The Sum of the 1D Magnifications Along the Axis of Positive Curvature for a Smooth Gravitational Potential with N Point Perturbations

by

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Abstract

Gravitational lensing is an important tool for determining the matter content of the universe. The locations of gravitationally lensed images tend to give us information about the overall structure of a lensing galaxy, whereas the magnifications of the images tell us about small scale structure of the galaxy such as the abundance of stars and dark matter condensations. In particular, flux ratio anomalies– disparities between predicted and observed magnifications of images– have led astronomers to study the role of perturbations in determining image brightness. In this paper, we explore the limits of demagnification due to point perturbations. We look at configurations of perturbations that are extremely improbable but that nonetheless illustrate interesting patterns in magnifications. Ultimately, we prove that for any number of point perturbations the total one dimensional magnification along the axis of curvature is constant and independent of perturbation size and location.

Thesis Supervisor: Paul L. Schechter
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Chapter 1

Introduction

Local perturbational corrections to the gravitational potentials of lensing galaxies have been shown to be a promising explanation for the puzzling flux ratio anomalies exhibited by many quadruple systems. However, as of yet this subject has only been investigated for physically reasonable situations involving random distributions of many perturbations and small numbers of deliberately placed perturbations. In this paper, we look at large, deliberately placed configurations of perturbations that are extremely improbable but that nonetheless illustrate exciting and useful patterns. These patterns in turn give us new insight into the more physically reasonable configurations of perturbations.

In chapter two we calculate the limits of demagnification due to a single perturbation and prove that the maximum demagnification occurs when the perturbation lies directly over the image, reproducing the 2002 work of Schechter and Wambsganss [13]. We then arrange point masses in a pyramid of direct hits. We show that with thus configuration we can demagnify a saddlepoint by any arbitrary amount, and we observe that a distinct pattern arises in the one dimensional magnifications.

In chapter three we prove more generally that the total one dimensional magnification along the axis of positive curvature is 1 for n=1,2,3 and 4 point perturbations.

Finally, in chapter four we show that for any number of point perturbations, and regardless of the placements and masses of these perturbations, the total one dimensional magnification along the axis of positive curvature is a constant.

1.1 Scientific Context

1.1.1 Gravitational Lensing as a Tool

The study of dark matter is one of the most important areas of research in astronomy today. Mapping out the distribution of dark matter in the universe will verify or disprove current models of galaxy formation, and finding the total amount of dark matter in existence will allow us to piece together the beginnings of the universe--and the nature of its end. In short, the currently accepted model of the universe will be made or broken by our observations of dark matter in the coming years.

However, because by definition dark matter does not absorb, reflect, or emit photons of any frequency, we must observe it indirectly through its gravitational potential. Dark matter makes its presence known by distorting spacetime and consequently altering the path of any passing photons. In particular, when a photon-emitting body such as a quasar passes behind a galaxy containing dark matter, the photons will be deflected around the galaxy on their way to our telescope in a way that tells us about the distribution of mass in that galaxy. This effect is known as gravitational lensing.

In gravitational lensing, most of the light deflection occurs when a photon is within a distance $\pm B$ of its point of closest approach to the galaxy, where B is the photon's impact parameter (capitalized here to distinguish it from the Einstein radius, b). Since B is small compared to distance the photon is traveling, we can treat the galaxy as thin and replace its 3D gravitational potential with a two dimensional projection. In situations where the 2D-potential is perfectly circular (and the photon source lies directly behind the galaxy), the photons will be deflected in such a way that we observe a bright ring, called an Einstein ring. More commonly, the 2D-potential will be oblong due either to a gravitational tide or to the intrinsic flatness of the galaxy and the photons will be deflected in such a way that they form multiple images of the photon source; in the case of four images this is known as a quadruple system.

The locations of the images in these systems can be found by calculating the stationary points in the time delay surface, τ. The time delay surface gives the travel time from the photon source to the observer as a function of angular position in the sky:

$$\tau = K\left[\frac{1}{2}(\vec{\theta}-\vec{\beta})^2 - \psi_{2D}\right]; K \equiv \frac{1+z_L}{c}\frac{D_L D_S}{D_{LS}} \tag{1.1}$$

Here ψ_{2D} is the 2D-potential of the lens, β is the angular position of the photon source, θ is the angular position of a given image, c is the speed of light, z_L is the redshift of the lensing galaxy, D_L, D_S, and D_{LS} are the distance to the lens, the distance to the source, and the distance between the lens and source respectively. In the case of quadruple systems, two of the observed images correspond to saddlepoints in the time delay surface, two correspond to minima, and a typically unobservable fifth image behind the lens itself corresponds to a steep maximum.

The magnification of a given image, μ, can likewise be found by calculating the inverse magnification matrix, $\frac{\partial\vec{\beta}}{\partial\vec{\theta}}$:

$$\mu^{-1} = \frac{\partial\vec{\beta}}{\partial\vec{\theta}} = \begin{pmatrix} 1-\frac{\partial^2\psi_{2D}}{\partial x^2} & -\frac{\partial^2\psi_{2D}}{\partial x\partial y} \\ -\frac{\partial^2\psi_{2D}}{\partial x\partial y} & 1-\frac{\partial^2\psi_{2D}}{\partial y^2} \end{pmatrix}$$

Where ψ_{2D} is again the 2D-potential of the lensing galaxy.

Since the time delay surface and the inverse magnification matrix both depend on the 2D-potential, by experimentally determining the location and brightness of the images in a quadruple system we can work backward to find an simple but accurate model of the 2D gravitational potential of the lensing galaxy. In this way gravitational lensing can be used as a tool to determine how much of the mass of a galaxy is in a smooth dark matter portion and how much is found in stars and compact stellar mass objects such as white dwarfs, black holes, and neutron stars. However, added complications arise when the simple 2D-potential that correctly predicts the image locations fails to reproduce the image magnifications. These inconsistencies in image brightness are known as flux ratio anomalies, and are the subject of this paper.

1.1.2 Flux Ratio Anomalies and Small Scale Structure

Gaudi and Petters (2002) [10] give a proof of a theorem that in a quadruple system two closely spaced images will be of similar brightness. This theorem relies on the assumption that the 2D gravitational potential is smooth on the scale of the image separation. The quadruple system PG1115+080 (Weymann et al. 1980)[7] was originally cited as experimental verification of this theory. In 2002 Zhao and Metcalf [14] explored an extensive suite of models of PG1115 and predicted that the difference in brightness between the two images should be approximately 10%. However, in 2006, Pooley et al. [1] noted that the difference in brightness between the two images in PG1115 seems to have varied with time, ranging from a 5% to a 50% difference (Vanderriest et al. 1986; Kristian et al. 1993; Courbin et al. 1997; Iwamuro et al 2000)[8][5][3][4]. This was well beyond what Zhao and Metcalf's models of the system could account for. Furthermore, it is unlikely that the discrepancy in image brightness came from the difference in travel time between the images since travel time would only vary by a few days between the images, whereas the brightness of the quasar acting as a photon source would vary much more slowly.[11]

This discrepancy between the observed and predicted image brightness is not limited to the PG1115 system.The system SDSS0924+0219 looks very much like PG1115, with a close pair of images where one is nearly an order of magnitude brighter than the other (Inada et al. 2003) [6].

At the same time, these simple models for the lensing potentials predict the image *locations* very nicely. This suggests that the discrepancy between the observed and predicted brightness might be due to small scale structure in the lensing potential. In fact, we will see that these flux ratio anomalies arise from the incorrect assumption that the 2D gravitational potential is smooth on the scale of the image separation. In reality, the 2D potential of the intervening galaxy is "grainy" due to the deep local wells of stars and compact objects perturbing the smooth 2D gravitational potential of the dark matter.

This graininess becomes significant when one observes that saddlepoints and minima in the travel time behave differently when the lensing potential is perturbed. In 1979 Chang and Refsdal [2] demonstrated that a saddlepoint can be strongly demagnified by micro-lensing, whereas a minimum cannot. In 2002, Schechter and Wambsganss [13] calculated that hypothetically one could strategically place stars in the 2D potential such that the saddlepoint would appear fainter than the unlensed photon source.

Thus, it has been established that small scale structure in the 2D-potential could account for the flux ratio anomalies exhibited by many quadruple systems. However, as of yet this subject has only been investigated for random distributions of perturbations [12][9] and situations involving one or two perturbations to the 2D-potential.[11] In this paper, we build on previous work by taking an analytic look at the effects of multiple perturbations.

1.2 Topics Covered in this Paper

In this paper we seek to explore the limits of demagnification due to small scale structure. We create a hierarchy of successively smaller perturbations, each one placed directly over the images spawned by the previous tier. The hierarchy begins with a photon source directly behind a macro-lens, which creates two macro-saddle images and two macro-minimum images. A milli-lens is placed in the line of sight to one of the two macro-saddles, creating two milli-saddles. A micro-lens then put in front of each milli-saddle, creating four micro-saddles, and so on. Though these situations are highly contrived, they allow us to see interesting patterns in demagnification. We will find that by strategically placing stars we can demagnify a macro-saddle by any arbitrary amount, but that the total one dimensional magnification along the axis of curvature will be constant. We will prove that this is the case for one, two, three, and four perturbing masses. Finally we will show that it is constant for any number of perturbations and regardless of the mass of those perturbations.

1.3 A History of this Project

This project was started by Lucia Tian who worked on it as an undergraduate research project in 2005. Tain proved that the total 1D magnification along the axis of curvature is 1 for the special cases of n=1 and n=2, and her work led to the general conjecture proved in this paper. Tian's work was continued by Stephanie Chan in the summer of 2006. Chan succeeded in verifying Tian's work but she did not have time to document the progresses she made.

The present author began work in January of 2007.

Chapter 2

Demagnification of a Macro-saddle with a Hierarchy of Perturbations

We begin our exploration of demagnification by creating a typical macro-saddle. We macro-lens a point photon source (for instance a quasar) with a macro-lens of smooth mass distribution (typically a galaxy), creating a standard quadruple system. In the vicinity of one of the four resultant macro-images the 2D-potential can be approximated to second order as:

$$\psi = \frac{\kappa_0}{2}(x^2 + y^2) + \frac{\gamma_0}{2}(x^2 - y^2), \tag{2.1}$$

where κ_0 represents the convergence (the dimensionless local mass density) and γ_0 represents the shear (the combined effects of all tides). Typical values for a highly magnified saddlepoint macro-image would be:

$$\kappa_0 = \gamma_0 = 0.53. \tag{2.2}$$

We can find the magnification of a given unperturbed macro-image using the inverse magnification matrix:

$$\frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \begin{pmatrix} 1 - \frac{\partial^2 \psi}{\partial x^2} & -\frac{\partial^2 \psi}{\partial x \partial y} \\ -\frac{\partial^2 \psi}{\partial x \partial y} & 1 - \frac{\partial^2 \psi}{\partial y^2} \end{pmatrix}.$$

To find the matrix elements we take the derivatives of the unperturbed local potential (Equation 2.1):

$$\frac{\partial^2 \psi}{\partial x^2} = \kappa_0 + \gamma_0 \tag{2.3}$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = 0 \tag{2.4}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \kappa_0 - \gamma_0 \tag{2.5}$$

Plugging these derivatives into the inverse magnification matrix we get:

$$\frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \begin{pmatrix} 1 - \frac{\partial^2 \psi}{\partial x^2} & -\frac{\partial^2 \psi}{\partial x \partial y} \\ -\frac{\partial^2 \psi}{\partial x \partial y} & 1 - \frac{\partial^2 \psi}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 1 - (\kappa_0 + \gamma_0) & 0 \\ 0 & 1 - (\kappa_0 - \gamma_0) \end{pmatrix}$$

$$\left| \frac{\partial \vec{\beta}}{\partial \vec{\theta}} \right|^{-1} = \mu_{macro} \tag{2.6}$$

$$\mu_{macro} = \frac{1}{((1 - \kappa_0) - \gamma_0)((1 - \kappa_0) + \gamma_0)} \tag{2.7}$$

A typical unperturbed macro-saddle will therefore have magnification:

$$\mu_{macro} = \frac{1}{-.06} = -16.67 \tag{2.8}$$

2.1 Demagnification of a Macro-saddle: Single Perturbation Case

To find the maximum demagnification achievable with one perturbation we optimize the perturbation position and then calculate the demagnification for that position.

2.1.1 Optimizing Perturbation Placement

In this section we present, in brief, a calculation of the optimum perturbation location to demagnify a saddlepoint image. The full calculation is available in appendix A.

We optimize perturbation placement by perturbing the macro-saddle with a point mass at position a along the y-axis and then minimizing the magnification of the two resultant milli-images with respect to a.

To perturb the local potential of a given macro-image with a point mass we add a term that goes as $ln(r)$ to the local 2D-potential (equation 2.1). We center this point mass at some unspecified point $y = a$:

$$\psi_{milli} = \frac{\kappa_0}{2}(x^2 + y^2) + \frac{\gamma_0}{2}(x^2 - y^2) + b^2 ln((x^2 + (y - a)^2)^{1/2}) \tag{2.9}$$

where κ_0 represents the convergence, γ_0 represents the shear, and b represents the Einstein radius, a constant that depends on the mass of the perturbation, M:

$$b = \left[\frac{4GM}{c^2}\frac{D_{ds}}{D_d D_s}\right]^{1/2} \tag{2.10}$$

To minimize the magnification, μ_{milli}, we must maximize the determinant of the inverse magnification matrix, μ^{-1}_{milli}:

$$\mu^{-1}_{milli} = (1 - \frac{\partial^2\psi_{milli}}{\partial x^2})(1 - \frac{\partial^2\psi_{milli}}{\partial y^2}) - (\frac{\partial^2\psi_{milli}}{\partial y \partial x})^2 \tag{2.11}$$

To maximize this we find the derivatives and plug them into equation 2.11, giving us:

$$\begin{aligned}\mu^{-1}_{milli} = &\left(1 - \kappa_0 - \gamma_0 - \frac{b^2}{(x^2 + (y-a)^2)} - \frac{-2b^2x^2}{(x^2 + (y-a)^2)^2}\right) \\ &\times \left(1 - \kappa_0 + \gamma_0 - \frac{b^2}{x^2 + (y-a)^2} - \frac{-2b^2(y-a)^2}{(x^2 + (y-a)^2)^2}\right) \\ &- \left(\frac{(-2b^2x(y-a))^2}{(x^2 + (y-a)^2)^4}\right)\end{aligned} \tag{2.12}$$

We cannot yet maximize with respect to a because (x,y) is image position and will clearly depend on a. Thus we must find y(a) and x(a) for the two images. We find (x,y) by finding the stationary points in the time delay surface:

$$\tau = K\left[\frac{1}{2}(\vec{\theta} - \vec{\beta})^2 - \psi_{milli}\right], K \equiv \frac{1 + z_L}{c}\frac{D_L D_S}{D_{LS}}, \tag{2.13}$$

where ψ_{milli} is the 2D-potential of the milli-lens, $\vec{\beta}$ is the position of the source, $\vec{\theta}$ is the position of a given image, c is the speed of light, and z_L, D_L, D_S, and D_{LS} are geometric factors having to do with the distances between the source, lens, and observer.

We plug in for ψ_{milli} and find the gradient, $\vec{\nabla}\tau$, noting that since the source here is the macro-image, the source location will be $\beta_x = \beta_y = 0$. Since the gradient of the time delay is zero at the image positions, we set $\vec{\nabla}\tau$ equal to 0 and find:

$$\begin{aligned} 0 &= \left[x(1-\kappa_0-\gamma_0-\frac{b^2}{(x^2+(y-a)^2)})\right]\hat{x} \\ 0 &= \left[y(1-\kappa_0+\gamma_0)-\frac{b^2(y-a)}{(x^2+(y-a)^2)}\right]\hat{y} \end{aligned} \tag{2.14}$$

By symmetry the images are expected along the y-axis. Choosing $x = 0$, we find two solutions:

$$0 = y(1-\kappa_0+\gamma_0)-\frac{b^2}{(y-a)} \tag{2.15}$$

Rearranging and applying the quadratic formula we find:

$$x(a) = 0 \tag{2.16}$$

$$y(a) = \frac{a \pm \sqrt{a^2+\frac{4b^2}{1-\kappa_0+\gamma_0}}}{2} \tag{2.17}$$

Since perturbed saddlepoints produce only two milli-images, it is safe to assume these are the only two real answers. Now we can continue with our minimization of the magnification. Equation 2.12 becomes:

$$\begin{aligned} \mu_{milli}^{-1} = &\left(1-\kappa_0-\gamma_0-\frac{b^2}{(y-a)^2}\right) \\ &\times\left(1-\kappa_0+\gamma_0+\frac{b^2}{(y-a)^2}\right) \end{aligned} \tag{2.18}$$

After rearranging equation 2.18 and taking the derivative with respect to a we find:

$$\partial\mu_{milli}^{-1}/\partial a = \left(\gamma_0 + \frac{b^2}{(y-a)^2}\right)\frac{b^2}{(y-a)^3}(-2+\frac{a}{(y-\frac{a}{2})}) \tag{2.19}$$

To minimize the total magnification with respect to a, we must minimize the magnifications of the two images. Adding $\partial\mu_{milli}^{-1}/\partial a$ at the two image locations, setting the sum equal to 0, and defining $\frac{\sqrt{a^2+\frac{4b^2}{1-\kappa_0+\gamma_0}}}{2} \equiv \eta$ for convenience, we get:

$$\begin{aligned} 0 = \partial\mu_{total}^{-1}/\partial a = &\left(\frac{\gamma_0(\eta-a/2)^2+b^2}{(\eta-a/2)^5}\right)(-2+\frac{a}{(\eta)}) \\ &+\left(\frac{\gamma_0(-\eta-a/2)^2+b^2}{(-\eta-a/2)^5}\right)(-2+\frac{a}{(-\eta)}) \end{aligned} \tag{2.20}$$

Guessing $a = 0$:

$$\begin{aligned} 0 = \partial\mu_{total}^{-1}/\partial a = &\left(\frac{\gamma_0(\eta)^2+b^2}{(\eta)^5}\right) \\ &-\left(\frac{\gamma_0(\eta)^2+b^2}{(\eta)^5}\right) \end{aligned} \tag{2.21}$$

And therefore $a = 0$ is a solution. Looking at equation 2.20, we see that $\partial\mu_{total}^{-1}/\partial a$ is third order in a, and therefore there will be two more solutions. However, the term that is third order in a is positive, which means that $\partial\mu_{total}^{-1}/\partial a$ goes to negative infinity as a goes to negative infinity and positive infinity as a goes to positive infinity. Therefore, at the two remaining solutions, μ_{total}^{-1} will go from a negative slope to a positive slope, which means they are both minima. In contrast, at a=0, μ_{total}^{-1} will go from negative slope to positive slope, implying a is the maximum in the inverse magnification we have been looking for. Thus the maximum demagnification with one perturbation will occur when the perturbation is placed at $a = 0$, directly in the line of sight to the macro-image.

Now that we know to put the perturbation in the line of sight we can find the maximum demagnification of the macro-image attainable with two different types of perturbations: perturbation with a point-mass and perturbation with an isothermal sphere.

2.1.2 Maximum Demagnification of a Macro-saddle: Point-Mass Perturbation

Repeating the calculation of Schechter and Wambsganss [13], we now find the maximum demagnification achievable with one point mass perturbation. To perturb the local potential of a given macro-image with a point-mass (star or black hole) placed directly in the line of sight, we modify the potential as follows:

$$\psi_{micro} = \frac{\kappa_0}{2}(x^2 + y^2) + \frac{\gamma_0}{2}(x^2 - y^2) + b^2 ln((x^2 + y^2)^{1/2}), \tag{2.22}$$

where b again depends on the mass of the perturbation. The image locations can again be found by setting the gradient of the time delay surface equal to zero:

$$\begin{aligned} 0 = \vec{\nabla}\tau = K & \left[(x - x_\beta) - (\kappa_0 x + \gamma_0 x + \frac{b^2 x}{(x^2 + y^2)})\right] \hat{x} \\ +K & \left[(y - y_\beta) - (\kappa_0 y - \gamma_0 y + \frac{b^2 y}{(x^2 + y^2)})\right] \hat{y} \end{aligned} \tag{2.23}$$

Since the source is the macro-saddle, $\beta_x = \beta_y = 0$, and $\vec{\nabla}\tau$ becomes:

$$\begin{aligned} 0 &= \left[x(1 - \kappa_0 - \gamma_0 - \frac{b^2}{(x^2 + y^2)}))\right] \hat{x} \\ 0 &= \left[y(1 - \kappa_0 + \gamma_0 - \frac{b^2}{(x^2 + y^2)})\right] \hat{y} \end{aligned} \tag{2.24}$$

One trivial solution to this equation is:

$$x = y = 0. \tag{2.25}$$

which corresponds to a maximum in the time delay surface.

If the macro-image we are perturbing has values of κ_0 and γ_0 corresponding to a macro-saddle, we will find only three images (this maximum at (0,0) and two additional saddlepoints), but if the values of κ_0 and γ_0 are chosen such that the macro-image is a macro-minimum, we will find a total of five images (this maximum as well as two minima and two saddlepoints). In this section we will solve for the case of a general macro-image and wait until the end to choose values for κ_0 and γ_0.

Rearranging equation 2.24 we get:

$$x^3 = x\left(\frac{b^2}{1-\kappa_0-\gamma_0} - y^2\right) \tag{2.26}$$

$$yx^2 = y\left(\frac{b^2}{1-\kappa_0+\gamma_0} - y^2\right) \tag{2.27}$$

Since $\gamma_0 \neq 0$, it is impossible to solve this system of equations unless either x or y is zero. This gives solutions:

$$x = 0, y = \pm\frac{b}{(1-\kappa_0+\gamma_0)^{1/2}} \tag{2.28}$$

$$x = \pm\frac{b}{(1-\kappa_0-\gamma_0)^{1/2}}, y = 0 \tag{2.29}$$

We find the corresponding magnifications by taking the appropriate derivatives of ψ_{micro} and plugging them into the inverse magnification matrix. This gives us:

$$\frac{\partial\vec{\beta}}{\partial\vec{\theta}} = \begin{pmatrix} 1-(\kappa_0+\gamma_0+\frac{b^2}{(x^2+y^2)}+\frac{-2b^2x^2}{(x^2+y^2)^2}) & \frac{2b^2xy}{(x^2+y^2)^2} \\ \frac{2b^2xy}{(x^2+y^2)^2} & 1-(\kappa_0-\gamma_0+\frac{b^2}{(x^2+y^2)}+\frac{-2b^2y^2}{(x^2+y^2)^2}) \end{pmatrix}$$

For the two images on the x-axis at $y = 0, x = \pm\frac{b}{(1-\kappa_0-\gamma_0)^{1/2}}$, the inverse magnification matrix will then become:

$$\frac{\partial\vec{\beta}}{\partial\vec{\theta}} = \begin{pmatrix} 2-2\kappa_0-2\gamma_0 & 0 \\ 0 & 2\gamma_0 \end{pmatrix}$$

Finally, to find the magnification of the two images on the x-axis we take the inverse of the determinant of $\frac{\partial\vec{\beta}}{\partial\vec{\theta}}$ and find:

$$\mu_{micro} = \frac{1}{(4\gamma_0)(1-\kappa_0-\gamma_0)}. \tag{2.30}$$

For the two images on the y-axis at $x = 0, y = \pm\frac{b}{(1-\kappa_0+\gamma_0)^{1/2}}$, the inverse magnification matrix will become:

$$\frac{\partial\vec{\beta}}{\partial\vec{\theta}} = \begin{pmatrix} -2\gamma_0 & 0 \\ 0 & 2 - 2\kappa_0 + 2\gamma_0 \end{pmatrix}$$

To solve for the magnification of the two images on the y-axis we take the inverse of the determinant of $\frac{\partial\vec{\beta}}{\partial\vec{\theta}}$ and find that:

$$\mu_{micro} = \frac{1}{(-4\gamma_0)(1 - \kappa_0 + \gamma_0)}, \tag{2.31}$$

which successfully reproduces the 2002 calculation by Schechter and Wambsganss.[13]

A minimum is defined by values of the local convergence and shear where $(1-\kappa_0-\gamma_0) > 0$. Furthermore, for macro-lensing by galaxies, typically κ_0 will approximately equal γ_0 and therefore $(1 - \kappa_0 + \gamma_0) = 1$. Under these conditions the macro-minimum will be split into in four micro-images, two micro-minima on the x-axis (equation 2.29) with magnifications given by equation 2.30, and two micro-saddles on the y-axis (equation 2.28), with magnifications given by equation 2.31.

If the perturbed macro-image is a saddlepoint, κ_0 will still approximately equal γ_0 and therefore $(1 - \kappa_0 + \gamma_0)$ will still equal one, however by definition $(1 - \kappa_0 - \gamma_0)$ will negative, and the perturbation of the potential will result in only two micro-saddles. We can see why by looking at the equations we found for the image locations. Since $(1 - \kappa_0 - \gamma_0) < 0$ for a saddlepoint macro-image, $x = \pm\frac{b}{(1-\kappa_0-\gamma_0)^{1/2}}, y = 0$ (equation 2.29) is imaginary, and thus these images will not exist. By the same argument, $x = 0, y = \pm\frac{b}{(1-\kappa_0+\gamma_0)^{1/2}}$ (equation 2.28) will still be a valid image location since $(1 - \kappa_0 + \gamma_0)$ is still positive. Thus, when saddlepoint macro-image is micro-lensed there will only be two saddlepoint micro-images, and they will both be on the y-axis. Their magnifications are given by equation 2.31.

The images locations and magnifications are summarized in table 2.1.

Plugging in typical saddlepoint values of $\gamma_0 = \kappa_0 = 0.53$, we find that for the macro-saddle case both micro-images have a magnification of $\mu_{micro} = -1/2.12 =$

Table 2.1: Milli-Image Position for Perturbed Minimum: Point Mass

Position	Magnification
$x = 0,\ y = \pm\frac{b}{(1-\kappa_0+\gamma_0)^{1/2}}$	$\mu_{micro} = \frac{1}{(-4\gamma_0)(1-\kappa_0+\gamma_0)}$
$x = \pm\frac{b}{(1-\kappa_0-\gamma_0)^{1/2}},\ y = 0,$	$\mu_{micro} = \frac{1}{(4\gamma_0)(1-\kappa_0-\gamma_0)}$

-0.47. Thus, the perturbed macro-saddle has a magnification of $\mu_{total} = 2\mu_{micro} = -0.94$, a factor of 16 dimmer than the unperturbed macro-saddle ($\mu_{macro} = -16.67$) and, remarkably, even dimmer than the original source! Since observed flux ratio anomalies typically involve saddlepoints about an order of magnitude dimmer than predicted, this result is in line with experimental evidence.

2.1.3 Maximum Demagnification of a Macro-saddle: Isothermal Perturbation

An interesting variation on the point perturbation case is the isothermal perturbation case. Whereas the presence of point perturbations in the 2D-potential implies a galaxy rich in stars and black holes, the presence of isothermal perturbations implies the lensing galaxy contains dark matter condensations.

To perturb the local potential of a given macro-image with an isothermal sphere placed directly in the line of sight, we modify the potential as follows:

$$\psi_{milli} = \frac{\kappa_0}{2}(x^2 + y^2) + \frac{\gamma_0}{2}(x^2 - y^2) + b(x^2 + y^2)^{1/2}, \tag{2.32}$$

where κ_0 is the convergence, γ_0 is the shear, and b is a constant that depends on the mass of the perturbation. As before, we can find the image locations by solving for the stationary points in the time delay surface, τ. We plug in for ψ_{milli}, take the gradient of τ, and set the whole thing equal to zero, giving us:

$$\begin{aligned} 0 = \vec{\nabla}\tau = K &\left[(x - \beta_x) - (\kappa_0(x) + \gamma_0(x) + b(x)(x^2 + y^2)^{-1/2})\right]\hat{x} \\ +K &\left[(y - \beta_y) - (\kappa_0(y) - \gamma_0(y) + b(y)(x^2 + y^2)^{-1/2})\right]\hat{y} \end{aligned} \tag{2.33}$$

Since the source here is the macro-image and the macro-image is defined to be at the origin, as before the source location is $\beta_x = \beta_y = 0$. This gives us a system of two equations and two unknowns:

$$\begin{aligned} 0 &= \left[x(1-\kappa_0-\gamma_0-b(x^2+y^2)^{-1/2})\right]\hat{x} \\ 0 &= \left[y(1-\kappa_0+\gamma_0-b(x^2+y^2)^{-1/2})\right]\hat{y} \end{aligned} \tag{2.34}$$

Again, one trivial solution is the maximum:

$$x = y = 0, \tag{2.35}$$

As with the point mass case we can also find two minima and two saddlepoints. By symmetry we expect $y = 0$, reducing equation 2.34 to:

$$\begin{aligned} 0 &= \left[x(1-\kappa_0-\gamma_0-b(x^2)^{-1/2})\right]\hat{x} \\ 0 &= 0\hat{y} \end{aligned} \tag{2.36}$$

Which has solutions:

$$y = 0, x = \pm\frac{b}{1-\kappa_0-\gamma_0}. \tag{2.37}$$

Guessing $x = 0$ equation 2.34 becomes:

$$\begin{aligned} 0 &= 0\hat{x} \\ 0 &= \left[y(1-\kappa_0+\gamma_0-b(y^2)^{-1/2})\right]\hat{y} \end{aligned} \tag{2.38}$$

Which has solutions:

$$x = 0, y = \pm\frac{b}{1-\kappa_0+\gamma_0} \tag{2.39}$$

If the perturbed macro-image is a minimum in the time delay surface, $(1-\kappa_0-\gamma_0) > 0$. This allows all four image locations (equations 2.37 and 2.39).

In contrast, if the perturbed macro-image is a saddlepoint in the time delay surface, $(1-\kappa_0-\gamma_0) < 0$. Looking at equation 2.34 we see that equation 2.39 is still a valid solution, but that equation 2.37 results in the system of equations:

$$0 = \left[(\pm\frac{b}{1-\kappa_0-\gamma_0})(1-\kappa_0-\gamma_0-|1-\kappa_0-\gamma_0|) \right] \hat{x} \tag{2.40}$$

$$0 = 0\hat{y} \tag{2.41}$$

Since two negative numbers can never sum to zero, this system of equations cannot be solved. Thus, only two milli-saddles are created by perturbing a macro-saddle.

We find the total magnification of these milli-images by taking the appropriate derivatives of ψ_{milli} and plugging them into the inverse magnification matrix. We find:

$$\frac{\partial\vec{\beta}}{\partial\vec{\theta}} = \begin{pmatrix} 1-(\kappa_0+\gamma_0+\frac{b}{(x^2+y^2)^{1/2}}-\frac{bx^2}{(x^2+y^2)^{3/2}}) & \frac{bxy}{(x^2+y^2)^{3/2}} \\ \frac{bxy}{(x^2+y^2)^{3/2}} & 1-(\kappa_0-\gamma_0+\frac{b}{(x^2+y^2)^{1/2}}-\frac{by^2}{(x^2+y^2)^{3/2}}) \end{pmatrix}$$

For the two milli-images on the x-axis at $y=0, x=\pm\frac{b}{1-\kappa_0-\gamma_0}$ the inverse magnification matrix becomes:

$$\frac{\partial\vec{\beta}}{\partial\vec{\theta}} = \begin{pmatrix} 1-(\kappa_0+\gamma_0) & 0 \\ 0 & 1-(\kappa_0-\gamma_0+|1-\kappa_0-\gamma_0|) \end{pmatrix}$$

Taking the inverse of the determinant we find the magnification of the two images on the x-axis:

$$\mu_{milli} = \frac{1}{((1-\kappa_0-\gamma_0)(1-\kappa_0+\gamma_0-|1-\kappa_0-\gamma_0|))} \tag{2.42}$$

For the two milli-images on the y-axis at $x=0, y=\pm\frac{b}{1-\kappa_0+\gamma_0}$ the inverse magnification matrix becomes:

$$\frac{\partial\vec{\beta}}{\partial\vec{\theta}} = \begin{pmatrix} 1-(\kappa_0+\gamma_0+|1-\kappa_0+\gamma_0|) & 0 \\ 0 & 1-(\kappa_0-\gamma_0) \end{pmatrix}$$

Taking the inverse of the determinant we find the magnification of the two images on the y-axis:

$$\mu_{milli} = \frac{1}{(1-\kappa_0-\gamma_0-|1-\kappa_0+\gamma_0|)(1-\kappa_0+\gamma_0)} \tag{2.43}$$

If the perturbed macro-image is a minimum, $(1-\kappa_0-\gamma_0) > 0$ and $(1-\kappa_0+\gamma_0) = 1$, so for the two images on the x-axis we get:

$$\mu_{milli} = \frac{1}{(2\gamma_0)(1-\kappa_0-\gamma_0)} \tag{2.44}$$

and for the two images on the y-axis we get:

$$\mu_{milli} = \frac{1}{(-2\gamma_0)(1-\kappa_0+\gamma_0)} \tag{2.45}$$

If the perturbed macro-image is a saddlepoint, $(1-\kappa_0-\gamma_0) < 0$ but $(1-\kappa_0+\gamma_0)$ will still be approximately equal to 1. There will only be two images, both on the y-axis, and they will have magnification:

$$\mu_{milli} = \frac{1}{(-2\gamma_0)(1-\kappa_0+\gamma_0)}, \tag{2.46}$$

which is the same value we found for the milli-images on the y-axis when the macro-image was a minimum. (Mathematically, this comes from the fact that $|1-\kappa_0-\gamma_0|$ only appears in the magnifications of milli-images on the x-axis.) Plugging in standard saddlepoint values $\gamma_0 = \kappa_0 = 0.53$, we find that both milli-images have a magnification of $\mu_{milli} = -1/1.06 = -0.94$. Thus, the saddlepoint perturbed with an isothermal sphere directly in the line of sight has a magnification of $\mu_{total} = 2\mu_{milli} = -1.89$. This is a factor of 2 brighter than the micro-lensed macro-saddle ($\mu_{total} = -0.94$), but still a respectable factor of 8 dimmer than the unperturbed saddlepoint ($\mu_{macro} = -16.67$).

The positions and magnifications of the images are summarized in table 2.2:

Table 2.2: Milli-Image Position for Perturbed Minimum: Isothermal Mass

Position	Magnification
$x=0,\ y=\pm\frac{b}{(1-\kappa_0+\gamma_0)}$	$\mu_{milli}=\frac{1}{(-2\gamma_0)(1-\kappa_0+\gamma_0)}$
$x=\pm\frac{b}{(1-\kappa_0-\gamma_0)},\ y=0,$	$\mu_{milli}=\frac{1}{(2\gamma_0)(1-\kappa_0-\gamma_0)}$

2.2 Demagnification of a Macro-saddle: Hierarchies of Point Perturbations

Now we explore the limits of demagnification of a macro-saddle by creating a hierarchy of perturbations. As described in the introduction, we create a pyramid of successively smaller perturbations, each one placed directly over the images spawned by the previous tier. Since a perturbed saddlepoint produces only two images, each successive tier of the pyramid doubles in size.

We begin by creating a macro-saddle with local potential:

$$\psi_{macro}=\frac{\kappa_0}{2}(x^2+y^2)+\frac{\gamma_0}{2}(x^2-y^2). \tag{2.47}$$

A milli-lens point perturbation is then placed directly on top of it so as to maximally demagnify it (see section 2.1.1), changing the local potential to

$$\psi_{macro}=\frac{\kappa_0}{2}(x^2+y^2)+\frac{\gamma_0}{2}(x^2-y^2)+b_1^2 ln((x^2+y^2)^{1/2}) \tag{2.48}$$

and creating two milli-saddles. This process is repeated for two point mass micro-lenses, creating four micro-saddles, which are in turn perturbed with four point mass nano-lenses, producing eight nano-saddles, and so forth. Each time a saddlepoint is perturbed with a point mass, a new local shear and convergence are defined in the vicinity of the two resulting images, according to the formula:

$$\begin{pmatrix} (1-\kappa_{n+1}-\gamma_{n+1}) & 0 \\ 0 & (1-\kappa_{n+1}+\gamma_{n+1}) \end{pmatrix} \equiv \mu^{-1}(\kappa_n,\gamma_n)=\begin{pmatrix} 1-\frac{\partial^2\psi}{\partial x^2}(\kappa_n,\gamma_n) & -\frac{\partial^2\psi}{\partial x\partial y}(\kappa_n,\gamma_n) \\ -\frac{\partial^2\psi}{\partial x\partial y}(\kappa_n,\gamma_n) & 1-\frac{\partial^2\psi}{\partial y^2}(\kappa_n,\gamma_n) \end{pmatrix}$$

And the equation for the potential in the vicinity of each of the two resulting images will be:

$$\psi = \frac{\kappa_{n+1}}{2}(x^2+y^2) + \frac{\gamma_{n+1}}{2}(x^2-y^2) \tag{2.49}$$

2.2.1 One Perturbation Pyramid

We can use our work from section 2.1.2 to find the new effective κ and γ after the macro-image is milli-lensed once with a point mass. In the immediate vicinity of each milli-image, we define an effective convergence, κ_1 and and effective shear, γ_1, such that:

$$\begin{pmatrix} (1-\kappa_1-\gamma_1) & 0 \\ 0 & (1-\kappa_1+\gamma_1) \end{pmatrix} = \begin{pmatrix} 1-\frac{\partial^2\psi}{\partial x^2}(\kappa_0,\gamma_0) & -\frac{\partial^2\psi}{\partial x\partial y}(\kappa_0,\gamma_0) \\ -\frac{\partial^2\psi}{\partial x\partial y}(\kappa_0,\gamma_0) & 1-\frac{\partial^2\psi}{\partial y^2}(\kappa_0,\gamma_0) \end{pmatrix}$$

From section 2.1.2 the inverse magnification matrix is:

$$\frac{\partial\vec{\beta}}{\partial\vec{\theta}} = \begin{pmatrix} -2\gamma_0 & 0 \\ 0 & 2-2\kappa_0+2\gamma_0 \end{pmatrix},$$

so we can set up the equality:

$$\begin{pmatrix} (1-\kappa_1-\gamma_1) & 0 \\ 0 & (1-\kappa_1+\gamma_1) \end{pmatrix} = \begin{pmatrix} -2\gamma_0 & 0 \\ 0 & 2-2\kappa_0+2\gamma_0 \end{pmatrix}.$$

This gives us a system of two equations and two unknowns:

$$\begin{aligned} 1-\kappa_1-\gamma_1 &= -2\gamma_0 \\ 1-\kappa_1+\gamma_1 &= 2-2\kappa_0+2\gamma_0 \end{aligned} \tag{2.50}$$

Solving this system we get:

$$\begin{aligned} \kappa_1 &= \kappa_0 \\ \gamma_1 &= 1-\kappa_0+2\gamma_0 \end{aligned} \tag{2.51}$$

And the total magnification for our typical macro-saddle is $\mu_{total} = -0.94$ as we found in section 2.1.2.

2.2.2 Three Perturbation Pyramid

Now in the immediate vicinity of each milli-saddle the potential will be:

$$\psi_{milli} = \frac{\kappa_1}{2}(x^2 + y^2) + \frac{\gamma_1}{2}(x^2 - y^2). \tag{2.52}$$

If we then use a micro-lens to perturb each milli-saddle, the potential in the vicinity of each milli-image will become:

$$\psi_{milli} = \frac{\kappa_1}{2}(x^2 + y^2) + \frac{\gamma_1}{2}(x^2 - y^2) + b_1^2 ln((x^2 + y^2)^{1/2}, \tag{2.53}$$

and four micro-images will be created. Now in the immediate vicinity of each micro-image we can define an effective convergence, κ_2 and an effective shear, γ_2, such that:

$$\begin{pmatrix} (1 - \kappa_2 - \gamma_2) & 0 \\ 0 & (1 - \kappa_2 + \gamma_2) \end{pmatrix} \equiv \mu_{micro}^{-1}(\kappa_1, \gamma_1) = \begin{pmatrix} -2\gamma_1 & 0 \\ 0 & 2 - 2\kappa_1 + 2\gamma_1 \end{pmatrix}$$

Where we have reused use our work from section 2.1.2 by replacing κ_0, γ_0, b_0 with κ_1, γ_1, b_1 in the potential. This gives us a system of two equations and two unknowns:

$$\begin{aligned} 1 - \kappa_2 - \gamma_2 &= -2\gamma_1 \\ 1 - \kappa_2 + \gamma_2 &= 2 - 2\kappa_1 + 2\gamma_1 \end{aligned} \tag{2.54}$$

Solving this for κ_2 and γ_2 we find:

$$\begin{aligned} \kappa_2 &= \kappa_1 \\ \gamma_2 &= 1 - \kappa_1 + 2\gamma_1 \end{aligned} \tag{2.55}$$

And with $\kappa_0 = \gamma_0 = .53$ the magnification of each micro-saddle will be:

$$\mu_{micro} = \frac{1}{(-4\gamma_1)(1 - \kappa_1 + \gamma_1)} = -0.082 \tag{2.56}$$

Which gives us a total magnification of $\mu_{total} = 4\mu_{micro} = -0.33$, fifty times dimmer than the unperturbed saddlepoint ($\mu_{total} = -16.67$).

2.2.3 Seven Perturbation Pyramid

Now in the immediate vicinity of each micro-saddle the potential will be:

$$\psi_{micro} = \frac{\kappa_2}{2}(x^2 + y^2) + \frac{\gamma_2}{2}(x^2 - y^2). \tag{2.57}$$

If we then use a nano-lens to perturb each micro-saddle, the potential in the vicinity of each micro-image will become:

$$\psi_{micro} = \frac{\kappa_2}{2}(x^2 + y^2) + \frac{\gamma_2}{2}(x^2 - y^2) + b_2^2 ln((x^2 + y^2)^{1/2}, \tag{2.58}$$

and eight nano-images will be created. In the immediate vicinity of each micro-saddle we can define an effective convergence, κ_3 and and effective shear, γ_3:

$$\begin{pmatrix} (1 - \kappa_3 - \gamma_3) & 0 \\ 0 & (1 - \kappa_3 + \gamma_3) \end{pmatrix} \equiv \mu_{micro}^{-1}(\kappa_2, \gamma_2) = \begin{pmatrix} -2\gamma_2 & 0 \\ 0 & 2 - 2\kappa_2 + 2\gamma_2 \end{pmatrix}$$

This gives us a system of two equations and two unknowns with solutions:

$$\kappa_3 = \kappa_2 \tag{2.59}$$

$$\gamma_3 = 1 - \kappa_2 + 2\gamma_2 \tag{2.60}$$

With $\kappa_0 = \gamma_0 = .53$ the magnification of each nano-saddle will be:

$$\mu_{nano} = \frac{1}{(-4\gamma_2)(1 - \kappa_2 + \gamma_2)} = -0.018 \tag{2.61}$$

Which gives us a total magnification of $\mu_{total} = 8\mu_{nano} = -0.14$, two orders of magnitude dimmer than the unperturbed macro-saddle ($\mu_{total} = -16.67$).

2.2.4 Infinite Pyramid of Perturbations

We can conclude that, in general:

$$\kappa_{n-1} = \kappa_n \tag{2.62}$$

$$1 - \kappa_{n-1} + 2\gamma_{n-1} = \gamma_n \tag{2.63}$$

for $n \geq 0$. Thus, κ_n remains constant while γ_n increases without limit. Since, for this particular configuration of perturbations, the magnification is:

$$\mu_n^{-1} = \begin{vmatrix} 2\gamma_n & 0 \\ 0 & 2 - 2\kappa_n + 2\gamma_n \end{vmatrix}$$

,

$$\mu_n = \frac{1}{-2(2 - 2\kappa_n + 2\gamma_n)(\gamma_n)} \tag{2.64}$$

$$\mu_{ntotal} = \frac{2^{n+1}}{-4(1 - \kappa_n + \gamma_n)(\gamma_n)} \tag{2.65}$$

for $n \geq 1$.

To find μ_{total} in terms of κ_0 and γ_0 we look at the cases of n=0, 1, and 2, and we quickly see a pattern emerge:

$$\begin{aligned} \mu_{0total} &= \frac{2}{-4(1 - \kappa_0 + \gamma_0)(\gamma_0)} & (2.66) \\ &= \frac{1}{-2(1 - \kappa_0 + \gamma_0)(\gamma_0)} & (2.67) \end{aligned}$$

$$\begin{aligned} \mu_{1total} &= \frac{4}{-4(1 - \kappa_1 + \gamma_1)(\gamma_1)} & (2.68) \\ &= \frac{4}{-4(1 - \kappa_0 + (1 - \kappa_0 + 2\gamma_0))(1 - \kappa_0 + 2\gamma_0)} & (2.69) \end{aligned}$$

$$= \frac{1}{-2(1-\kappa_0+\gamma_0)(1-\kappa_0+2\gamma_0)} \tag{2.70}$$

$$\mu_{2total} = \frac{8}{-4(1-\kappa_2+\gamma_2)(\gamma_2)} \tag{2.71}$$

$$= \frac{1}{-(1-\kappa_0+(1-\kappa_0+2\gamma_0))(1-\kappa_0+2(1-\kappa_0+2\gamma_0))} \tag{2.72}$$

$$= \frac{1}{-2(1-\kappa_0+\gamma_0)(3-3\kappa_0+4\gamma_0)} \tag{2.73}$$

We can easily extrapolate to the case of n tiers of perturbations:

$$\mu_{ntotal} = \frac{1}{-2(1-\kappa_0+\gamma_0)((2^n-1)(1-\kappa_0)+2^n\gamma_0)} \tag{2.74}$$

Therefore we can conclude that, by increasing the number of tiers of perturbations, the magnification can be made arbitrarily small. Furthermore, for large numbers of tiers the 1 in the $(2^n - 1)$ term becomes negligibly small and we can say that the total magnification will look like:

$$\mu_{ntotal} \approx \frac{1}{-(2^{n+1})(1-\kappa_0+\gamma_0)^2}. \tag{2.75}$$

The information from sections 2.2.1, 2.2.2, 2.2.3, and 2.2.4 is summarized in table 2.3:

Table 2.3: Summary of Pyramid Data

Perturbations:	n=1	n=3	n=7
μ_{total}	$\frac{-1}{(2\gamma_0)(1-\kappa_0+\gamma_0)}$	$\frac{-1}{2(1-\kappa+2\gamma_0)(1-\kappa_0+\gamma_0)}$	$\frac{-1}{2(3-3\kappa+4\gamma_0)(1-\kappa_0+\gamma_0)}$
λ_1	$-2\gamma_0$	$-2(1-\kappa+2\gamma_0)$	$-2(3-3\kappa+4\gamma_0)$
λ_2	$2(1-\kappa_0+\gamma_0)$	$4(1-\kappa_0+\gamma_0)$	$8(1-\kappa_0+\gamma_0)$
κ_n	κ_0	κ_0	κ_0
γ_n	$1-\kappa_0+2\gamma_0$	$3-3\kappa_0+4\gamma_0$	$7-7\kappa_0+8\gamma_0$

In this table μ_{total} is the total magnification of the macro-image after all the lensing events, λ_1 is the one dimensional inverse magnification along the axis of negative curvature, and λ_2 is the one dimensional inverse magnification along the axis of positive curvature. For the limiting case of infinite magnification ($\kappa_0 = \gamma_0 = 0.5$) table 2.3 becomes table 2.4.

Table 2.4: Summary of Pyramid Data for Limiting Case of $\kappa_0 = \gamma_0 = 0.5$

Perturbations:	n=1	n=3	n=7
$\mu_{total}(\kappa_0 = \gamma_0 = 0.5)$	-1	$-\frac{1}{3}$	$-\frac{1}{7}$
$\lambda_1(\kappa_0 = \gamma_0 = 0.5)$	-1	-3	-7
$\lambda_2(\kappa_0 = \gamma_0 = 0.5)$	2	4	8
$\kappa_n(\kappa_0 = \gamma_0 = 0.5)$	0.5	0.5	0.5
$\gamma_n(\kappa_0 = \gamma_0 = 0.5)$	1.5	3.5	7.5

Looking at the two tables we begin to notice interesting patterns in the magnifications. Whereas the total one dimensional magnification along the axis of positive curvature is a constant, $\frac{1}{1-\kappa_0+\gamma_0}$, The total one dimensional magnification along the axis of negative curvature is getting smaller by about half with each new tier. In fact, this is true no matter where the masses are placed and no matter how large they are, as we will prove in the following chapters.

Chapter 3

Total 1D Magnification Along the Axis of Curvature for n=1,2,3,4

In this chapter we will explicitly solve for the total one dimensional magnification along the axis of positive curvature for the cases of $n = 1, 2, 3$, and 4 perturbations. We will accomplish this by first finding a general equation for the 1D magnification at a given image position (x_j, y_j), and then, looking at the case of each value of n separately, by summing up the 1D magnification given by the general equation at all image locations.

3.1 Deriving a General Equation for the 1D Magnification

To find a general equation for the 1D magnification at a given image position (x_j, y_j) we begin with the 2D-potential:

$$\psi = \frac{\kappa_0}{2}(x^2+y^2)+\frac{\gamma_0}{2}(x^2-y^2)+b_1^2 ln((x^2+(y-q_1)^2)^{1/2})+b_2^2 ln((x^2+(y-q_2)^2)^{1/2})+b_3^2 ln((x^2+(y-a_3)^2)^{1/2})+... \tag{3.1}$$

Here $(0, q_j)$ is the perturbation location and b_j is the Einstein radius, a constant that depends on the mass, M_j, of perturbation j:

$$b_j = \left[\frac{4GM_j}{c^2}\frac{D_{ds}}{D_d D_s}\right]^{1/2} \tag{3.2}$$

Without loss of generality we can set $\kappa_0 = \gamma_0 = 0$. Then:

$$\psi = b_1^2 ln((x^2 + (y - q_1)^2)^{1/2}) + b_2^2 ln((x^2 + (y - q_2)^2)^{1/2}) + ... \tag{3.3}$$

To find the total one-dimensional magnification along the axis of positive curvature we must find the positive eigenvalue, λ_2, of the inverse magnification matrix, μ^{-1}, for the two images:

$$\lambda_2 = 1 - \frac{\partial^2 \psi}{\partial y^2} \tag{3.4}$$

$\mu_{1Dtotal}$ will then be the sum of the inverses of the eigenvalues:

$$\mu_{1Dtotal} = (\lambda_2(x_1, y_1))^{-1} + (\lambda_2(x_2, y_2))^{-1} \tag{3.5}$$

The time delay surface will be described by the equation:

$$\tau = K\left[\frac{1}{2}(\vec{\theta} - \vec{\beta})^2 - \psi\right], K \equiv \frac{1 + z_L}{c}\frac{D_L D_S}{D_{LS}} \tag{3.6}$$

The potential ψ has been constructed such that the source is at the origin, so $\vec{\beta} = 0$

$$\tau = K\left[\frac{1}{2}(\vec{\theta})^2 - \psi\right] \tag{3.7}$$

$$\tau = K\left[\frac{1}{2}(x^2 + y^2) - (b_1^2 ln((x^2 + (y - q_1)^2)^{1/2}) + b_2^2 ln((x^2 + (y - q_2)^2)^{1/2}) + ...)\right] \tag{3.8}$$

Differentiating:

$$\frac{\partial \tau}{\partial y} = K\left[y - \left(\frac{b_1^2(y - q_1)}{(x^2 + (y - q_1)^2)} + \frac{b_2^2(y - q_2)}{(x^2 + (y - q_2)^2)} + ...\right)\right] \tag{3.9}$$

Since every perturbed saddlepoint will split into two saddlepoints images, both on the y-axis, after a macro-saddle has been split by n arbitrarily placed perturbations all the resultant images will lie along the y-axis. Therefore we can set x = 0:

$$\frac{\partial \tau}{\partial y} = K\left[y - \frac{\partial \psi}{\partial y}\right] = K\left[y - \left(\frac{b_1^2}{(y-q_1)} + \frac{b_2^2}{(y-q_2)} + ...\right)\right] \tag{3.10}$$

Next we multiply both sides by the product of the denominators:

$$\begin{aligned} \frac{\partial \tau}{\partial y}(y-q_1)(y-q_2)...(y-q_n) &= Ky((y-q_1)(y-q_2)...(y-q_n)) \\ &- Kb_1^2(y-q_2)(y-q_3)...(y-q_n) \\ &- Kb_2^2(y-q_1)(y-q_3)...(y-q_n) \\ &- ... \end{aligned} \tag{3.11}$$

Conveniently $\frac{\partial \tau}{\partial y}(y-q_1)(y-q_2)...(y-q_n)$ is a polynomial, which means we can also write it as a product of its roots:

$$\frac{\partial \tau}{\partial y}(y-q_1)(y-q_2)...(y-q_n) = K(y-y_1)(y-y_2)(y-y_3)...(y-y_{n+1}), \tag{3.12}$$

where we have included the factor of K on the right side to account for the fact that the highest order term in the left side is Ky^{n+1}. Plugging in we get:

$$K\left[y - \frac{\partial \psi}{\partial y}\right](y-q_1)(y-q_2)...(y-q_n) = K(y-y_1)(y-y_2)(y-y_3)...(y-y_{n+1}) \tag{3.13}$$

The K's cancel, and differentiating equation 3.13 we get:

$$\begin{aligned} &\left[y - \frac{\partial \psi}{\partial y}\right]((y-q_2)(y-q_3)...(y-q_n) + (y-q_1)(y-q_3)...(y-q_n) + ...) \\ &\qquad + \left[1 - \frac{\partial^2 \psi}{\partial y^2}\right]((y-q_1)(y-q_2)...(y-q_n)) \\ &= ((y-y_2)(y-y_3)...(y-y_{n+1})) + ((y-y_1)(y-y_3)...(y-y_{n+1})) + ... \end{aligned} \tag{3.14}$$

Now to find $\left[1-\frac{\partial^2\psi}{\partial y^2}\right]$ we simply need to evaluate this at root y_j. Conveniently, since $\left[y-\frac{\partial\psi}{\partial y}\right]$ is the gradient of the time delay, it will be zero for all y_j, since y_j is, by definition, a stationary point in the time delay surface (so long as $y_j \neq$ any q_j). Thus we can effectively say:

$$\left[1-\frac{\partial^2\psi}{\partial y^2}\right]=\frac{((y-y_2)(y-y_3)...(y-y_{n+1}))+((y-y_1)(y-y_3)...(y-y_{n+1}))+...}{((y-q_1)(y-q_2)...(y-q_n))} \tag{3.15}$$

To find $\mu_{1D,total}$ we evaluate the inverse of equation 3.15 for all y_j and sum the results.

Now we use equation 3.15 to find the total 1D magnification for the cases of n=1,2,3 and 4 perturbations.

3.2 Total 1D Magnification: One Perturbation Case

We can use our results from section 3.1 to find the total 1D magnification for one perturbation.

With just one perturbation ($n=1$) equation 3.15 becomes:

$$\left[1-\frac{\partial^2\psi_1}{\partial y^2}\right]=\frac{(y-y_2)+(y-y_1)}{(y-q_1)} \tag{3.16}$$

To find $\mu_{1D,total}$ we evaluate the inverse of equation 3.16 at image positions y_1 and y_2 and sum the results. We get:

$$\mu_{1D,total} = \left[1-\frac{\partial^2\psi_1}{\partial y^2}(x_1,y_1)\right]^{-1}+\left[1-\frac{\partial^2\psi_1}{\partial y^2}(x_2,y_2)\right]^{-1} \tag{3.17}$$

$$= \frac{(y_1-q_1)}{(y_1-y_1)+(y_1-y_2)}+\frac{(y_2-q_1)}{(y_2-y_1)+(y_2-y_2)} \tag{3.18}$$

$$\mu_{1D,total}=\frac{(y_1-q_1)}{(y_1-y_2)}+\frac{(y_2-q_1)}{(y_2-y_1)} \tag{3.19}$$

$$\mu_{1D,total}=\frac{((y_1-q_1)-(y_2-q_1))}{(y_1-y_2)} \tag{3.20}$$

$$\mu_{1D,total}=\frac{(y_1-y_2)}{(y_1-y_2)}=1 \tag{3.21}$$

And therefore the total 1D magnification along the axis of positive curvature for one perturbation will remarkably be one- a constant completely independent of perturbation location and size.

3.3 Total 1D Magnification: Two Perturbation Case

To find the total 1D magnification for two perturbations we follow the same basic procedure. With two perturbations ($n = 2$) equation 3.15 becomes:

$$\left[1 - \frac{\partial^2 \psi_1}{\partial y^2}\right] = \frac{(y-y_2)(y-y_3) + (y-y_1)(y-y_3) + (y-y_1)(y-y_2)}{(y-q_1)(y-q_2)} \tag{3.22}$$

To find $\mu_{1D,total}$ we evaluate the inverse of equation 3.22 at image positions y_1, y_2, and y_3 and sum the results. We get:

$$\begin{aligned} \mu_{1D,total} &= \left[1 - \frac{\partial^2 \psi_1}{\partial y^2}(x_1, y_1)\right]^{-1} + \left[1 - \frac{\partial^2 \psi_1}{\partial y^2}(x_2, y_2)\right]^{-1} + \left[1 - \frac{\partial^2 \psi_1}{\partial y^2}(x_3, y_3)\right]^{-1} && (3.23) \\ &= \frac{(y_1-q_1)(y_1-q_2)}{(y_1-y_2)(y_1-y_3) + (y_1-y_1)(y_1-y_3) + (y_1-y_1)(y_1-y_2)} \\ &+ \frac{(y_2-q_1)(y_2-q_2)}{(y_2-y_2)(y_2-y_3) + (y_2-y_1)(y_2-y_3) + (y_2-y_1)(y_2-y_2)} \\ &+ \frac{(y_3-q_1)(y_3-q_2)}{(y_3-y_2)(y_3-y_3) + (y_3-y_1)(y_3-y_3) + (y_3-y_1)(y_3-y_2)} && (3.24) \end{aligned}$$

For convenience we define:

$$\varrho(y) \equiv (y-q_1)(y-q_2) \tag{3.25}$$

$$\begin{aligned} \mu_{1D,total} &= \frac{\varrho(y_1)}{(y_1-y_2)(y_1-y_3)} + \frac{\varrho(y_2)}{(y_2-y_1)(y_2-y_3)} + \frac{\varrho(y_3)}{(y_3-y_1)(y_3-y_2)} && (3.26) \\ &= \frac{\varrho(y_1)(y_3-y_2) - \varrho(y_2)(y_3-y_1) + \varrho(y_3)(y_2-y_1)}{(y_1-y_2)(y_2-y_3)(y_3-y_1)} && (3.27) \end{aligned}$$

Looking at just the numerator:

$$numerator = \varrho(y_1)(y_3-y_2) - \varrho(y_2)(y_3-y_1) + \varrho(y_3)(y_2-y_1) \tag{3.28}$$

To show that the total 1D magnification is one, the first thing we must show is that the numerator does not explicitly depend on perturbation location.[1] To do this we look at the powers of y in $\varrho(y)$ one at a time and show that all terms containing factors of q_j will cancel out. Expanding $\varrho(y)$ we see that it will have terms of order y^0, y^1, and y^2:

$$\varrho(y) = (y - q_1)(y - q_2) \tag{3.29}$$
$$= y^2 - q_1 y - q_2 y + q_1 q_2 \tag{3.30}$$

Looking at just the y^0 terms:

$$(q_1 q_2)(y_2 - y_3) + (q_1 q_2)(y_3 - y_1) + (q_1 q_2)(y_1 - y_2) \tag{3.31}$$
$$= (q_1 q_2)(y_2 - y_3 + y_3 - y_1 + y_1 - y_2) \tag{3.32}$$
$$= 0 \tag{3.33}$$

So the terms containing $q_1 q_2$ will cancel.

Now looking at just the y^1 terms:

$$(q_1 + q_2) y_1 (y_2 - y_3) + (q_1 + q_2) y_2 (y_3 - y_1) + (q_1 + q_2) y_3 (y_1 - y_2) \tag{3.34}$$
$$= (q_1 + q_2)(y_1(y_2 - y_3) + y_2(y_3 - y_1) + y_3(y_1 - y_2)) \tag{3.35}$$
$$= 0 \tag{3.36}$$

So the terms containing $(q_1 + q_2)$ will also cancel, leaving us with:

$$\mu_{1D,total} = \frac{y_1^2(y_3 - y_2) - y_2^2(y_3 - y_1) + y_3^2(y_2 - y_1)}{(y_3 - y_2)(y_3 - y_1)(y_2 - y_1)}. \tag{3.37}$$

As expected, we have found that the 1D magnification does not explicitly depend on

[1] To be more precise, we must show that the total 1D magnification can be reduced to a fraction such that the numerator and denominator are each a polynomial in image position of order $n(n+1)/2$, where n is the number of perturbations. But for the sake of brevity, henceforth we will simply refer to this form of the 1D magnification as "a form where the numerator does not explicitly depend on perturbation location."

q. Now we must show that the remaining expression is equal to one.

To show that the numerator and the denominator are equal in equation 3.37, we use a Taylor expansion proof. We choose to expand $(y_3 - y_2)(y_3 - y_1)(y_2 - y_1)$ with respect to $y = y_3$:

$$f(y) = (y - y_2)(y - y_1)(y_2 - y_1) \tag{3.38}$$

We take the appropriate derivatives and write out the Taylor expansion, g(y), which will be exact since f(y) is a polynomial:

$$g(y) = y_2 y_1 (y_2 - y_1) + (y_1^2 - y_2^2)y + \frac{2(y_2 - y_1)y^2}{2} \tag{3.39}$$

Rearranging and plugging in $y = y_3$ we get:

$$g(y) = y_1^2(y_3 - y_2) - y_2^2(y_3 - y_1) + y_3^2(y_2 - y_1) \tag{3.40}$$

And therefore we can return to equation 3.37 and say that the total 1D magnification along the axis of curvature is 1.

3.4 Total 1D Magnification: Three Perturbation Case

Now we outline the calculation of the total 1D magnification for n=3 perturbations. The complete calculation can be found in appendix B.

With three perturbations ($n = 3$) equation 3.15 becomes:

$$\left[1 - \frac{\partial^2 \psi_1}{\partial y^2}\right] = \tag{3.41}$$

$$+ \quad \frac{(y - y_2)(y - y_3)(y - y_4)}{(y - q_1)(y - q_2)(y - q_3)} + \frac{(y - y_1)(y - y_3)(y - y_4)}{(y - q_1)(y - q_2)(y - q_3)} \tag{3.42}$$

$$+ \quad \frac{(y - y_1)(y - y_2)(y - y_4)}{(y - q_1)(y - q_2)(y - q_3)} + \frac{(y - y_1)(y - y_2)(y - y_3)}{(y - q_1)(y - q_2)(y - q_3)}. \tag{3.43}$$

To find $\mu_{1D,total}$ we evaluate the inverse of equation B.1 for image positions $y = y_1$, y_2, y_3, and y_4 and then sum the results. Defining a function $\varrho(y_j) \equiv (y_j - q_1)(y_j - q_2)(y_j - q_3)$ for convenience, the 1D magnification can more simply be written:

$$\begin{aligned}\mu_{1D,total} &= \\ &- \frac{\varrho(y_1)}{(y_2 - y_1)(y_3 - y_1)(y_4 - y_1)} + \frac{\varrho(y_2)}{(y_2 - y_1)(y_3 - y_2)(y_4 - y_2)} \\ &- \frac{\varrho(y_3)}{(y_3 - y_1)(y_3 - y_2)(y_4 - y_3)} + \frac{\varrho(y_4)}{(y_4 - y_1)(y_4 - y_2)(y_4 - y_3)}. \end{aligned} \tag{3.44}$$

Combining these fractions and looking at just the numerator we get:

$$\begin{aligned} numerator &= \\ &- \varrho(y_1)(y_4 - y_3)(y_4 - y_2)(y_3 - y_2) \\ &+ \varrho(y_2)(y_4 - y_3)(y_4 - y_1)(y_3 - y_1) \\ &- \varrho(y_3)(y_4 - y_2)(y_4 - y_1)(y_2 - y_1) \\ &+ \varrho(y_4)(y_3 - y_2)(y_3 - y_1)(y_2 - y_1) \end{aligned} \tag{3.45}$$

Where the denominator is equal to:

$$denominator = (y_4 - y_3)(y_4 - y_2)(y_4 - y_1)(y_3 - y_2)(y_3 - y_1)(y_2 - y_1). \tag{3.46}$$

We can simplify this further using relation 3.37 derived in section 3.3:

$$(c - b)(c - a)(b - a) = +a^2(c - b) - b^2(c - a) + c^2(b - a) \tag{3.47}$$

This gives us:

$$\begin{aligned} numerator &= \\ &- \varrho(y_1)(y_2^2(y_4 - y_3) - y_3^2(y_4 - y_2) + y_4^2(y_3 - y_2)) \\ &+ \varrho(y_2)(y_1^2(y_4 - y_3) - y_3^2(y_4 - y_1) + y_4^2(y_3 - y_1)) \\ &- \varrho(y_3)(y_1^2(y_4 - y_2) - y_2^2(y_4 - y_1) + y_4^2(y_2 - y_1)) \end{aligned}$$

$$+ \quad \varrho(y_4)(y_1^2(y_3-y_2) - y_2^2(y_3-y_1) + y_3^2(y_2-y_1)) \qquad (3.48)$$

To simplify this we must compare the different components of $\varrho(y_j)$ in the numerator:

$$\varrho(y_j) \equiv (y_j - q_1)(y_j - q_2)(y_j - q_3)$$
$$= y_j^3 - (q_1+q_2+q_3)y_j^2 + (q_1q_2 + q_1q_3 + q_2q_3)y_j^1 - (q_1q_2q_3)y_j^0 \qquad (3.49)$$

First looking at the y_j^0 terms in the numerator:

$$\begin{aligned}\sum y_j^0 terms \quad &= \\ q_1q_2q_3(\quad &+ \quad (y_2^2(y_4-y_3) - y_3^2(y_4-y_2) + y_4^2(y_3-y_2)) \\ &- \quad (y_1^2(y_4-y_3) - y_3^2(y_4-y_1) + y_4^2(y_3-y_1)) \\ &+ \quad (y_1^2(y_4-y_2) - y_2^2(y_4-y_1) + y_4^2(y_2-y_1)) \\ &- \quad (y_1^2(y_3-y_2) - y_2^2(y_3-y_1) + y_3^2(y_2-y_1))) \qquad (3.50)\end{aligned}$$

We find they will sum to 0 if we group them by the squared coefficients:

$$\sum y_j^0 terms = q_1q_2q_3(y_1^2(0) + y_2^2(0) + y_3^2(0) + y_4^2(0)) = 0 \qquad (3.51)$$

So the terms containing $q_1q_2q_3$ in equation 3.48 will cancel.

Now looking at just the y_j^1 terms of equation 3.48:

$$\begin{aligned}\sum y_j^1 terms \quad &= \qquad (3.52) \\ = (q_1q_3 + q_2q_3 + q_1q_2)(\quad &- \quad (y_2^2y_1(y_4-y_3) - y_3^2y_1(y_4-y_2) + y_4^2y_1(y_3-y_2)) \\ &+ \quad (y_1^2y_2(y_4-y_3) - y_3^2y_2(y_4-y_1) + y_4^2y_2(y_3-y_1)) \\ &- \quad (y_1^2y_3(y_4-y_2) - y_2^2y_3(y_4-y_1) + y_4^2y_3(y_2-y_1)) \\ &+ \quad (y_1^2y_4(y_3-y_2) - y_2^2y_4(y_3-y_1) + y_3^2y_4(y_2-y_1))) \qquad (3.53)\end{aligned}$$

We find that they too will sum to zero if we group them by the squared coefficients.

$$\sum y_j^1 terms = (q_1q_3 + q_2q_3 + q_1q_2)(y_1^2(0) + y_2^2(0) + y_3^2(0) + y_4^2(0)) = 0 \qquad (3.54)$$

So the terms containing $(q_1q_3 + q_2q_3 + q_1q_2)$ will also cancel out in equation 3.48.

Now we look at just the y_j^2 terms of equation 3.48:

$$\begin{aligned}
\sum y_j^2 terms &= && (3.55)\\
= (q_1 + q_2 + q_3)(&+ y_1^2y_2^2(y_4 - y_3) - y_2^2y_1^2(y_4 - y_3)\\
&+ y_1^2y_4^2(y_3 - y_2) - y_4^2y_1^2(y_3 - y_2)\\
&- y_1^2y_3^2(y_4 - y_2) + y_3^2y_1^2(y_4 - y_2)\\
&+ y_2^2y_3^2(y_4 - y_1) - y_3^2y_2^2(y_4 - y_1)\\
&- y_2^2y_4^2(y_3 - y_1) + y_4^2y_2^2(y_3 - y_1)\\
&+ y_3^2y_4^2(y_2 - y_1) - y_4^2y_3^2(y_2 - y_1))\\
&= (q_1 + q_2 + q_3)(0) = 0 && (3.56)
\end{aligned}$$

So the terms containing $(q_1 + q_2 + q_3)$ will also cancel to zero and we are left with:

$$\begin{aligned}
numerator = &- y_1^3(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)\\
&+ y_2^3(y_3 - y_1)(y_4 - y_1)(y_4 - y_3)\\
&- y_3^3(y_2 - y_1)(y_4 - y_1)(y_4 - y_2)\\
&+ y_4^3(y_2 - y_1)(y_3 - y_1)(y_3 - y_2) \qquad (3.57)
\end{aligned}$$

Notice this leaves us with an equation for the 1D Magnification that does not explicitly depend on perturbation location or size.

By inspection we see that the y_j^3 terms cannot cancel to zero; If we expand the expression y_j^3 only appears once for each j and the coefficient of a given y_j^3 is only zero when $y_j = y_i$ for $i \neq j$. So now to prove that the total 1D magnification is one, we must show that the numerator of μ_{total} (equation 3.57) is equal to the denominator (equation 3.46). We can see this by Taylor expanding the denominator with respect to $y = y_1$. (The complete proof that equation 3.57 is equal to equation 3.46 is given in appendix B, equations B.32 through B.66.) After a lengthy calculation we find that the numerator equals the denominator, and therefore that the total 1D magnification is equal to 1 for the case of three perturbations.

3.5 Total One-Dimensional Magnification: Four Perturbation Case

Now we outline the calculation of the total 1D magnification for n=4 perturbations. The complete calculation can be found in appendix C.

With four perturbations ($n = 4$) equation 3.15 becomes:

$$\begin{aligned}
&\left[1 - \frac{\partial^2 \psi_1}{\partial y^2}\right] = \\
+ &\frac{(y-y_2)(y-y_3)(y-y_4)(y-y_5)}{(y-q_1)(y-q_2)(y-q_3)(y-q_4)} + \frac{(y-y_1)(y-y_3)(y-y_4)(y-y_5)}{(y-q_1)(y-q_2)(y-q_3)(y-q_4)} \\
+ &\frac{(y-y_1)(y-y_2)(y-y_4)(y-y_5)}{(y-q_1)(y-q_2)(y-q_3)(y-q_4)} + \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_5)}{(y-q_1)(y-q_2)(y-q_3)(y-q_4)} \\
+ &\frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y-q_1)(y-q_2)(y-q_3)(y-q_4)}
\end{aligned} \tag{3.58}$$

To find $\mu_{1D,total}$ we evaluate the inverse of equation C.1 for image locations y_1, y_2, y_3, y_4 and y_5 and sum the results. We get:

$$\begin{aligned}
\mu_{1D,total} &= \sum_{i=1}^{5} \left[1 - \frac{\partial^2 \psi_1}{\partial y^2}(x_i, y_i)\right]^{-1} \\
&= \frac{((y_1-q_1)(y_1-q_2)(y_1-q_3)(y_1-q_4))}{(y_5-y_1)(y_4-y_1)(y_3-y_1)(y_2-y_1)} \\
&- \frac{((y_2-q_1)(y_2-q_2)(y_2-q_3)(y_2-q_4))}{(y_5-y_2)(y_4-y_2)(y_3-y_2)(y_2-y_1)} \\
&+ \frac{((y_3-q_1)(y_3-q_2)(y_3-q_3)(y_3-q_4))}{(y_5-y_3)(y_4-y_3)(y_3-y_2)(y_3-y_1)} \\
&- \frac{((y_4-q_1)(y_4-q_2)(y_4-q_3)(y_4-q_4))}{(y_5-y_4)(y_4-y_3)(y_4-y_2)(y_4-y_1)} \\
&+ \frac{((y_5-q_1)(y_5-q_2)(y_5-q_3)(y_5-q_4))}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)}
\end{aligned} \tag{3.59}$$

We define $(y_i - q_1)(y_i - q_2)(y_i - q_3)(y_i - q_4) \equiv \varrho(y_i)$ for convenience and combine these fractions into one large fraction with numerator:

$$
\begin{aligned}
& numerator = \\
& \varrho(y_1)(y_5 - y_4)(y_5 - y_3)(y_5 - y_2)(y_4 - y_3)(y_4 - y_2)(y_3 - y_2) \\
- \; & \varrho(y_2)(y_5 - y_4)(y_5 - y_3)(y_5 - y_1)(y_4 - y_3)(y_4 - y_1)(y_3 - y_1) \\
+ \; & \varrho(y_3)(y_5 - y_4)(y_5 - y_2)(y_5 - y_1)(y_4 - y_2)(y_4 - y_1)(y_2 - y_1) \\
- \; & \varrho(y_4)(y_5 - y_3)(y_5 - y_2)(y_5 - y_1)(y_3 - y_2)(y_3 - y_1)(y_2 - y_1) \\
+ \; & \varrho(y_5)(y_4 - y_3)(y_4 - y_2)(y_4 - y_1)(y_3 - y_2)(y_3 - y_1)(y_2 - y_1),
\end{aligned}
\tag{3.60}
$$

and denominator:

$$
(y_5 - y_4)(y_5 - y_3)(y_5 - y_2)(y_5 - y_1)(y_4 - y_3)(y_4 - y_2)(y_4 - y_1)(y_3 - y_2)(y_3 - y_1)(y_2 - y_1).
\tag{3.61}
$$

Since we proved in section 3.4, that equation 3.46 is equal to equation 3.57), we can say that:

$$
\begin{aligned}
& & & (d - c)(d - b)(d - a)(c - b)(c - a)(b - a) \\
& = & - \; & a^3(c - b)(d - b)(d - c) \\
& & + \; & b^3(c - a)(d - a)(d - c) \\
& & - \; & c^3(b - a)(d - a)(d - b) \\
& & + \; & d^3(b - a)(c - a)(c - b),
\end{aligned}
\tag{3.62}
$$

and since we proved in section 3.3, that equation 3.37 was equal to one, we can say that:

$$
(c - b)(c - a)(b - a) = +a^2(c - b) - b^2(c - a) + c^2(b - a).
\tag{3.63}
$$

Therefore, we can rewrite the numerator (equation 3.60) as:

$numerator$

$$
\begin{aligned}
= \varrho(y_1)[& - y_2^3(y_5^2(y_4-y_3) - y_4^2(y_5-y_3) + y_3^2(y_5-y_4)) \\
& + y_3^3(y_5^2(y_4-y_2) - y_4^2(y_5-y_2) + y_2^2(y_5-y_4)) \\
& - y_4^3(y_5^2(y_3-y_2) - y_3^2(y_5-y_2) + y_2^2(y_5-y_3)) \\
& + y_5^3(y_4^2(y_3-y_2) - y_3^2(y_4-y_2) + y_2^2(y_4-y_3))] \\
\\
-\varrho(y_2)[& - y_1^3(y_5^2(y_4-y_3) - y_4^2(y_5-y_3) + y_3^2(y_5-y_4)) \\
& + y_3^3(y_5^2(y_4-y_1) - y_4^2(y_5-y_1) + y_1^2(y_5-y_4)) \\
& - y_4^3(y_5^2(y_3-y_1) - y_3^2(y_5-y_1) + y_1^2(y_5-y_3)) \\
& + y_5^3(y_4^2(y_3-y_1) - y_3^2(y_4-y_1) + y_1^2(y_4-y_3))] \\
\\
+\varrho(y_3)[& - y_1^3(y_5^2(y_4-y_2) - y_4^2(y_5-y_2) + y_2^2(y_5-y_4)) \\
& + y_2^3(y_5^2(y_4-y_1) - y_4^2(y_5-y_1) + y_1^2(y_5-y_4)) \\
& - y_4^3(y_5^2(y_2-y_1) - y_2^2(y_5-y_1) + y_1^2(y_5-y_2)) \\
& + y_5^3(y_4^2(y_2-y_1) - y_2^2(y_4-y_1) + y_1^2(y_4-y_2))] \\
\\
-\varrho(y_4)[& - y_1^3(y_5^2(y_3-y_2) - y_3^2(y_5-y_2) + y_2^2(y_5-y_3)) \\
& + y_2^3(y_5^2(y_3-y_1) - y_3^2(y_5-y_1) + y_1^2(y_5-y_3)) \\
& - y_3^3(y_5^2(y_2-y_1) - y_2^2(y_5-y_1) + y_1^2(y_5-y_2)) \\
& + y_5^3(y_3^2(y_2-y_1) - y_2^2(y_3-y_1) + y_1^2(y_3-y_2))] \\
\\
+\varrho(y_5)[& - y_1^3(y_4^2(y_3-y_2) - y_3^2(y_4-y_2) + y_2^2(y_4-y_3)) \\
& + y_2^3(y_4^2(y_3-y_1) - y_3^2(y_4-y_1) + y_1^2(y_4-y_3)) \\
& - y_3^3(y_4^2(y_2-y_1) - y_2^2(y_4-y_1) + y_1^2(y_4-y_2)) \\
& + y_4^3(y_3^2(y_2-y_1) - y_2^2(y_3-y_1) + y_1^2(y_3-y_2))]
\end{aligned} \tag{3.64}
$$

$\varrho(y_i) \equiv (y_i - q_1)(y_i - q_2)(y_i - q_3)(y_i - q_4)$ has terms of order y_i^0, y_i^1, y_i^2, y_i^3, and y_i^4. To ultimately prove that the sum of the one dimensional magnifications is one, we must look at these groups of terms one at a time and show that all but the y_i^4 terms will cancel to zero. This is an algebraically involved process, and so we have relegated the details of the calculation to the appendix. We refer the reader to appendix C, equations C.18 through C.39.

At the end of a lengthy calculation we will find that the y_i^3, y_i^2, y_i^1, and y_i^0 terms will all cancel to zero, and we will be left with only the y_i^4 terms. Therefore the total one dimensional magnification will be:

$$\begin{aligned}
\mu_{1D,total} = & \\
& \frac{y_1^4(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_4-y_3)(y_4-y_2)(y_3-y_2)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
- & \frac{y_2^4(y_5-y_4)(y_5-y_3)(y_5-y_1)(y_4-y_3)(y_4-y_1)(y_3-y_1)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
+ & \frac{y_3^4(y_5-y_4)(y_5-y_2)(y_5-y_1)(y_4-y_2)(y_4-y_1)(y_2-y_1)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
- & \frac{y_4^4(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
+ & \frac{y_5^4(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)}
\end{aligned} \tag{3.65}$$

Which again does not explicitly depend on the perturbation location q_j.

Taking a step back, we notice that this looks very similar to the 'q-independent form'[2] of the 1D magnifications for the cases of one two and three perturbations. In each case, the 1D magnification took on the form of:

$$\begin{aligned}
& \prod(y_i-y_j), forallunique combinations of(i,j) where i,j\epsilon[1,2,3...n,n+1] and i\neq j. \\
= & + \; y_1^n(\prod(y_i-y_j), forallunique combinations of(i,j) where i,j\epsilon[2,3...n,n+1] and i\neq j.) \\
& - \; y_2^n(\prod(y_i-y_j), forallunique combinations of(i,j) where i,j\epsilon[1,3,4...n,n+1] and i\neq j.) \\
& + \; y_3^n(\prod(y_i-y_j), forallunique combinations of(i,j) where i,j\epsilon[1,2,4...n,n+1] and i\neq j.) \\
& + \; ... \\
& - \; y_4^n(\prod(y_i-y_j), forallunique combinations of(i,j) where i,j\epsilon[1,2,3...n+1] and i\neq j.)
\end{aligned}$$

[2]Here q-independent means not *explicitly* dependent on the perturbation location, q. In this form the total 1D magnification will still depend implicitly on q through the image locations.

$$+ \quad y_{n+1}^n(\prod(y_i - y_j), forall unique combinations of (i,j) where i,j\epsilon[1,2,4...n] and i \neq j) \quad (3.66)$$

We show in the next section that not only does this relation hold for n=4, completing the above proof that for four perturbations the total 1D magnification (equation 3.65) is 1, but that this pattern will allow us to prove this relation holds for all n.

3.6 Mathematical Patterns

A pattern had begun to emerge in the q-independent 1D magnifications. Each time we have cancelled out the q-terms and shown that the 1D magnification does not depend explicitly on perturbation location, the remaining terms have followed equation 3.66. We summarize this pattern for n = 1,2,3 in section 3.6.1 and prove that the pattern continues for n=4 in section 3.6.2.

3.6.1 A Summary of the Pattern for n = 1, 2, 3

So far we have proven relation 3.66 for the case of n=1,2,3.

In the case of one perturbation, we proved that:

$$\begin{aligned} &(y_2 - y_1) \\ &= -y_1^1 \\ &\quad +y_2^1 \end{aligned} \quad (3.67)$$

In the case of two perturbations, we proved that:

$$\begin{aligned} &(y_3 - y_2)(y_3 - y_1)(y_2 - y_1) \\ &= y_1^2(y_3 - y_2) \\ &\quad -y_2^2(y_3 - y_1) \\ &\quad +y_3^2(y_2 - y_1) \end{aligned} \quad (3.68)$$

In the case of three perturbations, we proved that:

$$\begin{aligned} & (y_4 - y_3)(y_4 - y_2)(y_4 - y_1)(y_3 - y_2)(y_3 - y_1)(y_2 - y_1) \\ = & - y_1^3(y_3 - y_2)(y_4 - y_2)(y_4 - y_3) \\ & + y_2^3(y_3 - y_1)(y_4 - y_1)(y_4 - y_3) \\ & - y_3^3(y_2 - y_1)(y_4 - y_1)(y_4 - y_2) \\ & + y_4^3(y_2 - y_1)(y_3 - y_1)(y_3 - y_2) \end{aligned} \tag{3.69}$$

And for four perturbations we found but did not prove that:

$$\mu_{1D,total} =$$

$$\begin{aligned} & \frac{y_1^4(y_5 - y_4)(y_5 - y_3)(y_5 - y_2)(y_4 - y_3)(y_4 - y_2)(y_3 - y_2)}{(y_5 - y_4)(y_5 - y_3)(y_5 - y_2)(y_5 - y_1)(y_4 - y_3)(y_4 - y_2)(y_4 - y_1)(y_3 - y_2)(y_3 - y_1)(y_2 - y_1)} \\ - & \frac{y_2^4(y_5 - y_4)(y_5 - y_3)(y_5 - y_1)(y_4 - y_3)(y_4 - y_1)(y_3 - y_1)}{(y_5 - y_4)(y_5 - y_3)(y_5 - y_2)(y_5 - y_1)(y_4 - y_3)(y_4 - y_2)(y_4 - y_1)(y_3 - y_2)(y_3 - y_1)(y_2 - y_1)} \\ + & \frac{y_3^4(y_5 - y_4)(y_5 - y_2)(y_5 - y_1)(y_4 - y_2)(y_4 - y_1)(y_2 - y_1)}{(y_5 - y_4)(y_5 - y_3)(y_5 - y_2)(y_5 - y_1)(y_4 - y_3)(y_4 - y_2)(y_4 - y_1)(y_3 - y_2)(y_3 - y_1)(y_2 - y_1)} \\ - & \frac{y_4^4(y_5 - y_3)(y_5 - y_2)(y_5 - y_1)(y_3 - y_2)(y_3 - y_1)(y_2 - y_1)}{(y_5 - y_4)(y_5 - y_3)(y_5 - y_2)(y_5 - y_1)(y_4 - y_3)(y_4 - y_2)(y_4 - y_1)(y_3 - y_2)(y_3 - y_1)(y_2 - y_1)} \\ + & \frac{y_5^4(y_4 - y_3)(y_4 - y_2)(y_4 - y_1)(y_3 - y_2)(y_3 - y_1)(y_2 - y_1)}{(y_5 - y_4)(y_5 - y_3)(y_5 - y_2)(y_5 - y_1)(y_4 - y_3)(y_4 - y_2)(y_4 - y_1)(y_3 - y_2)(y_3 - y_1)(y_2 - y_1)} \end{aligned} \tag{3.70}$$

We will prove this in the following section.

3.6.2 Extending the Pattern to n=4

To prove that equation 3.70 is equal to 1, we begin with the denominator and arrange it in a revealing form:

$$denominator =$$

$$[(y_5 - y_4)(y_5 - y_3)(y_5 - y_2)(y_4 - y_3)(y_4 - y_2)(y_3 - y_2)][(y_5 - y_1)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1)]$$

$$(3.71)$$

In this form it becomes clear that we can substitute in the relation we found for n=3 perturbations (equation 3.69). This gives us:

$$\begin{aligned} numerator = [& - y_2^3(y_4 - y_3)(y_5 - y_3)(y_5 - y_4) \\ & + y_3^3(y_4 - y_2)(y_5 - y_2)(y_5 - y_4) \\ & - y_4^3(y_3 - y_2)(y_5 - y_2)(y_5 - y_3) \\ & + y_5^3(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)][(y_5 - y_1)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1)] \end{aligned} \tag{3.72}$$

Next we move all terms with the same subscript as the leading factor, y_j^3, to the front and rearrange the equation:

$$\begin{aligned} & numerator = \\ - & y_2^3(y_2 - y_1)[(y_4 - y_3)(y_5 - y_3)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_3 - y_1)] \\ + & y_3^3(y_3 - y_1)[(y_4 - y_2)(y_5 - y_2)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_2 - y_1)] \\ - & y_4^3(y_4 - y_1)[(y_3 - y_2)(y_5 - y_2)(y_5 - y_3)(y_5 - y_1)(y_3 - y_1)(y_2 - y_1)] \\ + & y_5^3(y_5 - y_1)[(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1)] \end{aligned} \tag{3.73}$$

Distributing the front term we get:

$$\begin{aligned} & numerator = \\ - & y_2^4[(y_4 - y_3)(y_5 - y_3)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_3 - y_1)] \\ + & y_3^4[(y_4 - y_2)(y_5 - y_2)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_2 - y_1)] \\ - & y_4^4[(y_3 - y_2)(y_5 - y_2)(y_5 - y_3)(y_5 - y_1)(y_3 - y_1)(y_2 - y_1)] \\ + & y_5^4[(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1)] \\ + & y_1 y_2^3[(y_4 - y_3)(y_5 - y_3)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_3 - y_1)] \\ - & y_1 y_3^3[(y_4 - y_2)(y_5 - y_2)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_2 - y_1)] \end{aligned}$$

$$
\begin{aligned}
+ &\quad y_1 y_4^3[(y_3-y_2)(y_5-y_2)(y_5-y_3)(y_5-y_1)(y_3-y_1)(y_2-y_1)] \\
- &\quad y_1 y_5^3[(y_3-y_2)(y_4-y_2)(y_4-y_3)(y_4-y_1)(y_3-y_1)(y_2-y_1)] \qquad (3.74)
\end{aligned}
$$

In section 3.5 and in appendix C (equation C.37), in the process of showing that the sum of the 1D magnifications was not explicitly dependent on perturbation location for n=4, we already proved that:

$$
\begin{aligned}
- &\quad y_1^3[(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_4-y_3)(y_4-y_2)(y_3-y_2)] \\
+ &\quad y_2^3[(y_4-y_3)(y_5-y_3)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_3-y_1)] \\
- &\quad y_3^3[(y_4-y_2)(y_5-y_2)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_2-y_1)] \\
+ &\quad y_4^3[(y_3-y_2)(y_5-y_2)(y_5-y_3)(y_5-y_1)(y_3-y_1)(y_2-y_1)] \\
- &\quad y_5^3[(y_3-y_2)(y_4-y_2)(y_4-y_3)(y_4-y_1)(y_3-y_1)(y_2-y_1)] \qquad (3.75) \\
= &\quad 0 \qquad (3.76)
\end{aligned}
$$

Therefore we can say that:

$$
\begin{aligned}
+ &\quad y_1 y_2^3[(y_4-y_3)(y_5-y_3)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_3-y_1)] \\
- &\quad y_1 y_3^3[(y_4-y_2)(y_5-y_2)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_2-y_1)] \\
+ &\quad y_1 y_4^3[(y_3-y_2)(y_5-y_2)(y_5-y_3)(y_5-y_1)(y_3-y_1)(y_2-y_1)] \\
- &\quad y_1 y_5^3[(y_3-y_2)(y_4-y_2)(y_4-y_3)(y_4-y_1)(y_3-y_1)(y_2-y_1)] \\
= &\quad y_1^4(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_4-y_3)(y_4-y_2)(y_3-y_2) \qquad (3.77)
\end{aligned}
$$

and thus:

$$
\begin{aligned}
&numerator = \\
+ &\quad y_1^4[(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_4-y_3)(y_4-y_2)(y_3-y_2)] \\
- &\quad y_2^4[(y_4-y_3)(y_5-y_3)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_3-y_1)] \\
+ &\quad y_3^4[(y_4-y_2)(y_5-y_2)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_2-y_1)] \\
- &\quad y_4^4[(y_3-y_2)(y_5-y_2)(y_5-y_3)(y_5-y_1)(y_3-y_1)(y_2-y_1)]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad y_5^4[(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1)] \\
&= \quad denominator \qquad (3.78)
\end{aligned}
$$

And we have proven that relation 3.66 holds for n=4, which completes our proof from section 3.5 that for n=4 perturbations the total 1D magnification is 1.

Chapter 4

Total 1D Magnification: N Perturbation Case

Now we outline the calculation of the total 1D magnification for n perturbations. The complete calculation can be found in appendix B.

As the cases of one, two, three, and four perturbations have demonstrated, the proof that the total 1D magnification is 1 can be broken down into two parts: a proof that the total 1D magnification does not explicitly depend on the locations, q_j, of the perturbations, and a proof that the numerator of the q-independent 1D magnification is equal to the denominator. We prove these two parts in sections 4.2 and 4.1 respectively. Note that the order of the two parts of the argument is inverted in this chapter.

4.1 Proof: The Perturbation-Position-Independent Total 1D Magnification Sums to 1

Now we show that the q-independent total 1D magnification sums to 1 for n perturbations. The argument we just made in section 3.6 for four perturbations informs how we can extend the proof to n perturbations. Following the pattern described in equation 3.66, we guess that the n perturbation case will look like:

$$
\begin{aligned}
&s\mu_{1Dtotal} = \\
&((y_{n+1}-y_n)(y_{n+1}-y_{n-1})...(y_{n+1}-y_2)(y_{n+1}-y_1)) \\
&\times((y_n-y_{n-1})(y_n-y_{n-2})...(y_n-y_2)(y_n-y_1)) \\
&\times((y_{n-1}-y_{n-2})(y_{n-1}-y_{n-3})...(y_{n-1}-y_2)(y_{n-1}-y_1)) \\
&\times... \\
&\times((y_4-y_3)(y_4-y_2)(y_4-y_1)) \\
&\times((y_3-y_2)(y_3-y_1)) \\
&\times(y_2-y_1) \\
&=
\end{aligned}
$$

$$
\begin{aligned}
(-1)^n(\quad &+y_1^n((y_{n+1}-y_n)(y_{n+1}-y_{n-1})...(y_{n+1}-y_2))((y_n-y_{n-1})(y_n-y_{n-2})...(y_n-y_2))(...)((y_3 - \\
&-y_2^n((y_{n+1}-y_n)(y_{n+1}-y_{n-1})...(y_{n+1}-y_1))((y_n-y_{n-1})(y_n-y_{n-2})...(y_n-y_1))(...)((y_3 - \\
&+y_3^n((y_{n+1}-y_n)(y_{n+1}-y_{n-1})...(y_{n+1}-y_2)(y_{n+1}-y_1))((y_n-y_{n-1})(y_n-y_{n-2})...(y_n-y_2 \\
&\vdots \\
&\pm y_n^n((y_{n+1}-y_{n-1})...(y_{n+1}-y_2)(y_{n+1}-y_1))(...)((y_3-y_2)(y_3-y_1))((y_2-y_1)) \\
&\mp y_{n+1}^n((y_n-y_{n-1})...(y_n-y_2)(y_n-y_1))(...)((y_3-y_2)(y_3-y_1))((y_2-y_1)))
\end{aligned}
$$

For odd and even n, respectively. This can be written more compactly using product and summation notation:

$$
\begin{aligned}
&\prod_{\ell=n+1}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell-y_j) = \\
&\left((-1)^n\sum_{\ell=1}^{\ell=n+1}\left(y_\ell^n(y_\ell-y_\ell)\prod_{i=n+1}^{i=1}\frac{1}{(y_i-y_\ell)}\right)\right)\times\left(\prod_{\ell=n+1}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell-y_j)\right) \quad (4.3)
\end{aligned}
$$

In this compact form we can more easily prove that it is true for any n. As is suggested by the proof for the case of n=4, one way to prove this is with a recursive argument– in other words by showing both that equation 4.3 is true for n=1 and that if it is true for n=k, it will be true for n=k+1.

First we will show that the base case is true. For n=1 equation 4.3 becomes:

$$
\begin{aligned}
&(y_2 - y_1) = \\
&(-\left(y_1(y_1 - y_1)\frac{1}{(y_2 - y_1)(y_1 - y_1)}\right) \\
&+\left(y_2(y_2 - y_2)\frac{1}{(y_2 - y_2)(y_1 - y_2)}\right)) \times ((y_2 - y_1))
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
&(y_2 - y_1) = \\
&-(y_1) \\
&+(y_2)
\end{aligned} \tag{4.5}
$$

Which is certainly true.

Now to complete the proof we must show that if equation 4.3 holds for n = k, it will hold for n= k+1. In other words, we want to show that for n = k+1

$$
\begin{aligned}
&\prod_{\ell=k+2}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell - y_j) = \\
&\left((-1)^{k+1}\sum_{\ell=1}^{\ell=k+2}\left(y_\ell^{k+1}(y_\ell - y_\ell)\prod_{i=k+2}^{i=1}\frac{1}{(y_i - y_\ell)}\right)\right) \times \left(\prod_{\ell=k+2}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell - y_j)\right)
\end{aligned} \tag{4.6}
$$

given that for n=k:

$$
\begin{aligned}
&\prod_{\ell=k+1}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell - y_j) = \\
&\left((-1)^{k}\sum_{\ell=1}^{\ell=k+1}\left(y_\ell^{k}(y_\ell - y_\ell)\prod_{i=k+1}^{i=1}\frac{1}{(y_i - y_\ell)}\right)\right) \times \left(\prod_{\ell=k+1}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell - y_j)\right).
\end{aligned} \tag{4.7}
$$

To begin, we use equation 4.7 to rewrite the left side of equation 4.6 as:

$$
\begin{aligned}
&leftside = \\
&\left(\prod_{j=k+1}^{j=1}(y_{k+2} - y_j)\right)\left(\prod_{\ell=k+1}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell - y_j)\right) =
\end{aligned} \tag{4.8}
$$

$$\left(\prod_{j=k+1}^{j=1}(y_{k+2}-y_j)\right)\left((-1)^k\sum_{\ell=1}^{\ell=k+1}\left(y_\ell^k(y_\ell-y_\ell)\prod_{i=k+1}^{i=1}\frac{1}{(y_i-y_\ell)}\right)\right)\times\left(\prod_{\ell=k+1}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell-y_j)\right) \tag{4.9}$$

Expanding the sum:

$$\begin{aligned}
&leftside = \\
(\quad +\ &\left(y_1^k(y_1-y_1)\prod_{i=k+1}^{i=1}\frac{1}{(y_i-y_1)}\right)\\
+\ &\left(y_2^k(y_2-y_2)\prod_{i=k+1}^{i=1}\frac{1}{(y_i-y_2)}\right)\\
+\ &\left(y_3^k(y_3-y_3)\prod_{i=k+1}^{i=1}\frac{1}{(y_i-y_3)}\right)\\
&\vdots\\
+\ &\left(y_k^k(y_k-y_k)\prod_{i=k+1}^{i=1}\frac{1}{(y_i-y_k)}\right)\\
+\ &\left(y_{k+1}^k(y_{k+1}-y_{k+1})\prod_{i=k+1}^{i=1}\frac{1}{(y_i-y_{k+1})}\right))\\
&\times(-1)^k\left(\prod_{\ell=k+1}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell-y_j)\right)\left(\prod_{j=k+1}^{j=1}(y_{k+2}-y_j)\right)
\end{aligned} \tag{4.10}$$

Pulling out a factor of $(y_{k+2}-y_j)$, distributing, and simplifying with summation notation we get:

$$\begin{aligned}
leftside = &\\
&(\left(y_1^{k+1}(y_1-y_1)\prod_{i=k+2}^{i=1}\frac{1}{(y_i-y_1)}\right)\\
&\left(y_2^{k+1}(y_2-y_2)\prod_{i=k+2}^{i=1}\frac{1}{(y_i-y_2)}\right)\\
&\left(y_3^{k+1}(y_3-y_3)\prod_{i=k+2}^{i=1}\frac{1}{(y_i-y_3)}\right)\\
&\vdots\\
&\left(y_k^{k+1}(y_k-y_k)\prod_{i=k+2}^{i=1}\frac{1}{(y_i-y_k)}\right)
\end{aligned}$$

$$\left(y_{k+1}^{k+1}(y_{k+1} - y_{k+1}) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_{k+1})} \right)$$

$$-y_{k+2}(\quad \sum_{p=1}^{p=k+1} \left(y_p^k (y_p - y_p) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_p)} \right)$$

$$\times (-1)^{k+1} \left(\prod_{\ell=k+2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right) \tag{4.11}$$

Following our strategy for n=4, we now attempt to show that:

$$\sum_{p=1}^{p=k+2} \left(y_p^k (y_p - y_p) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_p)} \right) \times \left(\prod_{\ell=k+2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right) = 0 \tag{4.12}$$

If we expand out equation 4.12, we see that the p^{th} term of the sum is independent of y_p, except for the leading y_p^k factor, and that furthermore the coefficient of y_p^k contains only $k+1$ distinct variables. Therefore we can write the coefficients of y_p^k in the form of equation 4.7:

$$\begin{aligned}
y_1^k[\quad & + \quad (y_2^k(y_2 - y_2) \prod_{i=k+2,k+1,\ldots 3,2} \frac{1}{(y_i - y_2)}) \\
& + \quad (y_3^k(y_3 - y_3) \prod_{i=k+2,k+1,\ldots 3,2} \frac{1}{(y_i - y_3)}) \\
& + \quad (y_4^k(y_4 - y_4) \prod_{i=k+2,k+1,\ldots 3,2} \frac{1}{(y_i - y_4)}) \\
& \quad \vdots \\
& + \quad (y_k^k(y_k - y_k) \prod_{i=k+2,k+1,\ldots 3,2} \frac{1}{(y_i - y_k)}) \\
& + \quad (y_{k+1}^k(y_{k+1} - y_{k+1}) \prod_{i=k+2,k+1,\ldots 3,2} \frac{1}{(y_i - y_{k+1})}) \\
& + \quad (y_{k+2}^k(y_{k+2} - y_{k+2}) \prod_{i=k+2,k+1,\ldots 3,2} \frac{1}{(y_i - y_{k+2})})] \\
& \times \quad (\prod_{\ell=k+2,k+1\ldots 4,3} \prod_{j=\ell-1,\ell-2\ldots 3,2} (y_\ell - y_j))
\end{aligned}$$

$$+y_2^k[\quad - \quad (y_1^k(y_1 - y_1) \prod_{i=k+2,k+1,\ldots 3,1} \frac{1}{(y_i - y_1)})]$$

$$- \quad (y_3^k(y_3 - y_3) \prod_{i=k+2,k+1\ldots 3,1} \frac{1}{(y_i - y_3)})]$$

$$- \quad (y_4^k(y_4 - y_4) \prod_{i=k+2,k+1,\ldots 3,1} \frac{1}{(y_i - y_4)})]$$

$$\vdots$$

$$- \quad (y_k^k(y_k - y_k) \prod_{i=k+2,k+1,\ldots 3,1} \frac{1}{(y_i - y_k)})]$$

$$- \quad (y_{k+1}^k(y_{k+1} - y_{k+1}) \prod_{i=k+2,k+1,\ldots 3,1} \frac{1}{(y_i - y_{k+1})})]$$

$$- \quad (y_{k+2}^k(y_{k+2} - y_{k+2}) \prod_{i=k+2,k+1,\ldots 3,1} \frac{1}{(y_i - y_{k+2})})]$$

$$\times \quad (\prod_{\ell=k+2,k+1\ldots 4,3} \prod_{j=\ell-1,\ell-2\ldots 3,1} (y_\ell - y_j))$$

$$+y_3^k[\quad + \quad (y_1^k(y_1 - y_1) \prod_{i=k+2,k+1,\ldots 4,2,1} \frac{1}{(y_i - y_1)})$$

$$+ \quad (y_2^k(y_2 - y_2) \prod_{i=k+2,k+1,\ldots 4,2,1} \frac{1}{(y_i - y_2)})$$

$$+ \quad (y_4^k(y_4 - y_4) \prod_{i=k+2,k+1,\ldots 4,2,1} \frac{1}{(y_i - y_4)})$$

$$\vdots$$

$$+ \quad (y_k^k(y_k - y_k) \prod_{i=k+2,k+1,\ldots 4,2,1} \frac{1}{(y_i - y_k)})$$

(4.13)

$$+ \quad (y_{k+1}^k(y_{k+1} - y_{k+1}) \prod_{i=k+2,k+1,\ldots 4,2,1} \frac{1}{(y_i - y_{k+1})})$$

$$+ \quad (y_{k+2}^k(y_{k+2} - y_{k+2}) \prod_{i=k+2,k+1,\ldots 4,2,1} \frac{1}{(y_i - y_{k+2})})]$$

$$\times \quad (\prod_{\ell=k+2,k+1\ldots 4,2} \prod_{j=\ell-1,\ell-2\ldots 2,1} (y_\ell - y_j))$$

$$+ \vdots$$

$$+y_k^k[\quad \pm \quad (y_1^k(y_1 - y_1) \prod_{i=k+2,k+1,\ldots 3,2,1} \frac{1}{(y_i - y_1)})$$

$$\pm \quad (y_2^k(y_2 - y_2) \prod_{i=k+2,k+1,\dots 3,2,1} \frac{1}{(y_i - y_2)})$$

$$\pm \quad (y_3^k(y_3 - y_3) \prod_{i=k+2,k+1,\dots 3,2,1} \frac{1}{(y_i - y_3)})$$

$$\vdots$$

$$\pm \quad (y_{k-1}^k(y_{k-1} - y_{k-1}) \prod_{i=k+2,k+1,\dots 3,2,1} \frac{1}{(y_i - y_{k-1})})$$

$$\pm \quad (y_{k+1}^k(y_{k+1} - y_{k+1}) \prod_{i=k+2,k+1,\dots 3,2,1} \frac{1}{(y_i - y_{k+1})})$$

$$\pm \quad (y_{k+2}^k(y_{k+2} - y_{k+2}) \prod_{i=k+2,k+1,\dots 3,2,1} \frac{1}{(y_i - y_{k+2})})]$$

$$\times \quad (\prod_{\ell=k+2,k+1\dots 3,2} \prod_{j=\ell-1,\ell-2\dots 2,1} (y_\ell - y_j))$$

$$+y_{k+1}^k[\quad \mp \quad (y_1^k(y_1 - y_1) \prod_{i=k+2,k,\dots 4,2,1} \frac{1}{(y_i - y_1)})$$

$$\mp \quad (y_2^k(y_2 - y_2) \prod_{i=k+2,k,\dots 4,2,1} \frac{1}{(y_i - y_2)})$$

$$\mp \quad (y_3^k(y_3 - y_3) \prod_{i=k+2,k,\dots 4,2,1} \frac{1}{(y_i - y_3)})$$

$$\vdots$$

$$\mp \quad (y_{k-1}^k(y_{k-1} - y_{k-1}) \prod_{i=k+2,k,\dots 4,2,1} \frac{1}{(y_i - y_{k-1})})$$

$$\mp \quad (y_k^k(y_k - y_k) \prod_{i=k+2,k,\dots 4,2,1} \frac{1}{(y_i - y_k)})$$

$$\mp \quad (y_{k+2}^k(y_{k+2} - y_{k+2}) \prod_{i=k+2,k,\dots 4,2,1} \frac{1}{(y_i - y_{k+2})})]$$

$$\times \quad (\prod_{\ell=k+2,k\dots 4,2} \prod_{j=\ell-1,\ell-2\dots 2,1} (y_\ell - y_j))$$

$$+y_{k+2}^k[\quad \pm \quad (y_1^k(y_1 - y_1) \prod_{i=k+1,k,\dots 4,2,1} \frac{1}{(y_i - y_1)})$$

$$\pm \quad (y_2^k(y_2 - y_2) \prod_{i=k+1,k,\dots 4,2,1} \frac{1}{(y_i - y_2)})$$

$$\pm \quad (y_3^k(y_3 - y_3) \prod_{i=k+1,k,\dots 4,2,1} \frac{1}{(y_i - y_3)})$$

$$\vdots$$

$$\begin{aligned}
&\pm \quad (y_{k-1}^k(y_{k-1} - y_{k-1}) \prod_{i=k+1,k,\ldots 4,2,1} \frac{1}{(y_i - y_{k-1})}) \\
&\pm \quad (y_k^k(y_k - y_k) \prod_{i=k+1,k,\ldots 4,2,1} \frac{1}{(y_i - y_k)}) \\
&\pm \quad (y_{k+1}^k(y_{k+1} - y_{k+1}) \prod_{i=k+1,k,\ldots 4,2,1} \frac{1}{(y_i - y_{k+1})}) \\
&\times \quad (\prod_{\ell=k+1,k\ldots 4,2} \prod_{j=\ell-1,\ell-2\ldots 2,1} (y_\ell - y_j))
\end{aligned} \tag{4.14}$$

If we look at only the $-y_j^k y_i^k$ factors, we see that each combination of i and j will appear exactly twice.

In fact, the $y_i^k y_j^k$ pairs will always cancel. We can see this by noting that out of the k+2 possible variables, the coefficients will each contain only k variables, and that neither coefficient will contain the variables y_i or y_j. Since all the pairs cancel we can say:

$$\begin{aligned}
\textstyle\sum_{p=1}^{p=k+2} &\left(y_p^k(y_p - y_p) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_p)} \right) \times \left(\prod_{\ell=k+2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right) \\
= &\quad 0
\end{aligned} \tag{4.15}$$

and therefore:

$$\begin{aligned}
-\textstyle\sum_{p=1}^{p=k+1} &\left(y_p^k(y_p - y_p) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_p)} \right) \\
= &\left(y_{k+2}^k(y_{k+2} - y_{k+2}) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_{k+2})} \right)
\end{aligned} \tag{4.16}$$

The left side of equation 4.6 will therefore equal:

$leftside =$

$$\begin{aligned}
&(\left(y_1^{k+1}(y_1 - y_1) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_1)} \right) \\
&\left(y_2^{k+1}(y_2 - y_2) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_2)} \right)
\end{aligned}$$

$$
\begin{aligned}
&\left(y_3^{k+1}(y_3 - y_3) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_3)} \right) \\
&\vdots \\
&\left(y_k^{k+1}(y_k - y_k) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_k)} \right) \\
&\left(y_{k+1}^{k+1}(y_{k+1} - y_{k+1}) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_{k+1})} \right) \\
&\left(y_{k+2}^{k+1}(y_{k+2} - y_{k+2}) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_{k+2})} \right)) \\
&\times(-1)^{k+1} \left(\prod_{\ell=k+2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right)
\end{aligned}
$$

$$
= \quad (-1)^{k+1} \sum_{\ell=1}^{\ell=k+2} \left(y_\ell^{k+1}(y_\ell - y_\ell) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_\ell)} \right) \times \left(\prod_{\ell=k+2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right) \tag{4.17}
$$

Which completes the proof that equation 4.6 holds for any n.

4.2 Proof: 1D Magnification Does Not Explicitly Depend on Perturbation Position

To prove that the total one dimensional magnification does not explicitly depend on perturbation position for n perturbations, we write another recursive proof. The numerator of the total 1D magnification for n perturbations can be written as:

$$
\begin{aligned}
numerator \;=\; & \\
(-1)^n(\;+\; & \varrho(y_1)(y_1 - y_1) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_1)} \\
+\; & \varrho(y_2)(y_2 - y_2) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_2)} \\
+\; & \varrho(y_3)(y_3 - y_3) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_3)}
\end{aligned}
$$

$$\vdots$$

$$+ \quad \varrho(y_n)(y_n - y_n) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_n)}$$

$$+ \quad \varrho(y_{n+1})(y_{n+1} - y_{n+1}) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_{n+1})})$$

$$\times \left(\prod_{\ell=n+1}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right), \tag{4.18}$$

where $\varrho(y_j) \equiv (y_j - q_1)(y_j - q_2)...(y_j - q_n)$.

We have already proven the base case in section 3.2, now we need only show that what holds for n=k-1 holds for n=k. Let's say that we have proven equation 4.18 does not explicitly depend on perturbation location q_j for n=k-1 perturbations:

$$(-1)^{k-1}(\quad + \quad y_1^m(y_1 - y_1) \prod_{i=k}^{i=1} \frac{1}{(y_i - y_1)}$$

$$+ \quad y_2^m(y_2 - y_2) \prod_{i=k}^{i=1} \frac{1}{(y_i - y_2)}$$

$$+ \quad y_3^m(y_3 - y_3) \prod_{i=k}^{i=1} \frac{1}{(y_i - y_3)}$$

$$\vdots$$

$$+ \quad y_{k-1}^m(y_{k-1} - y_{k-1}) \prod_{i=k}^{i=1} \frac{1}{(y_i - y_{k-1})}$$

$$+ \quad y_k^m(y_k - y_k) \prod_{i=k}^{i=1} \frac{1}{(y_i - y_k)})$$

$$\times \left(\prod_{\ell=k}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right)$$

$$= \quad 0, for (0 \leq m < k - 1) \tag{4.19}$$

Then we must show that for n=k perturbations equation 4.18 also does not explicitly depend on perturbation location:

$$(-1)^k(\quad + \quad y_1^m(y_1 - y_1) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_1)}$$

$$+ \quad y_2^m(y_2 - y_2) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_2)}$$

$$
\begin{aligned}
&+ \quad y_3^m(y_3 - y_3) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_3)} \\
&+ \quad \ldots \\
&+ \quad y_k^m(y_k - y_k) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_k)} \\
&+ \quad y_{k+1}^m(y_{k+1} - y_{k+1}) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_{k+1})}) \\
&\quad \times \left(\prod_{\ell=k+1}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right) \\
&= \quad 0, for(0 \leq m < k) \qquad (4.20)
\end{aligned}
$$

Looking at expression 4.20, we see that a given y_j^m will have a coefficient that contains all possible y terms except y_j itself. For this reason each coefficient can be rewritten using identity 4.3 from section 4.1. This gives us:

$$
\begin{aligned}
& numerator = \\
& (-1)^k(-1)^{k-1}(\\
+ \quad & y_1^m \left(\sum_{\ell=2,3\ldots k,k+1} y_\ell^{k-1}(y_\ell - y_\ell) \prod_{i=k+1,k\ldots 3,2} \frac{1}{(y_i - y_\ell)} \right) \times \left(\prod_{\ell=k+1,k..4,3} \prod_{j=\ell-1\ldots 3,2} (y_\ell - y_j) \right) \\
- \quad & y_2^m \left(\sum_{\ell=1,3\ldots k,k+1} y_\ell^{k-1}(y_\ell - y_\ell) \prod_{i=k+1,k\ldots 3,1} \frac{1}{(y_i - y_\ell)} \right) \times \left(\prod_{\ell=k+1,k..4,3} \prod_{j=\ell-1\ldots 3,1} (y_\ell - y_j) \right) \\
+ \quad & y_3^m \left(\sum_{\ell=1,2\ldots k,k+1} y_\ell^{k-1}(y_\ell - y_\ell) \prod_{i=k+1,k\ldots 2,1} \frac{1}{(y_i - y_\ell)} \right) \times \left(\prod_{\ell=k+1,k..4,2} \prod_{j=\ell-1\ldots 2,1} (y_\ell - y_j) \right) \\
& \vdots \\
\pm \quad & y_k^m \left(\sum_{\ell=1,2\ldots k-1,k+1} y_\ell^{k-1}(y_\ell - y_\ell) \prod_{i=k+1,k-1\ldots 2,1} \frac{1}{(y_i - y_\ell)} \right) \times \left(\prod_{\ell=k+1,k-1..4,2} \prod_{j=\ell-1\ldots 2,1} (y_\ell - y_j) \right) \\
\mp \quad & y_{k+1}^m \left(\sum_{\ell=1,2\ldots k-1,k} y_\ell^{k-1}(y_\ell - y_\ell) \prod_{i=k,k-1\ldots 2,1} \frac{1}{(y_i - y_\ell)} \right) \times \left(\prod_{\ell=k,k-1..4,2} \prod_{j=\ell-1\ldots 2,1} (y_\ell - y_j) \right)) \qquad (4.21)
\end{aligned}
$$

For k odd and k even respectively. Expanding and grouping this by leading factor, y_l^{k-1}, we get:

$$numerator =$$

$$
\begin{aligned}
-(y_1^{k-1} \quad & [-y_2^m \left((y_2 - y_2) \prod_{i=k+1,k\ldots 3,2} \frac{1}{(y_i - y_2)} \right) \\
& -y_3^m \left((y_3 - y_3) \prod_{i=k+1,k\ldots 3,2} \frac{1}{(y_i - y_3)} \right) \\
& \vdots \\
& -y_k^m \left((y_k - y_k) \prod_{i=k+1,k\ldots 3,2} \frac{1}{(y_i - y_k)} \right) \\
& -y_{k+1}^m \left((y_{k+1} - y_{k+1}) \prod_{i=k+1,k\ldots 3,2} \frac{1}{(y_i - y_{k+1})} \right)] \times \left(\prod_{\ell=k+1,k\ldots 3} \prod_{j=\ell-1\ldots 2} (y_\ell - y_j) \right)
\end{aligned}
$$

$$
\begin{aligned}
+y_2^{k-1} \quad & [+y_1^m \left((y_1 - y_1) \prod_{i=k+1,k\ldots 3,1} \frac{1}{(y_i - y_1)} \right) \\
& +y_3^m \left((y_3 - y_3) \prod_{i=k+1,k\ldots 3,1} \frac{1}{(y_i - y_3)} \right) \\
& \ldots \\
& +y_k^m \left((y_k - y_k) \prod_{i=k+1,k\ldots 3,1} \frac{1}{(y_i - y_k)} \right) \\
& +y_{k+1}^m \left((y_{k+1} - y_{k+1}) \prod_{i=k+1,k\ldots 3,1} \frac{1}{(y_i - y_{k+1})} \right)] \times \left(\prod_{\ell=k+1,k\ldots 4,3} \prod_{j=\ell-1\ldots 3,1} (y_\ell - y_j) \right)
\end{aligned}
$$

$$
\begin{aligned}
+y_3^{k-1} \quad & [+y_1^m \left((y_1 - y_1) \prod_{i=k+1,k\ldots 4,2,1} \frac{1}{(y_i - y_1)} \right) \\
& +y_2^m \left((y_2 - y_2) \prod_{i=k+1,k\ldots 4,2,1} \frac{1}{(y_i - y_2)} \right) \\
& \ldots \\
& \pm y_k^m \left((y_k - y_k) \prod_{i=k+1,k\ldots 4,2,1} \frac{1}{(y_i - y_k)} \right) \\
& \mp y_{k+1}^m \left((y_{k+1} - y_{k+1}) \prod_{i=k+1,k\ldots 4,2,1} \frac{1}{(y_i - y_{k+1})} \right)] \times \left(\prod_{\ell=k+1,k\ldots 4,2} \prod_{j=\ell-1\ldots 4,2,1} (y_\ell - y_j) \right)
\end{aligned}
\tag{4.22}
$$

$+\vdots$

$$
\begin{aligned}
+y_k^{k-1} \quad & [+y_1^m\left((y_1-y_1)\prod_{i=k+1,k-1\ldots 2,1}\frac{1}{(y_i-y_1)}\right) \\
& +y_2^m\left((y_2-y_2)\prod_{i=k+1,k-1\ldots 2,1}\frac{1}{(y_i-y_2)}\right) \\
& +y_3^m\left((y_3-y_3)\prod_{i=k+1,k-1\ldots 2,1}\frac{1}{(y_i-y_3)}\right) \\
& \vdots \\
& +y_{k+1}^m\left((y_{k+1}-y_{k+1})\prod_{i=k+1,k-1\ldots 2,1}\frac{1}{(y_i-y_{k+1})}\right)] \times \left(\prod_{\ell=k+1,k-1\ldots 4,2}\ \prod_{j=\ell-1,k+1,k-1\ldots 2,1}(y_\ell-y_j)\right)
\end{aligned}
$$

$$
\begin{aligned}
+y_{k+1}^{k-1} \quad & [+y_1^m\left((y_1-y_1)\prod_{i=k,k-1\ldots 2,1}\frac{1}{(y_i-y_1)}\right) \\
& +y_2^m\left((y_2-y_2)\prod_{i=k,k-1\ldots 2,1}\frac{1}{(y_i-y_2)}\right) \\
& +y_3^m\left((y_3-y_3)\prod_{i=k,k-1\ldots 2,1}\frac{1}{(y_i-y_3)}\right) \\
& \vdots \\
& +y_k^m\left((y_k-y_k)\prod_{i=k,k-1\ldots 2,1}\frac{1}{(y_i-y_k)}\right)] \times \left(\prod_{\ell=k,k-1\ldots 4,2}\ \prod_{j=\ell-1\ldots 2,1}(y_\ell-y_j)\right))
\end{aligned} \tag{4.23}
$$

Written in this form, we see that each coefficient of y_j^{k-1} is in the form of equation 4.19 and that we can therefore say that for any $0 \leq m < k-1$ they will be zero. But we already showed in section 4.1, equation 4.15 that the $m = k-1$ terms will be zero! So equation 4.20 will be zero for any $0 \leq m < k$ and we have completed the recursive proof. This proves that the 1D magnification does not explicitly depend on perturbation location for n perturbations.

Together sections 4.1 and 4.2 prove that the total 1D magnification will be 1 for n perturbations.

Chapter 5

Conclusions

In this paper we showed that the greatest demagnification is achieved when the perturbation is placed directly on top of the saddlepoint. Following Schechter et al. [13] we showed that for a single point mass perturbation a macro-saddle could be demagnified by a factor of sixteen from $\mu_{macro} = -16.67$ to $\mu_{total} = -0.94$. We also showed, as an interesting variant, that an isothermal sphere perturbation can demagnify a macro-saddle from $\mu_{macro} = -16.67$ to $\mu_{total} = -1.89$. Next we looked at the case of infinite point perturbations, and showed that with enough direct hits we can make the magnification of a macro-saddle arbitrarily small. We also found that in our hierarchy of point perturbations each time a saddlepoint image is split, the new local values of the shear and convergence are $\kappa' = \kappa$ and $\gamma' = 1 - \kappa + 2\gamma$. We then showed that with κ_0 and γ_0 set to zero for convenience, the total 1D magnification along the axis of curvature is one for any number of perturbations and irregardless of perturbation size and location.

In this study, the assumption that κ_0 and γ_0 can be set to zero without loss of generality was taken as a given, but for future study we suggest that this assumption be investigated. Another useful extension to the project would be to derive an expression for the total 1D magnification along the axis of negative curvature. Finally, we suggest statistical studies of small numbers of randomly distributed perturbations, in particular focusing on the likelihood of various magnifications for a given number of perturbations.

Appendix A

Optimizing Milli-lens Placement

In this section we calculate where along the y-axis a point perturbation should be placed to maximally demagnify a macro-saddle. This is accomplished by minimizing the sum of the magnifications of the two images, μ_{total}, with respect to the perturbation location.

We begin by modifying ψ_{milli} to include a point perturbation centered at some unspecified point $y = a$:

$$\psi_{milli} = \frac{\kappa_0}{2}(x^2 + y^2) + \frac{\gamma_0}{2}(x^2 - y^2) + b^2 ln((x^2 + (y-a)^2)^{1/2}) \quad \text{(A.1)}$$

To minimize the magnification of one image, μ_{milli}, we must maximize the determinant of the inverse magnification matrix, μ_{milli}^{-1}:

$$\mu_{milli}^{-1} = (1 - \frac{\partial^2\psi_{milli}}{\partial x^2})(1 - \frac{\partial^2\psi_{milli}}{\partial y^2}) - (\frac{\partial^2\psi_{milli}}{\partial y \partial x})^2 \quad \text{(A.2)}$$

To maximize this we must first find the derivatives:

$$\frac{\partial\psi_{milli}}{\partial x} = \kappa_0 x + \gamma_0 x + \frac{b^2}{(x^2 + (y-a)^2)^{1/2}}(x)(x^2 + (y-a)^2)^{-1/2} \quad \text{(A.3)}$$

$$\frac{\partial\psi_{milli}}{\partial x} = \kappa_0 x + \gamma_0 x + \frac{b^2 x}{(x^2 + (y-a)^2)} \quad \text{(A.4)}$$

$$\frac{\partial^2\psi_{milli}}{\partial x^2} = \kappa_0 + \gamma_0 + \frac{b^2}{(x^2+(y-a)^2)} + \frac{-2b^2x^2}{(x^2+(y-a)^2)^2} \tag{A.5}$$

$$\frac{\partial^2\psi_{milli}}{\partial x\partial y} = \frac{-b^2x(2(y-a))}{(x^2+(y-a)^2)^2} \tag{A.6}$$

$$\frac{\partial\psi_{milli}}{\partial y} = \kappa_0 y - \gamma_0 y + \frac{b^2}{(x^2+(y-a)^2)^{1/2}}((x^2+(y-a)^2)^{-1/2})(y-a) \tag{A.7}$$

$$\frac{\partial\psi_{milli}}{\partial y} = \kappa_0 y - \gamma_0 y + \frac{b^2(y-a)}{x^2+(y-a)^2} \tag{A.8}$$

$$\frac{\partial^2\psi_{milli}}{\partial y^2} = \kappa_0 - \gamma_0 + \frac{b^2}{x^2+(y-a)^2} + \frac{b^2(y-a)}{(x^2+(y-a)^2)^2}(-2(y-a)) \tag{A.9}$$

$$\frac{\partial^2\psi_{milli}}{\partial y^2} = \kappa_0 - \gamma_0 + \frac{b^2}{x^2+(y-a)^2} + \frac{-2b^2(y-a)^2}{(x^2+(y-a)^2)^2} \tag{A.10}$$

Plugging in the expressions for the derivatives we get:

$$\begin{aligned} \mu_{milli}^{-1} = &\left(1 - (\kappa_0 + \gamma_0 + \frac{b^2}{(x^2+(y-a)^2)} + \frac{-2b^2x^2}{(x^2+(y-a)^2)^2})\right) \\ &\times \left(1 - (\kappa_0 - \gamma_0 + \frac{b^2}{x^2+(y-a)^2} + \frac{-2b^2(y-a)^2}{(x^2+(y-a)^2)^2})\right) \\ &- \left(\frac{-b^2x(2(y-a))}{(x^2+(y-a)^2)^2}\right)^2 \end{aligned} \tag{A.11}$$

$$\begin{aligned} \mu_{milli}^{-1} = &\left(1 - \kappa_0 - \gamma_0 - \frac{b^2}{(x^2+(y-a)^2)} - \frac{-2b^2x^2}{(x^2+(y-a)^2)^2}\right) \\ &\times \left(1 - \kappa_0 + \gamma_0 - \frac{b^2}{x^2+(y-a)^2} - \frac{-2b^2(y-a)^2}{(x^2+(y-a)^2)^2}\right) \\ &- \left(\frac{(-2b^2x(y-a))^2}{(x^2+(y-a)^2)^4}\right) \end{aligned} \tag{A.12}$$

We cannot yet maximize with respect to a because (x, y) is the image position and will clearly depend on a. Thus we must find y(a) and x(a):

$$\tau = K\left[\frac{1}{2}(\vec{\theta}-\vec{\beta})^2 - \psi_{milli}\right], K \equiv \frac{1+z_L}{c}\frac{D_L D_S}{D_{LS}} \tag{A.13}$$

$$\tau = K\left[\frac{1}{2}((x-\beta_x)^2 + (y-\beta_y)^2 - (\frac{\kappa_0}{2}(x^2+y^2) + \frac{\gamma_0}{2}(x^2-y^2) + b^2 ln((x^2+(y-a)^2)^{1/2}))\right] \tag{A.14}$$

$$\vec{\nabla}\tau = K\left[(x-\beta_x) - (\kappa_0 x + \gamma_0 x + \frac{b^2}{(x^2+(y-a)^2)^{1/2}}(x)(x^2+(y-a)^2)^{-1/2})\right]\hat{x}$$
$$+K\left[(y-\beta_y) - (\kappa_0 y - \gamma_0 y + \frac{b^2}{(x^2+(y-a)^2)^{1/2}}(y-a)(x^2+(y-a)^2)^{-1/2})\right]\hat{y} \tag{A.15}$$

$$0 = \vec{\nabla}\tau = \left[(x-\beta_x) - (\kappa_0 x + \gamma_0 x + \frac{b^2 x}{(x^2+(y-a)^2)})\right]\hat{x}$$
$$+\left[(y-\beta_y) - (\kappa_0 y - \gamma_0 y + \frac{b^2(y-a)}{(x^2+(y-a)^2)})\right]\hat{y} \tag{A.16}$$

Since the source here is the macro-image, the source location is simply:

$$\beta_x = \beta_y = 0. \tag{A.17}$$

$$0 = \left[x - \kappa_0 x - \gamma_0 x - \frac{b^2 x}{(x^2+(y-a)^2)})\right]\hat{x}$$
$$0 = \left[y - \kappa_0 y + \gamma_0 y - \frac{b^2(y-a)}{(x^2+(y-a)^2)}\right]\hat{y} \tag{A.18}$$

$$0 = \left[x(1-\kappa_0-\gamma_0 - \frac{b^2}{(x^2+(y-a)^2)})\right]\hat{x}$$
$$0 = \left[y(1-\kappa_0+\gamma_0 - \frac{b^2}{(x^2+(y-a)^2)}) + \frac{b^2 a}{(x^2+(y-a)^2)}\right]\hat{y} \tag{A.19}$$

If we choose $x = 0$, we find one solution:

$$0 = y(1-\kappa_0+\gamma_0 - \frac{b^2}{(y-a)^2}) + \frac{b^2 a}{(y-a)^2} \tag{A.20}$$

$$0 = y(1 - \kappa_0 + \gamma_0) - \frac{b^2(y-a)}{(y-a)^2} \tag{A.21}$$

$$\frac{b^2}{(y-a)} = y(1 - \kappa_0 + \gamma_0) \tag{A.22}$$

$$\frac{b^2}{1 - \kappa_0 + \gamma_0} = y(y-a) \tag{A.23}$$

$$0 = y^2 - ay - \frac{b^2}{1 - \kappa_0 + \gamma_0} \tag{A.24}$$

$$\begin{aligned} x(a) &= 0 \\ y(a) &= \frac{a \pm \sqrt{a^2 + \frac{4b^2}{1-\kappa_0+\gamma_0}}}{2} \end{aligned} \tag{A.25}$$

Since perturbed saddlepoints produce only two milli-images, it is safe to assume these are the only two real answers. Now we can continue with our minimization of the magnification. Setting $x = 0$, equation A.12 becomes:

$$\begin{aligned} \mu_{milli}^{-1} &= \left(1 - \kappa_0 - \gamma_0 - \frac{b^2}{(y-a)^2}\right) \\ &\times \left(1 - \kappa_0 + \gamma_0 - \frac{b^2}{(y-a)^2} + \frac{2b^2}{(y-a)^2}\right) \end{aligned} \tag{A.26}$$

$$\begin{aligned} \mu_{milli}^{-1} &= \left(1 - \kappa_0 - \gamma_0 - \frac{b^2}{(y-a)^2}\right) \\ &\times \left(1 - \kappa_0 + \gamma_0 + \frac{b^2}{(y-a)^2}\right) \end{aligned} \tag{A.27}$$

Let $\gamma_0 + \frac{b^2}{(y-a)^2} \equiv \gamma'$, then:

$$\mu_{milli}^{-1} = (1 - \kappa_0 - \gamma')(1 - \kappa_0 + \gamma') \tag{A.28}$$

$$\mu_{milli}^{-1} = (1 - \kappa_0)^2 - (\gamma')^2 \tag{A.29}$$

$$\partial\mu_{milli}^{-1}/\partial a = -\frac{\partial}{\partial a}(\gamma')^2 \tag{A.30}$$

$$\partial\mu_{milli}^{-1}/\partial a = -\frac{\partial}{\partial a}\left((\gamma_0 + \frac{b^2}{(y-a)^2})\right)^2 \tag{A.31}$$

$$\partial\mu_{milli}^{-1}/\partial a = -2\left((\gamma_0 + (b^2)(y-a)^{-2})\right)(-2)(b^2)(y-a)^{-3}(dy/da - 1) \tag{A.32}$$

$$\partial\mu_{milli}^{-1}/\partial a = 4\left(\gamma_0 + \frac{b^2}{(y-a)^2}\right)\frac{b^2}{(y-a)^3}(dy/da - 1) \tag{A.33}$$

We can simplify this expression by finding $\frac{\partial y}{\partial a}$ as a function of y. Starting with equation A.25 we get:

$$y = \frac{a}{2} \pm \left(\frac{a^2}{4} + \frac{b^2}{1-\kappa_0+\gamma_0}\right)^{1/2} \tag{A.34}$$

$$\frac{\partial y}{\partial a} = \frac{1}{2} \pm \left(\frac{a}{4}\right)\left(\frac{a^2}{4} + \frac{b^2}{1-\kappa_0+\gamma_0}\right)^{-1/2} \tag{A.35}$$

Using equation A.34 we can say:

$$\left(y - \frac{a}{2}\right)^{-1} = \pm\left(\frac{a^2}{4} + \frac{b^2}{1-\kappa_0+\gamma_0}\right)^{-1/2}, \tag{A.36}$$

and therefore

$$\frac{\partial y}{\partial a} = \frac{1}{2} + \left(\frac{a}{4}\right)(y - \frac{a}{2})^{-1}. \tag{A.37}$$

Plugging this in to equation A.33 we get:

$$\partial\mu_{milli}^{-1}/\partial a = 4\left(\gamma_0 + \frac{b^2}{(y-a)^2}\right)\frac{b^2}{(y-a)^3}(\frac{1}{2} + \frac{\left(\frac{a}{4}\right)}{(y-\frac{a}{2})} - 1) \tag{A.38}$$

$$\partial\mu_{milli}^{-1}/\partial a = \left(\gamma_0 + \frac{b^2}{(y-a)^2}\right)\frac{b^2}{(y-a)^3}(-2 + \frac{a}{(y-\frac{a}{2})}) \tag{A.39}$$

To minimize the total magnification, μ_{total}, with respect to a, we must minimize the magnifications of the two images. We add derivatives at the two image locations and set this equal to 0:

$$\begin{aligned} 0 = \partial\mu_{total}^{-1}/\partial a = \\ = \frac{\partial\mu_{milli}^{-1}(x_1,y_1)}{\partial a} + \frac{\partial\mu_{milli}^{-1}(x_2,y_2)}{\partial a} = \\ \left(\frac{\gamma_0(\frac{a+\sqrt{a^2+\frac{4b^2}{1-\kappa_0+\gamma_0}}}{2} - a)^2 + b^2}{(\frac{a+\sqrt{a^2+\frac{4b^2}{1-\kappa_0+\gamma_0}}}{2} - a)^5}\right)(-2 + \frac{a}{(\frac{a+\sqrt{a^2+\frac{4b^2}{1-\kappa_0+\gamma_0}}}{2} - \frac{a}{2})}) \\ + \left(\frac{\gamma_0(\frac{a-\sqrt{a^2+\frac{4b^2}{1-\kappa_0+\gamma_0}}}{2} - a)^2 + b^2}{(\frac{a-\sqrt{a^2+\frac{4b^2}{1-\kappa_0+\gamma_0}}}{2} - a)^5}\right)(-2 + \frac{a}{(\frac{a-\sqrt{a^2+\frac{4b^2}{1-\kappa_0+\gamma_0}}}{2} - \frac{a}{2})}) \end{aligned} \tag{A.40}$$

Letting $\frac{\sqrt{a^2+\frac{4b^2}{1-\kappa_0+\gamma_0}}}{2} \equiv \eta$:

$$\begin{aligned} 0 = \partial\mu_{total}^{-1}/\partial a = \left(\frac{\gamma_0(\eta - a/2)^2 + b^2}{(\eta - a/2)^5}\right)(-2 + \frac{a}{(\eta)}) \\ + \left(\frac{\gamma_0(-\eta - a/2)^2 + b^2}{(-\eta - a/2)^5}\right)(-2 + \frac{a}{(-\eta)}) \end{aligned} \tag{A.41}$$

Guessing $a = 0$:

$$\begin{aligned} 0 = \partial\mu_{total}^{-1}/\partial a = \left(\frac{\gamma_0(\eta)^2 + b^2}{(\eta)^5}\right)(-2) \\ + \left(\frac{\gamma_0(-\eta)^2 + b^2}{(-\eta)^5}\right)(-2) \end{aligned} \tag{A.42}$$

$$\begin{aligned} 0 = \partial\mu_{total}^{-1}/\partial a = \left(\frac{\gamma_0(\eta)^2 + b^2}{(\eta)^5}\right) \\ - \left(\frac{\gamma_0(\eta)^2 + b^2}{(\eta)^5}\right) \end{aligned} \tag{A.43}$$

And therefore $a = 0$ is a solution. Looking at equation A.41, we see that $\partial\mu_{total}^{-1}/\partial a$ is third order in a, and therefore there will be two more solutions. However, the term that is third order in a is positive, which means that $\partial\mu_{total}^{-1}/\partial a$ goes to negative infinity as a goes to negative infinity and positive infinity as a goes to positive infinity. Therefore, at the two remaining solutions, μ_{total}^{-1} will go from a negative slope to a positive slope, which means they are both minima. In contrast, at a=0, μ_{total}^{-1} will go from negative slope to positive slope, implying a is the maximum in the inverse magnification we have been looking for. Thus the maximum demagnification with one perturbation will occur when the perturbation is placed at $a = 0$, directly in the line of sight to the macro-image.

Appendix B

Total One-Dimensional Magnification: Three Perturbation Case

Here we prove that the sum of the one dimensional magnification along the axis of curvature is 1 for 3 perturbations. To prove this we find μ_{total}, show that all the terms involving perturbation location, q, will cancel out, and then show that the remaining terms sum to 1.

With three perturbations ($n = 3$) equation 3.15 becomes:

$$\left[1 - \frac{\partial^2 \psi_1}{\partial y^2}\right] = \tag{B.1}$$

$$+ \frac{(y-y_2)(y-y_3)(y-y_4)}{(y-q_1)(y-q_2)(y-q_3)} + \frac{(y-y_1)(y-y_3)(y-y_4)}{(y-q_1)(y-q_2)(y-q_3)} \tag{B.2}$$

$$+ \frac{(y-y_1)(y-y_2)(y-y_4)}{(y-q_1)(y-q_2)(y-q_3)} + \frac{(y-y_1)(y-y_2)(y-y_3)}{(y-q_1)(y-q_2)(y-q_3)}. \tag{B.3}$$

To find $\mu_{1D,total}$ we evaluate the inverse of equation ?? for image positions y_1, y_2, y_3, and y_4 and then sum the results. This gives us:

$$\mu_{1D,total} = \left[1 - \frac{\partial^2 \psi_1}{\partial y^2}(x_1, y_1)\right]^{-1} + \left[1 - \frac{\partial^2 \psi_1}{\partial y^2}(x_2, y_2)\right]^{-1}$$

$$+ \left[1 - \frac{\partial^2\psi_1}{\partial y^2}(x_3, y_3)\right]^{-1} + \left[1 - \frac{\partial^2\psi_1}{\partial y^2}(x_4, y_4)\right]^{-1} \tag{B.4}$$

$$\begin{aligned} &= \frac{(y_1 - q_1)(y_1 - q_2)(y_1 - q_3)}{(y_1 - y_2)(y_1 - y_3)(y_1 - y_4)} + \frac{(y_2 - q_1)(y_2 - q_2)(y_2 - q_3)}{(y_2 - y_1)(y_2 - y_3)(y_2 - y_4)} \\ &+ \frac{(y_3 - q_1)(y_3 - q_2)(y_3 - q_3)}{(y_3 - y_1)(y_3 - y_2)(y_3 - y_4)} + \frac{(y_4 - q_1)(y_4 - q_2)(y_4 - q_3)}{(y_4 - y_1)(y_4 - y_2)(y_4 - y_3)} \end{aligned} \tag{B.5}$$

Rewriting:

$$\begin{aligned} \mu_{1D,total} &= \\ &- \frac{(y_1 - q_1)(y_1 - q_2)(y_1 - q_3)}{(y_4 - y_1)(y_3 - y_1)(y_2 - y_1)} \\ &+ \frac{(y_2 - q_1)(y_2 - q_2)(y_2 - q_3)}{(y_4 - y_2)(y_3 - y_2)(y_2 - y_1)} \\ &- \frac{(y_3 - q_1)(y_3 - q_2)(y_3 - q_3)}{(y_4 - y_3)(y_3 - y_2)(y_3 - y_1)} \\ &+ \frac{(y_4 - q_1)(y_4 - q_2)(y_4 - q_3)}{(y_4 - y_3)(y_4 - y_2)(y_4 - y_1)} \end{aligned} \tag{B.6}$$

We can simplify this by defining a function $\varrho(y_j)$:

$$\varrho(y_j) \equiv (y_j - q_1)(y_j - q_2)(y_j - q_3) \tag{B.7}$$

Then the 1D magnification can more simply be written:

$$\begin{aligned} \mu_{1D,total} &= \\ &- \frac{\varrho(y_1)}{(y_2 - y_1)(y_3 - y_1)(y_4 - y_1)} \\ &+ \frac{\varrho(y_2)}{(y_2 - y_1)(y_3 - y_2)(y_4 - y_2)} \\ &- \frac{\varrho(y_3)}{(y_3 - y_1)(y_3 - y_2)(y_4 - y_3)} \\ &+ \frac{\varrho(y_4)}{(y_4 - y_1)(y_4 - y_2)(y_4 - y_3)} \end{aligned} \tag{B.8}$$

Combining these fractions we get:

$$\begin{aligned}
\mu_{1D,total} &= \\
&- \frac{\varrho(y_1)(y_4-y_3)(y_4-y_2)(y_3-y_2)}{(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
&+ \frac{\varrho(y_2)(y_4-y_3)(y_4-y_1)(y_3-y_1)}{(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
&- \frac{\varrho(y_3)(y_4-y_2)(y_4-y_1)(y_2-y_1)}{(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
&+ \frac{\varrho(y_4)(y_3-y_2)(y_3-y_1)(y_2-y_1)}{(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \qquad \text{(B.9)}
\end{aligned}$$

We can simplify this further using the relation 3.37 derived in section 3.3:

$$(a-b)(a-c)(b-c) = c^2(a-b) - b^2(a-c) + a^2(b-c) \qquad \text{(B.10)}$$

$$\begin{aligned}
\mu_{1D,total} &= \\
&- \frac{\varrho(y_1)(y_2^2(y_4-y_3) - y_3^2(y_4-y_2) + y_4^2(y_3-y_2))}{(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
&+ \frac{\varrho(y_2)(y_1^2(y_4-y_3) - y_3^2(y_4-y_1) + y_4^2(y_3-y_1))}{(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
&- \frac{\varrho(y_3)(y_1^2(y_4-y_2) - y_2^2(y_4-y_1) + y_4^2(y_2-y_1))}{(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
&+ \frac{\varrho(y_4)(y_1^2(y_3-y_2) - y_2^2(y_3-y_1) + y_3^2(y_2-y_1))}{(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \qquad \text{(B.11)}
\end{aligned}$$

Now, looking at just the numerator:

$$\begin{aligned}
numerator &= \\
&- \varrho(y_1)(y_2^2(y_4-y_3) - y_3^2(y_4-y_2) + y_4^2(y_3-y_2)) \\
&+ \varrho(y_2)(y_1^2(y_4-y_3) - y_3^2(y_4-y_1) + y_4^2(y_3-y_1)) \\
&- \varrho(y_3)(y_1^2(y_4-y_2) - y_2^2(y_4-y_1) + y_4^2(y_2-y_1)) \\
&+ \varrho(y_4)(y_1^2(y_3-y_2) - y_2^2(y_3-y_1) + y_3^2(y_2-y_1)) \qquad \text{(B.12)}
\end{aligned}$$

To simplify this we must compare the different components of $\varrho(y_4)$ in the numerator:

$$\varrho(y_4) \equiv (y_j - q_1)(y_j - q_2)(y_j - q_3) \tag{B.13}$$

$$= y_j^3 - q_1 y_j^2 - q_2 y_j^2 + q_1 q_2 y_j - y_j^2 q_3 + q_1 q_3 y_j + q_2 q_3 y_j - q_1 q_2 q_3 \tag{B.14}$$

$$= y_j^3 - (q_1 + q_2 + q_3) y_j^2 + (q_1 q_2 + q_1 q_3 + q_2 q_3) y_j^1 - (q_1 q_2 q_3) y_j^0 \tag{B.15}$$

First looking at the y_j^0 terms in the numerator:

$$\begin{aligned}
\sum y_j^0 terms \;&= \\
q_1 q_2 q_3(\;&+\; (y_2^2(y_4 - y_3) - y_3^2(y_4 - y_2) + y_4^2(y_3 - y_2)) \\
&-\; (y_1^2(y_4 - y_3) - y_3^2(y_4 - y_1) + y_4^2(y_3 - y_1)) \\
&+\; (y_1^2(y_4 - y_2) - y_2^2(y_4 - y_1) + y_4^2(y_2 - y_1)) \\
&-\; (y_1^2(y_3 - y_2) - y_2^2(y_3 - y_1) + y_3^2(y_2 - y_1)))
\end{aligned} \tag{B.16}$$

Grouping by the squared coefficients we get:

$$\begin{aligned}
\sum y_j^0 terms \;&= \\
q_1 q_2 q_3(\;&-\; y_1^2(y_4 - y_3) + y_1^2(y_4 - y_2) - y_1^2(y_3 - y_2) \\
&+\; y_2^2(y_4 - y_3) - y_2^2(y_4 - y_1) + y_2^2(y_3 - y_1) \\
&-\; y_3^2(y_2 - y_1) + y_3^2(y_4 - y_1) - y_3^2(y_4 - y_2) \\
&+\; y_4^2(y_2 - y_1) - y_4^2(y_3 - y_1) + y_4^2(y_3 - y_2))
\end{aligned} \tag{B.17}$$

$$\begin{aligned}
= q_1 q_2 q_3(\;&\; y_1^2(-(y_4 - y_3) + (y_4 - y_2) - (y_3 - y_2)) \\
&+\; y_2^2((y_4 - y_3) - (y_4 - y_1) + (y_3 - y_1)) \\
&+\; y_3^2(-(y_2 - y_1) + (y_4 - y_1) - (y_4 - y_2)) \\
&+\; y_4^2((y_2 - y_1) - (y_3 - y_1) + (y_3 - y_2)))
\end{aligned} \tag{B.18}$$

$$= q_1 q_2 q_3(\quad y_1^2(0) + y_2^2(0) + y_3^2(0) + y_4^2(0)) \tag{B.19}$$

$$= 0$$

So we can get rid of the terms containing $q_1q_2q_3$ in equation B.12, and our new expression will be:

$$\begin{aligned}
&numerator = \\
- &(y_1^3 - (q_1+q_2+q_3)y_1^2 + (q_1q_3+q_2q_3+q_1q_2)y_1)(y_3-y_2)(y_4-y_2)(y_4-y_3) \\
+ &(y_2^3 - (q_1+q_2+q_3)y_2^2 + (q_1q_3+q_2q_3+q_1q_2)y_2)(y_3-y_1)(y_4-y_1)(y_4-y_3) \\
- &(y_3^3 - (q_1+q_2+q_3)y_3^2 + (q_1q_3+q_2q_3+q_1q_2)y_3)(y_2-y_1)(y_4-y_1)(y_4-y_2) \\
+ &(y_4^3 - (q_1+q_2+q_3)y_4^2 + (q_1q_3+q_2q_3+q_1q_2)y_4)(y_2-y_1)(y_3-y_1)(y_3-y_2)
\end{aligned} \tag{B.20}$$

Now looking at just the y_j^1 terms of equation B.12:

$$\begin{aligned}
&\sum y_j^1 terms = \\
(- &(q_1q_3+q_2q_3+q_1q_2)y_1(y_2^2(y_4-y_3) - y_3^2(y_4-y_2) + y_4^2(y_3-y_2)) \\
+ &(q_1q_3+q_2q_3+q_1q_2)y_2(y_1^2(y_4-y_3) - y_3^2(y_4-y_1) + y_4^2(y_3-y_1)) \\
- &(q_1q_3+q_2q_3+q_1q_2)y_3(y_1^2(y_4-y_2) - y_2^2(y_4-y_1) + y_4^2(y_2-y_1)) \\
+ &(q_1q_3+q_2q_3+q_1q_2)y_4(y_1^2(y_3-y_2) - y_2^2(y_3-y_1) + y_3^2(y_2-y_1)))
\end{aligned} \tag{B.21}$$

$$\begin{aligned}
&= (q_1q_3+q_2q_3+q_1q_2)(\\
- &(y_2^2y_1(y_4-y_3) - y_3^2y_1(y_4-y_2) + y_4^2y_1(y_3-y_2)) \\
+ &(y_1^2y_2(y_4-y_3) - y_3^2y_2(y_4-y_1) + y_4^2y_2(y_3-y_1)) \\
- &(y_1^2y_3(y_4-y_2) - y_2^2y_3(y_4-y_1) + y_4^2y_3(y_2-y_1)) \\
+ &(y_1^2y_4(y_3-y_2) - y_2^2y_4(y_3-y_1) + y_3^2y_4(y_2-y_1)))
\end{aligned} \tag{B.22}$$

Again grouping by the squared coefficients:

$$\sum y_j^1 terms \;\; =$$

$$
\begin{aligned}
& (q_1q_3 + q_2q_3 + q_1q_2)(\\
+ \quad & y_1^2y_2(y_4 - y_3) - y_1^2y_3(y_4 - y_2) + y_1^2y_4(y_3 - y_2) \\
- \quad & y_2^2y_4(y_3 - y_1) + y_2^2y_3(y_4 - y_1) - y_2^2y_1(y_4 - y_3) \\
+ \quad & y_3^2y_4(y_2 - y_1) - y_3^2y_2(y_4 - y_1) + y_3^2y_1(y_4 - y_2) \\
- \quad & y_4^2y_1(y_3 - y_2) + y_4^2y_2(y_3 - y_1) - y_4^2y_3(y_2 - y_1)) \qquad \text{(B.23)}
\end{aligned}
$$

$$
\begin{aligned}
& = (q_1q_3 + q_2q_3 + q_1q_2)(\\
+ \quad & y_1^2(y_2(y_4 - y_3) - y_3(y_4 - y_2) + y_4(y_3 - y_2)) \\
+ \quad & y_2^2(-y_4(y_3 - y_1) + y_3(y_4 - y_1) - y_1(y_4 - y_3)) \\
+ \quad & y_3^2(y_4(y_2 - y_1) - y_2(y_4 - y_1) + y_1(y_4 - y_2)) \\
+ \quad & y_4^2(-y_1(y_3 - y_2) + y_2(y_3 - y_1) - y_3(y_2 - y_1))) \qquad \text{(B.24)}
\end{aligned}
$$

Canceling:

$$
\begin{aligned}
& \sum y_j^1 terms = \\
= \quad & (q_1q_3 + q_2q_3 + q_1q_2)(y_1^2(0) + y_2^2(0) + y_3^2(0) + y_4^2(0)) = 0 \qquad \text{(B.25)}
\end{aligned}
$$

So the terms containing $(q_1q_3+q_2q_3+q_1q_2)$ will also cancel out in equation B.20. This leaves us with:

$$
\begin{aligned}
& numerator = \\
- \quad & (y_1^3 - (q_1 + q_2 + q_3)y_1^2)(y_3 - y_2)(y_4 - y_2)(y_4 - y_3) \\
+ \quad & (y_2^3 - (q_1 + q_2 + q_3)y_2^2)(y_3 - y_1)(y_4 - y_1)(y_4 - y_3) \\
- \quad & (y_3^3 - (q_1 + q_2 + q_3)y_3^2)(y_2 - y_1)(y_4 - y_1)(y_4 - y_2) \\
+ \quad & (y_4^3 - (q_1 + q_2 + q_3)y_4^2)(y_2 - y_1)(y_3 - y_1)(y_3 - y_2) \qquad \text{(B.26)}
\end{aligned}
$$

Now we look at just the y_j^2 terms of equation B.12:

$$\sum y_j^2 terms =$$

$$
\begin{aligned}
(\quad + \quad & (q_1+q_2+q_3)y_1^2(y_2^2(y_4-y_3)-y_3^2(y_4-y_2)+y_4^2(y_3-y_2)) \\
- \quad & (q_1+q_2+q_3)y_2^2(y_1^2(y_4-y_3)-y_3^2(y_4-y_1)+y_4^2(y_3-y_1)) \\
+ \quad & (q_1+q_2+q_3)y_3^2(y_1^2(y_4-y_2)-y_2^2(y_4-y_1)+y_4^2(y_2-y_1)) \\
- \quad & (q_1+q_2+q_3)y_4^2(y_1^2(y_3-y_2)-y_2^2(y_3-y_1)+y_3^2(y_2-y_1)))
\end{aligned} \tag{B.27}
$$

$$
\begin{aligned}
& = (q_1+q_2+q_3)(\\
+ \quad & y_1^2y_2^2(y_4-y_3)-y_1^2y_3^2(y_4-y_2)+y_1^2y_4^2(y_3-y_2) \\
- \quad & y_2^2y_1^2(y_4-y_3)+y_2^2y_3^2(y_4-y_1)-y_2^2y_4^2(y_3-y_1) \\
+ \quad & y_3^2y_1^2(y_4-y_2)-y_3^2y_2^2(y_4-y_1)+y_3^2y_4^2(y_2-y_1) \\
- \quad & y_4^2y_1^2(y_3-y_2)+y_4^2y_2^2(y_3-y_1)-y_4^2y_3^2(y_2-y_1))
\end{aligned} \tag{B.28}
$$

$$
\begin{aligned}
= (q_1+q_2+q_3)(& \\
+ \quad & y_1^2y_2^2(y_4-y_3)-y_2^2y_1^2(y_4-y_3) \\
+ \quad & y_1^2y_4^2(y_3-y_2)-y_4^2y_1^2(y_3-y_2) \\
- \quad & y_1^2y_3^2(y_4-y_2)+y_3^2y_1^2(y_4-y_2) \\
+ \quad & y_2^2y_3^2(y_4-y_1)-y_3^2y_2^2(y_4-y_1) \\
- \quad & y_2^2y_4^2(y_3-y_1)+y_4^2y_2^2(y_3-y_1) \\
+ \quad & y_3^2y_4^2(y_2-y_1)-y_4^2y_3^2(y_2-y_1))
\end{aligned} \tag{B.29}
$$

Canceling:

$$
\sum y_j^2 terms = (q_1+q_2+q_3)(0) = 0 \tag{B.30}
$$

So the terms containing $(q_1+q_2+q_3)$ will also cancel and equation B.26 will become:

$$
\begin{aligned}
& numerator = \\
- \quad & y_1^3(y_3-y_2)(y_4-y_2)(y_4-y_3) \\
+ \quad & y_2^3(y_3-y_1)(y_4-y_1)(y_4-y_3)
\end{aligned}
$$

$$
\begin{aligned}
&- \quad y_3^3(y_2 - y_1)(y_4 - y_1)(y_4 - y_2) \\
&+ \quad y_4^3(y_2 - y_1)(y_3 - y_1)(y_3 - y_2)
\end{aligned} \tag{B.31}
$$

By inspection we see that the y_j^3 terms cannot cancel to zero; if we expand the expression y_j^3 only appears once for each j and the coefficient of a given y_j^3 is only zero when $y_j = y_i$ for $i \neq j$. Notice this leaves us with an equation for the 1D magnification that does not explicitly depend on the perturbation locations or sizes.

We can derive an even more interesting result if we continue simplifying. We will find that equation B.31 is in fact equal to the denominator, $(y_2 - y_1)(y_3 - y_1)(y_3 - y_2)(y_4 - y_1)(y_4 - y_2)(y_4 - y_3)$, which we can see by Taylor expanding the denominator with respect to $y = y_1$.

Let $f(y) = (y_2 - y)(y_3 - y)(y_3 - y_2)(y_4 - y)(y_4 - y_2)(y_4 - y_3)$. Instead of just expanding f(y) directly as we did last time, here we must be sneaky to avoid excessive algebra. We will expand F(y), a function defined to be f(y) minus the last three terms in the numerator. Since F(y) is a polynomial, our Taylor expansion will be exact. If we find that F(y) equals the first term in numerator, we will have shown that the denominator f(y) is equal to the numerator.

$$
\begin{aligned}
F(y) = f(y) \quad &- \quad y_4^3(y_2 - y)(y_3 - y)(y_3 - y_2) \\
&+ \quad y_3^3(y_2 - y)(y_4 - y)(y_4 - y_2) \\
&- \quad y_2^3(y_4 - y)(y_3 - y)(y_4 - y_3)
\end{aligned} \tag{B.32}
$$

We find the values of the derivatives of F(y) at y=0:

$$
f(y) = (y_2 - y)(y_3 - y)(y_3 - y_2)(y_4 - y)(y_4 - y_2)(y_4 - y_3) \tag{B.33}
$$

$$
f(0) = y_2 y_3 y_4 (y_3 - y_2)(y_4 - y_2)(y_4 - y_3) \tag{B.34}
$$

$$
\begin{aligned}
F(0) = y_2 y_3 y_4 (y_3 - y_2)(y_4 - y_2)(y_4 - y_3) \\
-y_4^3 y_2 y_3 (y_3 - y_2) + y_3^3 y_2 y_4 (y_4 - y_2) - y_2^3 y_4 y_3 (y_4 - y_3)
\end{aligned} \tag{B.35}
$$

$$\begin{aligned}\frac{df}{dy} = &-(y_3-y)(y_4-y)(y_3-y_2)(y_4-y_2)(y_4-y_3)\\ &-(y_2-y)(y_4-y)(y_3-y_2)(y_4-y_2)(y_4-y_3)\\ &-(y_2-y)(y_3-y)(y_3-y_2)(y_4-y_2)(y_4-y_3)\end{aligned} \tag{B.36}$$

$$\begin{aligned}\frac{dF}{dy} = &-(y_3-y)(y_4-y)(y_3-y_2)(y_4-y_2)(y_4-y_3)\\ &-(y_2-y)(y_4-y)(y_3-y_2)(y_4-y_2)(y_4-y_3)\\ &-(y_2-y)(y_3-y)(y_3-y_2)(y_4-y_2)(y_4-y_3)\\ &+y_4^3(y_3-y)(y_3-y_2)-y_3^3(y_4-y)(y_4-y_2)+y_2^3(y_3-y)(y_4-y_3)\\ &+y_4^3(y_2-y)(y_3-y_2)-y_3^3(y_2-y)(y_4-y_2)+y_2^3(y_4-y)(y_4-y_3)\end{aligned} \tag{B.37}$$

$$\begin{aligned}\frac{dF(0)}{dy} = &-y_3y_4(y_3-y_2)(y_4-y_2)(y_4-y_3)\\ &-y_2y_4(y_3-y_2)(y_4-y_2)(y_4-y_3)\\ &-y_2y_3(y_3-y_2)(y_4-y_2)(y_4-y_3)\\ &+y_4^3y_3(y_3-y_2)-y_3^3y_4(y_4-y_2)+y_2^3y_3(y_4-y_3)\\ &+y_4^3y_2(y_3-y_2)-y_3^3y_2(y_4-y_2)+y_2^3y_4(y_4-y_3)\end{aligned} \tag{B.38}$$

$$\begin{aligned}\frac{dF^2}{dy^2} = &(y_4-y)(y_3-y_2)(y_4-y_2)(y_4-y_3)\\ &+(y_3-y)(y_3-y_2)(y_4-y_2)(y_4-y_3)\\ &+(y_4-y)(y_3-y_2)(y_4-y_2)(y_4-y_3)\\ &+(y_2-y)(y_3-y_2)(y_4-y_2)(y_4-y_3)\\ &+(y_3-y)(y_3-y_2)(y_4-y_2)(y_4-y_3)\\ &+(y_2-y)(y_3-y_2)(y_4-y_2)(y_4-y_3)\\ &+y_4^3(y_3-y_2)-y_3^3(y_4-y_2)+y_2^3(y_4-y_3)\\ &+y_4^3(y_3-y_2)-y_3^3(y_4-y_2)+y_2^3(y_4-y_3)\end{aligned} \tag{B.39}$$

$$\frac{d^2F}{dy^2} = 2(y_4 - y)(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)$$
$$+2(y_3 - y)(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)$$
$$+2(y_2 - y)(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)$$
$$+2y_4^3(y_3 - y_2) - y_3^3(y_4 - y_2) + y_2^3(y_4 - y_3) \tag{B.40}$$

$$\frac{d^2F(0)}{dy^2} = 2y_4(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)$$
$$+2y_3(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)$$
$$+2y_2(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)$$
$$+2y_4^3(y_3 - y_2) - y_3^3(y_4 - y_2) + y_2^3(y_4 - y_3) \tag{B.41}$$

$$\frac{d^3F}{dy^3} = -6(y_3 - y_2)(y_4 - y_2)(y_4 - y_3) \tag{B.42}$$

Using the Taylor expansion formula and our results from equations B.35, B.38, B.38 and B.38 we write out a Maclaurin series for Fy:

$$G(y) = [y_2y_3y_4(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)$$
$$-y_4^3y_2y_3(y_3 - y_2) + y_3^3y_2y_4(y_4 - y_2) - y_2^3y_4y_3(y_4 - y_3)]$$
$$+y[-y_3y_4(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)$$
$$-y_2y_4(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)$$
$$-y_2y_3(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)$$
$$+y_4^3y_3(y_3 - y_2) - y_3^3y_4(y_4 - y_2) + y_2^3y_3(y_4 - y_3)$$
$$+y_4^3y_2(y_3 - y_2) - y_3^3y_2(y_4 - y_2) + y_2^3y_4(y_4 - y_3)]$$
$$+y^2/2[2y_4(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)$$
$$+2y_3(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)$$
$$+2y_2(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)$$
$$+2y_4^3(y_3 - y_2) - y_3^3(y_4 - y_2) + y_2^3(y_4 - y_3)]$$

$$+y^3/6[-6(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)] \tag{B.43}$$

$$\begin{aligned}
G(y) = & [-y_2y_3y_4(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
& -y_2y_3y_4(y_4^2(y_3 - y_2) + y_3^2(y_2 - y_4) + y_2^2(y_4 - y_3))] \\
& -y[y_3y_4(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
& +y_2y_4(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
& +y_2y_3(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
& +y_4^3y_3(y_3 - y_2) + y_3^3y_4(y_2 - y_4) + y_2^3y_3(y_4 - y_3) \\
& +y_4^3y_2(y_3 - y_2) + y_3^3y_2(y_2 - y_4) + y_2^3y_4(y_4 - y_3)] \\
& +y^2[-y_4(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
& -y_3(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
& -y_2(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
& +y_4^3(y_3 - y_2) + y_3^3(y_2 - y_4) + y_2^3(y_4 - y_3)] \\
& +y^3[(y_3 - y_2)(y_2 - y_4)(y_4 - y_3)]
\end{aligned} \tag{B.44}$$

Plugging y_1 back in for y and looking at just the coefficients of y_1^0:

$$\begin{aligned}
& [-y_2y_3y_4(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
& -y_2y_3y_4(y_4^2(y_3 - y_2) + y_3^2(y_2 - y_4) + y_2^2(y_4 - y_3))]
\end{aligned} \tag{B.45}$$

$$\begin{aligned}
& = [-y_2y_3y_4(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
& -y_2y_3y_4(-(y_3 - y_2)(y_2 - y_4)(y_4 - y_3))]
\end{aligned} \tag{B.46}$$

$$= 0 \tag{B.47}$$

(Here we used relation B.10) So now G(y) is:

$$\begin{aligned}
G(y) = & -y_1[y_3y_4(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
& +y_2y_4(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
& +y_2y_3(y_3 - y_2)(y_2 - y_4)(y_4 - y_3)
\end{aligned}$$

$$
\begin{aligned}
&+y_4^3 y_3(y_3-y_2)+y_3^3 y_4(y_2-y_4)+y_2^3 y_3(y_4-y_3)\\
&+y_4^3 y_2(y_3-y_2)+y_3^3 y_2(y_2-y_4)+y_2^3 y_4(y_4-y_3)]\\
&+y_1^2[-y_4(y_3-y_2)(y_2-y_4)(y_4-y_3)\\
&-y_3(y_3-y_2)(y_2-y_4)(y_4-y_3)\\
&-y_2(y_3-y_2)(y_2-y_4)(y_4-y_3)\\
&+y_4^3(y_3-y_2)+y_3^3(y_2-y_4)+y_2^3(y_4-y_3)]\\
&+y_1^3[(y_3-y_2)(y_2-y_4)(y_4-y_3)] \qquad \text{(B.48)}
\end{aligned}
$$

Looking at just the coefficients of y_1^1:

$$
\begin{aligned}
- & \ y_1[y_3 y_4(y_3-y_2)(y_2-y_4)(y_4-y_3)\\
+ & \ y_2 y_4(y_3-y_2)(y_2-y_4)(y_4-y_3)\\
+ & \ y_2 y_3(y_3-y_2)(y_2-y_4)(y_4-y_3)\\
+ & \ y_4^3 y_3(y_3-y_2)+y_3^3 y_4(y_2-y_4)+y_2^3 y_3(y_4-y_3)\\
+ & \ y_4^3 y_2(y_3-y_2)+y_3^3 y_2(y_2-y_4)+y_2^3 y_4(y_4-y_3)]
\end{aligned}
\qquad \text{(B.49)}
$$

$$
\begin{aligned}
= - & \ y_1[(y_2 y_3+y_3 y_4+y_4 y_2)(y_3-y_2)(y_2-y_4)(y_4-y_3)\\
+ & \ (y_4 y_2+y_4 y_3)y_4^2(y_3-y_2)+(y_3 y_2+y_3 y_4)y_3^2(y_2-y_4)\\
+ & \ (y_2 y_3+y_2 y_4)y_2^3(y_4-y_3)] \qquad \text{(B.50)}
\end{aligned}
$$

$$
\begin{aligned}
= - & \ y_1[(-y_2 y_3-y_3 y_4-y_4 y_2)(y_4^2(y_3-y_2)+y_3^2(y_2-y_4)+y_2^2(y_4-y_3))\\
+ & \ (y_4 y_2+y_4 y_3)y_4^2(y_3-y_2)+(y_3 y_2+y_3 y_4)y_3^2(y_2-y_4)\\
+ & \ (y_2 y_3+y_2 y_4)y_2^3(y_4-y_3)] \qquad \text{(B.51)}
\end{aligned}
$$

$$
\begin{aligned}
- & \ y_1[(-y_2 y_3-y_3 y_4-y_4 y_2)(y_4^2(y_3-y_2)+y_3^2(y_2-y_4)+y_2^2(y_4-y_3))\\
+ & \ (y_4 y_2+y_4 y_3)y_4^2(y_3-y_2)+(y_3 y_2+y_3 y_4)y_3^2(y_2-y_4)+(y_2 y_3+y_2 y_4)y_2^3(y_4-y_3)] \qquad \text{(B.52)}\\
= - & \ y_1[(-y_2 y_3-y_3 y_4-y_4 y_2)y_4^2(y_3-y_2)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad (-y_2y_3 - y_3y_4 - y_4y_2)y_3^2(y_2 - y_4) \\
&+ \quad (-y_2y_3 - y_3y_4 - y_4y_2)y_2^2(y_4 - y_3) \\
&+ \quad (y_4y_2 + y_4y_3)y_4^2(y_3 - y_2) + (y_3y_2 + y_3y_4)y_3^2(y_2 - y_4) + (y_2y_3 + y_2y_4)y_2^3(y_4 - y_3)]
\end{aligned}
\tag{B.53}
$$

$$
\begin{aligned}
= \quad - \quad & y_1[(-y_2y_3)y_4^2(y_3 - y_2) \\
+ \quad & (-y_4y_2)y_3^2(y_2 - y_4) \\
+ \quad & (-y_3y_4)y_2^2(y_4 - y_3)]
\end{aligned}
\tag{B.54}
$$

$$
= \quad - \quad y_1[-y_2y_3^2y_4^2 + y_2^2y_3y_4^2 - y_4y_2^2y_3^2 + y_4^2y_2y_3^2 - y_3y_4^2y_2^2 + y_3^2y_4y_2^2] \tag{B.55}
$$

$$
= \quad 0 \tag{B.56}
$$

So now G(y) is:

$$
\begin{aligned}
G(y) = \quad + \quad & y_1^2[-y_4(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
- \quad & y_3(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
- \quad & y_2(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
+ \quad & y_4^3(y_3 - y_2) + y_3^3(y_2 - y_4) + y_2^3(y_4 - y_3)] \\
+ \quad & y_1^3[(y_3 - y_2)(y_2 - y_4)(y_4 - y_3)]
\end{aligned}
\tag{B.57}
$$

Finally, looking at just the coefficients of y_1^2:

$$
\begin{aligned}
y_1^2terms = y_1^2[\quad - \quad & y_4(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
- \quad & y_3(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
- \quad & y_2(y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
+ \quad & y_4^3(y_3 - y_2) + y_3^3(y_2 - y_4) + y_2^3(y_4 - y_3)]
\end{aligned}
\tag{B.58}
$$

$$
\begin{aligned}
= y_1^2[(-y_4 - y_3 - y_2) \quad \times \quad & (y_3 - y_2)(y_2 - y_4)(y_4 - y_3) \\
+ \quad & y_4^3(y_3 - y_2) + y_3^3(y_2 - y_4) + y_2^3(y_4 - y_3)]
\end{aligned}
\tag{B.59}
$$

$$
\begin{aligned}
= y_1^2[(y_4+y_3+y_2) &\times (y_4^2(y_3-y_2)+y_3^2(y_2-y_4)+y_2^2(y_4-y_3)) \\
&+ y_4^3(y_3-y_2)+y_3^3(y_2-y_4)+y_2^3(y_4-y_3)]
\end{aligned} \tag{B.60}
$$

$$
\begin{aligned}
= &\quad y_1^2[(y_4+y_3+y_2)y_4^2(y_3-y_2)+(y_4+y_3+y_2)y_3^2(y_2-y_4) \\
&+ (y_4+y_3+y_2)y_2^2(y_4-y_3)) \\
&+ y_4^3(y_3-y_2)+y_3^3(y_2-y_4)+y_2^3(y_4-y_3)]
\end{aligned} \tag{B.61}
$$

$$
\begin{aligned}
= y_1^2[&+ y_4^3(y_3-y_2)+(y_3+y_2)y_4^2(y_3-y_2) \\
&+ y_3^3(y_2-y_4)+(y_4+y_2)y_3^2(y_2-y_4) \\
&+ y_2^3(y_4-y_3)+(y_4+y_3)y_2^2(y_4-y_3)) \\
&+ y_4^3(y_3-y_2)+y_3^3(y_2-y_4)+y_2^3(y_4-y_3)]
\end{aligned} \tag{B.62}
$$

$$
\begin{aligned}
= y_1^2[&- y_4^3(y_3-y_2)-y_3^3(y_2-y_4)-y_2^3(y_4-y_3) \\
&+ y_4^3(y_3-y_2)+y_3^3(y_2-y_4)+y_2^3(y_4-y_3)]
\end{aligned} \tag{B.63}
$$

$$
= 0
$$

Thus the coefficient of y_1^2 is also 0, and G(y)=F(y) is simply

$$
F(y) = y_1^3[(y_3-y_2)(y_2-y_4)(y_4-y_3)], \tag{B.64}
$$

which is the fourth term in the numerator. Thus:

$$
\begin{aligned}
F(y) &= y_1^3[(y_3-y_2)(y_2-y_4)(y_4-y_3)] \\
&= f(y)-y_4^3(y_2-y_1)(y_3-y_1)(y_3-y_2) \\
&\quad +y_3^3(y_2-y_1)(y_4-y_1)(y_4-y_2) \\
&\quad -y_2^3(y_4-y_1)(y_3-y_1)(y_4-y_3)
\end{aligned} \tag{B.65}
$$

$$f(y) =$$

$$\begin{aligned} & y_4^3(y_2 - y_1)(y_3 - y_1)(y_3 - y_2) \\ & -y_3^3(y_2 - y_1)(y_4 - y_1)(y_4 - y_2) \\ & +y_2^3(y_4 - y_1)(y_3 - y_1)(y_4 - y_3) \\ & -y_1^3[(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)] \end{aligned} \tag{B.66}$$

Which proves our original claim and finishes the proof that the total 1D magnification is equal to 1 for the case of three perturbations.

Appendix C

Total One-Dimensional Magnification: Four Perturbation Case

Here we prove that the sum of the one dimensional magnification along the axis of curvature is 1 for 4 perturbations. To prove this we find μ_{total}, show that all the terms involving perturbation location, q, will cancel out, and then show that the remaining terms sum to 1.

First we find μ_{total}. With four perturbations ($n = 4$) equation 3.15 becomes:

$$\begin{aligned}
&\left[1 - \frac{\partial^2 \psi_1}{\partial y^2}\right] = \\
+ \quad &\frac{(y-y_2)(y-y_3)(y-y_4)(y-y_5)}{(y-q_1)(y-q_2)(y-q_3)(y-q_4)} + \frac{(y-y_1)(y-y_3)(y-y_4)(y-y_5)}{(y-q_1)(y-q_2)(y-q_3)(y-q_4)} \\
+ \quad &\frac{(y-y_1)(y-y_2)(y-y_4)(y-y_5)}{(y-q_1)(y-q_2)(y-q_3)(y-q_4)} + \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_5)}{(y-q_1)(y-q_2)(y-q_3)(y-q_4)} \\
+ \quad &\frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y-q_1)(y-q_2)(y-q_3)(y-q_4)}
\end{aligned} \tag{C.1}$$

To find $\mu_{1D,total}$ we evaluate the inverse of equation ?? for image locations y_1, y_2, y_3,

y_4 and y_5 and sum the results. We get:

$$\begin{aligned}
\mu_{1D,total} &= \sum_{i=1}^{5}\left[1-\frac{\partial^2\psi_1}{\partial y^2}(x_i,y_i)\right]^{-1} \\
&= \frac{((y_1-q_1)(y_1-q_2)(y_1-q_3)(y_1-q_4))}{(y_1-y_2)(y_1-y_3)(y_1-y_4)(y_1-y_5)} \\
&+ \frac{((y_2-q_1)(y_2-q_2)(y_2-q_3)(y_2-q_4))}{(y_2-y_1)(y_2-y_3)(y_2-y_4)(y_2-y_5)} \\
&+ \frac{((y_3-q_1)(y_3-q_2)(y_3-q_3)(y_3-q_4))}{(y_3-y_1)(y_3-y_2)(y_3-y_4)(y_3-y_5)} \\
&+ \frac{((y_4-q_1)(y_4-q_2)(y_4-q_3)(y_4-q_4))}{(y_4-y_1)(y_4-y_2)(y_4-y_3)(y_4-y_5)} \\
&+ \frac{((y_5-q_1)(y_5-q_2)(y_5-q_3)(y_5-q_4))}{((y_5-y_1)(y_5-y_2)(y_5-y_3)(y_5-y_4))} \qquad \text{(C.2)}
\end{aligned}$$

Rewriting:

$$\begin{aligned}
\mu_{1D,total} &= \\
&= \frac{((y_1-q_1)(y_1-q_2)(y_1-q_3)(y_1-q_4))}{(y_5-y_1)(y_4-y_1)(y_3-y_1)(y_2-y_1)} \\
&+ \frac{((y_2-q_1)(y_2-q_2)(y_2-q_3)(y_2-q_4))}{-(y_5-y_2)(y_4-y_2)(y_3-y_2)(y_2-y_1)} \\
&+ \frac{((y_3-q_1)(y_3-q_2)(y_3-q_3)(y_3-q_4))}{(y_5-y_3)(y_4-y_3)(y_3-y_2)(y_3-y_1)} \\
&+ \frac{((y_4-q_1)(y_4-q_2)(y_4-q_3)(y_4-q_4))}{-(y_5-y_4)(y_4-y_3)(y_4-y_2)(y_4-y_1)} \\
&+ \frac{((y_5-q_1)(y_5-q_2)(y_5-q_3)(y_5-q_4))}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)} \qquad \text{(C.3)}
\end{aligned}$$

Defining $(y_i-q_1)(y_i-q_2)(y_i-q_3)(y_i-q_4) \equiv \varrho(y_i)$ and combining these fractions we get:

$$\begin{aligned}
&\mu_{1D,total} = \\
&+ \frac{\varrho(y_1)(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_4-y_3)(y_4-y_2)(y_3-y_2)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
&- \frac{\varrho(y_2)(y_5-y_4)(y_5-y_3)(y_5-y_1)(y_4-y_3)(y_4-y_1)(y_3-y_1)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
&+ \frac{\varrho(y_3)(y_5-y_4)(y_5-y_2)(y_5-y_1)(y_4-y_2)(y_4-y_1)(y_2-y_1)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)}
\end{aligned}$$

$$- \frac{\varrho(y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)}$$
$$+ \frac{\varrho(y_5)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \quad (C.4)$$

We add up these fractions and look at just the numerator:

$$\begin{aligned} numerator = &+ \varrho(y_1)(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_4-y_3)(y_4-y_2)(y_3-y_2) \\ &- \varrho(y_2)(y_5-y_4)(y_5-y_3)(y_5-y_1)(y_4-y_3)(y_4-y_1)(y_3-y_1) \\ &+ \varrho(y_3)(y_5-y_4)(y_5-y_2)(y_5-y_1)(y_4-y_2)(y_4-y_1)(y_2-y_1) \\ &- \varrho(y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1) \\ &+ \varrho(y_5)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1) \end{aligned} \quad (C.5)$$

In the case of three perturbations we proved that:

$$\begin{aligned} &(d-c)(d-b)(d-a)(c-b)(c-a)(b-a) \\ = &- a^3(c-b)(d-b)(d-c) \\ &+ b^3(c-a)(d-a)(d-c) \\ &- c^3(b-a)(d-a)(d-b) \\ &+ d^3(b-a)(c-a)(c-b) \end{aligned} \quad (C.6)$$

we can use this relation to expand the numerator:

$$\begin{aligned} &numerator = \\ &\varrho(y_1)[- y_2^3(y_4-y_3)(y_5-y_3)(y_5-y_4) \\ &\quad + y_3^3(y_4-y_2)(y_5-y_2)(y_5-y_4) \\ &\quad - y_4^3(y_3-y_2)(y_5-y_2)(y_5-y_3) \\ &\quad + y_5^3(y_3-y_2)(y_4-y_2)(y_4-y_3)] \end{aligned} \quad (C.7)$$

$$-\varrho(y_2)[- y_1^3(y_4-y_3)(y_5-y_3)(y_5-y_4)$$

$$
\begin{aligned}
&+ \; y_3^3(y_4-y_1)(y_5-y_1)(y_5-y_4)\\
&- \; y_4^3(y_3-y_1)(y_5-y_1)(y_5-y_3)\\
&+ \; y_5^3(y_3-y_1)(y_4-y_1)(y_4-y_3)]
\end{aligned}
\tag{C.8}
$$

$$
\begin{aligned}
+\varrho(y_3)[\; &- \; y_1^3(y_4-y_2)(y_5-y_2)(y_5-y_4)\\
&+ \; y_2^3(y_4-y_1)(y_5-y_1)(y_5-y_4)\\
&- \; y_4^3(y_2-y_1)(y_5-y_1)(y_5-y_2)\\
&+ \; y_5^3(y_2-y_1)(y_4-y_1)(y_4-y_2)]
\end{aligned}
\tag{C.9}
$$

$$
\begin{aligned}
-\varrho(y_4)[\; &- \; y_1^3(y_3-y_2)(y_5-y_2)(y_5-y_3)\\
&+ \; y_2^3(y_3-y_1)(y_5-y_1)(y_5-y_3)\\
&- \; y_3^3(y_2-y_1)(y_5-y_1)(y_5-y_2)\\
&+ \; y_5^3(y_2-y_1)(y_3-y_1)(y_3-y_2)]
\end{aligned}
\tag{C.10}
$$

$$
\begin{aligned}
+\varrho(y_5)[\; &- \; y_1^3(y_3-y_2)(y_4-y_2)(y_4-y_3)\\
&+ \; y_2^3(y_3-y_1)(y_4-y_1)(y_4-y_3)\\
&- \; y_3^3(y_2-y_1)(y_4-y_1)(y_4-y_2)\\
&+ \; y_4^3(y_2-y_1)(y_3-y_1)(y_3-y_2)]
\end{aligned}
\tag{C.11}
$$

We can write it in even simpler components using the relation we proved for two perturbations:

$$
\begin{aligned}
&(a-b)(c-b)(c-a)\\
&= -a^2(c-b)+b^2(c-a)+c^2(a-b)
\end{aligned}
\tag{C.12}
$$

$$
numerator =
$$

$$
\varrho(y_1)[\; - \; y_2^3(y_5^2(y_4-y_3)-y_4^2(y_5-y_3)+y_3^2(y_5-y_4))
$$

$$
\begin{aligned}
&+ \quad y_3^3(y_5^2(y_4-y_2)-y_4^2(y_5-y_2)+y_2^2(y_5-y_4)) \\
&- \quad y_4^3(y_5^2(y_3-y_2)-y_3^2(y_5-y_2)+y_2^2(y_5-y_3)) \\
&+ \quad y_5^3(y_4^2(y_3-y_2)-y_3^2(y_4-y_2)+y_2^2(y_4-y_3))]
\end{aligned} \tag{C.13}
$$

$$
\begin{aligned}
-\varrho(y_2)[\quad &- \quad y_1^3(y_5^2(y_4-y_3)-y_4^2(y_5-y_3)+y_3^2(y_5-y_4)) \\
&+ \quad y_3^3(y_5^2(y_4-y_1)-y_4^2(y_5-y_1)+y_1^2(y_5-y_4)) \\
&- \quad y_4^3(y_5^2(y_3-y_1)-y_3^2(y_5-y_1)+y_1^2(y_5-y_3)) \\
&+ \quad y_5^3(y_4^2(y_3-y_1)-y_3^2(y_4-y_1)+y_1^2(y_4-y_3))]
\end{aligned} \tag{C.14}
$$

$$
\begin{aligned}
+\varrho(y_3)[\quad &- \quad y_1^3(y_5^2(y_4-y_2)-y_4^2(y_5-y_2)+y_2^2(y_5-y_4)) \\
&+ \quad y_2^3(y_5^2(y_4-y_1)-y_4^2(y_5-y_1)+y_1^2(y_5-y_4)) \\
&- \quad y_4^3(y_5^2(y_2-y_1)-y_2^2(y_5-y_1)+y_1^2(y_5-y_2)) \\
&+ \quad y_5^3(y_4^2(y_2-y_1)-y_2^2(y_4-y_1)+y_1^2(y_4-y_2))]
\end{aligned} \tag{C.15}
$$

$$
\begin{aligned}
-\varrho(y_4)[\quad &- \quad y_1^3(y_5^2(y_3-y_2)-y_3^2(y_5-y_2)+y_2^2(y_5-y_3)) \\
&+ \quad y_2^3(y_5^2(y_3-y_1)-y_3^2(y_5-y_1)+y_1^2(y_5-y_3)) \\
&- \quad y_3^3(y_5^2(y_2-y_1)-y_2^2(y_5-y_1)+y_1^2(y_5-y_2)) \\
&+ \quad y_5^3(y_3^2(y_2-y_1)-y_2^2(y_3-y_1)+y_1^2(y_3-y_2))]
\end{aligned} \tag{C.16}
$$

$$
\begin{aligned}
+\varrho(y_5)[\quad &- \quad y_1^3(y_4^2(y_3-y_2)-y_3^2(y_4-y_2)+y_2^2(y_4-y_3)) \\
&+ \quad y_2^3(y_4^2(y_3-y_1)-y_3^2(y_4-y_1)+y_1^2(y_4-y_3)) \\
&- \quad y_3^3(y_4^2(y_2-y_1)-y_2^2(y_4-y_1)+y_1^2(y_4-y_2)) \\
&+ \quad y_4^3(y_3^2(y_2-y_1)-y_2^2(y_3-y_1)+y_1^2(y_3-y_2))]
\end{aligned} \tag{C.17}
$$

Now that we have an expression for the numerator of μ_{total}, we can show that all the terms involving perturbation location, q, will cancel out. Looking at $\varrho(y_i)$, we see that $\varrho(y_i) \equiv (y_i - q_1)(y_i - q_2)(y_i - q_3)(y_i - q_4)$ has terms of order y_i^0, y_i^1, y_i^2, y_i^3, and

y_i^4. To prove that the sum of the one dimensional magnifications is one, we must look at these groups of terms one at a time.

First, looking at just the y_i^0 terms we find:

$$
\begin{aligned}
\sum y_i^0 terms = & \\
+[\ - \ & y_2^3(y_5^2(y_4-y_3) - y_4^2(y_5-y_3) + y_3^2(y_5-y_4)) \\
+ \ & y_3^3(y_5^2(y_4-y_2) - y_4^2(y_5-y_2) + y_2^2(y_5-y_4)) \\
- \ & y_4^3(y_5^2(y_3-y_2) - y_3^2(y_5-y_2) + y_2^2(y_5-y_3)) \\
+ \ & y_5^3(y_4^2(y_3-y_2) - y_3^2(y_4-y_2) + y_2^2(y_4-y_3))]
\end{aligned} \tag{C.18}
$$

$$
\begin{aligned}
+[\ + \ & y_1^3(y_5^2(y_4-y_3) - y_4^2(y_5-y_3) + y_3^2(y_5-y_4)) \\
- \ & y_3^3(y_5^2(y_4-y_1) - y_4^2(y_5-y_1) + y_1^2(y_5-y_4)) \\
+ \ & y_4^3(y_5^2(y_3-y_1) - y_3^2(y_5-y_1) + y_1^2(y_5-y_3)) \\
- \ & y_5^3(y_4^2(y_3-y_1) - y_3^2(y_4-y_1) + y_1^2(y_4-y_3))]
\end{aligned} \tag{C.19}
$$

$$
\begin{aligned}
+[\ - \ & y_1^3(y_5^2(y_4-y_2) - y_4^2(y_5-y_2) + y_2^2(y_5-y_4)) \\
+ \ & y_2^3(y_5^2(y_4-y_1) - y_4^2(y_5-y_1) + y_1^2(y_5-y_4)) \\
- \ & y_4^3(y_5^2(y_2-y_1) - y_2^2(y_5-y_1) + y_1^2(y_5-y_2)) \\
+ \ & y_5^3(y_4^2(y_2-y_1) - y_2^2(y_4-y_1) + y_1^2(y_4-y_2))]
\end{aligned} \tag{C.20}
$$

$$
\begin{aligned}
+[\ + \ & y_1^3(y_5^2(y_3-y_2) - y_3^2(y_5-y_2) + y_2^2(y_5-y_3)) \\
- \ & y_2^3(y_5^2(y_3-y_1) - y_3^2(y_5-y_1) + y_1^2(y_5-y_3)) \\
+ \ & y_3^3(y_5^2(y_2-y_1) - y_2^2(y_5-y_1) + y_1^2(y_5-y_2)) \\
- \ & y_5^3(y_3^2(y_2-y_1) - y_2^2(y_3-y_1) + y_1^2(y_3-y_2))]
\end{aligned} \tag{C.21}
$$

$$
\begin{aligned}
+[\ - \ & y_1^3(y_4^2(y_3-y_2) - y_3^2(y_4-y_2) + y_2^2(y_4-y_3)) \\
+ \ & y_2^3(y_4^2(y_3-y_1) - y_3^2(y_4-y_1) + y_1^2(y_4-y_3))
\end{aligned}
$$

$$
\begin{aligned}
&- \quad y_3^3(y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2)) \\
&+ \quad y_4^3(y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2))] \qquad \text{(C.22)}
\end{aligned}
$$

Grouping the terms by the y_i^3 factor in front (henceforth called a "leading factor"), we get:

$$
\begin{aligned}
\sum y_i^0 terms = &+ \quad y_1^3(y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4)) \\
&- \quad y_1^3(y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4)) \\
&+ \quad y_1^3(y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3)) \\
&- \quad y_1^3(y_4^2(y_3 - y_2) - y_3^2(y_4 - y_2) + y_2^2(y_4 - y_3)) \\
&- \quad y_2^3(y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4)) \\
&+ \quad y_2^3(y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4)) \\
&- \quad y_2^3(y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3)) \\
&+ \quad y_2^3(y_4^2(y_3 - y_1) - y_3^2(y_4 - y_1) + y_1^2(y_4 - y_3)) \\
&+ \quad y_3^3(y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4)) \\
&- \quad y_3^3(y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4)) \\
&+ \quad y_3^3(y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2)) \\
&- \quad y_3^3(y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2)) \\
&- \quad y_4^3(y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3)) \\
&+ \quad y_4^3(y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3)) \\
&- \quad y_4^3(y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2)) \\
&+ \quad y_4^3(y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2)) \\
&+ \quad y_5^3(y_4^2(y_3 - y_2) - y_3^2(y_4 - y_2) + y_2^2(y_4 - y_3)) \\
&- \quad y_5^3(y_4^2(y_3 - y_1) - y_3^2(y_4 - y_1) + y_1^2(y_4 - y_3)) \\
&+ \quad y_5^3(y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2)) \\
&- \quad y_5^3(y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2)) \qquad \text{(C.23)}
\end{aligned}
$$

$$\begin{aligned}
\sum y_i^0 terms = y_1^3(&+ y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4) \\
&- y_5^2(y_4 - y_2) + y_4^2(y_5 - y_2) - y_2^2(y_5 - y_4) \\
&+ y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3) \\
&- y_4^2(y_3 - y_2) + y_3^2(y_4 - y_2) - y_2^2(y_4 - y_3)) \\
+y_2^3(&- y_5^2(y_4 - y_3) + y_4^2(y_5 - y_3) - y_3^2(y_5 - y_4) \\
&+ y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4) \\
&- y_5^2(y_3 - y_1) + y_3^2(y_5 - y_1) - y_1^2(y_5 - y_3) \\
&+ y_4^2(y_3 - y_1) - y_3^2(y_4 - y_1) + y_1^2(y_4 - y_3)) \\
+y_3^3(&+ y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4) \\
&- y_5^2(y_4 - y_1) + y_4^2(y_5 - y_1) - y_1^2(y_5 - y_4) \\
&+ y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2) \\
&- y_4^2(y_2 - y_1) + y_2^2(y_4 - y_1) - y_1^2(y_4 - y_2)) \\
+y_4^3(&- y_5^2(y_3 - y_2) + y_3^2(y_5 - y_2) - y_2^2(y_5 - y_3) \\
&+ y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3) \\
&- y_5^2(y_2 - y_1) + y_2^2(y_5 - y_1) - y_1^2(y_5 - y_2) \\
&+ y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2)) \\
+y_5^3(&+ y_4^2(y_3 - y_2) - y_3^2(y_4 - y_2) + y_2^2(y_4 - y_3) \\
&- y_4^2(y_3 - y_1) + y_3^2(y_4 - y_1) - y_1^2(y_4 - y_3) \\
&+ y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2) \\
&- y_3^2(y_2 - y_1) + y_2^2(y_3 - y_1) - y_1^2(y_3 - y_2)) \qquad \text{(C.24)}
\end{aligned}$$

Canceling:

$$\begin{aligned}
\sum y_i^0 terms = y_1^3(&+ y_5^2(0) - y_4^2(0) + y_3^2(0) \\
&- y_5^2(0) + y_4^2(0) - y_2^2(0) \\
&+ y_5^2(0) - y_3^2(0) + y_2^2(0) \\
&- y_4^2(0) + y_3^2(0) - y_2^2(0))
\end{aligned}$$

$$
\begin{aligned}
+y_2^3(& - y_5^2(0) + y_4^2(0) - y_3^2(0) \\
& + y_5^2(0) - y_4^2(0) + y_1^2(0) \\
& - y_5^2(0) + y_3^2(0) - y_1^2(0) \\
& + y_4^2(0) - y_3^2(0) + y_1^2(0)) \\
+y_3^3(& + y_5^2(0) - y_4^2(0) + y_2^2(0) \\
& - y_5^2(0) + y_4^2(0) - y_1^2(0) \\
& + y_5^2(0) - y_2^2(0) + y_1^2(0) \\
& - y_4^2(0) + y_2^2(0) - y_1^2(0)) \\
+y_4^3(& - y_5^2(0) + y_3^2(0) - y_2^2(0) \\
& + y_5^2(0) - y_3^2(0) + y_1^2(0) \\
& - y_5^2(0) + y_2^2(0) - y_1^2(0) \\
& + y_3^2(0) - y_2^2(0) + y_1^2(0)) \\
+y_5^3(& + y_4^2(0) - y_3^2(0) + y_2^2(0) \\
& - y_4^2(0) + y_3^2(0) - y_1^2(0) \\
& + y_4^2(0) - y_2^2(0) + y_1^2(0) \\
& - y_3^2(0) + y_2^2(0) - y_1^2(0)) = 0 \qquad \text{(C.25)}
\end{aligned}
$$

So the y_i^0 terms will all cancel out.

Repeating this process for the y_i^1 terms we get:

$$
\begin{aligned}
& \sum y_i^1 terms = \\
y_1[& - y_2^3(y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4)) \\
& + y_3^3(y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4)) \\
& - y_4^3(y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3)) \\
& + y_5^3(y_4^2(y_3 - y_2) - y_3^2(y_4 - y_2) + y_2^2(y_4 - y_3))] \\
\\
+y_2[& + y_1^3(y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4))
\end{aligned}
$$

$$
\begin{aligned}
&- \; y_3^3(y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4)) \\
&+ \; y_4^3(y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3)) \\
&- \; y_5^3(y_4^2(y_3 - y_1) - y_3^2(y_4 - y_1) + y_1^2(y_4 - y_3))]
\end{aligned}
$$

$$
\begin{aligned}
+y_3[\; &- \; y_1^3(y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4)) \\
&+ \; y_2^3(y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4)) \\
&- \; y_4^3(y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2)) \\
&+ \; y_5^3(y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2))]
\end{aligned}
$$

$$
\begin{aligned}
+y_4[\; &+ \; y_1^3(y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3)) \\
&- \; y_2^3(y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3)) \\
&+ \; y_3^3(y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2)) \\
&- \; y_5^3(y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2))]
\end{aligned}
$$

$$
\begin{aligned}
+y_5[\; &- \; y_1^3(y_4^2(y_3 - y_2) - y_3^2(y_4 - y_2) + y_2^2(y_4 - y_3)) \\
&+ \; y_2^3(y_4^2(y_3 - y_1) - y_3^2(y_4 - y_1) + y_1^2(y_4 - y_3)) \\
&- \; y_3^3(y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2)) \\
&+ \; y_4^3(y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2))]
\end{aligned}
\tag{C.26}
$$

If we look at this expression we notice that within the square brackets the largest possible power of y_i for a given i is three. Furthermore, the y_j^1 contribution of from the $\varrho(y_j)$ term outside the square brackets will never increase this power. Since the y_j^1 factor will never be multiplied with a y_j^2 factor, the only source of third power terms for a given subscript is the leading terms. Thus, it makes sense to group the expression by leading term as we did for the y_j^0 case:

$$
\begin{aligned}
\sum y_i^1 terms = [\; &- \; y_2^3(y_1 y_5^2(y_4 - y_3) - y_1 y_4^2(y_5 - y_3) + y_1 y_3^2(y_5 - y_4)) \\
&+ \; y_3^3(y_1 y_5^2(y_4 - y_2) - y_1 y_4^2(y_5 - y_2) + y_1 y_2^2(y_5 - y_4))
\end{aligned}
$$

$$
\begin{aligned}
- \quad & y_4^3(y_1y_5^2(y_3-y_2) - y_1y_3^2(y_5-y_2) + y_1y_2^2(y_5-y_3)) \\
+ \quad & y_5^3(y_1y_4^2(y_3-y_2) - y_1y_3^2(y_4-y_2) + y_1y_2^2(y_4-y_3))]
\end{aligned}
$$

$$
\begin{aligned}
+[\quad + \quad & y_1^3(y_2y_5^2(y_4-y_3) - y_2y_4^2(y_5-y_3) + y_2y_3^2(y_5-y_4)) \\
- \quad & y_3^3(y_2y_5^2(y_4-y_1) - y_2y_4^2(y_5-y_1) + y_2y_1^2(y_5-y_4)) \\
+ \quad & y_4^3(y_2y_5^2(y_3-y_1) - y_2y_3^2(y_5-y_1) + y_2y_1^2(y_5-y_3)) \\
- \quad & y_5^3(y_2y_4^2(y_3-y_1) - y_2y_3^2(y_4-y_1) + y_2y_1^2(y_4-y_3))]
\end{aligned}
$$

$$
\begin{aligned}
+[\quad - \quad & y_1^3(y_3y_5^2(y_4-y_2) - y_3y_4^2(y_5-y_2) + y_3y_2^2(y_5-y_4)) \\
+ \quad & y_2^3(y_3y_5^2(y_4-y_1) - y_3y_4^2(y_5-y_1) + y_3y_1^2(y_5-y_4)) \\
- \quad & y_4^3(y_3y_5^2(y_2-y_1) - y_3y_2^2(y_5-y_1) + y_3y_1^2(y_5-y_2)) \\
+ \quad & y_5^3(y_3y_4^2(y_2-y_1) - y_3y_2^2(y_4-y_1) + y_3y_1^2(y_4-y_2))]
\end{aligned}
$$

$$
\begin{aligned}
+[\quad + \quad & y_1^3(y_4y_5^2(y_3-y_2) - y_4y_3^2(y_5-y_2) + y_4y_2^2(y_5-y_3)) \\
- \quad & y_2^3(y_4y_5^2(y_3-y_1) - y_4y_3^2(y_5-y_1) + y_4y_1^2(y_5-y_3)) \\
+ \quad & y_3^3(y_4y_5^2(y_2-y_1) - y_4y_2^2(y_5-y_1) + y_4y_1^2(y_5-y_2)) \\
- \quad & y_5^3(y_4y_3^2(y_2-y_1) - y_4y_2^2(y_3-y_1) + y_4y_1^2(y_3-y_2))]
\end{aligned}
$$

$$
\begin{aligned}
+[\quad - \quad & y_1^3(y_5y_4^2(y_3-y_2) - y_5y_3^2(y_4-y_2) + y_5y_2^2(y_4-y_3)) \\
+ \quad & y_2^3(y_5y_4^2(y_3-y_1) - y_5y_3^2(y_4-y_1) + y_5y_1^2(y_4-y_3)) \\
- \quad & y_3^3(y_5y_4^2(y_2-y_1) - y_5y_2^2(y_4-y_1) + y_5y_1^2(y_4-y_2)) \\
+ \quad & y_4^3(y_5y_3^2(y_2-y_1) - y_5y_2^2(y_3-y_1) + y_5y_1^2(y_3-y_2))] \quad \text{(C.27)}
\end{aligned}
$$

$$\sum y_i^1 terms =$$

$$
\begin{aligned}
(\quad + \quad & y_1^3(y_2y_5^2(y_4-y_3) - y_2y_4^2(y_5-y_3) + y_2y_3^2(y_5-y_4)) \\
+ \quad & y_1^3(-y_3y_5^2(y_4-y_2) + y_3y_4^2(y_5-y_2) - y_3y_2^2(y_5-y_4))
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad y_1^3(y_4y_5^2(y_3-y_2) - y_4y_3^2(y_5-y_2) + y_4y_2^2(y_5-y_3)) \\
&+ \quad y_1^3(-y_5y_4^2(y_3-y_2) + y_5y_3^2(y_4-y_2) - y_5y_2^2(y_4-y_3)) \\
&+ \quad y_2^3(-y_1y_5^2(y_4-y_3) + y_1y_4^2(y_5-y_3) - y_1y_3^2(y_5-y_4)) \\
&+ \quad y_2^3(y_3y_5^2(y_4-y_1) - y_3y_4^2(y_5-y_1) + y_3y_1^2(y_5-y_4)) \\
&+ \quad y_2^3(-y_4y_5^2(y_3-y_1) + y_4y_3^2(y_5-y_1) - y_4y_1^2(y_5-y_3)) \\
&+ \quad y_2^3(y_5y_4^2(y_3-y_1) - y_5y_3^2(y_4-y_1) + y_5y_1^2(y_4-y_3)) \\
&+ \quad y_3^3(y_1y_5^2(y_4-y_2) - y_1y_4^2(y_5-y_2) + y_1y_2^2(y_5-y_4)) \\
&+ \quad y_3^3(-y_2y_5^2(y_4-y_1) + y_2y_4^2(y_5-y_1) - y_2y_1^2(y_5-y_4)) \\
&+ \quad y_3^3(y_4y_5^2(y_2-y_1) - y_4y_2^2(y_5-y_1) + y_4y_1^2(y_5-y_2)) \\
&+ \quad y_3^3(-y_5y_4^2(y_2-y_1) + y_5y_2^2(y_4-y_1) - y_5y_1^2(y_4-y_2)) \\
&+ \quad y_4^3(-y_1y_5^2(y_3-y_2) + y_1y_3^2(y_5-y_2) - y_1y_2^2(y_5-y_3)) \\
&+ \quad y_4^3(y_2y_5^2(y_3-y_1) - y_2y_3^2(y_5-y_1) + y_2y_1^2(y_5-y_3)) \\
&+ \quad y_4^3(-y_3y_5^2(y_2-y_1) + y_3y_2^2(y_5-y_1) - y_3y_1^2(y_5-y_2)) \\
&+ \quad y_4^3(y_5y_3^2(y_2-y_1) - y_5y_2^2(y_3-y_1) + y_5y_1^2(y_3-y_2)) \\
&+ \quad y_5^3(y_1y_4^2(y_3-y_2) - y_1y_3^2(y_4-y_2) + y_1y_2^2(y_4-y_3)) \\
&+ \quad y_5^3(-y_2y_4^2(y_3-y_1) + y_2y_3^2(y_4-y_1) - y_2y_1^2(y_4-y_3)) \\
&+ \quad y_5^3(y_3y_4^2(y_2-y_1) - y_3y_2^2(y_4-y_1) + y_3y_1^2(y_4-y_2)) \\
&+ \quad y_5^3(-y_4y_3^2(y_2-y_1) + y_4y_2^2(y_3-y_1) - y_4y_1^2(y_3-y_2))) \quad \text{(C.28)}
\end{aligned}
$$

$$
\sum y_i^1 terms =
$$

$$
\begin{aligned}
(y_1^3(\quad &+ \quad y_2y_5^2(y_4-y_3) - y_2y_4^2(y_5-y_3) + y_2y_3^2(y_5-y_4) \\
&- \quad y_3y_5^2(y_4-y_2) + y_3y_4^2(y_5-y_2) - y_3y_2^2(y_5-y_4) \\
&+ \quad y_4y_5^2(y_3-y_2) - y_4y_3^2(y_5-y_2) + y_4y_2^2(y_5-y_3) \\
&- \quad y_5y_4^2(y_3-y_2) + y_5y_3^2(y_4-y_2) - y_5y_2^2(y_4-y_3))
\end{aligned}
$$

$$
+y_2^3(\quad - \quad y_1y_5^2(y_4-y_3) + y_1y_4^2(y_5-y_3) - y_1y_3^2(y_5-y_4)
$$

$$
\begin{aligned}
& + \quad y_3y_5^2(y_4-y_1) - y_3y_4^2(y_5-y_1) + y_3y_1^2(y_5-y_4) \\
& - \quad y_4y_5^2(y_3-y_1) + y_4y_3^2(y_5-y_1) - y_4y_1^2(y_5-y_3) \\
& + \quad y_5y_4^2(y_3-y_1) - y_5y_3^2(y_4-y_1) + y_5y_1^2(y_4-y_3))
\end{aligned}
$$

$$
\begin{aligned}
+y_3^3(& + \quad y_1y_5^2(y_4-y_2) - y_1y_4^2(y_5-y_2) + y_1y_2^2(y_5-y_4) \\
& - \quad y_2y_5^2(y_4-y_1) + y_2y_4^2(y_5-y_1) - y_2y_1^2(y_5-y_4) \\
& + \quad y_4y_5^2(y_2-y_1) - y_4y_2^2(y_5-y_1) + y_4y_1^2(y_5-y_2) \\
& - \quad y_5y_4^2(y_2-y_1) + y_5y_2^2(y_4-y_1) - y_5y_1^2(y_4-y_2))
\end{aligned}
$$

$$
\begin{aligned}
+y_4^3(& - \quad y_1y_5^2(y_3-y_2) + y_1y_3^2(y_5-y_2) - y_1y_2^2(y_5-y_3) \\
& + \quad y_2y_5^2(y_3-y_1) - y_2y_3^2(y_5-y_1) + y_2y_1^2(y_5-y_3) \\
& - \quad y_3y_5^2(y_2-y_1) + y_3y_2^2(y_5-y_1) - y_3y_1^2(y_5-y_2) \\
& + \quad y_5y_3^2(y_2-y_1) - y_5y_2^2(y_3-y_1) + y_5y_1^2(y_3-y_2))
\end{aligned}
$$

$$
\begin{aligned}
+y_5^3(& + \quad y_1y_4^2(y_3-y_2) - y_1y_3^2(y_4-y_2) + y_1y_2^2(y_4-y_3) \\
& - \quad y_2y_4^2(y_3-y_1) + y_2y_3^2(y_4-y_1) - y_2y_1^2(y_4-y_3) \\
& + \quad y_3y_4^2(y_2-y_1) - y_3y_2^2(y_4-y_1) + y_3y_1^2(y_4-y_2) \\
& - \quad y_4y_3^2(y_2-y_1) + y_4y_2^2(y_3-y_1) - y_4y_1^2(y_3-y_2)))
\end{aligned}
\tag{C.29}
$$

If we wanted to go about this systematically we could further group the terms by the y_i^2 factors (henceforth referred to as secondary factors), but it is easy enough to see how the terms cancel in the current form:

$$
\begin{aligned}
& \sum y_i^1 terms = \\
& \quad (y_1^3(\quad + \quad y_2y_5^2(0) - y_2y_4^2(0) + y_2y_3^2(0) \\
& \qquad\qquad - \quad y_3y_5^2(0) + y_3y_4^2(0) - y_3y_2^2(0) \\
& \qquad\qquad + \quad y_4y_5^2(0) - y_4y_3^2(0) + y_4y_2^2(0)
\end{aligned}
$$

$$
\begin{aligned}
& - \quad y_5 y_4^2(0) + y_5 y_3^2(0) - y_5 y_2^2(0)) \\
\\
+y_2^3(& - \quad y_1 y_5^2(0) + y_1 y_4^2(0) - y_1 y_3^2(0) \\
& + \quad y_3 y_5^2(0) - y_3 y_4^2(0) + y_3 y_1^2(0) \\
& - \quad y_4 y_5^2(0) + y_4 y_3^2(0) - y_4 y_1^2(0) \\
& + \quad y_5 y_4^2(0) - y_5 y_3^2(0) + y_5 y_1^2(0)) \\
\\
+y_3^3(& + \quad y_1 y_5^2(0) - y_1 y_4^2(0) + y_1 y_2^2(0) \\
& - \quad y_2 y_5^2(0) + y_2 y_4^2(0) - y_2 y_1^2(0) \\
& + \quad y_4 y_5^2(0) - y_4 y_2^2(0) + y_4 y_1^2(0) \\
& - \quad y_5 y_4^2(0) + y_5 y_2^2(0) - y_5 y_1^2(0)) \\
\\
+y_4^3(& - \quad y_1 y_5^2(0) + y_1 y_3^2(0) - y_1 y_2^2(0) \\
& + \quad y_2 y_5^2(0) - y_2 y_3^2(0) + y_2 y_1^2(0) \\
& - \quad y_3 y_5^2(0) + y_3 y_2^2(0) - y_3 y_1^2(0) \\
& + \quad y_5 y_3^2(0) - y_5 y_2^2(0) + y_5 y_1^2(0)) \\
\\
+y_5^3(& + \quad y_1 y_4^2(0) - y_1 y_3^2(0) + y_1 y_2^2(0) \\
& - \quad y_2 y_4^2(0) + y_2 y_3^2(0) - y_2 y_1^2(0) \\
& + \quad y_3 y_4^2(0) - y_3 y_2^2(0) + y_3 y_1^2(0) \\
& - \quad y_4 y_3^2(0) + y_4 y_2^2(0) - y_4 y_1^2(0))) \\
& \quad = 0 \qquad \text{(C.30)}
\end{aligned}
$$

So the y_i^1 terms will all cancel out.

Repeating this process for the y_i^2 terms we get:

$$\sum y_i^2 terms =$$

$$
\begin{aligned}
(y_1^2[\quad - \quad & y_2^3(y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4)) \\
+ \quad & y_3^3(y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4)) \\
- \quad & y_4^3(y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3)) \\
+ \quad & y_5^3(y_4^2(y_3 - y_2) - y_3^2(y_4 - y_2) + y_2^2(y_4 - y_3))] \\
\\
+(y_2^2[\quad + \quad & y_1^3(y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4)) \\
- \quad & y_3^3(y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4)) \\
+ \quad & y_4^3(y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3)) \\
- \quad & y_5^3(y_4^2(y_3 - y_1) - y_3^2(y_4 - y_1) + y_1^2(y_4 - y_3))] \\
\\
+(y_3^2[\quad - \quad & y_1^3(y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4)) \\
+ \quad & y_2^3(y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4)) \\
- \quad & y_4^3(y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2)) \\
+ \quad & y_5^3(y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2))] \\
\\
+(y_4^2[\quad + \quad & y_1^3(y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3)) \\
- \quad & y_2^3(y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3)) \\
+ \quad & y_3^3(y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2)) \\
- \quad & y_5^3(y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2))] \\
\\
+(y_5^2[\quad - \quad & y_1^3(y_4^2(y_3 - y_2) - y_3^2(y_4 - y_2) + y_2^2(y_4 - y_3)) \\
+ \quad & y_2^3(y_4^2(y_3 - y_1) - y_3^2(y_4 - y_1) + y_1^2(y_4 - y_3)) \\
- \quad & y_3^3(y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2)) \\
+ \quad & y_4^3(y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2))] \qquad \text{(C.31)}
\end{aligned}
$$

Following the same algebraic steps as in the y_j^1 case we get:

$$
\begin{aligned}
\sum y_i^2 terms = \\
(y_1^3(&+ y_2^2y_5^2(y_4-y_3) - y_2^2y_4^2(y_5-y_3) + y_2^2y_3^2(y_5-y_4) \\
&- y_3^2y_5^2(y_4-y_2) + y_3^2y_4^2(y_5-y_2) - y_3^2y_2^2(y_5-y_4) \\
&+ y_4^2y_5^2(y_3-y_2) - y_4^2y_3^2(y_5-y_2) + y_4^2y_2^2(y_5-y_3) \\
&- y_5^2y_4^2(y_3-y_2) + y_5^2y_3^2(y_4-y_2) - y_5^2y_2^2(y_4-y_3)) \\
\\
+y_2^3(&- y_1^2y_5^2(y_4-y_3) + y_1^2y_4^2(y_5-y_3) - y_1^2y_3^2(y_5-y_4) \\
&+ y_3^2y_5^2(y_4-y_1) - y_3^2y_4^2(y_5-y_1) + y_3^2y_1^2(y_5-y_4) \\
&- y_4^2y_5^2(y_3-y_1) + y_4^2y_3^2(y_5-y_1) - y_4^2y_1^2(y_5-y_3) \\
&+ y_5^2y_4^2(y_3-y_1) - y_5^2y_3^2(y_4-y_1) + y_5^2y_1^2(y_4-y_3)) \\
\\
+y_3^3(&+ y_1^2y_5^2(y_4-y_2) - y_1^2y_4^2(y_5-y_2) + y_1^2y_2^2(y_5-y_4) \\
&- y_2^2y_5^2(y_4-y_1) + y_2^2y_4^2(y_5-y_1) - y_2^2y_1^2(y_5-y_4) \\
&+ y_4^2y_5^2(y_2-y_1) - y_4^2y_2^2(y_5-y_1) + y_4^2y_1^2(y_5-y_2) \\
&- y_5^2y_4^2(y_2-y_1) + y_5^2y_2^2(y_4-y_1) - y_5^2y_1^2(y_4-y_2)) \\
\\
+y_4^3(&- y_1^2y_5^2(y_3-y_2) + y_1^2y_3^2(y_5-y_2) - y_1^2y_2^2(y_5-y_3) \\
&+ y_2^2y_5^2(y_3-y_1) - y_2^2y_3^2(y_5-y_1) + y_2^2y_1^2(y_5-y_3) \\
&- y_3^2y_5^2(y_2-y_1) + y_3^2y_2^2(y_5-y_1) - y_3^2y_1^2(y_5-y_2) \\
&+ y_5^2y_3^2(y_2-y_1) - y_5^2y_2^2(y_3-y_1) + y_5^2y_1^2(y_3-y_2)) \\
\\
+y_5^3(&+ y_1^2y_4^2(y_3-y_2) - y_1^2y_3^2(y_4-y_2) + y_1^2y_2^2(y_4-y_3) \\
&- y_2^2y_4^2(y_3-y_1) + y_2^2y_3^2(y_4-y_1) - y_2^2y_1^2(y_4-y_3) \\
&+ y_3^2y_4^2(y_2-y_1) - y_3^2y_2^2(y_4-y_1) + y_3^2y_1^2(y_4-y_2) \\
&- y_4^2y_3^2(y_2-y_1) + y_4^2y_2^2(y_3-y_1) - y_4^2y_1^2(y_3-y_2))) \quad \text{(C.32)}
\end{aligned}
$$

This clearly cancels:

$$\sum y_i^2 terms = (y_1^3(0) + y_2^3(0) + y_3^3(0) + y_4^3(0) + y_5^3(0)) = 0 \tag{C.33}$$

So the y_i^2 terms will all cancel out.

Repeating this process for the y_i^3 terms we get:

$$\begin{aligned}
\sum y_i^3 terms = & \\
(y_1^3[& - y_2^3(y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4)) \\
& + y_3^3(y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4)) \\
& - y_4^3(y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3)) \\
& + y_5^3(y_4^2(y_3 - y_2) - y_3^2(y_4 - y_2) + y_2^2(y_4 - y_3))] \\
\\
+y_2^3[& + y_1^3(y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4)) \\
& - y_3^3(y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4)) \\
& + y_4^3(y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3)) \\
& - y_5^3(y_4^2(y_3 - y_1) - y_3^2(y_4 - y_1) + y_1^2(y_4 - y_3))] \\
\\
+y_3^3[& - y_1^3(y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4)) \\
& + y_2^3(y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4)) \\
& - y_4^3(y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2)) \\
& + y_5^3(y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2))] \\
\\
+y_4^3[& + y_1^3(y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3)) \\
& - y_2^3(y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3)) \\
& + y_3^3(y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2)) \\
& - y_5^3(y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2))]
\end{aligned}$$

$$
\begin{aligned}
+y_5^3[\quad - \quad & y_1^3(y_4^2(y_3 - y_2) - y_3^2(y_4 - y_2) + y_2^2(y_4 - y_3)) \\
+ \quad & y_2^3(y_4^2(y_3 - y_1) - y_3^2(y_4 - y_1) + y_1^2(y_4 - y_3)) \\
- \quad & y_3^3(y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2)) \\
+ \quad & y_4^3(y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2))]
\end{aligned} \qquad \text{(C.34)}
$$

Now that we have third order $\varrho(y_j)$ terms we cannot just group by leading term as we did in the previous three cases. Instead we must distribute y_j^3:

$$
\begin{aligned}
\sum y_i^3 terms = & \\
(\quad - \quad & y_1^3y_2^3(y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4)) \\
+ \quad & y_1^3y_3^3(y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4)) \\
- \quad & y_1^3y_4^3(y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3)) \\
+ \quad & y_1^3y_5^3(y_4^2(y_3 - y_2) - y_3^2(y_4 - y_2) + y_2^2(y_4 - y_3)) \\
+ \quad & y_1^3y_2^3(y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4)) \\
- \quad & y_2^3y_3^3(y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4)) \\
+ \quad & y_2^3y_4^3(y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3)) \\
- \quad & y_2^3y_5^3(y_4^2(y_3 - y_1) - y_3^2(y_4 - y_1) + y_1^2(y_4 - y_3)) \\
- \quad & y_1^3y_3^3(y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4)) \\
+ \quad & y_2^3y_3^3(y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4)) \\
- \quad & y_3^3y_4^3(y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2)) \\
+ \quad & y_3^3y_5^3(y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2)) \\
+ \quad & y_1^3y_4^3(y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3)) \\
- \quad & y_2^3y_4^3(y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3)) \\
+ \quad & y_3^3y_4^3(y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2)) \\
- \quad & y_4^3y_5^3(y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2)) \\
- \quad & y_1^3y_5^3(y_4^2(y_3 - y_2) - y_3^2(y_4 - y_2) + y_2^2(y_4 - y_3)) \\
+ \quad & y_2^3y_5^3(y_4^2(y_3 - y_1) - y_3^2(y_4 - y_1) + y_1^2(y_4 - y_3))
\end{aligned}
$$

$$
\begin{aligned}
- &\; y_3^3 y_5^3 (y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2)) \\
+ &\; y_4^3 y_5^3 (y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2))) \qquad \text{(C.35)}
\end{aligned}
$$

Rearranging:

$$
\sum y_i^3 terms =
$$

$$
\begin{aligned}
(- &\; y_1^3 y_2^3 (y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4)) \\
+ &\; y_1^3 y_2^3 (y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4)) \\
+ &\; y_1^3 y_3^3 (y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4)) \\
- &\; y_1^3 y_3^3 (y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4)) \\
- &\; y_1^3 y_4^3 (y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3)) \\
+ &\; y_1^3 y_4^3 (y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3)) \\
+ &\; y_1^3 y_5^3 (y_4^2(y_3 - y_2) - y_3^2(y_4 - y_2) + y_2^2(y_4 - y_3)) \\
- &\; y_1^3 y_5^3 (y_4^2(y_3 - y_2) - y_3^2(y_4 - y_2) + y_2^2(y_4 - y_3)) \\
- &\; y_2^3 y_3^3 (y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4)) \\
+ &\; y_2^3 y_3^3 (y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4)) \\
+ &\; y_2^3 y_4^3 (y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3)) \\
- &\; y_2^3 y_4^3 (y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3)) \\
- &\; y_2^3 y_5^3 (y_4^2(y_3 - y_1) - y_3^2(y_4 - y_1) + y_1^2(y_4 - y_3)) \\
+ &\; y_2^3 y_5^3 (y_4^2(y_3 - y_1) - y_3^2(y_4 - y_1) + y_1^2(y_4 - y_3)) \\
- &\; y_3^3 y_4^3 (y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2)) \\
+ &\; y_3^3 y_4^3 (y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2)) \\
+ &\; y_3^3 y_5^3 (y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2)) \\
- &\; y_3^3 y_5^3 (y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2)) \\
- &\; y_4^3 y_5^3 (y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2)) \\
+ &\; y_4^3 y_5^3 (y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2))) \qquad \text{(C.36)}
\end{aligned}
$$

Which clearly cancels:

$$\sum y_i^3 terms = 0 \tag{C.37}$$

So the y_i^3 terms will all cancel out.

Repeating this process for the y_i^4 terms we get:

$$\begin{aligned}
&\sum y_i^4 terms = \\
&(y_1^4[\ - \ y_2^3(y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4)) \\
&\qquad + \ y_3^3(y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4)) \\
&\qquad - \ y_4^3(y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3)) \\
&\qquad + \ y_5^3(y_4^2(y_3 - y_2) - y_3^2(y_4 - y_2) + y_2^2(y_4 - y_3))] \\
&+(y_2^4[\ + \ y_1^3(y_5^2(y_4 - y_3) - y_4^2(y_5 - y_3) + y_3^2(y_5 - y_4)) \\
&\qquad - \ y_3^3(y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4)) \\
&\qquad + \ y_4^3(y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3)) \\
&\qquad - \ y_5^3(y_4^2(y_3 - y_1) - y_3^2(y_4 - y_1) + y_1^2(y_4 - y_3))] \\
&+(y_3^4[\ - \ y_1^3(y_5^2(y_4 - y_2) - y_4^2(y_5 - y_2) + y_2^2(y_5 - y_4)) \\
&\qquad + \ y_2^3(y_5^2(y_4 - y_1) - y_4^2(y_5 - y_1) + y_1^2(y_5 - y_4)) \\
&\qquad - \ y_4^3(y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2)) \\
&\qquad + \ y_5^3(y_4^2(y_2 - y_1) - y_2^2(y_4 - y_1) + y_1^2(y_4 - y_2))] \\
&+(y_4^4[\ + \ y_1^3(y_5^2(y_3 - y_2) - y_3^2(y_5 - y_2) + y_2^2(y_5 - y_3)) \\
&\qquad - \ y_2^3(y_5^2(y_3 - y_1) - y_3^2(y_5 - y_1) + y_1^2(y_5 - y_3)) \\
&\qquad + \ y_3^3(y_5^2(y_2 - y_1) - y_2^2(y_5 - y_1) + y_1^2(y_5 - y_2)) \\
&\qquad - \ y_5^3(y_3^2(y_2 - y_1) - y_2^2(y_3 - y_1) + y_1^2(y_3 - y_2))]
\end{aligned}$$

$$
\begin{aligned}
+(y_5^4[\; - \; & y_1^3(y_4^2(y_3-y_2) - y_3^2(y_4-y_2) + y_2^2(y_4-y_3)) \\
+ \; & y_2^3(y_4^2(y_3-y_1) - y_3^2(y_4-y_1) + y_1^2(y_4-y_3)) \\
- \; & y_3^3(y_4^2(y_2-y_1) - y_2^2(y_4-y_1) + y_1^2(y_4-y_2)) \\
+ \; & y_4^3(y_3^2(y_2-y_1) - y_2^2(y_3-y_1) + y_1^2(y_3-y_2))]
\end{aligned} \tag{C.38}
$$

These terms cannot possible cancel to zero because the five y_j^4 factors from the $\varrho(y_j)$ are the only fourth order factors that appear in the expression, and each has a different subscript. Therefore, the y_j^4 terms will remain, and the total one dimensional magnification will be:

$$
\begin{aligned}
&\mu_{1D,total} = \\
&\frac{y_1^4(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_4-y_3)(y_4-y_2)(y_3-y_2)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
-&\frac{y_2^4(y_5-y_4)(y_5-y_3)(y_5-y_1)(y_4-y_3)(y_4-y_1)(y_3-y_1)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
+&\frac{y_3^4(y_5-y_4)(y_5-y_2)(y_5-y_1)(y_4-y_2)(y_4-y_1)(y_2-y_1)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
-&\frac{y_4^4(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)} \\
+&\frac{y_5^4(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)}{(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1)}
\end{aligned} \tag{C.39}
$$

Which again does not depend on the perturbation location q_j.

Now that we have shown that all the terms involving perturbation location, q, will cancel out, we must show that the remaining terms sum to 1. To prove that equation C.39 is equal to 1, we show that the denominator equals the numerator:

$$
\begin{aligned}
&denominator = \\
&(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_5-y_1)(y_4-y_3)(y_4-y_2)(y_4-y_1)(y_3-y_2)(y_3-y_1)(y_2-y_1) \\
&= [(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_4-y_3)(y_4-y_2)(y_3-y_2)] \\
&\times[(y_5-y_1)(y_4-y_1)(y_3-y_1)(y_2-y_1)]
\end{aligned} \tag{C.40}
$$

Substituting in the relation we found for n=3 perturbations:

$$
\begin{aligned}
& (y_5 - y_4)(y_5 - y_3)(y_5 - y_2)(y_4 - y_3)(y_4 - y_2)(y_3 - y_2) \\
= & - y_2^3(y_4 - y_3)(y_5 - y_3)(y_5 - y_4) \\
& + y_3^3(y_4 - y_2)(y_5 - y_2)(y_5 - y_4) \\
& - y_4^3(y_3 - y_2)(y_5 - y_2)(y_5 - y_3) \\
& + y_5^3(y_3 - y_2)(y_4 - y_2)(y_4 - y_3),
\end{aligned}
\tag{C.41}
$$

we can write:

$$
\begin{aligned}
& denominator = \\
([& - y_2^3(y_4 - y_3)(y_5 - y_3)(y_5 - y_4) \\
& + y_3^3(y_4 - y_2)(y_5 - y_2)(y_5 - y_4) \\
& - y_4^3(y_3 - y_2)(y_5 - y_2)(y_5 - y_3) \\
& + y_5^3(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)] \\
& \times [(y_5 - y_1)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1)]
\end{aligned}
\tag{C.42}
$$

$$
\begin{aligned}
= & - y_2^3(y_4 - y_3)(y_5 - y_3)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1) \\
& + y_3^3(y_4 - y_2)(y_5 - y_2)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1) \\
& - y_4^3(y_3 - y_2)(y_5 - y_2)(y_5 - y_3)(y_5 - y_1)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1) \\
& + y_5^3(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)(y_5 - y_1)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1)
\end{aligned}
\tag{C.43}
$$

$$
\begin{aligned}
= & - y_2^3(y_2 - y_1)[(y_4 - y_3)(y_5 - y_3)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_3 - y_1)] \\
& + y_3^3(y_3 - y_1)[(y_4 - y_2)(y_5 - y_2)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_2 - y_1)] \\
& - y_4^3(y_4 - y_1)[(y_3 - y_2)(y_5 - y_2)(y_5 - y_3)(y_5 - y_1)(y_3 - y_1)(y_2 - y_1)] \\
& + y_5^3(y_5 - y_1)[(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1)]
\end{aligned}
\tag{C.44}
$$

$$
\begin{aligned}
= & - (y_2^4 - y_1 y_2^3)[(y_4-y_3)(y_5-y_3)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_3-y_1)] \\
& + (y_3^4 - y_1 y_3^3)[(y_4-y_2)(y_5-y_2)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_2-y_1)] \\
& - (y_4^4 - y_1 y_4^3)[(y_3-y_2)(y_5-y_2)(y_5-y_3)(y_5-y_1)(y_3-y_1)(y_2-y_1)] \\
& + (y_5^4 - y_1 y_5^3)[(y_3-y_2)(y_4-y_2)(y_4-y_3)(y_4-y_1)(y_3-y_1)(y_2-y_1)]
\end{aligned} \tag{C.45}
$$

$$
\begin{aligned}
= & - y_2^4[(y_4-y_3)(y_5-y_3)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_3-y_1)] \\
& + y_3^4[(y_4-y_2)(y_5-y_2)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_2-y_1)] \\
& - y_4^4[(y_3-y_2)(y_5-y_2)(y_5-y_3)(y_5-y_1)(y_3-y_1)(y_2-y_1)] \\
& + y_5^4[(y_3-y_2)(y_4-y_2)(y_4-y_3)(y_4-y_1)(y_3-y_1)(y_2-y_1)] \\
& + y_1 y_2^3[(y_4-y_3)(y_5-y_3)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_3-y_1)] \\
& - y_1 y_3^3[(y_4-y_2)(y_5-y_2)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_2-y_1)] \\
& + y_1 y_4^3[(y_3-y_2)(y_5-y_2)(y_5-y_3)(y_5-y_1)(y_3-y_1)(y_2-y_1)] \\
& - y_1 y_5^3[(y_3-y_2)(y_4-y_2)(y_4-y_3)(y_4-y_1)(y_3-y_1)(y_2-y_1)]
\end{aligned} \tag{C.46}
$$

But in equation C.37 we already proved that $\sum y_i^3 terms = 0$:

$$
\begin{aligned}
& - y_1^3[(y_5-y_4)(y_5-y_3)(y_5-y_2)(y_4-y_3)(y_4-y_2)(y_3-y_2)] \\
& + y_2^3[(y_4-y_3)(y_5-y_3)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_3-y_1)] \\
& - y_3^3[(y_4-y_2)(y_5-y_2)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_2-y_1)] \\
& + y_4^3[(y_3-y_2)(y_5-y_2)(y_5-y_3)(y_5-y_1)(y_3-y_1)(y_2-y_1)] \\
& - y_5^3[(y_3-y_2)(y_4-y_2)(y_4-y_3)(y_4-y_1)(y_3-y_1)(y_2-y_1)] \\
& = 0
\end{aligned} \tag{C.47}
$$

Therefore we can say that:

$$
\begin{aligned}
& + y_1 y_2^3[(y_4-y_3)(y_5-y_3)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_3-y_1)] \\
& - y_1 y_3^3[(y_4-y_2)(y_5-y_2)(y_5-y_4)(y_5-y_1)(y_4-y_1)(y_2-y_1)] \\
& + y_1 y_4^3[(y_3-y_2)(y_5-y_2)(y_5-y_3)(y_5-y_1)(y_3-y_1)(y_2-y_1)]
\end{aligned}
$$

$$
\begin{aligned}
&- \quad y_1 y_5^3[(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1)] \\
&= \quad y_1^4(y_5 - y_4)(y_5 - y_3)(y_5 - y_2)(y_4 - y_3)(y_4 - y_2)(y_3 - y_2)
\end{aligned} \tag{C.48}
$$

And thus:

$$
\begin{aligned}
& denominator = \\
- \quad & y_2^4[(y_4 - y_3)(y_5 - y_3)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_3 - y_1)] \\
+ \quad & y_3^4[(y_4 - y_2)(y_5 - y_2)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_2 - y_1)] \\
- \quad & y_4^4[(y_3 - y_2)(y_5 - y_2)(y_5 - y_3)(y_5 - y_1)(y_3 - y_1)(y_2 - y_1)] \\
+ \quad & y_5^4[(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1)] \\
+ \quad & y_1 y_2^3[(y_4 - y_3)(y_5 - y_3)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_3 - y_1)] \\
- \quad & y_1 y_3^3[(y_4 - y_2)(y_5 - y_2)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_2 - y_1)] \\
+ \quad & y_1 y_4^3[(y_3 - y_2)(y_5 - y_2)(y_5 - y_3)(y_5 - y_1)(y_3 - y_1)(y_2 - y_1)] \\
- \quad & y_1 y_5^3[(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1)]
\end{aligned} \tag{C.49}
$$

$$
\begin{aligned}
= \quad + \quad & y_1^4[(y_5 - y_4)(y_5 - y_3)(y_5 - y_2)(y_4 - y_3)(y_4 - y_2)(y_3 - y_2)] \\
- \quad & y_2^4[(y_4 - y_3)(y_5 - y_3)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_3 - y_1)] \\
+ \quad & y_3^4[(y_4 - y_2)(y_5 - y_2)(y_5 - y_4)(y_5 - y_1)(y_4 - y_1)(y_2 - y_1)] \\
- \quad & y_4^4[(y_3 - y_2)(y_5 - y_2)(y_5 - y_3)(y_5 - y_1)(y_3 - y_1)(y_2 - y_1)] \\
+ \quad & y_5^4[(y_3 - y_2)(y_4 - y_2)(y_4 - y_3)(y_4 - y_1)(y_3 - y_1)(y_2 - y_1)] \\
& = numerator
\end{aligned} \tag{C.50}
$$

And we have proven that the numerator equals the denominator, which proves that for n=4 perturbations the total 1D magnification is 1.

Appendix D

Total One-Dimensional Magnification: N Perturbations

Now we prove that the total 1D magnification along the axis of curvature is 1 for n perturbations.

As the cases of one, two, three, and four perturbations have demonstrated, the proof that the total 1D magnification is 1 can be broken down into two parts: a proof that the total 1D magnification is independent of the locations, q_j, of all perturbations, and a proof that the remaining terms will sum to 1. We prove these two parts in sections D.2 and D.1 respectively.

D.1 Proof: The q-Independent Total 1D Magnification Sums to 1

Now we show that the position independent total 1D magnification sums to 1 for n perturbations. The argument we made in section 3.6 for four perturbations informs how we can extend the proof to n perturbations. Following the pattern described in equation 3.66, we guess that the n perturbation case will look like:

$$
\begin{aligned}
&((y_{n+1}-y_n)(y_{n+1}-y_{n-1})...(y_{n+1}-y_2)(y_{n+1}-y_1))\\
&\times((y_n-y_{n-1})(y_n-y_{n-2})...(y_n-y_2)(y_n-y_1))\\
&\times((y_{n-1}-y_{n-2})(y_{n-1}-y_{n-3})...(y_{n-1}-y_2)(y_{n-1}-y_1))\\
&\vdots\\
&\times((y_4-y_3)(y_4-y_2)(y_4-y_1))\\
&\times((y_3-y_2)(y_3-y_1))\\
&\times(y_2-y_1)\\
&=\\
&(-1)^n(\\
&+y_1^n((y_{n+1}-y_n)(y_{n+1}-y_{n-1})...(y_{n+1}-y_2))((y_n-y_{n-1})(y_n-y_{n-2})...(y_n-y_2))(...)\\
&\qquad\times((y_4-y_3)(y_4-y_2))((y_3-y_2))\\
&-y_2^n((y_{n+1}-y_n)(y_{n+1}-y_{n-1})...(y_{n+1}-y_1))((y_n-y_{n-1})(y_n-y_{n-2})...(y_n-y_1))(...)\\
&\qquad\times((y_4-y_3)(y_4-y_1))((y_3-y_1))\\
&+y_3^n((y_{n+1}-y_n)(y_{n+1}-y_{n-1})...(y_{n+1}-y_2)(y_{n+1}-y_1))((y_n-y_{n-1})(y_n-y_{n-2})...\\
&\qquad\times(y_n-y_2)(y_n-y_1))(...)((y_4-y_2)(y_4-y_1))((y_2-y_1))\\
&\vdots\\
&\pm y_n^n((y_{n+1}-y_{n-1})...(y_{n+1}-y_2)(y_{n+1}-y_1))(...)\\
&\qquad\times((y_4-y_3)(y_4-y_2)(y_4-y_1))((y_3-y_2)(y_3-y_1))((y_2-y_1))\\
&\mp y_{n+1}^n((y_n-y_{n-1})...(y_n-y_2)(y_n-y_1))(...)\\
&\qquad\times((y_4-y_3)(y_4-y_2)(y_4-y_1))((y_3-y_2)(y_3-y_1))((y_2-y_1)))
\end{aligned}
\tag{D.1}
$$

For odd and even n, respectively. This can be written more compactly using product and summation notation:

$$
\prod_{j=n}^{j=1}(y_{n+1}-y_j)\times\prod_{j=n-1}^{j=1}(y_n-y_j)\times(...)\times\prod_{j=2}^{j=1}(y_3-y_j)\times\prod_{j=1}^{j=1}(y_2-y_j)
$$
$$=$$

$$
\begin{aligned}
(\quad & -y_1^n(y_1 - y_1) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_1)} \\
& -y_2^n(y_2 - y_2) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_2)} \\
& -y_3^n(y_3 - y_3) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_3)} \\
& \vdots \\
& -y_n^n(y_n - y_n) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_n)} \\
& -y_{n+1}^n(y_{n+1} - y_{n+1}) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_{n+1})}) \\
& \times \left(\prod_{j=n}^{j=1}(y_{n+1} - y_j) \times \prod_{j=n-1}^{j=1} (y_n - y_j) \times (\ldots) \times \prod_{j=2}^{j=1}(y_3 - y_j) \times \prod_{j=1}^{j=1}(y_2 - y_j) \right)
\end{aligned}
\tag{D.2}
$$

Writing this even more compactly:

$$
\begin{aligned}
& \prod_{\ell=n+1}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) = \\
& \left((-1)^n \sum_{\ell=1}^{\ell=n+1} \left(y_\ell^n (y_\ell - y_\ell) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_\ell)} \right) \right) \times \left(\prod_{\ell=n+1}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right)
\end{aligned}
\tag{D.3}
$$

Now that we have derived a compact form we can prove the that it is true for any n. As is suggested by the proof for the case of n=4, one way to prove this is with a recursive argument– in other words by showing both that equation D.3 is true for n=1 and that if it is true for $n = k$, it will be true for $n = k + 1$.

First we will show that the base case is true. For n=1 equation D.3 becomes:

$$
\begin{aligned}
& \prod_{\ell=2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) = \\
& (-1)^1 \left(\sum_{\ell=1}^{\ell=2} \left(y_\ell (y_\ell - y_\ell) \prod_{i=2}^{i=1} \frac{1}{(y_i - y_\ell)} \right) \right) \times \left(\prod_{\ell=2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right)
\end{aligned}
\tag{D.4}
$$

$$
\prod_{j=2-1}^{j=1} (y_2 - y_j) =
$$

$$
\begin{aligned}
&(-\left(y_1(y_1-y_1)\prod_{i=2}^{i=1}\frac{1}{(y_i-y_1)}\right) \\
&-\left(y_2(y_2-y_2)\prod_{i=2}^{i=1}\frac{1}{(y_i-y_2)}\right)) \times \left(\prod_{j=2-1}^{j=1}(y_\ell-y_j)\right)
\end{aligned}
\tag{D.5}
$$

$$
\begin{aligned}
&(y_2-y_1)= \\
&(-\left(y_1(y_1-y_1)\frac{1}{(y_2-y_1)(y_1-y_1)}\right) \\
&+\left(y_2(y_2-y_2)\frac{1}{(y_2-y_2)(y_1-y_2)}\right)) \times ((y_2-y_1))
\end{aligned}
\tag{D.6}
$$

$$
(y_2-y_1) = -(y_1)+(y_2)
\tag{D.7}
$$

Which is certainly true. Now to complete the proof we must show that if equation D.3 holds for $n=k$, it will hold for $n=k+1$. In other words, we want to show that for $n=k+1$:

$$
\begin{aligned}
&\prod_{\ell=k+2}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell-y_j)= \\
&\left((-1)^{k+1}\sum_{\ell=1}^{\ell=k+2}\left(y_\ell^{k+1}(y_\ell-y_\ell)\prod_{i=k+2}^{i=1}\frac{1}{(y_i-y_\ell)}\right)\right) \times \left(\prod_{\ell=k+2}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell-y_j)\right)
\end{aligned}
\tag{D.8}
$$

given that for n=k:

$$
\begin{aligned}
&\prod_{\ell=k+1}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell-y_j)= \\
&\left((-1)^{k}\sum_{\ell=1}^{\ell=k+1}\left(y_\ell^{k}(y_\ell-y_\ell)\prod_{i=k+1}^{i=1}\frac{1}{(y_i-y_\ell)}\right)\right) \times \left(\prod_{\ell=k+1}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell-y_j)\right).
\end{aligned}
\tag{D.9}
$$

To begin, we use equation D.9 to rewrite the left side of equation D.8 as:

$$
\begin{aligned}
&leftside= \\
&\left(\prod_{j=k+1}^{j=1}(y_{k+2}-y_j)\right)\left(\prod_{\ell=k+1}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell-y_j)\right)
\end{aligned}
$$

$$= \left(\prod_{j=k+1}^{j=1} (y_{k+2} - y_j) \right) \left((-1)^k \sum_{\ell=1}^{\ell=k+1} \left(y_\ell^k (y_\ell - y_\ell) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_\ell)} \right) \right) \times \left(\prod_{\ell=k+1}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right) \quad \text{(D.10)}$$

Expanding the sum:

$$\begin{aligned} leftside = & \\ (& + \left(y_1^k (y_1 - y_1) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_1)} \right) \\ & + \left(y_2^k (y_2 - y_2) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_2)} \right) \\ & + \left(y_3^k (y_3 - y_3) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_3)} \right) \\ & + \vdots \\ & + \left(y_k^k (y_k - y_k) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_k)} \right) \\ & + \left(y_{k+1}^k (y_{k+1} - y_{k+1}) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_{k+1})} \right)) \\ & \times (-1)^k \left(\prod_{\ell=k+1}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right) \left(\prod_{j=k+1}^{j=1} (y_{k+2} - y_j) \right) \end{aligned} \quad \text{(D.11)}$$

$$\begin{aligned} leftside = & \\ (& + (y_{k+2} - y_1) \left(y_1^k (y_1 - y_1) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_1)} \right) \\ & + (y_{k+2} - y_2) \left(y_2^k (y_2 - y_2) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_2)} \right) \\ & + (y_{k+2} - y_3) \left(y_3^k (y_3 - y_3) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_3)} \right) \\ & + \vdots \\ & + (y_{k+2} - y_k) \left(y_k^k (y_k - y_k) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_k)} \right) \\ & + (y_{k+2} - y_{k+1}) \left(y_{k+1}^k (y_{k+1} - y_{k+1}) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_{k+1})} \right)) \end{aligned}$$

$$\times(-1)^k \left(\prod_{\ell=k+2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right) \tag{D.12}$$

Distributing the factor in front:

$$\begin{aligned}
leftside =& \\
(&+\left(y_1^{k+1}(y_1 - y_1) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_1)} \right) \\
&+\left(y_2^{k+1}(y_2 - y_2) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_2)} \right) \\
&+\left(y_3^{k+1}(y_3 - y_3) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_3)} \right) \\
&+ \vdots \\
&+\left(y_k^{k+1}(y_k - y_k) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_k)} \right) \\
&+\left(y_{k+1}^{k+1}(y_{k+1} - y_{k+1}) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_{k+1})} \right) \\
-y_{k+2}(&+\left(y_1^{k}(y_1 - y_1) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_1)} \right) \\
&+\left(y_2^{k}(y_2 - y_2) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_2)} \right) \\
&+\left(y_3^{k}(y_3 - y_3) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_3)} \right) \\
&\vdots \\
&+\left(y_k^{k}(y_k - y_k) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_k)} \right) \\
&+\left(y_{k+1}^{k}(y_{k+1} - y_{k+1}) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_{k+1})} \right))) \\
&\times \ (-1)^{k+1} \left(\prod_{\ell=k+2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right)
\end{aligned} \tag{D.13}$$

$$leftside =$$

$$
\begin{aligned}
(&+ \left(y_1^{k+1}(y_1 - y_1) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_1)} \right) \\
&+ \left(y_2^{k+1}(y_2 - y_2) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_2)} \right) \\
&+ \left(y_3^{k+1}(y_3 - y_3) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_3)} \right) \\
&+ \vdots \\
&+ \left(y_k^{k+1}(y_k - y_k) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_k)} \right) \\
&+ \left(y_{k+1}^{k+1}(y_{k+1} - y_{k+1}) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_{k+1})} \right) \\
&- y_{k+2} \sum_{p=1}^{p=k+1} \left(y_p^k(y_p - y_p) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_p)} \right)) \\
&\quad \times (-1)^{k+1} \left(\prod_{\ell=k+2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right)
\end{aligned}
\tag{D.14}
$$

Following our strategy for n=4, we now attempt to show that:

$$
\begin{aligned}
&\sum_{p=1}^{p=k+2} \left(y_p^k(y_p - y_p) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_p)} \right) \times \left(\prod_{\ell=k+2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right) \\
&= 0
\end{aligned}
\tag{D.15}
$$

To show this we expand the expression.

$$
\sum_{p=1}^{p=k+2} \left(y_p^k(y_p - y_p) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_p)} \right) \times \left(\prod_{\ell=k+2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right)
\tag{D.16}
$$

$$
\begin{aligned}
&= \sum_{p=1}^{p=k+2} y_p^k \frac{[(y_{k+2} - y_{k+1})(y_{k+2} - y_k)...(y_{k+2} - y_1)]}{(y_{k+2} - y_p)(y_{k+1} - y_p)(y_k - y_p)...(y_2 - y_p)(y_1 - y_p)} \\
&\times [(y_{k+1} - y_k)(y_{k+1} - y_{k-1})...(y_{k+1} - y_1)][...][(y_3 - y_2)(y_3 - y_1)][(y_2 - y_1)]
\end{aligned}
\tag{D.17}
$$

Since all y_p terms will cancel out except for the leading y_p^k factor, the coefficient of y_p^k contains only $k+1$ distinct variables. Therefore we can write the coefficient in the form of equation D.9:

$$= \sum_{p=1}^{p=k+2} y_p^k \frac{[(y_{k+2}-y_{k+1})(y_{k+2}-y_k)...(y_{k+2}-y_1)]}{(y_{k+2}-y_p)(y_{k+1}-y_p)(y_k-y_p)...(y_2-y_p)(y_1-y_p)}$$
$$\times[(y_{k+1}-y_k)(y_{k+1}-y_{k-1})...(y_{k+1}-y_1)][...][(y_3-y_2)(y_3-y_1)][(y_2-y_1)]$$

$$\begin{aligned}
= &+ y_1^k[(y_{k+2}-y_{k+1})(y_{k+2}-y_k)...(y_{k+2}-y_2)][(y_{k+1}-y_k)(y_{k+1}-y_{k-1})...(y_{k+1}-y_2)] \\
&\qquad \times[...][(y_3-y_2)] \\
&- y_2^k[(y_{k+2}-y_{k+1})(y_{k+2}-y_k)...(y_{k+2}-y_1)][(y_{k+1}-y_k)(y_{k+1}-y_{k-1})...(y_{k+1}-y_1)] \\
&\qquad \times[...][(y_3-y_1)][(y_3-y_1)] \\
&+ y_3^k[(y_{k+2}-y_{k+1})(y_{k+2}-y_k)...(y_{k+2}-y_1)][(y_{k+1}-y_k)(y_{k+1}-y_{k-1})...(y_{k+1}-y_1)] \\
&\qquad \times[...][(y_2-y_1)] \\
&\vdots \\
&\pm y_k^k[(y_{k+2}-y_{k+1})(y_{k+2}-y_k)...(y_{k+2}-y_1)][(y_{k+1}-y_k)(y_{k+1}-y_{k-1})...(y_{k+1}-y_1)] \\
&\qquad \times[...][(y_3-y_2)(y_3-y_1)][(y_3-y_1)][(y_2-y_1)] \\
&\mp y_{k+1}^k[(y_{k+2}-y_{k+1})(y_{k+2}-y_k)...(y_{k+2}-y_1)][(y_{k+1}-y_k)(y_{k+1}-y_{k-1})...(y_{k+1}-y_1)] \\
&\qquad \times[...][(y_3-y_2)(y_3-y_1)][(y_3-y_1)][(y_2-y_1)] \\
&\pm y_{k+2}^k[(y_{k+2}-y_{k+1})(y_{k+2}-y_k)...(y_{k+2}-y_1)][(y_{k+1}-y_k)(y_{k+1}-y_{k-1})...(y_{k+1}-y_1)] \\
&\qquad \times[...][(y_3-y_2)(y_3-y_1)][(y_3-y_1)][(y_2-y_1)]
\end{aligned} \tag{D.18}$$

$$\begin{aligned}
= &\quad (-1)^k(\\
&+ y_1^k[\sum_{\ell=2,3,...k+1,k+2}(y_\ell^k(y_\ell-y_\ell)\prod_{i=k+2,k+1,...3,2}\frac{1}{(y_i-y_\ell)})(\prod_{\ell=k+2,k+1...4,3}\prod_{j=\ell-1,\ell-2...3,2}(y_\ell-y_j))] \\
&- y_2^k[\sum_{\ell=1,3,...k+1,k+2}(y_\ell^k(y_\ell-y_\ell)\prod_{i=k+2,k+1,...3,1}\frac{1}{(y_i-y_\ell)})(\prod_{\ell=k+2,k+1...4,3}\prod_{j=\ell-1,\ell-2...3,1}(y_\ell-y_j))] \\
&+ y_3^k[\sum_{\ell=1,2,4...k+1,k+2}(y_\ell^k(y_\ell-y_\ell)\prod_{i=k+2,k+1,...4,2,1}\frac{1}{(y_i-y_\ell)})(\prod_{\ell=k+2,k+1...4,2}\prod_{j=\ell-1,\ell-2...2,1}(y_\ell-y_j))] \\
&\vdots \\
&\pm y_k^k[\sum_{\ell=1,2,3...k+1,k+2}(y_\ell^k(y_\ell-y_\ell)\prod_{i=k+2,k+1,...3,2,1}\frac{1}{(y_i-y_\ell)})(\prod_{\ell=k+2,k+1...3,2}\prod_{j=\ell-1,\ell-2...2,1}(y_\ell-y_j))]
\end{aligned}$$

$$\mp \; y_{k+1}^k[\sum_{\ell=1,2,4\ldots k,k+2}(y_\ell^k(y_\ell - y_\ell)\prod_{i=k+2,k,\ldots 4,2,1}\frac{1}{(y_i-y_\ell)})(\prod_{\ell=k+2,k\ldots 4,2}\;\prod_{j=\ell-1,\ell-2\ldots 2,1}(y_\ell - y_j))]$$

$$\pm \; y_{k+2}^k[\sum_{\ell=1,2,4\ldots k,k+1}(y_\ell^k(y_\ell - y_\ell)\prod_{i=k+1,k,\ldots 4,2,1}\frac{1}{(y_i-y_\ell)})(\prod_{\ell=k+1,k\ldots 4,2}\;\prod_{j=\ell-1,\ell-2\ldots 2,1}(y_\ell - y_j))]) \tag{D.19}$$

For k odd and k even, respectively. Dropping the overall sign and expanding again:

$$\begin{aligned}
y_1^k[\; & + \; (y_2^k(y_2-y_2)\prod_{i=k+2,k+1,\ldots 3,2}\frac{1}{(y_i-y_2)}) \\
& + \; (y_3^k(y_3-y_3)\prod_{i=k+2,k+1,\ldots 3,2}\frac{1}{(y_i-y_3)}) \\
& + \; (y_4^k(y_4-y_4)\prod_{i=k+2,k+1,\ldots 3,2}\frac{1}{(y_i-y_4)}) \\
& \quad \vdots \\
& + \; (y_k^k(y_k-y_k)\prod_{i=k+2,k+1,\ldots 3,2}\frac{1}{(y_i-y_k)}) \\
& + \; (y_{k+1}^k(y_{k+1}-y_{k+1})\prod_{i=k+2,k+1,\ldots 3,2}\frac{1}{(y_i-y_{k+1})}) \\
& + \; (y_{k+2}^k(y_{k+2}-y_{k+2})\prod_{i=k+2,k+1,\ldots 3,2}\frac{1}{(y_i-y_{k+2})})] \\
& \times \; (\prod_{\ell=k+2,k+1\ldots 4,3}\;\prod_{j=\ell-1,\ell-2\ldots 3,2}(y_\ell - y_j)) \\
\\
+y_2^k[\; & - \; (y_1^k(y_1-y_1)\prod_{i=k+2,k+1,\ldots 3,1}\frac{1}{(y_i-y_1)})] \\
& - \; (y_3^k(y_3-y_3)\prod_{i=k+2,k+1,\ldots 3,1}\frac{1}{(y_i-y_3)})] \\
& - \; (y_4^k(y_4-y_4)\prod_{i=k+2,k+1,\ldots 3,1}\frac{1}{(y_i-y_4)})] \\
& \quad \vdots \\
& - \; (y_k^k(y_k-y_k)\prod_{i=k+2,k+1,\ldots 3,1}\frac{1}{(y_i-y_k)})] \\
& - \; (y_{k+1}^k(y_{k+1}-y_{k+1})\prod_{i=k+2,k+1,\ldots 3,1}\frac{1}{(y_i-y_{k+1})})] \\
& - \; (y_{k+2}^k(y_{k+2}-y_{k+2})\prod_{i=k+2,k+1,\ldots 3,1}\frac{1}{(y_i-y_{k+2})})]
\end{aligned}$$

$$\times \quad (\prod_{\ell=k+2,k+1\ldots4,3}\ \prod_{j=\ell-1,\ell-2\ldots3,1}(y_\ell - y_j))$$

$$
\begin{aligned}
+y_3^k[\quad &+ \quad (y_1^k(y_1 - y_1)\prod_{i=k+2,k+1,\ldots4,2,1}\frac{1}{(y_i - y_1)}) \\
&+ \quad (y_2^k(y_2 - y_2)\prod_{i=k+2,k+1,\ldots4,2,1}\frac{1}{(y_i - y_2)}) \\
&+ \quad (y_4^k(y_4 - y_4)\prod_{i=k+2,k+1,\ldots4,2,1}\frac{1}{(y_i - y_4)}) \\
&\quad \vdots \\
&+ \quad (y_k^k(y_k - y_k)\prod_{i=k+2,k+1,\ldots4,2,1}\frac{1}{(y_i - y_k)})
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad (y_{k+1}^k(y_{k+1} - y_{k+1})\prod_{i=k+2,k+1,\ldots4,2,1}\frac{1}{(y_i - y_{k+1})}) \\
&+ \quad (y_{k+2}^k(y_{k+2} - y_{k+2})\prod_{i=k+2,k+1,\ldots4,2,1}\frac{1}{(y_i - y_{k+2})})] \\
&\times \quad (\prod_{\ell=k+2,k+1\ldots4,2}\ \prod_{j=\ell-1,\ell-2\ldots2,1}(y_\ell - y_j))
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
+y_k^k[\quad &\pm \quad (y_1^k(y_1 - y_1)\prod_{i=k+2,k+1,\ldots3,2,1}\frac{1}{(y_i - y_1)}) \\
&\pm \quad (y_2^k(y_2 - y_2)\prod_{i=k+2,k+1,\ldots3,2,1}\frac{1}{(y_i - y_2)}) \\
&\pm \quad (y_3^k(y_3 - y_3)\prod_{i=k+2,k+1,\ldots3,2,1}\frac{1}{(y_i - y_3)}) \\
&\quad \vdots \\
&\pm \quad (y_{k-1}^k(y_{k-1} - y_{k-1})\prod_{i=k+2,k+1,\ldots3,2,1}\frac{1}{(y_i - y_{k-1})}) \\
&\pm \quad (y_{k+1}^k(y_{k+1} - y_{k+1})\prod_{i=k+2,k+1,\ldots3,2,1}\frac{1}{(y_i - y_{k+1})}) \\
&\pm \quad (y_{k+2}^k(y_{k+2} - y_{k+2})\prod_{i=k+2,k+1,\ldots3,2,1}\frac{1}{(y_i - y_{k+2})})]
\end{aligned}
$$

$$\times \Big(\prod_{\ell=k+2,k+1\ldots 3,2} \prod_{j=\ell-1,\ell-2\ldots 2,1} (y_\ell - y_j)\Big)$$

$$
\begin{aligned}
+y_{k+1}^k\Big[\; & \mp \; \Big(y_1^k (y_1 - y_1) \prod_{i=k+2,k,\ldots 4,2,1} \frac{1}{(y_i - y_1)}\Big) \\
& \mp \; \Big(y_2^k (y_2 - y_2) \prod_{i=k+2,k,\ldots 4,2,1} \frac{1}{(y_i - y_2)}\Big) \\
& \mp \; \Big(y_3^k (y_3 - y_3) \prod_{i=k+2,k,\ldots 4,2,1} \frac{1}{(y_i - y_3)}\Big) \\
& \quad \vdots \\
& \mp \; \Big(y_{k-1}^k (y_{k-1} - y_{k-1}) \prod_{i=k+2,k,\ldots 4,2,1} \frac{1}{(y_i - y_{k-1})}\Big) \\
& \mp \; \Big(y_k^k (y_k - y_k) \prod_{i=k+2,k,\ldots 4,2,1} \frac{1}{(y_i - y_k)}\Big) \\
& \mp \; \Big(y_{k+2}^k (y_{k+2} - y_{k+2}) \prod_{i=k+2,k,\ldots 4,2,1} \frac{1}{(y_i - y_{k+2})}\Big)\Big] \\
& \times \Big(\prod_{\ell=k+2,k\ldots 4,2} \prod_{j=\ell-1,\ell-2\ldots 2,1} (y_\ell - y_j)\Big)
\end{aligned}
$$

$$
\begin{aligned}
+y_{k+2}^k\Big[\; & \pm \; \Big(y_1^k (y_1 - y_1) \prod_{i=k+1,k,\ldots 4,2,1} \frac{1}{(y_i - y_1)}\Big) \\
& \pm \; \Big(y_2^k (y_2 - y_2) \prod_{i=k+1,k,\ldots 4,2,1} \frac{1}{(y_i - y_2)}\Big) \\
& \pm \; \Big(y_3^k (y_3 - y_3) \prod_{i=k+1,k,\ldots 4,2,1} \frac{1}{(y_i - y_3)}\Big) \\
& \quad \vdots \\
& \pm \; \Big(y_{k-1}^k (y_{k-1} - y_{k-1}) \prod_{i=k+1,k,\ldots 4,2,1} \frac{1}{(y_i - y_{k-1})}\Big) \\
& \pm \; \Big(y_k^k (y_k - y_k) \prod_{i=k+1,k,\ldots 4,2,1} \frac{1}{(y_i - y_k)}\Big) \\
& \pm \; \Big(y_{k+1}^k (y_{k+1} - y_{k+1}) \prod_{i=k+1,k,\ldots 4,2,1} \frac{1}{(y_i - y_{k+1})}\Big) \\
& \times \Big(\prod_{\ell=k+1,k\ldots 4,2} \prod_{j=\ell-1,\ell-2\ldots 2,1} (y_\ell - y_j)\Big) \qquad \text{(D.20)}
\end{aligned}
$$

If we look at only the $-y_j^k y_i^k$ factors, we see that each combination of i and j will appear exactly twice. For instance, looking at just the $-y_1^k y_2^k$ terms:

$$
\begin{aligned}
&y_1^k[y_2^k(y_2 - y_2) \prod_{i=k+2,k+1,\ldots 3,2} \frac{1}{(y_i - y_2)}] \times (\prod_{\ell=k+2,k+1\ldots 4,3} \prod_{j=\ell-1,\ell-2\ldots 3,2} (y_\ell - y_j)) \\
&-y_2^k[(y_1^k(y_1 - y_1) \prod_{i=k+2,k+1,\ldots 3,1} \frac{1}{(y_i - y_1)})] \times (\prod_{\ell=k+2,k+1\ldots 4,3} \prod_{j=\ell-1,\ell-2\ldots 3,1} (y_\ell - y_j))
\end{aligned} \tag{D.21}
$$

$$
\begin{aligned}
&= y_1^k y_2^k (\\
&\quad [(y_{k+2} - y_{k+1})(y_{k+2} - y_k)\ldots(y_{k+2} - y_3)][(y_{k+1} - y_k)(y_{k+1} - y_{k-1})\ldots(y_{k+1} - y_3)][\ldots][(y_4 - y_3)] \\
&- [(y_{k+2} - y_{k+1})(y_{k+2} - y_k)\ldots(y_{k+2} - y_3)][(y_{k+1} - y_k)(y_{k+1} - y_{k-1})\ldots(y_{k+1} - y_3)][\ldots][(y_4 - y_3)] \\
&\quad)
\end{aligned} \tag{D.22}
$$

$$= 0$$

In fact, the $y_i^k y_j^k$ pairs will always cancel. We can see this by noting that out of the $k+2$ possible variables, the coefficients will each contain only k variables, and that neither coefficient will contain the variables y_i or y_j. Since all the pairs cancel we can say:

$$
\begin{aligned}
&\sum_{p=1}^{p=k+2} \left(y_p^k (y_p - y_p) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_p)} \right) \times \left(\prod_{\ell=k+2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right) \\
&= 0
\end{aligned} \tag{D.23}
$$

and therefore:

$$
\begin{aligned}
&- \sum_{p=1}^{p=k+1} \left(y_p^k (y_p - y_p) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_p)} \right) \times \left(\prod_{\ell=k+2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right) \\
&= \left(y_{k+2}^k (y_{k+2} - y_{k+2}) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_{k+2})} \right) \times \left(\prod_{\ell=k+2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right)
\end{aligned} \tag{D.24}
$$

$$
- \sum_{p=1}^{p=k+1} \left(y_p^k (y_p - y_p) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_p)} \right)
$$

$$= \left(y_{k+2}^{k}(y_{k+2} - y_{k+2}) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_{k+2})} \right) \tag{D.25}$$

Now we can go back to equation D.14 and substitute in equation D.25:

$$\begin{aligned}
leftside = \\
& \Bigg(\left(y_{1}^{k+1}(y_1 - y_1) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_1)} \right) \\
& \left(y_{2}^{k+1}(y_2 - y_2) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_2)} \right) \\
& \left(y_{3}^{k+1}(y_3 - y_3) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_3)} \right) \\
& \vdots \\
& \left(y_{k}^{k+1}(y_k - y_k) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_k)} \right) \\
& \left(y_{k+1}^{k+1}(y_{k+1} - y_{k+1}) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_{k+1})} \right) \\
+y_{k+2} \quad & \left(y_{k+2}^{k}(y_{k+2} - y_{k+2}) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_{k+2})} \right) \Bigg) \\
& \times (-1)^{k+1} \left(\prod_{\ell=k+2}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right)
\end{aligned} \tag{D.26}$$

$$\begin{aligned}
leftside = \\
& \Bigg(\left(y_{1}^{k+1}(y_1 - y_1) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_1)} \right) \\
& \left(y_{2}^{k+1}(y_2 - y_2) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_2)} \right) \\
& \left(y_{3}^{k+1}(y_3 - y_3) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_3)} \right) \\
& \vdots \\
& \left(y_{k}^{k+1}(y_k - y_k) \prod_{i=k+2}^{i=1} \frac{1}{(y_i - y_k)} \right)
\end{aligned}$$

$$\left(y_{k+1}^{k+1}(y_{k+1}-y_{k+1})\prod_{i=k+2}^{i=1}\frac{1}{(y_i-y_{k+1})}\right)$$

$$\left(y_{k+2}^{k+1}(y_{k+2}-y_{k+2})\prod_{i=k+2}^{i=1}\frac{1}{(y_i-y_{k+2})}\right))$$

$$\times(-1)^{k+1}\left(\prod_{\ell=k+2}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell-y_j)\right) \tag{D.27}$$

$$leftside =$$

$$(-1)^{k+1}\sum_{\ell=1}^{\ell=k+2}\left(y_\ell^{k+1}(y_\ell-y_\ell)\prod_{i=k+2}^{i=1}\frac{1}{(y_i-y_\ell)}\right)\times\left(\prod_{\ell=k+2}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell-y_j)\right) \tag{D.28}$$

QED.

D.2 Proof: 1D Magnification Does Not Explicitly Depend on Perturbation Position

To prove that the total one dimensional magnification does not explicitly depend on position for n perturbations we write another recursive proof. The numerator of the total 1D magnification for n perturbations can be written as:

$$numerator =$$

$$(-1)^n(\quad +\varrho(y_1)((y_{n+1}-y_n)(y_{n+1}-y_{n-1})...(y_{n+1}-y_2))((y_n-y_{n-1})(y_n-y_{n-2})...(y_n-y_2))$$

$$\times(...)((y_4-y_3)(y_4-y_2))((y_3-y_2)$$

$$-\varrho(y_2)((y_{n+1}-y_n)(y_{n+1}-y_{n-1})...(y_{n+1}-y_1))((y_n-y_{n-1})(y_n-y_{n-2})...(y_n-y_1))$$

$$\times(...)((y_4-y_3)(y_4-y_1))((y_3-y_1)$$

$$+\varrho(y_3)((y_{n+1}-y_n)(y_{n+1}-y_{n-1})...(y_{n+1}-y_2)(y_{n+1}-y_1))((y_n-y_{n-1})$$

$$\times(y_n-y_{n-2})...(y_n-y_2)(y_n-y_1))(...)((y_4-y_2)(y_4-y_1))((y_2-y_1))$$

$$\vdots$$

$$\pm\varrho(y_n)((y_{n+1}-y_{n-1})...(y_{n+1}-y_2)(y_{n+1}-y_1))(...)((y_4-y_3)(y_4-y_2)(y_4-y_1))$$

$$\times((y_3 - y_2)(y_3 - y_1))((y_2 - y_1))$$

$$\mp\varrho(y_{n+1})((y_n - y_{n-1})...(y_n - y_2)(y_n - y_1))(...)((y_4 - y_3)(y_4 - y_2)(y_4 - y_1))$$

$$\times((y_3 - y_2)(y_3 - y_1))((y_2 - y_1))) \tag{D.29}$$

For odd and even n, respectively, and where $\varrho(y_j) \equiv (y_j - q_1)(y_j - q_2)...(y_j - q_n)$. Equation D.29 can be written more compactly using product and summation notation:

$$\begin{aligned}
numerator &= (-1)^n(\\
&+ \varrho(y_1)(y_1 - y_1) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_1)} \\
&+ \varrho(y_2)(y_2 - y_2) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_2)} \\
&+ \varrho(y_3)(y_3 - y_3) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_3)} \\
&\vdots \\
&+ \varrho(y_n)(y_n - y_n) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_n)} \\
&+ \varrho(y_{n+1})(y_{n+1} - y_{n+1}) \prod_{i=n+1}^{i=1} \frac{1}{(y_i - y_{n+1})}) \\
&\times \left(\prod_{\ell=n+1}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right)
\end{aligned} \tag{D.30}$$

We have already proven the base case in section 3.2, now we need only show that what holds for $n = k-1$ holds for $n = k$. Lets say that we have proven equation D.30 does not explicitly depend on perturbation location q_j for $n = k - 1$ perturbations:

$$\begin{aligned}
(-1)^{k-1}(&+ y_1^m(y_1 - y_1) \prod_{i=k}^{i=1} \frac{1}{(y_i - y_1)} \\
&+ y_2^m(y_2 - y_2) \prod_{i=k}^{i=1} \frac{1}{(y_i - y_2)} \\
&+ y_3^m(y_3 - y_3) \prod_{i=k}^{i=1} \frac{1}{(y_i - y_3)}
\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
+ \quad & y_{k-1}^m (y_{k-1} - y_{k-1}) \prod_{i=k}^{i=1} \frac{1}{(y_i - y_{k-1})} \\
+ \quad & y_k^m (y_k - y_k) \prod_{i=k}^{i=1} \frac{1}{(y_i - y_k)}) \\
\times \quad & \left(\prod_{\ell=k}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right) \\
= \quad & 0, for(0 \leq m < k-1)
\end{aligned} \tag{D.31}
$$

Then we must show that for n=k perturbations equation D.30 also does not explicitly depend on perturbation location:

$$
\begin{aligned}
(-1)^k(\quad + \quad & y_1^m (y_1 - y_1) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_1)} \\
+ \quad & y_2^m (y_2 - y_2) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_2)} \\
+ \quad & y_3^m (y_3 - y_3) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_3)} \\
&\vdots \\
+ \quad & y_k^m (y_k - y_k) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_k)} \\
+ \quad & y_{k+1}^m (y_{k+1} - y_{k+1}) \prod_{i=k+1}^{i=1} \frac{1}{(y_i - y_{k+1})}) \\
\times \quad & \left(\prod_{\ell=k+1}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) \right) \\
= \quad & 0, for(0 \leq m < k)
\end{aligned} \tag{D.32}
$$

Looking at expression D.32, we see that a given y_j^m will have a coefficient that contains all possible y terms except y_j itself. For this reason each coefficient can be rewritten using identity D.3 from section D.1:

$$
\prod_{\ell=k}^{\ell=2} \prod_{j=\ell-1}^{j=1} (y_\ell - y_j) =
$$

$$\left((-1)^{k-1}\sum_{\ell=1}^{\ell=k}\left(y_\ell^{k-1}(y_\ell-y_\ell)\prod_{i=k}^{i=1}\frac{1}{(y_i-y_\ell)}\right)\right)\times\left(\prod_{\ell=k}^{\ell=2}\prod_{j=\ell-1}^{j=1}(y_\ell-y_j)\right) \quad \text{(D.33)}$$

Which gives us:

$$\begin{aligned}
& \mathit{numerator} = \\
& (-1)^k(-1)^{k-1}(\\
+ \;& y_1^m\left(\sum_{\ell=2,3\ldots k,k+1} y_\ell^{k-1}(y_\ell-y_\ell)\prod_{i=k+1,k\ldots 3,2}\frac{1}{(y_i-y_\ell)}\right)\times\left(\prod_{\ell=k+1,k..4,3}\prod_{j=\ell-1\ldots 3,2}(y_\ell-y_j)\right) \\
- \;& y_2^m\left(\sum_{\ell=1,3\ldots k,k+1} y_\ell^{k-1}(y_\ell-y_\ell)\prod_{i=k+1,k\ldots 3,1}\frac{1}{(y_i-y_\ell)}\right)\times\left(\prod_{\ell=k+1,k..4,3}\prod_{j=\ell-1\ldots 3,1}(y_\ell-y_j)\right) \\
+ \;& y_3^m\left(\sum_{\ell=1,2\ldots k,k+1} y_\ell^{k-1}(y_\ell-y_\ell)\prod_{i=k+1,k\ldots 2,1}\frac{1}{(y_i-y_\ell)}\right)\times\left(\prod_{\ell=k+1,k..4,2}\prod_{j=\ell-1\ldots 2,1}(y_\ell-y_j)\right) \\
- \;& \ldots \\
\pm \;& y_k^m\left(\sum_{\ell=1,2\ldots k-1,k+1} y_\ell^{k-1}(y_\ell-y_\ell)\prod_{i=k+1,k-1\ldots 2,1}\frac{1}{(y_i-y_\ell)}\right)\times\left(\prod_{\ell=k+1,k-1..4,2}\prod_{j=\ell-1\ldots 2,1}(y_\ell-y_j)\right) \\
\mp \;& y_{k+1}^m\left(\sum_{\ell=1,2\ldots k-1,k} y_\ell^{k-1}(y_\ell-y_\ell)\prod_{i=k,k-1\ldots 2,1}\frac{1}{(y_i-y_\ell)}\right)\times\left(\prod_{\ell=k,k-1..4,2}\prod_{j=\ell-1\ldots 2,1}(y_\ell-y_j)\right)
\end{aligned} \quad \text{(D.34)}$$

For k odd and k even respectively. Expanding and grouping this by leading factor we get:

$$\begin{aligned}
& \mathit{numerator} = \\
-(y_1^{k-1} \quad & [-y_2^m\left((y_1-y_1)\prod_{i=k+1,k\ldots 3,1}\frac{1}{(y_i-y_1)}\right)\times\left(\prod_{\ell=k+1,k..4,3}\prod_{j=\ell-1\ldots 3,1}(y_\ell-y_j)\right) \\
& +y_3^m\left((y_1-y_1)\prod_{i=k+1,k\ldots 2,1}\frac{1}{(y_i-y_1)}\right)\times\left(\prod_{\ell=k+1,k..4,2}\prod_{j=\ell-1\ldots 2,1}(y_\ell-y_j)\right) \\
& \vdots \\
& \pm y_k^m\left((y_1-y_1)\prod_{i=k+1,k-1\ldots 2,1}\frac{1}{(y_i-y_1)}\right)\times\left(\prod_{\ell=k+1,k-1..4,2}\prod_{j=\ell-1\ldots 2,1}(y_\ell-y_j)\right) \\
& \mp y_{k+1}^m\left((y_1-y_1)\prod_{i=k,k-1\ldots 2,1}\frac{1}{(y_i-y_1)}\right)\times\left(\prod_{\ell=k,k-1..4,2}\prod_{j=\ell-1\ldots 2,1}(y_\ell-y_j)\right)]
\end{aligned}$$

$$
\begin{aligned}
+y_2^{k-1} \quad & [+y_1^m \left((y_2-y_2)\prod_{i=k+1,k\ldots3,2}\frac{1}{(y_i-y_2)}\right)\times\left(\prod_{\ell=k+1,k..4,3}\prod_{j=\ell-1\ldots3,2}(y_\ell-y_j)\right)\\
& +y_3^m \left((y_2-y_2)\prod_{i=k+1,k\ldots2,1}\frac{1}{(y_i-y_2)}\right)\times\left(\prod_{\ell=k+1,k..4,2}\prod_{j=\ell-1\ldots2,1}(y_\ell-y_j)\right)\\
& \vdots\\
& \pm y_k^m \left((y_2-y_2)\prod_{i=k+1,k-1\ldots2,1}\frac{1}{(y_i-y_2)}\right)\times\left(\prod_{\ell=k+1,k-1..4,2}\prod_{j=\ell-1\ldots2,1}(y_\ell-y_j)\right)\\
& \mp y_{k+1}^m \left((y_2-y_2)\prod_{i=k,k-1\ldots2,1}\frac{1}{(y_i-y_2)}\right)\times\left(\prod_{\ell=k,k-1..4,2}\prod_{j=\ell-1\ldots2,1}(y_\ell-y_j)\right)]
\end{aligned}
$$

$$
\begin{aligned}
+y_3^{k-1} \quad & [+y_1^m \left((y_3-y_3)\prod_{i=k+1,k\ldots3,2}\frac{1}{(y_i-y_3)}\right)\times\left(\prod_{\ell=k+1,k..4,3}\prod_{j=\ell-1\ldots3,2}(y_\ell-y_j)\right)\\
& -y_2^m \left((y_3-y_3)\prod_{i=k+1,k\ldots3,1}\frac{1}{(y_i-y_3)}\right)\times\left(\prod_{\ell=k+1,k..4,3}\prod_{j=\ell-1\ldots3,1}(y_\ell-y_j)\right)\\
& \vdots\\
& \pm y_k^m \left((y_3-y_3)\prod_{i=k+1,k-1\ldots2,1}\frac{1}{(y_i-y_3)}\right)\times\left(\prod_{\ell=k+1,k-1..4,2}\prod_{j=\ell-1\ldots2,1}(y_\ell-y_j)\right)\\
& \mp y_{k+1}^m \left((y_3-y_3)\prod_{i=k,k-1\ldots2,1}\frac{1}{(y_i-y_3)}\right)\times\left(\prod_{\ell=k,k-1..4,2}\prod_{j=\ell-1\ldots2,1}(y_\ell-y_j)\right)]
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
+y_k^{k-1} \quad & [+y_1^m \left((y_k-y_k)\prod_{i=k+1,k\ldots3,2}\frac{1}{(y_i-y_k)}\right)\times\left(\prod_{\ell=k+1,k..4,3}\prod_{j=\ell-1\ldots3,2}(y_\ell-y_j)\right)\\
& -y_2^m \left((y_k-y_k)\prod_{i=k+1,k\ldots3,1}\frac{1}{(y_i-y_k)}\right)\times\left(\prod_{\ell=k+1,k..4,3}\prod_{j=\ell-1\ldots3,1}(y_\ell-y_j)\right)\\
& +y_3^m \left((y_k-y_k)\prod_{i=k+1,k\ldots2,1}\frac{1}{(y_i-y_k)}\right)\times\left(\prod_{\ell=k+1,k..4,2}\prod_{j=\ell-1\ldots2,1}(y_\ell-y_j)\right)\\
& \vdots\\
& \mp y_{k+1}^m \left((y_k-y_k)\prod_{i=k,k-1\ldots2,1}\frac{1}{(y_i-y_k)}\right)\times\left(\prod_{\ell=k,k-1..4,2}\prod_{j=\ell-1\ldots2,1}(y_\ell-y_j)\right)]
\end{aligned}
$$

$$
\begin{aligned}
+y_{k+1}^{k-1} \quad & [+y_1^m \left((y_{k+1} - y_{k+1}) \prod_{i=k+1,k\ldots3,2} \frac{1}{(y_i - y_{k+1})} \right) \times \left(\prod_{\ell=k+1,k..4,3} \prod_{j=\ell-1\ldots3,2} (y_\ell - y_j) \right) \\
& -y_2^m \left((y_{k+1} - y_{k+1}) \prod_{i=k+1,k\ldots3,1} \frac{1}{(y_i - y_{k+1})} \right) \times \left(\prod_{\ell=k+1,k..4,3} \prod_{j=\ell-1\ldots3,1} (y_\ell - y_j) \right) \\
& +y_3^m \left((y_{k+1} - y_{k+1}) \prod_{i=k+1,k\ldots2,1} \frac{1}{(y_i - y_{k+1})} \right) \times \left(\prod_{\ell=k+1,k..4,2} \prod_{j=\ell-1\ldots2,1} (y_\ell - y_j) \right) \\
& \vdots \\
& \pm y_k^m \left((y_{k+1} - y_{k+1}) \prod_{i=k+1,k-1\ldots2,1} \frac{1}{(y_i - y_{k+1})} \right) \times \left(\prod_{\ell=k+1,k-1..4,2} \prod_{j=\ell-1\ldots2,1} (y_\ell - y_j) \right)])
\end{aligned}
\tag{D.35}
$$

For k odd and k even respectively. Rewriting:

$$
\begin{aligned}
& numerator = \\
-(y_1^{k-1} \quad & [-y_2^m \left((y_2 - y_2) \prod_{i=k+1,k\ldots3,2} \frac{1}{(y_i - y_2)} \right) \\
& -y_3^m \left((y_3 - y_3) \prod_{i=k+1,k\ldots3,2} \frac{1}{(y_i - y_3)} \right) \\
& \vdots \\
& -y_k^m \left((y_k - y_k) \prod_{i=k+1,k\ldots3,2} \frac{1}{(y_i - y_k)} \right) \\
& -y_{k+1}^m \left((y_{k+1} - y_{k+1}) \prod_{i=k+1,k\ldots3,2} \frac{1}{(y_i - y_{k+1})} \right)] \times \left(\prod_{\ell=k+1,k\ldots3} \prod_{j=\ell-1\ldots2} (y_\ell - y_j) \right) \\
+y_2^{k-1} \quad & [+y_1^m \left((y_1 - y_1) \prod_{i=k+1,k\ldots3,1} \frac{1}{(y_i - y_1)} \right) \\
& +y_3^m \left((y_3 - y_3) \prod_{i=k+1,k\ldots3,1} \frac{1}{(y_i - y_3)} \right) \\
& \vdots \\
& +y_k^m \left((y_k - y_k) \prod_{i=k+1,k\ldots3,1} \frac{1}{(y_i - y_k)} \right) \\
& +y_{k+1}^m \left((y_{k+1} - y_{k+1}) \prod_{i=k+1,k\ldots3,1} \frac{1}{(y_i - y_{k+1})} \right)] \times \left(\prod_{\ell=k+1,k\ldots4,3} \prod_{j=\ell-1\ldots3,1} (y_\ell - y_j) \right)
\end{aligned}
$$

$$
\begin{aligned}
+y_3^{k-1} \quad & [+y_1^m \left((y_1 - y_1) \prod_{i=k+1,k\ldots 4,2,1} \frac{1}{(y_i - y_1)} \right) \\
& +y_2^m \left((y_2 - y_2) \prod_{i=k+1,k\ldots 4,2,1} \frac{1}{(y_i - y_2)} \right) \\
& \vdots \\
& \pm y_k^m \left((y_k - y_k) \prod_{i=k+1,k\ldots 4,2,1} \frac{1}{(y_i - y_k)} \right) \\
& \mp y_{k+1}^m \left((y_{k+1} - y_{k+1}) \prod_{i=k+1,k\ldots 4,2,1} \frac{1}{(y_i - y_{k+1})} \right)] \times \left(\prod_{\ell=k+1,k\ldots 4,2} \prod_{j=\ell-1\ldots 4,2,1} (y_\ell - y_j) \right)
\end{aligned} \tag{D.36}
$$

$$
+ \vdots
$$

$$
\begin{aligned}
+y_k^{k-1} \quad & [+y_1^m \left((y_1 - y_1) \prod_{i=k+1,k-1\ldots 2,1} \frac{1}{(y_i - y_1)} \right) \\
& +y_2^m \left((y_2 - y_2) \prod_{i=k+1,k-1\ldots 2,1} \frac{1}{(y_i - y_2)} \right) \\
& +y_3^m \left((y_3 - y_3) \prod_{i=k+1,k-1\ldots 2,1} \frac{1}{(y_i - y_3)} \right) \\
& \vdots *** fixsigns? \\
& +y_{k+1}^m \left((y_{k+1} - y_{k+1}) \prod_{i=k+1,k-1\ldots 2,1} \frac{1}{(y_i - y_{k+1})} \right)] \times \left(\prod_{\ell=k+1,k-1\ldots 4,2} \prod_{j=\ell-1,k+1,k-1\ldots 2,1} (y_\ell - y_j) \right) \\
+y_{k+1}^{k-1} \quad & [+y_1^m \left((y_1 - y_1) \prod_{i=k,k-1\ldots 2,1} \frac{1}{(y_i - y_1)} \right) \\
& +y_2^m \left((y_2 - y_2) \prod_{i=k,k-1\ldots 2,1} \frac{1}{(y_i - y_2)} \right) \\
& +y_3^m \left((y_3 - y_3) \prod_{i=k,k-1\ldots 2,1} \frac{1}{(y_i - y_3)} \right) \\
& \vdots \\
& +y_k^m \left((y_k - y_k) \prod_{i=k,k-1\ldots 2,1} \frac{1}{(y_i - y_k)} \right)] \times \left(\prod_{\ell=k,k-1\ldots 4,2} \prod_{j=\ell-1\ldots 2,1} (y_\ell - y_j) \right))
\end{aligned} \tag{D.3}
$$

Written in this form, we see that each coefficient of y_j^{k-1} is in the form of (equation D.31)and that we can therefore say that for any $0 \leq m < k-1$ they will be zero. But we already showed in section D.1, equation D.23 that the $m = k-1$ terms will be zero! So equation D.32 will be zero for any $0 \leq m < k$ and we have completed the recursive proof. This illustrates that the 1D magnification is does not explicitly depend on perturbation location for n perturbations.

Together sections D.1 and D.2 prove that the total 1D magnification will be one for n perturbations!

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