarantees that the standard tableaux (technically their images under $\psi_{\underline{a},\underline{b}}$) span C($\underline{a}$,$\underline{b}$). As an immediate consequence of the preceding lemma, we observe (via the usual reasoning) that the products

$$\prod_{j=1}^{d} \mathrm{Tab}(l_{j,1}, \ldots, l_{j,n}),$$

where $(l_{j,i})$ ranges over all doubly flagged standard tableaux, are linearly independent.

□

In practice, the confluent SAGBI bases I consider can be viewed as a SAGBI basis together with a Groebner basis on the ideal $\mathrm{init}_{\prec_\phi}(I)$ appearing in Theorem 2.4.1. In particular, if init $(\prod_l \mathrm{init}\,(\phi(x_{i_l})))$ is the given initial term, then the desired unique product is $\prod_l \phi(x_{j_l})$ where $\prod_l x_{j_l}$ is the reduction of $\prod_l x_{i_l}$ via a Groebner basis for $\mathrm{init}_{\prec_\phi}(I)$. This provides something of an "explanation" for the appearance in Chapter I of algorithms which stabilize the columns of a row-standard tableau–these algorithms are working modulo the deformed ideal $\mathrm{init}_{\prec_\phi}(I)$ rather than modulo the ideal I itself.

2.7 The Cohen-Macaulayness of some rings

One of the philosophical reasons that Groebner bases should be expected to be useful is the fact (see for instance [Ei]) that if S is a polynomial ring over a field and I an ideal in S then $S/\mathrm{init}\,(I)$ is the special fiber of a flat family of algebras whose other fibers are S. This implies, via semicontinuity of Ext groups and other machinery, the following folk theorem introduced to me by David Eisenbud.

Fact 2.7.1 *Given a nontrivial graded ideal $\bigoplus_{j\geq 0} I_j$ in a ordinary polynomial ring S over a field, and a weight order $\prec$ on monomials in S, we find that if $S/\mathrm{init}_{\prec}(I)$ is Cohen-Macaulay then so is I.*

For the applications of this section it suffices to know that there exists a diagonal term pre-order given by weights on the variables. It is a general fact due to Robbiano that term total orders can be approximated by weight pre-orders, and Sturmfels in [Stu94, Stu96] explicitly provides a set of "Vandermonde weights" that accomplishes the desired task. Sturmfels' machinery (here generalized in Theorem 2.4.1) associates to any variable the weight of the initial term of the subalgebra generator it maps to.

In this section I will apply the preceding fact to provide easy proofs that the rings studied in the previous sections are Cohen-Macaulay. Cohen-Macaulayness is a well-known property for many of the rings $R^{\mathcal{D}}$, but I believe that the following propositions are new.

The method of proof involves deforming the rings via Corollary 2.4.2 to commutative semigroup rings. To this end I generalize the results of the Sections 2.2 and 2.6 to provide SAGBI bases for the rings considered in the first of those sections. I will then apply a result of Hochster's [Ho72] providing a criterion for the Cohen-Macaulayness of certain semigroup rings.

Fact 2.7.2 (Hochster) *Let $\mathbf{k}$ be a field. Suppose M is a finite set of monomials in $\mathbf{k}[x_1,\dots,x_n]$ and suppose that $\mathbf{k}[M]$ is the commutative semigroup ring of the monoid $\mathbf{Z}\{M\}$ generated by M. The ring $\mathbf{k}[M]$ is Cohen-Macaulay whenever $\mathbf{Z}\{M\}$ satisfies the following property:* For $b, c, d \in \mathbf{Z}\{M\}$, where $bd^k = c^k$, there exists $a \in \mathbf{Z}\{M\}$ such that $b = a^k$.

Since we have the monoid $\mathbf{Z}\{M\}$ embedded in the group generated by x_i, x_i^{-1} for all $1 \leq i \leq n$, we can rewrite the condition in Hochster's theorem to read that if

$b = \left(\frac{c}{d}\right)^k$ then b must have a kth root in $\mathbf{Z}\{M\}$. Obviously this condition holds whenever $\mathbf{Z}\{M\}$ is saturated in $\mathbf{k}[x_i, x_i^{-1} : 1 \le i \le n]$, that is whenever $a^k \in \mathbf{Z}\{M\}$ imply $a \in \mathbf{Z}\{M\}$. We shall show that the commutative semigroup rings arising from deforming certain rings generated by determinants are indeed saturated.

Fix an alphabet $\mathcal{L} = \mathcal{L}^-$ and a field $\mathbf{k}$. Anticipating the results (or at least their proofs) I will define the *column monoid* $M(\mathcal{D})$ associated to a collection $\mathcal{D}$ of row-convex diagrams. In particular, inside the monoid of monomials in the letterplaces define $M(\mathcal{D})$ to be the submonoid generated by by $\prod_{(i,j)\in D}(T_{i,j}|j)$ for all row-standard tableaux of all shapes $D \in \mathcal{D}$. When $\mathcal{D}$ is the set of all one-rowed, row-convex shapes starting at 1 and ending at some λ_i in a partition λ, then the semigroup ring $\mathbf{k}[M(\mathcal{D})]$ is the *column algebra* for flag varieties studied by Rippel in [Ri94]. The column algebra is a useful object in the row-convex case precisely because the column algebra can frequently be seen to agree with the semigroup ring generated by all initial terms of $R^{\mathcal{D}}$ under a diagonal term order. Suppose $\underline{a}, \underline{b}$ are two weakly increasing sequences of $\mathcal{L}$ indexed by $\mathcal{P}$, and if $\phi_{\underline{a},\underline{b}}$. Define the *flagged column monoid* $M_{\underline{a},\underline{b}}(\mathcal{D})$ to be the submonoid of the letterplace monomials generated by $\prod_{(i,j)\in D}(T_{i,j}|j)$ for all doubly flagged row-standard tableaux of all shapes $D \in \mathcal{D}$.

Proposition 2.7.1 *Let* $\mathbf{k}$ *be a field and let* $\mathcal{L}^-$ *be a finite set of negative letters. Let* D *be a row-convex shape. The ring* $\mathbf{k} \otimes_{\mathbf{Z}} R^D(\mathcal{L}^-)$ *is Cohen-Macaulay.*

Further, if $\underline{a}, \underline{b}$ *are two weakly increasing sequences of* $\mathcal{L}$ *indexed by* $\mathcal{P}$*, and if* $\phi_{\underline{a},\underline{b}}$ *is defined as in Chapter 1, Section 1.8 then the image of* $\mathbf{k} \otimes_{\mathbf{Z}} R^D(\mathcal{L}^-)$ *under* $\phi_{\underline{a},\underline{b}}$ *is also Cohen-Macaulay.*

The proof requires the following result whose proof follows precisely the lines of those in the preceding section.

Proposition 2.7.2 *The ring* $R^D_{\underline{a},\underline{b}}(\mathcal{L})$ *has a* SAGBI *basis, under any diagonal term order, consisting of* $[T]$ *for all straight tableaux* T *of shape* D*. This can be made coherent by requiring that* init $([T])$ *be reduced using* $[T]$ *for any straight tableaux of* *shape $d \circ D$.* □

Proof.(of 2.7.1) By Proposition 2.7.2 and Corollary 2.4.2 it suffices to show that the flagged column-content monoid $M_{\underline{a},\underline{b}}(D)$ is saturated in the abelian group generated by all letterplaces under multiplication.

Consider an element of this monoid; it is formed by multiplication of say j elements corresponding to row-standard tableaux of shape D and hence it the monomial corresponding to some row-standard tableaux T of shape $j \circ D$. But by the proof of Algorithm `Straight-Filling1` we can follow this algorithm to write down a straight tableaux S of shape $j \circ D$ having the same modified column word as T and hence corresponding to the same monomial in the column algebra.

Suppose that b is the monomial corresponding to S and that $b = (c/d)^k$ where c, d are monomials in the column algebra. This means that k divides j and k divides the number of times any letter $l \in \mathcal{L}$ appears in any given column column of S. But when we apply Algorithm `Straight-Filling1` to w_S we find that the algorithm does everything in blocks of k. That is, if we put the $ik+1$ copy of some letter x into row $rk+1$ of column c then we must put copies $ik+2, \ldots, (i+1)k$ into rows $rk+1 \cdots (r+1)k$ of that column. But that means that separating S into k tableaux by extracting rows $l, k+l, \ldots, (j/k-1)k+l$ for $1 \leq l \leq k$ we find that the resulting k tableaux of shape $(j/k) \circ D$ are identical.

Since S is flagged iff T was, the flagged version of this result follows similarly. □

The multihomogeneous generalization of the preceding result relies on the following proposition.

Proposition 2.7.3 *Suppose the ring $R^{\mathcal{D}}_{\underline{a},\underline{b}}(\mathcal{L})$ is multihomogeneous with respect to $\mathcal{D}$. The ring $R^{\mathcal{D}}_{\underline{a},\underline{b}}$ then has a* SAGBI *basis consisting of all $[T]$ where T is straight and has shape $D \in \mathcal{D}$.*

Coherency is achieved by reducing init $([T])$ *using $[T]$ for all straight tableaux of shape $D_1 \circ \cdots \circ D_k$ with each $D_i \in \mathcal{D}$. Multi-homogeneity guarantees that (given an* *ordering on $\mathcal{D}$ and assuming $D_i < D_{i+1}$) $[T]$ factors uniquely into a product $\prod[T_i]$ with each T_i straight of shape D_i and $T = T_1 \circ \cdots \circ T_k$.* □

Proposition 2.7.4 *With the assumptions of the preceding proposition, if $\mathcal{D}$ is a set of one-rowed, row-convex tableaux such that no two rows end in the same column then the ring $\phi_{\underline{a},\underline{b}}\left(\mathbf{k} \otimes_{\mathbf{Z}} R^{\mathcal{D}}(\mathcal{L}^-)\right)$ is Cohen-Macaulay.*

Proof. As in the previous proposition, the flagged statement follows immediately.

Again we will show that the column monoid is saturated in the letterplace algebra. Start with a monomial $a^k \in M(\mathcal{D})$ with a in the multiplicative group generated by the letterplaces. Since the shapes in $\mathcal{D}$ all end in different columns, we know that a^k can be read off from the columns of some row-standard tableaux T of a uniquely determined shape D where the rows of D all come from $\mathcal{D}$ and where any such row shape appears a multiple of k times in D.

Now given a letter x and a column i, $(x|i)$ must divide a^k some multiple of k times– i.e. x appears a multiple of k times in column i. Use Algorithm `straight-filling1` to write down a straight tableau S with the same modified column word as T. By the analysis in the preceding proof, we can split S up into k identical tableaux S_0 such that ${S_0}^{\circ k} = S$. □

Proposition 2.7.5 *Suppose $\mathcal{D}$ is a finite collection of shapes such that $\mathbf{Z}\{\mathcal{D}\}$ is closed under compression, then $R^{\mathcal{D}}_{\underline{a},\underline{b}}$ has a* SAGBI *basis consisting of all $[T]$ with T anti-straight of shape $D \in \mathcal{D}$.* □

Proposition 2.7.6 *Again suppose $\mathbf{k}$ is a field and $\underline{a}, \underline{b}$ are as above. Fix two finite index sets I_0, I_1 of columns. If $\mathcal{D}$ is the set of all one-rowed, row-convex tableaux starting in a column of I_0 and ending in a column of I_1, then the ring $\phi_{\underline{a},\underline{b}}\left(\mathbf{k} \otimes_{\mathbf{Z}} R^{\mathcal{D}}(\mathcal{L}^-)\right)$ is Cohen-Macaulay.*

Proof. The proof goes much as those of the two preceding propositions. It suffices to show that if given a flagged, row-standard tableau T whose rows are of shape chosen from $\mathcal{D}$ and such that any given letter x appears a multiple of k times in any given column i, then there exists a flagged, row-standard tableau S such that $S = {S_0}^{\circ k}$ for some S_0. Examination of Algorithm `anti-straight filling` verifies this fact. $\square$

Chapter 3

Quantum straightening laws

Despite the vast recent interest in quantum groups, there remains much to say about straightening laws for representations quantum gl_n–even in cases where the gl_n-representations are completely understood. In Section 3.3, I state the natural definition of a q-Schur module associated to a general shape. Following the letter-place philosophy, this module is constructed inside $Mat_q(t_{i,j})$, the quantized version of the algebra of polynomial functions on a matrix. Section 3.2 reviews some facts, commonly known to quantum group theorists and some combinatorialists, about this algebra–I have provided references or proofs for each result. Sections 3.4 and 3.5 generalize the basis results and straightening laws of Chapter 1 to the appropriate q-Schur modules. Section 3.6 simplifies the proof of the Huang-Zhang standard basis theorem for quantum bitableaux with the result that I obtain for quantum bitableaux the the dominance order results available in the classical case. A corresponding straightening law remains to be found however. Finally Section 3.7 proves the existence of certain short-exact sequences for quantum Schur modules. All of these results are characteristic-free and should generalize to quantum Weyl modules and to quantum super-Schur modules. Unfortunately, more groundwork needs to be laid in formalizing the "right" notion of a q-deformation of the super-bialgebra $Super([\mathcal{L} \mid \mathcal{P}])$, before this program can be carried out.

3.1 Notation

Throughout this chapter, it will be assumed that all "letters" in $\mathcal{L}$ are negatively signed. Indeed, if $|\mathcal{L}| = m$ then I will identify $\mathcal{L}$ with the set $\{1, 2, \ldots, m\}$ under the usual ordering.

3.2 Quantum matrices and quantum determinants

In the introduction to Chapter 1, I suggested that instead of considering variables $x_{i,j}$ as functions on the space of matrices, it would be useful to do away with the matrices entirely and simply consider the algebra $\mathbf{k}[x_{i,j}]$. Part of the motivation for this approach comes from the philosophy behind "quantum spaces" where the algebra of functions on a space is replaced with a suitably deformed algebra.

For the remainder of this section let $\mathbf{k}$ be a commutative ring containing a distinguished invertible element q. Suppose we start with vectors in $\mathbf{k}^n$. We replace the algebra $\mathbf{k}[x_1, \ldots, x_n]$ of functions on these vectors with the deformed algebra $\mathbf{k}_q[x_1, \ldots, x_n]$ where the x_i are subject to the commutation rule $x_i x_j = q^{-1} x_j x_i$ for all $i < j$. A *quantum matrices* $(t_{i,j})$ arises as a matrix inducing an algebra homomorphism by sending $x_i \mapsto \sum_j t_{i,j} x_j$.

For our purposes, it will be more convenient to work with the q-deformation of the exterior rather than the symmetric algebra. Define $\Lambda_q(\xi_1, \ldots, \xi_n)$ to be the free noncommutative k-algebra generated by the ξ_i's and quotiented by the relations $\xi_i^2 = 0$ and $-q^{-1}\xi_i\xi_j = \xi_j\xi_i$ for $i < j$.

Proposition 3.2.1 *Let $M = (a_{i,j})$ by an $n \times n$ matrix whose entries commute with the ξ_i. The two maps $\phi_M(\xi_i) = \sum_j a_{i,j}\xi_j$ and $\phi_{M^t}(\xi_i) = \sum_j a_{j,i}\xi_j$ will both induce algebra homomorphisms iff the $a_{i,j}$ satisfy the following commutation relations:*

$$a_{i,k}a_{i,l} = q^{-1}a_{i,l}a_{i,k}, \qquad a_{i,k}a_{j,k} = q^{-1}a_{j,k}a_{i,k},$$
$$\left(q - q^{-1}\right)\left(a_{i,l}a_{j,k} - a_{j,k}a_{i,l}\right) = 0,$$

$$a_{i,k}a_{j,l} - a_{j,l}a_{i,k} = (q^{-1} - q)a_{i,l}a_{j,k},$$

for $i < j$ and $k < l$. □

The proof is a direct calculation.

The preceding result leads to the following definition.

Definition 3.2.1 *Let R be a* $\mathbf{k}$*-algebra. An R-matrix $A = (a_{i,j})$ is called a* quantum matrix *if the $a_{i,j}$ satisfy the relations*

$$a_{i,k}a_{i,l} = q^{-1}a_{i,l}a_{i,k}, \qquad a_{i,k}a_{j,k} = q^{-1}a_{j,k}a_{i,k}, \qquad a_{i,l}a_{j,k} = a_{j,k}a_{i,l},$$
$$a_{i,k}a_{j,l} - a_{j,l}a_{i,k} = (q^{-1} - q)a_{i,l}a_{j,k},$$

for $i < j$ and $k < l$. Such a quantum matrix is generic *when the subalgebra of R generated by the $a_{i,j}$ satisfies no additional relations. When the matrix $t_{i,j}$ is generic, call this subalgebra $Mat_q^{\mathbf{k}}(t_{i,j})$.*

Definition 3.2.2 *A monomial $\prod_k t_{i_k,j_k}$ is* ordered *if the sequence j_k weakly increases and if $i_k > i_{k+1}$ whenever $j_k = j_{k+1}$.*

Obviously the ordered monomials span the algebra generated by the entries of a quantum matrix. We will require the fact that for generic matrices the ordered monomials are linearly independent. A proof can be found in for instance [Ma].

Fact 3.2.1 *Assume $\mathbf{k} = \mathbf{Q}(q)$ where q is transcendental over $\mathbf{Q}$. The ordered monomials in the $t_{i,j}$ for $1 \leq i, j \leq n$ form a* $\mathbf{k}$*-vector-space basis for $Mat_q^{\mathbf{k}}(t_{i,j})$.*

As noted in [HZ94], this fact can be checked directly, albeit laboriously, by application of Theorem 1.2 of [Be78]. Indeed this theorem implies that the preceding fact for $n = 3$ is equivalent to the result for general n. We can apply the relations in Definition 3.2.1 to rewrite any $\prod_k t_{i_k,j_k}$ into a sum of monomials so that if one of these monomials is indexed by say $\underline{i}', \underline{j}'$, then $\underline{j}' < \underline{j}$ or they are equal and $\underline{i}' > \underline{i}$ (both lexicographically). This proves the following.

Corollary 3.2.2 *Assuming $\mathbf{k} = \mathbf{Q}$, the $\mathbf{Z}[q, q^{-1}]$-algebra generated inside $Mat_q(t_{i,j})$ by the $t_{i,j}$ is free as a $\mathbf{Z}[q, q^{-1}]$-module with basis given by the ordered monomials in the $t_{i,j}$. Call this algebra $Mat_q(t_{i,j})$* □

Proposition 3.2.3 *Suppose X is a generic $n \times n$ quantum matrix over $\mathbf{k}$. Let $\underline{a}, \underline{b}$ be weakly increasing sequences of $1 \ldots n$ each having length n and such that $b_j \leq a_j$ for each j. Let $\mathcal{I} = \{t_{i,j} | i < b_j \text{ or } i > a_j\}$. Let R be the algebra $Mat_q(t_{i,j})/(\mathcal{I})$. Let $X_{\underline{a},\underline{b}}$ be a matrix with entries in R formed by setting to 0 all entries (i, j) of X where $i < b_j$ or $i > a_j$. By construction, $X_{\underline{a},\underline{b}}$ is a quantum matrix. A basis for the algebra $Mat_q(t_{i,j})/(\mathcal{I})$ is given by all ordered monomials containing only variables $t_{i,j}$ outside of $\mathcal{I}$.*

Proof. It suffices to check that using the relations in Definition 3.2.1 and the relations $t_{i,j} = 0$ for $t_{i,j} \in \mathcal{I}$ as a reduction method (analogously to the proof of Corollary 3.2.2) reduces any monomial in the free algebra generated by the $t_{i,j}$'s to a unique ordered monomial. Then by the Diamond Lemma (Theorem 1.2 of [Be78]) we know that the ordered monomials are linearly independent. Since we already know that reduction is unique without imposing the relations setting variables $\mathcal{I}$ to 0, it suffices to show that under this original reduction, any monomial divisible by some element of $\mathcal{I}$ reduces only to linear combinations of monomials each divisible by some element of $\mathcal{I}$. Only reduction using the last relation requires any work. The desired result follows immediately from the fact that given $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$ if either $X_{\underline{a},\underline{b}_{i,k}}$ or $X_{\underline{a},\underline{b}_{j,l}}$ is in $\mathcal{I}$ then at least one of $X_{\underline{a},\underline{b}_{i,l}}$ or $X_{\underline{a},\underline{b}_{j,k}}$ lies in $\mathcal{I}$. □

Definition 3.2.3 *Let X be a quantum matrix. The q-determinant, $\det_q(X)$, of the minor of X indexed by rows $i_1 < \cdots < i_k$ and columns $j_1 < \cdots j_k$ is*

$$\sum_{\sigma \in S_k} (-q)^{-l(\sigma)} t_{i_1, j_{\sigma(1)}} \cdots t_{i_k, j_{\sigma(k)}},$$

where the length of σ, $l(\sigma)$, is the number of inversions in $\sigma_1, \ldots, \sigma_k$. This quantum *determinant will be written either as*

$$\text{Column \#:} \quad \begin{array}{c} j_1 \\ \boxed{i_1} \end{array} \qquad \begin{array}{ccc} j_2 & \cdots & j_k \\ \boxed{i_2} & \cdots & \boxed{i_k} \end{array}$$

or

$$Tab_q(i_1, \ldots, i_k | j_1, \ldots, j_k).$$

We directly extend the definition to all sequences $\underline{i}$, maintaining the requirement that $\underline{j}$ strictly increase. For all sequences $\underline{j}$ and all increasing sequences $\underline{i}$ we define

$$Tab_q(i_1, \ldots, i_k | j_1, \ldots, j_k) = \sum_{\sigma \in S_k} (-q)^{-l(\sigma)} t_{i_{\sigma^{-1}(1)}, j_1} \cdots t_{i_{\sigma^{-1}(k)}, j_k}.$$

This is justified by the following proposition.

Proposition 3.2.4 *If $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$ then*

$$\sum_{\sigma \in S_k} (-q)^{-l(\sigma)} t_{i_1, j_{\sigma(1)}} \cdots t_{i_k, j_{\sigma(k)}} = \sum_{\sigma \in S_k} (-q)^{-l(\sigma)} t_{i_{\sigma^{-1}(1)}, j_1} \cdots t_{i_{\sigma^{-1}(k)}, j_k}.$$

In other words, $\det_q(X) = \det_q(X^t)$.

Proof. In each monomial, successively move $t_{\sigma^{-1}(1), j_1}$ then $t_{\sigma^{-1}(2), j_2}$, etc. to the front of the monomial by applying the fact that $t_{i,l}$ and $t_{j,k}$ commute for $i < j$ and $k < l$. □

Definition 3.2.4 *If S and T are tableau of the same shape where at least one of S and T is row-standard, then define $[S|T]_q = \prod_r Tab_q(S_r|T_r)$ where S_r, T_r are respectively the rth rows of S and T.*

We now recall several facts from [HZ94, LeTh96] mostly originating in [TfTo91].

Proposition 3.2.5 *Suppose that $j_1 < \cdots < j_k$, then $Tab_q(i_1, \ldots, i_k | j_1, \ldots, j_k) = 0$ whenever the $i_1, \ldots i_k$ are not distinct. If they are distinct, then*

$$Tab_q(i_{\rho(1)}, \ldots, i_{\rho(k)} | j_1, \ldots, j_k) = (-q)^{-l(\rho)} Tab_q(i_1, \ldots, i_k | j_1, \ldots, j_k).$$

Likewise, if $i_1 < \cdots < i_k$, *then* $Tab_q(i_1, \ldots, i_k | j_1, \ldots, j_k) = 0$ *whenever the* $j_1, \ldots j_k$ *are not distinct. If they are distinct, then*

$$Tab_q(i_1, \ldots, i_k | j_{\rho(1)}, \ldots, j_{\rho(k)}) = (-q)^{-l(\rho)} Tab_q(i_1, \ldots, i_k | j_1, \ldots, j_k).$$

This fact is proved by induction on the length of ρ combined with directly checking its truth for $k = 2$. □

The following q-analogue of Laplace expansion is taken from page 221 of [HZ94] where it is attributed to [TfTo91]. It is only necessary to observe that our bitableau $[S|T]_q$ is equal to $(-1)^{\binom{n}{2}}(S|T)$ where $(S|T)$ is the bitableau of [HZ94] and n is the number of cells in S.

Fact 3.2.2 *Suppose the sequences* $\underline{u}, \underline{v}, \underline{\mu}, \underline{\nu}$ *strictly increase, then*

$$Tab_q(u_1 \cdots u_k | \mu_1 \cdots \mu_i \nu_{i+1} \cdots \nu_k) = $$
$$= \sum_{\text{shuffles, } \sigma} (-q)^{-l(\sigma)} \left[\begin{array}{c|c} u_{\sigma(1)} \cdots u_{\sigma(i)} & \mu_1 \cdots \mu_i \\ u_{\sigma(i+1)} \cdots u_{\sigma(k)} & \nu_{i+1} \cdots \nu_k \end{array} \right]_q .$$

Likewise,

$$Tab_q(u_1 \cdots u_i v_{i+1} \cdots v_k | \mu_1 \cdots \mu_k) = $$
$$= \sum_{\text{shuffles, } \sigma} (-q)^{-l(\sigma)} \left[\begin{array}{c|c} u_1 \cdots u_i & \mu_{\sigma(1)} \cdots \mu_{\sigma(i)} \\ u_{i+1} \cdots u_k & \nu_{\sigma(i+1)} \cdots \nu_{\sigma(k)} \end{array} \right]_q .$$

□

The following "exchange lemma" is the key step in the Huang-Zhang proof of the quantum standard basis theorem.

Corollary 3.2.6 (Lemma 10, [HZ94]) *Define $t = k + j + i - s$. The identity*

$$\sum_{\text{shuffles, } \sigma} (-q)^{-l(\sigma)} \left[\begin{array}{c|c} u_1 \cdots u_i v_{\sigma(1)} \cdots v_{\sigma(s-i)} & \mu_1 \cdots \mu_s \\ v_{\sigma(s-i+1)} \cdots v_{\sigma(j)} w_1 \cdots w_k & \nu_1 \cdots \nu_t \end{array} \right]_q =$$

$$= \sum_{\text{shuffles, } \sigma, \tau} (-q)^{-l(\sigma)-l(\tau)} \left[\begin{array}{c|c} u_1 \cdots u_i & \mu_{\sigma(1)} \cdots \mu_{\sigma(i)} \\ v_1 \cdots v_j & \mu_{\sigma(i+1)} \cdots \mu_{\sigma(s)} \nu_{\tau(1)} \cdots \nu_{\tau(i+j-s)} \\ w_1 \cdots w_k & \nu_{\tau(i+j-s+1)} \cdots \nu_{\tau(t)} \end{array} \right]_q$$

holds as long as the right side of the first bitableau and the left side of the second bitableau are row-standard.

The proof from [HZ94] uses the Laplace expansion to split each bitableau on the left-hand-side into a sum over bitableaux containing four rows. Laplace expansion is then reversed to combine the middle two rows and obtain the right-hand-side. □

The next proposition, the quantum analogue of the two-rowed Akin-Buchsaum inclusion is an easy corollary of the exchange lemma, although I have not seen it previously in the literature.

Proposition 3.2.7 *Suppose that $\{1, \ldots, j\} \supseteq \{\nu_1, \ldots, \nu_t\} \supseteq \{i_1, \ldots, i_r\}$ that $s = j - r$, that $k = t - r$ and that each sequence, $\underline{v}, \underline{w}, \underline{i}, \underline{\nu}$, strictly increases. The exchange lemma implies*

$$\sum_{\text{shuffles, } \sigma} (-q)^{-l(\sigma)} \left[\begin{array}{c|c} v_{\sigma(1)} \cdots\cdots\cdots\cdots v_{\sigma(s)} & 1 \cdots \hat{i}_1 \cdots \hat{i}_r \cdots j \\ v_{\sigma(s+1)} \cdots v_{\sigma(j)} w_1 \cdots w_k & \nu_1 \cdots\cdots\cdots\cdots \nu_t \end{array} \right]_q$$

$$= (-q)^{-l(\tau)} \left[\begin{array}{c|c} v_1 \cdots\cdots v_j & 1 \cdots \hat{i}_1 \cdots \hat{i}_r \cdots j \, i_1 \cdots i_r \\ w_1 \cdots w_k & \nu_1 \cdots \hat{i}_1 \cdots \hat{i}_r \cdots \nu_t \end{array} \right]_q$$

$$= (-q)^{-l(\rho)-l(\tau)} \left[\begin{array}{c|c} v_1 \cdots\cdots v_j & 1 \cdots\cdots\cdots\cdots\cdots\cdots j \\ w_1 \cdots w_k & \nu_1 \cdots \hat{i}_1 \cdots \hat{i}_r \cdots \nu_t \end{array} \right]_q$$

where $\rho = 1 \cdots \hat{i}_1 \cdots \hat{i}_r \cdots j i_1 \cdot i_r$ and $\tau = i_1 \cdots i_r \nu_1 \cdots \hat{i}_1 \cdots \hat{i}_r \cdots \nu_t$. □

If the containment $\{1,\ldots,j\} \supseteq \{\nu_1,\ldots,\nu_t\}$ in this proposition is an equality, then we recover (as on page 223 of [HZ94]) the commutation relation of [TfTo91].

I next present some results on quantum determinants of matrix products. These are implicit in the exercises of [Ka] who references [ReTaFa89]. We will need the following lemma.

Lemma 3.2.8 *If A and B are two $n \times n$ quantum matrices such that the entries of A commute with the entries of B then BA is a quantum matrix.*

Proof. Recall that A is a quantum matrix iff the two maps $\phi_A(\xi_i) = \sum_j a_{i,j}\xi_j$ and $\phi_{A^t}(\xi_i) = \sum_j a_{j,i}\xi_j$ both induce algebra homomorphisms. Here the entries of A are assumed to commute with the ξ_i's. But if the entries of A and B commute, then A, B induce homomorphisms of the algebra generated the ξ_i's and the entries of respectively B and A. Thus $\phi_A \circ \phi_B$ is a homomorphism. Since it is the homomorphism induced by BA and since likewise $\phi_{(BA)^t} = \phi_{B^t} \circ \phi_{A^t}$, the matrix BA must be a quantum matrix. □

Technically, the preceding proof should be carried out when $q \neq q^{-1}$, but direct calculation shows that the (i,l) and (j,k) entries commute for $i < j$ and $k < l$. In any event, the above proof holds over $\mathbf{Z}[q, q^{-1}]$, the case we will be interested in.

Proposition 3.2.9 *If A, B are $n \times n$ quantum matrices such the entries of A commute with those of B, then $\det_q(AB) = \det_q(A)\det_q(B)$.*

We first need an alternate characterization of the quantum determinant.

Lemma 3.2.10 *Let A be an $m \times n$ quantum matrix with $m < n$. If variables $\xi_1,\ldots,\xi_n$ which commute with the $a_{i,j}$, generate the quantum exterior algebra, then*

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} a_{i,j}\xi_j\right) = \sum_{1\le p_1<\cdots<p_m\le n} Tab_q(1,\ldots,m|p_1,\ldots,p_m)\xi_{p_1}\cdots\xi_{p_m}.$$

Proof. Looking at the coefficient of each product $\xi_{p_1}\cdots\xi_{p_m}$ separately, it suffices to prove the result for the case $m = n$. But this follows from the facts that each $\xi_j^2 = 0$ and $\prod_j a_{j,\sigma(j)}\xi_{\sigma(j)} = (-q)^{-l(\sigma)}\prod_j a_{j,\sigma(j)}\xi_j$. □

If $X = (x_{i,j})$ is an $n \times n$ quantum matrix, and if ϕ_x is as in Definition 3.2.1, then $\prod_{i=1}^n \left(\sum_{j=1}^n x_{i,j}\xi_j\right) = \phi_X(\xi_1\cdots\xi_n)$, Proposition 3.2.9 follows immediately. □

I close this section with the Cauchy-Binet rule for quantum matrices. This is another easy generalization of known methods that I have been unable to locate in the literature.

Proposition 3.2.11 *Let A and B be $n \times n$ quantum matrices for which the entries of A commute with those of B. Fix row indices $1 \le l_1 < \cdots < l_k \le n$ and column indices $1 \le p_1 < \cdots < p_k \le n$. The q-determinant of the indicated $k \times k$ quantum minor satisfies the expansion.*

$$\begin{aligned}&\mathrm{Tab}_q(l_1,\ldots,l_k|p_1,\ldots,p_k) =\\ &\qquad = \sum_{1\le r_1<\cdots<r_k\le n} \mathrm{Tab}_q^A(l_1,\ldots,l_k|r_1,\ldots,r_k)\,\mathrm{Tab}_q^B(r_1,\ldots,r_k|p_1,\ldots,p_k)\end{aligned}$$

where $\mathrm{Tab}_q^A(\underline{l}|\underline{r})$ and $\mathrm{Tab}_q^B(\underline{l}|\underline{r})$ are the quantum determinants of the indicated minors in respectively matrices A and B.

Proof. Expanding $\phi_A(\xi_{l_1}\cdots\xi_{l_k})$ gives $\sum_{\underline{r}} \mathrm{Tab}_q^A(l_1\cdots l_k|r_1\cdots r_k)\xi_{r_1}\cdots\xi_{r_k}$. Applying ϕ_B and expanding yields

$$\sum_{\underline{r}}\sum_{\underline{s}} \mathrm{Tab}_q^A(l_1\cdots l_k|r_1\cdots r_k)\mathrm{Tab}_q^B(r_1\cdots r_k|s_1\cdots s_k)\xi_{s_1}\cdots\xi_{s_k}.$$

Comparing the coefficient of $\xi_{s_1}\cdots\xi_{s_k}$ above with the coefficient of $\xi_{s_1}\cdots\xi_{s_k}$ in $\phi_{AB}(\xi_{l_1}\cdots\xi_{l_k})$ completes the proof. □

The proposition of course holds for non-square matrices by setting the appropriate entries of A and B to zero.

3.3 q-Constructions

In this section I will present q-analogues of the now-familiar constructs from Chapters 1 and 2. I first show that the generalized SAGBI basis formalism that I introduced in Chapter 2 can be applied to the ring $Mat_q(t_{i,j})$. Next, replacing the polynomial ring $\mathcal{S}uper([\mathcal{L}^- \mid \mathcal{P}^-])$ with its q-deformation, $Mat_q(t_{i,j})$, I directly generalize the construction of the Schur modules to a construction of q-Schur modules.

Definition 3.3.1 *A term order compatible (in the sense of Chapter 2, Subsection 2.3.1) with a set $\mathcal{T}$ of ordered monomials in $Mat_q(t_{i,j})$ is* diagonal *if the initial monomial of the q-determinant of any minor indexed by rows $i_1 < \cdots < i_k$ and columns $j_1 < \cdots < j_k$ is* init $\left(\prod_{s,r} t_{i_r,j_s}\right)$.

For our purposes the *ordered monomials*, $\mathcal{T}$ in $Mat_q(t_{i,j})$ will be all products $\prod_l t_{i_l,j_l}$ such that $\underline{j}$ weakly increases and $j_l = j_{l+1}$ implies $i_l \geq i_{l+1}$. Define the partial order $\prec_{\text{diag}}$ by $\prod_l t_{i_l,j_l} \prec_{\text{diag}} \prod_l t_{i'_l,j'_l}$ when $\underline{j} = \underline{j}'$ and $\underline{i} < \underline{i}'$ in lexicographic order. Caveat: The partial order $\prec_{\text{diag}}$ does not agree with the total order that was imposed on variables for the purpose of defining the ordered monomials, nor is it required to agree.

Proposition 3.3.1 *The order $\prec_{\text{diag}}$ on $\mathcal{T}$ is a diagonal term order compatible with the generators $t_{i,j}$ of $Mat_q(t_{i,j})$.* □

Next it is necessary to define a quantized Schur module $\mathcal{S}_q^D$ for a shape D. Essentially, we generalize the ordinary definitions in the obvious fashion.

Definition 3.3.2 *If S, T are tableaux define $[S|T]_q$ to be $\prod_i [S_i|T_i]$ where S_i, T_i are the ith rows S and T respectively.*

Define $[T]_q = [T|Der^-(D)]_q$ where D is the shape of T.

Define $\mathcal{S}_q^D(\mathcal{L})$, the q-Schur module *of shape D, to be the $\mathbf{Z}[q, q^{-1}]$-span of $[T]_q$ where T runs over all T of shape D, filled with elements of $\mathcal{L}$.*

Notice that permuting the rows of a shape D need no longer leave the q-Schur module invariant. Thus it is essential that for the remainder of the chapter we extend the definition of a row-convex shape. A row-convex shape is *sorted* if its right-hand edge is partition-like, i.e. as we move down rows, the right-hand edge moves weakly left.

For any diagonal term order $\prec$ compatible with a set of monomials $\mathcal{T}$, define $\mathrm{Tab}_q^{\prec}(T) = (q)^{N_T}[T]_q$ such that $\mathrm{LT}_{\prec}\left(\mathrm{Tab}_q^{\prec}(T)\right) = \mathrm{init}_{\prec}\left(\mathrm{Tab}_q^{\prec}(T)\right)$. We may omit the superscript if the diagonal term order is clear.

3.4 Straight bases

The object of this section will be to generalize to the q-Schur modules the straight basis theorem and straightening law of Chapter I.

In Chapter 1, I restricted the definition of straightness to sorted tableaux of row-convex shape D; i.e. for rows $i < j$ of D, row i must end at least as far right as row j. If instead for each pair of rows $i < j$, the end of row i is farther *left* than the end of row j, I will say that D is *reverse-sorted.*

Definition 3.4.1 *A two-rowed, row-convex tableaux T is* capsized-straight *if interchanging the rows of T yields a straight tableaux.*

The straightening law of Chapter 1 relied on being able to straighten a pair of rows at a time. First we need to define the appropriate straightening relation.

Lemma 3.4.1 *Let D be a two-rowed shape with cells in fewer than s columns. For distinct letters $u_1, \ldots, u_s$ there exist integers n_σ such that*

$$\begin{bmatrix} v_1 & \ldots & u_1 & \ldots & u_2 & \ldots\ldots & u_{s'} & \ldots & v_r & \\ w_1 & \ldots & u_{s'+1} & \ldots & u_{s'+2} & \ldots\ldots & u_s & \ldots & w_{r'} \end{bmatrix}_q = \sum_{\text{nontriv. shuffles, } \sigma} (-q)^{n_\sigma}[T_\sigma]_q$$

where T_σ is the unique row-standard tableau of shape D whose top row contains $v_1, \ldots, v_r$; $\sigma(u_1), \ldots, \sigma(u_{s'})$ and whose bottom row contains the letters $w_1, \ldots, w_{r'}$

and $\sigma(u_{s'+1}), \ldots, \sigma(u_s)$. If there is no such row-standard tableau then just require the rows to weakly increase; by Proposition 3.2.5 its image under $[\ ]_q$ vanishes.

Proof. Apply Proposition 3.2.5 and Lemma 3.2.6. Since s is larger than the number of columns is D, the middle minor on the right-hand side of Lemma 3.2.6 vanishes by another application of Proposition 3.2.5. □

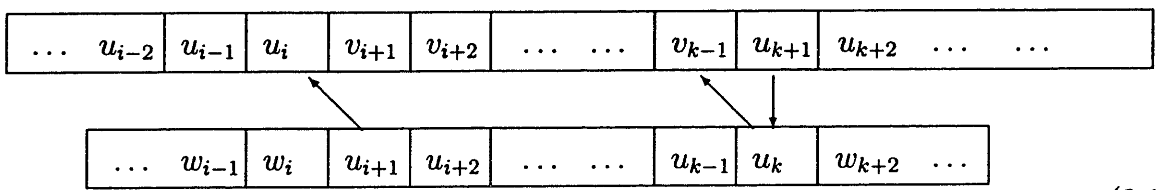
Proposition 3.4.2 *Let D be a two-rowed, row-convex shape. If D is sorted (respectively reverse-sorted) then $S_q^D(\mathcal{L}^-)$ has a basis consisting of all elements $[T]_q$ indexed by the straight (resp. capsized-straight) tableaux of shape D.*

Now working in $Mat_q(t_{i,j})$ is analogous to having only negative letters in the classical case. More explicitly, the determinant of a quantum minor with a repeated row vanishes. So when handling $S_q^D(\mathcal{L})$ we can ignore all tableaux not row-standard; by Proposition 3.2.5 either there exists some unit α such that $[T] = \alpha[S]$ where S is row-standard of shape D and each row of S has the same contents as the corresponding row of T, or alternately, there exists no such S and $[T] = 0$.

The fact that I am handling only the q-analog of the Schur modules allows me to provide a much simpler straightening algorithm than that of Chapter 1. Indeed this is fortunate since there is no known q-analogue of the straightening syzygy syzygy of [DRS76].

Proof. I will prove the theorem for sorted shapes. The reverse-sorted case follows symmetrically by reversing all constructions across the x-axis. I will execute the proof assuming that the top row contains the bottom row, but the proof goes through all the more easily if the shape D is skew.

I start by briefly describing my simplified straightening algorithm. Suppose that T is a row-standard but non-straight tableaux. Let k be the column in which T has a flippable inversion. Let $i+1$ be the column of the first cell in the bottom row of T where the contents of that cell are bigger than the contents of the cell in column i of the top row. If the bottom row ends at least as left as the top row, then $i+1$ will be the first cell of the bottom row. Now use Lemma 3.4.1 to replace $[T]_q$ with a linear combination of $[T_\sigma]_q$ by q-symmetrization of the entries in all cells of the top row, except those in columns $i+1$ through $k-1$, together with the contents of all cells in the bottom row that fall in columns $i+1$ through k. Suppose T is as pictured below,

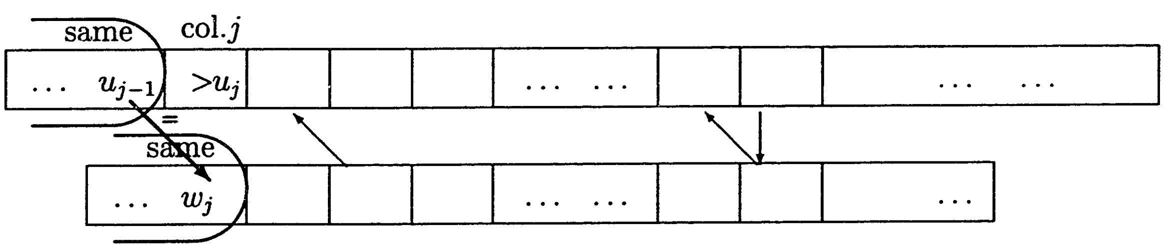


(3.1)

then the strictly increasing sequence $\ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_k, u_{k+1}, \ldots$ is being q-symmetrized. To finish the proof of Proposition 3.4.2, it suffices to prove that each such application of Lemma 3.4.1 produces only T_σ with strictly larger column word than the column word of T. Let $j = \min_{\sigma(u_c)\neq u_c} c$. The proof decomposes into three cases.

Case I. $(j \leq i)$

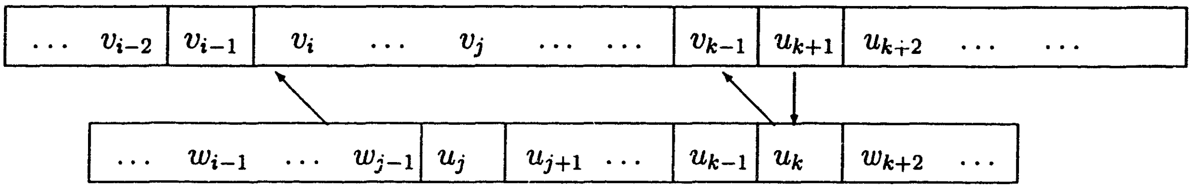
The tableau T_σ is partly described by,



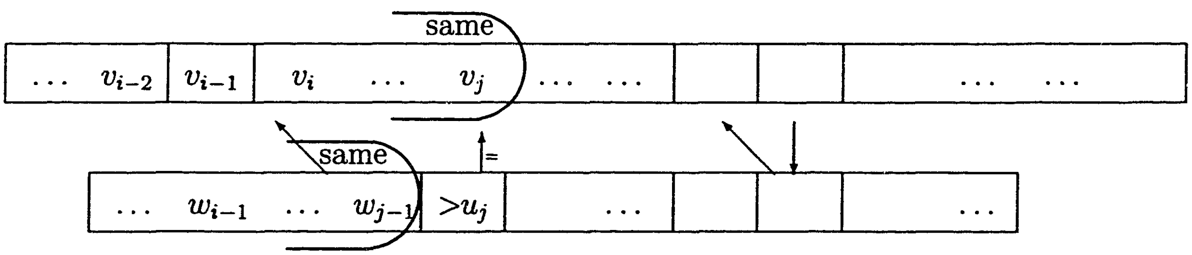
where *the arrows of equation 3.1 have been copied solely to aid comparison.* The circled entries in the top row remain the same because of the inequality indicated by the left-hand northwest arrow in equation 3.1. The circled entries in the bottom row remain the same because to change them, the permutation σ would have had to bring some element smaller than w_j down from the top row. But because the aforementioned northwest arrow indicates the leftmost such inequality occurring in T, each element that could have been brought down must be strictly larger than w_j. As the entry in cell j of the top row increased upon permuting and only further increased on resorting the rows, we find that the column word increases.

Case II. $(i < j < k)$

In this case we are only considering the tableaux T_σ arising from shuffling $u_j, u_{j+1}, \cdots$ in the following picture,



From this information we know that in forming T_σ, row-sorting does not move the first $j-1$ entries in the bottom row of T. Further since k was the leftmost flippable inversion, we must have had that the top cell in column j of T had a value less than or equal to the value of the bottom cell. This information is recorded below in the picture of T_σ,



where the arrows indicate the relations that held in T. The inequality holding in column j guarantees that none of the u's that are moved up to the top row can be resorted into the first j cells of that row–thus we can guarantee that those cells are indeed unchanged. Since the bottom cell of column j had to increase before resorting and this increase is preserved, we conclude that the column word increases.

Case III. $(j = k)$

The last case is particularly simple. We are guaranteed that T and T_σ agree in their first $k-1$ columns. Since the bottom element of column k increases even before row-sorting, we are done. $\square$

Porism 3.4.3 *If T is a row-standard tableau of two-rowed, row-convex shape D and if $\sum_\iota [T^{(\iota)}]_q$ is the expansion of $[T]_q$ in terms of straight tableaux $T^{(\iota)}$, then $w_T \leq w_{T^{(\iota)}}$*

in lexicographic order. If D is sorted then $c_T < c_{T^{(\iota)}}$. □

At this stage in the proof that straight tableaux span, we were essentially done–apply the preceding syzygy to any non-straight tableau and iterate. However, unlike straightening for skew tableaux, row-convex straightening requires that the straightening syzygies be applied when two not-necessarily adjacent rows form a non-straight tableau. In the quantum world however, the fact that $ac = \sum_i a_i c_i$ need not imply that $abc = \sum_i \alpha_i a_i b c_i$ even if some extra scalars α_i are factored in. The next several lemmas are designed to circumvent this problem. I will show that if we choose the rows forming a non-straight subdiagram to be sufficiently close together, then we can essentially pull out the intervening rows, apply the straightening syzygy, and then put the relevant rows back.

Lemma 3.4.4 *Suppose that $T = \begin{smallmatrix}\cdots T_1 \cdots \\ \cdots T_2 \cdots\end{smallmatrix}$ is a row-standard tableau of two-rowed, row-convex shape D. Further suppose that one row of D contains the other, that D' is the shape obtained by interchanging the rows of D and that T' is the tableau obtained by interchanging the rows of T. Under these assumptions, $\mathrm{Tab}_q^{\prec_{\mathrm{diag}}}(T)$ can be written as $\mathrm{Tab}_q(T') + \sum_\iota \alpha_\iota \mathrm{Tab}_q(T^{(\iota)})$ where each tableau $T^{(\iota)}$ is row-standard of shape D' and has strictly larger modified column word than has T.*

Proof. Suppose that D is sorted (respectively reverse-sorted) Because all straight (capsized-straight) tableaux have different modified column words, the basis elements $[T]_q$, or alternately $\mathrm{Tab}_q(T)$, found in Proposition 3.4.2 must have different initial terms. By the comment after Proposition 3.2.7 we can write $\mathrm{Tab}_q(T)$ as a linear combination $\mathrm{Tab}_q(T^{(\iota)})$ where the tableaux $T^{(\iota)}$ have shape D'. Since $[T']_q$ has an expansion indexed by capsized-straight (straight) tableaux, we may as well assume that all the tableaux with modified column words distinct from that of T are capsized-straight (straight) and that the tableau with the same modified column word is T'. Taking initial terms it is clear that $\mathrm{Tab}(T')$ appears with the same coefficient as did

Tab(T) and that all other tableau have lexicographically larger column words. □

In the above argument, any two-rowed basis indexed by tableau which enjoy Woodcock's property of being row-standard and having distinct modified column words can fill in for the straight/capsized straight tableaux. The price paid for this generality is that the resulting expansion algorithm works only in $Mat_q(t_{i,j})$ rather than in the presentation of $S_q(D)$.

Lemma 3.4.5 *Suppose that D is a diagram in which two rows $r < s$ form a skew subdiagram and suppose there exists no intermediate row $r < i < s$ such that those three rows form a skew subdiagram of D. There exists a diagram D' obtained by permuting the rows of D such that $\mathcal{S}_q^D = \mathcal{S}_q^{D'}$ and such that under this permutation the rows r and s end up adjacent with r still preceding s.*

Proof. The rows lying between r and s fall come in two (not necessarily disjoint) types. Rows of type A are contained in row r and rows of type B contain row s. This implies that any row of type B appearing above a row of type A contains the row of type A. Thus by the remark after Proposition 3.2.7, we can permute the rows of type A up to before row r; start with the northmost such row and move it up by successive interchanges, then move up the second northmost row of type A etc. Once this is done, carry out an analogous process (starting with the southernmost row of type B) to move the rows of type B south of row s. □

Lemma 3.4.6 *Suppose D is a sorted, row-convex diagram. Suppose $r < s$ are rows of D where row r strictly contains row s. There exists a shape D' obtained by permuting the rows of D such that rows r and s end up adjacent in D' and such that $\mathcal{S}_q^D = \mathcal{S}_q^{D'}$.*

Proof. As in the preceding lemma, the proof is accomplished through constructing a permutation by sequentially interchanging adjacent rows for which one row contains the other. Again, the intervening rows are of type A–which are contained in row r

and type B–which fail to be contained in row r. Any row of type B appearing above a row of type A contains that row and any row of type B certainly contains row s. We permute the rows of type A up and then permute the rows of type B down just as in the previous lemma. □

Porism 3.4.7 *Given diagrams D and D' as in either Lemma 3.4.5 or Lemma 3.4.6, any tableau T of shape D (respectively D') can be rewritten as a linear combination $T' + \sum_\iota \alpha_\iota T^{(\iota)}$ of tableaux of shape D' (respectively D) where each $T^{(\iota)}$ has modified column word strictly larger than w_T, and where T' is obtained by permuting the rows of T by the same permutation taking D to D' (resp. D' to D) that was produced in the proof of Lemma 3.4.5.*

Proof. Apply Lemma 3.4.4. □

This result says that while quantum minors do not quasi-commute, in the preceding cases they almost quasi-commute. In particular, in these cases we achieve enough quasi-commutation modulo an appropriate filtration to push through the straightening algorithm.

Theorem 3.4.8 *Suppose D is a sorted row-convex shape. The ring elements $[T]_q$ indexed by all straight tableaux of shape D with entries in $\mathcal{L}^-$ form a basis for $\mathcal{S}_q^D(\mathcal{L}^-)$.*

Proof. Independence is an immediate consequence of the fact that the initial term of a row-standard tableaux T under a diagonal term order is still determined by the modified column word of T– and since the combinatorics is unchanged, distinct straight tableau of course have distinct modified column words.

The spanning algorithm is now easy. Present any element of $\mathcal{S}_q^D$ as a linear combination of $[T]_q$ (or $\mathrm{Tab}_q(T)$) where T ranges over some collection of row-standard tableaux of shape D. For any non-straight tableaux T pick a pair of rows $r < s$ that form a non-straight subtableaux. If the upper row contains the lower row, then use Lemma 3.4.6 and Porism 3.4.7 to rewrite $[T]_q$ as a linear combination indexed by a permutation T' of T and tableaux with larger modified column word. Since r, s are now adjacent, we can apply Proposition 3.4.2 to rewrite these rows in T' as a linear combination where these rows have larger column word and weakly larger modified column word. Reapplying Lemma 3.4.6 and Porism 3.4.7 to all of these tableaux gives a linear combination of tableaux where all but one have modified column words that are strictly larger than that of T. The tableau with the same modified column word has had its column word increased.

Precisely the same kind of argument goes through if non-straight rows $r < s$ form a strictly skew subtableau. The only difference is that we need to use Lemma 3.4.5 rather than Lemma 3.4.6. We are guaranteed that we can meet the extra conditions of this lemma since any three-rowed skew subtableaux of D in which the first and third rows are non-straight (equals non-standard) must have either the first and second rows or the second and third rows forming a non-straight subtableaux. $\square$

Porism 3.4.9 *If T is a row-standard tableau of sorted, row-convex shape D and if $\sum_\iota [S^{(\iota)}]_q$ is the expansion of $[T]_q$ in terms of shape D straight tableaux, $S^{(\iota)}$, then $w_T \leq w_{S^{(\iota)}}$ and $c_T < c_{S^{(\iota)}}$ in lexicographic order.* $\square$

Corollary 3.4.10 *Consider an element $p = \sum_{\iota \in \mathcal{I}} \alpha_\iota Tab_1(T^{(\iota)})$ of $Mat_q(t_{i,j})$ where each $T^{(\iota)}$ is row-standard. Suppose that $\sum_\kappa \alpha_\kappa = 0$ where the sum is over all $\kappa \in \mathcal{I}$ such that $w_{T^{(\kappa)}}$ is lexically minimal. Every tableau appearing in the expansion of p into the straight basis must have strictly larger modified column word than $w_{T^{(\kappa)}}$.* $\square$

Proof. By the preceding porism this inequality must hold weakly. If equality ever held, then, by the distinctness of modified column words for straight tableaux, the initial term of p must equal $w_{T^{(\kappa)}}$. But applying the porism again together with the vanishing of $\sum_\kappa \alpha_\kappa = 0$, this cannot happen. $\square$

3.5 Anti-straight bases for q-Schur modules

The purpose of this section is to build the few extra tools necessary to generalize to q-Schur modules the results of Chapter 1 on superSchur modules determined by collections of row-convex shapes. Most of the hard work has already been done in Section 3.4. Essentially, it remains to show that there is a quantum James-Peel inclusion of the q-Schur module, determined by a compression of a shape D, into the q-Schur module of D.

Lemma 3.5.1 *Suppose that D is a sorted row-convex shape whose rows start at $m_1, \ldots, m_d$ and end in columns $\lambda_1, \ldots, \lambda_d$. Suppose that the first and last rows of D form a skew subdiagram and further that $m_1 - 1 \leq \lambda_d$. Finally, suppose that all of the intervening rows are contained in the top row, i.e. $m_1 \leq m_i$ for all $i < d$. If D' is the diagram whose rows respectively start in columns $m_d, m_2, \ldots, m_{d-1}, m_1$ and end in columns $\lambda_1, \ldots, \lambda_d$ then $\mathcal{S}_q^{D'} \subseteq \mathcal{S}_q^D$.*

The condition that $m_1 - 1 \leq \lambda_d$ is necessary to guarantee that D' is row-convex.

Proof. Let E be a (no longer sorted) diagram–its rows start at $m_2, \ldots, m_{d-1}, m_1, m_d$ and end at $\lambda_2, \ldots, \lambda_{d-1}, \lambda_1, \lambda_d$ Let E' be a diagram whose rows start at $m_2, \ldots, m_d, m_1$ and end at $\lambda_2, \ldots, \lambda_{d-1}, \lambda_1, \lambda_d$.

By the comment following Proposition 3.2.7 we observe that permuting the middle rows of D past the top row in order to form E preserves the module, namely $\mathcal{S}_q^D = \mathcal{S}_q^E$. But Proposition 3.2.7 gives the inclusion map for $\mathcal{S}_q^E \supseteq \mathcal{S}_q^{E'}$. Finally, we can permute the intermediate rows back inside (since the next to bottom row only got longer in the passage from E to E'.) Thus $\mathcal{S}_q^D \supseteq \mathcal{S}_q^{E'} = \mathcal{S}_q^{D'}$. □

Proposition 3.5.2 *Suppose that D is a sorted, row-convex diagram with rows starting at $\underline{m}$ and ending at $\underline{\lambda}$. Suppose that for some $r < s$ the subdiagram formed by row r and row s is skew and that $m_r - 1 \leq \lambda_s$. Let D' be the new diagram whose rows start and end in the same places as the rows of D save for rows r and s which* *start and end respectively in columns m_s, λ_r and in columns m_r, λ_s. The q-analogue $\mathcal{S}_q^{D'} \subseteq \mathcal{S}_q^D$ of the James-Peel inclusion holds.*

Proof. The proof will be an induction on the number of rows intervening between r and s.

Suppose that r and s fail to be adjacent. If there is no row $r < j < s$ such that rows r, j, s define a skew subdiagram of D (in particular if r and s are adjacent) then Lemma 3.5.1 shows $\mathcal{S}_q^{D'} \subseteq \mathcal{S}_q^D$.

So assume that such a j exists. We now define a diagram $D^{(1)}$ by performing a James-Peel lift from row s to row j in diagram D and we define a diagram $D^{(2)}$ by lifting row j to row r in diagram $D^{(1)}$. More explicitly $D^{(1)}$ agrees with D except in rows j and s which run respectively from columns m_s through λ_j and from m_j through λ_s. Likewise $D^{(2)}$ agrees with $D^{(1)}$ except in rows r and j which run from m_s through λ_r and from m_r through λ_j. But since D' arises by lifting row s of $D^{(2)}$ to row j, we find by induction that

$$\mathcal{S}_q^D \supseteq \mathcal{S}_q^{D^{(1)}} \supseteq \mathcal{S}_q^{D^{(2)}} \supseteq \mathcal{S}_q^{D'}.$$

□

Definition 3.5.1 *Suppose that $\mathcal{D}$ is a collection of shapes. Define $\mathcal{S}_q^{\mathcal{D}}(\mathcal{L}^-)$ to be the $\mathbf{Z}[q, q^{-1}]$-span of all $[T]_q$ where T is filled from $\mathcal{L}^-1$ and the shape of T lies in $\mathcal{D}$.*

Theorem 3.5.3 *Let $\mathcal{D}$ be any finite collection of row-convex shapes. A basis for $\mathcal{S}_q^{\mathcal{D}}(\mathcal{L}^-)$ is given by all $[T]_q$ where T is anti-straight on letters in $\mathcal{L}^-$ and the shape of T is a compression of some $D \in \mathcal{D}$.*

Proof. Since such tableaux have distinct modified column words, they have distinct initial terms under a diagonal term order and hence are linearly independent. Since Proposition 3.5.2 tells us that if D' is a compression of D then $\mathcal{S}_q^{D'} \subseteq \mathcal{S}_q^D$ then, using

Proposition 3.4.9 to guarantee that modified column words strictly increase, the proof of Theorem 1.7.7 goes through unchanged. □

3.6 Dominance order for straightening quantum bitableaux

In cite HZ, Huang and Zhang provided a quantum straightening law for quantum linear supergroups. This section aims at strengthening their result for the case of quantum linear *groups.* Huang and Zhang prove a basis theorem for the quantum general linear group by proving spanning algorithmically and then appealing to a dimension count to provide the independence résult. Here I take the opposite approach. I prove independence by showing how a suitable leading term of a product of quantum minors determines those minors. Ideally, I would have a straightening algorithm to prove spanning, but I also have been forced to appeal to a dimension count. The proof I provide does in fact produce an algorithm for straightening. However, like most algorithms inspired by SAGBI bases or other triangularity arguments, the algorithm is merely a byproduct of the theorem. Since, as mentioned earlier, a q-analog of the classical straightening syzygies is not yet known, Huang and Zhang were forced to rely on a cruder straightening law based on quantum Laplace expansion and hence showed that bitableaux of a given shape expand into bitableaux of lexicographically longer shape. The novelty of my approach is that it provides a proof that bitableaux of a given shape expand into bitableaux of longer shape with respect to the *dominance* partial order. Since this recovers the full strength provided by the classical straightening syzygies, it strongly suggests that q-analogues of such syzygies exist.

Let $A = (a_{i,j})$ and $B = (b_{k,l})$ be $n \times n$ generic quantum matrices in which the $a_{i,j}$'s commute with the $b_{k,l}$'s. By Lemma 3.2.8, AB is itself a quantum matrix *call it* Y.

Fix an order on monomials in the $a_{i,j}$ and $b_{i,j}$. The set $\mathcal{T}$ of ordered monomials is a $\mathbf{Z}[q,q^{-1}]$-basis for $Mat_q(a_{i,j};b_{i,j}) = Mat_q(a_{i,j}) \otimes Mat_q(b_{i,j})$. A term (partial) order, $\prec$, compatible (as defined in Chapter 2) with $\mathcal{T}$ is *bidiagonal* if for all $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$ we have

$$\mathrm{init}_{\prec}\left(\mathrm{Tab}_q(i_1,\ldots,i_k|j_1,\ldots,j_k)\right) = \mathrm{init}_{\prec}\left(\prod_{l=1}^{k} a_{i_l,l}b_{l,j_l}\right).$$

For such a $\prec$, define $\mathrm{Tab}_q^{\prec}(S|T)$ for any pair of row-standard tableaux of the same shape to equal $q^{N_{S,T}}[S|T]_q$ where $N_{S,T}$ is the unique integer so that $\mathrm{LT}_{\prec}\left(\mathrm{Tab}_q(S|T)\right) = \mathrm{init}_{\prec}\left(\mathrm{Tab}_q(S|T)\right)$.

For the remainder of this section, define the set $\mathcal{T}$ to be all monomials $\prod_l a_{i_l,h_l} \prod_l a_{h'_l,j_l}$ where the sequences $\underline{h}$ and $\underline{h}'$ weakly increase and where $h_l = h_{l+1}$ implies $i_l > i_{l+1}$ and $h'_l = h'_{l+1}$ implies $j_l > j_{l+1}$.

Definition 3.6.1 *Choose $m \in \{1,\ldots,n\}$. Define a total order $<_m$ on sequences in $\{1,\ldots,n\}$ by $\underline{i} <_m \underline{j}$ when there are more letters less than or equal to m in $\underline{i}$ than there are in $\underline{j}$ or, if these numbers are equal, when $\underline{i} < \underline{j}$ lexicographically.*

Define a partial order $\prec_m$ on $\mathcal{T}$ by $\prod_{l=1}^k a_{i_l,h_l}b_{h_l,j_l} \prec_m \prod_{l=1}^k a_{i'_l,h'_l}b_{h'_l,j'_l}$ when $\underline{h} <_m \underline{h}'$ or $\underline{h} = \underline{h}'$ and in lexicographic order both $\underline{i} < \underline{i}'$ and $\underline{j} < \underline{j}'$.

Proposition 3.6.1 *The order $\prec_m$ forms a bidiagonal term order compatible with the ordered monomials $\mathcal{T}$ in the $a_{i,j}$'s and $b_{i,j}$'s.*

Proof. It is an easy exercise to verify that $\prec_m$ is a term order. The bidiagonal property follows from Proposition 3.2.11; since this tells us that

$$\begin{aligned}\mathrm{init}_{\prec_m}\left(\mathrm{Tab}_q(i_1,\ldots,i_k|j_1,\ldots,j_k)\right) &= \\ &= \mathrm{init}_{\prec_m}\left(\mathrm{Tab}_q^A(i_1,\ldots,i_k|1,\ldots,k)\mathrm{Tab}_q^B(1,\ldots,k|j_1,\ldots,j_k)\right),\end{aligned}$$

which is $\prod_{l=1}^k a_{i_l,l} \prod_{l=1}^k b_{l,j_l}$. $\square$

Corollary 3.6.2 *The elements $[S|T]_q$ of $Mat_q(a_{i,j}; b_{i,j})$ are linearly independent over $\mathbf{Z}[q, q^{-1}]$ where the pairs (S, T) run over all pairs of standard tableaux of the same partition shape. All minors are taken from the quantum matrix AB.*

Proof. The initial term of $[S|T]_q$ where S, T are standard of the same partition-shape, determines the tableaux. The number of times i appears in the lth column of S (respectively T) is the number of times $a_{i,l}$ (respectively $b_{l,i}$ appears in the initial monomial. □

Corollary 3.6.3 *Let $X = (t_{i,j})$ be a generic quantum matrix. Since there is an algebra homomorphism from $Mat_q(t_{i,j})$ to the algebra generated by the entries of AB, we conclude that the elements $[S|T]_q$ of $Mat_q(t_{i,j})$ are linearly independent over $\mathbf{Z}[q, q^{-1}]$ where again the pairs (S, T) run over standard tableaux of the same partition shape.* □

We now prove (in the negative letter case!) the central theorem of [HZ94].

Theorem 3.6.4 *Let $X = (x_{i,j})$ be an arbitrary quantum matrix. Any element of $\mathbf{Q}(q) \otimes_{\mathbf{Z}} Mat_q(x_{i,j})$ may be expressed as a linear combination of $[S|T]_q$ where S, T are standard tableaux of the same partition shape.*

This spanning set forms a basis when X is generic.

Proof. Spanning over $\mathbf{Q}(q)$ is a direct dimension count: choose your favorite proof that the number of semistandard bitableaux with k cells and entries in $1, \ldots, n$ equals $\left(\binom{n^2+k-1}{k}\right)$, the number of degree k ordered monomials in n^2 variables. To verify the dimension count while still roughly following the philosophy of this thesis, use the classical bitableaux straightening algorithm from [DRS76]. □

As an immediate corollary we realize that if X is generic then the initial terms of the elements $\mathrm{Tab}_q(S|T)$, where S, T are standard of the same partition shape, must be precisely the ordered monomials in $Mat_q(x_{i,j})$. This permits the preceding result to be lifted to $Mat_q(x_{i,j})$.

Corollary 3.6.5 *With the above assumptions, any element of $Mat_q(x_{i,j})$ may be expressed as a $\mathbf{Z}[q, q^{-1}]$-linear combination of $[S|T]_q$ with S, T standard.*

Again this is a $\mathbf{Z}[q, q^{-1}]$-basis when X is generic.

Proof. Assume $(t_{i,j})$ is a generic quantum matrix. I exhibit an algorithm along the lines of SAGBI-bases for performing the expansion. All polynomials in $Mat_q(t_{i,j})$ are assumed to be expanded into the basis of ordered monomials. For any $p \in Mat_q(t_{i,j})$, let M be a minimal ordered monomial appearing with coefficient $\alpha \in \mathbf{Z}[q, q^-1]$ in p. Replace αM with $\alpha \cdot (S, T) - \alpha \cdot \mathrm{tail}\,(\mathrm{Tab}_q(S|T))$ where S, T are the unique bitableaux such that $\mathrm{LT}\,(\mathrm{Tab}_q(S|T)) = M$. Iterate. The bitableaux, such as (S, T) above, are considered formal variables and do not show up as leading terms.

Since $p - \mathrm{Tab}_q(S|T)$ is still in $Mat_q(t_{i,j})$ and has a smaller maximal term than M (or at least fewer maximal terms incomparable with M) this algorithm terminates in a formal combination of bitableaux. When each formal $(S|T)$ is replaced by $\mathrm{Tab}_q(S|T)$ we obtain p again. $\square$

I conclude with the promised strengthening of the Huang-Zhang basis theorem.

Theorem 3.6.6 *Assume S, T are row-standard tableau of partition-shape λ and that $Tab_q(S|T) = \sum_\iota \alpha_\iota Tab_q(S_\iota|T_\iota)$ where each pair (S_ι, T_ι) is a standard bitableau of shape λ_ι. Denote the conjugate of a partition μ by μ'.*

The shape λ' dominates $\lambda_\iota{}'$, that is $\sum_{j=1}^k \lambda'_j \geq \sum_{j=1}^k \lambda'_{\iota_j}$ for all k.

Proof. Suppose not, then there exists a ι such that λ_ι violates the dominance condition. In particular choose κ so that λ'_κ is maximal in the dominance partial order. Let m be the smallest integer such that $\sum_{j=1}^m \lambda'_j < \sum_{j=1}^m \lambda'_{\kappa_j}$. But this implies that $\mathrm{init}\,(\mathrm{Tab}_q(S_\kappa|T_\kappa))$ is $(\prec_m)$-smaller than $\mathrm{init}\,(\mathrm{Tab}_q(S|T))$. The maximality of λ'_κ says that, in the expansion of $\mathrm{Tab}_q(S|T)$ into ordered monomials, no other copies of $\mathrm{init}\,(\mathrm{Tab}_q(S_\kappa|T_\kappa))$ appear. Thus the minimal terms of $\mathrm{Tab}_q(S|T)$ must include $\mathrm{init}\,(\mathrm{Tab}_q(S_\kappa|T_\kappa))$. Contradiction. $\square$

This technique generalizes directly to the following result

Proposition 3.6.7 *Copy the assumptions of the preceding theorem.*

Given integers $j, k \geq 1$, the number of entries less than j appearing the first k columns of S_ι (respectively T_ι) cannot exceed the number of entries less than j appearing the first k columns of S (respectively T.)

Proof. The method of the preceding proof translates directly. Simply refine the term order by replacing the lexicographic comparison $\underline{i} <_{m'} \underline{i}'$ in Definition 3.6.1 by the condition that

$$|\{(i_l|h_l) : i_l \leq j, h_l \leq k\}| > |\{(i'_l|h'_l) : i'_l \leq j, h'_l \leq k\}|$$

or these cardinalities are equal and $\underline{i} < \underline{i}'$. (Respectively, make the corresponding changes for $\underline{j} < \underline{j}'$.) $\square$

3.7 Short exact sequences of q-Schur modules

The Akin-Buchsbaum short exact sequences described in [AB85] are a fundamental tool in constructing characteristic-free resolutions of Schur modules over V by direct sums of tensor products of anti-symmetric powers of V. The original proof of exactness relied on the use of spectral sequences. In recent years, Klucznik [K96] and Woodcock [W94] have given more direct proofs of exactness. Klucznik's proof is particularly nice in that it provides him the opportunity to prove exact a much broader class of complexes. In this section, I employ the combinatorics of straight and anti-straight tableaux to quantize some of these short-exact sequences–including all of the Akin-Buchsbaum sequences. I also produce some short-exact sequences whose kernels are defined by unions of tableaux; these give new results even in the classical case. I conclude with two splittings of these sequences.

3.7.1 The quantized universal enveloping algebra of gl_n

Assume that $\mathcal{L} = \mathcal{L}^-$ has n elements. Assume that $t_{i,j}$ is an $n \times n$ generic quantum matrix. In order to establish that the desired complexes exist it will be desirable to possess an analogue of the place polarizations in the classical case. To this end, and to demonstrate that the sequences here presented are indeed sequences of representations, I will recall the definition of $U_q(gl_n)$, the *quantized enveloping algebra of gl_n*. This definition has been lifted from [LeTh96] who reference [Ji86]. The definition in [LeTh96] is essentially over $\mathbf{Q}(q)$, we will be using an action (also taken from [LeTh96]) on $\mathbf{Q}(q) \otimes_{\mathbf{Z}[q,q^{-1}]} Mat_q(t_{i,j})$. However I will only be applying elements of $U_q(gl_n)$ which preserve $1 \otimes Mat_q(t_{i,j})$.

We will not be using the structure of $U_q(gl_n)$, but for completeness a presentation of the algebra follows. The algebra $U_q(gl_n)$ is generated by noncommuting variables e_i, f_i for $i = 1, \ldots, n-1$ and $q^{\epsilon_i}, q^{-\epsilon_i}$ for $i = 1, \ldots, n$ subject to the relations, $q^{\epsilon_i} q^{-\epsilon_i} = 1$, $q^{-\epsilon_i} q^{\epsilon_i} = 1$, $q^{\epsilon_i} q^{\epsilon_j} = q^{\epsilon_j} q^{\epsilon_i}$; $q^{\epsilon_i} e_i q^{-\epsilon_i} = q e_i$, $q^{\epsilon_i} e_{i-1} q^{-\epsilon_i} = q^{-1} e_{i-1}$, $q^{\epsilon_i} e_j q^{-\epsilon_i} = e_j$; $q^{\epsilon_i} f_i q^{-\epsilon_i} = q^{-1} f_i$, $q^{\epsilon_i} f_{i-1} q^{-\epsilon_i} = q e_{i-1}$, $q^{\epsilon_i} f_j q^{-\epsilon_i} = e_j$, where $j \neq i-1$. Further, impose the relation $e_i f_j - f_j e_i = \delta_{i,j} \left(q^{\epsilon_i} q^{-\epsilon_{i+1}} - q^{-\epsilon_i} q^{\epsilon_{i+1}}\right) / (q - q^{-1})$ where $|i-j| > 1$ and $\delta_{i,j}$ is the Kronecker delta. Finally, require that all e_i's with indices differing by at least two commute (similarly for the f_j's) and when $|i-j| = 1$ impose the relation $e_j e_i^2 - (q + q^{-1}) e_i e_j e_i + e_i^2 e_j$, (again similarly for the f_j's.)

What we do require is the following action as described in [LeTh96].

Fact 3.7.1 *There is a representation of $U_q(gl_n)$ on $\mathbf{Q}(q) \otimes_{\mathbf{Z}[q,q^{-1}]} Mat_q(t_{i,j})$ defined by the action of the generators on the quantum variables,*

$$q^{\epsilon_i} t_{k,l} = q^{\delta_{i,l}} t_{k,l}, \qquad q^{-\epsilon_i} t_{k,l} = q^{-\delta_{i,l}} t_{k,l}, \qquad e_i t_{k,l} = \delta_{i+1,l} t_{k,l-1}, \qquad f_i t_{k,l} = \delta_{i,l} t_{k,l+1},$$

where $\delta_{i,j}$ is the Kronecker delta and by the quantized Leibniz formulas,

$$q^{\epsilon_i}(gh) = (q^{\epsilon_i} g)(q^{\epsilon_i} h), \qquad\qquad q^{-\epsilon_i}(gh) = (q^{-\epsilon_i} g)(q^{-\epsilon_i} h),$$

$$e_i(gh) = (e_i g)h + (q^{-\epsilon_i} q^{\epsilon_{i+1}} g)(e_i h), \qquad f_i(gh) = (f_i g)(q^{\epsilon_i} q^{-\epsilon_{i+1}} h) + g(f_i h).$$

The above action only effects the second index (the "places") of each $t_{i,j}$. If $U_q(gl_n)$ is also given an action on $\mathbf{Q}(q) \otimes_{\mathbf{Z}[q,q^{-1}]} Mat_q(t_{i,j})$ by acting likewise on the first index (the "letters") of each $t_{i,j}$, then these two actions commute.

The following definition will prove useful. It is analogous to the $\mathbf{Z}$-subalgebra of the (ordinary) universal enveloping algebra that I utilized in Chapter 1. Let $U_q(n)$ be the $\mathbf{Z}[q, q^{-1}]$-subalgebra of $U_q(gl_n)$ generated by the e_i's, f_i's, q^{ϵ_i}'s and $q^{-\epsilon_i}$'s.

The following is an immediate consequence.

Corollary 3.7.1 *Suppose $|\mathcal{L}| = n$. Each q-Schur module, $\mathcal{S}_q^D(\mathcal{L})$, is a module over $U_q(n)$ with action descending from the action of $U_q(n)$ on letters.*

Any map between q-Schur modules (constructed as submodules of $Mat_q(t_{i,j})$) and defined by an element of $U_q(n)$ acting on places must be a map of $U_q(n)$-modules. $\square$

From [LeTh96] we also take the following property, verifiable by direct computation.

Fact 3.7.2 *Choose integers $1 \le i_1 < \cdots < i_k \le n$ and $1 \le j_1 < \cdots < j_k \le n$. The generators of $U_q(n)$ act on quantum minors as follows,*

$$q^{\pm\epsilon_j} Tab_q(i_1, \ldots, i_k | j_1, \ldots, j_k) = \begin{cases} q^{\pm 1} Tab_q(i_1, \ldots, i_k | j_1, \ldots, j_k) & \text{if } j \in \underline{j} \\ Tab_q(i_1, \ldots, i_k | j_1, \ldots, j_k) & \text{otherwise} \end{cases}$$

$$e_j Tab_q(i_1, \ldots, i_k | j_1, \ldots, j_k) = \begin{cases} Tab_q(i_1, \ldots, i_k | j_1, \ldots, j_{l-1}, j, j_{l+1}, \ldots) & \text{if } j+1 = j_l \text{ for some } l \\ 0 & \text{otherwise} \end{cases}$$

$$f_j Tab_q(i_1, \ldots, i_k | j_1, \ldots, j_k) = \begin{cases} Tab_q(i_1, \ldots, i_k | j_1, \ldots, j_{l-1}, j+1, j_{l+1}, \ldots) & \text{if } j = j_l \text{ for some } l \\ 0 & \text{otherwise} \end{cases}$$

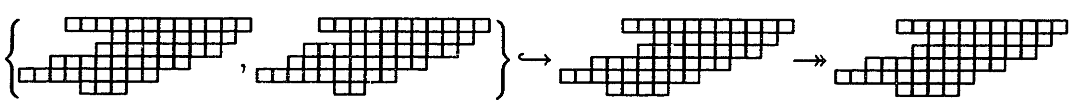
3.7.2 Sequences and tableaux

The techniques of this subsection generalize those used by Woodcock in his proof of the Akin-Buchsbaum short exact sequences.

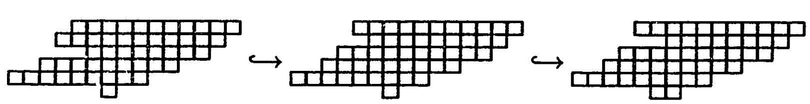
I will present short-exact sequences of the form

$$\mathcal{S}_q^{\mathcal{D}} \hookrightarrow \mathcal{S}_q^{D} \twoheadrightarrow \mathcal{S}_q^{D''},$$

which I will abbreviate by replacing each q-Schur module with the (set of) shape(s) indexing it. A typical example is



Observe that to get the projected shape, we have pushed the bottom row of the middle shape a single cell to the right. To get the kernel, we have performed two different James-Peel lifts. In general, one needs to include as many tableaux in this collection as there are rows to lift to. In this case, I have eliminated one these shapes (corresponding to a James-Peel lift from the bottom row to the second row) because the inclusion



shows that its module is already contained in the kernel.

Theorem 3.7.2 *Let $\mathcal{L} = \mathcal{L}^-$ have n elements. Let D'' be a sorted row-convex shape. Let D be the row-convex shape formed by shifting the entire bottom row of D'' one cell to the left. Let $\mathcal{D}$ be the set of all nontrivial direct compressions of D obtained by lifting the bottom row to some other row. There is an exact sequence*

$$\mathcal{S}_q^{\mathcal{D}}(\mathcal{L}) \hookrightarrow \mathcal{S}_q^{D}(\mathcal{L}) \twoheadrightarrow \mathcal{S}_q^{D''}(\mathcal{L}), \tag{3.2}$$

of $U_q(n)$-modules.

The proof is contained in Proposition 3.7.3 and 3.7.5. $\square$

Proposition 3.7.3 *The sequence in Equation 3.2 is a complex, the first map is an injection, the second is a surjection, and both are $U_q(n)$-modules homomorphisms.*

Proof. To show that the first map is an injection, it suffices to show that for all $D' \in \mathcal{D}$, we have $\mathcal{S}_q^{D'} \subset \mathcal{S}_q^{D}$. But this is a direct consequence of Proposition 3.5.2. Suppose the last row of D starts in column c and ends in column c', the desired projection can be achieved by applying the composition $f_c \circ f_{c+1} \circ \cdots \circ f_{c'}$ acting on places. Obviously this commutes with $U_q(n)$ action on letters, so it remains only to prove that $\mathcal{S}_q^{D'} \subset \ker(f_c \circ \cdots \circ f_{c'})$ for all $D' \in \mathcal{D}$. But since there is at most one row in D' lacking a cell at $c'+1$ (it has to be the bottom row) and that row must also lack a cell in column c, the composition must vanish on $\mathcal{S}_q^{D'}$. $\square$

To prove exactness, it will be convenient to generalize some of the definitions of Chapter 1. These generalizations follow the lines set out (independently) in [W94] and [Ri94], though as noted above the application is along the lines of [W94].

I start by defining the notions of left and right flippability.

Definition 3.7.1 *Fix a row-standard tableaux of two-rowed, row-convex shape. A column c in that tableaux is* left-flippable *if cutting the tableaux between column $c-1$ and c, flipping the left half of the tableau, and gluing back together produces another row-standard, row-convex tableaux. Similarly the column is* right-flippable *if this procedure goes through after cutting between column c and $c+1$.*

We say a pair of cells in column c (for instance an inversion or an anti-inversion) are left (or right) flippable when the two-rowed sub-tableaux determined by those cells is left (or right) flippable at column c.

Observe that the condition of being a flippable inversion (flippable anti-inversion) is equivalent to being a left-flippable inversion (right-flippable anti-inversion). Indeed suitable combinations of these conditions are easily seen to induce the relations presented in [W94].

Definition 3.7.2 *We say that a* flippable segment *in a two-rowed, row-convex, row-standard tableaux T is a set of cells in T occupying both rows of a segment from c through c' of the columns with where the cells in column c are left-flippable and those in column c' are right-flippable. The segment is* irreducible *if it contains no proper subsegment that is flippable. A flippable segment in an arbitrary tableau is a flippable segment in a two-rowed subtableau.*

The following lemma was implicit in Chapter 1.

Lemma 3.7.4 *Again work in a row-standard tableaux T of two-rowed, row-convex shape D.*

Any irreducible flippable segment consists entirely of inversions or entirely of anti-inversions.

If D is sorted, then T contains a left-flippable inversion iff it contains a flippable segment which includes inversions.

If D is reverse-sorted, then T contains a right-flippable inversion at column C iff it contains a flippable segment including inversions or *its top row extends further left than the bottom row.* □

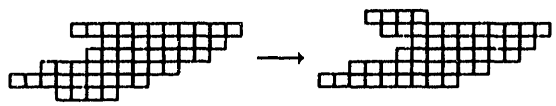
Finally, I define a class of tableaux specifically for this proof.

Definition 3.7.3 *A tableaux of row-convex shape is* nearly straight *if it is row-standard, has no flippable segments containing inversions and if no two-row sub-tableaux that fails to be sorted is flippable (i.e. possesses a left-flippable or a right-flippable pair).*

With this machinery in place, we finish the proof of exactness.

Proposition 3.7.5 *The sequence in Equation 3.2 is exact at its middle term.*

Proof. As before, let c, c' be respectively the first and last cells in the bottom row of D. Define a new shape $\tilde{D}$ from D by moving the bottom row of D to the top of the diagram as in the following example

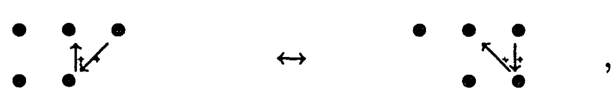


A tableau T of shape D is *$\tilde{D}$-nearly straight* if on permuting its rows to shape $\tilde{D}$, the resulting tableaux is nearly straight.

Let $\mathcal{B}$ be a set of row-standard tableaux each having sorted row-convex shape D or whose shape is some compression of D. Suppose every modified column word of a row-standard (equivalently straight) tableaux of shape D appears as the modified column word of some tableau in $\mathcal{B}$. We know that the $[T]_q$ for $T \in \mathcal{B}$ are a basis for $\mathcal{S}_q^D$.

Now by successively flipping those irreducible flippable segments containing inversions, we can reduce any row-standard tableau of shape $\tilde{D}$ either to a nearly straight tableau with the same shape and the same modified column word, *or* to a row-standard tableau which, on permuting to shape D, is formed by a James-Peel lift of the bottom row and which again has the same modified column word. We conclude that $\mathcal{S}_q^{\mathcal{D}}$ together with all $[T]_q$, where T is $\tilde{D}$-nearly straight of shape D, spans $\mathcal{S}_q^D$.

Now by Lemma 3.7.4, any pair of rows (besides the bottom row) in a nearly-straight tableau T must be straight. Further, any pair containing the bottom row must be anti-straight. But, as shown below,



after shifting, the anti-straight rows become straight. We conclude that if T is $\tilde{D}$-nearly straight, then the image of $[T]_q$ under the projection map is the $[S]_q$ for some straight tableau S of shape D''. Since distinct $\tilde{D}$-nearly-straight tableaux correspond to distinct straight tableaux under this shifting, we conclude that the sequence is exact and that the $[T]_q$, for all $\tilde{D}$-nearly-straight tableaux of shape D, are linearly independent. □

Porism 3.7.6 *The $\tilde{D}$-nearly straight tableaux above have distinct modified column words.* □

Porism 3.7.7 *The $\mathbf{Z}[q, q^{-1}]$-module $\mathcal{S}_q^D$ above splits as follows. A basis for the kernel is given by $[T]_q$ for all anti-straight tableaux T whose shape is a compression of D and where a row contains a cell in column c only if it contains a cell in column $c' + 1$.*

There is a free $\mathbf{Z}[q, q^{-1}]$-module with basis given by all $[T]_q$ where T is a $\tilde{D}$-nearly straight tableaux, and this module maps isomorphically (as a $\mathbf{Z}[q, q^{-1}]$-module) onto $\mathcal{S}_q^{D''}$. □

Porism 3.7.8 *Alternately, write down the anti-straight basis for $\mathcal{S}_q^D$. The elements indexed by tableaux whose shapes have a cell in column c without a cell in column $c+1$ form a basis for a $\mathbf{Z}[q, q^{-1}]$-module which maps isomorphically onto $\mathcal{S}_q^{D''}$. The remaining basis elements form a basis for the kernel of this map.* □

We now derive the quantized Akin-Buchsbaum sequences.

Corollary 3.7.9 *Suppose that D'' is* almost-skew *i.e. a sorted row-convex shape which on removal of its bottom row becomes skew. The collection $\mathcal{D}$ in Theorem 3.7.2 can be replaced by the James-Peel lift, D', of the bottom row from D to the lowest row, s, of D such that row s and the bottom row form a strictly skew subtableau.*

Proof. It suffices to show that if we James-Peel lift from the bottom row to some row $r < s$ then the resultant shape, D''', is a compression of D'. But this is easily seen by first lifting from the bottom row of D' to row r and then lifting from row s to row r in the resulting shape. □

The above results and their proofs generalize directly to super-Schur modules.

Proposition 3.7.10 *If $\mathcal{L}$ is an arbitrary signed alphabet and $\mathcal{D}$, D, and D' are as in Theorem 3.7.2 then*

$$\mathcal{S}^{\mathcal{D}}(\mathcal{L}) \hookrightarrow \mathcal{S}^{D}(\mathcal{L}) \twoheadrightarrow \mathcal{S}^{D''}(\mathcal{L})$$

is short exact. □

Chapter 4

The Robinson-Schensted-Knuth algorithm and combinatorics of initial terms.

The Robinson-Schensted-Knuth correspondence and related combinatorics plays an important role in the representation theory of the symmetric and general linear groups. The connection is partly "explained" by the fact that the Schensted correspondence shows up (see [LeTh96]) as quantum bitableaux straightening at $q = 0$. The recent work of Reiner and Shimozono on representations associated to general shapes–column convex, northwest, and %-avoiding in order of increasing generality–relies extensively on describing bases and character formulas in terms of the insertion properties of the row words of tableaux having these shapes. The usefulness of the Schensted correspondence is made all the more intriguing by the fact that it remains unclear how to generalize this approach much beyond the %-avoiding shapes studied by Magyar.

Remarkably, it turns out that straight tableaux are somewhat more subtle objects then the Reiner-Shimozono decomposable tableaux or the tableaux appearing in the corresponding Lakshmibai-Magyar basis. All of these bases satisfy an insertion property (see [RS96c]); roughly, you can tell if a tableaux belongs to the basis by looking at the recording tableaux of its row-word. The straight tableaux satisfy no such property. It turns out however, that the modified column word is a good substitute. This means that instead of studying bases via Schensted, one is studying the initial terms. This chapter develops some of these combinatorial connections, conjectures some generalizations, and shows algebraically how to view the Reiner-Shimozono decomposable tableaux as being "compatibly" dual to the straight tableaux of transpose shape.

4.1 Decomposable tableaux

The following results make clear some of the connections between straight tableaux of Chapter 1 and and the decomposable tableau defined by Reiner and Shimozono in [RS95]. I show that the modified column word of a shape D straight tableau can be identified by examining its recording tableau under the Schensted algorithm. In particular, the criteria identifying such a *realizable* modified column word are precisely the criteria for *decomposability* introduced in [RS95]. As immediate consequence we gain the ability to distinguish among the various possible decompositions of a recording tableau.

4.1.1 Basic definitions and lemmas.

The techniques of [RS95] rely heavily on the Robinson-Schensted-Knuth insertion algorithm. (See [Sa91] for an extremely readable exposition.) Since I want to handle supersymmetric tableaux I'll define a simple modification of the insertion algorithm to handle signed letters. Further [RS95] uses column insertion and I use row insertion. I will repeat several of Reiner and Shimozono's definitions with the appropriate modifications.

Recall that Robinson-Schensted-Knuth acts on a biword, $\mathbf{w} = (\hat{w}, \check{w})$ by row inserting $\check{w}$ from the left and recording insertions by corresponding entries of $\hat{w}$. Since ordinary Robinson-Schensted-Knuth produces column-strict semistandard tableau a given letter will not bump another instance of that letter during the insertion process. From a supersymmetric point of view these should be positively signed letters. So if we want to insert negatively signed letters, it is natural to require that they bump each other in the insertion process. The following example should make the insertion process clear.

Example 4.1.1 Inserting the word $\check{w} = 1^-2^+2^+1^-$ gives

$$1^- \longrightarrow 1^-2^+ \longrightarrow 1^-2^+2^+ \longrightarrow \begin{array}{lll} 1^- & 2^+ & 2^+ \\ 1^- & & \end{array}$$

successively.

Observe that the above insertion algorithm ignores the content of the recording word. We fix this by requiring a biword to be "sorted" in order to perform the signed insertion algorithm.

Definition 4.1.1 *A biword is said to be* sorted *when its upper row weakly increases and $\hat{w}_i = \hat{w}_{i+1}$ implies that $\check{w}_i <+ \check{w}_{i+1}$ when $\hat{w}_i$ is positive and implies that $\check{w}_i -> \check{w}_{i+1}$ when $\hat{w}_i$ is negative.*

If $\mathbf{w}$ is sorted, let $P_r(\mathbf{w})$ be the row insertion tableau of $\mathbf{w}$ and let $Q_r(\mathbf{w})$ be its row recording tableau.

Let $\mathbf{w}$ be a biword. If for no $i \neq j$, do we find that the ith and jth components of $\mathbf{w}$ are equal and the signs of $\hat{w}_i$ and $\check{w}_i$ are opposite then there exists a unique sorted biword, denoted $\mathrm{sort}(\mathbf{w})$ whose columns are a permutation of the the columns of $\mathbf{w}$. Conversely, if such a pair i, j of columns does exist, then there is no sorted biword whose columns are a permutation of the columns of $\mathbf{w}$. In this case let $\mathrm{sort}(\mathbf{w}) = 0$.

Example 4.1.2 If all letters in the top word are positive, then

$$\mathrm{sort}\left(\begin{array}{c} 211343 \\ 121132 \end{array} \right) = \begin{array}{c} 112334 \\ 121123 \end{array} .$$

If all letters in the top word are negative, then

$$\mathrm{sort}\left(\begin{array}{c} 211343 \\ 121132 \end{array}\right) = \begin{array}{c} 112334 \\ 211213 \end{array}.$$

Finally we have

$$\mathrm{sort}\left(\begin{array}{c} 1^-1^- \\ 2^+2^+ \end{array}\right) = 0.$$

Equation 4.1 defines a bijection, ϕ, between monomials in $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ and sorted biwords with letters in the upper word coming from $\mathcal{P}$ and letters in the lower word coming from $\mathcal{L}$.

$$\phi\Big(\prod_i (l_i|p_i)\Big) = \mathrm{sort}\left(\begin{array}{c} p_1\ p_2\ p_3 \cdots \\ l_1\ l_2\ l_3 \ldots \end{array}\right). \tag{4.1}$$

If M is a monomial in $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$, then the function Ψ on page 40 is defined by $\Psi(M) = \phi(\check{M})$.

The next few results relate the insertion algorithm for signed sorted biwords to ordinary Robinson-Schensted-Knuth. None of these are particularly novel, being analogous to standard results about distinguishing letters, but they are sufficiently useful to be stated explicitly.

Definition 4.1.2 *Given a signed alphabet $\mathcal{L}$, define an alphabet $\mathcal{L}'$ to be the unsigned set*

$$\{x', x'', x''', \ldots | x \in \mathcal{L}\}.$$

Denote the ith copy of x by $x^{\{i\}}$. Order L' by saying $x^{\{i\}} \leq y^{\{j\}}$ when $x < y$ and $x^{\{i\}} \leq x^{\{j\}}$ when $i < j$.

Define a function dist *which maps words in $\mathcal{L}$ to words in $\mathcal{L}'$ by distinguishing the letters as follows. Require that if a letter x appears in w say $n > 0$ times then in* $\mathrm{dist}(w)$ *the length n subsequence $xx\cdots x$ of w has been replaced by $x'x''\cdots x^{\{n\}}$ if x is positive or $x^{\{n\}}\cdots x''x'$ if x is negative.*

Further define $\mathrm{dist}_0(w)$ *to be an unsigned version of* dist *that treats all letters as if they were positive.*

Example 4.1.3

$$\mathrm{dist}(\ 1^- 2^+ 1^- 1^- 3^+ 2^+\ ) = \ 1^{-'''}\ 2^{+'}\ 1^{-''}\ 1^{-'}\ 3^{+'}\ 2^{+''}$$

Finally, define a monoid homomorphism, forget, from words in $\mathcal{L}'$ to words in $\mathcal{L}$ by requiring that $\mathrm{forget}(x^{(i)}) = x$ for all i and all $x \in \mathcal{L}$.

Definition 4.1.3 *The operation* dist *extends to a sorted biword* $\mathbf{v}$ *by defining* $\mathrm{dist}(\mathbf{v})$ *to be* $(\mathrm{dist}_0(\hat{v}), \mathrm{dist}(\check{v}))$.

We extend forget *to biwords by setting* $\mathrm{forget}(v) = (\mathrm{forget}(\hat{v}), \mathrm{forget}(\check{v}))$.

The following propositions are evident from the above definitions and basic properties of the Robinson-Schensted algorithm.

Proposition 4.1.1 *Given a sorted signed biword,* $\mathbf{w}$ *the insertion and recording tableaux resulting from inserting* $\mathbf{w}$ *are the same as the insertion and recording tableaux obtained by inserting* $\mathbf{w}' = \mathrm{dist}(w)$ *and then "forgetting" the distinguishing marks.*

Furthermore, $P_r(\mathbf{w}')$ *and* $Q_r(\mathbf{w}')$ *can be recovered from* $Q_r(\mathbf{w})$ *and* $P_r(\mathbf{w})$ *respectively by making copies of a negative letter increase as they appear in rows from top to bottom and by making copies of a positive letter increase from left to right.*

Proof. The only difficulty here is recovering $Q_r(\mathbf{w}')$ and $P_r(\mathbf{w}')$. The hardest part, for positive letters in Q, is essentially a non-crossing lemma for row insertion of an increasing sequence which may be derived from part 2 of Lemma 3.4.1 in [Sa91]. $\square$

Proposition 4.1.2 *Given a sorted signed biword,* $\mathbf{w}$ *the insertion and recording tableaux resulting from inserting* $\mathbf{w}$ *are standard. (That is columns ($\lessdot_-$)-increase downwards and rows $\lessdot_+$ increase to the right.)* $\square$

Observe that if $\mathbf{w}$ is the modified column biword of T (as on page 62 of Chapter 1) then $\mathbf{w}$ is sorted unless there are multiple copies of some positive letter in some column of T. If this obstruction occurs then $\mathrm{sort}(\mathbf{w}) = 0$.

4.1.2 Row-convex tableau

Assume from here on that D is a row-convex shape. We are now in a position to establish some connections between row-D-decomposability and straight tableau. Recall that the straight tableau of shape D are constructed so that for any initial monomial of an element of S^D there exists a unique straight tableau with the same initial monomial.

A modified standard biword, $\mathbf{w}$, is *D-realizable* if there exists a row-standard tableau T such that $\mathbf{w}_T = \mathbf{w}$. This T is said to be a (D) realization of C. By Proposition 1.7.3, T could be taken to be straight.

We will need the following

Lemma 4.1.3 *Suppose $\mathbf{w}$ is a sorted biword. Let $\mathbf{w}' = \text{dist}(\mathbf{w})$ and let $\mathbf{v}'$ be the inverted biword $\text{sort}((\check{w}', \hat{w}'))$. The word, $\mathbf{v}$, obtained by forgetting distinguishing marks in $\mathbf{v}'$ is also a sorted biword.*

Proof. Since w was sorted, v will contain no two identical columns each with one positive and one negative entry. Hence it suffices to check that if $x \neq y$ and if $\begin{matrix} a^{\{i\}} \\ x \end{matrix}$ and $\begin{matrix} a^{\{j\}} \\ y \end{matrix}$ appear as columns of v' with $i < j$, then $x < y$ when a is positive and $x > y$ when a is negative. But this is immediate from the definition of dist. □

Definition 4.1.4 *Given a biword $\mathbf{w}$ and a biword $\mathbf{w}' = \text{dist}(w)$, abuse notation and call w' D-realizable when the biword $(\text{forget}(\hat{w}'), \check{w}')$ is D-realizable. A D-realization of $\mathbf{w}'$ will be a D-realization of $(\text{forget}(\hat{w}'), \check{w}')$.*

This definition is justified by the following lemma.

Lemma 4.1.4 *Let D be a row-convex diagram. Let $\mathbf{w}$ be a biword and let $\mathbf{w}' = \text{dist}(\mathbf{w})$. The biword $\mathbf{w}$ is D-realizable iff $(\text{forget}(\hat{w}'), \check{w}')$ is D-realizable.*

Proof. Suppose D has k rows, then D-realizability for a biword $\mathbf{w}$ is equivalent to being able to write $\mathbf{w}$ as a disjoint union of k subwords $\mathbf{r}_1, \ldots, \mathbf{r}_k$ such that $\check{r}_i$ is

($\lessdot$)-increasing and $\hat{r}_i$ equals the ith row of $Der^-(D)$. The process of distinguishing $\mathbf{w}$ replaces sequences of a given negative letter in $\check{w}$ by a *decreasing* sequence of distinguished copies of that letter. Thus a subword of $\check{w}$ is $\lessdot$-increasing iff the same subword in $\check{w}'$ is strictly (equivalently $\lessdot$) increasing. Hence a D-realization of w is a D-realization of $(\text{forget}(\hat{w}'), \check{w}')$. $\square$

The following closely related notion is adapted from [RS95].

Definition 4.1.5 *A* pseudo horizontal strip *in a tableau T is a subset of the cells of T with at most one cell per column such that the contents of the subset $\lessdot$-increase from left to right.*

Definition 4.1.6 *A tableau Q is* row D-decomposable *if there exists a sequence $(s_1, s_2, \ldots, s_k)$ of pseudo horizontal strips in Q whose disjoint union equals Q and such that the content of s_i is the content of the ith row of $Der^-(D)$. Such a set is called* a row D-decomposition *of Q.*

A hint at the usefulness of this kind of construction is provided by the following proposition.

Proposition 4.1.5 *(Reiner and Shimozono [RS95] Theorem 1) Let D be a column convex diagram. Let S^D be the vector space spanned by all tableau of shape D filled with distinct positive letters. The elements $[T]$ indexed by row standard tableau T, where $Q_r(c_{\tilde{T}})$ is row-$\tilde{D}$-decomposable form a basis of S^D. Here $\tilde{T}$ is the transpose of the tableau T.* $\square$

The results of [RS95] are actually stated in terms of column-D-decomposability and column insertion. Also, to be technical, I differ by a reflection about the x-axis from their NorthWest condition. Shimozono [S96] has since provided a characteristic-free extension allowing repeated letters.

We now clarify the connection between realizability and decomposability. Let D be any row-convex shape. Let Q be a standard (implies partition-shaped) tableau with the same content as $Der^-(D)$. Let $\mathbf{c}_Q$ be the column biword of Q. Let $\mathbf{c}'_Q = \mathrm{dist}(\mathbf{c}_Q)$. Let $\mathbf{v}'_Q = \mathrm{sort}((\check{w}'_Q, \hat{w}'_Q))$. Finally let $\mathbf{v}_Q = \mathrm{forget}(\mathbf{v}'_Q)$. Since $P_r(\mathbf{c}_Q) = Q$ we conclude, by Proposition 4.1.1 and the fact (see [Sa91] that inverting a permutation switches P and Q), that $Q_r(\mathbf{v}_Q) = Q$. Observe that all letters involved are negative.

Proposition 4.1.6 *Preserving the immediately preceding definitions, there exists a bijection between the D-realizations of* $\mathbf{v}_Q$ *and the row-D-decompositions of* Q.

Proof. Let k be the number of rows in D. A row-D-decomposition of Q is equivalent to a decomposition of $\mathbf{c}_Q$ into k disjoint subwords (interleaving allowed), $s_1, \ldots, s_k$ such that the upper and lower words of each s_i ($<_+$)-increase and such that $\check{s}_i$ equals the ith row of $Der^-(D)$. This is equivalent to a decomposition of $\mathbf{c}'_Q$ into disjoint subwords, s'_i such that the the upper and lower word of s'_i strictly increase and $\mathrm{forget}(\check{s}'_i)$ is the ith row of $Der^-(D)$. This decomposition is equivalent (after inverting $\mathbf{c}'_Q$ and sorting it) to a decomposition of v'_Q into r'_i which again strictly increase in their upper and lower words and such that $\mathrm{forget}(\hat{r}'_i)$ is the ith row of $Der^-(D)$. But this is the same as specifying a D-realization of $\mathbf{v}'_Q$ (recall Definition 4.1.4) which we have seen in the proof of Lemma 4.1.4 to be equivalent to specifying a D-realization of $\mathbf{v}_Q$. □

The idea underlying this result appears in the proof of Theorem 14 of [RS95].

Remark: In [RS95] no means was provided for distinguishing particular D-decompositions of a recording tableau. The preceding proposition picks out precisely those decompositions that correspond to straight D-realizations. The filling procedures thus provide algorithms for finding this decomposition.

A condition guaranteeing that two words on the same set of distinct letters have the same insertion tableau (i.e. are *Knuth-equivalent* written $\leftrightarrow$) is easily described. (The proof is somewhat harder than the statement–see [Sa91]) The equivalence relations amongst length three words are as follows,

$$j, i, k \leftrightarrow j, k, i \qquad k, i, j \leftrightarrow i, k, j,$$

where $i < j < k$. If two longer words differ only in a length 3 *consecutive* subword and these subwords are Knuth equivalent, then so are the longer words. In general, Knuth equivalence is defined as the transitive closure of these relations.

Because inverting a biword reverses its P and Q tableaux, two words have the same Q tableaux (are *dual-Knuth equivalent*, written $\overset{*}{\leftrightarrow}$) when they are formed from the transitive closure of the relations

$$j, j-1, j+1 \overset{*}{\leftrightarrow} j+1, j-1, j \qquad j, j+1, j-1 \overset{*}{\leftrightarrow} j-1, j+1, j \tag{4.2}$$

on *not necessarily consecutive* subsequences.

Two modified standard biwords $\mathbf{w}, \mathbf{w}'$ with the same upper word are Knuth (dual-Knuth) equivalent iff $\mathrm{dist}(\check{v}), \mathrm{dist}(\check{w})$ are Knuth (dual-Knuth) equivalent. By Proposition 4.1.1 this definition is equivalent to the recording tableaux for $\mathbf{v}, \mathbf{w}$ being identical.

Proposition 4.1.7 *Let D be a row-convex shape. The set of lower words for D-realizable modified standard biwords is closed under dual-Knuth equivalence.*

Proof. It suffices to prove the result for distinct letters.

So we must show that if $\mathbf{w}$ has a D-realization T, then after applying any of the relations (4.2), the resulting word has a D-realization T'. If we view a dual-Knuth move as giving a permutation T'' of the entries of T, then this says that we must be able to form T' from some column-stabilizing permutation of the entries of T''. If the letters $j-1, j, j+1$ appear in distinct rows then the result is immediate. They cannot all appear in the same row since row-standardness would prevent the dual-Knuth moves from applying. So we may as well assume that precisely two of $j-1, j, j+1$ appear in the same row. By row-convexity they must be adjacent. If the entries appear in three different columns then the dual-Knuth permutations all preserve row-standardness. So we are left with the checking that dual-Knuth moves take row-standard tableaux to other row-standard tableaux with the same modified column words as follows,

$$\begin{array}{ll} \begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix} \overset{*}{\leftrightarrow} \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} & \begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix} \overset{*}{\leftrightarrow} \begin{smallmatrix} 2 & 3 \\ 1 & \end{smallmatrix} \\ \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} \overset{*}{\rightarrow} \begin{smallmatrix} 2 & 3 \\ 1 & \end{smallmatrix} & \begin{smallmatrix} 2 & 3 \\ 1 & \end{smallmatrix} \overset{*}{\leftrightarrow} \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} . \end{array}$$

Here (since $j-1, j, j+1$ are adjacent) we have eliminated all entries not involved in the dual-Knuth moves and we have disregarded the order of the rows involved. □

Corollary 4.1.8 *Let D be a row-convex tableaux. Let $\prec$ be a diagonal term order on $Super([\mathcal{L} \mid \mathcal{P}])$. The set $\{\Psi(\text{init}_{\prec}(p)) : p \in \mathcal{S}^D(\mathcal{L})\}$ is closed under dual-Knuth equivalence.* □

The combinatorial connection with straight tableaux is given by the next result. An algebraic connection will be provided in Section 4.3.

Theorem 4.1.9 *A word $\breve{w}$ is the modified column word of a (necessarily unique) shape D straight tableau iff the biword $\mathbf{w} = (c_{Der^-(D)}, \breve{w})$ is a modified standard biword and $Q_r(\mathbf{w})$ is row-D-decomposable.*

Proof. By Proposition 4.1.7, it suffices to show that for each standard tableau Q with the same content as $Der^-(D)$, there exists a biword, $\mathbf{w}$, such that $Q_r(\mathbf{w}) = Q$ with the following property. The lower word, $\breve{w}$ is the modified column word of a shape D straight tableau iff Q is row-D-decomposable. But the proof of Proposition 4.1.6 says it suffices to choose $\mathbf{w} = \text{sort}\begin{pmatrix} c_Q \\ c_{Der^-(D)} \end{pmatrix}$. □

4.2 A bijection of two-rowed tableaux

The central step in the straightening algorithms of Chapter 1 is the expansion of $[T]$ for a two-rowed tableau T into a linear combination of $[S_i]$ where each S_i is closer to being straight. The following combinatorial fact is an immediate consequence of this straightening process.

Proposition 4.2.1 *The number of standard tableaux of partition shape* $(\lambda_1, \lambda_2 - m)$ *with given content equals the number of straight tableaux of skew shape* $(\lambda_1, \lambda_2)/(0, m)$ *with the same content.* □

In this section, I provide a bijective proof of the above fact. This, combined with suitable initial-term arguments, provides a shorter proof of the straight basis theorem–though not of the straightening algorithm proper.

I will describe a content preserving map, pushright, from standard tableaux of shape $(\lambda_1, \lambda_2 - m)$ to straight tableau of shape $(\lambda_1, \lambda_2)/(0, m)$. I will also describe a content preserving map, pushleft, which inverts pushright.

We define pushright on standard tableau by a three step algorithm. The first step slides parts of the bottom row right (while preserving the order.) The second step flips some of the columns top to bottom. The third step slides the new bottom row further right to fit into the shape $(\lambda_1, \lambda_2)/(0, m)$. The details are contained in Figure 4-1.

Proof.(of Algorithm pushright's correctness) I claim that the flips in part 2 can be done in place. That is, after part 2 ends the top row and the bottom row will still be row standard–i.e. ordered by $<_+$. So consider a typical tableau appearing at the end of part 1.

$$\begin{bmatrix} \dots & t & u & v & \dots & w & x & \dots \\ \dots & & a & b & \dots & c & & \dots \end{bmatrix} .$$

Here $a, b, \dots, c$ is the content of a block of nonempty cells. By assumption $a \rightarrow u$ hence $a > t$. Additionally $c <_+ x$ since if not, c would have been slid under x (or further right) in part 1. Thus the tableau

$$\begin{bmatrix} \dots & t & a & b & \dots & c & x & \dots \\ \dots & & u & v & \dots & w & & \dots \end{bmatrix} .$$

is still row standard.

Input: Standard tableau $\left[\begin{array}{ccccc} x_1 & x_2 & x_3 & \cdots & x_{\lambda_1} \\ y_1 & y_2 & \cdots & y_{\lambda_2-m} & \end{array}\right]$.
Output: Straight tableau $\left[\begin{array}{cccccc} x_1 & \cdots & x_{m+1} & & \cdots & x_{\lambda_1} \\ & & y_{m+1} & y_{m+2} & \cdots & y_{\lambda_2} \end{array}\right]$.

1 $i \leftarrow \lambda_2 - m$ ▷ Column containing next letter to slide.
  $j \leftarrow \lambda_2$ ▷ Largest spot available to slide into.
  while $i > 0$
    Let k be the largest integer such that $k \leq j$ and y_i -> x_k.
      (We know k exists since standardness implies y_i -> x_i.)
    $y_k \leftrightarrow y_i$ ▷ Slide y_i into column k.
    $j \leftarrow k - 1$
    $i \leftarrow i - 1$
2 Let $j \leq \lambda_2$ be the largest column containing an empty spot in the bottom row.
  for $1 \leq i < j$
    if y_i exists then
      $x_i \leftrightarrow y_i$ ▷ Flip x_i and y_i when y_i is not flush right.
3 $k \leftarrow \lambda_2$ ▷ This step pushes the bottom row flush to column λ_2.
  while $k > m$
    Let $k \leq \lambda_2$ be the largest column empty in the bottom row.
    Let i be the first column left of k with entry in the bottom row.
    $y_k \leftrightarrow y_i$
    $k \leftarrow i$

Figure 4-1: **Algorithm** pushright

Now, recall that part 2 only flips those columns that are not flush right. So the columns that *are* flush right still have (*bottomvalue*) -> (*topvalue*) and thus can't violate standardness. The columns that are not flush right will have inversions, but part 3 will move each of them right by at least one. But since the inversions existed before the move in part 3, these inversions will not be flippable. Thus the output tableau is row standard and has no flippable inversions. Hence it is straight. □

The function pushleft which is to be pushright^{-1} is defined by a similar three step algorithm. Each element in the bottom row that is part of an inversion is slid left by the first step. In fact, it is slid as far left as possible subject to the restriction that it still forms an inversion. The second step flips all inversions. The third step left justifies the bottom row. The details to this algorithm are in Figure 4-2.

Input: Straight tableau $\left[\begin{array}{ccccccc} x_1 & \cdots & x_{m+1} & & \cdots & & x_{\lambda_1} \\ & & y_{m+1} & y_{m+2} & \cdots & y_{\lambda_2} & \end{array} \right]$.

Output: Standard tableau $\left[\begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & & x_{\lambda_1} \\ y_1 & y_2 & \cdots & y_{\lambda_2 - m} & & \end{array} \right]$.

1 $i \leftarrow m+1$ $\rhd$ Column containing next letter to slide.
  $j \leftarrow 1$ $\rhd$ Smallest spot available to slide into.
  while $i \leq \lambda_2$ and y_i <- x_i
      (Straightness implies that if y_r -> x_r, then y_s -> x_s for all $s \geq r$.)
    Let k be the smallest integer such that $k \geq j$ and y_i <- x_k.
    $y_k \leftrightarrow y_i$ $\rhd$ Slide y_i into column k.
    $j \leftarrow k+1$
    $i \leftarrow i+1$
2 Let $j \leq \lambda_2$ be the largest column containing an empty spot in the bottom row.
      (Straightness implies that all inversions move left at least one spot.)
  for $1 \leq i < j$
    if y_i exists **then**
      $x_i \leftrightarrow y_i$ $\rhd$ Flip all inversions.
3 $k \leftarrow 1$ $\rhd$ This step flushes the bottom row left.
  while $k \leq \lambda_2 - m$
    Let k be the smallest column with an empty spot in the bottom row.
    Let i be the first column right of k with an entry in the bottom row.
    $y_k \leftrightarrow y_i$
    $k \leftarrow i$

Figure 4-2: **Algorithm** pushleft

Proof.(of Algorithm pushleft's correctness) As noted in the algorithm, if $x_i > y_i$ in the input tableau, then part 1 of the algorithm moves y_i down to a column strictly left of i. Thus the flips in part 2 remove all inversions. Assuming that part 2 leaves the tableau row-standard, then part 3 maintains the lack of inversions hence the output tableau is standard.

To check that part 2 preserves row standardness we again consider a section of the tableau provided by part 1 in which there are no missing letters in the sequence $a, b, \ldots, c$,

$$\left[\begin{array}{ccccccccc} \ldots & u & v & w & \ldots & x & y & \ldots \\ \ldots & & a & b & \ldots & c & & \ldots \end{array} \right]$$

Assume that y exists. I.E. x is not in the rightmost column. Now every two-element column in this tableau to the left (not necessarily immediate left) of some one-element column satisfies $(bottomelement) \leftarrow (topelement)$. Hence $y \not\rightarrow x \rightarrow c$. Further, $a \not\rightarrow u$ since otherwise step 1 would not have placed a further left than underneath v. We conclude that

$$\begin{bmatrix} \ldots & u & a & b \ldots c & y & \ldots \\ \ldots & & v & w \ldots x & & \ldots \end{bmatrix}$$

is row standard. □

Theorem 4.2.2 *The function* pushright *is a content-preserving bijection from standard tableau of shape* $(\lambda_1, \lambda_2 - m)$ *to straight tableau of shape* $(\lambda_1, \lambda_2)/(0, m)$. *Its inverse is* pushleft.

Proof. The only difficulty is to show that part 1 of each algorithm reverses part 2 of the other. Suppose so. Then applying part 2 inverts the original application of part 2 since they flip the same columns. Finally, part 3 just sweeps the spread-out bottom row back to the left or right justified format it started in.

Part 3 of pushleft followed by part 1 of pushright is the identity: Observe that after applying part 2, no element y in the bottom row can be moved into an empty space to its right while preserving the fact that y is not involved in an inversion. This follows since all y that have an empty space to their immediate right had been flipped down from the top row i.e. a tableau like

$$\begin{bmatrix} \ldots & x & z & \ldots \\ \ldots & y & & \ldots \end{bmatrix} \quad \text{came from} \quad \begin{bmatrix} \ldots & y & z & \ldots \\ \ldots & x & & \ldots \end{bmatrix}.$$

so $y \not\leftarrow z$.

Part 3 of pushleft sweeps the bottom row flush left, but the conditions on part 1 of pushright are precisely that we move elements as far right as possible while preserving the order and prohibiting inversions. Thus we recover the tableau presented at the end of part 2 of pushleft.

That part 3 of pushright followed by part 1 of pushleft is the identity follows similarly: The tableau formed by parts 1 and 2 of pushright is defined by the contents of its top row, the contents of its bottom row, and the following facts. One: if i is a column where (at this stage) $x_i \rightarrow y_i$ then y_i cannot be moved left while preserving this property. That is $x_{i-1} \not\rightarrow y_i$. Two: if a two-celled column, i, does not contain an inversion then there is no one-element column between i and λ_2 inclusive. Part 3 of pushright sweeps this bottom row into columns $m+1 \ldots \lambda_2$ while maintaining their order. But part 1 of pushleft simply spreads the bottom row back out so that it satisfies the above conditions. □

Remark. The bijection pushright is also accomplished by moving elements one at a time from left to right in the bottom row. In particular, take the rightmost element, move it right as long as it does not form an inversion. When moving it further right would form an inversion, invert it and the element above it and then move that element as far right as possible (stopping of course by column λ_2.) This interprets the preceding bijection as a process of interchanging columns in the two-rowed shape while preserving the Knuth-equivalence class of the modified column word.

Proposition 4.2.3 *The bijections* pushright *and* pushleft *preserve the Knuth-equivalence classes of the modified column word.* □

4.3 Compatible duality

Let D be a row-convex shape and let $\tilde{D}$ be its transpose. This section establish a concrete algebraic relationship between the supersymmetric version of the Reiner-Shimozono basis ([RS95] [S96]) of decomposable tableaux for the Weyl module of shape $\tilde{D}$ and the straight basis for the Schur module of shape D over a space V. Since the Schur module of any shape is dual to the Weyl module of transpose shape built on a dual vector space V^*, it makes sense to ask for elements of one space to act on the other. The main result below says essentially that if we fill the straight tableaux with entries from a basis of V and if we fill the decomposable tableaux with entries from a dual basis, then the matrix describing how the basis of straight tableaux is acted upon by the basis of decomposable tableau must be unitriangular. In fact this result is true if the straight tableaux are replaced by any basis with the same initial terms. In the case of almost skew diagrams this covers all bases considered in [W94]. All of the results presented below work for super-Schur modules, so the first subsection concretely establishes the above bilinear form and then shows that it is indeed invariant under the action of the general linear Lie superalgebra.

4.3.1 Bilinear forms

In this section we will start by recalling an invariant bilinear form on $\mathcal{S}uper([\mathcal{L}^* \mid \mathcal{P}^*]) \times \mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ and use this to establish duality.

Recall from Chapter 1 that the general linear Lie superalgebra, $pl_{\mathcal{L}}$ is the free **Z**-module with generators $E_{a,b}$ for $a, b \in \mathcal{L}$ and bracket

$$[E_{a,b}, E_{c,d}] = \delta_{b,c} E_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)} \delta_{d,a} E_{c,a} \quad .$$

If L is a set of letters, then $pl_{\mathcal{L}}$ acts on $\mathcal{S}uper(\mathcal{L})$ and $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ by $E_{a,b} \mapsto D_{a,b}$.

It will be useful for the sequel to define the *dual*, $\mathcal{L}^*$, of an alphabet $\mathcal{L}$ and to define the action of $pl_{\mathcal{L}}$ on $\mathcal{S}uper(\mathcal{L}^*)$ and $\mathcal{S}uper([\mathcal{L}^* \mid \mathcal{P}])$.

Definition 4.3.1 *If $\mathcal{L}$ is an alphabet, define $\mathcal{L}^* = \{a^* | a \in \mathcal{L}\}$ with $|a^*| = |a| + 1$ and $a^* \leq b^*$ iff $a \leq b$.*

Now, given $a, b \in L$, define $E_{a,b}$ on $V_{\mathcal{L}^*}$ by

$$E_{a,b}(c^*) = \begin{cases} -(-1)^{|b||c^*|} b^* = -(-1)^{(|a|+|b|)|c^*|} b^* & \text{if } a = c \\ 0 & \text{otherwise} \end{cases}$$

Extend the action of $E_{a,b}$ to $\mathcal{S}uper(\mathcal{L}^*)$ by requiring that it be a left superderivation (with sign equal to $(-1)^{|a|+|b|}$). Similarly the action of $E_{a,b}$ on $\mathcal{S}uper([\mathcal{L}^* \mid \mathcal{P}])$ is defined on $\mathcal{S}uper^1(\mathcal{L}^*|\mathcal{P})$ by

$$E_{a,b}(c^*|x) = (E_{a,b}c^*|x)$$

and extended to a left superderivation such that $E_{a,b}(u|w) = (E_{a,b}u|w)$ for all $u \in \mathcal{S}uper(L^*)$ and $w \in \mathcal{S}uper(P)$. The preceding constructions may be justified by appealing to the techniques of Section 1.4.2.

Following [GRS87] a bilinear form $\langle , \rangle$ satisfying Laplace-type expansion identities may be defined on $\mathcal{S}uper(\mathcal{L}^*|\mathcal{P}^*) \times \mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$. In particular the bilinear form satisfies the equations,

$$\langle uv, w\rangle = \sum_{(w)} (-1)^{|v||w_{(1)}|} \langle u, w_{(1)}\rangle \langle v, w_{(2)}\rangle$$

$$\langle u, vw\rangle = \sum_{(u)} (-1)^{|u_{(2)}||v|} \langle u_{(1)}, v\rangle \langle u_{(2)}, w\rangle$$

$$\langle (a^*|x^*), (b|y)\rangle = (-1)^{|x^*||b|} \delta_{a,b}\delta_{x,y} \quad .$$

The coproduct $\Delta(w) = \sum_{(w)} w_{(1)} \otimes w_{(2)}$ is defined to be $\prod_i 1 \otimes w_i + w_i \otimes 1$ when each w_i is a single letterplace and $w = \prod_i w_i$. It extends to the whole superalgebra by linearity.

This bilinear form is invariant in the sense of [Sc79] (3.46).

Proposition 4.3.1 *For $a, b \in L$, and r, s homogeneous in respectively $\mathcal{S}uper(\mathcal{L}^*|\mathcal{P}^*)$, and $\mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$, we find*

$$\langle Lr, s\rangle + (-1)^{|E_{b,a}||r|} \langle r, Ls\rangle = 0 \quad .$$

Proof. It suffices to show the identity for the case where r, s are monomials, $a \neq b$, and $r = (a^*|x^*)^i (b^*|x^*)^j w^*$ and $s = (a|x)^{i+1}(b|x)^{j-1} w$ for some monomials $w \in \mathcal{S}uper([\mathcal{L} \mid \mathcal{P}])$ and $w^* \in \mathcal{S}uper([\mathcal{L}^* \mid \mathcal{P}^*])$ neither divisible by $(a|x)$ or $(b|x)$ or respectively by $(a^*|x^*)$ or $(b^*|x^*)$. Furthermore, we can assume that if $i \geq 1$ then $|(a|x)| = 0$ and if $j \geq 2$ then $|(b|x)| = 0$.

Verifying the proposition then comes down to checking the equality of the equations,

$$(-1)^{|E_{b,a}|\left(|(a^*|x^*)^i|+|b^*|\right)+|(a|x)^{i+1}|\left(|(b^*|x^*)^{j-1}|+|w^*|\right)+|(b|x)^{j-1}||w^*|}\times$$
$$\times j\langle (a^*|x^*)^{i+1}, (a|x)^{i+1}\rangle\langle (b^*|x^*)^{j-1}, (b|x)^{j-1}\rangle\langle w^*, w\rangle.$$

and

$$(-1)^{|E_{b,a}||(a^*|x^*)^i(b^*|x^*)^j w^*|+|E_{b,a}||(a^*|x^*)^i|+|(a|x)^i||(b^*|x^*)^j|+|(a|x)^i||w^*|+|(b|x)^j||w^*|}\times$$
$$\times(i+1)\langle (a^*|x^*)^i, (a|x)^i\rangle\langle (b^*|x^*)^j, (b|x)^j\rangle\langle w^*, w\rangle.$$

These equations simplify to

$$(-1)^{|E_{b,a}||b^*|+|E_{b,a}||(a^*|x^*)^i|+|(a|x)^{i+1}|\left(|(b^*|x^*)^{j-1}|+|w^*|\right)+|(b|x)^{j-1}||w^*|}\times$$
$$\times j(i+1)!(-1)^{(i+1)|x^*||a|}(j-1)!(-1)^{(j-1)|x^*||b|}\langle w^*, w\rangle.$$

and

$$(-1)^{|E_{b,a}||(a^*|x^*)^i(b^*|x^*)^j w^*|+|E_{b,a}||(a^*|x^*)^i|+|(a|x)^i||(b^*|x^*)^j|+|(a|x)^i||w^*|+|(b|x)^j||w^*|}\times$$
$$\times(i+1)i!(-1)^{i|x^*||a|}j!(-1)^{j|x^*||b|}\langle w^*, w\rangle.$$

Checking this equality amounts to checking that

$$|E_{b,a}||(a^*|x^*)^i(b^*|x^*)^j w^*| \equiv |b^*||E_{b,a}| + (j-i-1)|(a|x)||(b^*|x^*)|+$$
$$+|(a|x)||w^*| + |(b|x)||w^*| + |x^*|(|a|+|b|) \qquad (\text{mod } 2).$$

To check this it suffices to observe that the assumptions on i, j relative to the signs of $(a|x)$ and $(b|x)$ imply $|(b^*|x^*)| \equiv |(b^*|x^*)^j|$, $0 \equiv |(a^*|x^*)^i|$ and $(j-i-1)|(a|x)||(b^*|x^*)| \equiv 0$ all mod 2. $\square$

Suppose that $\mathcal{L} = \mathcal{L}^-$ or $\mathcal{L} = \mathcal{L}^+$. Then the general linear Lie superalgebra $pl_{\mathcal{L}}$ equals the general linear Lie algebra $gl(\mathcal{L})$ and Proposition 4.3.1 says that the Grosshans-Rota-Stein inner product is invariant in the usual sense of a Lie algebra action.

An example worth keeping in mind is the case where $\mathcal{L} = \mathcal{L}^-$ and $\mathcal{P} = \mathcal{P}^+$. In this case, both $Super([\mathcal{L} \mid \mathcal{P}])$ and $Super([\mathcal{L}^* \mid \mathcal{P}^*])$ are exterior algebras and

$$\left\langle \prod_{i=1}^{n}(x_i^*|y_i^*), \prod_{i=1}^{n}(l_i|p_i) \right\rangle = \begin{cases} c(-1)^{\binom{n}{2}} & \text{if } \prod_{i=1}^{n}(x_i|y_i) = c\prod_{i=1}^{n}(l_i|p_i), \qquad c = \pm 1 \\ 0 & \text{otherwise} \end{cases}.$$

For the remainder of this section I will use $\tilde{T}$ to denote the following "dual" transpose of T.

Definition 4.3.2 *Given a tableau T of shape D with entries chosen from $\mathcal{L}$, define $\tilde{T}$ to be the tableau of transpose shape $\tilde{D}$ with entries in $\mathcal{L}^*$ arrived at by transposing T and "starring" each entry of the transpose. We can make $\tilde{}$ an involution by defining $a^{**} = a$.*

Given a tableau T of sorted row-convex shape D on letters $\mathcal{L}$ and a tableau $\tilde{T}'$ of shape $\tilde{D}$ on the letters $\mathcal{L}^*$, we would like to be able to intelligently define

$$\left\langle (T|Der^-(D)), (\tilde{T}'|Der^-(\tilde{D}) \right\rangle$$

and thus obtain a relationship between row-convex tableaux and column-convex tableaux. Now directly applying the preceding bilinear form is useless; the places on the left-hand side fail to be dual to the place on the right-hand side so the result is automatically zero. The solution is to define a new bilinear form, $\langle\, , \rangle_D$ on

$\mathcal{S}^D(\mathcal{L}) \times \mathcal{S}^{\tilde{D}}(\mathcal{L}^*)$ by

$$\left\langle [T], [\tilde{T}'] \right\rangle_D = \left\langle [T|Der^+(D)], [\tilde{T}'|Der^-(\tilde{D})] \right\rangle. \tag{4.3}$$

This sort of duality holds in general by suitably abstract arguments. For the case we are concerned with, the result may be seen directly.

Proposition 4.5.2 *Suppose that D is a row-convex shape. The bilinear form $\langle p, q\rangle_D$ of Equation 4.3 does not depend on the presentation of p.*

Proof. It suffices to show that if $[T] = \sum_\iota \alpha_\iota [T_\iota]$ with all tableaux of shape D, then $[T|Der^+(D)] = \sum_\iota \alpha_\iota [T_\iota|Der^+(D)] + z$ where $\left\langle z, [\tilde{T}] \right\rangle = 0$ for all $\tilde{T}$ of shape $\tilde{D}$. In particular, it suffices to show that the straightening law Proposition 1.6.10 and its skew counterpart,

$$\left[\begin{array}{c|ccc} a^{(i)}b^{(j)} & \quad l & \ldots & \ldots \; i{+}j{+}l{-}1 \\ b^{(i+l)}c^{(k)} & 1 \; \ldots & \ldots \; i{+}l{+}k & \end{array} \right] = 0, \tag{4.4}$$

with $l \geq 1$ and $j \geq k+1$ must hold modulo some $z \in \ker\left(\left\langle \;, [\tilde{T}] \right\rangle\right)$ for all $\tilde{T}$. The straightening law of [GRS87] guarantees that any bitableau expands into a linear combination of bitableaux having weakly longer (in dominance order) shape. In particular, suppose that a, b, c, x, y are positive letters, the straightening law used polarizations of the identities

$$\left[\begin{array}{c|c} a^{(i+l)}b^{(j)} & x^{(i+j+l)} \\ b^{(l)}c^{(k)} & y^{(l+k)} \end{array} \right] = (-1)^l \left[\begin{array}{c|c} b^{(j+l)}a^{(i)} & x^{(i+j+l)} \\ a^{(l)}c^{(k)} & y^{(l+k)} \end{array} \right] + \sum_{t>0} (-1)^{l-t} \left[\begin{array}{c|c} b^{(j+l)}a^{(i+t)} & x^{(i+j+l)}y^{(t)} \\ a^{(l-t)}c^{(k)} & y^{(l+k-t)} \end{array} \right]$$

and $$\left[\begin{array}{c|c} a^{(i)}b^{(j)} & x^{(i+j)} \\ b^{(i+l)}c^{(k)} & y^{(i+k+l)} \end{array} \right] = \sum_{t\geq 0} (-1)^{i-t} \left[\begin{array}{c|c} b^{(i+j+l)}a^{(t)} & x^{(i+j+l+t)} \\ a^{(i-t)}c^{(k)} & y^{(i+k-t)} \end{array} \right].$$

So in either case, it is enough to show that for all tableaux S, T,

$$\left\langle \left[T \left| \begin{array}{c} \cdots \\ x^{(r)}y^{(l+t)} \\ \cdots \\ y^{(s-l-t)} \\ \cdots \end{array} \right. \right], \left[S \left| \begin{array}{ccccc} & x^* & & & \\ & \vdots & & & \\ & x^* & & y^* & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & x^* & & y^* & \\ & \vdots & & \vdots & \\ & x^* & & y^* & \\ & & & \left.\begin{array}{c}\vdots \\ y^*\end{array}\right\} & l \text{ entries} \end{array} \right. \right] \right\rangle = 0,$$

where $t \geq 1$, there are r x^*'s and s y^*'s in the second bitableau and x, y have positive sign. But this is equivalent to

$$\left\langle \left[T \left| \begin{array}{c} \cdots \\ x^{(r+l+t)} \\ \cdots \\ y^{(s-l-t)} \\ \cdots \end{array} \right. \right], \left[S \left| \begin{array}{ccccc} & x^* & & & \\ & \vdots & & & \\ & x^* & & y^* & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & x^* & & y^* & \\ & \vdots & & \vdots & \\ & x^* & & y^* & \\ & & & \left.\begin{array}{c}\vdots \\ y^*\end{array}\right\} & l \text{ entries} \end{array} \right. \right] E_{x,y}^{(l+t)} \right\rangle = \left\langle \left[T \left| \begin{array}{c} \cdots \\ x^{(r+l+t)} \\ \cdots \\ y^{(s-l-t)} \\ \cdots \end{array} \right. \right], 0 \right\rangle = 0,$$

where the super Lie algebra is acting via a right-representation on places. □

As promised, I sketch a justification of Equation 4.3 for arbitrary choice of shape as follows. Consider $\mathcal{S}^D(\mathcal{L})$ as the image of a composite map f as in Proposition 1.5.1 of Chapter 1. Since the image of dual map f^* is $\mathcal{S}^{\tilde{D}}(\mathcal{L}^*)$, we define $\langle f(x), f^*(y^*) \rangle = y^*(x)$ and check that if $f^*(y^*) = 0$ then so does $y^*(x)$. But $w^* \in \ker(w^*)$ iff $w^* \circ f = 0$, i.e. $\ker(w^*) \supseteq \operatorname{Im}(f)$.

4.3.2 Pairings of row-convex and column-convex tableaux

Before showing that straight bases act compatibly on the generalized Reiner-Shimozono basis, I introduce the following (misuse of) notation.

Definition 4.3.3 *A signed tableaux T of column convex shape D is* decomposable *when $Q_r(c_{\tilde{T}})$ is row-$\tilde{D}$-decomposable.*

By Theorem 4.2.1, this is equivalent to the existence of a row-stabilizing permutation of T such that the columns of the permuted tableau (->)-increase.

If D is a row-convex shape, this obviously puts the decomposable tableaux on $\mathcal{L}^*$ of shape $\tilde{D}$ into natural bijection with the straight tableaux of shape D on $\mathcal{L}$–the row-word of a decomposable tableau equals (after un-($*$'ing) each letter) the modified column word of the corresponding straight tableau. (The row reading word is the column reading word of the transposed tableau.) In fact the following is true.

Theorem 4.3.3 *Order the decomposable tableaux of shape $\tilde{D}$ by lexicographic order on their row-words. Order the straight tableau of shape D by lexicographic order on their modified column words. When comparing a word in $\mathcal{L}$ with a word in $\mathcal{L}^*$ we will assume that $a = a^*$ for all $a \in \mathcal{L}$. If T_i is the ith straight tableau and $\tilde{T}_j$ the jth decomposable tableau under the above order, then the matrix $\left(\left\langle [T_i], [\tilde{T}_j] \right\rangle_D\right)$ is upper triangular with ± 1's on the diagonal.*

The proof will involve following notion of an *interpolant* adopted from [GRS87].

Definition 4.3.4 *A pair of tableaux (R, T) of the same shape is said to be* interpolated *by a matrix S if the contents of the rows of R agree with the contents of the respective rows of S and if the contents of the columns of S agree with the contents of the respective columns of T.*

Proposition 4.3.4 *If $\left\langle [T|Der^+(D)], (\tilde{T}'|Der^-(\tilde{D})] \right\rangle \neq 0$, then T and T' admit an interpolant of shape D.*

Proof. Suppose that the (i, j) entry of T is $w_{i,j}$ and that $\mathcal{P}^- = \{1, 2, 3, \ldots\}$. The function $\langle [T|Der^+(D)], \rangle$ is nonzero only on the monomial $\prod_{(i,j)\in D}(w^*_{i,j}|i)$. This monomial can appear in $[\tilde{T}'|Der^-(D)]$ only if there exists a row-stabilizing permutation of $\tilde{T}'$ such that, for all i, j, $w^*_{i,j}$ ends up in column i. In other words, there exists a column-stabilizing permutation of the entries of T' in which row i^* contains precisely the $w_{i,j}$. But then this permutation of T' (necessarily of shape D) interpolates the pair (T, T').

□

Proposition 4.3.5 *Suppose that the tableau T on $\mathcal{L}$ is straight of shape D and that $\tilde{T}'$ on $\mathcal{L}^*$ is decomposable of shape $\tilde{D}$. If (T, T') have an interpolant S, then the modified column word of T is lexicographically smaller than the column word of T'. If these words are equal then there is a unique interpolant.*

Proof. Look at the largest letter, say x, in the first column of T'. Let us says x appears in row j of T'. Let y be the largest letter, say y, in the the first column of T. Suppose that y appears in row i of T. Let z be the (i, j)th entry of S. Now since T is row-standard and S is a (T, T')-interpolant, $y \lessdot z$. We conclude that $y \lessdot z \lessdot x$. If $y \neq x$ we are done. Suppose $y = z = x$, then z has shown up in the same cell of S as of T. Iterating this process shows that $w_T \leq w_{T'}$ and that if they are equal, the interpolant is T. □

Proof.(of Theorem 4.3.3.) But using Proposition 4.3.4, we find that the first half of Proposition 4.3.5 gives us the triangularity property in Theorem 4.3.3. The second half of Proposition 4.3.5, together with the proof of Proposition 4.3.4, shows that the diagonal contains only ± 1's. □

Examination of the proof of Proposition 4.3.4 reveals that any basis for row-convex super-Schur modules satisfying the generalized Woodcock condition must be compatible with the basis for the transpose column-convex module given by the decomposable tableau.

Porism 4.3.6 *Let D be a row-convex shape. Choose any set of row-standard tableau $\mathcal{B}$ such that the modified column words appearing in the straight tableaux each arise once from tableaux in $\mathcal{B}$. Again ordering tableaux T_i in $\mathcal{B}$ by their modified column words and ordering decomposable tableaux $\tilde{T}'_j$ of shape $\tilde{D}$ by their row words, we find that the matrix $\left(\left\langle [T_i], [\tilde{T}'_j] \right\rangle\right)$ is upper unitriangular.* □

4.4 Conjectures on tableaux and initial terms.

The results of this chapter, assorted computations, the work of Magyar, and the results of Reiner and Shimozono suggest some conjectures linking initial terms with the Lakshmibai-Magyar basis for %-avoiding shapes.

Definition 4.4.1 *A shape is %-avoiding (respectively dual %-avoiding) if it contains no subshape, determined by picking two rows and two columns, of the form (respectively of the form .)*

The sorted row-convex tableaux are all dual %-avoiding shapes.

I present these conjectures in order of increasing uncertainty. First, I believe that the insertion property enjoyed by the words indexing initial monomials for row-convex shapes also holds for the initial monomials corresponding to dual %-avoiding shapes.

Conjecture 4.4.1 *Suppose that D is a dual %-avoiding shape and that Ψ is as defined on page 40. If $\prec$ is a diagonal term order then the set of words*

$$\left\{\Psi\left(\text{init}_{\prec}(p)\right) : p \in \mathcal{S}^D(\mathcal{L})\right\}$$

is closed under dual Knuth equivalence.

The following conjecture would generalize the bijection presented in Section 4.2

Conjecture 4.4.2 *Suppose that D, D' are both dual %-avoiding shape and $\Psi, \prec$ are as above. If D' arises from permuting the columns of D, then there exists a bijection between $\left\{\Psi\left(\text{init}_{\prec}(p)\right) : p \in \mathcal{S}^D(\mathcal{L})\right\}$ and $\left\{\Psi\left(\text{init}_{\prec}(p)\right) : p \in \mathcal{S}^D(\mathcal{L})\right\}$ which preserves Knuth equivalence class.*

Conjecture 4.4.3 *If D is a dual %-avoiding shape, the set $\mathcal{L}^-$ consists entirely of negative letters, and $\mathcal{L}^+$ is the same set but with positive sign, then the set $\left\{\Psi\left(\text{init}_{\prec}(p)\right) : p \in \mathcal{S}^D(\mathcal{L}^+)\right\}$ equals the set of column words appearing in the Magyar-Lakshmibai basis for $\mathcal{S}^{\bar{D}}(\mathcal{L}^-)$. (Magyar and Lakshmibai index this module by the shape D reflected across the y-axis.)*

I would predict that if this conjecture holds then the type of compatible duality result I proved in Section 4.3 will also hold.

A simple example shows that the first two conjectures do not hold outside the dual %-avoiding case. The initial terms of the submodule generated by all $[T]$ where T is filled with distinct letters from the set $\{1, 2, 3\}$ and has shape are indexed by the modified column words of the tableaux

$$\begin{matrix}1 & & 2\\ & 3 & \end{matrix}\,, \qquad \begin{matrix}1 & & 3\\ & 2 & \end{matrix}\,, \qquad \begin{matrix}2 & & 3\\ & 1 & \end{matrix}\,.$$

But this set is not closed under dual Knuth equivalence. In particular, the words $3, 1, 2$ and $2, 3, 1$ do not index initial terms. Nor does the sequence $2, 3, 1$ (Knuth equivalent to $2, 1, 3$) correspond to an initial term for the shape .

In closing, I would like to remark that just as the property of column-standardness needed to be sacrificed to define the straight tableaux (while maintaining row-standardness), it is impossible to significantly generalize the class of shapes for which one has a basis while preserving both row-standardness of the indexing tableaux and giving these tableaux modified column words which correspond to initial terms. In other words, for more general shapes, the initial terms, under a diagonal term order, of products of determinants *do not* exhaust the initial terms of the module generated by these products. For example, under the default diagonal term order of page 40, the Schur module $\mathcal{S}(\{1, 2, 3, 4, 5, 6, 7, 8\})$ has $x_{1,1}x_{4,1}x_{2,2}x_{3,3}x_{6,3}x_{7,4}x_{5,5}x_{8,5}$ as a leading term. This is not realizable as the leading term of $[T]$ where T is some tableau of the given shape. Hence for this shape *any* diagonal term order, $\prec$, will produce some leading terms that do not arise as $\mathrm{init}_{\prec}([T])$.

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