Essays on Supply Chain Contracting and Tactical Decisions for Inter-generational Product Transitions

by

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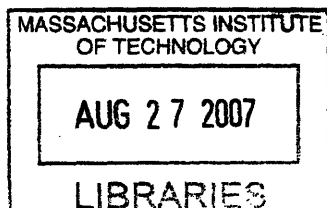at the
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Abstract

In this dissertation, we explore problems in two areas of Supply Chain Management. The first relates to strategic supplier management. The second focuses on tactical decisions on inventory and pricing during inter-generational product transition.

In many industries, manufacturing firms use multiple competing suppliers in their component or product sourcing strategy. Chapter 2 studies optimal history-dependent contracts with multiple suppliers in a dynamic, uncertain, imperfect-information environment. The results provide an optimal contract structure for the manufacture and optimal performance and effort paths for the suppliers. We compare incentives in the form of product margin and that of business volume. Our results suggest that a volume contract may increase the total profit for the supply chain, partly due to its ability to allocate higher volume to the supplier that is more likely to input high effort, and partly through relative performance evaluation. However, for two suppliers with large asymmetry, it is better to contract independently with each supplier using margin incentive, rather than forcing them into a volume race.

Chapter 3 studies the inventory planning decisions in the context of a technology product transition, i.e., when a new generation product replaces an old one. High uncertainties in a new product introduction coupled with long lead-time often lead to extreme cases of demand and supply mismatches. When a company runs out of the old product, a customer may be offered the new product as a substitute. We show that the optimal substitution decision is a time-varying threshold policy and establish the optimal planning policy. Further, we determine the optimal delay in new product introduction, given the initial inventory of the old product.

In Chapter 4, we study the optimal pricing decisions during a product transition. We restrict the new product price to be constant and formulate the dynamic pricing problem for the old product. We derive a closed-form solution for the optimal price under non-homogeneous Poisson demands. In addition, we compare three heuristic pricing policies: fixed-price, two-price, and myopic rolling-horizon policies. The results suggest that changing price once during the transition (the two-price policy) improves the profit dramatically and is near optimal.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

In this dissertation, we explore problems in two areas of Supply Chain Management. The first relates to incentive designs in long-term supply contracts (Chapter 2). The second focuses on tactical decisions in production, inventory and pricing during inter-generational product transition (Chapter 3 and 4).

Throughout the last few decades, the U.S. buyer-supplier dynamic has evolved from an arms-length transaction-based relationship to a more collaborative trust-based one, partly driven by the success of its Japanese manufacturing competitors such as Toyota. Many U.S. companies have reduced the number of suppliers and focused on improving the quality of the supplier relationships as a way to improve efficiency. What is common today is that manufacturers are establishing long-term ties with a handful of suppliers. Several questions arise in this context: How should a manufacturer manage such a small group of suppliers over the long run? What incentive structures shall be put in place to ensure that the manufacturer maximizes the benefits from these long-term relationships, especially when dealing with multiple suppliers? What types of reward or punishment work well in such supply relationships? These questions motivate Chapter 2 of this dissertation.

Some firms use volume-based incentives implicitly. In one telecommunications equipment manufacturing firm we interviewed, a procurement manager asserted that increasing the volume of business to a supplier was the best incentive lever he had to encourage good supplier performance. Furthermore, discussions with the sales representatives of the suppliers suggested that they, too, believed that volume incentives were the most effective way to incentivize good supplier performance. However, in the telecommunications firm, the formal, written contracts (many of which ran to 100 pages or more) we examined made no mention of volume incentive structures, but did outline margin agreements, as well as the monetary rewards and punishments on quality, delivery, and response time. So, these volume rewards and punishments may be executed outside of the formal legal agreements.

Other firms use volume incentive more explicitly. In an automotive setting, Chrysler Corporation, during its very successful expansion in the 1990's, had a procurement strategy that often relied on volume incentives. According to Tom Stallkamp, former President of Chrysler and former head of Chrysler Procurement and Supply, this strategy worked very well for them in sourcing tires. Chrysler had two main tire suppliers, Goodyear and Michelin, who competed head to head for Chrysler's business through a volume-based incentive. Each quarter, the share of business each would receive depended on its own performance and that of the other supplier in the previous quarters – on cost, quality, delivery, and other metrics.

These phenomena have prompted us to examine both volume and monetary rewards and punishments as long-term incentives for driving supplier performance. In particular, we compare the impact of volume and margin incentives in the long run. In Chapter 2, we model the performance-based incentive contract as an infinite-horizon repeated-game problem to compare the optimal reward scheme using volume incentives with those using margin incentives and to explore the strengths and weaknesses of each contract type.

In Chapters 3 and 4 we focus on another important area in Supply Chain Management – tactical planning decisions for inter-generational product transition. In high-tech industries, a newer generation product replaces an old product periodically. In many cases, such a transition does not occur instantaneously but involves a transitional period during which the company sells both products simultaneously. Such a time period involves high uncertainty and high risk.

This research is motivated by collaborative work with a telecommunications equipment manufacturer in the U.S. In this industry, the lead-time for critical components is about 13 weeks, sometimes even longer. The production lead-time is usually 5 weeks. As a result, planning decisions need to be made as early as 4-6 months before the actual transition. In addition, a long lead-time makes replenishment during the transition impossible or very expensive. Introduction of a new product to the market tends to create high uncertainties in both the demand side and the supply side. If many new features are added to the new generation product, it is difficult to predict how well it will be accepted by the customers, as well as how well the suppliers will be able to handle the technological or production changes that the transition would entail. The launch of a new product involves many sequential steps and depends on many factors: technology, production, supplies, etc. As a result, the new product release date often fluctuates dramatically. Long lead-time exacerbates the problem of demand forecasting because the managers need to predict demand 18 weeks ahead. Consequently, the company often runs out of the product that customers want while having excess of the other during a product transition. Because replenishment during the transition is not possible, the managers are left with very limited options.

Substitution is one way of coping with the volatile demand situation during a product transition. When a shortage of the old product occurs, the new product can be used as a substitute. Substitution provides risk pooling, but it adds further complexity to the inventory planning problem. The model in Chapter 3 analyzes the optimal planning decisions with such a dynamic substitution policy. We consider both uncertainties in demand and uncertainties in the new product release date.

Pricing is another option for dealing with demand and supply mismatch during transition: The managers can manipulate the prices of the two products to mitigate the demand risk. For example, if sales of the old product are sluggish during the transition, they could put in a promotion and discount it. On the other hand, if the new product does not sell well, managers may consider increasing the price of the old product to make the new one appear more attractive. In Chapter 4, we study the dynamic pricing problem for the transitional period when there is no reorder option. Specifically, we look at a situation

when the price of the new product is market-driven and stays constant during the transitional period and we solve for the optimal dynamic price of the old product. We also explore simple pricing heuristics that are easier to implement in practice.

Chapter 2

Performance-based Contracting in Supply Chain

2.1 Introduction

In this chapter, we analyze multi-period supply chain relationships, between a manufacturer and one or more suppliers, to assess the relative advantages of using different contract types. In particular, we compare the outcomes from a contractual form that rewards (and/or punishes) supplier performance by giving a greater (lesser) volume of business in subsequent periods with a contractual form that pays the supplier(s) more (less) margin for good (poor) performance.

Throughout the last few decades, the U.S. buyer-supplier dynamic has evolved from an arms-length transaction-based relationship to a more collaborative trust-based one, partly driven by the success of its Japanese manufacturing competitors such as Toyota. Many U.S. companies have reduced the number of suppliers and focused on improving the quality of the supplier relationship as a way to improve efficiency. For instance, Chrysler has since 1989 shrunk its production supplier base from 2,500 companies to roughly 1,000 and has fundamentally changed the way it works with those that remain (Dyer 2000): On one hand Chrysler emphasized long-term commitment to a small number of suppliers; on the other it still tried hard to foster competition among the suppliers. For example, in sourcing tires, Chrysler established long-term ties to two suppliers: Michelin and Goodyear. It promised to buy tires only from these two suppliers. However, the suppliers are kept on their toes because the good performer over time would potentially get a better margin, or earn a larger share of business from Chrysler. Tom Stallkamp, former President of Chrysler and former head of Chrysler Procurement and Supply, cited volume as the most effective incentive lever for driving supplier performance.

The goal of this research is to model and explore the use of business share allocations as incentives for driving supplier performance. We compare the optimal reward scheme using volume incentives with those using margin incentives to explore the strengths and weaknesses of each contract type. Because we are interested in supplier relationships with repeated interaction, the optimal performance-based reward scheme sheds light on how current and past performance should be considered in supplier incentives. We also examine longitudinal supplier behavior under the optimal reward scheme. We use simulation to explore the trajectories of supplier performance and compensation.

An Example

In a repeated contracting relationship, rewards can come from payment in the current business period and/or from promised expected future gains. In practice, rating systems or grading classifications are sometimes used to represent the parties' expectations about

the future performance and payouts to suppliers. To provide more intuitive understanding, consider a hypothetical pay and grading system for employees in a consulting firm (Table 2.1)[1].

An employee entering the firm as an Analyst receives $50K in annual salary and no bonus. The only way for the Analyst to move up from that level is to get a graduate degree (MBA or Ph.D.), after which she can get "regraded" at the Associate level, which pays $100K and on average $25K in bonus (awarded contingently each year based on that year's performance). If an employee performs well as an Associate, she can be promoted to the next level, Manager, which increases the salary to $150K and the expected bonus to $50K. Therefore, the incentive for someone in the Associate level to work hard includes the annual bonus (current incentive) plus the opportunity to be promoted to a higher level position (future payoff).

Position Level or Employment Rank	Annual Salary (base)	Annual Bonus (performance-based)	Promotion (performance-based)
Analyst	$50K	None	No
Associate	$100K	$25K	Yes
Manager	$150K	$50K	Yes
Associate Principal	$250K	$100K	Yes
Principal (Partner)	$500K- $1M in total pay.		Yes
Director (Partner)	Partners split the firm's total profit		No

Table 2.1: Hypothetical Salary and Bonus Structure in a Consulting Firm

In a buyer-supplier relationship, the current incentive for a supplier may include cash payments and/or immediate reward of more business. The future incentive, similar to the various rank levels of the consulting firm, could simply be a rating or ranking system that categorizes the suppliers to different levels, with each level representing a certain expected future payoff. For instance, in the "SCORE" ("Supplier Cost Reduction Effort") model that Chrysler implemented in an effort to reduce the total cost of a vehicle, suppliers were encouraged to propose cost saving ideas which Chrysler would selectively implement (Stallkamp 2005). When the cost saving materialized, a supplier could claim its share of the savings (the immediate reward) or could opt to yield the savings to Chrysler, which would boost its overall performance rating at Chrysler and potentially lead to more business (future incentive).

Structure and Overview

In this chapter, we examine optimal contracts for both risk-neutral and risk-averse suppliers. For the margin contract, we consider both single-supplier and two-supplier contractual relationships. For the volume contracting setup, we consider two-supplier contracts only in order to capture the zero-sum nature of the volume allocations

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[1] Source: Tor Schoenmeyr, MIT Sloan School, formerly an employee at a major consulting firm.

suggested in the Michelin-Goodyear competition mentioned in Chapter 1. Building on techniques developed in the repeated game literature, we derive the equilibrium outcomes for our models.

Our results suggest that volume contracts can either outperform or under-perform the margin contracts, depending on the scenario. The margin contract in general works better when suppliers' ranks are low and the volume contract is superior when the suppliers have high and comparable ranks. In addition, we find that the optimal contract is not always "fair": The manufacturer may promote/demote a supplier regardless of current period performance in order to prevent that supplier from reaching a trapping low-effort state.

The rest of the chapter is organized as follows: Section 2.2 reviews some of the relevant contracting literature from both economics and operations management. Section 2.3 presents the problem formulations for margin and volume-incentive contracts, respectively. We also provide analytical results for risk-neutral suppliers in Section 2.3. For the case of risk-averse suppliers, we rely on computation to obtain the optimal contract and derive comparative statics, which are presented in Section 2.4. The last section discusses the results and potential applications in addition to future research.

2.2 Literature Review

This research builds on two streams of literature, one in economics, and one in operations management. A key paper in the modern incentives literature in economics by Holmstrom (1979) framed a basic contracting relationship between a principal and an agent. This paper triggered a raft of work in labor relationships, with significant theoretical and empirical advances, for example on wage growth, performance bonuses and promotions (e.g. Baker, Gibbs and Holmstrom 1994a and 1994b, Gibbons and Waldman 1999a, and Fairburn and Malcomson 1997). Gibbons and Waldman (1999b) provide a comprehensive review on work in this area. Many incentive models developed in economics have been extended to describe governance mechanisms among various companies in the supply chain.

Papers on incentive schemes in a dual/multiple-sourcing environment mostly focus on single-period models or multi-period models without history dependencies. For example, Anton and Yao (1989) compare the split-award auction with a winner-take-all auction in a single-stage Nash equilibrium setting. In their model volume allocation is based on price. In a split-award auction, each supplier submits a menu of price and volume splits and the buyer selects a split that minimizes its procurement costs. They show that winner-take-all dominates split-award since the split-award auction allows implicit collusion and hence the buyer pays more in equilibrium. However, when upstream investment actions are considered, split-award auctions can become superior to winner-take-all for the buyer as potential profit from a split-award auction provides an incentive for the supplier to invest in innovations. Seshadri (1995) studies a dual-sourcing model with a cost-plus contest (using third-price bidding competition) that awards each supplier its actual audited cost and a constant fraction f ($0 \leq f \leq 1$) of a fixed incentive pool c to the lower-

cost provider. The optimal design is a tradeoff between f and c: as f increases, the incentive for effort increases, but to offset the increased risk, the buyer needs to raise c. The paper solves for the optimal risk-incentive tradeoff and compares it with a single-source incentive contract to derive conditions under which cost-plus contest yields lower total cost for the buyer. Klotz and Chatterjee (1995) consider a two-period dual-sourcing model where the buyer reserves a fixed volume share for both suppliers and puts out the rest in a competitive bidding in which the lower-cost provider takes all. In their model, the only connection between the first and second period is that the cost estimate of a supplier in the second period depends on its first period production quantity. Benjaafar, Elahi and Donohue (2005) consider the role of competition in eliciting service quality from supplier in a Supplier-Allocation model (each supplier gets a performance-based market share) and a Supplier-Selection model (each supplier is probabilistically chosen as the sole-source provider with the winning probability dependent upon performances). They focus on a proportional allocation mechanism $\alpha(s_i, s_{-i}) = s_i^\gamma / \sum_{i=1}^{N} s_i^\gamma$ where α is supplier i's demand share in the Supplier-Allocation approach (or probability of winning in the Supplier-Selection approach), s_i is the performance of supplier I, and γ signifies the intensity of the competition of the allocation function. They solve the symmetric Nash equilibrium and thus the equilibrium service levels for both models. Their model is a single-period model in which the buyer's only involvement is through α. In comparison, we develop a repeated game model where the buyer explicitly takes on the role of a principal and solves a profit maximization problem. Cachon and Zhang (2005) compare several performance-based allocation policies that assign incoming jobs to suppliers in a repeated capacity game of multiple-server queuing system. Their method is akin to that of Benjaafar, Elahi and Donohue (2005) in that both papers assume a policy first and then establish the equilibrium outcome. Although they study the problem in a repeated game setting, past actions and payoffs are disregarded. In comparison, we identify the history-dependent optimal policy and allow more general performance measures.

A dynamic contract can use both current compensation and future promises as incentives to induce the desired behavior. Thus dynamic contracts are potentially more powerful than static contracts. However, an optimal dynamic contract generally needs to keep track of the entire performance history, making for intractable problems due to the curse of dimensionality. In this chapter we adopt a method developed by economists to reduce the complexity of history-dependent contracts. Abreu, Pearce and Stacchetti (1986, 1990, hereafter APS) study an optimal cartel model and the central technique employed in their analysis is the reduction of the repeated game to a family of static games. The APS model is extended by Spear and Srivastava (1987) to the Principal/Agent framework. They study a dynamic contract in an infinite horizon with risk-neutral principal and risk-averse agents. They show that there is a stationary representation of the optimal contract using the agent's expected discounted utility as the state variable. Essentially, the incentive for the agent is decomposed into current compensation and continuation payoff. Value iteration methods are used to compute the optimal policies. Using a similar formulation but allowing public randomization of strategies (assuming a continuum of agents), Phelan and Townsend (1991) develop a computation technique that solves the equilibrium as a linear programming problem. Yeltekin (1999) applies the Phelan and Townsend method

18

in a multiple-agent setting and finds that the optimal dynamic contract displays a tournament feature when the agents' performances are affected by a common noise. Such a result corroborates a finding by Holmstrom (1982) that relative evaluation among agents is only valuable in filtering out common uncertainties in outputs. Our results indicate that relative performance evaluation (RPE) can be optimal whenever the suppliers are forced to compete for a limited-size reward, with or without common noise. Wang (1997) uses the Spear and Srivastava model to explain the low correlation between CEO compensation and performances. Our analysis also exploits the methods of Spear and Srivastava.

2.3 Model Formulations

In this section, we develop the motivation and structure of the model formulations for both margin and volume contracts. The two subsequent sections provide the analysis.

2.3.1 Single-Supplier Margin Contract

In the static problem, the buyer, or original equipment manufacturer (OEM), delegates the production of one product or component to a supplier. The supplier chooses its effort input a from a finite choice set A and incurs cost $\psi(a)$. The effort contributes to performance metrics such as on-time delivery, defect rate, yield, and so on. We aggregate these into one single term x that is measured in money units and captures the value of the supplier's performance to the OEM. Such aggregation may seem arbitrary, but firms have developed ways to do this. At Sun Microsystems, if a supplier receives a total score of 86 from the scorecard calculation, the commodity manager may inform the supplier that every dollar Sun spends with the supplier actually costs Sun $1.14 (Farlow, Schmidt and Tsay 1996).

We assume that a supplier's actual effort input is not observable by the OEM. The OEM, however, can make inferences about the supplier's choice by observing the performance score, which is an imperfect signal of the effort level. In other words, the effort input affects the performance x through its probability distribution function. The OEM observes this value and pays the supplier $w(x)$ after the supplier delivers. Let $f(x|a)$ be the density of the distribution of x under a given effort level a and $\phi()$ be the utility function of the supplier.

The OEM maximizes its total value by solving the following single-period (SP) problem:

$$\max E[x - w(x)] \mid a]$$
$$\text{s.t.} \quad E[\phi(w(x) \mid a)] - \psi(a) \geq u \qquad \text{(IR)}$$
$$E[\phi(w(x) \mid a)] - \psi(a)$$
$$\geq E[\phi(w(x) \mid \hat{a})] - \psi(\hat{a}) \quad \forall \hat{a} \in A \qquad \text{(IC)}$$
$$0 \leq w(x) \leq x \qquad \text{(NN \& BC)}$$

The first constraint is the supplier's individual rationality (IR) constraint, which dictates that the supplier will participate only if its expected payoff is higher than its outside

option u. The second constraint is the supplier's incentive compatibility (IC) constraint, requiring the supplier to respond optimally to the OEM's payment scheme. The last two constraints are the non-negativity constraint (NN) and the budget constraint (BC), requiring the payment be bounded. Such requirement ensures that the contract is self-sustaining.

The SP problem is a well-studied formulation (Mirrlees 1976, Milgrom 1981, Rogerson 1985, Innes 1990). It is a simple constrained nonlinear optimization problem. The OEM first finds the optimal incentive scheme that implements action $a \in A$, and then chooses the action that gives it the highest expected payoff. Milgrom (1981) shows that in the optimal static contract, the compensation $w(x)$ is non-decreasing in performance x under the following assumptions: [2]

Assumption 2.1 $f(x|a)$ satisfies the monotone likelihood ratio property (MLRP)[3] where f is the probability density function of x for a given effort a.
Assumption 2.2 $F(x|a)$ is convex in effort where F is the cumulative distribution function of x for a given effort a.

Without further assumption on the utility and cost function, not much more can be said about the optimal static contract other than the monotonicity property. Innes (1990) shows that the optimal contract is a threshold policy if the agent (supplier) is risk-neutral: The supplier receives the entire output when the performance is above a threshold and nothing when it is below the threshold.

Since our goal is to study supplier governance over the long term, we shall look beyond the static problem and search for optimal contract in repeated plays. In this chapter we study the dynamic contract in a repeated game over an infinite horizon. From the modeling point of view, the infinite horizon assumption provides tractability and allows computation of optimal contracts. In the auto industry, long-lived supply chain relationships are not uncommon. In the Chrysler example, Tom Stallkamp once stated: "I don't care if we don't buy a tire from anyone other than Goodyear or Michelin. In fact, I've told those companies we won't as long as they're doing what we want"[4]. Moreover, the life expectancy of the relationship is endogenous rather than exogenous to the problem we consider since often breakups are a result of failures to comply with contractual agreements.

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[2] Assumption 2.2 is often referred to as the Convex Distribution Function Condition (CDFC). Rogerson 1985 shows that MLRP and CDFC guarantee that the agent's optimization problem is concave (second-order derivative w.r.t. a is negative) and first-order conditions fully identify global optima for the agent. It is also called the Mirrlees-Rogerson condition. See also "Contract Theory" by Bolton and Dewatripont 2005.

[3] In the continuous-action case, MLRP implies $\dfrac{d}{dx}(\dfrac{f_a(x\,|\,a)}{f(x\,|\,a)}) \geq 0$. This corresponds to the intuitive requirement that a high-performance realization indicates a high effort choice by the supplier. Many ordinary distributions satisfy such property, e.g. the normal distribution.

[4] Articles in July 1997 issue of Ward's Auto World, "Sharing warranty costs: the new frontier - includes interviews with Ford's Carlos Mazzorin, Chrysler's Thomas T. Stallkamp and General Motors' Harold Kutner - Managing the Supply Chain" by Greg Gardner

With an infinite horizon, the OEM designs the contract to maximize its total discounted expected profit. In each contract period, the supplier chooses its effort level a_t from a finite choice set A and incurs cost $\psi(a_t)$. The value that the supplier generates for the OEM, which is denoted by x_t, depends stochastically, through $f(x|a)$, on the supplier's effort choice. At any time period t, the OEM bases compensation decisions on the entire history. Let $h^t = \{x_1, x_2, ..., x_t\}$ denote the history up to time t. A dynamic contract can be represented by $\sigma = \{a_t(h^{t-1}), w_t(h^t)\}_{t=1 \text{ to } \infty}$, which is a very complex problem, cursed by its dimensionality.

By using the perfect public equilibrium (PPE) formulation of APS (1986), the complexity of history dependencies can be significantly reduced. In a PPE, the supplier's strategy depends only on the public history h^t (past performance outcomes), not on its own past actions (past choices of a). A PPE is sequentially optimal: Following every public history the remaining subgame forms a Nash Equilibrium. In theory, the supplier could adopt strategies that are dependent on its past private action choice, but this will not benefit it since the performance outcome only depends on current period action and the contract payment is based only on the performances. [5]

A PPE is recursive, i.e., every public history induces a strategically identical game. Therefore, the equilibrium payoff can be decomposed into the current-period payoff u and the continuation values U, which are themselves payoffs of a PPE. That is,

$$u = E[\varphi(w) - \psi(a) + \delta U] \qquad (2\text{-}1)$$

where $u \in \mathcal{U}, U \in \mathcal{U}$, and $\mathcal{U}$ denotes the set of equilibrium payoffs in a PPE.
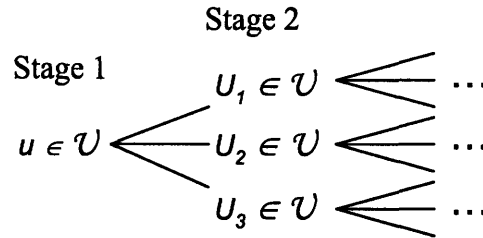
Figure 2-1: Game Tree of a PPE

APS 1986 defines this property as self-generation. In fact, if we view the continuation value promised to the supplier as the state variable, equation (2-1) is essentially the Bellman equation for an infinite horizon discounted problem (Bertsekas 2000). The set of PPE payoffs then forms a set of Markov states. Therefore, if the optimal strategy profile at stage 1 is $\sigma(u)$ where $u \in \mathcal{U}$, then at stage 2, as long as the state at stage 2 belongs to the same set $\mathcal{U}$ (Figure 2-1), the same optimal strategy profile applies. Spear and Srivastava (1987) use this concept to reduce the optimal history-dependent contract problem to a

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[5] Given a pure-strategy equilibrium where players' strategies may depend on private information, we can find an equivalent equilibrium where the strategies are public since each player perfectly forecasts how each opponent will play in each period in a pure strategy equilibrium. See also "Game Theory" by Fudenberg and Tirole (1991).

dynamic programming problem with a single-dimension state variable – u. Intuitively, the expected payoff u promised to the supplier in each period becomes a proxy of the past history $h^{t-1} = \{x_1, x_2, ..., x_{t-1}\}$. It "stores" the minimum necessary amount of information of the past (just enough for the OEM to know how to compensate the supplier in the immediate period and how much to promise for the future payoff).

Referring back to the compensation scheme at the consulting firm, the state variable u corresponds to the position level, e.g., Associate, Manager[6]. Each value of u represents the current "state" of a consultant and captures the relevant information about past performance.[7] In addition, u provides information about future payoffs: A consultant at the Principal level will have higher expected lifetime earnings than a person at the Manager level. Furthermore, the value of u provides information about how a consultant would be paid and promoted in the current contract period. Extending the analogy to a supply chain context, a supplier's current state u could be interpreted as its current standing (e.g. a number-coded rating) at the OEM. To generalize it further, u could also represent the various classes the suppliers are categorized into. At GM, suppliers are categorized into color-coded classes: green, yellow and red (Sherefkin 2006). At other companies, it may be gold, silver and bronze, or one-star to five-star ratings.

In the dynamic program formulation proposed by Spear and Srivastava, the control variables include the action choice $a(u)$, the current compensation $w(u, x)$, and the continuation payoff $U(u,x)$. In contrast to conventional dynamic programming formulation, the state transition is itself a decision variable. Let $V(u)$ denote the expected payoff to the OEM when the payoff promised to the supplier is u. We also assume that the OEM and the supplier have the same discount factor δ. The OEM's optimization problem is to find the optimal a, w and U that solve the following:

$$V(u) = \max_{a,w,U} E[x - w(u, x) + \delta V(U(u,x)) \mid a(u)]$$

s.t. $\quad E[\phi(w(u, x)) + \delta U(u,x) \mid a(u)] - \psi(a(u)) = u \quad \forall u \in \mathcal{U} \qquad$ (PK)

$\quad E[\phi(w(u, x)) + \delta U(u,x) \mid a(u)] - \psi(a(u))$

$$\geq E[\phi(w(u,x)) + \delta U(u,x) \mid \hat{a}] - \psi(\hat{a}) \quad \forall \hat{a} \in A \qquad \text{(IC)}$$

$\quad 0 \leq w(u,x) \leq x \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (BC)

Note that the first constraint is the promise-keeping (PK) constraint for the supplier's equilibrium payoff[8]. It specifies that the supplier's equilibrium payoff u is realized through the current period compensation $w(u,x)$ and continuation value $U(u,x)$. Potentially, suppliers with different u's are paid differently, i.e., a Gold-ranked supplier is

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[6] In the consultant example, u takes on only a few discrete values such as Analyst, Associates, etc. More generally, u can be continuous, as in our model.

[7] Strictly speaking, for the consultant analogy to work, we need to make additional assumptions: For the Markov property to hold, we assume that a consultant's promotion does not depend on how many years she has been at the current rank. In addition, for the game to be recursive, we assume that a consultant can be demoted, as well as promoted.

[8] It is called the "self-generation" constraint in APS and Spear and Srivastava.

treated and paid differently than a Silver-level supplier. As u and $U(u,x)$ both take on values in the same set $\mathcal{U}$, it is a self-generation set.

To provide further intuition of the concept of self-generation, we illustrate with the following equation that the continuation value $U(u,x)$ is itself a PPE payoff.

$$U(u,x) = E[\phi(w(u,\hat{x})) + \delta U(U,\hat{x})) \mid a(U(u,x))] - \psi(a(U(u,x)))$$

where $\hat{x}$ has distribution $f(\hat{x} \mid a(U(u,x)))$ (PK)

Alternatively $U()$ can be viewed as a fixed point of the set $\mathcal{U}$.

Consider a contracting problem where the transfer price between the supplier and the buyer (the OEM) is determined through a cost-plus model. That is, the buyer promises to pay the supplier the cost of the product c plus a certain margin p for each unit for an agreed production volume q_0. In an incentive contract, the margin is dependent on a supplier's current standing (i.e., the u value, which we also refer to as the "rank") and current period performance x. Figure 2-2 shows the sequence of events under such a cost-plus contract. Suppose at period t, the supplier's rank starts at u. The supplier then chooses an effort level based on its current rank u. Given the effort input, an output x is generated and observed. The buyer then decides the supplier's next period margin $P(u,x)$ and next period rank $U(u,x)$ based on both its current rank u and the performance in this period x. We assume that the parties settle the payment $P(u,x)q_0$ within period t so that we would not need to keep track of the margins as part of the state variable. In the next period, the supplier's rank becomes $U(u,x)$ and the game repeats.
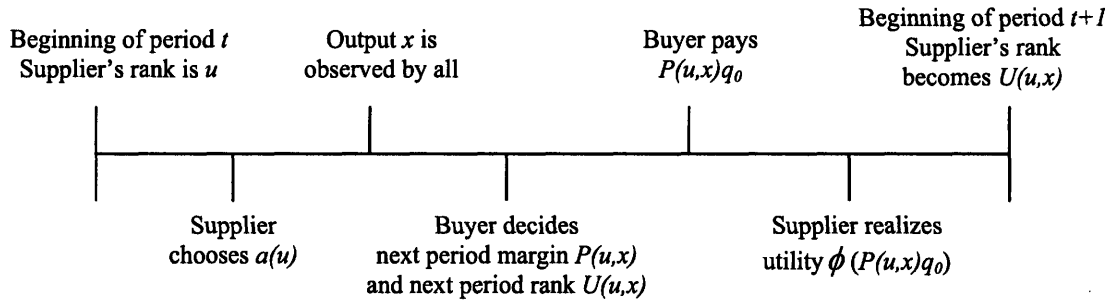
Figure 2-2: Sequence of Events for a Single-Supplier Margin Contract

Reformulating it in the context of a cost-plus contract, we get the following dynamic single-supplier problem (DSP):

$$V(u) = \max_{a,w,U} E[q_0 x - q_0 P(u,x) + \delta V(U(u,x)) \mid a(u)]$$

s.t. $E[\phi(q_0 P(u,x)) + \delta U(u,x) \mid a(u)] - \psi(a(u)) = u$ $\forall u \in \mathcal{U}$ (PK)

$E[\phi(q_0 P(u,x)) + \delta U(u,x) \mid a(u)] - \psi(a(u))$

$\geq E[\phi(q_0 P(u,x)) + \delta U(u,x) \mid \hat{a}] - \psi(\hat{a})$ $\forall \hat{a} \in A$ (IC)

$\underline{p} \leq P(u,x) \leq \bar{p}$ (BC)

23

Since the buyer pays for the cost of product cq_0 anyway, it can be viewed as a sunk cost. For the supplier, the amount of cq_0 is used to cover the material cost and is sometimes paid directly by the buyer to the suppliers further upstream. Therefore, we can leave it out of the optimization problem without affecting the implications of our results.

As mentioned earlier, in the buyer's objective function, we count payments to the supplier in the next period $P(u,x)q_0$ as a cost in the current period to reduce the state space.

The margins are bounded from both above and below. The lower bound may reflect the supplier's reservation utility and the upper bound might reflect the buyer's maximal acceptable margin level.

Definition 2.1
Define the optimal solution of the DSP problem to be $P^S(u,x)$ and $U^S(u,x)$.

Without loss of generality, we also make the following assumption to indicate that performance increases in the supplier's effort input. Let $F(x|a)$ be the cumulative distribution function of the performance output for a given effort level a and $F_a(x|a)$ be the first order derivative of $F(x|a)$ with respect to a.

Assumption 2.3 $F_a(x|a) \leq 0$ (with strict inequality for some x)

Under *Assumptions 2.1-2.3*, the (IC) constraint can be replaced by the first order condition. Therefore the incentive-compatibility constraint can be rewritten as:

$$\int (\phi(q_0 P(u,x)) + \delta U(u,x)) f_a(x\,|\,a)dx = \psi'(a(u)) \quad \text{(IC)}$$

We can then easily obtain the Kuhn-Tucker condition of the DSP problem through point-wise maximization.

Proposition 2.1
$P^S(u,x)$ and $U^S(u,x)$ solve the DSP problem if and only if there exists $\lambda(u)$ and $\mu(u)$ such that the following conditions are satisfied:

$$V'(U) + \lambda(u) + \mu(u)\frac{f_a(x\,|\,a)}{f(x\,|\,a)} = 0$$

$$-1 + \phi'(q_0 P)[\lambda(u) + \mu(u)\frac{f_a(x\,|\,a)}{f(x\,|\,a)}] = 0$$

(PK), (IC).

Proof. See Appendix.□

Lemma 2.1
With a risk-neutral supplier, $V'(u) = -1$.

Proof. See Appendix.□

Lemma 2.1 states that the Pareto frontier of the value function is linear. One additional unit of utility promised to the supplier implies one less unit of profit for the OEM. In *Proposition 2.2*, we show that the linearity of the value function leads to a bang-bang optimal policy.

Proposition 2.2

With a risk-neutral supplier, a bang-bang policy can be optimal. That is, a buyer promises

a constant continuation payoff $U = \dfrac{\delta q_0[\bar{p} - (\bar{p} - \underline{p})F(\bar{x} \mid a^*)] - \psi(a^*)}{1 - \delta}$

and $P = \begin{cases} \bar{p} \text{ if } x \geq \bar{x} \\ \underline{p} \text{ if } x < \bar{x} \end{cases}$ as the next period margin. The optimal action input a^* and the

critical performance threshold $\bar{x}$ can be determined jointly by $\dfrac{-\psi(a)}{F_a(\bar{x} \mid a)} = \bar{p} - \underline{p}$ and

$f_a(\bar{x} \mid a) = 0$.

Proof. See Appendix.□

2.3.2 Contracting Problems with Two Suppliers

With more than one supplier, the buyer has two alternative means for inducing performance: reward the supplier with higher (lower) margin – margin contract, or reward it with higher (lower) business volume – volume contract.

We describe a situation when the buyer is dedicated to two suppliers in a long-run relationship and would like to design incentives that elicit as much supplier effort as possible. Volume seems to be a cheap way of doing this – reward or punish the suppliers without yielding on margins, which is often the focus of contract negotiation (as it is easier to specify margin than business volume).

Contract design becomes much more complex when dealing with dual or multiple suppliers because there is an opportunity for relative performance evaluation (RPE). In other words, the OEM may base a supplier's compensation on its own absolute level of performance as well as on its performance relative to that of other suppliers. Some of the seminal papers on RPE, for example Holmstrom (1982) and Green and Stokey (1983), conclude that RPE is only useful when common industry noise is present. Yeltekin (2003) draws the same conclusion in a dynamic contracting context.

In the model that follows, we assume the suppliers are identical, with regard to their effort choices, utility functions, and cost functions. We also assume that the production functions of the suppliers are independent and have independent noise terms.

2.3.21 Margin Contract

We first examine margin incentive. To do so, we hold the volume allocation constant: we assume that the total volume is fixed and that each supplier receives a constant award of production quantity in each period - q_1 and q_2 respectively. Without loss of generality, we assume that $q_1 = q_2 = q_0$, which further simplifies the model. Additional notations are listed below.

$u = (u_1, u_2)$ represents the vector of continuation values for both suppliers.

$x = (x_1, x_2)$ denotes the vector of performance output.

$P_i(u,x)$ is the next period margin to the ith supplier given u and x

$U_i(u,x)$ is the next period rank to the ith supplier given u and x; it implies the promised continuation payoff from next period on.

For clarity, we refer to the vector u as the state of the system and each u_i as the supplier's rank level.

At the beginning of a contract period, each supplier starts with a rank and margin that was determined from last period. Each supplier chooses an effort level that maximizes its expected utility.

The buyer solves the following dynamic two-supplier margin contract program (DTM) to determine the margin and rank for each supplier in the next period.

$$V^M(u) \equiv \max_{a,P,U} E[q_0(x_1 + x_2) - (q_0 P_1(u,x) + q_0 P_2(u,x)) + \delta V^M(U_1(u,x), U_2(u,x)) \mid a_1(u), a_2(u)]$$

s.t. $\quad E[\phi(q_0 P_i(u,x)) + \delta U_i(u,x) \mid a_i(u), a_j(u)] - \psi(a_i(u)) = u_i \quad \forall u_i \in \mathcal{U}_i \qquad (PK_i) i = 1,2$

$\qquad E[\phi(q_0 P_i(u,x)) + \delta U_i(u,x) \mid a_i(u), a_j(u)] - \psi(a_i(u))$

$\qquad\qquad \geq E[\phi(q_0 P_i(u,x)) + \delta U_i(u,x) \mid \hat{a}_i, a_j(u)] - \psi(\hat{a}_i) \forall \hat{a}_i \in A \qquad (IC_i)$

$\qquad \underline{p} \leq P_i(u,x) \leq \overline{p} \qquad\qquad\qquad\qquad\qquad\qquad\qquad (BC_i)$

Comparing the two-supplier formulation with the single-supplier one, it is clear that the two-supplier problem is a more general problem because it allows relative performance evaluation. However, we show that the single-supplier solution is optimal in the DTM problem.

Proposition 2.3
$P_1(u,x) = P^S(u_1, x_1), P_2(u,x) = P^S(u_2, x_2)$ and
$U_1(u,x) = U^S(u_1, x_1), U_2(u,x) = U^S(u_2, x_2)$ solves the DTM problem.

Proof. See Appendix.□

Proposition 2.3 implies that the margin-based contract can be decoupled into two independent single-supplier problems. That is, relative performance evaluation does not add value. Such result corroborates the findings by Holmstrom 1982.

Intuitively, if a buyer can induce a supplier to input a preferred effort using independent margin incentive (margin reward is entirely based on the supplier's own performance and rank), it will not be cheaper to do so through a competitive margin reward scheme where a supplier's margin may also depend on the other supplier's rank and/or performance.

2.3.22 Competing Suppliers: Volume Contract

One potential advantage of volume as an incentive mechanism compared to margin is that the OEM may assign higher volume to the supplier that is more likely to work hard and thus leads to higher output. In addition, as the suppliers are splitting a fixed amount of business, volume-incentive contracts can potentially induce higher effort and more competition among the suppliers.

In a volume-incentive contract, both the current and future incentives are provided in the form of business shares. Each period the OEM divides the total business volume $2q_0$ between the suppliers' allocations, respectively, q_1 and q_2, where $q_1 + q_2 = 2q_0$. As in the money-reward contracting model, each supplier's past is aggregated into a single variable u_i, which represents a promised expected payoff. In each period, the OEM reviews supplier performances and updates the suppliers' rank level. Each supplier is then allocated a certain share of the total size for the immediate period. Again we assume that the suppliers' tastes and production functions are identical and that the production outcomes are independent and identically distributed. We also assume constant returns to scale in the supplier production functions; otherwise one supplier who got a high, early stochastic performance outcome could quickly race ahead of the other, leaving little opportunity for the laggard to recover. We discuss the implications of increasing or decreasing returns in the last section of this chapter.

Without loss of generality, we can assume $p_1 = p_2 = p_0$, thus the OEM solves the dynamic two-supplier volume contract problem (DTV):

$$V^V(q,u) \equiv \max_{a,Q,U} E[q_1 x_1 + q_2 x_2 - 2p_0 q_0 + \delta V^V(Q(u,x), U(u,x)) \mid a_1(u), a_2(u)]$$

s.t. $\quad E[\phi(p_0 Q_i(u,x)) + \delta U_i(u,x) \mid a_i(u), a_j(u)] - \psi(a_i(u)) = u_i \quad \forall u_i \in \mathcal{V}_i \quad (PK_i) \, i = 1,2$

$\quad E[\phi(p_0 Q_i(u,x)) + \delta U_i(u,x) \mid a_i(u), a_j(u)] - \psi(a_i(u))$

$$\geq E[\phi(p_0 Q_i(u,x)) + \delta U_i(u,x) \mid \hat{a}_i, a_j(u)] - \psi(\hat{a}_i) \, \forall \hat{a}_i \in A \quad (IC_i)$$

$\quad Q_1(u,x) + Q_2(u,x) = 2q_0 \qquad\qquad\qquad\qquad\qquad\qquad (BC)$

$\quad Q_i(u,x) \geq 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (NN_i)$

Note several differences compared to the margin-incentive contract:

i). The objective function of the buyer depends not only on the ranks of the two suppliers, but also on the current volume allocations q. This is intuitively quite straightforward: The value to the buyer from each supplier's performance output

27

depends on how the volume is split between the two. Thus to maintain the Markov property, the state variable has to include q.

ii). The total volume to be split is bounded by a constant $-2q_0$, which can be interpreted as a certain fixed business volume that the buyer uses as performance incentive.

iii). Since the impact of a supplier's volume on its performance value can not be overlooked, we consider a case when the value from a supplier's performance is proportional to the volume it receives.

Proposition 2.4
With risk-neutral suppliers, the optimal volume-incentive contract can be implemented using a series of static contracts. In each period, the optimal split is "all-or-nothing":

$$Q_i^*(u,x) = \begin{cases} 2q_0 & \text{if } a_1(U(u,x)) > a_2(U(u,x)) \\ 0 & \text{otherwise} \end{cases}$$

Proof. See Appendix.□

Such a volume split scheme is an extreme form of relative performance evaluation (RPE) (Lazear and Rosen 1981). This may strike one as a severe outcome, but it is a consequence of the risk neutrality assumption, which we will relax in Section 2.4.

Definition 2.2

$$V^V(u_1,u_2) \equiv \max_{q_1,q_2} V^V(q_1,q_2,u_1,u_2)$$

In general (at least for the case when the fixed margin $p_0 \leq \dfrac{\overline{p}}{2}$), the volume contract has a smaller feasible set than the margin contract. However, this does not imply $V^M(u_1,u_2) \geq V^V(u_1,u_2)$. In a volume contract, the allocation of volume affects the expected output because a supplier's output is proportional to its business share. Therefore, by allowing the volume to be dynamically adjusted along with the ranks, the buyer can potentially improve the expected output even if the suppliers input the same effort level in equilibrium as in the margin contract.

2.4 Numerical Studies (Risk-Averse Supplier(s))

A risk-averse supplier prefers to avoid the kind of all-or-nothing schemes of the previous section and desires more predictable and stable rewards. In the case of volume-incentive contracts, risk aversion translates to preference for production smoothing, i.e., companies would prefer stable and smooth demands for their services over lumpy and sparse demand.

Since our model does not yield closed-form solutions for the case of risk-averse suppliers, we use computation of numerical examples to explore optimal dynamic contracts and we use simulations to obtain the suppliers' dynamic performance paths under the optimal contract.

In Section 2.3, we derive some of the theoretical properties with assumptions of continuity and differentiability. However, in order to explore our model computationally, we use discrete action inputs and outputs. Such discretizations serve well to illustrate the key findings of this research.

2.4.1 Margin Contract

Because the two-supplier margin contract can be decoupled into two independent single-supplier contracts, it suffices to analyze the solutions for the single-supplier contract.

Spear and Srivastava 1987 show that with strictly risk-averse agent, V is strictly concave in u and the optimal policy cannot be a simple bang-bang policy. They establish that the optimal contract uses both current and future compensation to incentivize the agent. More specifically, the optimal contract shall equate the marginal rate of substitution of the future and present compensation for the OEM ($V'(U)$) with that for the supplier ($-\dfrac{1}{\phi'(w(u,x))}$). Since the dynamic contract allows the OEM to reward the supplier using both immediate payment and future promises, the optimal contract requires both be increasing in performance. Wang 1997 points out that the reason that the highest-powered incentive (bang-bang policy) is not optimal is due to the agent's need for consumption smoothing. Thus a very high reward or a very severe punishment is delayed to allow for consumption smoothing. Wang 1997 applies the self-generation concept from APS 1990 to the principal/agent model to compute the set of the equilibrium payoffs and then use value iteration methods to solve the Bellman equation and obtain the optimal contract. The self-generation computation involves two stages. In the first stage, the set of equilibrium payoffs for the supplier is calculated. In the second stage, value iteration is used to compute the OEM's optimal value and the optimal compensation scheme. We use a similar computation procedure to obtain the optimal solution.

As in a single-supplier margin contract, the set of equilibrium payoffs for each supplier in a two-supplier margin contract is a closed interval on the real line. The general algorithm for computing the equilibrium set of payoffs for the supplier is as follows:
1. Start with some generous $[\underline{U},\overline{U}]$ and discretize it to a finite set $\{U_k\}$
2. Add an additional constraint $\underline{U} \leq U(u,x) \leq \overline{U}$ to the buyer's optimization problem
3. Find the sequence $\{u_i\} \in \{U_k\}$ s.t. constraints (IC), (PK), (BC), (NN) and the constraint constructed in 2 have feasible solutions. Let n = cardinality of $\{u_i\}$
4. Let $\underline{U} = u_1$ and $\overline{U} = u_n$ and repeat 2 until $\{u_i\}$ converges

The set of feasible equilibrium payoffs is then characterized by the interval $[u_1, u_n]$.

We apply the above algorithm to the following example:

Two possible action inputs: a_H, a_L (high effort, low effort)

Two possible outputs: x_H =$25, x_L =$5

q_0 = 1 million

The output depends on the action input probabilistically:

$P(x=x_H|a=a_H)=0.67$

$P(x=x_H|a=a_L)=0.33$

Cost of effort $\psi(a_L) = 0$, $\psi(a_H) = 1$ million (supplier disutility of effort)

Discount factor $\delta=0.85$

$\underline{p} = 0$ and $\bar{p} = \$25$

The supplier's utility function is $\phi(w) = \sqrt{w}$

In this example, high output implies a value of $25 million to the buyer whereas a low output a mere $5 million.

Step 1 is to choose the initial starting interval. We let $\underline{U}$ be the total expected payoff of the supplier when it inputs high effort (a_H) but receives minimal margin in each period. Let $\bar{U}$ be the payoff when the supplier inputs low effort (a_L) but is rewarded with a margin that is equal to .67x_H +.33 x_L in every period[9]. These payoff values serve respectively as the initial lower and upper bounds of the feasible set. To facilitate the computation, we discretize the interval into 30 points; the initial lower and upper bounds become states 1 and 30 respectively. Recalling the consulting firm compensation system example from Section 2.1, we can think of these discrete states as the relative ranking of the supplier by the manufacturer.

Applying self-generation and the algorithm described above, we can solve for the fixed point solution of the equilibrium set. For this example, the set of equilibrium payoffs that satisfies the self-generating property is the interval from state 6 to state 29, which corresponding to an expected utility payoff of 0.23 million and 31.95 million respectively. Thus the feasible region for the two-supplier margin contract is a square region.

Across this region, the optimal action for the supplier is to expend a high level of effort with the exception of state 6 and 21-29. Therefore this numerical example describes a supplier that is "pessimistic" at low utility promise and "lazy" at high utility promises, or equivalently, a buyer that finds it too expensive to induce buyers in such states to input high effort.

Figure 2-3 describes the efficient frontier of a money contract. In general, as more is promised to the supplier, the less value the OEM retains. Note that the horizontal axis represents a relative scale obtained by discretizing $[\underline{U},\bar{U}]$ into 30 points. Thus 5 and 30 correspond to an expected payoff of -1.15 million and 33.3 million respectively. For

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[9] .67x_H +.33 x_L is the expected unit output for a supplier that inputs high effort. Hence $\bar{U}$ stands for the case when the supplier exerts low effort but is paid an amount that is equal to the expected output under high effort.

consistency, throughout the rest of this chapter, we use the same scale for a supplier's expected payoff (or rank).
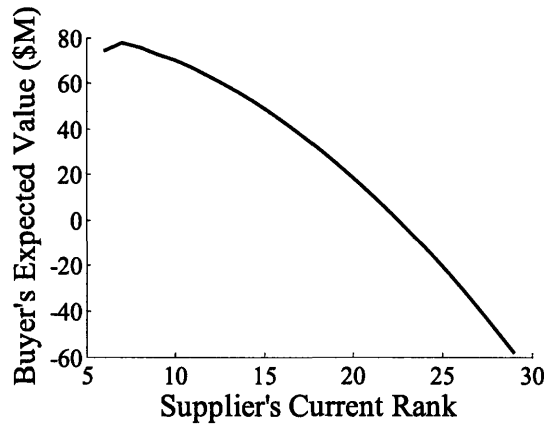
Figure 2-3: Pareto Frontier of a Margin Contract

Figures 2-4a and 2-4b show that the optimal contract utilizes both current and future incentives. The solid curve is the margin (or future rank) assignment when high performance is achieved and the dashed curve is that when low performance is achieved. In the low effort region (rank 6, 21-29), there is no differentiation for current period performance, i.e., the buyer allocates next period margin and rank regardless of the performance outcome. Rank 6 is a "threat" for suppliers. For a supplier starts in a higher rank than 6, if its performance deteriorates over time, there is a chance of falling into such a low-rank trap and never gets out. Ranks 21-29 are "temporary": The supplier receives high margin award but is demoted continuously to a rank where they would have to exert high effort. If a supplier starts below rank 20, it will never get above 20. In other words, these high ranks are not sustainable even if a supplier can negotiate itself into one. The supplier must either agree to move to the low ranks after taking advantage of high margins for some limited number of periods, or break out of the contract.
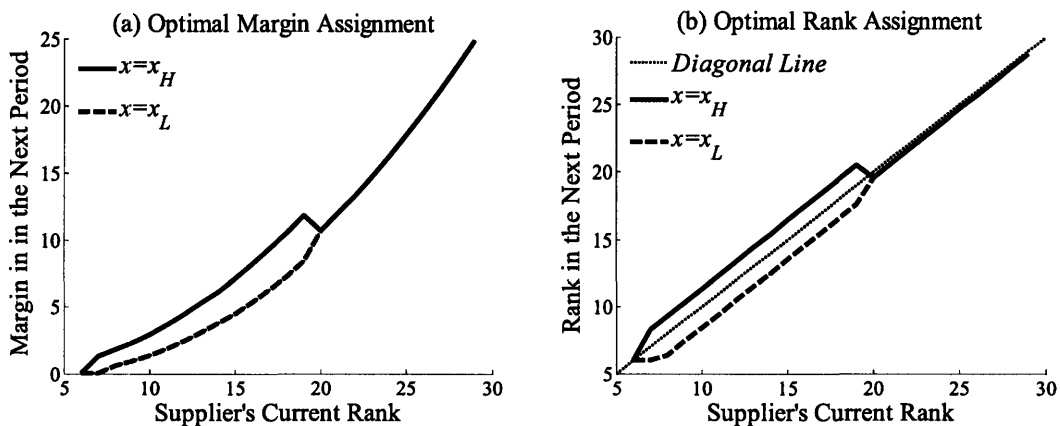
Figure 2-4: Optimal Contract Structure of a Single-Supplier Dynamic Contract

Under the optimal policy, a contract could start from any initial u (expected utility promised to the supplier) that is within the set $\mathcal{V}$. Each state corresponds to a unique compensation scheme as shown in Figure 2-4. In each period, the OEM measures the supplier's performance, which is probabilistically influenced by the choice of effort level. Based on this period's performance and the current state u, an immediate margin assignment is made (Figure 2-4a) and a new continuation value is promised to the supplier, which moves the supplier to a new rank (Figure 2-4b). Over time, a supplier moves from rank to rank – and one may infer from the supplier's state path how well it has been performing. The supplier's "performance path" is analogous to the career path of an individual over her lifetime. The equilibrium performance path of a supplier is determined only by future payoff promises, i.e., $U(u,x)$. Figure 2-5(a) shows a sample path of a supplier entering the contract at state 6. Figures 2-5(b) and 2-5(c) are the sample paths for a supplier starting from state 14 and 29 respectively. Note again that state 6 is a trapping state: If a supplier starts from this state, it stays there forever, resulting in equilibrium with low effort input in all subsequent periods. When starting from states 14 or 29, the supplier quickly settles into an oscillating mode around rank 20, although there is always a small probability that it may fall into the trapping state 6. We also note that states 20 and above represent feasible, but not sustainable ranks. A buyer could credibly promise a supplier such a rank when the parties enter into a contract, but the only way to sustain the contract is for the buyer to offer high margins initially and then scale it down to a lower rank level.
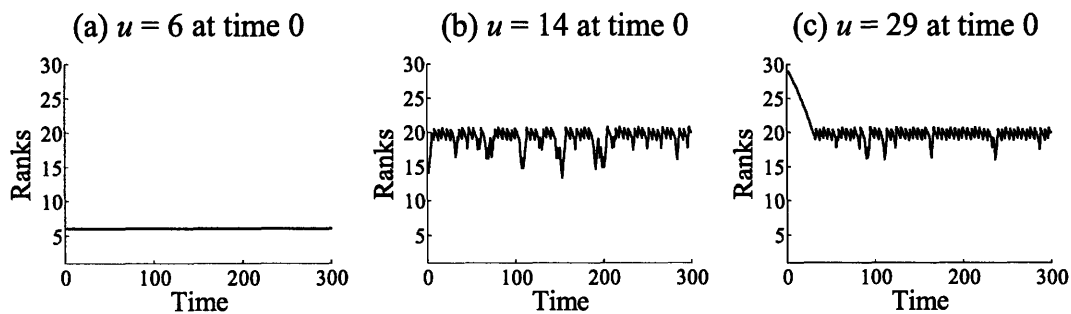
Figure 2-5: Sample Paths of State Transitions in a Single-supplier Margin Contract

Figure 2-6 shows the state distribution after 100 time periods for a supplier starting from rank 14. We observe two peaks - one around rank 20, the other at rank 6. For a supplier at state 14, therefore, the long-term threat for not performing is falling to rank 6 and never able to move up in ranks. The positive incentive is to over time move to the area around state 20 by merit of good performance. Therefore, these peaks serve as the carrot and stick in long-term supply relationships.
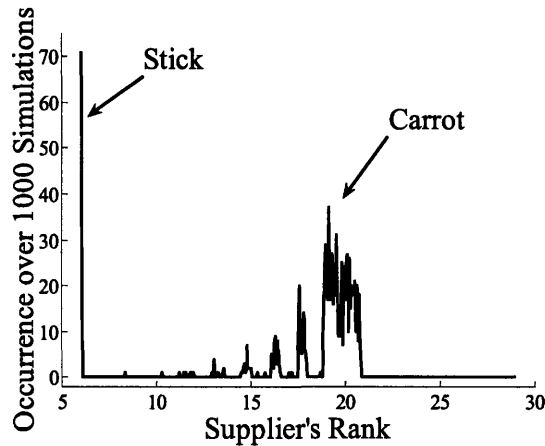
Figure 2-6: State Distribution after 100 Periods

There are some interesting lessons we could learn from the optimal margin contract. First, generosity pays off. Using a different set of parameters for the numerical example above, we observe that it may be optimal for the OEM to promise a supplier higher expected future payoff (i.e. a promotion) regardless of performance, especially when the supplier is at a low initial rank. This helps avoid a *no-incentive↔no-performance* equilibrium. Second, less differentiation at the higher end of the spectrum also makes sense. That is, in equilibrium, the OEM explicitly reduces a supplier's rank (i.e., a demotion) regardless of current performance in order to prevent them from slacking off in subsequent periods. These non-differentiation policies at the bottom and at the top seem to be the key in promoting healthy supply relationships. They give the low performer a chance to come back and prevent the high performer from slacking off.

2.4.2 Volume Contract

In a two-supplier volume contract, when the suppliers are risk averse, the winner-take-all policy is not optimal anymore because it results in very uncertain production volumes. With symmetric suppliers, production smoothing favors an allocation policy such that each gets an equal share of the business, but the moral hazard problem due to hidden effort requires incentive. Consequently our expectation of the optimal policy is that it should fall somewhere in between, i.e., a compromise of a winner-take-all policy and an equal-sharing policy.

Figure 2-7 illustrates the set of feasible equilibrium payoffs of the suppliers under the dynamic volume-incentive contract for various fixed unit margins. In comparison to the margin-reward contract, the feasible set takes on a different shape because of the equality constraint on volume. The feasible set changes with the fixed unit margin associated with each volume contract. When the unit margin is low, the feasible set is close to the origin; when the unit margin is high, the feasible set is further away from the origin. These feasible sets are each a subset of the feasible regions in the corresponding margin contract (which is the square region bounded by $u_1, u_2 \in [6,29]$.

33
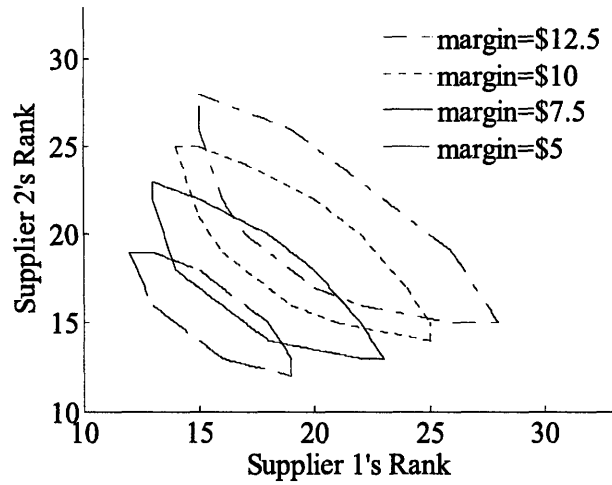
Figure 2-7: Set of Equilibrium Payoffs vs. Margin

In the volume contract, the effort choice of a supplier depends on many factors: the supplier's own rank level, the competitor's rank level, and the current volume allocation (Figure 2-8). In Figure 2-8(a), we assume a fixed unit margin $p_0 = \$10$; in (b), we assume a fixed unit margin $p_0 = \$5$.

(a) $p_0 = \$10$

(b) $p_0 = \$5$
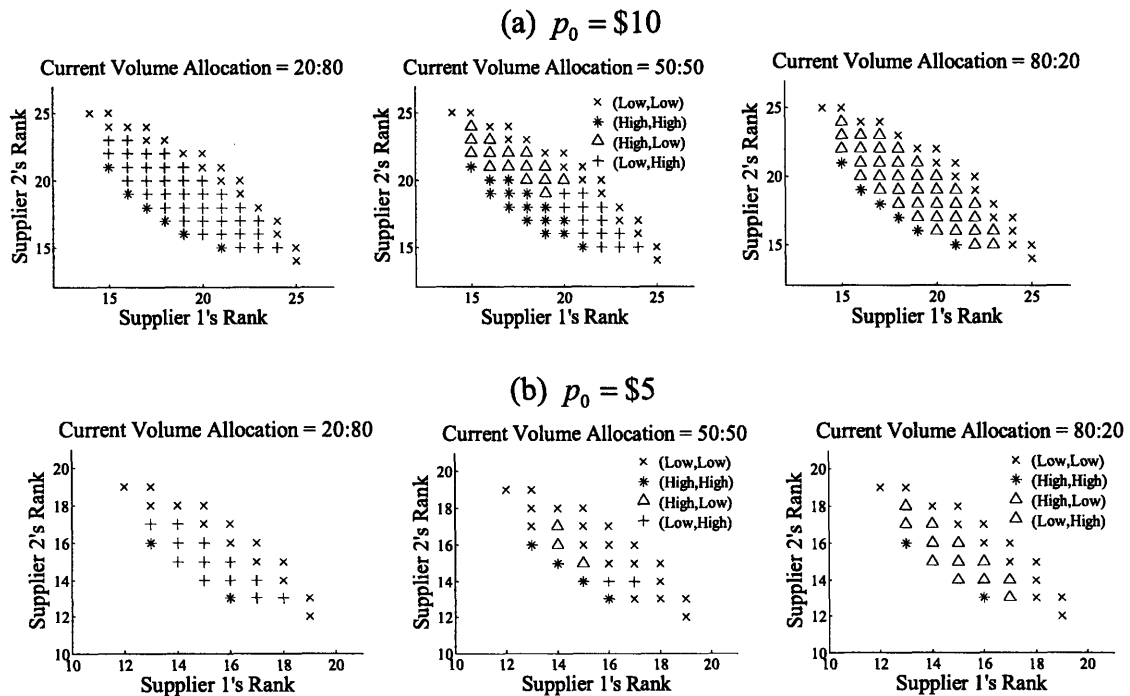
Figure 2-8: The Set of Equilibrium Payoffs and Optimal Efforts under a Volume Contract

34

Note that in the margin contract, states 21-29 are in the low effort region, where it is too expensive for the buyer to induce high effort input. In comparison, the buyer is able to induce at least one supplier to input high effort in many of these states under a volume contract with fixed unit margin of $10.

In comparison, states 12-20 are in the high effort region under a margin contract. However, in a volume contract with fixed unit margin of $5, the suppliers do not necessarily input high effort in equilibrium – in some cases only one supplier inputs high effort, in others both are expected to slack off.

Compared to the margin contract, the supplier's equilibrium effort level exhibits a common pattern within the feasible set: in states close to the origin, higher efforts are more likely and in states further away from the origin, lower efforts are more prevalent. This indeed is a direct result of the forced supplier competition in a volume contract. Each supplier's effort choice depends much more on the relative ranks and current volume splits, rather than on the absolute size of the incentive or rank.

The current volume allocation also affects the supplier's optimal effort choice. For example, Figure 2-8 shows that supplier 2 has an incentive to (and is expected to) input high effort much more frequently under the 20:80 split than the 50:50 split.

2.4.3 Comparison

To compare the margin contract with the volume contract, we need to establish some criteria for comparison. One option is to see which contract offers a larger total profit for the supply chain. To do that, we compare the optimal values achieved by the buyer when each supplier expects a given expected payoff. For a given pair of ranks promised to the two suppliers, the difference in the buyer's optimal value under a volume contract and under a margin contract is shown in Table 2.2.

a) $V^V(u_1,u_2) - V^M(u_1,u_2)$ (in $M) for a volume contract with fixed margin $5

|	12	13	14	15	16	17	18	19
12								-8.9
13					-2.1	2.6	1.2	-8.7
14				5.6	4.4	1.1	-6.0	
15			5.6	4.5	1.9	-4.4	-9.2	
16		-2.1	4.4	1.9	-1.6	-8.1		
17		2.6	1.1	-4.4	-8.1			
18		1.2	-6.0	-9.2				
19	-8.9	-8.7						

b) $V^V(u_1,u_2) - V^M(u_1,u_2)$ (in \$M) for volume contract with fixed margin \$10

	14	15	16	17	18	19	20	21	22	23	24	25
14												-5.0
15								-2.1	5.3	5.9	6.2	-3.1
16						1.7	7.7	8.8	8.3	6.4	-0.3	
17					2.0	8.6	9.9	9.8	8.3	7.8	-1.0	
18				2.0	8.6	10.5	10.8	9.6	8.8	1.0		
19			1.7	8.6	10.5	11.0	10.2	9.3	2.4			
20			7.7	9.9	10.8	10.2	7.9	3.0	-0.3			
21		-2.1	8.8	9.8	9.6	9.3	3.0	-0.1				
22		5.3	8.3	8.3	8.8	2.4	-0.3					
23		5.9	6.4	7.8	1.0							
24		6.2	-0.3	-1.0								
25	-5.0	-3.1										

Table 2.2: Value Difference to the Buyer under Volume and Margin Contracts

Apparently a volume contract may increase or decrease the total value for the contractual parties as opposed to a margin contract. For example, when the suppliers are at state (18,20), a volume contract with fixed margin of \$10 yields \$61M for the buyer, much higher than the \$50M realized under the margin contract (a difference of \$11M). However, when the suppliers are at state (14,25), the volume contract yields \$5M less than the margin contract. We also observe that the volume contract works best when the two suppliers have somewhat close ranks.

The lesson seems to be that if a buyer is able to negotiate the suppliers into lower ranks, then a margin contract is better. Otherwise, the buyer would be better off using a volume contract. In a margin contract, the suppliers will input high effort in the region of 7-20 and low effort in ranks above 20. A volume contract overcomes the incentive problem for suppliers with higher ranks and is often able to induce at least one of them to input high effort at these high ranks, thus leading to higher values to the buyer. In the low-rank region, the buyer is able to induce high effort in the margin contract, whereas the volume contract, due to lack of enough punishment power, can not always induce both suppliers to invest high effort. Therefore, the margin contract tends to outperform the volume contract in the low-rank region. In addition, if there is a big gap between the two suppliers, it is better to use a margin contract and incentivize the suppliers independently, rather than forcing them into a volume race.

Another reason that a volume contract can often achieve higher value is due to its ability to allocate higher volume to the supplier that is more likely to input high effort. Because the performance value is proportional to a supplier's business volume, it ought to benefit the buyer to be able to increase the impact of the high performer by giving it more business.

Further, when the comparison is based on the margin, i.e., when we compare a volume contract and margin contract with the same minimum margin requirement, the volume contract is clearly superior. For example, with a volume contract that promises a fixed

margin of $10, the buyer's least expected value is $22M, whereas the buyer's highest possible long-run value is only $4M under a margin contract with a minimum margin requirement of $10. This is because the buyer's ability to punish a supplier is significantly impeded when it has to offer a minimum margin of $10 under a margin contract. In contrast, a volume contract with a promised margin of $10 does a much better job in aligning the incentives by adjusting the volume allocated to the suppliers.

Compared to the money-reward contract, the volume incentive carries much stronger characteristics of relative performance evaluation (RPE). In Figure 2-9, in nearly all the states, supplier 1 receives significantly more (the vertical axis is the difference between the total expected pay under the two outcomes) when supplier 2 performs poorly than when it performs well.
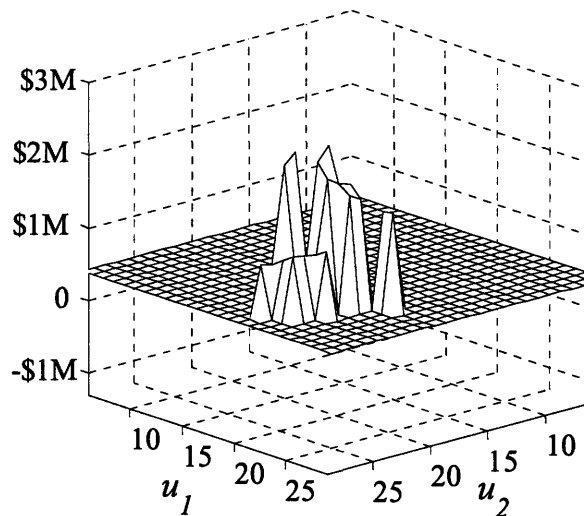
Figure 2-9: Difference in Supplier 1's Total Payoff under Performance Output
(H,L) and (H,H)
(with fixed unit margin $10 and volume split 50:50)

Under the optimal dynamic volume-incentive contract, the rank distribution forms a bimodal pattern over time, clustering around the two corners of the feasible set. There is a similar dichotomy pattern in the volume split. Figure 2-10 shows the volume and rank distribution after 50 contract periods. The volume split peaks around a bit over 0.2 and also around 0.8 (Figure 2-10a). The ranks peak around (16,24) and (24,16) (Figure 2-10b). Therefore, for supplier 1, the reward (carrot) for performing well relative to supplier 2 is the prospect of over time moving to a state with ranks around (24,16) and volume split around 80:20. Likewise, the threat (stick) for poor performance is to end up in a state with ranks around (16,24) and volume split close to 20:80. Also unique of the volume contract is that one supplier's carrot is exactly the other's stick, reflecting the relative performance evaluation (RPE) characteristic.

37

(a) Volume (b) Ranks
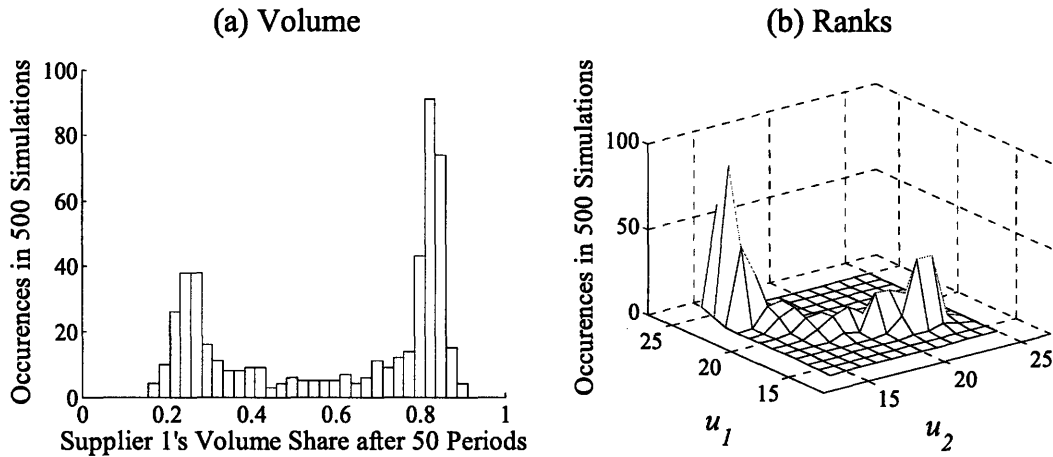
Figure 2-10: State Distribution

2.5 Discussion

In summary, we examine two options of dynamic performance-based contracting structures between a manufacturer and its suppliers. In particular, we develop a model and a computational approach to analyze optimal history-dependent dynamic contracts under a dual-sourcing strategy. We compare a volume-incentive contract with a margin-incentive contract. In addition to the optimal contract structure, we also obtain the equilibrium performance and payment trajectories of the suppliers under the optimal contract. As a result, our model captures the evolutionary dynamics of the supplier relationships under each type of contract.

Volume contracts may increase the total profit for the supply chain, partly due to its ability to allocate higher volume to the supplier that is more likely to input high effort. In addition, the margin contract fails to provide incentive when the suppliers have high ranks (promised high expected payoffs). A volume contract overcomes this through relative performance evaluation. However, the volume contract lacks the ability to punish both suppliers at the same time. Therefore, a margin contract tends to outperform the volume contract when the suppliers have low ranks (promised low expected payoffs). In addition, if there is a big gap between the two suppliers, it is better to use a margin contract and incentivize the suppliers independently, rather than forcing them into a volume race. Compared to the money-reward contract, the optimal volume-incentive contract carries stronger characteristics of relative performance evaluation (RPE).

Our results also indicate that the optimal contract is not always "fair". The structure of the optimal contract suggests that leniency toward the low performer is sometimes desirable. It may be optimal for the manufacturer to promise a supplier higher expected future payoff regardless of their performance to avoid being trapped to a *no incentive↔no performance* equilibrium. Likewise, it may be more profitable in some cases to reduce or cap a supplier's future payoff promises despite strong performance, in order to prevent that supplier from reaching a trapping "lazy" state.

The optimal contracts derived in this chapter require the manufacturer to be able to fine-tune the payments and promises to the suppliers at each possible state u. In practice this may be very difficult, and clearly impossible if a continuous state space is assumed. Therefore future research might productively consider incentive contract schemes that are easier to implement. Consider the incentive system in the consulting firm example from Section 2.1, which explicitly uses discrete levels and step-function bonuses. In a supply contracting context, the manufacturer might explicitly restrict the possible states to a few levels, say, bronze, silver, gold and platinum, each corresponding to some expected total utility payoff. In each contract period, a supplier may receive a fixed amount of bonus (margin/volume) or no bonus depending on her and her competitor's rank levels and performances. In addition, the manufacturer can decide to move the supplier from one rank level to another. Unlike the pay system for the consultant where only promotion is allowed, we allow both promotion and demotion. Given such a contract, we can explore 1) whether or not a PPE exists; 2) what the optimal restricted contract prescribes; 3) how the number of levels affects the efficiency of the contract.

For simplicity, we assume away the effect of scale economies when considering the volume-incentive contract. However, in many industries, production volume has a direct effect on cost. The supplier who is rewarded a higher volume often has a better chance of achieving lower unit cost in the next period. If the performance measure is cost, then the production function will depend on previous allocations and will not be memoryless. The model formulated in this chapter does not address such a dependency although we should be able to extend it to accommodate a one-period dependency easily. [10]

Also implicit in our model is that performance is solely an indication of effort but not of capability. The simplification allows us to focus on incentive issues in a dual or multiple-sourcing strategy. In future research, we may explore how inferences of supplier capabilities affect the optimal incentive scheme.

In this chapter we study margin and volume contracts as two distinct types of contract. It will be of interest to examine the effectiveness of a mixed contract, where both margin and volume incentives are employed. A mixed contract may offer potential for the most effective incentive contract design. However, the complexity of computing the optimal contract will increase dramatically. Resorting to a finite number of discrete levels may help simplify the task.

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[10] Same applies to the case of decreasing returns to scale.

Chapter 3

Optimal Planning Quantities for Product Transition

3.1 Introduction

This research is motivated by collaborative work with a telecommunications equipment manufacturer in the U.S. New products account for about one third of the company's revenue stream. A typical product lifecycle ranges from 15 to 24 months. Since the company carries very broad product lines, operations and purchasing managers constantly find themselves making planning decisions for product transitions. In this industry, lead-time for critical components is about 13 weeks, sometimes even longer. The production lead-time is usually 5 weeks. As a result, planning decisions need to be made as early as 4-6 months before the actual transition. A product transition starts when the new product is released and ready to be shipped to customers. The end of a transition is reached when the demand rate of the new product stabilizes and that of the old drops to a negligible level. This can take a few weeks or a few months. Long lead-time makes replenishment during the transition impossible or very expensive. Suppliers may double the price for expediting since they need to take resources away from other customers and have to pay their suppliers for expediting. The challenge for the purchasing organization is to plan the right amount for both the new and old products. Such decision is further complicated by the use of product substitution during the transition: The product manager needs to decide when to offer substitution to customers, as well as to understand how substitution affects the initial planning quantities. In cases when the company is already carrying large inventories of the old product, the managers have to determine whether and by how much to delay the new product release in order to avoid huge excess costs. This research sets out to address the above key planning issues.

As mentioned in Chapter 1, introduction of a new product to the market often creates high uncertainties in both the demand side and the supply side. The launch of a new product involves many sequential steps and depends on many factors including technology, production, supplies, etc. As a result, a new product release schedule often slips. When this happens, the inventory of the old product may run out, creating a supply gap. For example, one of our collaborating company's wireless access points (WAPs) products named Blofeld, was being replaced by its next generation successor Blofeld2. The transition was driven in part by an end-of-life (EOL) notice from a component supplier for Blofeld. The old product, Blofeld, did not meet the RoHS standards (an environmental legislation requirement that restricts the use of hazardous substance in electronics, which would become effective after specific dates, varied by country and region) and could not be sold after a certain date. Therefore, the company did not buy many old products on the EOL notice, hoping that Blofeld2 could quickly take over the demand. Unfortunately, the release of Blofeld2 was delayed, a supply gap became inevitable. Since it was impossible to replenish Blofeld, the company was scrambling to

complete Blofeld2 and trying to get it out to the customers sooner in order to narrow the supply gap.

To counteract such supply risks, operations managers tend to add large buffer inventories for the old product. The downside of a generous supply cushion is that the company may end up with excess inventories of the old product that eventually have to be written off. In another example of a product transition (Sultan→Sultan2) at the company we work with, the old product Sultan was selling very well before the transition. Unwilling to risk any supply gap and miss revenues, the managers decided to make large additional purchases of the old product as buffer inventory. This time, it turned out that they bought too much as the new Sultan2 project completed on schedule. Therefore, the real challenge in making planning decisions for a product transition is to balance the risk of shortage against the risk of excess. In this chapter we develop a dynamic programming model to find the optimal planning quantities of the old and the new product, taking into consideration uncertainties in demand and in the transition start date.

Risk of demand and supply mismatch is high during product transition. However, the fact that the old and new products share a common customer pool offers opportunities to mitigate the risk. Substitution is one way of coping with the volatile demand situation during a product transition. When shortage of the old product occurs, the new product can be used as a substitute. Substitution in the opposite direction is also possible: a customer of the new product might take the old when the new is not available. But substitution using the old for the new usually implies a temporary solution before the customer can be soon served with the new product. For this reason, we only consider substitution using the new for the old in this chapter. In comparison to the existing literature on substitution, we view each substitution decision as a dynamic problem: it depends on the current inventory levels of the old and new products and how much time is left for the transition. Substitution provides risk pooling, but it adds complexity to the inventory planning problem. Our model analyzes the optimal planning decisions, taking into consideration such a dynamic substitution policy.

Lastly, we address a closely related problem that product and operations managers constantly face. Sometimes companies can delay the release of the new product to postpone the transition for some period of time. In the second transition example mentioned above, the company delayed the release of Sultan2 in selected sales regions in order to consume some of the excess inventory of the old product. The decision to delay a new product release is primarily driven by the need to deplete excess inventory of the old product to avoid large excess and obsolescence cost. However, the new product typically has better margins and can in expectation bring in higher profit, thus such a delay can also hurt the bottom line. Therefore, the decision of when to launch the new product is a tradeoff of the two conflicting objectives.

The rest of this chapter is organized as follows: Section 3.2 is a review of the literature. In Section 3.3, we describe the problems and assumptions. In Section 3.4, we present the model with a deterministic transition start date (TSD). We extend the model to a case with a stochastic transition start date in Section 3.5. In Section 3.6 we solve a problem of

determining the optimal transition delay, given the inventory level of the old product. In the last section, we discuss the business implications and future research.

3.2 Literature Review

There has been much work done on last time buys that deals with the disposal of the old generation product. Goyal and Giri (2001) provide a review of the literature on managing deteriorating inventory. Rosenfield (1989, 1992) presents a marginal analysis model that solves for the optimal number of units to keep for products that are slow-moving or obsolescent when facing a one-time disposal opportunity. He assumes a Poisson demand with constant rate and obtains results under various perishability assumptions. He later shows that the optimal policy holds even if future disposal opportunities exist. Jain and Silver (1994) develop a dynamic programming model that determines the optimal ordering policy for a product with a random lifetime and stochastic but stationary demand. Several other papers study continuous inventory replenishment models for product with random lifetime including Krishnamoorthy and Varghese (1995), Kalpakam and Sapna (1994), and Ravichandram (1995). Song and Zipkin (1996) study the inventory control problem under deteriorating demand situation although in their model the demand is modeled as discrete Markov states. Their analysis is constrained to probabilistic transitions among low and high demand states. In comparison, we allow the expected demand rate to change continuously.

Wilson and Norton (1989) study the optimal entry timing for a product line extension. They show that the timing of the new product introduction affects the sales pattern of both products. The tradeoff that they focus on is the market growth stimulated by the introduction of the new product and the sales cannibalization of the old product due to the product line extension. We choose to ignore the impact of timing of the new product introduction on subsequent transitional demands as we are primarily interested in the operational tradeoffs (excess inventory vs. improved margin from the new product) when determining the optimal transition start date for a product transition.

The paper that examines a problem that is closest to ours is by Wilhelm and Xu (2002). They consider a multi-period production planning and pricing problem in the context of product upgrade when there are replenishment opportunities in each period. The decision variables in their dynamic programming formulation include whether or not to start an upgrade in that period, the production quantity for the product that is selling and the price of that product. Their analysis is made under the assumption that only one product is being sold at any point of time. In comparison, we look at a particular upgrade; allow both the old and new generation products to be sold throughout the duration of the transition period; and solve the optimal planning problem when there is no additional replenishment opportunity due to the long lead-time.

There are two streams of literature on inventory models with substitutions. One focuses on static substitutions in a single-period or repeated newsvendor problem. Parlar and Goyal (1984), Pasternack and Drezner (1991), Bassok et al. (1999) consider a single-period multiple-product inventory model with substitutions. Rao et al. (2002) study a

similar problem while incorporating the setup costs. Ignall and Veinott (1969) study ordering decisions for the repeated newsvendor problem and show that myopic order-up-to policy can be optimal. Bitran and Dasu (1992) model a multi-period production planning problem with demand substitutions considering uncertainties in both supply and demand. In the above mentioned substitution models, demand in each period is realized all at once and substitutions are essentially through ex post allocation of the inventories. Another stream of substitution literature couples the customers' choice model with the newsvendor problem to derive optimal order quantities. Smith and Agrawal (2000) consider a single-period multi-item stocking problem using demands modified by customers' substitution effects. Mahajan and van Ryzin (2001) analyze the same problem with dynamic substitution behaviors of the consumer, i.e., the customers' substitution choice is dependent upon the current inventory levels. Although this is in spirit similar to the substitution models developed in this research, we look at substitution decisions that are initiated by the firm, instead of the customers. It is assumed that customers are always willing to take a newer generation product as a substitute for an old product. However, they may be entitled to a compensation for not receiving their first choice, which is reflected in our model through the substitution cost.

3.3 Problem Description

In this chapter we develop models that solve the following three problems:

1. Given a deterministic transition start date, what are the optimal planning quantities for the old and new product?

   The decision variables are the optimal initial inventory positions for the old and new products and whether or not to offer substitution to customers. As substitution is a dynamic decision, we find conditions for substitution at any give time t during the transition.

2. Given a stochastic transition start date, what are the optimal planning quantities for the old and new product?

   We assume the distribution of the transition start date (TSD) to be discrete and bounded, i.e., the transition date can only take on discrete values and there is an upper and lower bound. We compare the optimal solutions with the case of deterministic transition date to explore the impact of uncertainties in transition date on optimal planning quantities and the total expected profit.

3. Given an initial scheduled transition date and the initial inventory liability of the old product, is it preferred to delay the transition? If so, by how much?

   As in problem 2, an upper bound on the transition date is assumed. We develop two alternative methods to obtain the optimal delay when facing excess inventories of the old product.

We look at a finite time horizon, from the time when the planning decisions need to be made to the end of the product transition (Figure 3-1). We define τ to be the date when transition starts and T to be the duration of the transition, i.e., the transition ends at $\tau+T$. Lead-time is longer than T, thus we make the assumption that there is only one order opportunity and it takes place at time 0. We refer to time 0 as the planning time.

44

We assume that demands of the new and old product are independent of each other. The demand of each product is assumed to be Poisson, with rate varying over time. The transition period is characterized by demand overlap of the new and old products. The company is selling both products during this time as customers gradually switch over from an older generation product to a new one. The expected demand rate of the old and new products at any given time is assumed to be known. For the new product, the expected demand rate before the transition starts is zero; it is non-decreasing during the transition period and then stabilizes at a constant level. For the old product, the expected demand rate before the transition starts is constant; it is non-increasing during the transition period and then falls to a negligible level after the transition. Figure 3-1 illustrates a typical pattern of the expected demand rate.
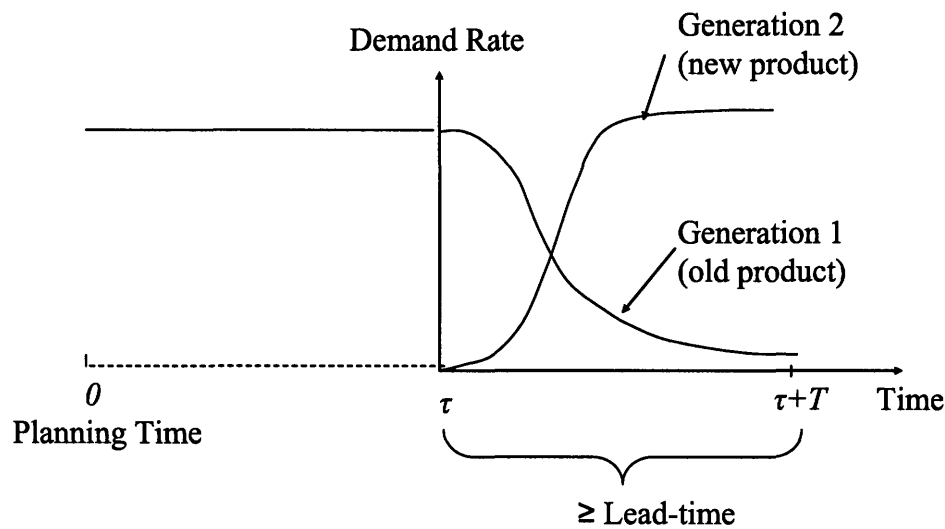
Figure 3-1: Demand Pattern during Product Transition

During the transition, if the old product runs out, the new product can be used as a substitute. However, the company incurs a cost on each substitution. The cost may be a result of discount given to the customer that is offered a substitute, or simply a loss of goodwill for giving a customer his/her second choice[11]. Unmet demand is lost. Excess inventory can only be salvaged after the transition.

The assumption of no salvaging before the end of the transition is appropriate in the high-tech industry. Products are frequently salvaged through the brokerage market. Once a company stops shipping a product due to product upgrade, it dumps the old products to a broker who stocks the product and sells them to the secondary market. The manufacturer can recoup about 10% of the original sale value of the product through this route. Therefore, one obvious reason that companies rarely salvage an old product before the transition starts is the fear of sales cannibalization (of both the old and new products).

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[11] We contend that one can discount the new product to a price below the old product when using it as a substitute. The implicit assumption in our model is that to a customer who desires the old product, the value of the new is no greater than that of the old. This may not be true in all contexts.

Consequently, in this model, we make the assumption that inventory can only decrease when a sale occurs and the option to salvage before the transition ends is excluded.

3.4 Model I (Deterministic Transition Start Date)

We start with a simple case when the new product release date (equivalently, the transition start date) is deterministic. Sometimes a product transition is driven by outside forces: A supplier is announcing end-of-life of a product or component; the government is issuing new regulations which the old product does not comply. In these cases, the company does not have much latitude in deciding when to start the transition, thus the transition start date is considered deterministic and given.

A transition starts when the new generation product is released (time τ) and ends when the demand of the new product is stabilized (time $\tau+T$). We study a case when there is no replenishment opportunity during the transition period. If the sum of the procurement lead-time and the production lead-time is L periods, then any inventory and production decisions regarding both the new and old products have to be made L periods ahead of the transition start date τ. As L is greater than T, replenishment orders placed after τ will not be available until after the transition. Our objective is to find the optimal starting inventory levels of the old and new products.

We use dynamic programming to formulate the problem as a discrete time model. Given that demand is Poisson, we define the time unit such that the probability of more than one demand arrival (old or new) during one time unit is negligible. The time unit might in practice corresponds to a week, a day or an hour. We are aware that the problem can be formulated in continuous time. However, we set it up using discrete time for the purpose of getting insights through computation. Given the time unit, let $\lambda_1(t)$ and $\lambda_2(t)$ be the demand rates for the old and new products respectively. The probability that a demand of the old product arrives during each time unit is $\lambda_1(t)$; the probability that a demand of the new product arrives is $\lambda_2(t)$ and the probability that no demand arrives is $1-\lambda(t)$ where $\lambda(t)=\lambda_1(t) + \lambda_2(t)$. Note that we assume $\lambda(t)<1$. Without loss of generality, we assume a sale (if any) always occurs at the beginning of each time unit. We also introduce the following notations:

r_1 – selling price of the old product

r_2 – selling price of the new product

p_1 – additional penalty for a shortage of the old product (in addition to the lost revenue)

p_2 – additional penalty for a shortage of the new product (in addition to the lost revenue)

$v_1 =r_1+p_1$ can be viewed as the total cost of a lost sale for the old product

$v_2 = r_2+p_2$ can be viewed as the total cost of a lost sale for the new product

g – substitution cost

h_1 – holding cost for the old product per unit time

h_2 – holding cost for the new product per unit time

s_1 – salvage value of the old product at the end of the transition

s_2 – salvage value of the new product at the end of the transition

c_1 – unit cost of the old product
c_2 – unit cost of the new product
T – length of the transition period
δ – discount factor
x_1 – inventory level of the old product
x_2 – inventory level of the new product
$V(x_1,x_2,t)$ – profit to go at time t given that the inventory levels are x_1 and x_2

Note that the salvage value for the new product is used to incorporate value depreciation of the new product. It does not necessarily imply that the company will salvage any left over units of the new product. In other words, one can interpret this value as the unit cost of producing (or procuring) the new product at the end of the transition.

3.4.1 Substitution Decision

We first note that if there were no demand interactions between the two products, inventory planning could be simplified to two independent problems, with each being a variation of the Newsvendor problem: The transition period $[\tau, \tau+T]$ can be considered a single selling-season. If the holding cost is taken to be a fixed value for all units sold and there are no incremental revenue discounting, we simply find the quantities that satisfy the newsboy ratio for the old and new products respectively.

However, as mentioned in Section 3.1, demand of the old and new generations of product share a common customer pool. A benefit of such overlap is risk pooling through substitution. We need to solve the dynamic substitution decision problem before we can obtain the optimal starting quantities. In our problem, substitution is only relevant when the old product inventory runs out. The substitution decision is made each time a demand for the old product cannot be satisfied. It is a dynamic decision because it depends on the current inventory level of the new product and how far we are into the transition period. If we substitute, we avoid the lost sale of an old product and some holding cost of a new product, but we may risk losing a sale of the new product if the new product inventory runs out before the transition ends. Therefore, we need to weigh the immediate benefit of substituting against the expected future loss from possible missed sales of the new.

At any time t during the transition, the profit-to-go is determined by the following recursive equations:

$$V(x_1,x_2,t) = (1-\lambda(t))[-h_1x_1 - h_2x_2 + \delta V(x_1,x_2,t+1)]$$
$$+\lambda_2(t)[-h_1x_1 - h_2(x_2-1) + r_2 + \delta V(x_1,x_2-1,t+1)] \quad \text{if } x_1 \geq 1, x_2 \geq 1$$
$$+\lambda_1(t)[-h_1(x_1-1) - h_2x_2 + r_1 + \delta V(x_1-1,x_2,t+1)]$$

$$V(x_1,0,t) = (1-\lambda(t))[-h_1x_1 + \delta V(x_1,0,t+1)]$$
$$+\lambda_2(t)[-h_1x_1 - p_2 + \delta V(x_1,0,t+1)] \quad \text{if } x_1 \geq 1$$
$$+\lambda_1(t)[-h_1(x_1-1) + r_1 + \delta V(x_1-1,0,t+1)]$$

$$V(0,x_2,t) = (1-\lambda(t))[-h_2 x_2 + \delta V(0,x_2,t+1)]$$
$$+\lambda_2(t)[-h_2(x_2-1)+r_2+\delta V(0,x_2-1,t+1)]$$
$$+\lambda_1(t)\max[-h_2 x_2 - p_1 + \delta V(0,x_2,t+1),$$
$$\quad -h_2(x_2-1)+r_1-g+\delta V(0,x_2-1,t+1)] \qquad \text{if } x_2 \geq 1$$

$$V(0,0,t) = (1-\lambda(t))\delta V(0,0,t+1)$$
$$+\lambda_2(t)[-p_2+\delta V(0,0,t+1)] \qquad t=\tau,\tau+1,\ \tau+2,\dots,\ \tau+T\text{-}1 \qquad (3\text{-}1)$$
$$+\lambda_1(t)[-p_1+\delta V(0,0,t+1)]$$

We define the terminal value $V(x_1,x_2,\tau+T) = s_1 x_1 + s_2 x_2$, which implies that at the end of the transition period, we recoup the salvage values of units left over. We use the DP formulation to prove our theoretical results, as well as to compute the optimal solutions.

Moreover, we make the following assumptions so that the value function is well-behaved.
   (i) $v_1 > s_1$ (3-2)
   (ii) $v_2 > s_2$ (3-3)
   (iii) $v_1 - g > s_2$ (3-4)
   (iv) $h_1 - \delta s_1 > h_2 - g - \delta s_2$ (3-5)
   (v) $v_2 > v_1 - g$ (3-6)

(i) and (ii) simply mean that the value of selling the old (new) product is greater than its salvage value. (iii) implies that the value of selling a new product as a substitute for the old product is no less than the new product's salvage value. To interpret (iv), we consider its equivalent form $v_1 + h_1 - \delta s_1 > v_1 - g + h_2 - \delta s_2$. This excludes the uninteresting case that it is optimal to use the new product to satisfy all demand, old or new. (v) rules out the possibility that it is more profitable to sell the new product as substitute to customers of the old product than to sell it to customers of the new product.

Since substitution decision is only relevant when the inventory of the old runs out, we can easily derive from equation (3-1) that the optimal policy is to substitute if and only if
$$v_1 - g + h_2 > \delta[V(0,x_2,t+1) - V(0,x_2-1,t+1)] \qquad (3\text{-}7)$$

Condition (3-7) has a simple interpretation: We substitute when the value of the substitution (the LHS term) is larger than the discounted future loss from having one less unit of the new product (the RHS term).

Lemma 3.1
$\alpha(x_2,t) \equiv V(0,x_2,t+1) - V(0,x_2-1,t+1)$ is non-increasing in x_2.

Proof. See Appendix. □

By definition $\alpha(x_2,t)$ is the difference function with respect to x_2 (slope of the discretized value function). *Lemma 3.1* implies that the value function $V(0,x_2,t)$ is concave in x_2 for

48

all t. That is, there is decreasing marginal return of having more inventory of the new product. *Lemma 3.1* and condition (3-7) lead to the optimality of a threshold policy.

Proposition 3.1 (Threshold Substitution Policy)

The optimal substitution policy is a time-varying threshold policy. There exists a threshold level $\bar{x}_2(t)$ such that at time t, when the old product runs out, it is optimal to substitute for the old product using a new product if the inventory level of the new product is above $\bar{x}_2(t)$ and no substitution should be allowed when it is below that level.

Mathematically, $\bar{x}_2(t)$ is defined as the largest inventory level x_2 such that

$\delta\alpha(x_2,t) \geq v_1 - g + h_2$. In the special case when $\delta\alpha(1,t) < v_1 - g + h_2$, we define $\bar{x}_2(t) = 0$.

It is easy to see that $\delta\alpha(x_2,t) < v_1 - g + h_2$ for $x_2 > \bar{x}_2(t)$ and $\delta\alpha(x_2,t) \geq v_1 - g + h_2$ for $x_2 \leq \bar{x}_2(t)$, which we use later in the proofs.

As $\alpha(x_2,t)$ is a function of t, $\bar{x}_2(t)$ should also vary with t. Intuitively, if it is optimal to substitute at period t for a given inventory level x_2, then we should substitute at period $t+1$ for the same inventory level x_2 since the chance of running out and missing a sale of the new product should become smaller as time goes on. Such intuition seems to suggest that $\bar{x}_2(t)$ ought to decrease in t. Indeed, this can be true, but only for certain special cases.

Proposition 3.2

With homogeneous Poisson demand, i.e., $\lambda_1(t) = \lambda_1$ and $\lambda_2(t) = \lambda_2$ $\forall t \in (\tau, \tau + T)$, the substitution threshold $\bar{x}_2(t)$ is non-increasing in t.

Proof. See Appendix. □

Homogeneous Poisson demand of the old and new products would imply that the expected demand for the old product drops instantaneously to some lower level after the new product is introduced and stays at that level throughout the transitional period. Correspondingly, the expected demand for the new product changes from 0 to a constant level after its release.

Figure 3-2 shows how the threshold changes over time under homogeneous Poisson demand. Note that the time axis here is the time elapsed since the start of the transition. The results indicate that if the demand is stationary, the threshold shall start high and may stay at a constant level for a while before it starts dropping in a precipitating manner. As we discussed earlier, further into the transition period, less time is left to sell a product, hence the motivation for substitution becomes stronger. Although the stationary demand scenario does not in general reflect the transitional demand, it offers insight on one particular aspect of the optimal policy. Moreover, it describes the optimal substitution policy for two substitutable products with stationary demands in general (not as a pair of overlapping generations of products).

49
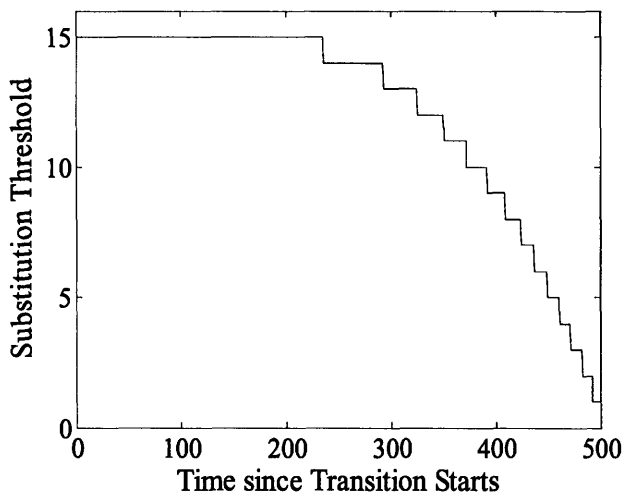
	Parameter Values
T	500
δ	0.998
v_1	68
v_2	50
g	3
h_1	0.005
h_2	0.005
s_1	5
s_2	18
$\lambda_1(t) = 0.08 \ \forall t \in (0,T)$	
$\lambda_2(t) = 0.08 \ \forall t \in (0,T)$	

Figure 3-2: Substitution Threshold under Homogeneous Poisson Demand

Proposition 3.3

If the demand is non-homogeneous Poisson, and the holding cost and revenue discounting are negligible, i.e., $h=0$ and $\delta=1$, then $\bar{x}_2(t)$ is non-increasing in t.

Proof. See Appendix. □

The substitution decision is a tradeoff of the immediate revenue against possible future sales loss of the new product. The expected future loss depends on the probability of running out of the new product and the time when it happens. The latter dependency matters because it affects the holding cost associated with the sale and how much revenue discounting shall be applied to the sale. When we ignore holding cost and revenue discounting, such dependency vanishes. Therefore, the risk of substitution is only affected by the probability of missing a future sale, which is non-increasing in t. As a result, $\bar{x}_2(t)$ is non-increasing in t.

When the holding cost or revenue discounting is significant, $\bar{x}_2(t)$ is not monotonic any more. Figure 3-3 (using parameter values and demand specified in Table 3.1) shows that for the expected demand pattern given in (a), the threshold increases during the early stage of the transition and decreases later during the transition. That is, substitution is favored at the beginning and the end of the transition. Early on during the transition, sales of the new product have not taken off, it is optimal to focus on meeting the old product demand and use substitution when the inventory of the old product runs out. If we instead hold the unit of new product and do not substitute, in expectation, it will take a long time for it to be sold toward a demand of the new, thus high holding cost will be incurred. As time goes on, the demand of the new product ramps up and the expected time to a sale of the new product decreases. Therefore, the substitution threshold increases. Later during the transition, the other dynamic starts dominating: Less and less time is left to sell anything, so we prefer to substitute than to hold as time elapses.

50
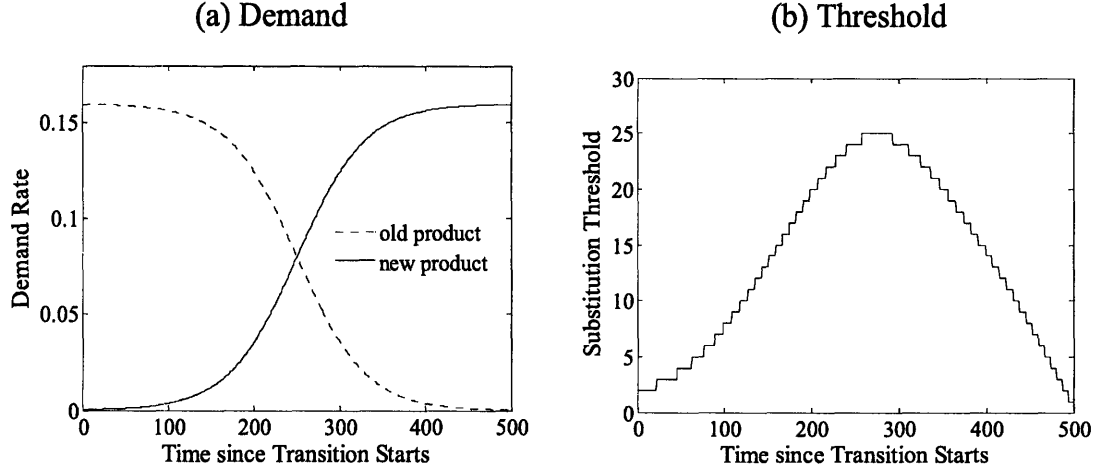
Figure 3-3: Substitution Threshold under Non-homogeneous Poisson Demand

T	500	v_1	68
g	3	v_2	50
s_1	5	h_1	0.005
s_2	18	h_2	0.005
δ	0.998		
$\lambda_1(t) = 0.16/(1+e^{0.025(t-250)})$ $\forall t \in (0,T)$		$\lambda_2(t) = 0.16/(1+e^{-0.025(t-250)})$ $\forall t \in (0,T)$	

Table 3.1: Parameter Values of the Example in Figure 3-3

We establish this formally in *Proposition 3.4.*

Proposition 3.4

If the total demand stays constant, $\lambda_1(t)$ is non-increasing in t and $\lambda_2(t)$ is non-decreasing in t, then $\bar{x}_2(t)$ is a unimodal function of t: It is increasing in t before a certain time $\hat{t}$ and decreasing in t after $\hat{t}$.

Proof. See Appendix. □

3.4.2 Optimal Initial Inventories

So far we have not accounted for the procurement and production costs since these can be considered sunk costs in making the substitution decision. However, such costs need to be taken into consideration when we solve for the optimal planning quantities. The optimal starting inventory levels should maximize the total discounted expected net profit, which is denoted by *NV*. Under the assumption of linear cost, the net profit is defined as

$$NV(x_1,x_2) \equiv V(x_1,x_2,0) - c_1x_1 - c_2x_2 \qquad (3\text{-}8)$$

51

Using equation (3-1) we show that $V(x_1, x_2, t)$ is jointly concave in x_1 and x_2 if conditions in equations (3-2)-(3-6) hold. It follows that NV will also be jointly concave in x_1 and x_2.

Proposition 3.5

$V(x_1,x_2,t)$ is jointly concave in x_1 and x_2. There exists a unique pair of (x_1,x_2) such that $V(x_1,x_2,t)$ is maximized.

Proof. See Appendix. □

The implication of the joint concavity of V with respect to the inventory levels of both products is twofold. First, there is decreasing marginal return to more inventories. That is, the value of an additional unit of old product is higher (lower) when the old product inventory is low (high); the value of an additional unit of the new product is higher (lower) when the new product inventory is low (high). Second, there is decreasing marginal value from substitution: The value of an additional unit of new product is higher (lower) when the old product inventory is low (high); the value of an additional unit of old product is higher (lower) when the new product inventory is low (high). Joint concavity guarantees unique optimal value of (x_1,x_2), allowing us to use simple search methods to find the optimal solution.

Results from several numerical examples we ran indicate that although the company's total expected profit increases with substitution, the total optimal quantities (old and new) can increase or decrease in comparison to the case of no substitutability. Therefore, substitution does not necessarily reduce inventory, contrary to the intuition of many operations managers. A similar result has been obtained in Pasternack and Drezner (1991)'s single-period model. Intuitively, substitution allows the demand of both the old and new product to be met by a unit of the new product inventory. Thus it may be cheaper to stock more inventory of the new product, leading to an increased total optimal inventory. Although the inventory cost might increase with substitution, the company is more than compensated for through increased sales.

3.5 Model II (Stochastic Transition Start Date)

Launch of a new product involves many sequential steps including engineering development and testing (hardware and software), material lineup, manufacturing, and delivery. With inherent uncertainties in each step, the actual date that the new product is released fluctuates, often dramatically. If the new product project slips and the inventory of the old product is depleted, supply gaps can be created. As a result, much larger supply cushion of the old product is needed compared to the case of deterministic transition start date (TSD).

3.5.1 Optimal Planning Quantities under Stochastic TSD

Let τ be the transition start time. τ is stochastic, with $p(\tau=k)$ representing the probability that $\tau=k$. We assume that τ has an upper and a lower bound $\bar{\tau}$ and $\underline{\tau}$ respectively. Such an

assumption on one hand facilitates the computation of the optimal policy and on the other reflects the reality in which the managers make decisions. At our collaborating company, many complain that the new product release is notoriously unpredictable. Nonetheless, nearly all point out that there are usually some kind of upper and lower bounds that they could work with. Figure 3-4 sketches the time horizon we look at. Time 0 is when the planning decisions are made and orders are placed. Note that τ is probabilistically distributed over a bounded range and the duration of the transition is still of fixed length T. We assume that for all realizations of τ, it is not possible to replenish before the transition ends.
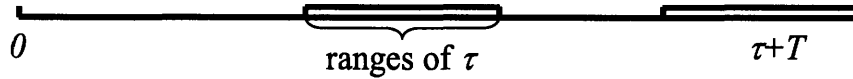

Figure 3-4: Planning Horizon

If we know what the transition date τ is, it is straightforward to compute the total profit. Let $W(\tau, x_1, x_2, t)$ denote the profit-to-go from time t till the end of the transition $(\tau+T)$ for a given τ where x_1 and x_2 are the inventory levels of the old and new product at time t. Since the transition has not started before τ, $\lambda_2(t)=0$ for $t \in (0, \tau)$. Therefore, for any $t \in (0, \tau)$, we can calculate $W(\tau, x_1, x_2, t)$ using:

$$W(\tau, x_1, x_2, t) = (1 - \lambda_1)[-h_1 x_1 + \delta W(\tau, x_1, x_2, t+1)]$$
$$+ \lambda_1[-h_1(x_1 - 1) + r_1 + \delta W(\tau, x_1 - 1, x_2, t+1)] - h_2 x_2 \quad \text{if } x_1 \geq 1$$
$$W(\tau, 0, x_2, t) = (1 - \lambda_1)\delta W(\tau, 0, x_2, t+1)$$
$$+ \lambda_1[-p_1 + \delta W(\tau, 0, x_2, t+1)] - h_2 x_2 \tag{3-9}$$

We define the terminal value $W(\tau, x_1, x_2, \tau) \equiv V(x_1, x_2, 0)$, which can be calculated as in Section 3.4. Thus $W(\tau, x_1, x_2, 0)$ is the expected profit from time 0 to $\tau+T$ for a given transition start date τ. To obtain the expected profit for a stochastic transition start date, we weigh each $W(\tau, x_1, x_2, 0)$ by the probability $p(\tau)$. That is,

$$EW(x_1, x_2) = \sum_{\tau = \underline{\tau}}^{\bar{\tau}} p(\tau)W(\tau, x_1, x_2, 0).$$ Finally, we subtract the procurement costs from W to

obtain the net profit $NW \ (x_1, x_2) = EW \ (x_1, x_2) - c_1 x_1 - c_2 x_2$ \qquad (3-10)
Since $V(x_1, x_2)$ is jointly concave in x_1 and x_2, it is easy to derive that $NW(x_1, x_2)$ is also jointly concave in x_1 and x_2.

Proposition 3.6

$NW(x_1, x_2)$ is jointly concave in x_1 and x_2. There exists a unique pair of (x_1^*, x_2^*) such that $NW(x_1, x_2)$ is maximized.

Given the concavity of NW, the optimal pair of (x_1, x_2) can be easily obtained.

53

3.5.2 Optimal Planning Quantities with Existing Inventories

In previous analysis, we assume that there are no restrictions on the planning quantities. However, in most cases, at the time of decision making (time 0 in Figure 3-4), companies have existing inventories of the old product. The committed inventory could be in the form of finished goods or unique components. These include inventories that the company physically holds, inventories in transit, and inventories at the suppliers, the distributors and other channel partners. Basically anything for which the supplier is liable must be included. Although in practice different types of liabilities often imply different sunk costs, we do not make such distinctions and treat all of the above as finished good inventories.

Let $NW_C(x_1, x_2, x_1^0)$ be the net value when the inventory liability of the old product at time 0 is x_1^0. Because we assume that the option to salvage before the transition ends is excluded (see Section 3.3), the objective function in (3-10) can be modified to:

$$NW_C(x_1, x_2, x_1^0) = \begin{cases} EW(x_1, x_2) - c_1(x_1 - x_1^0) - c_2 x_2 & \text{if } x_1 > x_1^0 \\ EW(x_1^0, x_2) - c_2 x_2 & \text{otherwise} \end{cases} \qquad (3\text{-}11)$$

Proposition 3.7 (Order-up-to Policy)
If the current inventory position at the time of planning is x_1^0, then the optimal order quantities are $(x_1^* - x_1^0, x_2^*)$ if $x_1^0 < x_1^*$ and $(0, x_2(x_1^0))$ if $x_1^0 \geq x_1^*$ where (x_1^*, x_2^*) maximizes $NW(x_1, x_2)$ as defined in equation (3-10). $x_2(x_1^0)$ is the optimal quantity of the new product given that the planning quantity for the old product is set to x_1^0. In addition, if $x_1^0 \geq x_1^*$, then $x_2(x_1^0) \leq x_2^*$

Proof. See Appendix. □

The implications of *Proposition 3.7* are simple: If the company is carrying much more inventories of the old product than needed, the optimal policy is to place no additional orders of the old product and plan the same or less for the new product. When there are excess old product inventories, the need of substitution using the new products decreases, thus requiring less new products.

3.5.3 Comparison to the Optimal Planning Decisions under Deterministic TSD

Figures 3-5 and 3-6 illustrate how the total profit and optimal planning quantities change with the variance of the transition start date. In the numerical example for these two figures, we let $\delta=1$ to exclude the complication from revenue discounting. Other parameter values are shown in Table 3.2. We model the transition start date (TSD) as a uniformly distributed variable and vary the lower and upper bounds to change the variance while keeping the mean value fixed at 500.
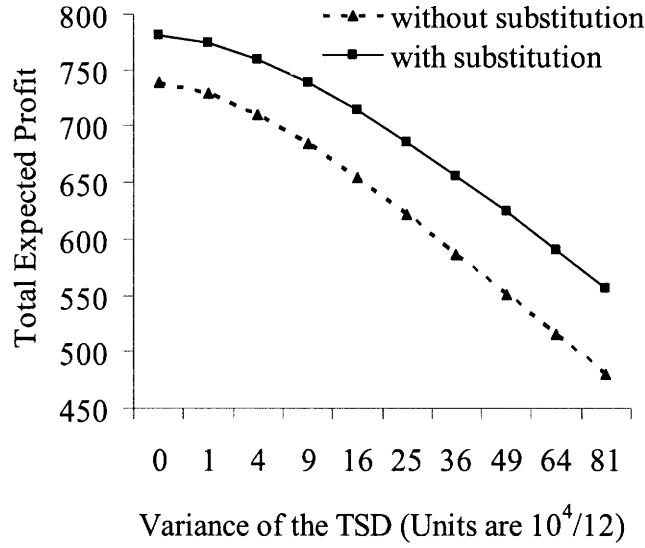
Figure 3-5: The Total Expected Profit vs. Uncertainties in TSD

T	500	v_1	58
s_1	5	v_2	50
s_2	18	g	4
c_1	15	h_1	0.005
c_2	18	h_2	0.005

$$\lambda_1(t) = \begin{cases} 0.08 & \forall t \in [0,\tau] \\ 0.08/(1+e^{0.01(t-250-\tau)}) & \forall t \in (\tau, \tau+T] \end{cases} \qquad \lambda_2(t) = \begin{cases} 0 & \forall t \in [0,\tau] \\ 0.08/(1+e^{-0.01(t-250-\tau)}) & \forall t \in (\tau, \tau+T] \end{cases}$$

Table 3.2: Parameter Values of the Example in Figures 3-5, 3-6 and 3-7

Apparently the expected profit deteriorates as the uncertainty in TSD increases. The option of dynamic substitution helps reduce such impact and substitution is most valued when the transition date is lest predictable (the gap between the two curves in Figure 3-5 grows with the variance).

Compared to the optimal planning decision under deterministic TSD (shown in Figure 3-6 as the data points with zero variance), the optimal quantity of the new product under stochastic TSD is higher and increases in the uncertainty of the TSD. The optimal policy prescribes higher ordering quantity of the new product because a new product can also be used to satisfy demand of the old, thus creating the need for additional inventories to cover the uncertainty in TSD.

Figure 3-6: Optimal Quantities vs. Variance in TSD

The impact of transition date uncertainty on the optimal quantity of the old product is ambiguous: In Figure 3-6, the optimal quantities of the old product is non-decreasing in variance with the exception of the small dip at variance $9 \times 10^4/12$. The explanation for the small dip is that we require the plan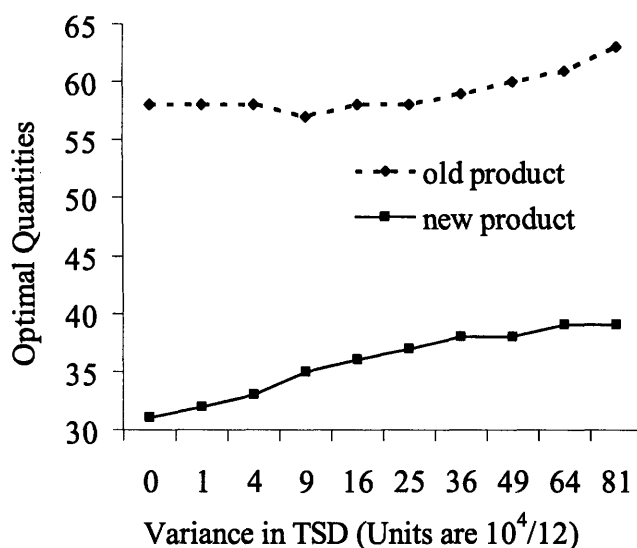ning quantities to be integers and that there is interdependency between the quantity choices for the old and new products. Had we allowed non-integral numbers for planning quantities, we would have obtained a smooth non-decreasing curve for the optimal quantities of the old product. However, we ran a series of numerical examples and the results indicate that uncertainty in the TSD may drive up as well as drive down the optimal quantity of the old product. We offer the following interpretation: Uncertainties in the TSD create the need for a supply cushion. In general, such a cushion is realized through increased quantities of both the new and old products. However, when the cost of excess of the new product and the cost of substitution are low enough, the need of a supply cushion due to uncertainty in the TSD is accomplished solely through increased inventories of the new product. Since such a cushion also helps to alleviate the demand risk (through substitution), it can cause a reduction in the optimal quantity of the old product.

To see the profit impact from the optimal policy derived in this chapter, we compare it with other simple alternatives. Figure 3-7 shows the profit realized using the Newsboy quantities and naïve substitution policies with parameter values shown in Table 3.2. Clearly both alternative policies fare worse than the optimal policy and the gap between the alternative policies and the optimal policy increases as the variation in the transition start date increases.
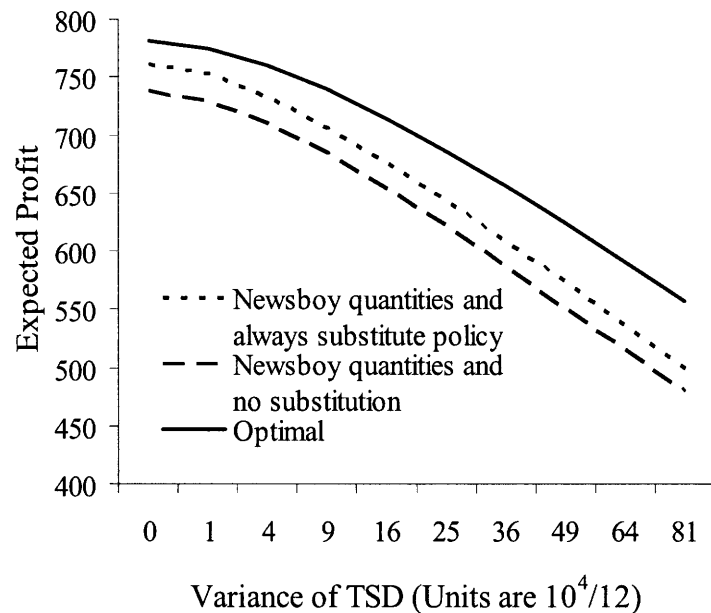
Figure 3-7: Comparison with Newsboy Quantities and Naïve Substitution Policies

3.6 Determine the Optimal TSD

In some cases, companies can delay the release of the new product to put off the transition for some period of time. In the second transition example mentioned in the introduction, the solution for the excess inventory of the old product was to delay the new product release in selected regions. It worked well in this case and saved the company tremendously on cost. The decision to delay a product transition when the new product is ready is primarily driven by the intent to deplete excess inventory of the old product and to avoid large excess and obsolescence cost. However, the new product typically has better margins and can in expectation bring in higher profit, thus such delay can also hurt the profits. Therefore, the optimal timing of a product upgrade is a tradeoff of the two conflicting objectives.

Figure 3-8 illustrates the impact of different transition start dates. Delaying the transition from τ to τ' increases the period of time that the old product is being sold and reduces the selling time (or delays the sales) of the new product.
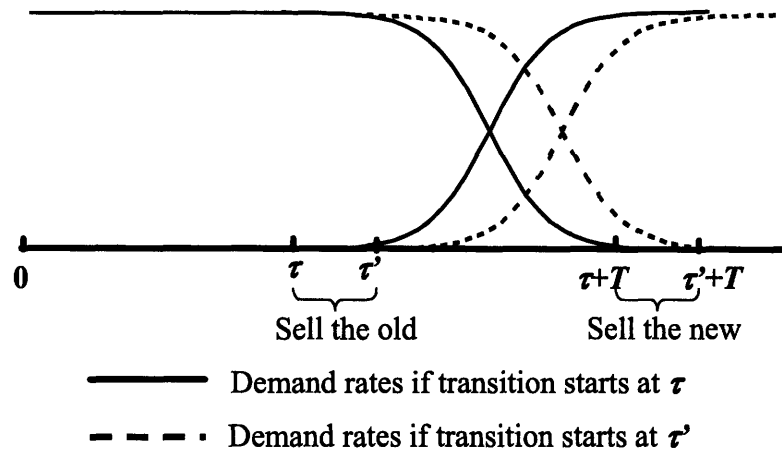
Figure 3-8: Impact of Delaying the TSD from τ to τ'

Another consideration in this problem is the market share competition. Early market entrants tend to grab larger market shares of the new product. As a result, an early TSD can benefit the company through increased market share. We acknowledge that this is an important factor to consider, especially in industries with intense competition. However, we leave it out of the model to limit the scope of our analysis.

As in the previous section, we only consider existing inventory commitment of the old product. At the time of planning, we look at the current inventory liability of the old product and determine the optimal time to release the new product. We demonstrate two alternative methods to model the tradeoff of margin difference and excess inventory, depending on the assumptions on the remaining time horizon. Both allow us to compare options of different transition start dates and choose an optimal one.

3.6.1 Fixed Value Method

We assume that the transition start date does not affect the remaining life time of the new product; hence the product will bring in the same revenue stream after the transition for all possible transition start date τ. Therefore, we can treat the expected profit from future sales of this product after the transition (mostly the new product) as a fixed value U, discounted to the end of the transition. Referring to Figure 3-7, the expected value of this product for the firm from time $\tau+T$ on *(or $\tau'+T$ if the transition starts at τ')* is a deterministic value U. Such an assumption can be problematic if there is a market share game being played: When an early release of the new product represents higher gains in market share, the revenue stream becomes very much dependent on the choice of τ. If this is the case, the method can be revised to accommodate such dependencies.

In the Fixed Value model, a larger value of τ allows the firm to sell more of the old product and avoid excess, at the price of delayed revenue realization from the new product. The impact of such delay is captured by the discounting of expected future values. Such a simplification allows us to decouple the planning for product transition

from the planning for stable demand. $\underline{\tau}$ represents the scheduled release date and $\bar{\tau}$ represents the latest dates for the new product release. Thus $\bar{\tau} - \underline{\tau}$ is the maximum delay allowed. Let $NV_{FV}(\tau, x_1, x_2, x_1^0)$ be the total expected profit for a given transition start date τ when the existing inventory level of the old product at the planning time (time 0) is x_1^0, then we have:

$$NV_{FV}(\tau, x_1, x_2, x_1^0) = \begin{cases} W(\tau, x_1, x_2, 0) + \delta^{\tau+T} U - c_1(x_1 - x_1^0) - c_2 x_2 & \text{if } x_1 > x_1^0 \\ W(\tau, x_1^0, x_2, 0) + \delta^{\tau+T} U - c_2 x_2 & \text{otherwise} \end{cases} \quad (3\text{-}12)$$

where $W(\tau, x_1, x_2, 0)$ is defined in equation (3-9).

Comparing this to equation (3-11), it is easy to see that for a given τ, the optimal solution again is an order-up-to policy.

The optimal transition start date τ is of a threshold pattern: for a given U, if inventory liability x_1^0 is below a threshold $\bar{x}_1$, no delay is necessary. Otherwise, the optimal policy prescribes a delay of the new product release that is increasing in the initial inventory of the old product. Figure 3-9 illustrates the dependency of the optimal delay on x_1^0 using parameter values in Table 3.3. When the initial inventory of the old product is between 0 and 52, we order 31 units for the new product, order up to 52 for the old product, and release the new product on the scheduled release date 450. For initial inventory between 52 and 56, no delay is necessary, and we do not place any order for the old product. In addition, less new products should be ordered. For inventory above 56, a delay is necessary and the optimal delay increases with the amount of initial inventory level. Again no order is necessary for the old product and the number of units of new products to purchase decreases with the old product inventory. Note that the threshold quantity for delay (56) is slightly higher than 52, the ideal quantity for the old product when there is no delay and no prior inventory commitment. This is because to obtain the latter, we need to consider the variable cost for each unit of product. However, to obtain the optimal delay, we treat the costs of committed inventory as sunk.
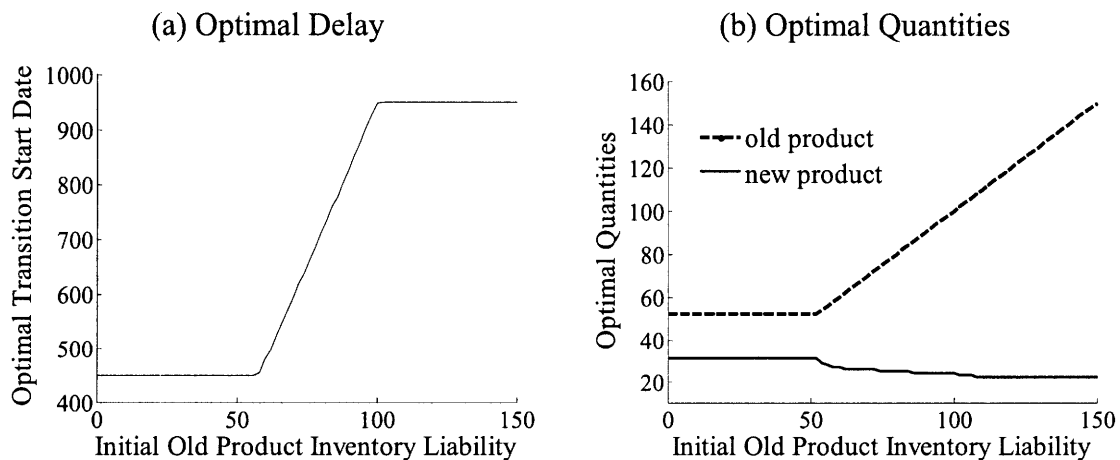

Figure 3-9: Optimal Delay vs. Initial Inventory Liability of the Old Product

T	500	v_1	68
δ	0.9997	v_2	50
c_1	15	h_1	0.005
c_2	16	h_2	0.005
$\underline{\tau}$	450	s_1	5
$\overline{\tau}$	950	s_2	18
U	2500	g	3

$$\lambda_1(t) = \begin{cases} 0.08 & \forall t \in [0,\tau] \\ 0.08/(1+e^{0.025(t-250-\tau)}) & \forall t \in (\tau,\tau+T] \end{cases} \qquad \lambda_2(t) = \begin{cases} 0 & \forall t \in [0,\tau] \\ 0.08/(1+e^{-0.025(t-250-\tau)}) & \forall t \in (\tau,\tau+T] \end{cases}$$

Table 3.3: Parameter Values for Figure 3-9.

3.6.2 Fixed Horizon Method

In this method, we pick a time H that satisfies $H > \overline{\tau} + T$ where $\overline{\tau}$ is the upper bound on the transition start date τ. H can signify certain fiscal target date such as end of a quarter or end of a year, thus $[0,H]$ represents a reasonable planning horizon (Figure 3-10). Assuming that the choice of τ does not affect what happens after time H, we only need to solve the optimization problem for the time period $[0, H]$. That is, we calculate the expected profit from the time of planning to time H and find the optimal quantities of x_1 and x_2, and the optimal choice of τ. Compared to the Fixed Value method, here delaying τ reduces the remaining selling time of the new product. Therefore this model reflects directly the tradeoffs of excess and margin difference, whereas in the Fixed Value method, the impact of margin difference is captured by the dependencies of U on the new product margins and the magnitude of the discount factor δ.

| 0 | τ | τ+T | H |
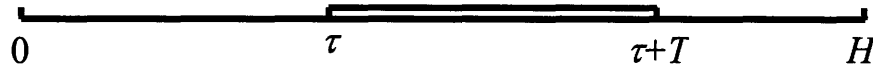
Figure 3-10: Fixed Time Horizon

Again we assume that the transition takes T time periods to finish. Therefore, from time 0 to time τ, only the old product is sold; from τ+T to H, only the new product is sold. We assume no reorder possibilities in the period $(\tau$+$T, H)$. At the end of H, units achieve their respective salvage values.

We already showed in Section 3.5 how to obtain the total expected profit from the time of planning to the end of the transition (time 0 to τ+T). Now we need to add the profit accrued during period $[\tau$+$T, H]$. Let $U(x_2,t)$ be the expected profit from time t to H where $t \in [\tau+T,H]$, we use the following recursive equations to compute U.

$$U(x_2, H) = s_2 x_2$$

$$U(x_2, t) = (1 - \lambda_2)[-h_2 x_2 + \delta U(x_2, t + 1)]$$
$$+ \lambda_2[-h_2(x_2 - 1) + r_2 + \delta U(x_2 - 1, t + 1)] \quad \text{if } x_2 \geq 1$$

$$U(0, t) = (1 - \lambda_2)\delta U(0, t + 1)]$$
$$+ \lambda_2[-p_2 + \delta U(0, t + 1)] \tag{3-13}$$

Next following equation (3-1) and using $V(x_1, x_2, \tau + T) = s_1 x_1 + U(x_2, \tau + T)$ as the terminal value for V (instead of $s_1 x_1 + s_2 x_2$), we can calculate the expected profit for the period $[\tau, H]$ as a function of x_1, x_2 and τ. Finally applying the recursion in equation (3-9), we obtain the total expected value $W'(\tau, x_1, x_2, 0)$ for the entire planning horizon $[0, H]$. Since different terminal value of V is used in the computation, we denote the expected value using $W'(\tau, x_1, x_2, 0)$ to differentiate from the original definition of $W(\tau, x_1, x_2, 0)$. Note that for a given τ, the objective function

$$NV_{FH}(\tau, x_1, x_2) = \begin{cases} W'(\tau, x_1, x_2, 0) - c_1(x_1 - x_1^0) - c_2 x_2 & \text{if } x_1 > x_1^0 \\ W'(\tau, x_1^0, x_2, 0) - c_2 x_2 & \text{otherwise} \end{cases} \tag{3-14}$$

is similar to that in equation (3-11). It is clear from *Proposition 3.7* that the order-up-to policy is optimal. We can then search for the optimal values of x_1, x_2 and τ.

Figure 3-11 shows results from a numerical example (parameter values shown in Table 3.4). We observe that the Fixed Horizon method yields a similar policy for the optimal delay: If inventory liability $x_1^0 < \overline{x}_1$, no delay is necessary. Otherwise, the optimal policy prescribes a delay of the new product release which is increasing in the amount of initial inventory of the old product.
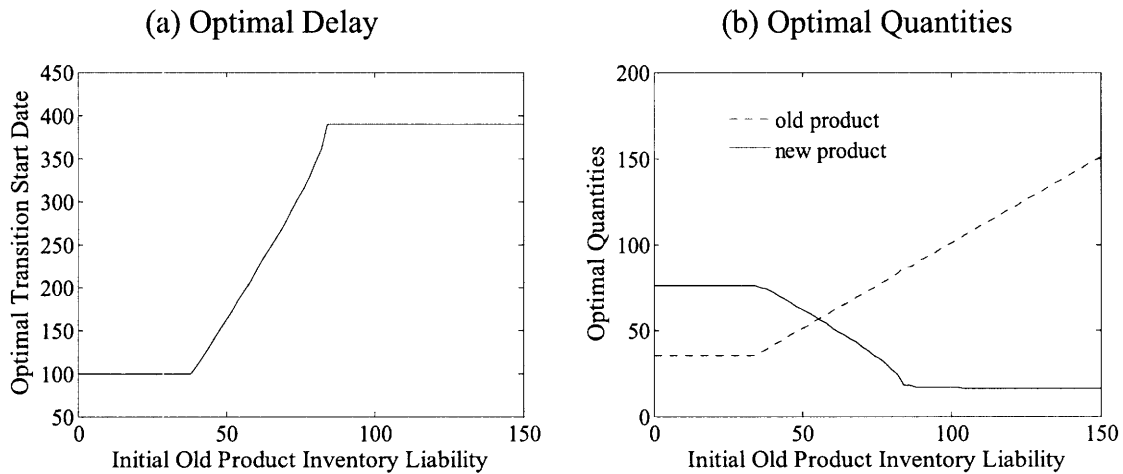
(a) Optimal Delay　　　　　　　　(b) Optimal Quantities

Figure 3-11: Optimal Delay and Quantities vs. Initial Inventory Liability of the Old Product

T	200	v_1	68
δ	0.998	v_2	50
c_1	15	h_1	0.0125
c_2	16	h_2	0.0125
$\underline{\tau}$	100	s_1	5
$\bar{\tau}$	390	s_2	10
H	600	g	3

$$\lambda_1(t) = \begin{cases} 0.2 & \forall t \in [0,\tau] \\ 0.2/(1+e^{0.025(t-100-\tau)}) & \forall t \in (\tau, \tau+T] \\ 0 & \forall t \in (\tau+T, H] \end{cases} \quad \lambda_2(t) = \begin{cases} 0 & \forall t \in [0,\tau] \\ 0.2/(1+e^{-0.025(t-100-\tau)}) & \forall t \in (\tau, \tau+T] \\ 0.20 & \forall t \in (\tau+T, H] \end{cases}$$

Table 3.4: Parameter Values for Figure 3-11.

The Fixed Horizon model can be computationally intensive. We use a heuristic procedure to simplify the computation. Suppose the planned release date is $\underline{\tau}$ and the latest possible release date is $\bar{\tau}$. Given the current inventory position of the old product (x_1^0), we need to determine whether and by how much to delay the transition. Using the methods developed in Section 3.5, we can easily obtain the optimal x_1 and x_2 for a given $\underline{\tau}$ and x_1^0. Let $x_1^*(\underline{\tau}, x_1^0)$ denote the optimal x_1. If $x_1^*(\underline{\tau}, x_1^0) > x_1^0$, no delay is necessary as the optimal quantity of the old product is higher than the initial inventory commitment. In this case the new product shall be released at the earliest possible time, which is $\underline{\tau}$. We denote the threshold above which delay is preferred as $\bar{x}_1$ which satisfies the fixed point condition $x_1^*(\underline{\tau}, \bar{x}_1) = \bar{x}_1$. To compute the value of the threshold, we simply drop the term $c_1(x_1 - x_1^0)$ (as $x_1^*(\underline{\tau}, \bar{x}_1) = \bar{x}_1$) in equation (3-14) and let τ take the value of its lower bound $\underline{\tau}$. In other words, the threshold $\bar{x}_1$ is the x_1 that maximizes $W'(\tau, x_1, x_2, 0) - c_2 x_2$. If $x_1^0 > \bar{x}_1$, delay is necessary. To obtain the optimal length of delay, we first find an upper bound on the delay. This could be $\bar{\tau}$ or a tighter bound. We then perform binary search to find the smallest τ such that $x_1^*(\tau, x_1^0) = x_1^0$, which is the optimal delay.

3.7 Summary

In this chapter we solve a planning problem for product upgrades when there is no replenishment opportunity during the transition period and we allow product substitutions. We show that the optimal substitution decision is a time-varying threshold policy. We prove that the substitution threshold $\bar{x}_2(t)$ is non-increasing in time under two special cases: a) Demand is homogeneous Poisson; b) Demand is non-homogeneous Poisson, but the holding cost and revenue discounting are negligible. Under more general conditions, $\bar{x}_2(t)$ is shown to be unimodal in t. That is, $\bar{x}_2(t)$ is increasing in t before a certain time $\hat{t}$ and is decreasing in t after $\hat{t}$.

We show that under both deterministic and stochastic transition start date, the total discounted net profit is a jointly concave function of the planning quantities of the old and new product. This implies that there are decreasing marginal returns to each additional unit of product and decreasing marginal rate of returns from substitution. We can then use simple search methods to find the optimal planning quantities under stochastic transition date and non-stationary demand. For a given initial inventory liability of the old product, the optimal policy is an order-up-to policy. Our result shows that variability in transition date adversely affects the company's profit and increases the optimal planning quantity of the new product. Substitution helps reduce such impact and the value of substitution increases in the variability of the transition start date.

Further, we determine the optimal transition delay given the inventory of old product. Both the Fixed Value method and the Fixed Horizon method yield similar policy for the optimal delay: If inventory liability is below a threshold, no delay is optimal. Otherwise, the optimal policy prescribes a delay of the new product release which is increasing in the amount of initial inventory liability.

Although the model developed in this chapter is motivated by examples in a particular industry, it can easily find applications in many other industries such as PCs and consumer electronics. The digital camera industry, for example, is typified by continuous introductions of new models driven by technology and competition. Canon, offers a new SLR model every 18 months. At Sony, a typical camera or camcorder model has a life cycle of 12 months or less. Its accessory life cycle is even shorter. At one camera company (Waltson and Olivia 2005), lack of appropriate planning costs the company $19.5 million during one camera model upgrade! Clearly there is potential for improvement using techniques developed in this chapter.

Limitations of our model lie in its assumptions. One of the key assumptions in our model is that there is no replenishment opportunity during the transition. As mentioned earlier, certain product upgrades may take a much longer time period. As a result, replenishment during the transition is allowed. We would like to explore in future research the optimal continuous-replenishment policy in these cases.

Also implicit in our model is that the price of the products stays constant throughout the transition. This is not always the case in practice. In fact, pricing is an important tool for demand management. During product transitions, product managers often can manipulate the price gap of the two products in order to push a particular product. For example, they sometimes lower the price of an old product to bleed off old product inventories or lower the price of the new product to speed up the transition process. In the next chapter, we present a dynamic pricing model that addresses such decisions during a product transition.

Chapter 4

Pricing Decisions during Inter-generational Product Transition

4.1 Introduction

In high-tech industries, a newer generation product replaces an old product periodically. In many cases, such a transition does not occur instantaneously but rather involves a transitional period during which the company sells both products simultaneously. Such a time period involves high uncertainty and high risk.

This research is motivated by cases of product transition in the same telecommunications equipment company mentioned in Chapter 3. This industry is characterized by frequent technology upgrades and long lead-times. Introduction of a new product to the market tends to create high uncertainties in demand. Long lead-time exacerbates the problem of demand forecasting because the managers needs to predict demand 18 weeks ahead. Consequently, the company often runs out of the product that customers want while having excess of the other. Because replenishment during the transition is not possible (again due to the long lead-time), the managers are left with very limited options. We study in Chapter 3 the option of product substitution: When one product is depleted, a company may offer the other one as a substitute. Pricing is another option: The managers can manipulate the prices of the two products to mitigate the risk of demand and supply mismatch. For example, if sales of the old product are sluggish during the transition, the management could put in a promotion and discount it. On the other hand, if the new product does not sell well, managers may consider increasing the price of the old product to make the new appear more attractive.

In practice, companies often adjust price during a product transition for two reasons - excess or shortage in inventory, and a change in their belief about demand. This chapter addresses the first case only. However, it is important to be aware that demand learning also has an important role in pricing decisions. When a company introduces a new product, often it is not clear what the demands of the new and old products will be. The company might underestimate or overestimate the market's acceptance of the new product. Being able to change price dynamically then serves as leverage against the lack of market knowledge. When the managers learn more about the demand as the transition progresses, they can change the price to reflect such updated demand belief.

In this chapter, we study the optimal pricing problem in the following context. We consider a finite time horizon - from the time a new product is introduced to a deterministic time after the transition finishes. That is, we consider a fixed interval of time starting from the launch of the new product. The choice of the interval length is made such that by the end of this time horizon, the demand for the old product has

dropped to a negligible level. At the beginning of the planning horizon, we are given the initial inventories of the old and new products. There is no option to reorder. During such a finite time horizon, the demand of the products changes over time – the new phases in while the old phases out. We decide the optimal price as a function of time and current inventories. Due to the close relationship of the two products, any pricing decision regarding one particular product will affect the demand of both products. For simplicity, we restrict the price of the new product to be constant and study the optimal pricing problem of the old product.

4.2 Literature Review

We do not intend to provide a comprehensive review of the vast amount of existing pricing literature. Rather, we consider papers that are directly related to our work. That is, we focus on dynamic pricing models with limited supply and no replenishment opportunities during the planning horizon.

The majority of these models focuses on a single product with stationary demand functions, i.e., demand is a function of price only. Gallego and van Ryzin (1994) consider a single-product dynamic pricing problem when there is no option for reordering. They study a base case of stationary demand function and then extend it to a case when demand's dependence on price and time is multiplicatively separable. Bitran and Mondschein (1997, 1993) study the pricing problem of perishable products in retail. They consider both continuous time model and periodic pricing model. They show that the percentage of price reduction in a retail store should increase over time. Further, if the number of reviews is appropriate, the periodic pricing policy loses little compared to the optimal continuous pricing solution. Monahan, Petruzzi and Zhao (2004) combine the dynamic pricing problem with the optimal stocking problem. They derive the optimal pricing and stocking solutions for a time-invariant exponential demand function.

Zhao and Zheng (2000) address non-homogeneous demand and derive certain structural properties for the optimal pricing solution that hold in general. For example, they show that the optimal price decreases with inventory and identify a condition under which the optimal price decreases over time for a given inventory level.

Because it is difficult to obtain closed-form pricing solutions in most cases, fixed-price (or equivalently, constant-price) heuristic is frequently used to approximate the optimal solution. It is shown to be asymptotically optimal with stationary demand function when there is infinite supply of inventory. Gallego and van Ryzin (1994), and later Bitran and Caldentey (2003) derive the performance bound of the fixed-price heuristic using similar methods. An extension of the fixed-price heuristic is the rolling-horizon approximation. That is, in each period, the optimal fixed-price for the remaining periods is applied. It is a myopic policy in that the decision maker determines the price assuming no future changes. Another heuristic widely studied is the deterministic approximation - apply the optimal prices solved for a deterministic problem to a problem with stochastic demand. Bitran and Caldentey (2003) derive the performance bound for the deterministic solution.

The difficulty for such deterministic approximation is that, for non-stationary demand, the deterministic approximation itself can be difficult to compute.

Since it is not always feasible for firms to change price continuously, a two-price policy is often studied. Feng and Gallego (1995) consider a case when a firm is restricted to two discrete prices and needs to decide the optimal time to switch from one price to the other. They show that a threshold policy is optimal - switch the price as soon as the time-to-go falls below or above a time threshold. Gallego and van Ryzin (1994) show that when restricted to a set of discrete prices, a similar stopping-time heuristic can be applied and such a heuristic is asymptotically optimal. Another type of two-price policy is when the time to switch is fixed and the decision is what price to switch to (Lazear 1986). In this chapter, we study a two-price policy where both the price and the time to switch are decision variables.

With multiple products, an appropriate demand model is necessary. Existing literature offers three ways to model demands. The Multinomial Logit (MNL) model is first proposed by Luce (1959). Under the MNL model, the customer's purchasing probability ρ_i depends on his/her utility from each product u_i through $\rho_i = \dfrac{e^{u_i}}{\sum\limits_{k} e^{u_k}}$. Another class of choice model incorporates the customer's decision more explicitly. The customer assigns a utility to each product and then solves a utility maximization problem under a given budget (Hauser and Urban 1986, Bitran, Caldentey and Vial 2004). In the reservation price model (Awad, Bitran and Mondschein 2000), a customer assigns a maximum amount of money he/she is willing to pay for each product (reservation price). He/She then decides among those that have a price lower than its corresponding reservation price which one to buy based on either the rank of preference or the rank of surplus.

Another stream of demand models for multiple products is the diffusion models. Fisher and Pry (1971) propose a technological substitution models in which the ratio of the fractional market share of the new and old technology follows an exponential growth pattern. Norton and Bass (1987) provide a diffusion-theory-based demand model for successive generations of products, extending from the seminal Bass diffusion model (1969). Compared to the choice models, these models have been widely tested with empirical data. Krishnan, Bass and Jain (1999) has applied the basic Bass diffusion model to a single-product pricing problem. Very few existing works address the dynamic pricing problem using the inter-generational diffusion models. One exception is Bayus (1992). He studies a pricing problem in the context of inter-generational product transition – the problem we are interested in. He uses the original Bass diffusion model for the new generation product demand and generates the old product demand by holding the total demand fixed. The difficulty for using the diffusion models is the analytical tractability. It is hard to obtain closed-form or structurally meaningful results, especially with two generations of products.

Most dynamic pricing models for multiple products use one of the three choice models, or a variation of them. Awad, Bitran and Mondschein (2000) study pricing policies for a

family of perishable products using the reservation price choice model. They propose several pricing heuristics to approximate the optimal solution. Bitran, Caldentey and Vial (2004) study a very similar problem. They combine the MNL model with a utility maximization model to describe the demand for substitutable products. They too resort to heuristic algorithms to approximate the optimal solution.

Gallego and van Ryzin (1997) study the dynamic pricing problem for multiple products in the context of a network flow problem. Their model formulation is generic enough to include a large range of applications. However, they do not model the demand relationships among the products but instead assumed a generic set of demand functions. They find the solution for a deterministic and time invariant problem and use heuristic policies to approximate the optimal policy for the stochastic problem. Bitran and Caldentey (2003) also give a generic formulation of the multiple-product problem and provide a general optimality condition. In general, these generic formulations can say very little about the optimal policy.

Kornish (2001) studies the pricing problem for a monopolist with frequent product upgrades. She assumes that the monopolist sells only the latest generation of product in any period. In contrast, we consider the pricing problem within the transitional period where a company sells two generations of products simultaneously.

More recently, Xu and Hopp (2004) consider a pricing problem for a single product with one or multiple retailers. Assuming that the demand process follows a geometric Brownian motion, they are able to obtain a closed-form solution for the monopoly case. When there are multiple retailers, they derive the equilibrium pricing policies for a retailer. In contrast, we study the centralized pricing decision, i.e., a company that sells multiple products and thus has to maximize the total expected profit from the two products.

4.3 Demand Model

We adopt the Multinomial Logit (MNL) consumer choice model to describe the demand functions of the two products. Assume that a customer's utility of purchasing product i ($i=1$ refers to the old product and $i=2$ refers to the new product) is $u_i(r_i,t) = a_i(t) - r_i$ where r_i is the selling price of product i and $a_i(t)$ is the time-varying attribute(s) that affects the customer's utility. The MNL model predicts the probability that a customer purchases product i to be $\rho_i(r,t) = \dfrac{e^{a_i(t)-r_i}}{e^{a_i(t)-r_i} + e^{a_j(t)-r_j}}$.

Assumption 4.1
For the ease of exposition, we assume that the total demand of the two products is a homogeneous Poisson process with rate λ_0.

Hence the demands of the old and new products are each a Poisson process with time-varying rates

68

$$\lambda_i(r,t) = \lambda_0 \rho_i(r,t) = \lambda_0 \frac{e^{a_i(t)-r_i}}{e^{a_i(t)-r_i} + e^{a_j(t)-r_j}} \quad i = 1,2 \qquad (4\text{-}1)$$

In general, the MNL model allows for the probability of no purchase. Since we restrict the total demand rate to be a constant, the no-purchase event also has a constant rate. The analysis in this chapter depends on this implicit assumption. In future research, we will consider the case when the total demand of the new and old products is non-stationary, but price-dependent. A MNL model with explicit non-purchase utility $u_0(r,t)$ can be incorporated in that case.

Assumption 4.2
Without loss of generality, we can assume that $a_1(t)$ is non-increasing in t and $a_2(t)$ is non-decreasing in t.

Such a demand model is simple, intuitive and symmetric with respect to the two products. Because the time factor and the price factor are separable within each exponent term, the analysis is tractable. In addition, it generates a logistic demand pattern that is often observed in practice. For example, when $a_1(t) = 4.5 - 0.1t$, $a_2(t) = 0.1t$, $r_1 = 35, r_2 = 40$ and $\lambda_0 = 0.1$, the above demand model generates a demand pattern shown in Figure 4-1.
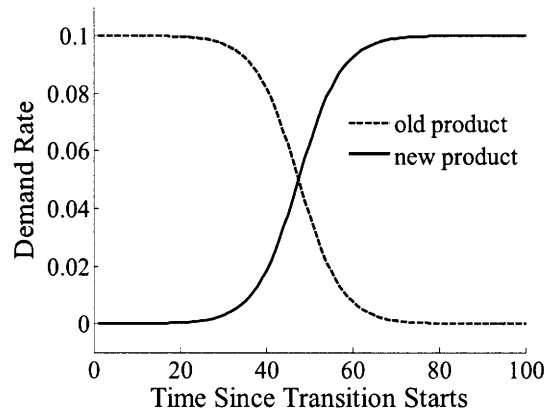
Figure 4-1: Demand Pattern under Equation (4-1)

To simplify the problem, we assume that the price of the new product is constant. During an interview, one product manager pointed out that the price of the new product is pretty much set by the market. Often what they have control over is the price of the old product. Therefore, we assume in the current model that the price of the new product is exogenously given and constant.

Assumption 4.3
The price of the new product r_2 is pre-determined and stays constant during the transitional period.

From equation (4-1), we obtain the following derivatives, which we will use later in finding the optimal solutions of the pricing problem.

$$\frac{\partial \lambda_1(r_1,t)}{\partial r_1} = -\lambda_0 \rho_1 \rho_2 \quad \text{and} \quad \frac{\partial \lambda_2(r_1,t)}{\partial r_1} = \lambda_0 \rho_1 \rho_2 \qquad (4\text{-}2)$$

$$\frac{\partial^2 \lambda_1(r_1,t)}{\partial r_1^2} = \lambda_0 \rho_1 \rho_2 (\rho_2 - \rho_1) \quad \text{and} \quad \frac{\partial^2 \lambda_2(r_1,t)}{\partial r_1^2} = \lambda_0 \rho_1 \rho_2 (\rho_1 - \rho_2) \qquad (4\text{-}3)$$

4.4 Optimal Dynamic Prices

Given that the demands are Poisson, we define the time unit such that the probability of more than one arrival in each time unit is zero. Within each time unit t,

Prob(a demand of old occurs) $= \lambda_1(r_1,t)$
Prob(a demand of new occurs) $= \lambda_2(r_1,t)$
Prob (no demand occurs) $= 1 - \lambda_0$

Let $V_t(x_1,x_2)$ be the value-to-go at the beginning of period t if the company has x_1 units of old products and x_2 units of new products.

Additional notations include:

p_i - penalty for a lost sale (in addition to revenue loss)
s_i - unit salvage value of a product at the end of the planning horizon T

$V_t(x_1,x_2)$ can be defined recursively:

$$V_t(x_1,x_2) = \max_{r_1} h_t(r_1,x_1,x_2)$$
where
$$
\begin{aligned}
h_t(r_1,x_1,x_2) \equiv\ & \lambda_1(r_1,t)(r_1 + V_{t+1}(x_1 - 1, x_2)) \\
& + \lambda_2(r_1,t)(r_2 + V_{t+1}(x_1, x_2 - 1)) \\
& + (1 - \lambda_0)V_{t+1}(x_1, x_2)
\end{aligned} \qquad (4\text{-}4)
$$

$$
\begin{aligned}
h_t(r_1,0,x_2) \equiv\ & \lambda_1(r_1,t)(-p_1 + V_{t+1}(0, x_2)) \\
& + \lambda_2(r_1,t)(r_2 + V_{t+1}(0, x_2 - 1)) \\
& + (1 - \lambda_0)V_{t+1}(0, x_2)
\end{aligned} \qquad (4\text{-}5)
$$

$$h_t(r_1, x_1, 0) \equiv \lambda_1(r_1, t)(r_1 + V_{t+1}(x_1 - 1, 0))$$
$$+ \lambda_2(r_1, t)(-p_2 + V_{t+1}(x_1, 0)) \qquad (4\text{-}6)$$
$$+ (1 - \lambda_0)V_{t+1}(x_1, 0)$$

$$h_t(r_1, 0, 0) \equiv \lambda_1(r_1, t)(-p_1 + V_{t+1}(0, 0))$$
$$+ \lambda_2(r_1, t)(-p_2 + V_{t+1}(0, 0)) + (1 - \lambda_0)V_{t+1}(0, 0) \qquad (4\text{-}7)$$

The terminal value is the salvage value of products left over after T:
$$V_{T+1}(x_1, x_2) = s_1 x_1 + s_2 x_2$$

Implicitly, we assume that even if a company runs out of a product, there is some base demand for that product that depends on the list price. For example, when $x_1 = 0$, the demand rate $\lambda_1(r_1, t)$ is not necessarily zero (unless the price is so high that demand approaches zero), indicating that there is still customer need for the products, which allows us to incorporate the impact of lost sales on expected profit.

The problem is to find the optimal r_1 for each (t, x_1, x_2) combination. That is, we would like to know the optimal price at any time t for any given inventory level (x_1, x_2). We adopt a marginal analysis approach that is similar to Bitran and Mondschein (1997, 1993). Consider the impact of a price increase dr_1 for the old product during a single period t. The benefit includes revenue increase from price increase of the old product $\lambda_1(r_1, t) \cdot dr_1$ and sales increase of the new product $r_2 \cdot \dfrac{\partial \lambda_2(r_1, t)}{\partial r_1} \cdot dr_1$. The loss is from sales decrease of the old product $r_1 \cdot \dfrac{\partial \lambda_1(r_1, t)}{\partial r_1} \cdot dr_1$.

Equating the marginal gains to the marginal loss yields
$$\lambda_1(r_1, t) + r_2 \frac{\partial \lambda_2(r_1, t)}{\partial r_1} + r_1 \frac{\partial \lambda_1(r_1, t)}{\partial r_1} = 0$$

With multiple periods, we need to consider the opportunity to sell a unit in the future, which leads to the revised marginal condition
$$\lambda_1(r_1, t) + \frac{\partial \lambda_2(r_1, t)}{\partial r_1}[r_2 - (V_{t+1}(x_1, x_2) - V_{t+1}(x_1, x_2 - 1))]$$
$$+ \frac{\partial \lambda_1(r_1, t)}{\partial r_1}[r_1 - (V_{t+1}(x_1, x_2) - V_{t+1}(x_1 - 1, x_2))] = 0$$

One can interpret the term $(V_{t+1}(x_1, x_2) - V_{t+1}(x_1, x_2 - 1))$ as following: The sales increase of the new product resulting from price increase of the old product is offset by the prospect to sell it in the future. If the company does not sell a unit of new product this

period, it will end up with $V_{t+1}(x_1,x_2)$ instead of $r_2 + V_{t+1}(x_1,x_2-1)$. The last term looks at the sales change in the old product.

From *Assumption 4.1*, we know that $\partial \lambda_1(r_1,t) = -\partial \lambda_2(r_1,t)$. Thus we can simplify the above equation to

$$\lambda_1(r_1,t) + \frac{\partial \lambda_1(r_1,t)}{\partial r_1}[r_1 - r_2 - V_{t+1}(x_1,x_2-1) + V_{t+1}(x_1-1,x_2)] = 0 \qquad (4\text{-}8)$$

Equation (4-8) is basically the first order condition of optimality for the value-to-go objective function $h_t(r_1,x_1,x_2)$ when $x_1,x_2 \geq 1$. $h_t(r_1,x_1,x_2)$ as defined in equations (4-4)-(4-7) is not necessarily concave in price. However, we establish quasi-concavity of the value function for the demand given in equation (4-1).

Proposition 4.1
$h_t(r_1,x_1,x_2)$ is quasi-concave in r_1 $\forall x_1 \geq 0, x_2 \geq 0, t \in [0,T]$.

Proof. See Appendix. □

Under the differentiability assumption, the Karush, Kuhn and Tucker (KKT) optimality condition for concave function can be extended to a quasi-concave objective function (Avriel et al. 1988).

Similar analysis can be done for the cases when one of the products runs out. Again referring back to equations (4-5)-(4-7), $\lambda_i(r_1,t)$ is the non-censored demand rate. Therefore, it is not necessarily zero when the inventory runs out. For consistency, we also assume that such a base demand is affected by the list price of the products even though the company may currently be out of a particular product.

If $x_2 = 0$, the first order condition becomes

$$\lambda_1(r_1,t) + \frac{\partial \lambda_1(r_1,t)}{\partial r_1}[r_1 + p_2 - V_{t+1}(x_1,0) + V_{t+1}(x_1-1,0)] = 0 \qquad (4\text{-}9)$$

Again we establish quasi-concavity by showing that the second order derivative at the zero-slope point is negative.

If $x_1 = 0$, the first order derivative is $\dfrac{\partial \lambda_1(r_1,t)}{\partial r_1}[-r_2 - p_1 + V_{t+1}(0,x_2) - V_{t+1}(0,x_2-1)] > 0$

Therefore, the optimal price in this case is always the maximum possible price. Essentially when the old product is depleted, the company ought to maximize revenues from the new product. Since in this model the price of the new product is constant, maximizing revenue is equivalent to maximizing demand. Therefore, it is optimal to price the old product as high as possible so as to make the new appear to be a good deal. This is arguably an artificial result due to the restrictions in this model. Nevertheless, it

suggests that when the inventory of the old product is running low, a company should consider a higher price for it, or remove any discount on the old product. The danger of setting a low price for the old product in this case is two-fold. On one hand, we are losing the old product margin without benefiting much from increased demand for the old (as the inventory is low); on the other, we are making the new product appear more expensive.

Definition 4.1
Define the optimal price of the old product at time t for a given inventory level (x_1, x_2) to be $r_t^*(x_1, x_2) \equiv \arg\max_{r_1} h_t(r_1, x_1, x_2)$

Substituting equations (4-2) and (4-3) into equation (4-8) yields
$$1 - \rho_2(r_1, t)[r_1 - r_2 - (V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2))] = 0 \qquad (4\text{-}8a)$$

Solving equation (4-8a) for r_1, we obtain the optimal price
$$r_t^*(x_1, x_2) = r_2 + V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2)$$
$$+ 1 + LambertW(e^{a_1(t) - a_2(t) - (V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2)) - 1})$$
where the *LambertW* function solves the equation $we^w = x$ for w as a function of x.

Clearly the optimal price of the old product depends on the price of the new product. The last term in the equation is non-negative. As t increases, this term can become very small and negligible. Intuitively, the optimal price is set such that the company is close to being indifferent between selling an old and selling a new. The term $r_2 + V_{t+1}(x_1, x_2 - 1)$ is the expected value from selling a new product at time t. $V_{t+1}(x_1 - 1, x_2)$ is the future value after selling an old product at time t. The term '+1' comes from the fact that a unit change in the price of the old product results in a unit change in the revenue of an old product.

Substituting equation (4-8a) into $h_t(r_1, x_1, x_2)$, which is defined in equations (4-4)-(4-7), we obtain the optimal solution for the dynamic pricing problem.

Proposition 4.2
$$V_t(x_1, x_2) = V_{t+1}(x_1, x_2)$$
$$+ \lambda_0[r_t^*(x_1, x_2) - 1 - (V_{t+1}(x_1, x_2) - V_{t+1}(x_1 - 1, x_2))] \qquad (4\text{-}10)$$
$$\forall x_1 \geq 1, x_2 \geq 0, t \in [0, T]$$

where the optimal price is given by
$$r_t^*(x_1, x_2) = r_2 + 1 + (V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2))$$
$$+ LambertW(e^{a_1(t) - a_2(t) - (V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2)) - 1}) \qquad (4\text{-}11)$$
and

$$r_t^*(x_1,0) = -p_2 + 1 + (V_{t+1}(x_1,0) - V_{t+1}(x_1-1,0))$$
$$+ LambertW(e^{a_1(t)-a_2(t)-(V_{t+1}(x_1,0)-V_{t+1}(x_1-1,0))-1}) \tag{4-12}$$

If $x_1 = 0$, the optimal price of the old product is the highest feasible price $r_t^*(0,x_2) = \bar{r}_1$ and

$$V_t(0,x_2) = V_{t+1}(0,x_2) - \lambda_0 p_1 + \lambda_0 \rho_2(\bar{r}_1,t)[r_2 + p_1 - (V_{t+1}(0,x_2) - V_{t+1}(0,x_2-1))] \quad \forall x_2 \geq 0$$

In addition, $V_{T+1}(x_1,x_2) = s_1 x_1 + s_2 x_2$.

Proof. See Appendix. □

General Structural Properties of the Optimal Price

Proposition 4.3
$r_t^*(x_1,x_2)$ decreases in x_1 and increases in x_2.

Proof.
Rewrite equation (4-8a) as:

$$V_{t+1}(x_1,x_2-1) - V_{t+1}(x_1-1,x_2) = r_1 - r_2 - \frac{1}{\rho_2(r_1,t)} \tag{4-8b}$$

For a given t, the RHS is increasing in r_1. We show in *Lemma 4.1* (see Appendix) that the LHS is decreasing in x_1 and increasing in x_2. As a result, optimal price of the old product decreases in x_1 and increases in x_2.□

As the inventory of the old product increases, a company should decrease the price of the old product. This fits naturally to our intuition. What happens when the inventory of the new product increases? The relative attractiveness of the new product depends on the price difference of the two products as $\rho_2(r,t) = \dfrac{e^{a_2(t)-r_2}}{e^{a_1(t)-r_1} + e^{a_2(t)-r_2}} = \dfrac{1}{1 + e^{a_1(t)-a_2(t)-(r_1-r_2)}}$.

r_2 is a constant, thus in order to sell more new products, it is necessary to make the old less attractive by jacking up the price. Although the result is obtained under these specific assumptions made in this model, it offers insight for the general case in practice. When facing potential excess of the new product, either due to bad planning or overly optimistic estimate of the new product demand, it makes sense for a company to remove any current promotion on the old product.

The next interesting question is how the optimal price changes with time. To isolate the impact of inventory, we consider the optimal price for a given inventory levels over time.

We first consider the following asymptotic cases.

Proposition 4.4
(i) If $x_1, x_2 \to \infty$, $r_t^*(x_1,x_2)$ decreases in t.

(ii) If $x_1 \to \infty$, $r_t^*(x_1, 0)$ decreases in t.

Proof.

From equation (4-11) and (4-12),

$r_t^*(\infty, \infty) = r_2 + s_1 - s_2 + 1 + LambertW(e^{a_1(t) - a_2(t) - s_1 + s_2 - 1})$ and

$r_t^*(\infty, 0) = -p_2 + s_1 - 1 + LambertW(e^{a_1(t) - a_2(t) - s_1 - 1})$

From *Assumption 4.2*, $a_1(t) - a_2(t)$ decreases in t. Because the *LambertW* function is an increasing function, both $r_t^*(\infty, \infty)$ and $r_t^*(\infty, 0)$ decreases in t. □

With finite inventories, the optimal price in general does not decrease monotonically in time, even for a fixed inventory level. In fact, we show that under certain conditions, the optimal price for a given inventory level (x_1, x_2) tends to increase with time later during the transition.

Proposition 4.5

If $\exists \bar{t} \in [0, T]$ such that $e^{a_1(\bar{t}) - a_2(\bar{t}) + r_2 + p_2 - e_1 - 1} < \varepsilon$, where ε is a very small positive number, then $r_{t-1}^*(x_1, x_2) \le r_t^*(x_1, x_2)$ $\forall x_1, x_2 \ge 0$ and $t \in [\bar{t}, T]$.

Proof. See Appendix. □

Proposition 4.5 indicates that the optimal price may increase in time towards the end of the transition. Moreover, our computation results indicate that the optimal price exhibits a "down-and-up" pattern under general conditions. The figure below shows how the optimal price changes with time when $a_i(t)$ takes on a linear function form of

$a_1(t) = a_0 - kt$ and $a_2(t) = kt$ where $k > 0$.

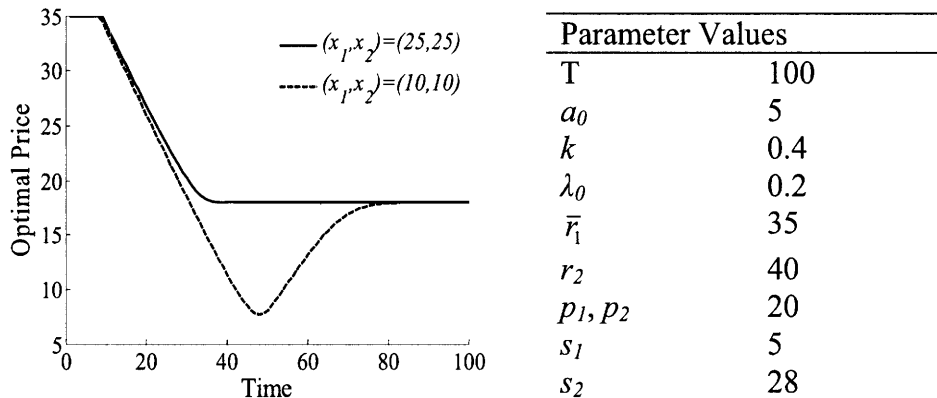| Parameter Values	
T	100
a_0	5
k	0.4
λ_0	0.2
$\bar{r}_1$	35
r_2	40
p_1, p_2	20
s_1	5
s_2	28

Figure 4-2: Optimal Price vs. Time for Given Inventory Level

Note that in this example, the total expected demand is $\lambda_0 T = 20$ (including new and old products) for the planning horizon. Therefore inventory level of (25,25) represents for all

75

practical purposes an infinite amount of inventory and that of (10,10) represents a more general case (with positive probability of running out of either product).

For a single product, it is a well-known result that the optimal dynamic price decreases with time for a given inventory level (see Gallego and van Ryzin 1994, Zhao and Zheng 2000). The intuitive explanation is that companies need to discount their product when there is less time left to sell. However, for multiple products, it is not obvious what that trend should be.

In this particular context, i.e., a product transition with two generations of products, the optimal price for the old product is affected by the demand and inventory of both products. If we just had the old product, then it makes sense to discount the price as we move closer to the end of the planning horizon for a given inventory level. On the other hand, from the perspective of the new product, we should try to make the old less attractive as we have less time to sell the new, i.e., we need to increase the price of the old over time. These two competing forces are driving the price of the old product in different directions over time. Because of the demand interactions of the two products, a price decrease of the old product can lead to both higher sales of the old and lower shortages of the new. Similarly a price increase of the old product can lead to both lower shortages of the old and higher sales of the new.

With infinite supply, the shortage costs become irrelevant, therefore we only need to consider the impact on sales. In Figure 4-1, we note that demand of the old product dominates initially and demand of the new dominates later. Therefore, as we move closer to the end of the transition, the loss in sales potential drops much faster for the old product than for the new product. In other words, a company should have more incentive to act on price if it is not able to sell the old product than if it is not able to sell the new within a certain time period. This holds because time is against the old product ($a_2(t)$ increases in time whereas $a_2(t)$ decreases in time). Hence, if neither product sells (as in Figure 4-2), the need for price decrease dominates the need for price increase. Therefore, the optimal price should be non-increasing over time. In Figure 4-2, the optimal price converges to a constant value later in the planning horizon. This is because the demand for the old product drops to a negligible level after certain time and the demand of the new product becomes stationary. Therefore, we are looking at the optimal price for the case of infinite supply and stationary demand. This is a well-studied problem and the optimal solution to such a problem is a constant price. From equation (4-11), when t is large, the last term is zero. With infinite supply, $V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2) \rightarrow s_1 - s_2$, thus this constant price is $r_2 + 1 + s_1 - s_2$. Given the parameters used in Figure 4-2, this price is 18.

With limited supply, there may be shortage risk for one or both products. For example, if the new product is facing shortage, we may decrease the price of the old to compensate, which may result in deeper discounts over time than the case of infinite supply. After the demand rate of the old product drops to a very low level, the optimal price becomes driven mostly by the new product sales, thus the optimal price increases over time until it

eventually converges to the same constant value 18. The convergence for this limited supply case is again due to the constant inventory level. However limited the inventory level may seem at the beginning of the transition, towards the end, it becomes a case of ample supply. Had this been a true price path where inventory decreases over time as products are sold, the convergence might not occur for the case of limited supply. Instead, the price may fluctuate below 18.

4.5 Fixed-Price Policy

Although continuously adjusting price over time can achieve the best value, it is not always feasible. Therefore, in this section we consider the case when a company has only one pricing opportunity at the beginning of the transition. Once they determine the price, it stays constant over the transitional period.

Because the price stays constant, we can view the period $[0,T]$ as a single period. The aggregate demands of the old and new products are Poisson with mean

$$\Lambda_1(r_1) = \int_0^T \lambda_1(r_1,t)dt \text{ and } \Lambda_2(r_1) = \int_0^T \lambda_2(r_1,t)dt \text{ respectively.}$$

The total profit as a function of the price of the old product and the inventories is

$$V^{FP}(r_1,x_1,x_2) = \underset{D_1}{E}[r_1 \min(D_1,x_1) - (D_1 - x_1)^+ p_1 + (x_1 - D_1)^+ s_1]$$

$$+ \underset{D_2}{E}[r_2 \min(D_2,x_2) - (D_2 - x_2)^+ p_2 + (x_2 - D_2)^+ s_2]$$

where D_1 and D_2 are Poisson random variables with mean $\Lambda_1(r_1)$ and $\Lambda_2(r_1)$ respectively.

Because λ_1 and λ_2 are correlated with each other and because of the Poisson distribution of demands, there is no simple closed-form solution even for this aggregated problem.

Taking derivative with respect to r_1 yields the first order condition for the fixed-price problem.

Lemma 4.2
The first order condition for the optimal fixed-price is

$$\Lambda_1(r_1) - G_{\Lambda_1}(x_1) + \frac{d\Lambda_1(r_1)}{dr_1}[(r_1 + V(r_1,x_1 - 1,x_2)) - (r_2 + V(r_1,x_1,x_2 - 1))] = 0 \qquad (4\text{-}13)$$

where $G_{\Lambda_1}(\cdot)$ is the Poisson loss function for the random variable D_1.

Proof. See Appendix.□

Similarly, we obtain the first order condition when the new product runs out to be

$$\Lambda_1(r_1) - G_{\Lambda_1}(x_1) + \frac{d\Lambda_1(r_1)}{dr_1}[r_1 + p_2 + V(r_1,x_1 - 1,0) - V(r_1,x_1,0)] = 0 \qquad (4\text{-}14)$$

When the old product runs out, i.e., $x_1 = 0$, the optimal price of the old product is the highest feasible price $r^{FP}_{t,0,x_2} = \bar{r}_1$. The intuition is similar to that for the optimal dynamic price when $x_1 = 0$.

To compare the fixed-price solution with the optimal prices derived previously, we consider the optimal rolling-horizon fixed prices, i.e., the optimal price at a particular time t during the transition assuming no future price changes, or equivalently, the optimal fixed price for $[t, T]$.

Let $\Lambda_{1t}(r_1) = \int_t^T \lambda_1(r_1, t)dt$ and $\Lambda_{2t}(r_1) = \int_t^T \lambda_2(r_1, t)dt$, then the total expected profit for period $[t, T]$ as a function of the price of the old product and the inventories is

$$V_t^{FP}(r_1, x_1, x_2) = \mathop{E}_{D_1}[r_1 \min(D_{1t}, x_1) - (D_{1t} - x_1)^+ p_1 + (x_1 - D_{1t})^+ s_1]$$

$$+ \mathop{E}_{D_2}[r_2 \min(D_{2t}, x_2) - (D_{2t} - x_2)^+ p_2 + (x_2 - D_{2t})^+ s_2]$$

where D_{1t} and D_{2t} are Poisson random variables with mean $\Lambda_{1t}(r_1)$ and $\Lambda_{2t}(r_1)$ respectively.

Definition 4.2
Define the optimal price of the old product for the fixed-price problem starting from time t to be $r_t^{FP}(x_1, x_2) \equiv \mathop{\arg\max}_{r_1} V_t^{FP}(r_1, x_1, x_2)$

Proposition 4.6
(i) If $x_1, x_2 \to \infty$, $r_t^{FP}(x_1, x_2)$ decreases in t.

(ii) If $x_1, x_2 \to \infty$, $r_t^{FP}(x_1, x_2) \le r_t^*(x_1, x_2)$

Proof. See Appendix.□

In Figure 4-3, each curve represents the price for a given inventory level. Again inventory level of (25,25) is for all practical purposes an infinite amount of inventory and (10,10) stands for the more general case. The red (solid and dashed) curves are the optimal prices of the DP problem (*Proposition 4.2*); the black (dotted and dash-dotted) curves are the optimal fixed prices with rolling horizon. Parameter values used to obtain Figure 4-3 are the same as those in Figure 4-2. With infinite supply of both products, the rolling-horizon fixed price decreases monotonically over time for a given inventory level and it is always lower than the optimal dynamic price. With limited supply, the fixed price starts out lower than the optimal price, but later becomes higher than the optimal dynamic price. Such a pattern is intuitive: From Section 4.4, the optimal dynamic price decreases over time initially and later increases. With a fixed price, we are making a pricing decision assuming no future opportunity to adjust price. Therefore, knowing that the optimal dynamic price should decrease in the subsequent periods, it makes sense to be "proactive" and set a price lower than the optimal dynamic price. Later during the transition, the optimal dynamic price would increase with time, thus it also makes sense

to use a fixed price that is higher than the optimal dynamic price knowing that we cannot increase it afterwards.
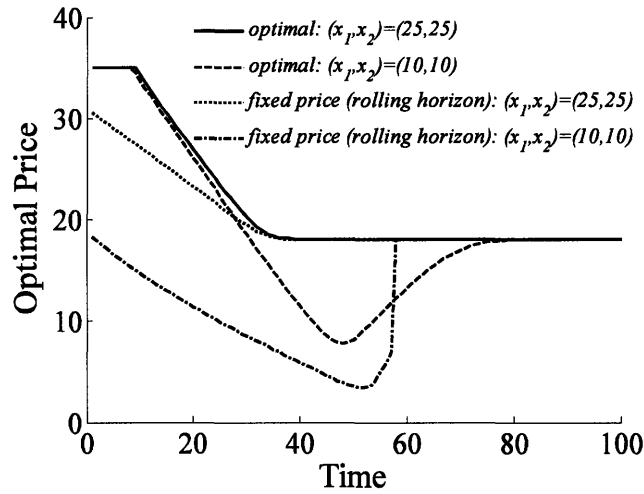

Figure 4-3: Fixed Price vs. Time for Given Inventory Level

Further, we observe that the fixed price also converges to the same value as the optimal price. As we mentioned earlier, later during the transition, both inventory (25, 25) and (10,10) become effectively cases of infinite supply and the optimal dynamic price is a constant price, i.e. the optimal dynamic price is the optimal fixed price

Fixed-Price with Deterministic Approximation

Equation (4-13) does not yield a closed-form solution for $r_t^{FP}(x_1, x_2)$. It has to be obtained through computation. Pricing literature often considers a deterministic approximation to the fixed-price problem. That is, one can find the optimal price when demand is deterministic and is equal to the expected demand and then use this price for the whole transitional period. Again we are lumping demand into one single period to find the optimal price, although the Poisson demand used to compute $r_t^{FP}(x_1, x_2)$ is now a deterministic value.

To maintain consistency in comparing prices, we follow the rolling-horizon calculation by finding the optimal deterministic price for the period $[t, T]$ at any $t \in [0, T]$.

The total profit for the period $[t, T]$ under the deterministic problem becomes

$$V_t^D(r_1, x_1, x_2) = r_1 \min(D_1, x_1) - (D_1 - x_1)^+ p_1 + (x_1 - D_1)^+ s_1]$$

$$+ r_2 \min(D_2, x_2) - (D_2 - x_2)^+ p_2 + (x_2 - D_2)^+ s_2]$$

where $D_1 = \int_t^T \lambda_1(r_1, \tau) d\tau$ and $D_2 = \int_t^T \lambda_2(r_1, \tau) d\tau$

79

Definition 4.3

Define the optimal fixed price of the old product for the deterministic problem starting from time t to be $r_t^D(x_1, x_2) \equiv \arg\max_{r_1} V_t^D(r_1, x_1, x_2)$

Throughout the rest of this chapter, we refer to $r_t^D(x_1, x_2)$ as the deterministic price. Note that in the dynamic pricing literature, researchers sometimes approximate the optimal solution by assuming that the demand within each small time period is deterministic (Bitran and Caldentey 2003). Such an approximation does not require the price to be fixed throughout the transitional period and is very different from the deterministic price $r_t^D(x_1, x_2)$.

In order to solve the deterministic price, we need to define certain critical prices.

Definition 4.4

Define the run-out price for the old product r_t^{old} to be the r_1 such that $\int_t^T \lambda_1(r_1, \tau) d\tau = x_1$, i.e., the price of the old product at which the total demand in period $[t, T]$ will be exactly x_1.

Similarly the run-out price for the new product r_t^{new} is defined as the r_1 that solves $\int_t^T \lambda_2(r_1, \tau) d\tau = x_2$. If the price of the old product is set at r_t^{new}, the demand of the new product during $[t, T]$ equals x_2.

To avoid confusion, note that both run-out prices refer to the price of the old product. r_t^{new} is the r_1 at which the new product runs out and r_t^{old} is the r_1 at which the new product runs out.

To find the optimal price for the deterministic problem, we specify a function form of $a_i(t)$.

Assumption 4.4

$a_1(t) = a_0 - kt$ and $a_2(t) = kt$ where $k > 0$.

Under *Assumption 4.4*, it is easy to derive that $D_1 = \lambda_0 \int_t^T \rho_1(r_1, \tau) d\tau = \frac{\lambda_0}{2k} \ln \frac{\rho_2(r_1, T)}{\rho_2(r_1, t)}$

and $D_2 = \lambda_0 \int_t^T \rho_2(r_1, \tau) d\tau = \lambda_0(T - t) - \frac{\lambda_0}{2k} \ln \frac{\rho_2(r_1, T)}{\rho_2(r_1, t)}$. Thus we obtain the run-out

prices by solving $\frac{\lambda_0}{2k} \ln \frac{\rho_2(r_1, T)}{\rho_2(r_1, t)} = x_1$ and $\lambda_0(T - t) - \frac{\lambda_0}{2k} \ln \frac{\rho_2(r_1, T)}{\rho_2(r_1, t)} = x_2$ respectively:

$$r_t^{old} = r_2 + a_0 - 2kT + \ln \frac{e^{2k(T-t)-\frac{2k}{\lambda_0}x_1} - 1}{1 - e^{-\frac{2k}{\lambda_0}x_1}}$$

$$r_t^{new} = r_2 + a_0 - 2kt - \ln \frac{e^{2k(T-t)-\frac{2k}{\lambda_0}x_2} - 1}{1 - e^{-\frac{2k}{\lambda_0}x_2}}$$

It is easy to see that the run-out price of the old product r_t^{old} decreases in x_1 and the run-out price of the new product r_t^{new} increases in x_2. This is intuitive: the higher the old product inventory, the lower the price needs to be to sell it off. On the contrary, to sell off a high new product inventory, the company needs to increase the price of the old (refer to *Proposition 4.3*).

In addition to the run-out prices, we also need to define the following optimal prices for $\max_{r_1} V_t^D(r_1, x_1, x_2)$.

Definition 4.5

Define $\hat{r}_t$ as the solution to $\max_{r_1} V_t^D(r_1, x_1, x_2)$ when $D_1 \le x_1, D_2 \le x_2$,

and $\tilde{r}_t$ as the solution to $\max_{r_1} V_t^D(r_1, x_1, x_2)$ when $D_1 \le x_1, D_2 \ge x_2$.

Note that D_1 and D_2 are both dependent on r_1.

Lemma 4.3

(i) $\hat{r}_t$ solves $\ln \frac{\rho_2(r_1, T)}{\rho_2(r_1, t)} - [(r_1 - s_1) - (r_2 - s_2)](\rho_1(r_1, t) - \rho_1(r_1, T)) = 0$.

(ii) $\tilde{r}_t$ solves $\ln \frac{\rho_2(r_1, T)}{\rho_2(r_1, t)} - (r_1 - s_1 + p_2)(\rho_1(r_1, t) - \rho_1(r_1, T)) = 0$

(iii) $\tilde{r}_t \le \hat{r}_t$.

Proof. See Appendix.□

Note that both $\hat{r}_t$ and $\tilde{r}_t$ are independent of the inventory levels.

Proposition 4.7 determines the optimal solution to the deterministic problem. For simplicity, we assume that the penalty cost for a lost sale is the same for both products.

Assumption 4.5

$p_1 = p_2$.

Proposition 4.7

If $r_1^{old}(t) > r_1^{new}(t)$, $r_t^D(x_1,x_2) = \max(r_1^{old}, \tilde{r}_t)$

If $r_1^{old}(t) \le r_1^{new}(t)$, then

$r_t^D(x_1,x_2) = \max(r_1^{new}, \tilde{r}_t)$ if $\hat{r}_t > r_1^{new}$ and $r_t^D(x_1,x_2) = \max(r_1^{old}, \hat{r}_t)$ otherwise.

Proof. See Appendix. □

Apparently, the optimal price in the deterministic problem can be one of the following: the run-out price of the old, the run-out price of the new, the interior optimum of the case for excess and the interior optimum of the case of excess for the old product only. Such a solution is very easy to obtain because all four values can be obtained with a one-step calculation. For example, for the parameter values used in Figure 4-2, we can compute the optimal deterministic price for the entire planning horizon $[0,T]$ as follows: For a given inventory level of (25,25), the run-out price of the old product $r^{old} = -\infty$, indicating that we would never be able to sell all 25 units of the old. The run-out price of the new product $r^{new} = \infty$, indicating that we would never be able to sell all 25 units of the new either. The interior solutions $\hat{r} = 30.6$ and $\tilde{r} = 14.6$. Therefore, from *Proposition 4.7*, it is clear that the optimal deterministic price is 30.6, which is the interior solution $\hat{r}$. However, when the inventory level is (10,10), we obtain $r^{old} = 4.2$ and $r^{new} = 5$. The values of $\hat{r}$ and $\tilde{r}$ are the same as those for inventory level (25, 25) because these prices are functions of time only. From *Proposition 4.7*, we should choose $\tilde{r} = 14.6$ as the optimal deterministic price in this case.

Proposition 4.8

(i) If $x_1, x_2 \to \infty$, $r_t^D(x_1,x_2)$ decreases in t.

(ii) If $x_1, x_2 \to \infty$, $r_t^{FP}(x_1,x_2) = r_t^D(x_1,x_2) \le r_t^*(x_1,x_2)$

Proof. See Appendix. □

Figure 4-4 compares the optimal dynamic price (red solid curves) with the optimal fixed price (black dashed curves) and the fixed price for the deterministic problem (blue dotted curves). When the amount of inventory is large (the case of inventory (25, 25)), the optimal price for the deterministic problem is equal to the optimal fixed price. This is reflected in Figure 4-4(a) by the overlapping of the blue dotted curve and the black dashed curve. Both the fixed price and the deterministic price are less than the optimal dynamic price. Parameter values used to obtain Figure 4-4 are the same as those in Figure 4-2.
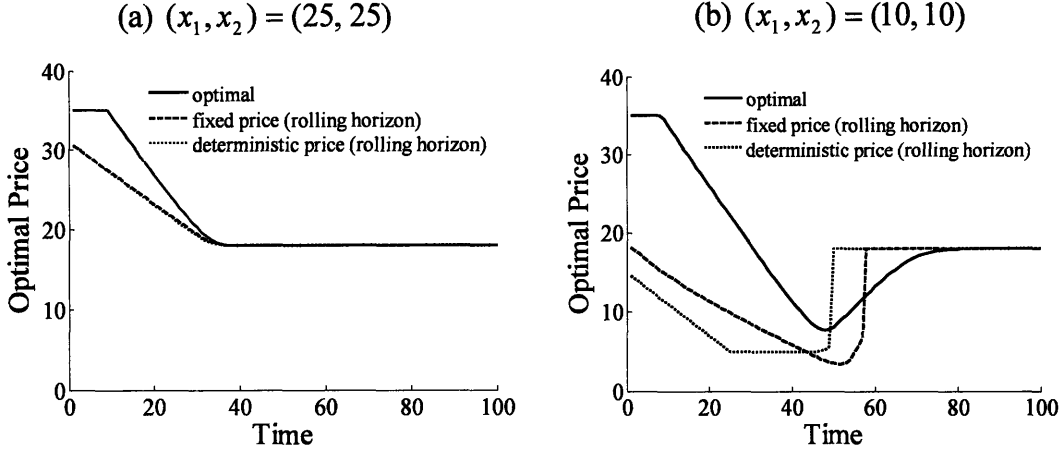
Figure 4-4: Prices. vs. Time for Given Inventory Level

In the case of limited inventory, all follow a "down-and-up" pattern although the fixed price and the deterministic price decreases slowly with time initially, but later converge much more quickly to a common price. From *Proposition 4.8*, the optimal fixed price is equal to the deterministic price when there is infinite supply. As a result, all converge to the same price 18.

Performance Bound Using $r_t^D(x_1, x_2)$

The solution for the deterministic problem is very easy to obtain and can be used as a heuristic price. But how much worse off is a company when it adopts such a heuristic? Following the method used in Gallego and van Ryzin (1994), we can determine the performance bounds of the deterministic approximation.

Definition 4.6

$$J^*(x_1, x_2) \equiv V_{t=1}(x_1, x_2), J^{FP}(x_1, x_2) \equiv V_{t=1}^{FP}(r_{t=1}^{FP}(x_1, x_2), x_1, x_2),$$

$$J^D(x_1, x_2) \equiv V_{t=1}^{FP}(r_{t=1}^D(x_1, x_2), x_1, x_2) \text{ and}$$

$$J^{LB}(x_1, x_2) \equiv x_1 s_1 + (r^D - s_1)\Lambda_1 - [\sqrt{\Lambda_1 + (\Lambda_1 - x_1)^2} - (x_1 - \Lambda_1)]/2$$

$$+ x_2 s_2 + (r_2 - s_2)\Lambda_2 - [\sqrt{\Lambda_2 + (\Lambda_2 - x_2)^2} - (x_2 - \Lambda_2)]/2$$

where $\Lambda_1 = \lambda_0 \int_0^T \rho_1(r^D, \tau)d\tau$ and $\Lambda_2 = \lambda_0 T - \int_0^T \rho_1(r^D, \tau)d\tau$.

$J^*(x_1, x_2)$ is the optimal value using the optimal dynamic pricing solution in *Proposition 4.2*. $J^{FP}(x_1, x_2)$ is the value using the optimal fixed price for the entire transitional period. $J^D(x_1, x_2)$ is the value achieved using the deterministic price heuristic. $J^{LB}(x_1, x_2)$, as we show next in *Proposition 4.9*, is a lower bound on the value achieved using the fixed deterministic price heuristic.

83

Proposition 4.9

$$1 \geq \frac{J^{FP}(x_1, x_2)}{J^*(x_1, x_2)} \geq \frac{J^D(x_1, x_2)}{J^*(x_1, x_2)} \geq \frac{J^{LB}(x_1, x_2)}{J^*(x_1, x_2)}$$

Proof. See Appendix. □

Table 4.1 compares the expected value of the fixed-price, deterministic-price with that of the optimal dynamic price. It also shows the performance lower bound when using the deterministic price. The parameter values used to obtain these results are the same as those used in Figure 4-2.

x_1	x_2	J^*	J^{FP}	J^D	J^{LB}	J^{FP}/J^*	J^D/J^*	J^{LB}/J^*
5	5	143	92	92	66	0.644	0.644	0.460
5	10	438	389	389	356	0.889	0.888	0.814
5	15	693	654	649	618	0.943	0.936	0.891
5	20	878	854	853	822	0.972	0.972	0.936
5	25	1024	1007	1004	977	0.983	0.980	0.954
10	5	293	167	166	139	0.569	0.565	0.475
10	10	562	459	454	423	0.817	0.809	0.753
10	15	763	704	703	674	0.923	0.921	0.883
10	20	918	888	887	860	0.968	0.966	0.937
10	25	1059	1037	1037	1016	0.979	0.979	0.959
15	5	377	197	197	174	0.523	0.523	0.463
15	10	610	487	486	458	0.798	0.795	0.751
15	15	792	730	728	703	0.921	0.920	0.888
15	20	943	913	912	887	0.968	0.967	0.941
15	25	1084	1062	1062	1043	0.980	0.980	0.962
20	5	421	222	222	202	0.527	0.527	0.480
20	10	638	512	511	486	0.803	0.800	0.761
20	15	817	755	753	730	0.924	0.922	0.893
20	20	968	938	937	913	0.969	0.967	0.944
20	25	1109	1087	1087	1069	0.980	0.980	0.964
25	5	449	247	247	228	0.550	0.550	0.509
25	10	664	537	536	512	0.810	0.807	0.772
25	15	842	780	778	755	0.926	0.924	0.897
25	20	993	963	962	939	0.970	0.968	0.945
25	25	1134	1112	1112	1095	0.981	0.981	0.965

Table 4.1: Performance for Different Starting Inventory Levels

Clearly the deterministic price-heuristic works the best when there are plenty of inventory for both products. With abundant supply (inventory level of (25,25)), the performance loss is less than 2% compared to the optimal dynamic price. It performs poorly when the inventory is unbalanced (one is in shortage and the other in excess). For example, consider the case (25,5), the ratio $J^{FP}/J^* = 0.55$, which indicates that the expected profit is 45% less than that of optimal dynamic price. In addition, the fixed-price heuristic does only slightly better than the fixed deterministic-price approach.

4.6 Two-Price Policy

In practice, companies do have opportunities to re-price their products during the transition. However, being able to change price continuously over time is limited to mainly the consumer goods that are sold online (computers, digital cameras and etc.) In most other cases, the number of times a company can re-price is limited to a couple. So we consider the pricing problem with one re-price opportunity during the transition. The interesting question in the two-price policy is when during the transition the company should re-price. Should a company re-price when it observes that there might be inventory shortage or excess? Or should it re-price at a fixed time during the transition?

Dynamic Programming (DP) Formulation

In the two-price problem, there are two decisions to be made - the initial price, denoted by r_0, and the second price. Let's focus on the latter price first.

Let $H_t(x_1, x_2)$ be the optimal profit-to-go from time t on for a fixed-price problem.

$$H_t(x_1, x_2) \equiv V_t^{FP}(r_t^{FP}(x_1, x_2), x_1, x_2)$$

Let $J_t(r_0, x_1, x_2)$ be the optimal profit-to-go for the re-price problem, i.e., given a re-price opportunity during $[t, T]$, and for a given initial price r_0.

Thus the choices are to re-price, incurring value $H_t(x_1, x_2)$, which will re-price to the best fixed-price for the remaining time $[t, T]$, or not re-price, in which case we maintain the price of the old product at r_0 and keep the option of re-pricing in $[t, T]$.

Then we have
$$J_t(r_0, x_1, x_2) = \max\{H_t(x_1, x_2), \lambda_1(r_0, t)(r_0 + J_{t+1}(r_0, x_1 - 1, x_2))$$
$$+ \lambda_2(r_0, t)(r_2 + J_{t+1}(r_0, x_1, x_2 - 1))$$
$$+ (1 - \lambda_0)J_{t+1}(r_0, x_1, x_2))\}$$

Therefore, it is optimal to re-price if and only if
$$H_t(x_1, x_2) > \lambda_1(r_0, t)(r_0 + J_{t+1}(r_0, x_1 - 1, x_2))$$
$$+ \lambda_2(r_0, t)(r_2 + J_{t+1}(r_0, x_1, x_2 - 1))$$
$$+ (1 - \lambda_0)J_{t+1}(r_0, x_1, x_2))$$

However, the computation for such a DP problem is prohibitive. We resort to heuristic solutions.

Since $J_{t+1}(r_0, x_1, x_2) \geq H_{t+1}(x_1, x_2)$, a necessary condition to re-price at time t is
$$H_t(x_1, x_2) > \lambda_1(r_0, t)(r_0 + H_{t+1}(x_1 - 1, x_2))$$
$$+ \lambda_2(r_0, t)(r_2 + H_{t+1}(x_1, x_2 - 1))$$
$$+ (1 - \lambda_0)H_{t+1}(x_1, x_2))$$

Or equivalently,

$$H_t(x_1,x_2) - H_{t+1}(x_1,x_2) > \lambda_1(r_0,t)(r_0 + H_{t+1}(x_1 - 1, x_2) - H_{t+1}(x_1,x_2))$$
$$+ \lambda_2(r_0,t)(r_2 + H_{t+1}(x_1, x_2 - 1) - H_{t+1}(x_1,x_2)) \quad (4\text{-}15)$$

Because it is relatively easy to obtain $H_t(x_1,x_2)$ computationally, we can find the necessary conditions for re-price.

Asymptotically, when $x_1,x_2 \to \infty$, such necessary condition translates to a simple criterion.

Proposition 4.10

If $x_1,x_2 \to \infty$, a necessary condition for re-price in the two-price policy is

$r_0 > r_t^*(x_1,x_2)$ or $r_0 < r_t^{FP}(x_1,x_2)$.

Proof. See Appendix. □

Proposition 4.10 implies that a re-price is only necessary if the optimal dynamic price falls below the initial price or if the optimal fixed price for the problem starting at t becomes greater than the initial price.

Corollary 4.1

If $x_1,x_2 \to \infty$, the optimal starting price r_0 satisfies $r_{t=1}^{FP}(x_1,x_2) \le r_0 \le r_{t=1}^*(x_1,x_2)$.

Proposition 4.10 along with *Corollary 4.1* suggests a simple heuristic to the two-price problem: set the initial price to r_0 and re-price to $r_t^{FP}(x_1,x_2)$ whenever $r_t^*(x_1,x_2)$ falls below r_0 or $r_t^{FP}(x_1,x_2)$ goes above r_0. The optimal r_0 can be determined through a simple search from $\min(r_{t=1}^{FP}(x_1,x_2), r_{t=1}^*(x_1,x_2))$ to $\max(r_{t=1}^{FP}(x_1,x_2), r_{t=1}^*(x_1,x_2))$. Further, *Proposition 4.8* indicates that $r_t^D(x_1,x_2)$ is asymptotically optimal for the fixed-price problem, we can use $r_t^D(x_1,x_2)$ instead of $r_t^{FP}(x_1,x_2)$ to further simply the policy.

The fact that *Proposition 4.10* is not a sufficient condition implies that if we follow the heuristic described as above, we might re-price too early. That is, we have not given it enough time for the uncertainty to play out before we readjust price to the inventory. Therefore, in an alternative heuristic, we consider re-pricing at the demand inflection point, i.e., when the demand rates of the two products cross over. Under the alternative heuristic, we wait til midway through the transition to adjust price. The intent is to pick a time where some uncertainty has realized and yet there is still enough time (demand) left in the transition to realize the benefits from price adjustment. A good candidate is the time when the two products have equal demand shares.

Table 4.2 compares the above two-price heuristics with the optimal dynamic pricing solution using simulations. Parameter values are as in Figure 4-2. We ran 300 simulations for each policy and compute the mean values. $J^*(x_1,x_2)$ is the optimal value using the

optimal dynamic pricing solution. $J^{2PP}(x_1, x_2)$ is the value using the two-price heuristic that re-prices when $r_t^*(x_1, x_2)$ falls below the initial price or $r_t^{FP}(x_1, x_2)$ goes above the initial price. $J^{2PD}(x_1, x_2)$ is the value achieved using the two-price heuristic that re-prices when the demand rates of the two product crosses over.

x_1	x_2	J^*	J^D	J^D(rolling-horizon)	J^{2PP}	J^{2PD}	J^D/J^*	J^D(rh)/J^*	J^{2PP}/J^*	J^{2PD}/J^*
5	5	169	115	137	142	155	0.684	0.815	0.844	0.921
5	10	461	409	426	436	442	0.886	0.922	0.946	0.957
5	15	697	655	667	673	676	0.940	0.957	0.965	0.970
5	20	860	839	843	845	851	0.975	0.980	0.982	0.989
5	25	998	980	980	986	990	0.982	0.982	0.988	0.992
10	5	288	180	175	238	239	0.625	0.607	0.825	0.831
10	10	552	465	440	498	503	0.841	0.796	0.901	0.911
10	15	741	697	716	718	718	0.940	0.965	0.968	0.969
10	20	890	867	873	879	884	0.975	0.981	0.988	0.994
10	25	1029	1019	1016	1022	1023	0.990	0.987	0.993	0.994
15	5	336	215	212	268	275	0.639	0.629	0.796	0.816
15	10	584	491	462	526	536	0.841	0.791	0.900	0.918
15	15	764	727	736	744	744	0.952	0.963	0.974	0.974
15	20	919	892	901	906	907	0.972	0.981	0.986	0.987
15	25	1053	1031	1043	1047	1050	0.979	0.991	0.994	0.997
20	5	363	239	230	298	302	0.658	0.635	0.822	0.833
20	10	605	519	490	551	557	0.857	0.809	0.910	0.919
20	15	794	739	765	764	766	0.931	0.964	0.963	0.965
20	20	949	916	923	932	932	0.965	0.973	0.983	0.982
20	25	1082	1060	1063	1072	1077	0.980	0.982	0.991	0.996
25	5	386	264	262	322	326	0.685	0.680	0.835	0.847
25	10	635	547	514	578	584	0.862	0.810	0.911	0.920
25	15	825	775	788	794	799	0.940	0.956	0.963	0.969
25	20	965	943	941	957	960	0.977	0.975	0.991	0.994
25	25	1113	1080	1085	1100	1101	0.970	0.975	0.988	0.989
						Average	0.882	0.884	0.936	0.945

Table 4.2: Performance of the Heuristic Policies

We observe that the additional pricing opportunity fills about 50% of the gap between a fixed-price policy and the optimal continuous pricing policy. Therefore, the additional pricing opportunity adds significant value. In fact, the two-price policies works better than the more expensive (i.e., more frequent price adjustment) myopic rolling-horizon approach. Such a myopic policy is often observed in practice: firms determine a price of the old product when they introduce the new product, without considering future price adjustment. When they observe inventory excess or shortage, they adjust price, again assuming no future changes. Our studies indicate that such a passive pricing strategy might not to be a wise one. Not only do they suffer from the cost of frequent price changes, but also the lack of foresight drives profit away.

Further, we observe that the value of the two-price policies is realized through initial higher prices and inventory-based adjustment at a later time. For example, when the starting inventory is (10,10), in the two-price policy derived from *Proposition 4.10* , the optimal initial price of the old product is 26, the average re-price time is around period 20 (with a total planning horizon of 100 periods). The average second price (the price we should change to) is 8. In the two-price policy that re-prices at the demand inflection point, the optimal initial price is 31, the average re-price time is around period 18, and the average second price is 8.6. These observations offer hints for a sensible pricing strategy during product transition: Offer a lower discount initially, and then wait for the uncertainty to play out before readjusting (usually further discounts, but may also be a price increase).

4.7 Discussions and Future Research

The main contribution of this research is that we address the pricing problem in a special albeit ubiquitous industry context – inter-generational product transition. The unique demand characteristics of the two products during the transitional period lead to an intriguing and managerially important problem. Under the assumption that total demand is stationary and the price of the new product is constant, we are able to find a closed-form solution for the dynamic pricing problem. We compare several simple heuristic policies that firms can easily employ in practice against the optimal continuous pricing policy. In particular, our findings indicate that the two-price heuristic works quite well. In contrast to existing two-price policies studied in the literature, we allow both the price and the time to re-price to be decision variables. Surprisingly, the rolling-horizon approach is only marginally better than a fixed deterministic price, and much worse than the simple two-price policy.

The current model does not address demand learning, which is often one of the most important reasons for price adjustment. A company learns more accurate demand information as the new product starts selling. Being able to adjust prices based on the updated demand information is certainly beneficial. However, if a company is limited by its ability to change price frequently, it may have to decide how to best take advantage of its limited pricing opportunities, that is, when to adjust price and what the new price should be. In the two-price policy discussed in this chapter, the main tradeoff in deciding when to re-price is the need to adjust to the current inventory situation and the need to wait for the uncertainty to play out. With demand learning, an important aspect we have to consider is the need to learn enough information on demand. Apparently, the longer we wait to adjust price, the more we will know about the actual demand. But if we wait for too long, we may miss the most advantage of a price adjustment. Existing learning models rarely consider non-stationary demand learning. It will be a challenge to find a simple and appropriate learning model for the demand patterns occurring during a product transition.

To simplify the model and produce meaningful results, we have assumed the price of the new product to be market driven and constant and that the total demand of the two products are fixed during the transitional period. There is certainly room for extension in

this regard. The demand model employed in this chapter allows such extension easily. We can incorporate the non-purchase option through the MNL choice model by assuming a non-purchase utility $u_0(t)$, thus the probability of non-purchase is

$$\rho_0(r,t) = \frac{e^{u_0(t)}}{e^{a_1(t)-r_1} + e^{a_2(t)-r_2} + e^{u_0(t)}}$$ and the probability of purchasing product i is

$$\rho_i(r,t) = \frac{e^{a_i(t)-r_j}}{e^{a_1(t)-r_1} + e^{a_2(t)-r_2} + e^{u_0(t)}} \quad i = 1,2$$

In addition, the base rate λ_0 should be interpreted as the Poisson rate of customer arrival. Such a model allows r_2 to be a decision variable as well. However, the dynamic programming computation will be much more intensive. Additional heuristic policies might be necessary.

The result in this chapter indicates that the optimal price of the old product might increase at the end of its life cycle. Indeed, there is some anecdotal evidence for such phenomenon. The following figure is the price history of two Canon digital cameras from March 2006 to May 2007. In September 2006, the new SD800 IS was introduced as an upgrade of the SD700 IS. Thus SD700 IS can be considered the old product and SD800 IS the new product in this series. We do observe a bit "tail up" of the old product price at the end some time after the new is introduced. In contrast to the assumptions made in our model, the price of the SD800 has been fluctuating quite dramatically over time.

PowerShot SD700 IS   PowerShot SD800 IS
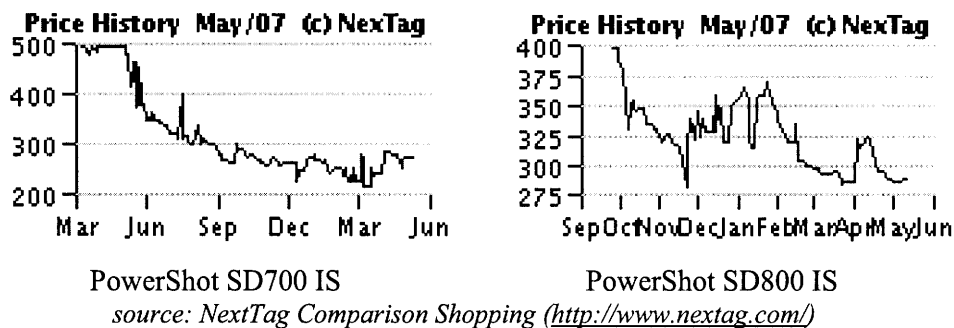*source: NextTag Comparison Shopping (http://www.nextag.com/)*

Figure 4-5: Price History of Canon Digital Cameras

Of course, such observation is no evidence that manufacturers or retailers are pricing their products based on the methods used in this model. But it creates interests for us to collect empirical price data, interpret the pricing dynamics, and come back to our analytical model to see if it has some practical ground.

Chapter 5

Conclusions

In this dissertation, we present two problem areas in Supply Chain Management.

In Chapter 2, we develop a model and a computational approach to analyze optimal history-dependent dynamic contracts under a dual-sourcing strategy. We compare a volume-incentive contract with a margin-reward contract. In addition to the optimal contract structure, we obtain the equilibrium performance and payment trajectories of the suppliers. As a result, our model captures the evolutionary dynamics of the supplier relationships under each type of contract.

The theoretically optimal contract structure requires continuous rank space and is difficult, if not impossible, to implement in practice. Therefore, restricting the supplier ranks to a few discrete levels may simplify the incentive scheme. Such practical consideration also inspires us to think about possible ways to obtain empirical validation for our model: What information can help us quantify incentive structures in practice? Is ranking supplier a common practice? What comes with a certain rank? How is business share split among suppliers? How does margin/volume change with performance? Who get rewarded more by performance, the low-ranked suppliers or the high-ranked ones? These and other related questions may help us define a more concrete long-term incentive structure.

In Chapter 3 we solve the inventory planning problem for inter-generational product transitions under uncertainties of demand and new product launch date. We consider a dynamic substitution decision and its impact on profit and optimal planning quantities. Further, we determine the optimal delay of new product launch when a company is facing large excess of the old product. In Chapter 4, we solve the dynamic pricing decisions during the transitional period and propose several heuristics policies. In particular, we find that a two-price policy with three decision variables (the initial price, the time to re-price, and the new price) works quite well.

In future research, we intend to consider two extensions.

One of the key assumptions in our model is that there is no replenishment opportunity during the transition. As mentioned earlier, certain product upgrades may take a much longer time period. As a result, replenishment during the transition is possible. We would like to explore in future research the optimal continuous-replenishment policy in these cases.

The current pricing model does not consider demand learning. However, demand learning plays an intrinsic role in manager's decision on price changes. One problem managers often struggle with is: When do they take action on price, knowing that they can only do it a couple of times? In addition to the tradeoff between the need to adjust to the current inventory situation and the need to wait for the uncertainty to play out, we have to consider the need to learn enough about demand. Existing learning models rarely consider non-stationary demand learning. To solve for the optimal pricing problem with demand learning, the important next step is to find a simple learning model that is appropriate for the demand patterns occurring during a product transition.

Appendix

Proposition 2.1

$P^S(u,x)$ and $U^S(u,x)$ solve the DSP problem if and only if there exists $\lambda(u)$ and $\mu(u)$ such that the following conditions are satisfied:

$$V'(U) + \lambda(u) + \mu(u)\frac{f_a(x\,|\,a)}{f(x\,|\,a)} = 0$$

$$-1 + \phi'(q_0 P)[\lambda(u) + \mu(u)\frac{f_a(x\,|\,a)}{f(x\,|\,a)}] = 0$$

(PK), (IC).

Proof.

$$L = \int \{q_0 x - q_0 P(u,x) + \delta V(U) + \lambda(u)[\delta\phi(q_0 P(u,x)) + \delta U]$$

$$+ \mu(u)[(\phi(q_0 P(u,x))) + \delta U)\frac{f_a(x\,|\,a)}{f(x\,|\,a)} - \psi'(a(u))]\} f(x\,|\,a)dx$$

$$= \int \{q_0 x - q_0 P(u,x) + \delta V(U) + [\lambda(u) + \mu(u)\frac{f_a(x\,|\,a)}{f(x\,|\,a)}][\phi(q_0 P(u,x)) + \delta U]$$

$$- \mu(u)\psi'(a(u))\} f(x\,|\,a)dx$$

Point-wise optimization with respect to U and P yields

$$V'(U) + \lambda(u) + \mu(u)\frac{f_a(x\,|\,a)}{f(x\,|\,a)} = 0$$

$$-1 + \phi'(q_0 P)[\lambda(u) + \mu(u)\frac{f_a(x\,|\,a)}{f(x\,|\,a)}] = 0. \quad \square$$

Lemma 2.1

With a risk-neutral supplier, $V'(u) = -1$.

Proof.

The supplier is risk-neutral, thus $\phi'(\cdot) = 1$
From *Proposition 2.1*, we have

$$V'(U) + \lambda(u) + \mu(u)\frac{f_a(x\,|\,a)}{f(x\,|\,a)} = 0$$

$$-1 + \lambda(u) + \mu(u)\frac{f_a(x\,|\,a)}{f(x\,|\,a)} = 0$$

which then gives $V'(U(u,x)) = -1$, i.e., $V(\cdot)$ is linear over the range of $U(u,x)$. Since the self-generation property implies that $U(u,x)$ and u take on values that belong to the same equilibrium set, we have $V'(u) = -1$. □

Proposition 2.2

With a risk-neutral supplier, a bang-bang policy can be optimal. That is, a buyer promises

a constant continuation payoff $U = \dfrac{\delta q_0[\bar{p} - (\bar{p} - \underline{p})F(\bar{x}\,|\,a^*)] - \psi(a^*)}{1-\delta}$

and $P = \begin{cases} \bar{p} \text{ if } x \geq \bar{x} \\ \underline{p} \text{ if } x < \bar{x} \end{cases}$ as the next period margin. The optimal action input a^* and the

critical performance threshold $\bar{x}$ can be determined jointly by $\dfrac{-\psi'(a)}{F_a(\bar{x}\,|\,a)} = q_0(\bar{p} - \underline{p})$ and

$f_a(\bar{x}\,|\,a) = 0$.

Proof.

Since $V'(u) = -1$, we have $V(u) + u = S$ where S is a constant. As a result, the objective function in (DSP) becomes $E[q_0 x - q_0 P(u,x) - \delta U(u,x) + \delta S \,|\, a(u)]$ instead.
We obtain the Lagrangian:

$L = \int \{q_0 x - q_0 P(u,x) - \delta U + \delta S + [\lambda(u) + \mu(u)\dfrac{f_a(x\,|\,a)}{f(x\,|\,a)}][q_0 P(u,x) + \delta U]$

$- \mu(u)\psi'(a(u))\} f(x\,|\,a)dx$

$= \int [-1 + \lambda(u) + \mu(u)\dfrac{f_a(x\,|\,a)}{f(x\,|\,a)}][q_0 P(u,x) + \delta U] f(x\,|\,a)dx + q_0 E(x\,|\,a) + \delta S - \mu(u)\psi'(a)$

Therefore, the optimization problem is linear in $q_0 P(u,x) + \delta U(u,x)$.

We define $C(u,x) \equiv q_0 P(u,x) + \delta U(u,x)$, the optimal policy prescribes

$C^*(u,x) = \begin{cases} \bar{C} \text{ if } \lambda(u) + \mu(u)\dfrac{f_a(x\,|\,a)}{f(x\,|\,a)} - 1 \geq 0 \\ \underline{C} \text{ if } \lambda(u) + \mu(u)\dfrac{f_a(x\,|\,a)}{f(x\,|\,a)} - 1 < 0 \end{cases}$

where $\bar{C}$ and $\underline{C}$ are the extreme values of $C(u,x)$

From the envelope theorem, we have $V'(u) = -\lambda(u) \quad \forall u \in \mathcal{U}$, thus $\lambda(u) = 1 \,\forall u \in \mathcal{U}$.
From *Assumption 2.3*, we have $\mu(u) > 0$. Hence for the subproblem starting from the second period on, the optimal policy is

94

$$C*(u,x) = \begin{cases} \overline{C} \text{ if } x \geq \overline{x} \\ \underline{C} \text{ if } x < \overline{x} \end{cases}$$

where $\overline{C}$ and $\underline{C}$ are the extreme values of $C(u,x)$ and $\overline{x}$ satisfies $f_a(\overline{x} \mid a(u)) = 0$.

Since in a PPE the set of possible continuation payoffs is the same at any time t, this applies to the original problem as well.

If we restrict the continuation payoff u to be a constant, then the optimal contract is reduced to a history independent form: $P = \begin{cases} \overline{p} \text{ if } x \geq \overline{x} \\ \underline{p} \text{ if } x < \overline{x} \end{cases}$

When u is constant, the optimal action a^* stays the same in each period.

From the (PK) constraints, we have $u = E[q_0 P + \delta U] - \psi(a) = E[q_0 P] + \delta u - \psi(a)$

Therefore $u = \dfrac{1}{1-\delta}[q_0 E(P) - \psi(a)] = \dfrac{1}{1-\delta}\{q_0[\overline{p} - F(\overline{x} \mid a)(\overline{p} - \underline{p})] - \psi(a)\}$. The optimal action a^* should maximize $q_0[\overline{p} - F(\overline{x} \mid a)(\overline{p} - \underline{p})] - \psi(a)$. We can derive the first-order condition to be $\dfrac{-\psi'(a)}{F_a(\overline{x} \mid a)} = q_0(\overline{p} - \underline{p})$. $\square$

Proposition 2.3

$P_1(u,x) = P^S(u_1,x_1), P_2(u,x) = P^S(u_2,x_2)$ and
$U_1(u,x) = U^S(u_1,x_1), U_2(u,x) = U^S(u_2,x_2)$ solves the DTM problem.

Proof.

It is easy to verify that $P_1(u,x) = P^S(u_1,x_1), P_2(u,x) = P^S(u_2,x_2)$ and
$U_1(u,x) = U^S(u_1,x_1), U_2(u,x) = U^S(u_2,x_2)$ is feasible for the DTM problem.

Let $\lambda_i(u)$ and $\mu_i(u)$ be the multipliers for the constraints (PK)$_i$ and (IC)$_i$, we obtain the Lagrangian:

$$L = \int \{[q_0 x_1 + q_0 x_2 - q_0 P_1(u,x) - q_0 P_2(u,x) + \delta V(U_1(u,x), U_2(u,x))]$$

$$+ \lambda_1(u)[\phi(q_0 P_1(u,x)) + \delta U_1(u,x) - \psi(a_1(u)) - u_1]$$

$$+ \lambda_2(u)[\phi(q_0 P_2(u,x)) + \delta U_2(u,x) - \psi(a_2(u)) - u_2]$$

$$+ \mu_1(u)[(\phi(q_0 P_1(u,x)) + \delta U_1(u,x))\frac{f_{a_1}(x_1 \mid a_1)}{f(x_1 \mid a_1)} - \psi'(a_1(u))]$$

$$+ \mu_2(u)[(\phi(q_0 P_2(u,x)) + \delta U_2(u,x))\frac{f_{a_2}(x_2 \mid a_2)}{f(x_2 \mid a_2)} - \psi'(a_2(u))]\}f(x_1 \mid a_1)f(x_2 \mid a_2)dx_1 dx_2$$

Point-wise optimization of L with respect to P_1, P_2, U_1 and U_2 yields

$$\frac{\partial V}{\partial U_1} + \lambda_1(u) + \mu_1(u)\frac{f_{a_1}(x_1 \mid a_1)}{f(x_1 \mid a_1)} = 0$$

$$\frac{\partial V}{\partial U_2} + \lambda_2(u) + \mu_2(u)\frac{f_{a_2}(x_2 \mid a_2)}{f(x_2 \mid a_2)} = 0$$

$$-1 + \phi'(q_0 P_1)[\lambda_1(u) + \mu_1(u)\frac{f_{a_1}(x_1 \mid a_1)}{f(x_1 \mid a_1)}] = 0$$

$$-1 + \phi'(q_0 P_2)[\lambda_2(u) + \mu_2(u)\frac{f_{a_2}(x_2 \mid a_2)}{f(x_2 \mid a_2)}] = 0$$

which then gives $\dfrac{\partial V}{\partial U_1} = -\dfrac{1}{\phi'(q_0 P_1)}$ and $\dfrac{\partial V}{\partial U_2} = -\dfrac{1}{\phi'(q_0 P_2)}$.

Therefore, given P_1 and P_2, V is separable in U_1 and U_2.

As a result, the DTM problem can be decoupled into two independent problems.

$$V_1(u) \equiv \max_{a,P,U} E[q_0 x_1 - q_0 P_1(u,x) + \delta V_2(U_1(u,x)) \mid a_1(u), a_2(u)]$$

$$\text{s.t.} \quad (PK_1), (IC_1), \quad \underline{p} \le P_1(u,x) \le \bar{p} \tag{P1}$$

and

$$V_2(u) \equiv \max_{a,P,U} E[q_0 x_2 - q_0 P_2(u,x) + \delta V_2(U_2(u,x)) \mid a_1(u), a_2(u)]$$

$$\text{s.t.} \quad (PK_2), (IC_2), \quad \underline{p} \le P_2(u,x) \le \bar{p} \tag{P2}$$

We then show that $P^S(u_1, x_1)$ and $U^S(u_1, x_1)$ solve problem (P1) and that $P^S(u_2, x_2)$ and $P^S(u_2, x_2)$ solve problem (P2).

Suppose $P_1(u,x)$ and $U_1(u,x)$ is the optimal solution to problem (P1). Then $P_1(u,x)$ and $U_1(u,x)$ satisfies the following Kuhn-Tucker conditions:

$$\frac{dV}{dU_1} + \lambda_1(u) + \mu_1(u)\frac{f_{a_1}(x_1 \mid a_1)}{f(x_1 \mid a_1)} = 0$$

$$-1 + \phi'(q_0 P_1)[\lambda_1(u) + \mu_1(u)\frac{f_{a_1}(x_1 \mid a_1)}{f(x_1 \mid a_1)}] = 0$$

$$\int [\phi(q_0 P_1(u,x)) + \delta U_1(u,x)]f(x_1 \mid a_1)f(x_2 \mid a_2)dx_1 dx_2 - \psi(a_1(u)) = u_1 \tag{PK_1}$$

$$\int [\phi(q_0 P_1(u,x)) + \delta U_1(u,x)]\frac{f_{a_1}(x_1 \mid a_1)}{f(x_1 \mid a_1)}f(x_1 \mid a_1)f(x_2 \mid a_2)dx_1 dx_2 - \psi'(a_i(u)) = 0 \tag{IC_1}$$

$$\underline{p} \le P_i(u,x) \le \bar{p}$$

Given $P_1(u,x)$ and $U_1(u,x)$, we can find $\hat{P}(u,x_1)$ and $\hat{U}(u,x_1)$ such that

$\phi(q_0\hat{P}(u,x_1)) + \hat{U}(u,x_1) = \int \phi(q_0 P_1(u,x)) + U_1(u,x)]f(x_2 \mid a_2)dx_2$. That is, we can find $\hat{P}(u,x_1)$ and $\hat{U}(u,x_1)$ such that

$$\frac{dV}{d\hat{U}} + \lambda_1(u) + \mu_1(u)\frac{f_{a_1}(x_1 \mid a_1)}{f(x_1 \mid a_1)} = 0$$

$$-1 + \phi'(q_0\hat{P})[\lambda_1(u) + \mu_1(u)\frac{f_{a_1}(x_1 \mid a_1)}{f(x_1 \mid a_1)}] = 0$$

$$\int [\phi(q_0\hat{P}(u,x_1)) + \delta\hat{U}(u,x_1)]f(x_1 \mid a_1)dx_1 - \psi(a_1(u)) = u_1 \quad (\text{PK}_1)$$

$$\int [\phi(q_0\hat{P}(u,x_1)) + \delta\hat{U}(u,x_1)]\frac{f_{a_1}(x_1 \mid a_1)}{f(x_1 \mid a_1)}f(x_1 \mid a_1)dx_1 - \psi'(a_1(u)) = 0 \quad (\text{IC}_1)$$

$$\underline{p} \le \hat{P}(u,x_1) \le \bar{p}$$

Compare the above with the Kuhn-Tucker conditions for the DSP problem and also observing that u_2 does not affect the optimal solution (since it does not affect the objective function or the constraints as we solve for $\hat{P}(u,x_1)$ and $\hat{U}(u,x_1)$), it is then clear that $\hat{P}(u,x_1) = P^S(u_1,x_1)$ and $\hat{U}(u,x_1) = U^S(u_1,x_1)$. $\square$

Proposition 2.4
With risk-neutral suppliers, the optimal volume-incentive contract can be implemented using a series of static contracts. In each period, the optimal split is "all-or-nothing":

$$Q_i^*(u,x) = \begin{cases} 2q_0 & \text{if } a_1(U(u,x)) > a_2(U(u,x)) \\ 0 & \text{otherwise} \end{cases}$$

Proof.

$$V(q,u) \equiv \max_{a,Q,U} E[q_1x_1 + q_2x_2 - 2p_0q_0 + \delta V(Q(u,x),U(u,x)) \mid a_1(u),a_2(u)]$$

$$\text{s.t.} \quad E[\phi(p_0Q_i(u,x)) + \delta U_i(u,x) \mid a_i(u),a_j(u)] - \psi(a_i(u)) = u_i \quad \forall u_i \in \mathcal{U}_i \ (\text{PK}_i)\,i = 1,2$$

$$E[\phi(p_0Q_i(u,x)) + \delta U_i(u,x) \mid a_i(u),a_j(u)] - \psi(a_i(u))$$

$$\ge E[\phi(p_0Q_i(u,x)) + \delta U_i(u,x) \mid \hat{a}_i,a_j(u)] - \psi(\hat{a}_i) \,\forall \hat{a}_i \in A \quad (\text{IC}_i),i = 1,2$$

$$Q_1(u,x) + Q_2(u,x) = 2q_0 \qquad\qquad\qquad\qquad (\text{BC})$$

$$Q_i(u,x) \ge 0 \qquad\qquad\qquad\qquad\qquad\qquad (\text{NN}_i)\,i = 1,2$$

Since (BC) is an equality constraint, we can reduce the number of variables by substituting Q_2 with $2q_0 - Q_1$.

Thus the Lagrangian becomes

$$L = \int\int [\delta V(Q,U) + (p_0 Q_1 + \delta U_1)(\lambda_1 + \mu_1 \frac{f_{a1}(x_1 \mid a_1)}{f(x_1 \mid a_1)})$$
$$+ (2p_0 q_0 - p_0 Q_1 + \delta U_2)(\lambda_2 + \mu_2 \frac{f_{a2}(x_2 \mid a_2)}{f(x_2 \mid a_2)})] f(x_1 \mid a_1) f(x_2 \mid a_2) dx_1 dx_2$$
$$+ q_1 E(x_1 \mid a_1) + q_2 E(x_2 \mid a_2) + \lambda_1(\psi(a_1) - u_1) - \mu_1 \psi'(a_1) + \lambda_2(\psi(a_2) - u_2) - \mu_2 \psi'(a_2)$$

Pointwise optimization w.r.t. Q_1, U_1 and U_2 yields:

$$\nabla_{U_i} V(U(u,x)) + \lambda_i(u) + \mu_i(u) \frac{f_{a_i}(x \mid a_i)}{f(x \mid a_i)} = 0$$

$$\lambda_1(u) + \mu_1(u) \frac{f_{a_1}(x_1 \mid a_1)}{f(x_1 \mid a_1)} = \lambda_2(u) + \mu_2(u) \frac{f_{a_2}(x_2 \mid a_2)}{f(x_2 \mid a_2)}$$

Therefore, $\nabla_{U_1} V(\cdot) = \nabla_{U_2} V(\cdot)$.

We can rewrite the Lagrangian as

$$L = \int\int [\delta V(Q,U) + (2p_0 q_0 + \delta U_1 + \delta U_2)(\lambda_1 + \mu_1 \frac{f_{a1}(x_1 \mid a_1)}{f(x_1 \mid a_1)})] f(x_1 \mid a_1) f(x_2 \mid a_2) dx_1 dx_2$$
$$+ q_1 E(x_1 \mid a_1) + q_2 E(x_2 \mid a_2) + \lambda_1(\psi(a_1) - u_1) - \mu_1 \psi'(a_1) + \lambda_2(\psi(a_2) - u_2) - \mu_2 \psi'(a_2)$$

Thus the objective function depends on Q only through $V(Q,U)$.
Since
$$V(Q,U) = Q_1 E(x_1 \mid a_1(U)) + Q_2 E(x_2 \mid a_2(U)) + ...$$
$$= 2q_0 E(x_2) + Q_1(E(x_1 \mid a_1(U)) - E(x_2 \mid a_2(U))) + ...$$
and $E(x_1 \mid a_1(U)) - E(x_2 \mid a_2(U)) > 0$ iff $a_1(U) > a_2(U)$
Thus the optimal volume award is

$$Q_i^*(u,x) = \begin{cases} 2q_0 & \text{if } a_1(U(u,x)) > a_2(U(u,x)) \\ 0 & \text{otherwise} \end{cases}$$

□

Lemma 3.1
$\alpha(x_2, t) \equiv V(0, x_2, t+1) - V(0, x_2 - 1, t+1)$ is non-increasing in x_2.

Proof.

We prove this by induction.
We first note that $\alpha(x_2, t)$ non-increasing in x_2 is equivalent to V being discretely concave.

For ease of exposition, we also define $\alpha(0,t) \equiv (v_2 + h_2)/\delta$. From the terminal value definition, $V(0,x_2,\tau+T) = s_2 x_2$, thus $\alpha(x_2,\tau+T-1) = s_2$. From assumption (ii), it is also easy to see that $\alpha(1,\tau+T-1) < \alpha(0,\tau+T-1)$. Therefore, $\alpha(x_2,\tau+T-1)$ is non-increasing in x_2.

Assume for induction that $\alpha(x_2, t)$ is non-increasing in x_2. Then condition (3-5) implies that $\exists\ \bar{x}_2(t)$ s.t. it is optimal to substitute at period t if and only if $x_2 > \bar{x}_2(t)$. $\bar{x}_2(t)$ is determined as the largest inventory level x_2 such that $\delta\alpha(x_2,t) \geq v_1 - g + h_2$ is true. In the special case when $\delta\alpha(1,t) < v_1 - g + h_2$, we define $\bar{x}_2(t) = 0$. Then from equation (3-1), we have:

$$\alpha(x_2,t-1) = (1 - \lambda(t))[-h_2 + \delta\alpha(x_2,t)] + \lambda_2(t)[-h_2 + \delta\alpha(x_2-1,t)]$$

$$+ \lambda_1(t)\begin{cases} -h_2 + \delta\alpha(x_2,t) & \text{if } x_2 \leq \bar{x}_2(t) \\ -h_2 + \delta\alpha(x_2-1,t) & \text{if } x_2 - 1 > \bar{x}_2(t) \\ v_1 - g & \text{if } x_2 - 1 \leq \bar{x}_2(t) < x_2 \end{cases} \quad (3\text{-}15)$$

Given that $\alpha(x_2,t)$ is non-increasing in x_2, it is clear from equation (3-15) that $\alpha(x_2,t-1)$ is non-increasing in x_2 for all $x_2 \geq 1$, finishing the induction proof. $\square$

Proposition 3.2
With homogeneous Poisson demands, the substitution threshold $\bar{x}_2(t)$ is non-increasing in t.

Proof.

For the ease of representation, we drop the subscripts of x_2 and $\bar{x}_2(t)$ in the proof.
Induction assumption: $\bar{x}(t) \geq \bar{x}(t+1)$ and $\alpha(x,t) \geq \alpha(x,t+1)$ $\forall x \in [1,\bar{x}(t)]$
We need to show $\bar{x}(t-1) \geq \bar{x}(t)$ and $\alpha(x,t-1) \geq \alpha(x,t)$ $\forall x \in [1,\bar{x}(t-1)]$.
From the definition of $\bar{x}(t)$, it is easy to see that $\delta\alpha(x,t) < v_1 - g + h_2$ if $x > \bar{x}(t)$ and $\delta\alpha(x,t) \geq v_1 - g + h_2$ if $x \leq \bar{x}(t)$.
Assume for contradiction that $\bar{x}(t-1) < \bar{x}(t)$, we then have $\delta\alpha(\bar{x}(t-1),t) \geq v_1 - g + h$.
From equation (3-15), we have:
$\alpha(\bar{x}(t),t-1) - \alpha(\bar{x}(t),t)$
$= (1 - \lambda)\delta[\alpha(\bar{x}(t),t) - \alpha(\bar{x}(t),t+1)] + \lambda_2\delta[\alpha(\bar{x}(t)-1,t) - \alpha(\bar{x}(t)-1,t+1)]$ By induction

$$+ \lambda_1\begin{cases} \delta[\alpha(\bar{x}(t),t) - \alpha(\bar{x}(t),t+1)] & \text{if } \bar{x}(t) \leq \bar{x}(t+1) \\ \delta[\alpha(\bar{x}(t),t) - \alpha(\bar{x}(t)-1,t+1)] & \text{if } \bar{x}(t+1) < \bar{x}(t)-1 \\ \delta\alpha(\bar{x}(t),t) - (v_1 - g + h_2) & \text{if } \bar{x}(t)-1 \leq \bar{x}(t+1) \end{cases}$$

assumption, the terms $\alpha(\bar{x}(t),t) - \alpha(\bar{x}(t),t+1)$, $\alpha(\bar{x}(t)-1,t) - \alpha(\bar{x}(t)-1,t+1)$ are nonnegative. We know from the definition of $\bar{x}(t)$ that $\delta\alpha(\bar{x}(t),t) \geq v_1 - g + h$. Under the condition $\bar{x}(t+1) < \bar{x}(t)-1$ we have $\alpha(\bar{x}(t),t) \geq (v_1 - g + h)/\delta > \alpha(\bar{x}(t)-1,t+1)$. Thus the term $\alpha(\bar{x}(t),t) - \alpha(\bar{x}(t)-1,t+1)$ is positive. Hence we obtain

99

$\alpha(\bar{x}(t), t-1) > \alpha(\bar{x}(t), t) \geq v_1 - g + h$ which implies that $\bar{x}(t-1) > \bar{x}(t)$. Therefore, we have a contradiction. Thus we have proved $\bar{x}(t-1) \geq \bar{x}(t)$.

Next we need to show $\alpha(x, t-1) \geq \alpha(x, t) \quad \forall x \in [1, \bar{x}(t-1)]$.

Given $\bar{x}(t-1) \geq \bar{x}(t) \quad \forall t = \tau + 1, \tau + 2, \ldots, \tau + T$, we can separate the interval $[1, \bar{x}(t-1)]$ into three subintervals :

i) $x \in [1, \bar{x}(t+1)]$; ii) $x \in (\bar{x}(t+1), \bar{x}(t)]$ and iii) $x \in (\bar{x}(t), \bar{x}(t-1)]$.

i) $\forall x \in [1, \bar{x}(t+1)]$

$\alpha(x, t-1) - \alpha(x, t)$

$= (1 - \lambda)\delta[\alpha(x, t) - \alpha(x, t+1)] + \lambda_2 \delta[\alpha(x-1, t) - \alpha(x-1, t+1)] + \lambda_1 \delta[\alpha(x, t) - \alpha(x, t+1)]$

By induction assumption, the RHS terms are all nonnegative, thus

$\alpha(x, t-1) \geq \alpha(x, t) \quad \forall x \in [1, \bar{x}(t+1)]$

ii) $\forall x \in (\bar{x}(t+1), \bar{x}(t)]$,

$\alpha(x, t-1) - \alpha(x, t) = (1 - \lambda)\delta[\alpha(x, t) - \alpha(x, t+1)] + \lambda_2 \delta[\alpha(x-1, t) - \alpha(x-1, t+1)]$

$\qquad + \lambda_1 \begin{cases} \delta[\alpha(x, t) - \alpha(x-1, t+1)] & \text{if } x - 1 > \bar{x}(t+1) \\ \delta\alpha(x, t) - (v_1 - g + h_2) & \text{if } x - 1 \leq \bar{x}(t+1) < x \end{cases}$

$\delta\alpha(x, t) - (v_1 - g + h_2)$ is nonnegative as $x \leq \bar{x}(t)$,

$\alpha(x, t) - \alpha(x-1, t+1)$ is nonnegative because for $x \leq \bar{x}(t)$ and $x - 1 > \bar{x}(t+1)$ we have

$\alpha(x, t) \geq (v_1 - g + h_2)/\delta$ and $\alpha(x-1, t+1) \leq (v_1 - g + h_2)/\delta$

iii) $x \in (\bar{x}(t), \bar{x}(t-1)]$

since $x > \bar{x}(t)$ and $x \leq \bar{x}(t-1)$, condition (5) implies $\alpha(x, t) \leq (v_1 - g + h_2)/\delta$ and

$\alpha(x, t-1) \geq (v_1 - g + h_2)/\delta$, thus $\alpha(x, t-1) - \alpha(x, t)$ is nonnegative.

Therefore, we have $\alpha(x, t-1) \geq \alpha(x, t) \quad \forall x \in [1, \bar{x}(t-1)]$, finishing the induction step.

Now we establish the base case. That is, we need to show $\bar{x}(\tau + T - 1) \geq \bar{x}(\tau + T)$ and that

$\alpha(x, \tau + T - 1) \geq \alpha(x, \tau + T) \quad \forall x \in [1, \bar{x}(\tau + T - 1)]$

Assumption (iii) $v_1 - g > s_2$ is equivalent to $\alpha(x, \tau + T) < v_1 - g$. Hence

$\delta\alpha(x, \tau + T) < v_1 - g + h_2$ for $x \geq 1$, Thus it is optimal to substitute at period T whenever inventory is positive, or equivalently $\bar{x}(\tau + T) = 0$. Hence $\bar{x}(\tau + T - 1) \geq \bar{x}(\tau + T)$ holds.

Since $\alpha(x, \tau + T - 1) = -h_2 + \delta s_2$ for $x \geq 2$, we have $\delta\alpha(x, \tau + T - 1) < s_2 < v_1 - g + h_2$ for $x \geq 2$. Thus $\bar{x}(\tau + T - 1) < 2$.

If $\bar{x}(\tau + T - 1) < 1$, the set $[1, \bar{x}(\tau + T - 1)]$ contains no integral points; if $\bar{x}(\tau + T - 1) \geq 1$,

$\alpha(1, \tau + T - 1) \geq (v_1 - g + h_2)/\delta > s_2 = \alpha(1, \tau + T)$. Therefore the base case is true. □

Proposition 3.3
If the demands are non-homogeneous Poisson, but the holding cost h=0 and the discount factor δ=1, then $\bar{x}_2(t)$ is non-increasing in t.

Proof.

For the ease of representation, we drop the subscripts of x_2 and $\bar{x}_2(t)$ in the proof. When there are no holding costs and $\delta=1$, equation (4-8) becomes:

$$\alpha(x,t-1) = (1-\lambda(t))\alpha(x,t) + \lambda_2(t)\alpha(x-1,t)$$

$$+ \lambda_1(t)\begin{cases} \alpha(x,t) & \text{if } x \le \bar{x}(t) \\ \alpha(x-1,t) & \text{if } x-1 > \bar{x}(t) \\ v_1 - g & \text{if } x-1 \le \bar{x}(t) < x \end{cases} \qquad (3\text{-}15a)$$

Thus

$$\alpha(x,t-1) - \alpha(x,t)$$

$$= \begin{cases} \lambda_2(t)[\alpha(x-1,t) - \alpha(x,t)] & \text{if } x \le \bar{x}(t) \\ \lambda(t)[\alpha(x-1,t) - \alpha(x,t)] & \text{if } x-1 > \bar{x}(t) \\ \lambda_2(t)[\alpha(x-1,t) - \alpha(x,t)] + \lambda_1(t)[v_1 - g - \alpha(x,t)] & \text{if } x-1 \le \bar{x}(t) < x \end{cases}$$

Since $\alpha(x,t)$ is non-increasing in x, $\alpha(x-1,t) - \alpha(x,t)$ is nonnegative, the term $v_1 - g - \alpha(x,t)$ is nonnegative for $x > \bar{x}(t)$. Therefore, we have $\alpha(x,t-1) \ge \alpha(x,t) \quad \forall t = \tau+1, \tau+2, ..., \tau+T$. □

Proposition 3.4
If the total demand stays constant, $\lambda_1(t)$ is non-increasing in t and $\lambda_2(t)$ is non-decreasing in t, then $\bar{x}_2(t)$ is a unimodal function of t: it is increasing in t before a certain time $\hat{t}$ and decreasing in t after $\hat{t}$.

Proof.

For the ease of representation, we drop the subscripts of x_2 and $\bar{x}_2(t)$ in the proof. In order to show that $\bar{x}(t)$ is a unimodal, it is sufficient to show the following:
a) $\bar{x}(t+1) \ge \bar{x}(t) \Rightarrow \bar{x}(t) \ge \bar{x}(t-1)$
b) $\bar{x}(t-1) \ge \bar{x}(t) \Rightarrow \bar{x}(t) \ge \bar{x}(t+1)$
We prove a) first.
Given $\bar{x}(t+1) \ge \bar{x}(t)$, we consider the following five cases:
i) $\bar{x}(t) \le x-1 < x \le \bar{x}(t+1)$
ii) $\bar{x}(t) \le \bar{x}(t+1) \le x-1 < x$
iii) $x-1 < x \le \bar{x}(t) \le \bar{x}(t+1)$
iv) $\bar{x}(t) \le x-1 \le \bar{x}(t+1) < x$
v) $x-1 \le \bar{x}(t) \le x \le \bar{x}(t+1)$
It is easy to show that for each case $\alpha(x,t-1) < \alpha(x,t)$, which in turn yields $\bar{x}(t) \ge \bar{x}(t-1)$
We then prove b) $\bar{x}(t-1) \ge \bar{x}(t) \Rightarrow \bar{x}(t) \ge \bar{x}(t+1)$.
Assume for contradiction that $\bar{x}(t) \le \bar{x}(t+1)$, from a), we have $\bar{x}(t-1) \le \bar{x}(t)$, which contradicts $\bar{x}(t-1) \ge \bar{x}(t)$.
From a) and b), it is easy to see that $\bar{x}(t)$ is unimodal in t. □

Proposition 3.5

$V(x_1,x_2,t)$ is jointly concave in x_1 and x_2. There exists a unique pair of (x_1,x_2) such that $V(x_1,x_2,t)$ is maximized.

Proof.

Dfine $\alpha(x_1,x_2,t) \equiv V(x_1,x_2,t+1) - V(x_1,x_2-1,t+1)$ and
$\beta(x_1,x_2,t) \equiv V(x_1,x_2,t+1) - V(x_1-1,x_2,t+1)$.

To show V is jointly concave, it is sufficient and necessary to show that $\alpha(x_1,x_2,t)$ and $\beta(x_1,x_2,t)$ are non-increasing in x_1 and x_2. We prove this using induction.

From equations (3-1)-(3-4),

For $x_1 \geq 1, x_2 \geq 1$

$$\alpha(x_1,x_2,t-1) = -h_2 + (1-\lambda(t)\Delta)\delta\alpha(x_1,x_2,t) + \lambda_2(t)\Delta\delta\alpha(x_1,x_2-1,t)$$
$$+ \lambda_1(t)\Delta\delta\alpha(x_1-1,x_2,t)$$

For $x_1 = 0, x_2 \geq 1$, from equation (3-15), we have

$$\alpha(x_1,x_2,t-1) = -h_2 + (1-\lambda(t)\Delta)\delta\alpha(0,x_2,t) + \lambda_2(t)\Delta\delta\alpha(0,x_2-1,t)$$

$$+ \lambda_1(t)\Delta\begin{cases} \delta\alpha(0,x_2,t) & \text{if } x_2 \leq \bar{x}(t) \\ \delta\alpha(0,x_2-1,t) & \text{if } x_2-1 > \bar{x}(t) \\ v_1 - g + h_2 & \text{if } x_2 - 1 \leq \bar{x}(t) < x_2 \end{cases}$$

For ease of exposition, we also define $\beta(0,x_2,t) \equiv (v_1 + h_2)/\delta$

For $x_1 \geq 2, x_2 \geq 1$

$$\beta(x_1,x_2,t-1) = -h_1 + (1-\lambda(t)\Delta)\delta\beta(x_1,x_2,t) + \lambda_2(t)\Delta\delta\beta(x_1,x_2-1,t)$$
$$+ \lambda_1(t)\Delta\delta\beta(x_1-1,x_2,t)$$

For $x_1 = 1, x_2 \geq 1$

$$\beta(x_1,x_2,t-1) = -h_1 + (1-\lambda(t)\Delta)\delta\beta(1,x_2,t) + \lambda_2(t)\Delta\delta\beta(1,x_2-1,t)$$

$$+ \lambda_1(t)\Delta\begin{cases} (v_1 + h_1) & \text{if } x_2 < \bar{x}(t) \\ \delta\alpha(0,x_2,t) - h_2 + g + h_1 & \text{o.w.} \end{cases}$$

For $x_1 \geq 1, x_2 = 0$

$$\beta(x_1,x_2,t-1) = -h_1 + (1-\lambda(t)\Delta)\delta\beta(x_1,0,t) + \lambda_2(t)\Delta\delta\beta(x_1,0,t) + \lambda_1(t)\Delta\delta\beta(x_1-1,0,t)$$

Therefore, we obtain the following recursive equations:

Regarding the partial difference $\alpha(x_1,x_2) - \alpha(x_1,x_2-1)$

For $x_1 \geq 1, x_2 \geq 2$

$$\alpha(x_1, x_2, t-1) - \alpha(x_1, x_2 - 1, t-1) = (1 - \lambda(t)\Delta)\delta[\alpha(x_1, x_2, t) - \alpha(x_1, x_2 - 1, t)]$$
$$+ \lambda_2(t)\Delta\delta[\alpha(x_1, x_2 - 1, t) - \alpha(x_1, x_2 - 2, t)]$$
$$+ \lambda_1(t)\Delta\delta[\alpha(x_1 - 1, x_2, t) - \alpha(x_1 - 1, x_2 - 1, t)]$$

For $x_1 = 0, x_2 \geq 2$

$$\alpha(0, x_2, t-1) - \alpha(0, x_2 - 1, t-1)$$
$$= (1 - \lambda(t)\Delta)\delta[\alpha(0, x_2, t) - \alpha(0, x_2 - 1, t)]$$
$$+ \lambda_2(t)\Delta\delta[\alpha(0, x_2 - 1, t) - \alpha(0, x_2 - 2, t)]$$
$$+ \lambda_1(t)\Delta \begin{cases} \delta[\alpha(0, x_2, t) - \alpha(0, x_2 - 1, t)] & \text{if } x_2 \leq \bar{x}(t) \\ \delta[\alpha(0, x_2 - 1, t) - \alpha(0, x_2 - 2, t)] & \text{if } x_2 - 2 > \bar{x}(t) \\ \delta\alpha(0, x_2 - 1, t) - (v_1 - g + h_2) & \text{if } x_2 - 2 \leq \bar{x}(t) < x_2 - 1 < x_2 \\ (v_1 - g + h_2) - \delta\alpha(0, x_2 - 1, t) & \text{if } x_2 - 2 < x_2 - 1 \leq \bar{x}(t) < x_2 \end{cases}$$

Regarding the partial difference $\beta(x_1, x_2) - \beta(x_1 - 1, x_2)$:

For $x_1 \geq 3, x_2 \geq 1$

$$\beta(x_1, x_2, t-1) - \beta(x_1 - 1, x_2, t-1) = (1 - \lambda(t)\Delta)\delta[\beta(x_1, x_2, t) - \beta(x_1 - 1, x_2, t)]$$
$$+ \lambda_2(t)\Delta\delta[\beta(x_1, x_2 - 1, t) - \beta(x_1 - 1, x_2 - 1, t)]$$
$$+ \lambda_1(t)\Delta\delta[\beta(x_1 - 1, x_2, t) - \beta(x_1 - 2, x_2, t)]$$

For $x_1 = 2, x_2 \geq 1$

$$\beta(2, x_2, t-1) - \beta(1, x_2, t-1) = (1 - \lambda(t)\Delta)\delta[\beta(2, x_2, t) - \beta(1, x_2, t)]$$
$$+ \lambda_2(t)\Delta\delta[\beta(2, x_2 - 1, t) - \beta(1, x_2 - 1, t)]$$
$$+ \lambda_1(t)\Delta \begin{cases} \delta\beta(1, x_2, t) - (v_1 + h_1) & \text{if } x_2 < \bar{x}(t) \\ \delta[\beta(1, x_2, t) - \alpha(0, x_2, t)] + h_2 - h_1 - g & \text{o.w.} \end{cases}$$

Regarding the cross partial difference $\alpha(x_1, x_2) - \alpha(x_1 - 1, x_2)$:

For $x_1 \geq 2, x_2 \geq 1$

$$\alpha(x_1, x_2, t-1) - \alpha(x_1 - 1, x_2, t-1) = (1 - \lambda(t)\Delta)\delta[\alpha(x_1, x_2, t) - \alpha(x_1 - 1, x_2, t)]$$
$$+ \lambda_2(t)\Delta\delta[\alpha(x_1, x_2 - 1, t) - \alpha(x_1 - 1, x_2 - 1, t)]$$
$$+ \lambda_1(t)\Delta\delta[\alpha(x_1 - 1, x_2, t) - \alpha(x_1 - 2, x_2, t)]$$

For $x_1 = 1, x_2 \geq 1$

$$\alpha(1, x_2, t-1) - \alpha(0, x_2, t-1) = (1 - \lambda(t)\Delta)\delta[\alpha(1, x_2, t) - \alpha(0, x_2, t)]$$
$$+ \lambda_2(t)\Delta\delta[\alpha(1, x_2 - 1, t) - \alpha(0, x_2 - 1, t)]$$
$$+ \lambda_1(t)\Delta \begin{cases} 0 & \text{if } x_2 < \bar{x}(t) \\ \delta[\alpha(0, x_2, t) - \alpha(0, x_2 - 1, t)] & \text{if } x_2 - 1 > \bar{x}(t) \\ \delta\alpha(0, x_2, t) - (v_1 - g + h_2) & \text{if } x_2 - 1 \leq \bar{x}(t) < x_2 \end{cases}$$

Regarding the cross partial difference $\beta(x_1,x_2)-\beta(x_1,x_2-1)$:

For $x_1 \geq 2, x_2 \geq 2$
$$\beta(x_1,x_2,t-1)-\beta(x_1,x_2-1,t-1) = (1-\lambda(t)\Delta)\delta[\beta(x_1,x_2,t)-\beta(x_1,x_2-1,t)]$$
$$+\lambda_2(t)\Delta\delta[\beta(x_1,x_2-1,t)-\beta(x_1,x_2-2,t)]$$
$$+\lambda_1(t)\Delta\delta[\beta(x_1-1,x_2,t)-\beta(x_1-1,x_2-1,t)]$$

For $x_1=1, x_2 \geq 2$
$$\beta(1,x_2,t-1)-\beta(1,x_2-1,t-1) = (1-\lambda(t)\Delta)\delta[\beta(1,x_2,t)-\beta(1,x_2-1,t)]$$
$$+\lambda_2(t)\Delta\delta[\beta(1,x_2-1,t)-\beta(1,x_2-2,t)]$$
$$+\lambda_1(t)\Delta\begin{cases} 0 & \text{if } x_2 < \bar{x}(t) \\ \delta\alpha(o,x_2,t)-(v_1-g+h_2) & \text{if } x_2-1<\bar{x}(t)<x_2 \\ \delta[\alpha(o,x_2,t)-\alpha(o,x_2-1,t)] & \text{if } x_2-1>\bar{x}(t) \end{cases}$$

For $x_1 \geq 1, x_2 = 1$
$$\beta(x_1,1,t-1)-\beta(x_1,0,t-1) = (1-\lambda(t)\Delta)\delta[\beta(x_1,1,t)-\beta(x_1,0,t)]$$
$$+\lambda_1(t)\Delta\delta[\beta(x_1-1,1,t)-\beta(x_1-1,0,t)]$$

Next we show by induction that these partial and cross partial differences are non-positive.

We first establish the base case:
Since $V(x_1,x_2,\tau+T+1) = s_1x_1 + s_2x_2$, it is easy to see that $\alpha(x_1,x_2,\tau+T) = s_2$ and
$\beta(x_1,x_2,\tau+T) = s_1$. Thus $\alpha(x_1,x_2,\tau+T)-\alpha(x_1,x_2-1,\tau+T) = 0$
$\beta(x_1,x_2,\tau+T)-\beta(x_1-1,x_2,\tau+T) = 0$
$\alpha(x_1,x_2,\tau+T)-\alpha(x_1-1,x_2,\tau+T) = 0$
$\beta(x_1,x_2,\tau+T)-\beta(x_1,x_2-1,\tau+T) = 0$
In addition, we have the following:
i) $\delta\alpha(x_1,1,\tau+T)-(v_2+h_2) \leq 0$
ii) $\delta\beta(1,x_2,\tau+T)-(v_1+h_1) \leq 0$
iii) $\delta[\beta(1,x_2,\tau+T)-\alpha(0,x_2,\tau+T)]-g+h_2 \leq 0$
Note that i)-iii) are direct results of Assumptions 1 i)-iv).

Induction assumption:
i) $\delta\alpha(x_1,1,t)-(v_2+h_2) \leq 0$
ii) $\delta\beta(1,x_2,t)-(v_1+h_1) \leq 0$
iii) $\delta[\beta(1,x_2,t)-\alpha(0,x_2,t)]-g+h_2 \leq 0$
iv) $\alpha(x_1,x_2,t)-\alpha(x_1,x_2-1,t) \leq 0$
v) $\alpha(x_1,x_2,t)-\alpha(x_1-1,x_2,t) \leq 0$

vi) $\beta(x_1, x_2, t) - \beta(x_1 - 1, x_2, t) \le 0$

vii) $\beta(x_1, x_2, t) - \beta(x_1, x_2 - 1, t) \le 0$

Induction step:

i)
$$\alpha(x_1, 1, t-1) = -h_2 + (1 - \lambda(t)\Delta)\delta\alpha(x_1, 1, t) + \lambda_2(t)\Delta(v_2 + h_2) + \lambda_1(t)\Delta\delta\alpha(x_1 - 1, 1, t)$$
$$\le -h_2 + (1 - \lambda(t)\Delta)(v_2 + h_2) + \lambda_2(t)\Delta(v_2 + h_2) + \lambda_1(t)\Delta(v_2 + h_2)$$
$$= v_2$$

Thus $\delta\alpha(x_1, 1, t-1) \le v_2 + h_2$

ii)
$$\beta(1, x_2, t-1) = -h_1 + (1 - \lambda(t)\Delta)\delta\beta(1, x_2, t) + \lambda_2(t)\Delta\delta\beta(1, x_2 - 1, t)$$
$$+ \lambda_1(t)\Delta \begin{cases} (v_1 + h_1) & \text{if } x_2 < \bar{x}(t) \\ \delta\alpha(0, x_2, t) - h_2 + g + h_1 & \text{o.w.} \end{cases}$$
$$\le -h_1 + (1 - \lambda(t)\Delta)\delta(v_1 + h_1) + \lambda_2(t)\Delta\delta(v_1 + h_1)$$
$$+ \lambda_1(t)\Delta \begin{cases} (v_1 + h_1) & \text{if } x_2 < \bar{x}(t) \\ (v_1 - g + h_2) - h_2 + g + h_1 & \text{o.w.} \end{cases}$$
$$= v_1$$

(since $x_2 > \bar{x}(t) \Leftrightarrow \delta\alpha(0, x_2, t) < v_1 - g + h_2$)

Thus $\delta\beta(1, x_2, t-1) \le v_1 + h_1$

iii)
$$\beta(1, x_2, t-1) - \alpha(0, x_2, t-1)$$
$$= -h_1 + h_2 + (1 - \lambda(t)\Delta)\delta[\beta(1, x_2, t) - \alpha(0, x_2, t)] + \lambda_2(t)\Delta\delta[\beta(1, x_2 - 1, t) - \alpha(0, x_2 - 1, t)]$$
$$+ \lambda_1(t)\Delta \begin{cases} v_1 + h_1 - \delta\alpha(0, x_2, t) & \text{if } x_2 < \bar{x}(t) \\ -h_2 + h_1 + g + \delta[\alpha(0, x_2, t) - \alpha(0, x_2 - 1, t)] & \text{if } x_2 - 1 > \bar{x}(t) \\ -h_2 + h_1 + g + \delta\alpha(0, x_2, t) - (v_1 - g + h_2) & \text{o.w.} \end{cases}$$
$$\le -h_1 + h_2 + (1 - \lambda(t)\Delta)(-h_2 + g) + \lambda_2(t)\Delta(-h_2 + g)$$
$$+ \lambda_1(t)\Delta \begin{cases} v_1 + h_1 - (v_1 - g + h_2) & \text{if } x_2 < \bar{x}(t) \\ -h_2 + h_1 + g & \text{if } x_2 - 1 > \bar{x}(t) \\ -h_2 + h_1 + g & \text{o.w.} \end{cases}$$
$$\le -h_1 + h_2 + (1 - \lambda(t)\Delta)(-h_2 + h_1 + g) + \lambda_2(t)\Delta(-h_2 + h_1 + g) + \lambda_1(t)\Delta(-h_2 + h_1 + g)$$
$$= g$$

Thus $\delta[\beta(1, x_2, t-1) - \alpha(0, x_2, t-1)] \le h_2 + g$

Inductions for iv)-vii) are straightforward from the recursion equations.

Therefore, we have shown that $\alpha(x_1,x_2,t)$ and $\beta(x_1,x_2,t)$ are nonincreasing in $x_1$and x_2.

□

Proposition 3.7 (Order-up-to Policy)
If the current inventory position at the time of planning is x_1^0, then the optimal order quantities are $(x_1^* - x_1^0, x_2^*)$ if $x_1^0 < x_1^*$ and $(0, x_2(x_1^0))$ if $x_1^0 \geq x_1^*$.

where (x_1^*, x_2^*) maximizes NW(x_1,x_2) and $(x_2(x_1^0))$ is defined as the optimal quantities of the new product given that the planning quantity for the old product is restricted to x_1^0.
If $x_1^0 \geq x_1^*$, then $x_2(x_1^0) \leq x_2^*$

Proof.

We can rewrite equation (3-11) as

$$NW_C(x_1,x_2,x_1^0) = c_1 x_1^0 + \begin{cases} NW(x_1,x_2) & \text{if } x_1^0 < x_1 \\ NW(x_1^0,x_2) & \text{otherwise} \end{cases}$$

where $NW(x_1,x_2)$ is as defined in equation (3-10).

Let (x_1^{**},x_2^{**}) be the optimal solution that maximizes $NW_C(x_1,x_2,x_1^0)$. We then have

$$NW_C(x_1^{**},x_2^{**},x_1^0) = c_1 x_1^0 + \begin{cases} NW(x_1^{**},x_2^{**}) & \text{if } x_1^0 < x_1^{**} \\ NW(x_1^0,x_2) & \text{otherwise} \end{cases}$$

Thus if $x_1^0 < x_1^{**}$, (x_1^{**},x_2^{**}) must be the optimal solution that maximizes $NW(x_1,x_2)$. That is, $(x_1^{**},x_2^{**})=(x_1^*,x_2^*)$

If $x_1^0 \geq x_1^{**}$, then we must have that $x_1^{**} = x_1^0$ and $x_2^{**} = x_2(x_1^0)$ where $x_2(x_1^0)$ is the optimal quantity that maximizes $NW(x_1,x_2)$ given $x_1 = x_1^0$

Next we show that if $x_1^0 \geq x_1^*$ then $x_2(x_1^0) \leq x_2^*$.
This follows directly from the joint concavity of NW. $x_2(x_1^0)$ is the optimal quantity given x_1^0, thus $\left.\dfrac{\partial NW}{\partial x_2}\right|_{(x_0,x_2(x_1^0))} = 0$. Since $x_1^0 \geq x_1^*$ and $\dfrac{\partial^2 NW}{\partial x_2 \partial x_1} \leq 0$, we have $\left.\dfrac{\partial NW}{\partial x_2}\right|_{(x_1^*,x_2(x_1^0))} \leq 0$.

$NW(x_1,x_2)$ is maximized at (x_1^*,x_2^*), thus $\left.\dfrac{\partial NW}{\partial x_2}\right|_{(x_1^*,x_2^*)} = 0$. Therefore,

$\left.\dfrac{\partial NW}{\partial x_2}\right|_{(x_1^*,x_2(x_1^0))} \leq \left.\dfrac{\partial NW}{\partial x_2}\right|_{(x_1^*,x_2^*)}$. Since $\dfrac{\partial^2 NW}{\partial x_2^2} \leq 0$, this implies $x_2(x_1^0) \leq x_2^*$. □

Proposition 4.1

$h_t(r_1, x_1, x_2)$ is quasi-concave in r_1 $\forall x_1 \geq 0, x_2 \geq 0, t \in [0, T]$.

Proof.

$$\frac{\partial^2 h_t(x_1, x_2)}{\partial r_1^2} = 2\frac{\partial \lambda_1(r_1, t)}{\partial r_1} + \frac{\partial^2 \lambda_1(r_1, t)}{\partial r_1^2}[r_1 - r_2 - V_{t+1}(x_1, x_2 - 1) + V_{t+1}(x_1 - 1, x_2)]$$

Substituting equation (4-8) into the second order derivative yields

$$\frac{\partial^2 h_t(x_1, x_2)}{\partial r_1^2} = 2\frac{\partial \lambda_1(r_1, t)}{\partial r_1} - \frac{\partial^2 \lambda_1(r_1, t)}{\partial r_1^2}\lambda_1(r_1, t)/\frac{\partial \lambda_1(r_1, t)}{\partial r_1}$$

Thus for the second order derivative to be negative, we need

$$2(\frac{\partial \lambda_1(r_1, t)}{\partial r_1})^2 \geq \frac{\partial^2 \lambda_1(r_1, t)}{\partial r_1^2}\lambda_1(r_1, t)$$

From equations (4-2) and (4-3), it is easy to verify that this condition is satisfied. □

Lemma 4.1

(i) $V_{t+1}(x_1, x_2) - V_{t+1}(x_1 - 1, x_2) \geq V_{t+1}(x_1 + 1, x_2 - 1) - V_{t+1}(x_1, x_2 - 1)$

(ii) $V_{t+1}(x_1, x_2) - V_{t+1}(x_1, x_2 - 1) \geq V_{t+1}(x_1 - 1, x_2 + 1) - V_{t+1}(x_1 - 1, x_2)$

Proof.

(i) For $t=T$, this is trivially true because $LHS=RHS=s_1$.
Assume for induction that

$V_{t+1}(x_1, x_2) - V_{t+1}(x_1 - 1, x_2) \geq V_{t+1}(x_1 + 1, x_2 - 1) - V_{t+1}(x_1, x_2 - 1)$ $\quad \forall x_1, x_2$

we need to show $V_t(x_1, x_2) - V_t(x_1 - 1, x_2) \geq V_t(x_1 + 1, x_2 - 1) - V_t(x_1, x_2 - 1)$ $\qquad$ (4-16)

$V_t(x_1, x_2) - V_t(x_1 - 1, x_2) \geq h_t(r_t^*(x_1 - 1, x_2), x_1, x_2) - h_t(r_t^*(x_1 - 1, x_2), x_1 - 1, x_2)$

$= \lambda_1(r_t^*(x_1 - 1, x_2), t)[V_{t+1}(x_1 - 1, x_2) - V_{t+1}(x_1 - 2, x_2)]$

$\quad + \lambda_2(r_t^*(x_1 - 1, x_2), t)[V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2 - 1)]$

$\quad + (1 - \lambda_0))[V_{t+1}(x_1, x_2) - V_{t+1}(x_1 - 1, x_2)]$

$= \lambda_0[V_{t+1}(x_1 - 1, x_2) - V_{t+1}(x_1 - 2, x_2)]$

$\quad + \lambda_2(r_t^*(x_1 - 1, x_2), t)\{[V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2 - 1)]$

$\qquad\qquad\qquad - [V_{t+1}(x_1 - 1, x_2) - V_{t+1}(x_1 - 2, x_2)]\}$

$\quad + (1 - \lambda_0))[V_{t+1}(x_1, x_2) - V_{t+1}(x_1 - 1, x_2)]$

$$V_t(x_1+1,x_2-1)-V_t(x_1,x_2-1)$$

$$\leq h_t(r_t^*(x_1+1,x_2-1),x_1+1,x_2-1)-h_t(r_t^*(x_1+1,x_2-1),x_1,x_2-1)$$

$$=\lambda_0[V_{t+1}(x_1,x_2-1)-V_{t+1}(x_1-1,,x_2-1)]$$
$$+\lambda_2(r_t^*(x_1+1,x_2-1),t)\{[V_{t+1}(x_1+1,x_2-2)-V_{t+1}(x_1,x_2-2)]$$
$$-[V_{t+1}(x_1,x_2-1)-V_{t+1}(x_1-1,x_2-1)]\}$$
$$+(1-\lambda_0)[V_{t+1}(x_1+1,x_2-1)-V_{t+1}(x_1,x_2-1)]$$

Define the term to the right of the above equality sign as LB_LHS, signifying an upper bound of the LHS of equation (4-16)

By induction assumption,
$$[V_{t+1}(x_1+1,x_2-2)-V_{t+1}(x_1,x_2-2)]-[V_{t+1}(x_1,x_2-1)-V_{t+1}(x_1-1,x_2-1)]\leq 0$$
Therefore
$$V_t(x_1+1,x_2-1)-V_t(x_1,x_2-1)$$
$$\leq \lambda_0[V_{t+1}(x_1,x_2-1)-V_{t+1}(x_1-1,,x_2-1)]+(1-\lambda_0)[V_{t+1}(x_1+1,x_2-1)-V_{t+1}(x_1,x_2-1)]$$
Define the term to the right of the above inequality sign as UB_RHS, signifying a lower bound of the RHS of equation (4-16)

$$LB\_LHS-UB\_RHS$$
$$=\lambda_0\{[V_{t+1}(x_1-1,x_2)-V_{t+1}(x_1-2,x_2)]-[V_{t+1}(x_1,x_2-1)-V_{t+1}(x_1-1,,x_2-1)]\}$$
$$+\lambda_2(r_t^*(x_1-1,x_2),t)\{[V_{t+1}(x_1,x_2-1)-V_{t+1}(x_1-1,x_2-1)]-[V_{t+1}(x_1-1,x_2)-V_{t+1}(x_1-2,x_2)]\}$$
$$+(1-\lambda_0)\{[V_{t+1}(x_1,x_2)-V_{t+1}(x_1-1,x_2)]-[V_{t+1}(x_1+1,x_2-1)-V_{t+1}(x_1,x_2-1)]\}$$
$$=\lambda_1(r_t^*(x_1-1,x_2),t)\{[V_{t+1}(x_1-1,x_2)-V_{t+1}(x_1-2,x_2)]-[V_{t+1}(x_1,x_2-1)-V_{t+1}(x_1-1,,x_2-1)]\}$$
$$+(1-\lambda_0))\{[V_{t+1}(x_1,x_2)-V_{t+1}(x_1-1,x_2)]-[V_{t+1}(x_1+1,x_2-1)-V_{t+1}(x_1,x_2-1)]\}$$

Again by induction assumption, the two terms within the curly bracket are both non-negative. Therefore, we have

$$V_t(x_1,x_2)-V_t(x_1-1,x_2)\geq LB\_LHS\geq UB\_RHS\geq V_t(x_1+1,x_2-1)-V_t(x_1,x_2-1)$$

Note the above analysis only considers the case $x_1,x_2\geq 2$. We also need to show
$$V_t(1,x_2)-V_t(0,x_2)\geq V_t(2,x_2-1)-V_t(1,x_2-1)$$
$$V_t(x_1,1)-V_t(x_1-1,1)\geq V_t(x_1+1,0)-V_t(x_1,0)$$
$$V_t(1,1)-V_t(0,1)\geq V_t(2,0)-V_t(1,0)$$
We follow similar inductive methods to show these are true.
We prove (ii) similarly.□

Proposition 4.2

$$V_t(x_1, x_2) = V_{t+1}(x_1, x_2)$$
$$+ \lambda_0 [r_t^*(x_1, x_2) - 1 - (V_{t+1}(x_1, x_2) - V_{t+1}(x_1 - 1, x_2))] \qquad (4\text{-}10)$$
$$\forall x_1 \geq 1, x_2 \geq 0, t \in [0, T]$$

where the optimal price is given by

$$r_t^*(x_1, x_2) = r_2 + 1 + (V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2))$$
$$+ LambertW(e^{a_1(t) - a_2(t) - (V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2)) - 1}) \qquad (4\text{-}11)$$

and

$$r_t^*(x_1, 0) = -p_2 + 1 + (V_{t+1}(x_1, 0) - V_{t+1}(x_1 - 1, 0))$$
$$+ LambertW(e^{a_1(t) - a_2(t) - (V_{t+1}(x_1, 0) - V_{t+1}(x_1 - 1, 0)) - 1}) \qquad (4\text{-}12)$$

If $x_1 = 0$, the optimal price of the old product is the highest feasible price $r_t^*(0, x_2) = \bar{r}_1$ and

$$V_t(0, x_2) = V_{t+1}(0, x_2) - \lambda_0 p_1 + \lambda_0 \rho_2(\bar{r}_1, t)[r_2 + p_1 - (V_{t+1}(0, x_2) - V_{t+1}(0, x_2 - 1))] \quad \forall x_2 \geq 0$$

In addition, $V_{T+1}(x_1, x_2) = s_1 x_1 + s_2 x_2$.

Proof.

Since $r_t^*(x_1, x_2)$ maximizes $h_t(r_1, x_1, x_2)$, we have from equation (4-4) that

$$V_t(x_1, x_2) = \lambda_1(r_t^*(x_1, x_2), t)(r_t^*(x_1, x_2) + V_{t+1}(x_1 - 1, x_2))$$
$$+ \lambda_2(r_t^*(x_1, x_2), t)(r_2 + V_{t+1}(x_1, x_2 - 1))$$
$$+ (1 - \lambda_0) V_{t+1}(x_1, x_2)$$

Because $\lambda_1(r_t^*(x_1, x_2), t) + \lambda_2(r_t^*(x_1, x_2), t) = \lambda_0$, we can rewrite the above equation as

$$V_t(x_1, x_2) = V_{t+1}(x_1, x_2)$$
$$+ [\lambda_0 - \lambda_2(r_t^*(x_1, x_2), t)](r_t^*(x_1, x_2) + V_{t+1}(x_1 - 1, x_2) - V_{t+1}(x_1, x_2))$$
$$+ \lambda_2(r_t^*(x_1, x_2), t)(r_2 + V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1, x_2))$$
$$= V_{t+1}(x_1, x_2) + \lambda_0(r_t^*(x_1, x_2) + V_{t+1}(x_1 - 1, x_2) - V_{t+1}(x_1, x_2))$$
$$+ \lambda_2(r_t^*(x_1, x_2), t)(-r_t^*(x_1, x_2) + r_2 - V_{t+1}(x_1 - 1, x_2) + V_{t+1}(x_1, x_2 - 1))$$
$$= V_{t+1}(x_1, x_2) + \lambda_0(r_t^*(x_1, x_2) + V_{t+1}(x_1 - 1, x_2) - V_{t+1}(x_1, x_2))$$
$$- \lambda_0 \rho_2(r_t^*(x_1, x_2), t)(r_t^*(x_1, x_2) - r_2 + V_{t+1}(x_1 - 1, x_2) - V_{t+1}(x_1, x_2 - 1))$$

From equation (4-8a), the last term is equal to $-\lambda_0$. Therefore

$$V_t(x_1, x_2) = V_{t+1}(x_1, x_2) + \lambda_0 [r_t^*(x_1, x_2) - 1 - (V_{t+1}(x_1, x_2) - V_{t+1}(x_1 - 1, x_2))]$$

Similarly, from equations (4-6) and (4-9), we obtain

$$V_t(x_1, 0) = V_{t+1}(x_1, 0) + \lambda_0 [r_t^*(x_1, 0) - 1 - (V_{t+1}(x_1, 0) - V_{t+1}(x_1 - 1, 0))].$$

If $x_1 = 0$, the optimal price of the old product is the highest feasible price $r_t^*(0, x_2) = \bar{r}_1$, thus from equation (4-5), we have

$$V_t(0, x_2) = \lambda_1(\bar{r}_1, t)(-p_1 + V_{t+1}(0, x_2)) + \lambda_2(\bar{r}_1, t)(r_2 + V_{t+1}(0, x_2 - 1)) + (1 - \lambda_0)V_{t+1}(0, x_2)$$

$$= V_{t+1}(0, x_2) + \lambda_1(\bar{r}_1, t)(-p_1) + \lambda_2(\bar{r}_1, t)(r_2 + V_{t+1}(0, x_2 - 1) - V_{t+1}(0, x_2))$$

$$= V_{t+1}(0, x_2) + (\lambda_0 - \lambda_2(\bar{r}_1, t))(-p_1) + \lambda_2(\bar{r}_1, t)(r_2 + V_{t+1}(0, x_2 - 1) - V_{t+1}(0, x_2))$$

$$= V_{t+1}(0, x_2) - \lambda_0 p_1 + \lambda_2(\bar{r}_1, t)(r_2 + p_1 + V_{t+1}(0, x_2 - 1) - V_{t+1}(0, x_2))$$

$$= V_{t+1}(0, x_2) - \lambda_0 p_1 + \lambda_0 \rho_2(\bar{r}_1, t)[r_2 + p_1 - (V_{t+1}(0, x_2) - V_{t+1}(0, x_2 - 1))]$$

$\square$

Proposition 4.5

If $\exists \bar{t} \in [0, T]$ such that $e^{a_1(\bar{t}) - a_2(\bar{t}) + r_2 + p_2 - e_1 - 1} < \varepsilon$, where ε is a very small positive number, then $r_{t-1}^*(x_1, x_2) \leq r_t^*(x_1, x_2) \quad \forall x_1, x_2 \geq 0$ and $t \in [\bar{t}, T]$.

Proof.

(i) First we consider the case when $x_1 \geq 1$ and $x_2 \geq 2$.

Note that $V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2) \in [-r_2 - p_2 + e_1, r_1 + p_1 - e_2]$

Thus $a_1(t) - a_2(t) - (V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2)) - 1 < a_1(t) - a_2(t) + r_2 + p_2 - e_1 - 1$

Thus if $e^{a_1(\bar{t}) - a_2(\bar{t}) + r_2 + p_2 - e_1 - 1} < \varepsilon$, then $e^{a_1(\bar{t}) - a_2(\bar{t}) - (V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2)) - 1} < \varepsilon$

From *Assumption 4.2*, $a_1(t) - a_2(t)$ decreases in t.

Therefore $e^{a_1(t) - a_2(t) - (V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2)) - 1} < \varepsilon \quad \forall t \in [\bar{t}, T]$.

When ε is small enough, the *LambertW* function term in equation (4-11) becomes negligible. We then have from equations (4-10) and (4-11) that

$$r_t^*(x_1, x_2) = r_2 + 1 + V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2) \quad \forall x_1, x_2 > 0 \qquad (4\text{-}11a)$$

$$V_t(x_1, x_2) = V_{t+1}(x_1, x_2) + \lambda_0[r_2 + V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1, x_2)] \quad \forall x_1, x_2 > 0 \qquad (4\text{-}10a)$$

Therefore,

$$V_t(x_1, x_2 - 1) - V_t(x_1 - 1, x_2) = V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2)$$
$$- \lambda_0\{[V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1, x_2 - 2)] - [V_{t+1}(x_1 - 1, x_2) - V_{t+1}(x_1 - 1, x_2 - 1)]\}$$
$$\forall x_1, x_2 > 0$$

The term within the curly bracket is non-negative from *Lemma 4.1*. Hence

$$V_t(x_1, x_2 - 1) - V_t(x_1 - 1, x_2) \leq V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2)$$

Substitute the above inequality into (4-11a), we have

$$r_{t-1}^*(x_1, x_2) \leq r_t^*(x_1, x_2) \quad \forall x_1 \geq 1, x_2 \geq 2, t \in [\bar{t}, T]$$

(ii) Now consider the case when $x_1 \geq 1$ and $x_2 = 0$.

110

As $V_{t+1}(x_1,0) - V_{t+1}(x_1 - 1,0) \in [e_1, r_1 + p_1]$

$a_1(t) - a_2(t) - (V_{t+1}(x_1,0) - V_{t+1}(x_1 - 1,0)) - 1 < a_1(t) - a_2(t) - e_1 - 1$

$$< a_1(t) - a_2(t) + r_2 + p_2 - e_1 - 1$$

Therefore, if $e^{a_1(t) - a_2(t) + r_2 + p_2 - e_1 - 1} < \varepsilon$, then clearly
$e^{a_1(t) - a_2(t) - (V_{t+1}(x_1,0) - V_{t+1}(x_1 - 1,0)) - 1} < \varepsilon$ $\forall t \in [\bar{t}, T]$. As a result, we ignore the *LambertW*
function term in equation (4-12) to obtain

$$r_t^*(x_1,0) = -p_2 + 1 + V_{t+1}(x_1,0) - V_{t+1}(x_1 - 1,0) \qquad (4\text{-}12a)$$

$$V_t(x_1,0) = V_{t+1}(x_1,0) - \lambda_0 p_2$$

It follows that

$$V_t(x_1,0) - V_t(x_1 - 1,0) = V_{t+1}(x_1,0) - V_{t+1}(x_1 - 1,0)$$

Substitute the above inequality into (4-12a), we have

$$r_{t-1}^*(x_1,0) = r_t^*(x_1,0) \quad \forall x_1 \geq 1, t \in [\bar{t}, T]$$

(iii) If $x_1 \geq 1$ and $x_2 = 1$

$$V_t(x_1, x_2 - 1) - V_t(x_1 - 1, x_2) = V_{t+1}(x_1, x_2 - 1) - V_{t+1}(x_1 - 1, x_2)$$

$$- \lambda_0 \{r_2 + p_2 - [V_{t+1}(x_1,1) - V_{t+1}(x_1,0)]\} \quad \forall x_1, x_2 > 0$$

The term within the curly bracket is non-negative because

$$V_{t+1}(x_1,1) - V_{t+1}(x_1,0) < r_2 + p_2.$$

Substitute the above inequality into (11a), we have

$$r_{t-1}^*(x_1,1) \leq r_t^*(x_1,1) \quad \forall x_1, x_2 \geq 2, t \in [\bar{t}, T]$$

(iv) If $x_1 = 0$, the optimal price of the old product is the highest feasible price
$r_t^*(0, x_2) = \bar{r}_1$, which satisfies $r_{t-1}^*(x_1, x_2) \leq r_t^*(x_1, x_2)$ $\forall t \in [\bar{t}, T]$ trivially, thus finishing
the proof. □

Lemma 4.2
The first order condition for the optimal fixed-price is

$$\Lambda_1(r_1) - G_{\Lambda_1}(x_1) + \frac{d\Lambda_1(r_1)}{dr_1}[(r_1 + V(r_1, x_1 - 1, x_2)) - (r_2 + V(r_1, x_1, x_2 - 1)] = 0 \qquad (4\text{-}13)$$

where $G_{\Lambda_1}(\cdot)$ is the Poisson loss function for the random variable D_1.

Proof.

$$V^{FP}(r_1, x_1, x_2) = \underset{D_1}{E}[r_1 \min(D_1, x_1) - (D_1 - x_1)^+ p_1 + (x_1 - D_1)^+ s_1]$$

$$+ \underset{D_2}{E}[r_2 \min(D_2, x_2) - (D_2 - x_2)^+ p_2 + (x_2 - D_2)^+ s_2]$$

where D_1 and D_2 are Poisson random variables with mean $\Lambda_1(r_1) = \int_0^T \lambda_1(r_1, t)dt$ and

$\Lambda_2(r_1) = \int_0^T \lambda_2(r_1, t)dt$ respectively.

Therefore, we can rewrite

$$V^{FP}(r_1, x_1, x_2) = \sum_{n=0}^{x_1} P_n(\Lambda_1)[nr_1 + (x_1 - n)s_1] + \sum_{n=x_1+1}^{\infty} P_n(\Lambda_1)[x_1 r_1 - (n - x_1)p_1]$$

$$+ \sum_{n=0}^{x_2} P_n(\Lambda_2)[nr_2 + (x_2 - n)s_2] + \sum_{n=x_2+1}^{\infty} P_n(\Lambda_2)[x_2 r_2 - (n - x_2)p_2]$$

where $P_n(\Lambda) = \dfrac{\Lambda^n e^{-\Lambda}}{n!}$ is Poisson density with mean Λ.

Note that $\dfrac{d}{d\Lambda} P_n(\Lambda) = P_{n-1}(\Lambda) - P_n(\Lambda)$, thus $\dfrac{d}{dr_1} P_n(\Lambda) = (P_{n-1}(\Lambda) - P_n(\Lambda))\dfrac{d\Lambda}{dr_1}$ (4-17)

Define $V_1(r_1, x_1) \equiv \sum_{n=0}^{x_1} P_n(\Lambda_1)[nr_1 + (x_1 - n)s_1] + \sum_{n=x_1+1}^{\infty} P_n(\Lambda_1)[x_1 r_1 - (n - x_1)p_1]$

and $V_2(r_1, x_2) \equiv \sum_{n=0}^{x_2} P_n(\Lambda_2)[nr_2 + (x_2 - n)s_2] + \sum_{n=x_2+1}^{\infty} P_n(\Lambda_2)[x_2 r_2 - (n - x_2)p_2]$

We then have,

$$\frac{\partial}{\partial r_1} V_1(r_1, x_1) = \sum_{n=0}^{x_1} \{\frac{dP_n(\Lambda_1)}{dr_1}[nr_1 + (x_1 - n)s_1] + P_n(\Lambda_1)n\}$$

$$+ \sum_{n=x_1+1}^{\infty} \{\frac{dP_n(\Lambda_1)}{dr_1}[x_1 r_1 - (n - x_1)p_1] + P_n(\Lambda_1)x_1\}$$

Substituting equation (4-17) into the above yields the following after some simple but tedious algebra:

$$\frac{\partial}{\partial r_1} V_1(r_1, x_1) = \frac{d\Lambda_1}{dr_1}[V_1(r_1, x_1 - 1) - V_1(r_1, x_1)] + \frac{d\Lambda_1}{dr_1} r_1 + \Lambda_1 - G_{\Lambda_1}(x_1)$$

Similarly, we obtain

$$\frac{\partial}{\partial r_1} V_2(r_1, x_2) = \frac{d\Lambda_2}{dr_1}[V_2(r_1, x_2 - 1) - V_2(r_1, x_2)] + \frac{d\Lambda_2}{dr_1} r_2$$

Therefore,

$$\frac{\partial}{\partial r_1} V^{FP}(r_1, x_1, x_2) = \frac{\partial}{\partial r_1} V_1(r_1, x_1) + \frac{\partial}{\partial r_1} V_2(r_1, x_2)$$

$$= \Lambda_1 - G_{\Lambda_1}(x_1) + \frac{d\Lambda_1}{dr_1}[V_1(r_1, x_1 - 1) - V_1(r_1, x_1)] + \frac{d\Lambda_1}{dr_1} r_1$$

$$+ \frac{d\Lambda_2}{dr_1}[V_2(r_1, x_2 - 1) - V_2(r_1, x_2)] + \frac{d\Lambda_2}{dr_1} r_2$$

We know that $d\Lambda_1 = -d\Lambda_2$, thus

$$\frac{\partial}{\partial r_1} V^{FP}(r_1, x_1, x_2) = \Lambda_1 - G_{\Lambda_1}(x_1) + \frac{d\Lambda_1}{dr_1} \{r_1 - r_2 - [V^{FP}(r_1, x_1, x_2 - 1) - V^{FP}(r_1, x_1 - 1, x_2)]\}$$

Therefore, the first order condition is

$$\Lambda_1 - G_{\Lambda_1}(x_1) + \frac{d\Lambda_1}{dr_1} \{r_1 - r_2 - [V^{FP}(r_1, x_1, x_2 - 1) - V^{FP}(r_1, x_1 - 1, x_2)]\} = 0. \quad \square$$

Proposition 4.6

(i) If $x_1, x_2 \to \infty$, $r_t^{FP}(x_1, x_2)$ decreases in t.

(ii) If $x_1, x_2 \to \infty$, $r_t^{FP}(x_1, x_2) \le r_t^*(x_1, x_2)$

Proof.

From equation (4-13), $r_t^{FP}(x_1 \to \infty, x_2 \to \infty)$ satisfies

$$\Lambda_1(r_1, t) + \frac{d\Lambda_1(r_1, t)}{dr_1}[(r_1 - s_1) - (r_2 - s_2)] = 0$$

Or equivalently, $\dfrac{\int_t^T \rho_1(r_1, \tau)\rho_2(r_1, \tau)d\tau}{\int_t^T \rho_1(r_1, \tau)d\tau}[(r_1 - s_1) - (r_2 - s_2)] = 1$ (4-18)

Note that the term $\dfrac{\int_t^T \rho_1(r_1, \tau)\rho_2(r_1, \tau)d\tau}{\int_t^T \rho_1(r_1, \tau)d\tau}$ is a weighted average of ρ_2 with weight equal

to $\rho_1(r_1, \tau)$ at each τ. Since $\rho_1(r_1, \tau)$ decreases in t and $\rho_2(r_1, \tau)$ increases in t, as t increases, more weight is given to larger values of ρ_2. Therefore,

$$\frac{\int_t^T \rho_1(r_1, \tau)\rho_2(r_1, \tau)d\tau}{\int_t^T \rho_1(r_1, \tau)d\tau}$$ increases in t. From equation (4-18), it is clear that

$r_t^{FP}(x_1 \to \infty, x_2 \to \infty)$ shall decrease in t.

From equation (4-2a), $r_t^*(x_1 \to \infty, x_2 \to \infty)$ satisfies $\rho_2(r_1, t)[r_1 - s_1 - (r_2 - s_2)] = 1$ (4-19)

Since $\rho_2(r_1, \tau) > \rho_2(r_1, t) \forall \tau > t$, we have $\dfrac{\int_t^T \rho_1(r_1, \tau)\rho_2(r_1, \tau)d\tau}{\int_t^T \rho_1(r_1, \tau)d\tau} > \rho_2(r_1, t)$.

Compare equation (4-18) with equation (4-19), we then have $r_t^{FP}(\infty, \infty) \le r_t^*(\infty, \infty)$. $\square$

Lemma 4.3

(i) $\hat{r}_t$ solves $\ln \dfrac{\rho_2(r_1, T)}{\rho_2(r_1, t)} - [(r_1 - s_1) - (r_2 - s_2)](\rho_1(r_1, t) - \rho_1(r_1, T)) = 0$.

(ii) $\tilde{r}_t$ solves $\ln \dfrac{\rho_2(r_1, T)}{\rho_2(r_1, t)} - (r_1 - s_1 + p_2)(\rho_1(r_1, t) - \rho_1(r_1, T)) = 0$

(iii) $\tilde{r}_t \le \hat{r}_t$.

Proof.

$$V_t^D(r_1,x_1,x_2) = r_1 \min(D_1,x_1) - (D_1 - x_1)^+ p_1 + (x_1 - D_1)^+ s_1]$$
$$+ r_2 \min(D_2,x_2) - (D_2 - x_2)^+ p_2 + (x_2 - D_2)^+ s_2]$$

(i) If $D_1 \le x_1, D_2 \le x_2$, then
$$V_t^D(r_1,x_1,x_2) = (r_1 - s_1 - (r_2 - s_2))D_1 + (r_2 - s_2)D_0 + x_1 s_1 + x_2 s_2$$
$$\frac{d}{dr_1}V_t^D(r_1,x_1,x_2) = D_1 + (r_1 - s_1 - (r_2 - s_2))\frac{dD_1}{dr_1} \qquad (4\text{-}20)$$

Substitute *Assumption 4.4* into the demand, we obtain first order condition
$$\ln\frac{\rho_2(T)}{\rho_2(t)} - [(r_1 - s_1) - (r_2 - s_2)](\rho_1(t) - \rho_1(T)) = 0$$
By *Definition 4.5*, $\hat{r}_t$ solves the above equation.

(ii) If $D_1 \le x_1, D_2 \ge x_2$, then
$$V_t^D(r_1,x_1,x_2) = (r_1 + p_2 - s_1)D_1 + r_2 x_2 + x_1 s_1 - p_2 D_0 + p_2 x_2$$
$$\frac{d}{dr_1}V_t^D(r_1,x_1,x_2) = D_1 + (r_1 - s_1 + p_2)\frac{dD_1}{dr_1}$$
Substitute *Assumption 4.4* into the demand, we obtain first order condition
$$\ln\frac{\rho_2(T)}{\rho_2(t)} - [(r_1 - s_1 + p_2)](\rho_1(t) - \rho_1(T)) = 0$$
By definition 4.5, $\tilde{r}_t$ solves the above equation.

(iii)
$r_1 - s_1 + p_2 > (r_1 - s_1) - (r_2 - s_2)$, thus it is obvious that
$$\ln\frac{\rho_2(\tilde{r}_t,T)}{\rho_2(\tilde{r}_t,t)} - (\tilde{r}_t - s_1 - (r_2 - s_2))(\rho_1(\tilde{r}_t,t) - \rho_1(\tilde{r}_t,T))$$
$$> \ln\frac{\rho_2(\tilde{r}_t,T)}{\rho_2(\tilde{r}_t,t)} - (\tilde{r}_t - s_1 + p_2)(\rho_1(\tilde{r}_t,t) - \rho_1(\tilde{r}_t,T)) = 0$$

As $\ln\frac{\rho_2(\hat{r}_t,T)}{\rho_2(\hat{r}_t,t)} - (\hat{r}_t - s_1 - (r_2 - s_2))(\rho_1(\hat{r}_t,t) - \rho_1(\hat{r}_t,T)) = 0$, and

$\ln\frac{\rho_2(T)}{\rho_2(t)} - [(r_1 - s_1) - (r_2 - s_2)](\rho_1(t) - \rho_1(T))$ is decreasing in r_1 (second order condition

of optimality), it should be that $\tilde{r}_t \le \hat{r}_t$. □

Proposition 4.7

If $r_1^{old}(t) > r_1^{new}(t)$, $r_t^D(x_1,x_2) = \max(r_1^{old}(t),\tilde{r}_t)$

If $r_1^{old}(t) \le r_1^{new}(t)$, then

$$r_t^D(x_1,x_2) = \max(r_1^{new},\tilde{r}_t) \text{ if } \hat{r}_t > r_1^{new} \text{ and } r_t^D(x_1,x_2) = \max(r_1^{old},\hat{r}_t) \text{ otherwise.}$$
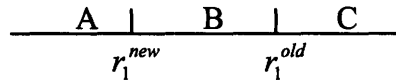
Proof.

For a given pair of (x_1,x_2) and a time t, we can easily obtain $r_1^{old}, r_1^{new}, \tilde{r}_t$, and $\hat{r}_t$. The problem is to determine $r_t^D(x_1,x_2)$ that maximizes the deterministic value

$$V_t^D(r_1,x_1,x_2) = r_1 \min(D_1,x_1) - (D_1 - x_1)^+ p_1 + (x_1 - D_1)^+ s_1]$$
$$+ r_2 \min(D_2,x_2) - (D_2 - x_2)^+ p_2 + (x_2 - D_2)^+ s_2]$$

We separate into 2 cases:

(i) $r_1^{old}(t) > r_1^{new}(t)$

$$\begin{array}{ccc} A & | \quad B \quad | & C \\ \hline & & \\ r_1^{new} & r_1^{old} & \end{array}$$

Consider 3 intervals A, B and C

For $r_1 \in A$, $D_1 > x_1, D_2 \le x_2$, thus

$$V_t^D(r_1,x_1,x_2) = r_1 x_1 - (D_1 - x_1)p_1 + r_2 D_2 + (x_2 - D_2)s_2$$
$$= r_1 x_1 - (r_2 - s_2 + p_1)D_1 + x_1 p_1 + x_2 s_2 - D_2 s_2$$

$$\frac{d}{dr_1}V_t^D(r_1,x_1,x_2) = x_1 - (r_2 - s_2 + p_1)\frac{dD_1}{dr_1}$$

Because $r_2 - s_2 + p_1 > 0$, $\frac{dD_1}{dr_1} < 0$, therefore the above first order derivative is always non-negative. Hence the optimal r_1 within A is r_1^{new}. Thus we can ignore interval A.

For $r_1 \in B$, $D_1 \ge x_1, D_2 \ge x_2$, thus

$$V_t^D(r_1,x_1,x_2) = r_1 x_1 - (p_1 - p_2)D_1 - p_2 D_0 + p_1 x_1 + r_2 x_2 + p_2 x_2$$

We assume $p_1 = p_2$, thus $\frac{d}{dr_1}V_t^D(r_1,x_1,x_2) = x_1 \ge 0$, therefore the optimal r_1 within B is r_1^{old}. Thus we can focus on interval C only.
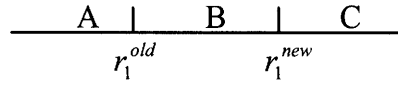
For $r_1 \in C$, $D_1 \le x_1, D_2 \ge x_2$, thus

$$V_t^D(r_1,x_1,x_2) = (r_1 + p_2 - s_1)D_1 + r_2 x_2 + x_1 s_1 - p_2 D_0 + p_2 x_2$$

From *Lemma 4.3*, $\tilde{r}_t$ solves the above equation. However, if the $\tilde{r}_t$ that satisfies this equation is not within interval C, i.e., $\tilde{r}_t < r_1^{old}$, then the optimal r_1 within interval C is r_1^{old}.

Therefore, the optimal r_1 when $r_1^{old}(t) > r_1^{new}(t)$ is $r_t^D(x_1, x_2) = \max(r_1^{old}(t), \tilde{r}_t)$.

(ii) $r_1^{old}(t) \le r_1^{new}(t)$

$$\underline{\quad A \quad | \quad B \quad | \quad C \quad}$$
$$\quad\quad r_1^{old} \quad\quad\quad r_1^{new}$$

Again consider 3 intervals A, B and C

For $r_1 \in A$, $D_1 > x_1, D_2 \le x_2$, thus

$$V_t^D(r_1, x_1, x_2) = r_1 x_1 - (D_1 - x_1)p_1 + r_2 D_2 + (x_2 - D_2)s_2$$
$$= r_1 x_1 - (r_2 - s_2 + p_1)D_1 + x_1 p_1 + x_2 s_2 - D_2 s_2$$

$$\frac{d}{dr_1} V_t^D(r_1, x_1, x_2) = x_1 - (r_2 - s_2 + p_1)\frac{dD_1}{dr_1}$$

Because $r_2 - s_2 + p_1 > 0$, $\frac{dD_1}{dr_1} < 0$, therefore the above first order derivative is always non-negative. Hence the optimal r_1 within A is r_1^{old}. Therefore, we can ignore interval A.

For $r_1 \in C$, $D_1 \le x_1, D_2 \ge x_2$, thus

$$V_t^D(r_1, x_1, x_2) = (r_1 + p_2 - s_1)D_1 + r_2 x_2 + x_1 s_1 - p_2 D_0 + p_2 x_2$$

By *Lemma 4.3*, $\tilde{r}_t$ solves the above equation.

For $r_1 \in B$, $D_1 \le x_1, D_2 \le x_2$, thus

$$V_t^D(r_1, x_1, x_2) = (r_1 - s_1 - (r_2 - s_2))D_1 + (r_2 - s_2)D_0 + x_1 s_1 + x_2 s_2$$

By *Lemma 4.3*, $\hat{r}_t$ solves the above equation.

Again we need to consider the scenarios when the above equations yield solutions that are not within the specified interval.

Since $\tilde{r}_t \le \hat{r}_t$, we only need to consider the following cases:

(a) $r_1^{new}(t) \le \tilde{r}_t \le \hat{r}_t$

Apparently r_1^{new} is optimal for interval B. Thus we only need to focus on interval C, the optimal solution in this case is $r_t^D(x_1, x_2) = \tilde{r}_t$

(b) $\tilde{r}_t \le r_1^{new}(t) < \hat{r}_t$

116

$\tilde{r}_t \leq r_1^{new}(t)$ means r_1^{new} is optimal for interval C and $r_1^{new}(t) < \hat{r}_t$ implies that r_1^{new} is optimal for interval B. Therefore $r_t^D(x_1, x_2) = r_1^{new}$.

(c) $\tilde{r}_t \leq \hat{r}_t \leq r_1^{new}(t)$

As in case (b), we only need to consider interval B. If $\hat{r}_t$ is also greater than r_1^{old}, $r_t^D(x_1, x_2) = \hat{r}_t$. Otherwise, r_1^{old} is optimal. That is, $r_t^D(x_1, x_2) = \max(r_1^{old}, \hat{r}_t)$

Therefore, we have:

If $r_1^{old}(t) > r_1^{new}(t)$, $r_t^D(x_1, x_2) = \max(r_1^{old}(t), \tilde{r}_t)$

If $r_1^{old}(t) \leq r_1^{new}(t)$, then

if $r_1^{new}(t) \leq \tilde{r}_t \leq \hat{r}_t$, $r_t^D(x_1, x_2) = \tilde{r}_t$

if $\tilde{r}_t \leq r_1^{new}(t) < \hat{r}_t$, $r_t^D(x_1, x_2) = r_1^{new}$

if $\tilde{r}_t \leq \hat{r}_t \leq r_1^{new}(t)$, $r_t^D(x_1, x_2) = \max(r_1^{old}, \hat{r}_t)$

This can be further simplified to:

If $r_1^{old}(t) > r_1^{new}(t)$, $r_{t,x_1,x_2}^D = \max(r_1^{old}(t), \tilde{r}_t)$

If $r_1^{old}(t) \leq r_1^{new}(t)$, then

$r_t^D(x_1, x_2) = \max(r_1^{new}, \tilde{r}_t)$ if $\hat{r}_t > r_1^{new}$ and $r_t^D(x_1, x_2) = \max(r_1^{old}, \hat{r}_t)$ otherwise. $\square$

Proposition 4.8

(i) If $x_1, x_2 \to \infty$, $r_t^D(x_1, x_2)$ decreases in t.

(ii) If $x_1, x_2 \to \infty$, $r_t^{FP}(x_1, x_2) = r_t^D(x_1, x_2) \leq r_t^*(x_1, x_2)$

Proof.

When $x_1, x_2 \to \infty$, equation (4-13) is equivalent to

$$\Lambda_1(r_1) + \frac{d\Lambda_1(r_1)}{dr_1}[(r_1 - s_1)) - (r_2 - s_2)] = 0 \text{ where } \Lambda_1(r_1) = \int_0^T \lambda_1(r_1, t)dt .$$

From equation (4-20), the first order condition for $r_t^D(x_1, x_2)$ when there are plenty of inventories is $D_1 + (r_1 - s_1 - (r_2 - s_2))\frac{dD_1}{dr_1} = 0$ where $D_1 = \int_t^T \lambda_1(r_1, \tau)d\tau$. Therefore $r_t^{FP}(x_1, x_2) = r_t^D(x_1, x_2)$. It follows then from *Proposition 4.6* that $r_t^D(x_1, x_2)$ is also decreasing in t and $r_t^D(x_1, x_2) \leq r_t^*(x_1, x_2)$. $\square$

Proposition 4.9

$$1 \geq \frac{J^{FP}(x_1, x_2)}{J^*(x_1, x_2)} \geq \frac{J^D(x_1, x_2)}{J^*(x_1, x_2)} \geq \frac{J^{LB}(x_1, x_2)}{J^*(x_1, x_2)}$$

Proof.

Clearly the solution to the fixed-price problem is a subset of the optimal dynamic pricing solution. Therefore $J^{FP}(x_1, x_2) \leq J^*(x_1, x_2)$. Because $r_{t=1}^{FP}(x_1, x_2)$ maximizes $V_{t=1}^{FP}(r_1, x_1, x_2)$, we have $J^{FP}(x_1, x_2) \geq J^D(x_1, x_2)$.

$$V^{FP}(r_1, x_1, x_2) = \underset{D_1}{E}[r_1 \min(D_1, x_1) - (D_1 - x_1)^+ p_1 + (x_1 - D_1)^+ s_1]$$

$$+ \underset{D_2}{E}[r_2 \min(D_2, x_2) - (D_2 - x_2)^+ p_2 + (x_2 - D_2)^+ s_2]$$

$$= \underset{D_1}{E}[(r_1 - s_1)D_1 + x_1 s_1 - (r_1 - s_1 + p_1)(D_1 - x_1)^+]$$

$$+ \underset{D_2}{E}[(r_2 - s_2)D_2 + x_2 s_2 - (r_2 - s_2 + p_2)(D_2 - x_2)^+]$$

where D_1 and D_2 are Poisson random variables with mean $\Lambda_1(r_1) = \int_0^T \lambda_1(r_1, t)dt$ and $\Lambda_2(r_1) = \int_0^T \lambda_2(r_1, t)dt$ respectively.

For a random variable D_1 with finite mean and variance Λ_1, we have (from Gallego 1992)

$$\underset{D_1}{E}[(D_1 - x_1)^+] \leq \frac{\sqrt{\Lambda_1 + (x_1 - \Lambda_1)^2} - (x_1 - \Lambda_1)}{2}$$

Similarly

$$\underset{D_2}{E}[(D_2 - x_2)^+] \leq \frac{\sqrt{\Lambda_2 + (x_2 - \Lambda_2)^2} - (x_2 - \Lambda_2)}{2}.$$

Therefore

$$J^D(x_1, x_2) \geq x_1 s_1 + (r^D - s_1)\Lambda_1 - [\sqrt{\Lambda_1 + (\Lambda_1 - x_1)^2} - (x_1 - \Lambda_1)]/2$$

$$+ x_2 s_2 + (r_2 - s_2)\Lambda_2 - [\sqrt{\Lambda_2 + (\Lambda_2 - x_2)^2} - (x_2 - \Lambda_2)]/2$$

$$= J^{LB}(x_1, x_2)$$

□

Proposition 4.10

If $x_1, x_2 \rightarrow \infty$, a necessary condition for re-price in the two-price policy is $r_0 > r_t^*(x_1, x_2)$ or $r_0 < r_t^{FP}(x_1, x_2)$.

Proof.

equation (4-15) gives a necessary condition for re-price.

$$H_t(x_1, x_2) - H_{t+1}(x_1, x_2) > \lambda_1(r_0, t)(r_0 + H_{t+1}(x_1 - 1, x_2) - H_{t+1}(x_1, x_2))$$

$$+ \lambda_2(r_0, t)(r_2 + H_{t+1}(x_1, x_2 - 1) - H_{t+1}(x_1, x_2))$$

From the definition of $H_t(x_1, x_2)$,

$$H_t(x_1,x_2) - V_{t+1}^{FP}(r_t^{FP}(x_1,x_2),x_1,x_2) \geq \lambda_1(r_0,t)(r_0 + H_{t+1}(x_1-1,x_2) - H_{t+1}(x_1,x_2))$$
$$+ \lambda_2(r_0,t)(r_2 + H_{t+1}(x_1,x_2-1) - H_{t+1}(x_1,x_2)) \quad (4\text{-}21)$$

If $x_1, x_2 \to \infty$,

LHS of (4-21) $\to \lambda_1(r_t^{FP}(x_1,x_2),t)(r_t^{FP}(x_1,x_2) - s_1) + \lambda_2(r_t^{FP}(x_1,x_2),t)(r_2 - s_2)$

RHS of (4-21) $\to \lambda_1(r_0,t)(r_0 - s_1) + \lambda_2(r_0,t)(r_2 - s_2)$

$LHS - RHS = \lambda_1(r_t^{FP}(x_1,x_2),t)[(r_t^{FP}(x_1,x_2) - s_1) - (r_2 - s_2)]$
$$- \lambda_1(r_0,t)[(r_0 - s_1) - (r_2 - s_2)]$$

Define $A(r_1,t) \equiv \lambda_1(r_1,t)[(r_1 - s_1) - (r_2 - s_2)]$, then

$LHS - RHS = A(r_t^{FP}(x_1,x_2),t) - A(r_0,t)$

Thus a necessary condition to re-price is $A(r_t^{FP}(x_1,x_2),t) > A(r_0,t)$

$$\frac{\partial}{\partial r_1} A(r_1,t) = \lambda_1(r_1,t) + \frac{\partial \lambda_1(r_1,t)}{\partial r_1}[(r_1 - s_1) - (r_2 - s_2)]$$

Observe that $\dfrac{\partial}{\partial r_1} A(r_1,t) = 0$ is the optimality condition for $r_t^*(x_1,x_2)$ when $x_1, x_2 \to \infty$.

Hence if $r_t^{FP}(x_1,x_2) \leq r_0 \leq r_t^*(x_1,x_2)$, $A(r_t^{FP}(x_1,x_2),t) \leq A(r_0,t)$, i.e., no re-price is necessary.

Therefore, a necessary condition to re-price is $r_0 > r_t^*(x_1,x_2)$ or $r_0 < r_t^{FP}(x_1,x_2)$. $\square$

119

Bibliography

1. Abreu, D., Pearce, D. and Stacchetti, E., 1986, "Optimal Cartel Equilibria with Imperfect Monitoring", *Journal of Economic Theory*, 39(1), pp.251-269
2. Abreu, D., Pearce, D. and Stacchetti, E., 1990, "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring", *Econometrica*, 58(5), pp.1041-1063
3. Baker, G.P., Gibbs, M. and Holmstrom, B., 1994a, "The Internal Economics of the Firm: evidence from personnel data", *Quarterly Journal of Economics* 109, pp.881-919.
4. Baker, G.P., Gibbs, M. and Holmstrom, B., 1994b, "The Wage Policy of a Firm", *Quarterly Journal of Economics*, 109, pp.921-955.
5. Benjaafar, S., Elahi, E., and Donohue, K., 2005 "Outsourcing via Service Competition", to appear in *Management Science*, 2006.
6. Bertsekas D., 2000, "Dynamic Programming and Optimal Control", Athena Scientific
7. Cachon, G. and Zhang, F., 2003, "Obtaining Fast Service in a Queuing System via Performance-Based Allocation of Demand", working paper
8. Dyer, J., 2000, "Collaborative Advantage: Winning through Extended Enterprise Supplier Networks", Oxford University Press
9. Fudenberg, D. and Tirole J., 1991, "Game Theory", The MIT Press
10. Gibbons, R. and Waldman, M., 1999a, "A Theory of Wage and Promotion Dynamics inside Firms", *Quarterly Journal of Economics*, 114, pp.1321-1358.
11. Gibbons, R. and Waldman, M., 1999b, "Careers in Organizations: Theory and Evidence", *Handbook of Labor Economics*, Volume 3b, Chapter 36.
12. Green J. and Stokey, N., 1983, "A Comparison of Tournaments and Contracts", *Journal of Political Economy*, 91(3), pp.349-364
13. Holmstrom B., 1979, "Moral Hazard and Observability", *The Bell Journal of Economics*, 10, pp.74-91.
14. Holmstrom B., 1982, "Moral Hazard in Teams", *The Bell Journal of Economics*, 13, pp.324-40.
15. Innes, R., 1990, "Limited Liability and Incentive Contracting with Ex-Ante Action Choices", *Journal of Economic Theory*, 52, pp.45-67
16. Klotz, D. and Chatterjee K., 1995, "Dual Sourcing in Repeated Procurement Competitions", *Management Science*, 41(8), pp.1317-1327
17. Lazear, E., Rosen, S., 1981, "Rank Order Tournament as Optimal Labor Contracts", *Journal of Political Economy*, 89(5), pp.841-864
18. Levin, J., 2003, "Relational Incentive Contracts", *The American Economic Review*, 93(3), pp.835-857
19. Levin, J., 2002, "Multilateral Contracting and the Employment Relationship", *Quarterly Journal of Economics*, 117(3), pp.1075-1103
20. Mirrlees, J., 1976, "The Optimal Structure of Incentives and Authority within an Organization", *The Bell Journal of Economics*, 7(1), pp.105-131.
21. Milgrom, P., 1981, "Good News and Bad News: Representation Theorems and Applications", *The Bell Journal of Economics*, 12, pp.380-391

22. Phelan,C. and Townsend,R., 1991, "Computing Multi-Period Information-Constrained Optima", *Review of Economic Studies*, 58, pp.853-881
23. Rogerson, W.P., 1985, "The First-Order Approach to Principal-Agent Problems.", *Econometrica*, 53, pp.1357-1367
24. Seshadri, S., 1995, "Bidding for Contests", *Management Science*, 41(4), pp.561-576
25. Sherefkin R. 2006, "Bo uses GM's clout to remake an industry", *Automotive News*, July 25
26. Spear, S. and Srivastava S., 1987, "On Repeated Moral Hazard with Discounting", *Review of Economic Studies* LIV, pp.599-717
27. Stallkamp, T., 2005, "SCORE!: A Better Way to Do Business: Moving from Conflict to Collaboration", Wharton School Publishing.
28. Wang, C., 1997, "Incentives, CEO Compensation, and Shareholder Wealth in a Dynamic Agency Model", Journal of Economic Theory, 76, pp.72-105
29. Yeltekin, S., 2003, "Dynamic Principal-Multiple Agent Contracts", Working Paper
30. Bassok, Y., Anupindi, R. and Akella, R., 1999, "Single-Period Multiproduct Inventory Models with Substitution", *Operations Research*, 47(4), pp.632-642.
31. Bitran, G. and Dasu, S., 1992, "Ordering Policies in an Environment of Stochastic Yields and Substitutable Demands", *Operations Research*, 40(5), pp.999-1017.
32. Goyal, S.K. and Giri, B.C., 2001, "Recent Trends in Modeling of Deteriorating Inventory", *European Journal of Operational Research*, 134, pp.1-16.
33. Ignall, E. and Veinott, A., 1969, "Optimality of Myopic Inventory Policies for Several Substitute Products", *Management Science*, 15(5), pp.284-304.
34. Jain, K. and Silver, E., 1994, "Lot Sizing for a Product Subject to Obsolescence or Perishability", *European Journal of Operational Research*, 75, pp.287-295.
35. Kalpakam, S. and Sapna, K., 1994, "Continuous Review (s,S) Inventory System with Random Lifetimes and Positive Leadtimes", *Operations Research Letters*, 16, pp.115-119.
36. Krishnamoorthy, A. and Varghese, T., 1995, "Inventory with Disaster", *Optimization*, 35, pp.85-93.
37. Mahajan, S. and van Ryzin, G., 2001, "Stocking Retail Assortment under Dynamic Consumer Substitution", *Operations Research*, 49(3), pp.334-351.
38. Parlar, M. and Goyal, S., 1984, "Optimal Ordering Decisions for Two Substitutable Products with Stochastic Demands", *Opsearch*, 21, pp.1-15.
39. Pasternack, B. and Drezner, Z., 1991, "Optimal Inventory Policies for Substitutable Commodities with Stochastic Demand", *Navel Research Logistics*, 38, pp.221-240.
40. Rao, U., Swaminathan, J. and Zhang, J., 2002, "Multi-product Inventory Planning with Downward Substitution, Stochastic Demand and Setup Costs", *IIE Transactions*, 36, pp.59-71.
41. Ravichandram, N., 1995, "Stochastic Analysis of a Continuous Review Perishable Inventory System with Positive Leadtime and Poisson Demand", *European Journal of Operational Research*, 84, pp.444-457.
42. Rosenfield, D., 1989, "Disposal of Excess Inventory", *Operations Research*, 37, pp.404-409.

43. Rosenfield, D., 1992, "Optimality of Myopic Policies in Disposing Excess Inventory", *Operations Research*, 40, pp.800-803.
44. Smith, S. and Agrawal, N., 2000, "Management of Multi-item Retail Inventory System with Demand Substitution", *Operations Research*, 48(1), pp.50-64.
45. Song, J. and Zipkin, P., 1996, "Managing Inventory with the Prospect of Obsolescence", *Operations Research*, 44, pp.215-222.
46. Waltson, N. and Olivia, R., 2005, "Leitax(A)", Harvard Business School case 9-606-002.
47. Wilson, L. and Norton, J., 1989, "Optimal Entry Timing for a Product Line Extension", *Marketing Science*, 8(1), pp.1-17.
48. Wilhelm, W. and Xu, K, 2002, "Prescribing Product Upgrades, Prices and Production Levels over Time in a Stochastic Environment", *European Journal of Operational Research*, 138, pp.601-621.
49. Avriel, M., Diewert, W.E., Schaible, S., and Zang, I., 1988, "Generalized Concavity", Plenum Press, MIT call number QA353.C64.G46
50. Awad, P., Bitran, G.R. and Mondschein, S.V., 2000, "Pricing Policies for a Family of Substitute Perishable. Products," working paper
51. Bass, F. M., 1969, "A New Product Growth Model for Consumer Durables," *Management Science*, 15, pp.215-227
52. Bayus, B.L., 1992, "The Dynamic Pricing of Next Generation Consumer Durables", *Marketing Science*, 11(3), pp.251-265
53. Bitran, G.R. and Mondschein, S.V., 1997, "Periodic Pricing of Seasonal Products in Retailing", *Management Science*, 43(1), pp.64-79.
54. Bitran, G.R. and Mondschein, S.V., 1993, "Pricing Perishable Products: An Application to the Retail Industry", MIT Working Paper #3592-93, Cambridge, MA, July 1993.
55. Bitran, G.R., Caldentey R. and Vial R., 2004, "Pricing Policies for Perishable Products with Demand Substitution", working paper
56. Bitran, G.R. and Caldentey, R., 2003, "An Overview of Pricing Models for Revenue Management", *Manufacturing & Service Operations Management*, 5(3), pp.203-229
57. Feng, Y. and Gallego, G., 1995, "Optimal Starting Times for End-of-Season Sales and Optimal Stopping Times for Promotional Fares", *Management Science*, 41(8), pp.1371-1391.
58. Fisher, J. and Pry, R., 1971, "A simple substitution model of technological change", *Technological Forecasting and Social Change*, 3, pp.75-88
59. Gallego, G. and van Ryzin, G., 1994, "Optimal Dynamic Pricing of Inventories with Stochastic Demand over Finite Horizons", *Management Science*, 40(8), pp.999-1018.
60. Gallego, G. and van Ryzin, G., 1997, "A Multiproduct Dynamic Pricing Problem and Its Applications to Network Yield Management", *Operations Research*, 45(1), pp.24-41
61. Hauser J.R. and Urban G.L., 1986, "The Value priority hypotheses for consumer budget plans", *Journal of Consumer Research*, 12, pp.446-462.
62. Kornish, L.J., 2001, "Pricing for a Durable-Goods Monopolist under Rapid Sequential Innovation", *Management Science*, 47(11), pp.1552-2561

63. Krishnan, T.V., Bass, F.M. and Jain, D.C., 1999, "Optimal Pricing Strategy for New Products", *Management Science*, 45(12), pp.1650-1663

64. Lazear, E.P., 1986, "Retail Pricing and Clearance Sales", *American Economic Review*, 76, pp.14-32

65. Luce, D., 1959,"Individual Choice Behavior; A Theoretical Analysis", New York, Wiley

66. Mahajan, V., Muller, E. and Wind, Y., 1998, "New-Product Diffusion Models", *International Series in Quantitative Marketing,* Kluwer Academic Publishers

67. Monohan, G.E., Petruzzi, N.C. and Zhao, W., 2004, "The Dynamic Pricing Problem from a Newsvendor's Perspective", *Manufacturing & Service Operations Management*, 6(1), pp.73-91.

68. Norton, J. and Bass, F., 1987, "A diffusion theory model of adoption and substitution for successive generations of high-technology products", *Management Science*, 33(9), pp.1069-1086

69. Wind, Y., Mahajan, V. and Cardozo, R. N., 1981, "New-Product Forecasting: Models and Applications", LexingtonBooks

70. Xu, X. and Hopp, W. J., 2005, "Dynamic Pricing and Inventory Control with Demand Substitution: The Value of Pricing Flexibility", working paper

71. Zhao, W. and Zheng, Y., 2000, "Optimal Dynamic Pricing for Perishable Assets with Nonhomogeneous Demand", *Management Science*, 46(3), pp.375-388.