Massachusetts Institute of Technology
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Working Paper 308 March 1988

Spurious Behaviors in Qualitative Prediction

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Abstract

I examine the scope and causes of the spurious behavior problem in two widely different approaches to qualitative prediction, Sacks' PLR and Kuipers' QSIM. QSIM's proliferation of spurious behaviors and PLR's limited applicability and problematic extensibility lead me to propose a third, intermediate approach to qualitative prediction called the Phase Space Geometry approach. This has the potential advantages of predicting far fewer spurious behaviors than QSIM-like approaches and being directly applicable to nonlinear systems of all orders.

1 Introduction

The world is full of dynamical systems with which humans and machines must interact. Many problems require the ability to predict the behavior of such systems. For example, design requires verification, common sense problem solving and planning require the ability to predict consequences, and troubleshooting demands a thorough understanding of the causal interactions among a device's components.

Unfortunately, the quantitive prediction of behavior is usually extremely demanding, often unnecessary, and sometimes even uninformative. First, it requires the ability to solve differential equations, most of which are unsolvable in closed form. Second, many naive reasoners, such as children, are able to predict the behavior of gadgets well enough to use them to solve problems, avoid danger, and even fix them when broken. Third, a particular numerical simulation of a system may not uncover all interesting potential behaviors of the system.

In this paper, I explore two different basic approaches to *qualitative* behavior prediction: simulation of qualitative constraints and the Piecewise Linear methodology of Sacks (1987). In particular, I examine the types of errors these approaches commit and suggest ways of improving their performance. Section 3 puts forth a proposal for a third methodology which, I believe, is intermediate in both performance and complexity, yet is potentially much more accurate than qualitative simulation and more widely applicable than PLR.

1.1 Goals and Assumptions of Qualitative Prediction

Simply stated, the goal of qualitative behavior prediction is to describe the possible behaviors of a dynamical system in qualitative terms, possibly restricting attention only to those behaviors resulting from a certain class of initial states. Qualitative behaviors are sequences (or graphs) of qualitative state transitions through which the device passes (or could possibly pass) after being coerced to a given initial state (or class of states). A system changes qualitative state either when one or more of its measurable parameters changes qualitative value or when the operating region of some component (or components) changes, causing the system's parameters to obey a different set of constraints.

deKleer & Brown (1984) go so far as to say that a goal of a qualitative physics should be to completely avoid the underlying quantitative physics. The idea is to avoid the computational difficulty inherent in quantitative approaches. Also, it is introspectively unrealistic to suppose that human naive reasoners are even aware of the differential equations underlying everyday experience.

Until the domain of "common sense devices" is better defined, however, the above view is extreme. It is rather doubtful that naive humans are any good at all at predicting the behavior of complex systems. To address the general problem of qualitative prediction, it is sufficient merely to require that the qualitative reasoner do significantly less work than a quantitative reasoner. This leaves open the possibility of using the far greater information in the quantitative equations to rule out some spurious behaviors. Thus, this paper is really about qualitative behavior prediction from exact quantitative differential equations.

1.2 Approaches to Qualitative Behavior Prediction

I have reviewed two basic approaches to the problem of behavior prediction. The first approach, on which by far the most artificial intelligence research has been done (Kuipers, 1986; deKleer & Brown, 1984; Forbus, 1984), is simulation using qualitative constraints. The idea is that the components of a device behave according to qualitative abstractions of their underlying quantitative laws. Parameter values are also abstracted from real numbers to their ordinal relations with a set of "landmark values." (I include derivatives as parameters of the system.) Behaviors are generated by starting in an initial state, generating all possible qualitative state transitions compatible with the qualitative constraints, and then iterating, forming a graph of possible behaviors. State transitions occur either when some parameter reaches (or leaves) a landmark value, or when the qualitative constraints governing device behavior change (due, for example, to a change in operating region of a device).

This approach is cleanly formalized by Kuipers (1986) in his QSIM program, so I will use it as the source of examples of this approach. While QSIM is not identical[1] to the systems of deKleer & Brown (1984) and Forbus (1984),

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[1] QSIM, in its basic form, creates new landmark values during the course of the simu-

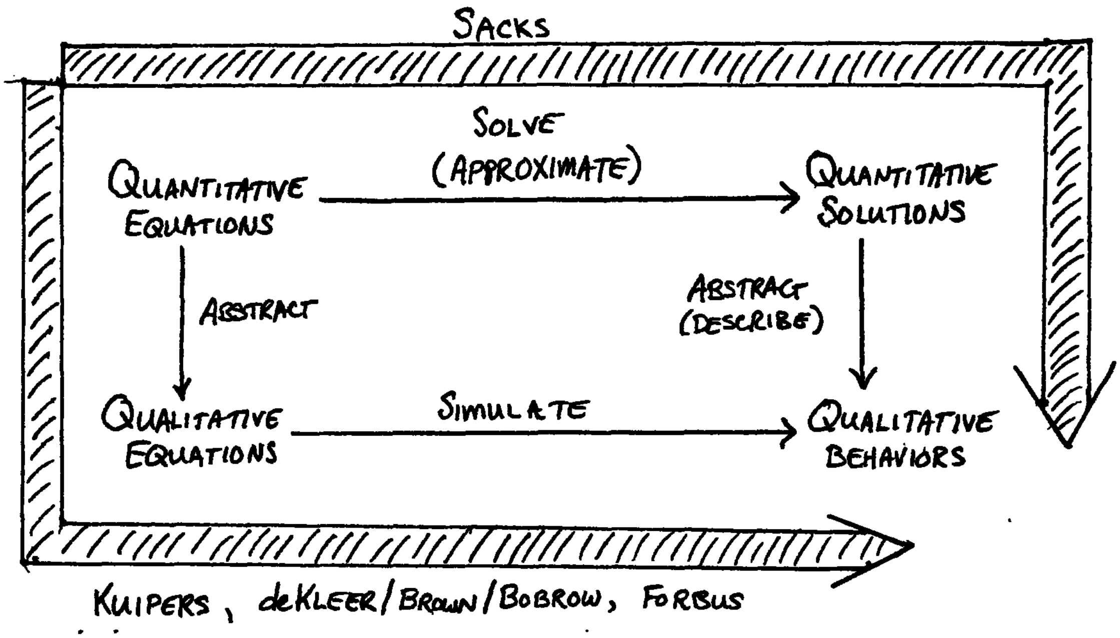

Figure 1: A diagram showing the differences in the two approaches to qualitative simulation.

it can be made compatible with either through use of the appropriate state transition table and landmark value set.

The other approach is the piecewise linear methodology of Sacks (1987). The idea here is that linear systems of differential equations are easy to solve exactly, and many nonlinear systems can be approximated by piecewise linear functions. Sacks' approach is to approximate the system of quantitative equations, solve for the local behaviors, and then patch the local solutions together into an approximate picture of the phase space of the system, with trajectories' qualitative behaviors indicated. To predict behavior from such a picture, one merely locates the initial conditions in the phase space and follows the appropriate qualitative trajectory. Figure 1 illustrates the difference in the approaches.

1.3 Types of Spurious Behaviors

All known systems for qualitative behavior prediction are imperfect in that they can either predict impossible (spurious) behaviors or fail to predict real

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lation (the others do not). I will discuss this feature when appropriate.

behaviors. There are basically four types of such errors. (These categories will be useful in reviewing existing systems.)

1. The system can fail to predict a correct qualitative behavior of the system. For example, the system could fail to predict the possibility of oscillation in a system.

2. The system can predict an impossible qualitative behavior. That is, there is *no dynamical system* which could exhibit a quantitative behavior which would be described qualitatively in that way. For example, it could predict that the derivative of a parameter is positive at $t = \infty$ with the parameter itself bounded from above.

3. The system can predict a behavior possible for some dynamical system, but not possible for one satisfying equations qualitatively like the given one. For example, it could predict oscillation of a parameter whose acceleration is constant.

4. The system can predict a behavior possible for some solution of some system of equations satisfying the same qualitative constraints, but impossible for the actual quantitative constraints. For example, the systems $y' = y^2$ and $y' = y^{1/2}$ both express y' as a monotonic increasing function of y, but one has y'' go to infinity with t, while the other has y'' go to zero. The prediction system could predict both possible behaviors for one of the systems. Thus, one such prediction would be spurious.

2 The Spurious Behavior Problem

This section reviews the scope of and proposed solutions to the spurious behavior problem in the two basic approaches.

2.1 Simulation of Qualitative Constraints

Type 1 errors. The basic QSIM algorithm (Kuipers, 1986) cannot commit Type 1 errors. The author shows that the correct qualitative behavior will always be among those produced, though the system will not know which is correct unless it finds only one.

Type 2 errors. QSIM can commit each of the other error types, however. Consider a rock falling from rest under the influence of constant gravity and subject to air friction. (See Example 3 in the Appendix.) The equation is $x'' = 1 - kx'$. QSIM predicts two impossible behaviors in addition to the correct one. The correct behavior is that the rock will asymptotically approach a finite "terminal velocity" and the rock will fall to infinity.

One impossible behavior is that the rock will fall to infinity with its acceleration strictly decreasing at infinity but bounded from below by 0. This is impossible because any parameter whose derivative does not approach zero as time goes to infinity must also go to (plus or minus) infinity. This type of error would be simple for QSIM to rule out.

The other impossible behavior is that the rock's velocity increases to and reaches the terminal velocity after finite time, remaining constant thereafter. No analytic solution to any equation can have this behavior, as the function must be represented by its Taylor series in a neighborhood around every point. Any system $y' = f(t, y)$ in which f is analytic and bounded in a neighborhood around the point (t_0, y_0) has a unique, analytic solution in that neighborhood.[2] Real-world dynamical systems tend to be described by analytic and bounded functions f, at least within their given operating ranges. (Obviously, when the constraints change, a parameter function may not be analytic at the change point.) Thus, it is quite reasonable to assume that the behavior of a physical system will be analytic. deKleer & Bobrow

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[2] This result is well-known in the theory of differential equations. It justifies the power-series method of solving differential equations.

(1984) incorporate this assumption into their "no change to constant" rule.[3] QSIM could easily incorporate such a rule as well.

Type 3 errors. The frictionless mass–spring system provides an example of QSIM producing Type 3 errors. (See Example 5 in the Appendix and Kuipers' (1986) discussion of this problem.) The essence of the problem is that QSIM predicts the possibility that the mass has a different velocity each time it passes through the rest-length point ($a = 0$). The correct behavior is that it has exactly the same velocity each time. The other behaviors are impossible for any qualitatively similar equation (*i.e.* replacing the linear dependency on x with any monotonic function of x that goes through (0,0) and goes to infinity with x). This can be shown via an energy conservation argument.

This proliferation of behaviors is seen in any of the qualitative simulation systems, but is exacerbated by the fact that QSIM generates a new landmark value each time a goes through 0. Each new landmark, l, creates three more distinct places v could be when a passes through 0 next ($l_i < v < l, v = l, l < v < l_{i+1}$). This fact calls into question the value of creating new landmarks at each critical point of a parameter. Kuipers (1986) argues that it is essential to do so in order to express the difference between stable, decaying, and increasing oscillation. On the other hand, the system usually lacks the ability to reason about when the parameters reach the landmarks relative to each other, thus generating several equally plausible possibilities instead of one, more abstract possibility.

A better approach might be to use domain knowledge to pick (in advance) the set of landmark values (possibly including critical values). For example, one might pick as an extra landmark the initial velocity of the mass. This is enough to express the different types of oscillation, so that no further landmarks need be defined. It does not solve the basic problem of ruling out non-stable oscillations, but it does at least reduce the branching at each point in the tree by about a factor of three. It also makes it possible to represent all behaviors finitely in a graph, rather than the infinitely branching tree necessary to QSIM's predictions.

Type 4 errors. If the system does not have access to the quantitative equations, then Type 4 errors are not errors at all. However, if the reasoning

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[3] Actually, they say they assume that the function f is C^{∞}, a weaker condition than analyticity. I believe this to be an error.

system does know the quantitative equations, there can be qualitative properties of the equations that are not captured by the qualitative constraints. For example, consider the system $x'' = 1 + x' - (x')^2$. (See Example 4 in the Appendix). QSIM would straight-forwardly encode this as $a = 1 + v - M^+(v)$. It could not deduce that a is bounded above, so it predicts the possibility of a increasing to $+\infty$.

The previous example illustrates a very fruitful source of Types 3 and 4 spurious behaviors: constraints of the form $y = M^+(x) - M^+(z)$ (or with various permutations of signs). This form of constraint is quite common in describing physical systems. Such a constraint leaves y virtually unconstrained when x and z are changing in the same direction; this requires QSIM to predict all possible behaviors for y. One way in which this indeterminacy is manifested in QSIM is called "chattering." (Kuipers & Chiu, 1987). This means that every critical point of y results in a three-way branch: either y and its derivative go up, go down, or stay the same. Examples 1 and 2 in the Appendix show this spurious "chattering" behavior when simulated by QSIM. In the Cascaded Tanks example (Example 1) this comes from the fact that $B' = M^+(A) - M^+(B)$.

Kuipers & Chiu (1987) propose two solutions for this particular type of spurious behavior. The first approach is based on the observation that if you don't care about the behavior of a parameter's derivative, then you don't have to keep explicit track of it. This alleviates spurious branching in the behavior tree because the partial qualitative states $(y, \Uparrow)$, $(y, \ominus)$, and $(y, \Downarrow)$ are regarded as the same. QSIM makes sure for any appearance of a don't-care derivative that some real derivative value could be substituted. This is a fine bookkeeping convention, but does not really address the deep issue at all. The issue of whether or not one is interested in some parameter's derivative is orthogonal to whether or not QSIM predicts spurious behaviors for it.

The second approach (the "HOD" method) is based on the idea of using higher-order derivatives of a function to determine the direction of change at a critical point. deKleer & Bobrow (1984) suggest this methodology, and Kuipers & Chiu (1987) incorporate it into the QSIM framework. The essence of their implementation is to symbolically derive the sign of the curvature of the chattering parameter at points at which the first derivative is zero by using only the qualitative constraints, together with rewrite rules. The ambiguity of the qualitative constraints is a major obstacle to such an endeavor. The key assumption which drives the algorithm is that for any monotone relationship $y = M(x)$, the curvature of M, d^2M/dx^2, is zero (or at least negligible compared with dM/dx). This implies that y'' has the same sign as x''. This assumption is valid for linear systems (which their examples happen to be), but flagrantly wrong for many nonlinear systems.

To see this, consider the simple nonlinear system,

$$\begin{aligned} B &= X^2 - A \\ A'' &= -4 \\ X'' &= -2 \\ X(1) &= 9 \\ X'(1) &= -2 \\ A(1) &= -2 \\ A'(1) &= -4 \end{aligned}$$

Starting the simulation at $t = 1$, QSIM quickly gets to the point ($t = 3$) where B' is zero. This results in a potential three-way branch. The HOD method deduces that the sign of the curvature (sd2) of B satisfies

$$\text{sd2}(B) = \text{sd2}(X) = \text{sgn}(-2) = -$$

Thus, it rules out branches where B remains constant or increases. However, the actual solution of the above system is

$$\begin{aligned} X(t) &= 10 - t^2 \\ A(t) &= -2t^2 \\ B(t) &= t^4 - 18t^2 + 100 \end{aligned}$$

Differentiation shows that $B''(3) = 72$, a positive number. In fact, B' is positive for all $t > 3$. Therefore, QSIM ruled out the correct behavior in favor of a spurious one.

The example above shows that the HOD method destroys one of the nicest features of QSIM: it can cause Type 1 errors. Another drawback of the HOD method is that it is not guaranteed to resolve the question of which way to branch.[4] On the other hand, the HOD method will not prune the

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[4] Consider $B = M^+(A) - M^+(C)$. If A's and C's second derivatives are known to be positive, we still have no information on the sign of B's second derivative. It would not be correct to assume $+ - + = 0$.

correct behavior if the system is linear. Thus, if the system has no access to the quantitative equations, but knows they are linear, this method might still be useful.

Possible improvements to QSIM. Following is a list of potential improvements to QSIM.

- To avoid some Type 2 errors, the system could easily incorporate the rules mentioned earlier: (1) parameters can not go from non-constant to constant, except when the operating region of some device changes; and (2) derivatives of bounded parameters must go to zero at infinity.

- Assuming access to the quantitative equations, the system could avoid many Type 3 and 4 errors by adopting a *piecewise monotonic* approach (analogous to the piecewise linear approach of Sacks (1987), see next subsection). If one can subdivide the parameter ranges into areas where dependencies are either monotonic increasing or decreasing, many spurious behaviors can be avoided. For example, the equation $y' = y - y^2, y > 0$ becomes hopelessly ambiguous when translated to $y' = M^+(y) - M^+(y)$. If it is split into two regions, $0 < y \leq 1/2$ and $1/2 \leq y$, the constraint becomes $y' = M^+(y)$ and $y' = M^-(y)$ respectively. My hand simulation produced all and only the behaviors compatible with these qualitative constraints (no Type 3 errors). In addition, the particular Type 4 error of y' going to infinity was avoided. Note also that this methodology still does not allow Type 1 errors, but does require the ability to find the criticial points of the functions. Also, some extra rules are needed for reasoning about allowable behaviors of functions on transitions between regions. These are straight-forwardly derived from continuity considerations.

- Domain knowledge could be brought to bear to reduce the spurious behaviors. The mass-spring system can be correctly simulated if the notion of total energy is added explicitly to the constraint set (*i.e.* add the constraint that total energy is constant; Kuipers (1985) shows how to do this). This seems to be special-case knowledge of the domain of conservative physical systems.

2.2 Sacks' Piecewise Linear Approach

Type 1 Errors. Sacks' approach, being based on piecewise linear approximations, can commit Type 1 errors when the approximation fails to capture the qualitative properties of the solutions of the exact equation. Consider the equation $y' = y^{-1}, y > 0$. (See Example 9.) The qualitative behavior of its solution flow is for y to go to infinity with t, with y' approaching zero asymptotically. However, any linear approximation to this equation either crosses $y' = 0$ at finite y, in which case y approaches a finite limit asymptotically, or else it fails to cross zero, in which case, y goes to infinity with y' either a positive constant or infinity. Neither case captures the qualitative behavior desired.

Another way such errors can, in principle, arise is that PLR operates by successively refining the linear approximation. It uses the heuristic that if no different qualitative properties are found between iterations n and $n + 1$, then it terminates. Presumably, this strategy can be defeated, though I know of no examples for which an adequate linearization exists that PLR fails to find.

Even when an example has an adequate piecewise linear approximation, such can be difficult to find. PLR's approach depends in part upon finding the critical points of the nonlinear function. This process involves first finding the (symbolic) derivative and then finding an expression for its zeros. This is impossible in general, though easy for low-degree polynomials and some trigonometric functions.

Sacks (1988) gives some examples in which his system fails to produce the correct behavior.

Type 2 Errors. It seems unlikely that PLR would commit Type 2 errors, as it is reasoning about the quantitative equations. That is, it solves the equations, then describes the solution. Thus, when it produces a qualitative description, it is of some solution. On the other hand, it can be overly conservative and leave in impossible transitions in the transition graph, so it is still possible it's qualitative behaviors are impossible for any reasonable functions.

Types 3 and 4. PLR can commit errors of Types 3 and 4. Sacks (1988) gives the example of the horseshoe magnet pendulum, where his system allows some state trajectories which gain overall energy. Spurious behaviors arise principally from a failure to rule out possible transitions in the global transition diagram. Two causes of this are (1) weak symbolic inequality reasoning failing to show that the phase space tangent vector can not point across a region boundary, and (2) the region splitting method (of ruling out trajectories) failing to recursively split upstream regions as needed.

Advantages.

- In principle, PLR solves exactly all linear systems. As many examples from nature are linear, this is an important strength.
- It exploits the greater information of the quantitative equations to generate far fewer spurious behaviors than qualitative simulation. In particular, it can exploit qualitative properties like "y' is bounded above" to avoid many Type 3 and 4 errors typical of the qualitative simulation systems.
- It generates a state diagram for the system covering all behaviors at once. Qualitative simulation programs only give the behaviors resulting from a single (qualitative) initial state. (Of course, QSIM could be run several times, starting once from each possible initial state.)

Limitations.

- The result is only as good as the piecewise linear approximation. For some systems, good piecewise linear approximations are difficult to find. For many systems, *no* such is adequate. In particular, the infamous pressure regulator example (deKleer & Bobrow, 1984; deKleer & Brown, 1984; Forbus, 1984) contains a nonlinear term due to the valve, across which the flow is the product of the pressure differential and the area: $F = PA$. If we connect this to a constant-flow device, then we have, as one of the system constraints, $k = PA$, k a constant. Since A' is proportional to the pressure on the output side of the valve, this is essentially the same example mentioned earlier, for which no adequate piecewise linear approximation exists.
- The system is limited by its inequality reasoner. Failure to prove inequalities among symbolic formulae leads directly to spurious behaviors.

- The current system is limited to two-dimensional phase spaces. This rules out many simple examples.
- The extension to higher orders and higher-dimensional phase spaces is problematic due to a lack of theoretical tools for constraining the spurious behaviors. For example, there is no known high-order analog of the Poincaré-Bendixson Theorem, which constrains the types of limiting behavior planar dynamical systems may have. There is no general analog of Liapunov functions for general dynamical systems.

To summarize, Sacks' piecewise linear approach is a major step toward avoiding spurious behavior errors, but its extension to higher orders may be fundamentally limited by a lack of mathematical tools. Also, many systems are either difficult or impossible to approximate linearly, so other approaches are needed.

3 A Proposal

The qualitative simulation approach is attractively simple, but seems to generate far too many spurious behaviors to be useful with complex systems. On the other hand, Sacks' approach produces very good results when it is applicable. Unfortunately, it is currently quite limited in applicability, and its prospects for extension are at best debatable. The two approaches represent ends of a spectrum; on one hand, the qualitative abstraction is performed at the outset, making behavior prediction relatively easy. The PLR approach tackles the complexity of the equations head on, only performing the qualitative description at the end. This section presents a proposal for qualitative behavior prediction that exploits quantitative information from the equations, but does not attempt to solve or even approximate a global solution to the equations.

Qualitative simulation expresses the qualitative states of a system by tessellating its phase space. Each parameter has a quantity space, or set of landmark values and intervals between them. The phase space is divided into regions corresponding to particular choices of the values (or intervals) taken on by each parameter and its derivative. The qualitative simulator attempts to form a diagram of possible transitions among these regions. (QSIM can dynamically refine the tessellation by creating new landmark values. For now, we ignore this and assume it remains fixed.)

Proposal. Use the quantitative equations to create this transition diagram directly, using only analytic geometry and inequality reasoning. I tentatively call this a "phase space geometry" approach.

3.1 An Example

To refine this idea, refer to Example 1, the Cascaded Tanks, in the Appendix. Two leaky tanks, A leaking into B, are being filled by a constant flow F^* into A. The flow out of a hole is proportional to the amount in the tank. Representing the amounts as A and B respectively, we get the equations

$$\begin{aligned} A' &= F^* - kA \\ B' &= kA - lB \end{aligned}$$

This system has a two-dimensional phase space (A versus B). For this example, I will work with quantity spaces containing only the single landmark, 0. The extension to any landmark set is straight-forward.

The proposed method first tessellates the phase space by graphing $A' = 0$, $B' = 0$, $A'' = 0$, and $B'' = 0$ using the defining differential equation. Referring to the phase space diagram of Example 1, these are the lines $A = F^*/k$, $A = (l/k)B$, $A = F^*/k$, and $A = (1/(l+k))[F^* + Bl^2/k]$ respectively. Ignore, for the moment, the line $B = F^*/l$. Each of the six regions, labelled 1 through 6 [5] corresponds to a different sign combination of A', B', A'', and B''. I have indicated the range of directions the phase space tangent vector may take on by two arrows joined at the tail. (Thus, for example, any state in region 3 must travel strictly downward and strictly rightward.) For each boundary, I have indicated which way the tangent vector points; either along the boundary, into, or out of that boundary, with the circle around the intersection point indicating that the tangent vector is zero. This is easy to determine analytically, because we know the equation of each boundary (*e.g.* $B' = 0$ implies $A = B(l/k)$), so we may find the normal vector; and we know the tangent vector (*e.g.* for the $B' = 0$ boundary, we get tangent vector $= (F^* - kA)\hat{A}$). We may therefore compute the dot product and determine whether it is positive, negative or zero.

The dashed line is introduced because any point starting in region 3a must eventually reach the 3a–4 boundary. This is because its tangent must point down and to the right. Note that a point in region 3 could either pass to 3a or approach region 7 asymptotically.

No point can enter the 2–3 boundary from outside of it. This is ruled out by uniqueness of solutions, since the point $(F^*/k, 0)$ travels through the entire 2–3 boundary to approach 7 asymptotically, and no point from region 2 or 3 can reach that initial point.

No point can traverse the 1–6a boundary from 1 to 6a because the tangent vector points the wrong way for such a transition. No point can start in 1 and have its trajectory meet that boundary by uniqueness of solutions and by the fact that the tangent vector field is transverse to the boundary. (We can form a flow box about any point on the boundary.)

We can further constrain the direction of the tangent vector by graphing the next higher-order phase space, A' versus B', and showing the $B'' = 0$

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[5] Include 3a as part of 3 and 6a as part of 6.

line. This shows that in region 1, the point must always move strictly away from the $B' = 0$ line, so any trajectory must cross into region 2 in finite time. This justifies removing the asymptotic approach arrow from 1 to 7.

The diagram is symmetrical about the $A' = 0$ line, so similar reasoning applies to the upper regions. All this reasoning leads to the qualitative state diagram shown. An arc with two hash marks across it indicates asymptotic approach. The semantics of the diagrams are that a point starting in a state must leave by one of the arrows leaving the state, with hashed arrows indicating that the transition requires t to go to infinity. The non-hashed transitions must take place in finite time.

This diagram contains no spurious behaviors and is therefore much better than the behavior tree produced by (unmodified) QSIM. Starting from (0,0) (region 1–6a), QSIM predicts all possible paths which traverse and touch the 1–2 boundary. It also defines many unimportant new landmark values. As the phase space diagram indicates, the one useful new landmark (not found by QSIM) would be F^*/l in B's quantity space, as this delimits regions 3a and 6a. This falls out naturally from consideration of allowable ranges of tangent vectors.

Of course, Sacks' system would solve this linear problem exactly and yield slightly more information than my proposal (in that it would also give information as to the shapes of the individual trajectories).

The Appendix shows the application of this approach to nine examples. Where appropriate, it compares the performance to that of the other approaches.

3.2 The Proposed Method

Of course, this is still only in the proposal stage, so the following will be a rough sketch. I will state it as if I were doing it myself, so it may be described rather geometrically. The bulk of the research will be translating the geometric operations into analytic terms. For example, deciding whether a tangent vector points into or out of a hypersurface involves deciding analytically whether $n \cdot t > 0$. Sacks has implemented code to do this task in his system.

1. *Given:* a dynamical system expressed in equation form as

$$x' = f(x), \quad x \in U \subset \mathbf{R}^n$$

2. *Tessellate:* For each x_i and each landmark value l_j in the quantity space of x'_i, sketch (in the phase space) the hypersurface defined by

$$x'_i(x) = l_j$$

   We may refine this tessellation by sketching other surfaces of interest, such as $B'' = 0$ in Example 1. (Because the tanks form a first order system, B' would not be a phase variable.) This divides the phase space into a finite number of regions, according to where in the quantity space each x'_i falls. Note that the intersections of the hypersurfaces correspond to two or more derivatives being determined. Thus, some of the regions can be of low dimension (for example, region 7 in Example 1 is a zero-dimensional point). Construct a qualitative state diagram whose states correspond to these regions. Initially, assume transitions are possible between any two regions which share a common boundary. The following steps eliminate impossible transitions.

3. *Constrain tangent vector:* Each region, by its definition, confines the direction (and magnitude) of the tangent vector field. (In Example 1, the tangent vector in region 3 must point down and to the right; The tangent vector in region 2–3 must point to the right; etc.) Remove any transition from one region to another where no point in the first region could possibly reach the boundary due to these constraints on the direction of the tangent vector. In Example 1, no transition is possible from region 2–3 to either region 2 or 3, because the tangent vector is constrained to point to the right. As in Example 1, it may be possible to further limit the direction of the tangent vector by using higher order derivative information (this is why the A'–B' phase diagram was introduced). In that example, the tangent vector in region 1 must go strictly rightward and at strictly steeper slope than that of the $B' = 0$ line.

4. *Identify new landmark surfaces.* It may be that points in some subregion can only move to some subset of the possible transitions for the whole region. (This can be partially determined by projecting the boundaries backwards in the opposite half of the cone of allowable trajectories for each region.) Subdivide regions for which this is true and replace them with the subregions in the state diagram. The surfaces

separating the new subregions might be termed "landmark surfaces" by analogy with landmark values because they delimit zones of qualitatively different behavior. In Example 1, this results in regions 3a and 6a splitting off of regions 3 and 6, because points in 3a can only go to 4 (not 7) and points in 6a can only go to 1 (not 7).

5. *Exploit Extra Knowledge.* Additional knowledge can be exploited by judicious choice of landmark surfaces. For example, spurious behaviors can be eliminated by introduction of surfaces of constant energy. Note that the system needs only know the form of the energy function, it can calculate $\mathbf{n} \cdot \mathbf{t}$ itself to show, for example, that trajectories always point into a surface of constant energy for dissipative systems. Examples 5 and 6 illustrate the use of energy level surfaces in eliminating spurious behaviors from the mass-spring and damped pendulum systems.

6. *Calculate* $\mathbf{n} \cdot \mathbf{t}$. A trajectory from region A to region B, across boundary AB is possible only if the tangent vector to the flow in AB has a positive component in the direction of the normal to the boundary which points into B. That is, only if $\mathbf{n} \cdot \mathbf{t} > 0$ on AB. In Example 1, a point may move from region 6a across 1–6a into 1, but not vice versa. Similarly, transitions across 3a–4 must go from 3a to 4. No transition is allowed across the 2–3 boundary because the tangent vector does not point across. If the sign of $\mathbf{n} \cdot \mathbf{t}$ is known to be both positive and negative on a boundary, another new landmark surface may be useful: the locus of points such that $\mathbf{n} \cdot \mathbf{t} = 0$. This subdivides the boundary and the regions bordering the boundary, so that transitions among the subregions are unidirectional. Example 8 illustrates the creation of such a new landmark surface: **S**.

7. *No-change-to-constant rule.* A transition in finite time to a boundary defined by a derivative going to zero is possible only if a transition across it is possible.[6] Thus, no finite-time transition is possible into regions 2–3, 5–6, or 7.

8. *Derivatives-zero-at-finite-limits.* No trajectory may approach a (finite) point asymptotically unless the tangent vector goes to zero. In Example

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[6] I have only been able to prove this in higher dimensions assuming analyticity and boundedness.

1, this rules out asymptotic transitions from 2 to 2–3, 3 to 2–3, 4 to 4–5, and 5 to 4–5. The only asymptotic approaches allowed are to 7.

See the Appendix for further discussion of the examples.

Types of Errors. Assuming the exact equations are used, no Type 1 errors are possible. This is because we only eliminate possible transitions when they are provably impossible. The method should not be subject to Type 2 errors, though I have no proof of this now. Types 3 and 4 are still possible, as Example 7 shows.

3.3 Potential Advantages and Disadvantages

Assuming this approach works out, it has several advantages over the other approaches to qualitative behavior prediction. It has some drawbacks, as well.

Spurious Behaviors. This approach completely eliminates the "chatter" problem which is the subject of (Kuipers & Chiu, 1987). In fact, this approach eliminates the general spurious behavior problem resulting from constraints of the form $y = M(a) \pm M(b)$. This is because such problems arise from the ambiguity introduced by replacing the quantitative constraints with monotone constraints. In particular, simple rules about allowed boundary traversals eliminate the chattering behaviors for the tanks cases.

In contrast with Sacks' approach, this method does not commit Type 1 errors. That is, the correct qualitative behavior(s) will be contained in the transition diagram. This may, of course, not remain true if the system needs to resort to approximations to deal with complexity of a particular geometry.

This approach still has difficulty with oscillatory systems, as do all approaches. It allows trajectories to violate energy conservation. The solution to these problems seems to require some form of special case knowledge, like explicit recognition that a damped pendulum loses energy. For example, sketching a few lines of constant energy in the phase space diagram of Example 7 would allow the system to rule out trajectories which start at low energy and move to high energy. Such an approach would stimulate subdivision of the regions. This is analogous to Sacks' use of Liapunov functions in global analysis. Other systems would require other specific theoretical insights to disallow certain transitions.

Computational Difficulty. This approach seems to lie somewhere between qualitative simulation and PLR in terms of computational difficulty. It is definitely more complex than the former, requiring analytic geometry and symbolic inequality reasoning. It shares many of the computational problems of PLR, but lacks a "global analysis" phase.

The difficulty of the analytic geometry should not be underestimated: techniques must be developed for handling the large number of symbolic constants (like k and l in Example 1, or g in Example 3), particularly with regard to bifurcations. This is probably the most serious potential obstacle to successful implementation.

Applicability. The approach is applicable uniformly, at least in principle, to a system with any number of dimensions in its phase space. This is in contrast with Sacks' approach which is currently limited to 2-D phase spaces. It does not attempt the sophisticated analyses which would be required of an extension of PLR to high dimensions. A corresponding price is paid in the number of spurious behaviors generated.

It also handles, again in principle, nonlinear equations without the need for approximation. We have seen that it can be difficult to piecewise linearize terms like PA where P and A are both phase variables. This approach handles those cases in the same way as the others.

The approach is not, of course, applicable to situations in which one has no access to the quantitative equations. Such a situation renders this as well as PLR inapplicable, while qualitative simulation approaches may still work.

Landmark Definition. The QSIM approach to landmark definition is simply to define a new one at every critical point of a parameter. This is not necessarily desirable in that it is not clear such values are of any qualitative interest, and extra landmark values tend to inflate the spurious behavior problem.

My approach, on the other hand, has potential to give much more insight on when to define qualitatively meaningful landmark values, such as F^*/l in Example 1. These values delimit regions of qualitatively different behavior in that points in either region have different sets of possible transitions.

In addition, this proposal introduces the more general notion of *landmark surface*. Rather than only being interested in rectilinear tessellations of the phase space, we might be interested in knowing when the system crosses some general surface of qualitative interest. (An example is the surface

$$\mathbf{S}: \quad x' - 2x/x' = 0$$

in the four-dimensional phase space of Example 8.)

The notion of landmark surface can give an easy way of incorporating domain knowledge such as energy conservation. Energy functions (more generally, Liapunov functions) provide one source of useful landmark surfaces.

4 Conclusion

4.1 Summary

I have discussed three approaches to the spurious behavior problem: techniques for improving the performance of qualitative simulation systems (QSIM), Sacks' piecewise linear methodology which is naturally less subject to the problem, and my own "phase-space geometry" proposal which uses quantitative information to achieve much better results than qualitative simulation, and appears to be more widely applicable and computationally somewhat simpler than PLR. The following list summarizes my main conclusions.

- Qualitative simulation suffers the worst from the problem, due to the inherent ambiguity of qualitative constraints.
  - The "ignore-the-derivative" method of Kuipers & Chiu (1987) does not address the deep issue of spurious behaviors, because whether one is interested in tracking a derivative bears no relation to whether QSIM predicts spurious behaviors for it.
  - The "HOD" method of Kuipers & Chiu (1987) is a highly questionable approach. It introduces the possibility of QSIM failing to predict the correct behavior of the system (a Type 1 error) and still seems rather likely not even to give an answer in many cases because of ambiguity. On the other hand, if the system is known to be linear, then Type 1 errors cannot occur and it does produce an unambiguous answer some of the time. Thus, it may be useful in this limited case.
  - The "higher order derivative method" of deKleer & Bobrow (1984) introduces little new other than their "no-change-to-constant" rule which is a very useful rule when the dynamical system is analytic and bounded.
  - QSIM's approach to landmark definition needlessly inflates the problem of spurious behaviors and should be avoided. Information about critical points should be deduced through other, more constrained, means.

  Overall, the outlook for solving the problem using only the qualitative constraints is grim. Section 2 suggests some alternative ways in which

QSIM could be improved, most of which involve the use of some quantitative information. These ways include using simple rules to avoid Type 2 errors, adopting a piecewise monotonic approach to avoid the $M - M$ problem, and incorporating domain knowledge such as knowledge about energy functions.

- PLR, while still subject to the problem, nevertheless is a more realistic approach for systems of any complexity. It can commit Type 1 errors (failure to find the correct behavior), but is far less subject to the other error types than either of the other two basic approaches. The price for this performance is paid in computational complexity and domain of applicability. Significant obstacles appear to lie in the way of extending this approach to higher dimensional phase spaces, such as a lack of mathematical tools to classify trajectories. This approach can not handle a significant class of non-linearizable equations, the simplest of which is $y' = 1/y, y > 0$. The prospects for fixing these problems are far from hopeless, however. These obstacles must be weighed against the natural way the other approaches handle both high-order systems and non-linearizable systems uniformly.

- My phase-space geometry proposal seems to have the potential to strike a balance between the two previous approaches. It exploits the information of the quantitative equations to achieve far better results than QSIM; yet it is more widely applicable and easier to extend to higher orders than the PLR approach. Of course a price is paid in that it is computationally more complex than QSIM and cannot achieve the same quality of predictions as PLR. The approach gives insight into what qualitative distinctions to make. In particular, it introduces the idea of *landmark surfaces*, regions of the phase space in which some important qualitative property changes, such as direction of the tangent field relative to a second surface. Of course, the phase space geometry approach suffers from the crucial drawback of being unimplemented. I believe it would be well worthwhile to attempt to implement it in light of its potential strong points.

4.2 Different Approaches for Different Problems?

The differences between qualitative simulation systems and the more quantitative approaches may indicate that they should not be directly compared. While QSIM-like systems do not perform as well as the others, they *can* reason without knowledge of the quantitative equations. It is undeniable that humans reason qualitatively about systems without such knowledge. On the other hand, humans are only good at such reasoning about simple, "common sense" systems. Quantitative approaches are better at prediction, but require the information of an expert. This suggests that the approaches be directed at two distinct problems: common sense reasoning and expert reasoning.

4.3 The Complexity Bound: Chaotic Behavior

Prediction of spurious behaviors is a problem for all current approaches to qualitative behavior prediction. Some systems avoid some of the four error types defined in Section 1, but all are subject to many errors, nevertheless. The problem gets much worse as complexity of the device's governing constraints increases and as its phase space increases in dimension.

For systems which exhibit *chaotic behavior*, i.e. exponential dependence on initial conditions, the outlook for qualitative behavior prediction is grim. Many surprisingly simple devices exhibit chaotic behaviors, such as coupled oscillators and the famous taffy pulling machine. Points which start out close in the phase space diverge exponentially quickly.

This class of devices is particularly troublesome to qualitative simulation systems like QSIM; attempting to simulate a chaotic system where the initial condition is represented as "$A = (l_i, l_{i+1})$" leads to total ambiguity in the predicted behavior.

Even quantitative approaches, such as my proposal and PLR, cannot hope eventually to handle such devices perfectly, as the qualitative state diagram for a chaotic system must account for the exponential dependence on initial conditions by refining regions endlessly.

There are two observations to make regarding chaotic systems. First, the way humans describe such systems indicates that current approaches to qualitative prediction may be at too fine a level of detail for common sense reasoning. A typical human description would be "the taffy gets all mixed up" rather than "a piece of taffy starting at x goes to y and one starting at

$x + \Delta x$ goes to z."

The second observation is that if we *are* interested in fine grain qualitative prediction, we should realize that the qualitative description of chaotic systems is currently an active area of research in the mathematics community. This suggests that qualitative reasoning research in AI should wait until more is known about the qualitative description of chaotic systems.

Appendix: Some Examples

Examples 1 through 4 (see following pages) all show how this proposal finds the correct qualitative state transition diagram in cases where QSIM-like approaches produce spurious behaviors. Example 5 is a case that all approaches have difficulty with. Example 6 shows how this approach can produce spurious behavior predictions. Example 7 shows how we handle a higher-than-2-dimensional phase space. Example 8 shows that this approach handles correctly the case on which the HOD technique produces a Type 1 error. It, too, has a high-dimensional phase space. Example 9 shows how the phase space geometry approach handles naturally the non-linearizable case that gives PLR trouble.

Example 1: The Cascaded, but Leaky, Tanks. QSIM (without HOD) predicts various classes of spurious behaviors for this example. The "chattering" behavior discussed by Kuipers & Chiu (1987) is the class of random transitions back and forth between regions 1 and 2. Another, similar spurious behavior class is the possibility of transitions back and forth between regions 1 and 6a. The HOD technique works, on this example, to eliminate both: for the system to attempt a 1–6a transition, it must first have B'' move through zero and become negative. Such a B'' transition is ruled out by the HOD technique.

Example 2: The Coupled, but Leaky, Tanks. This example has essentially the same issues as Example 1, except that QSIM with HOD comes up with a unique solution. The solution is ambiguous as to whether the equilibration happens in finite or infinite time, however.

Example 3: A falling rock with air friction. This example demonstrates Type 2 errors in QSIM. One type of error, a parameter becoming constant in finite time, could be ruled out be deKleer & Bobrow's (1984) no-change-to-constant rule. The other, a bounded parameter having positive derivative at infinity, could be ruled out by an equally simple rule.

Example 4: Flagrant Type 4 Errors in QSIM. Another qualitative property of equations is the relative growth rates of terms. This example shows how the fact that QSIM loses such information causes it to predict an obviously impossible behavior: x' going to infinity.

Example 5: A frictionless mass–spring system. Example 5 shows the behavior of a frictionless mass-spring system. While the result without landmarks is correct, it does not address the issue of decaying versus stable versus increasing oscillations. QSIM, on the other hand, does address the question; it predicts all possible combinations of fluctuating, rising, falling, stable oscillations.

If we introduce a landmark value representing the initial velocity of the mass (at $x = 0$), spurious behaviors appear. However, if domain knowledge of the conservation of energy is exploited by the introduction of the appropriate energy level surface, all spurious behaviors disappear.

This illustrates the only apparent solutions to this problem: (1) explicitly encode energy conservation in the given constraints (see (Kuipers, 1985)), or (2) derive energy-like constraints (Liapunov functions) directly from the equations (see (Sacks, 1988)).

Example 6: A damped gravitational pendulum. Example 6 shows how this approach can predict spurious behaviors. While it does correctly show that a point starting in region 1 (3) must decay into either 1a,1b,or 1c (3a, 3b, or 3c), it also allows a swinging trajectory to become a circular one and back again. (That is, a point is allowed to traverse 3a–4–1c–1a–1b–... The transition from 1a to 1b is not allowed in this context due to energy considerations.)

Since the system is dissipative, an energy landmark surface can be introduced to eliminate this spurious behavior. Note that a *qualitative bifurcation* appears here. If the damping is large, then a certain class of qualitative transitions is no longer possible, whereas in the low-friction regime they are. Detecting this and, in general, manipulating symbolically parameterized surfaces is a non-trivial task.

Example 7: Three coupled (not leaky) tanks. Example 7 (three coupled tanks) shows how this approach handles higher dimensional phase spaces correctly and naturally through the use of analytic geometry. Sacks' system currently can not handle high dimensional spaces, though it would be easy to extend his approach to handle this linear example.

The diagram is intended to depict the 3-dimensional phase space of the 3 tanks. As an example of the analytic technique, I will demonstrate the crossability test for the 1–2 boundary. First, the tangent vector on that boundary is given by

$$\mathbf{t} = (C - B)\hat{B} + (B - C)\hat{C}$$

The normal (pointing from 1 into 2) to the boundary is given by

$$\mathbf{n} = -\hat{A} + \hat{B}$$

This can be determined from differentiation of the defining equation, $A - B = 0$, of the plane.

The dot product is

$$\mathbf{n} \cdot \mathbf{t} = C - B$$

This is strictly greater than zero, because the 1–2 boundary lies completely within the $C' < 0$ half space, and $C' = B - C$. Thus, a transition is possible from 1 to 2 but not the opposite.

Example 8: HOD Counterexample. This example shows how the HOD technique's suspect "smoothness assumption" can cause Type 1 errors in QSIM. The idea is that the curvature of $(X(t))^2$ needn't be the same as the curvature of $X(t)$. In this example, the curvature of $X(t)$ is the constant, -2, while $(X(t))^2$ has positive curvature. The HOD technique rules out the correct behavior at the critical point, $t = 3$.

The phase space geometry approach does predict correctly. A portion of the four-dimensional phase space is shown. This example also shows how a new landmark surface is suggested: the $B' = 0$ surface has the phase space tangent pass through one way in some places and the other in other places. A new landmark surface is introduced to represent those points where the phase tangent dotted with the normal is zero. This subdivides the space and disambiguates the possible transition directions. The initial condition of interest starts in region 2, goes to 3, and then to 4. The passage from 3 to 4 is when B' goes from negative to positive. QSIM with HOD predicts that 3–4 transitions are impossible.

Example 9: Non-linearizable equation. This shows how the phase space geometry approach handles easily and correctly the non-linearizable example on which PLR fails.

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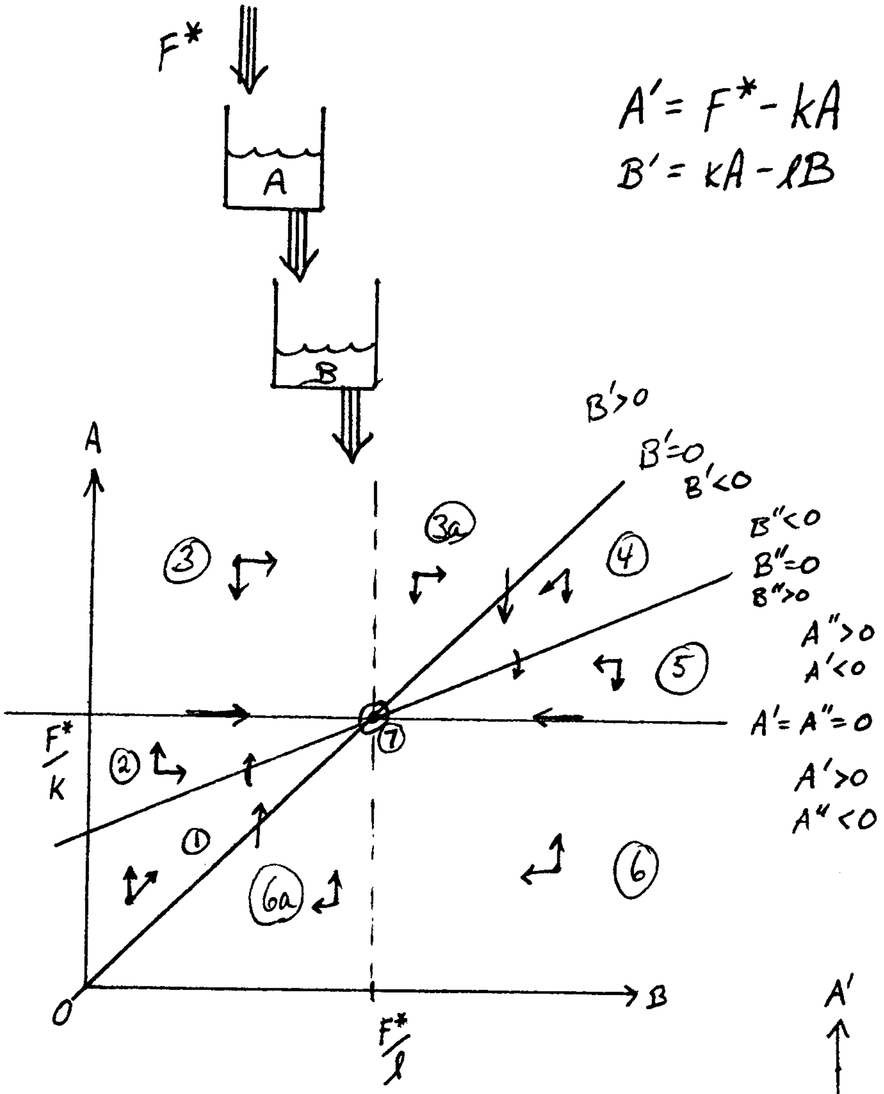
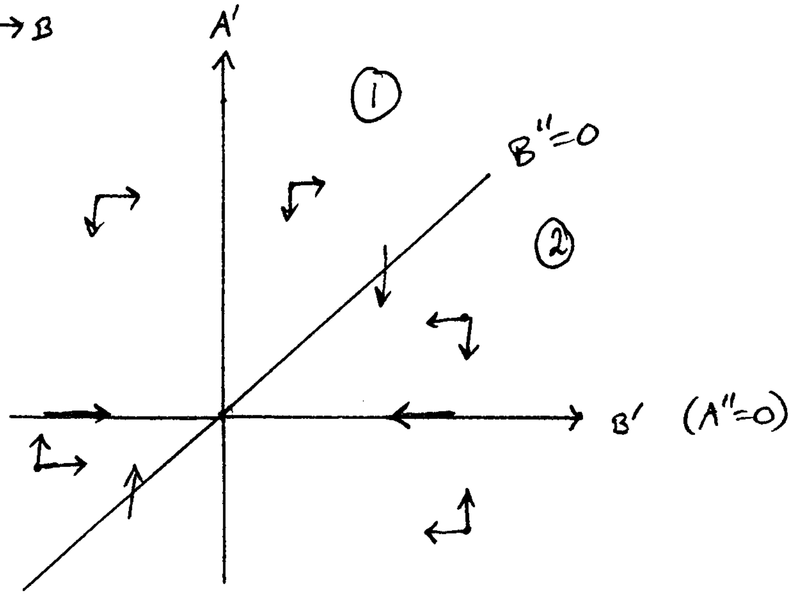
EXAMPLE 1. CASCADED TANKS

$$A' = F^* - kA$$
$$B' = kA - lB$$

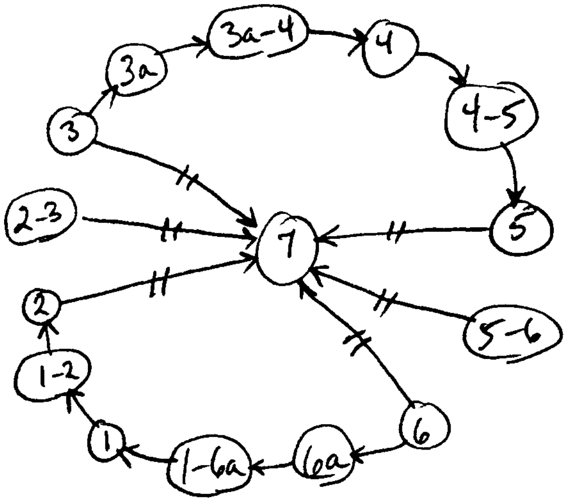


PHASE SPACE



A'–B' PHASE SPACE

EXAMPLE 1 (cont)



QUALITATIVE STATE DIAGRAM

QSIM (WITHOUT HOD TECHNIQUE) ALLOWS THESE SPURIOUS TRANSITIONS:

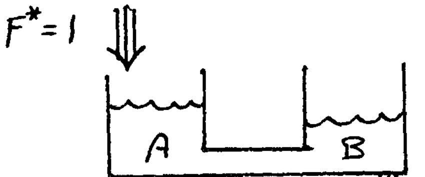
(PLUS SYMMETRIC CASES)

(TRANSITIONS TO ⑦ IN FINITE TIME)

QSIM (WITH HOD TECHNIQUE) ALLOWS THESE:

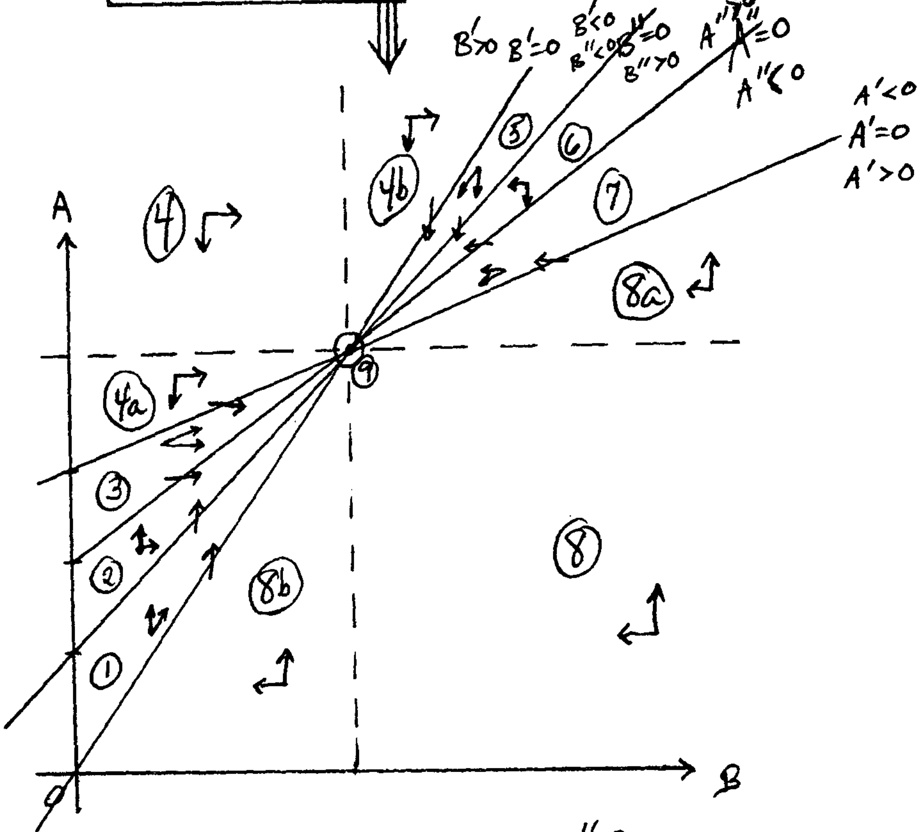
PLUS SYMMETRIC CASES, AS WELL AS TRANSITIONS TO ⑦ IN FINITE TIME.

EXAMPLE 2. COUPLED TANKS

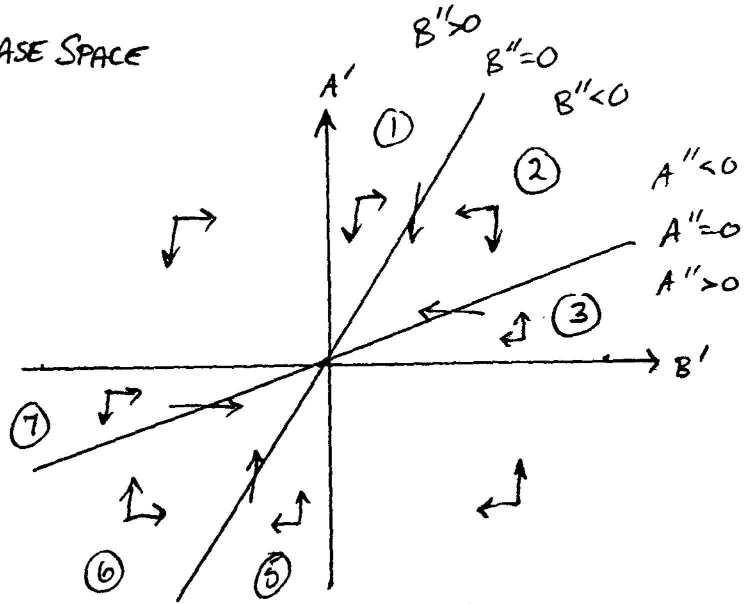


$$A' = 1 - A + B$$

$$B' = A - kB \qquad (k > 1)$$

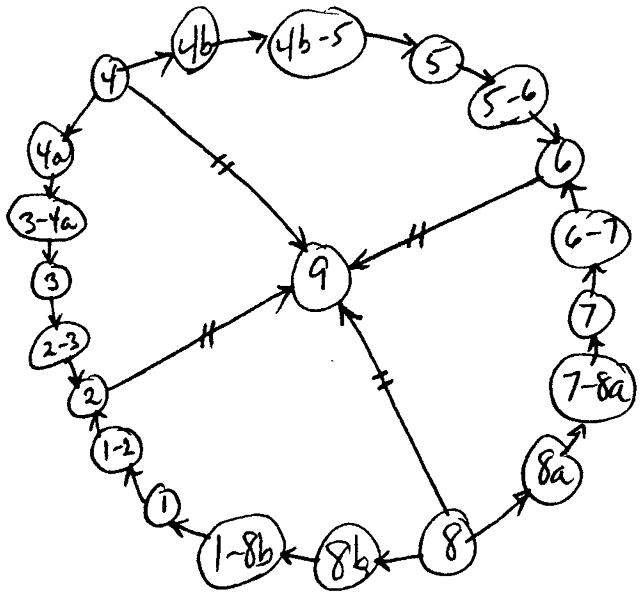


PHASE SPACE



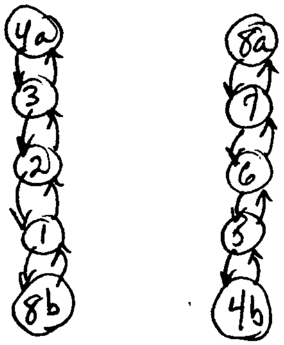
A'-B' PHASE SPACE

EXAMPLE 2 (CONT.)



QUALITATIVE STATE DIAGRAM

QSIM (WITHOUT HOD) ALLOWS THESE SPURIOUS TRANSITIONS:



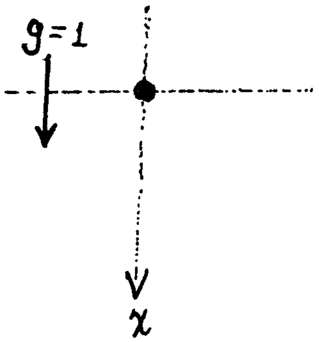
(BOUNDARY REGIONS SUPPRESSED)

PLUS FINITE TRANSITIONS TO ⑨

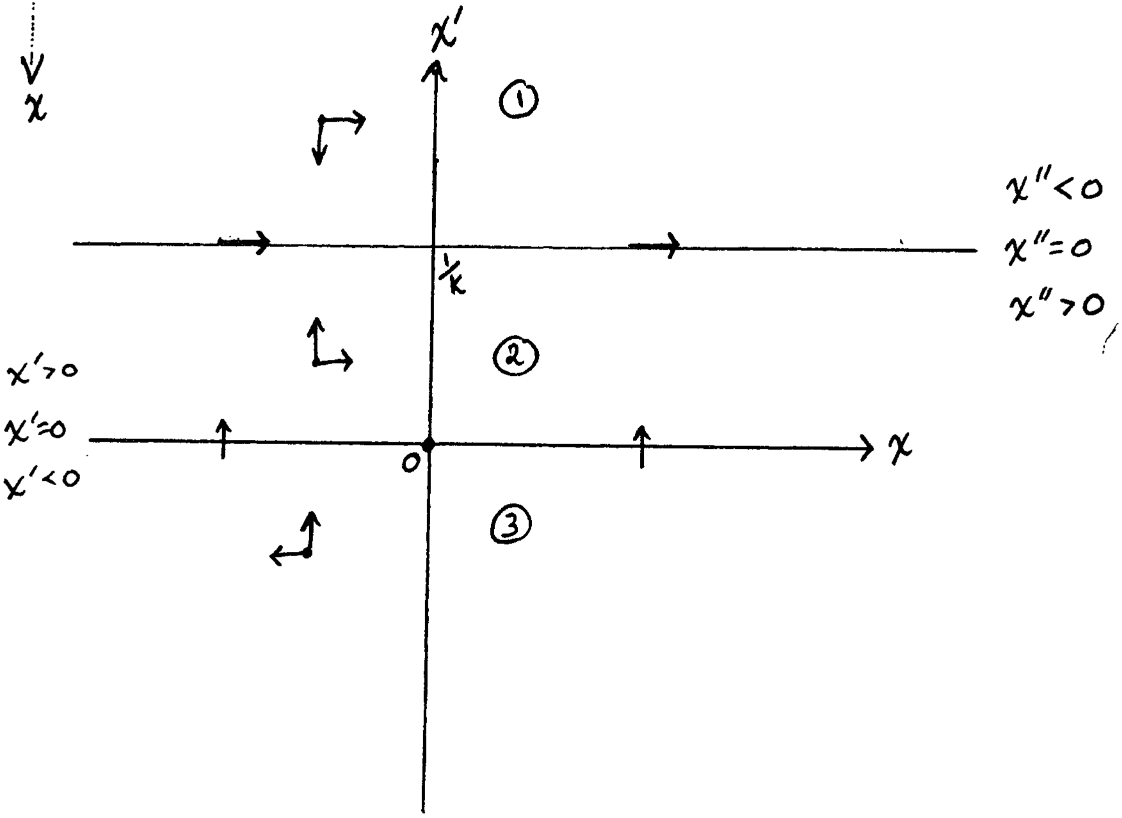
QSIM (WITH HOD)

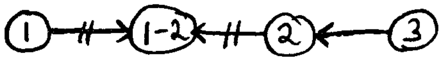
(NO SPURIOUS BEHAVIORS, EXCEPT AMBIGUITY OVER WHETHER EQUILBRATION HAPPENS IN FINITE TIME)

EXAMPLE 3. FALLING ROCK WITH AIR FRICTION



$$x'' = 1 - kx'$$





EXAMPLE 3 (CONT)

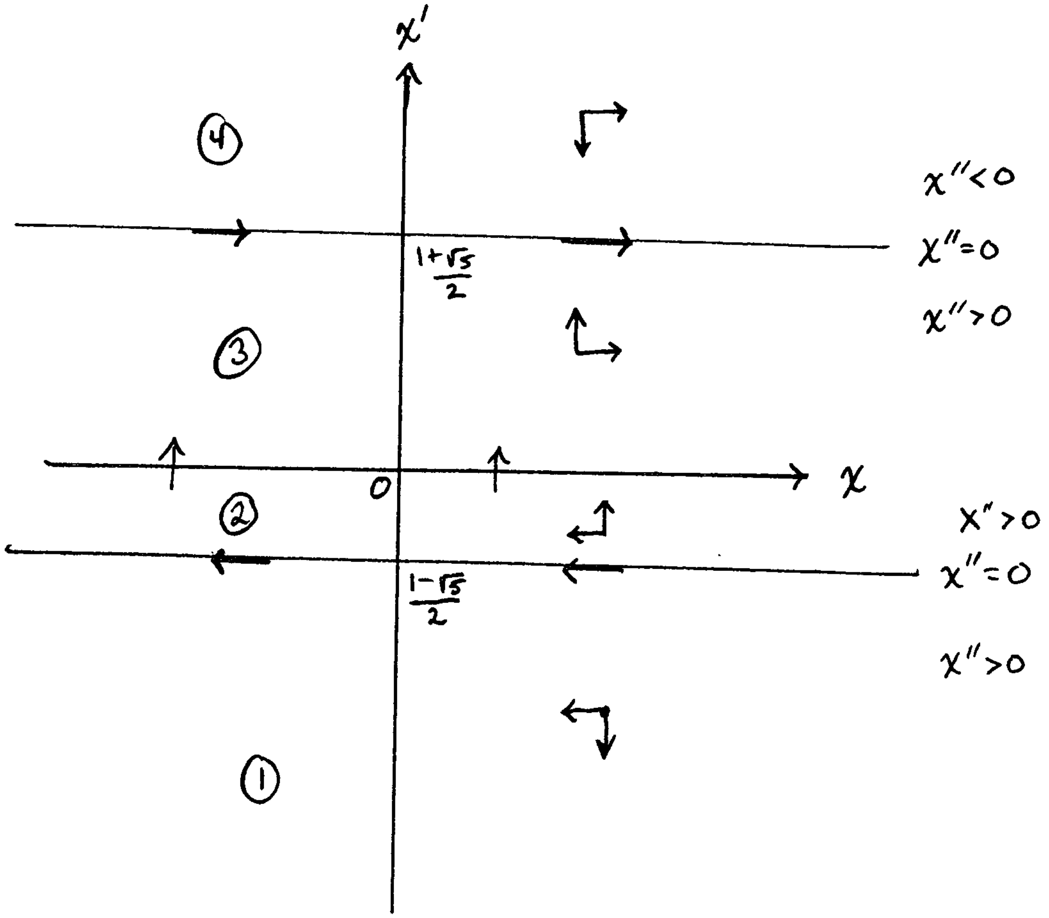
QSIM (WITH OR WITHOUT HOD) PREDICTS THESE SPURIOUS BEHAVIORS:

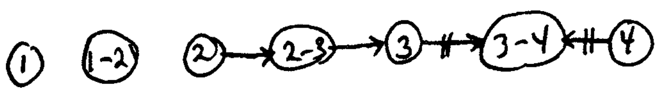
(FINITE TIME)

(NEVER LEAVE ②, BUT x GOES TO INFINITY)

EXAMPLE 4.

$$x'' = 1 + x' - (x')^2$$





EXAMPLE 4 (CONT)

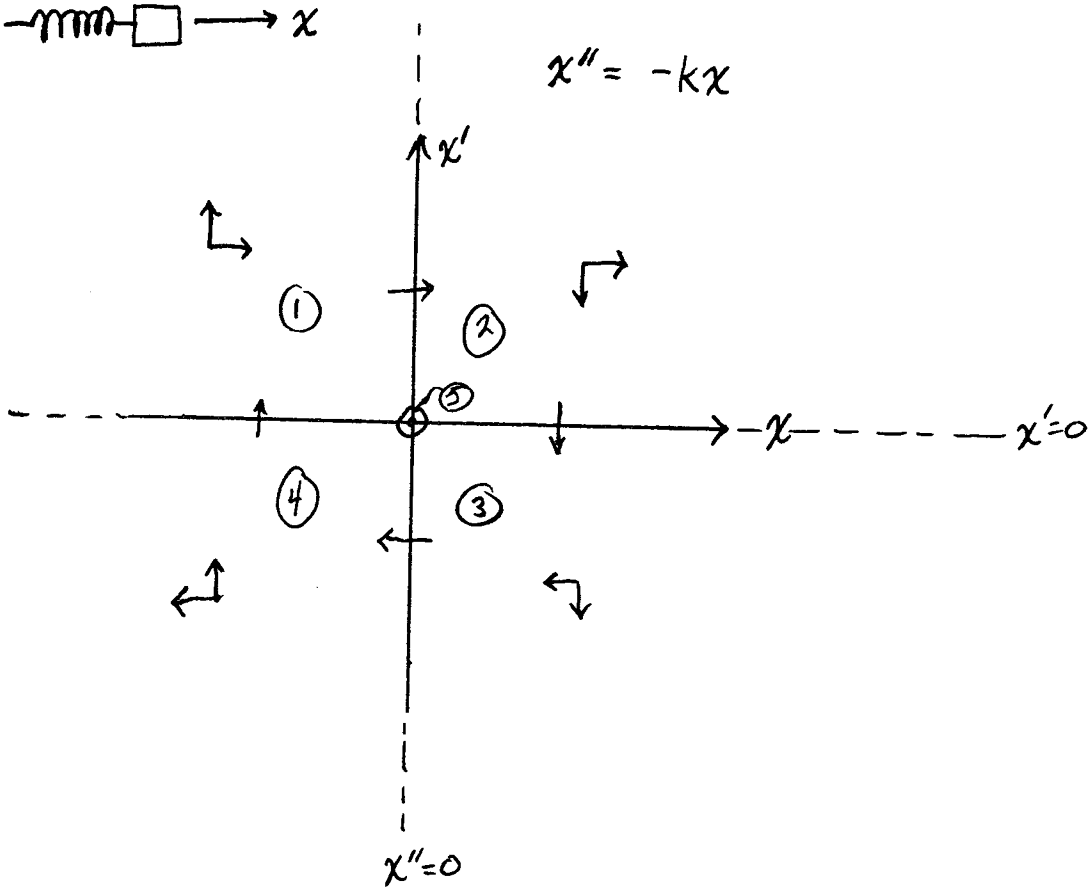
QSIM (WITH OR WITHOUT HOD) PREDICTS THESE SPURIOUS BEHAVIORS: (THERE MAY BE OTHERS AS WELL)

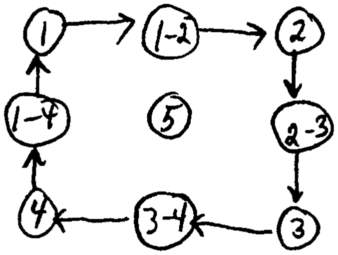
(FINITE TIME)

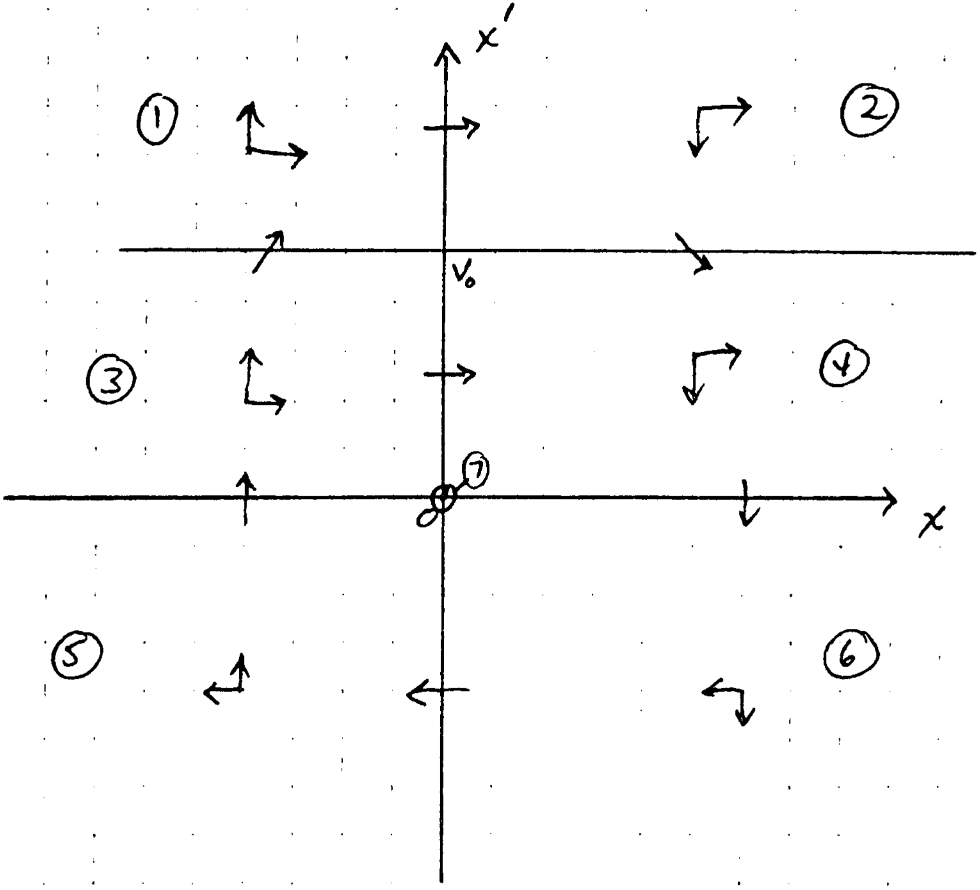
(NEVER LEAVES ③)

(x' EITHER GOING TO ∞ OR SOME LIMIT OTHER THAN $\frac{1+\sqrt{5}}{2}$)

5. FRICTIONLESS MASS-SPRING

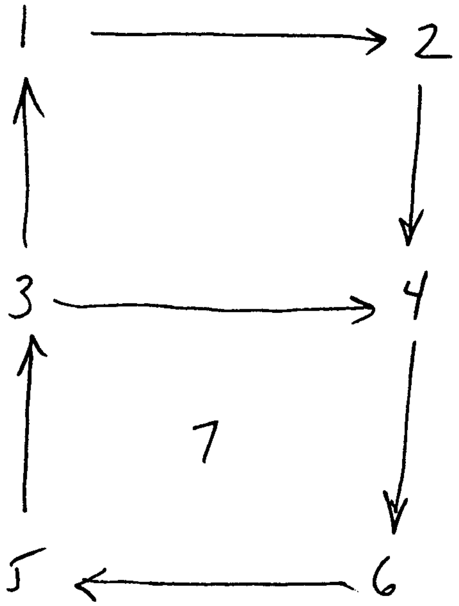






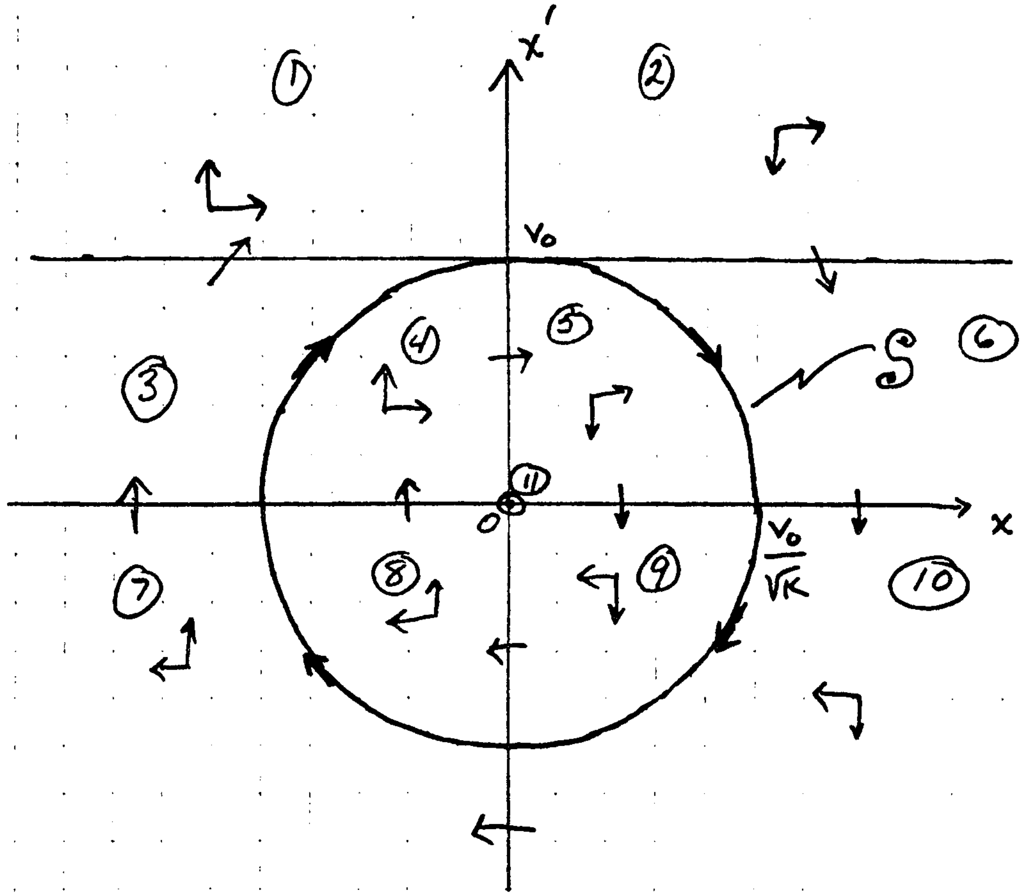
FRICTIONLESS MASS-SPRING WITH

LANDMARK VALUE : $x' = V_0$



FRICTIONLESS MASS-SPRING

WITH LANDMARK VALUE: $X' = V_0$



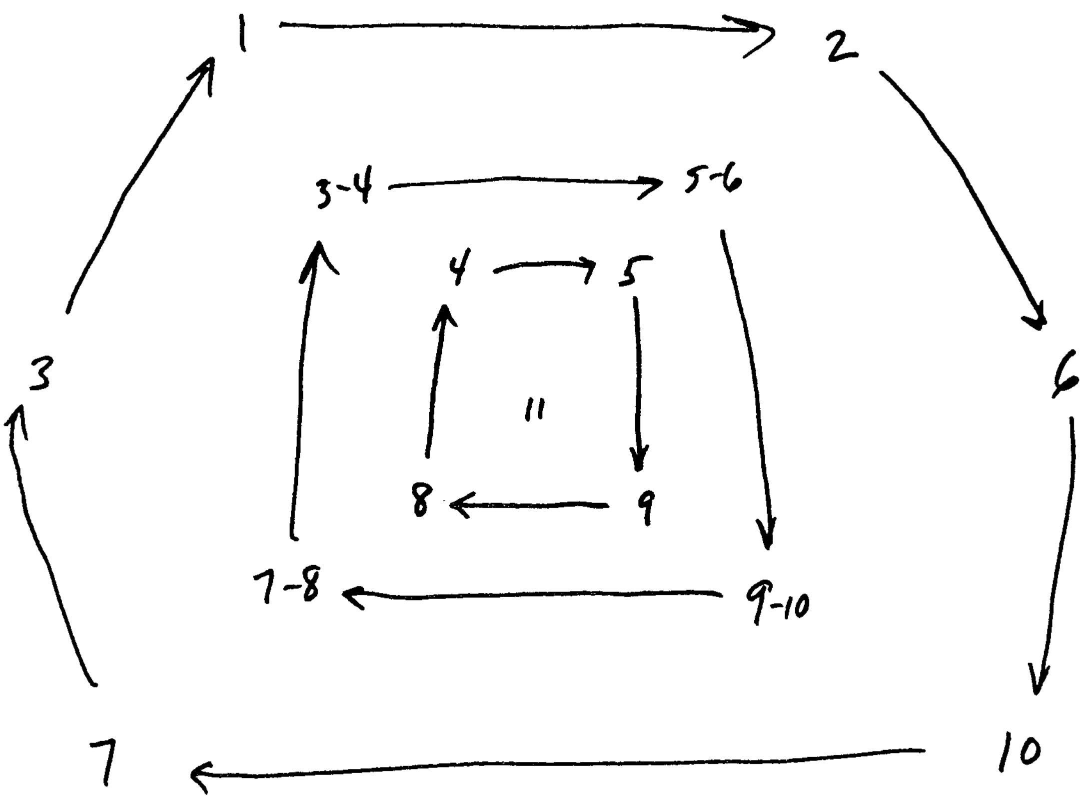
FRICTIONLESS MASS-SPRING WITH

LANDMARK VALUE : $X' = V_0$, AND

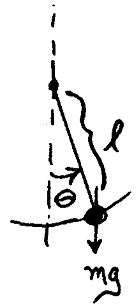
LANDMARK SURFACE: $E = \frac{1}{2}V_0^2 = \frac{1}{2}(x')^2 + \frac{1}{2}Kx^2$

($\underset{\sim}{n} \cdot \underset{\sim}{t} = 0$ everywhere on $\mathcal{S}$)

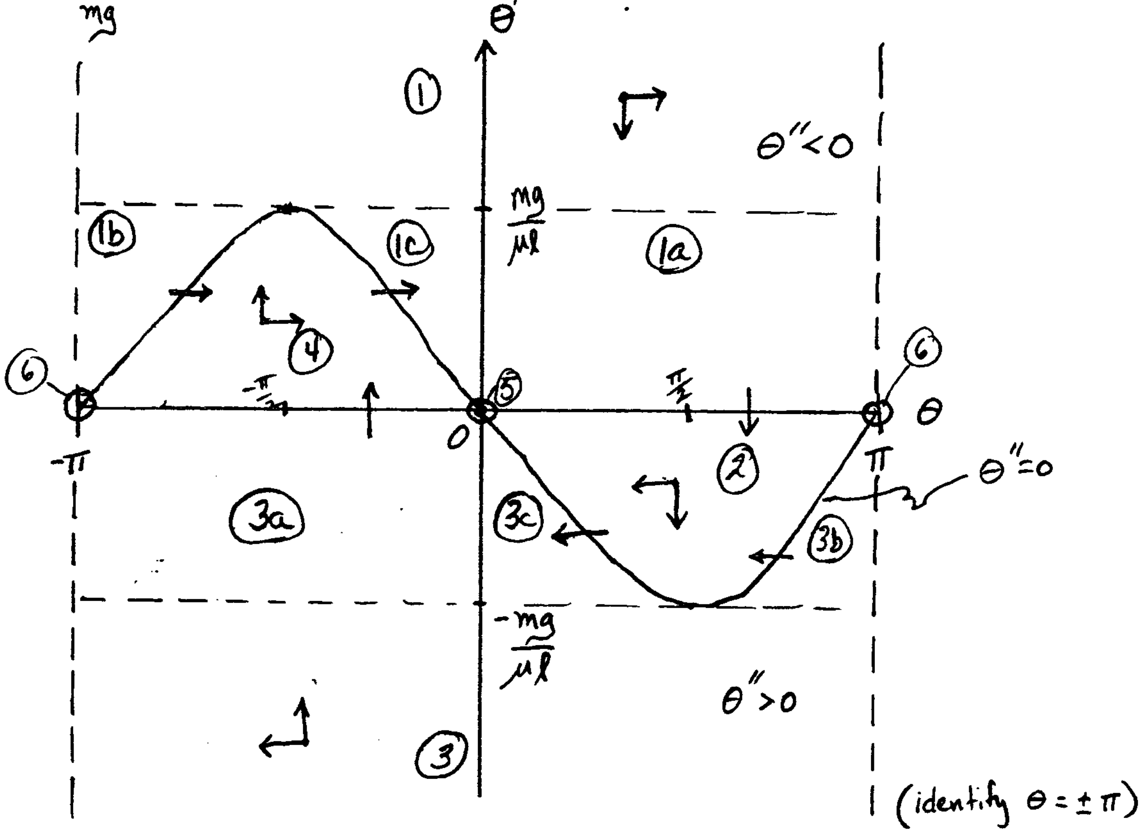
FRICTIONLESS MASS-SPRING (CONT)

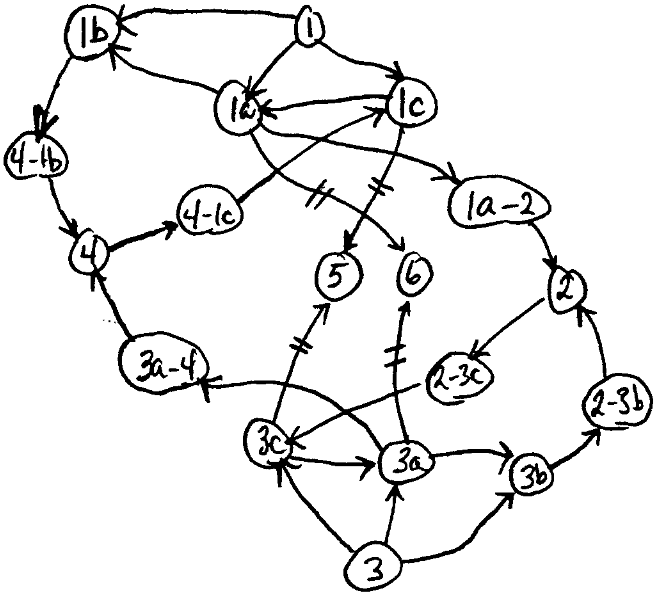


6. GRAVITATIONAL PENDULUM



$$\theta'' + \frac{\mu}{m}\theta' + \frac{g}{\ell}\sin\theta = 0$$





$\frac{u}{n}\theta' + \frac{g}{\ell}\sin = 0$

PENDULUM: LOW FRICTION REGIME

$\frac{m}{\mu} > 2\sqrt{\frac{\ell}{g}}$

$\theta' = \frac{-mg}{\mu\ell}\sin\theta$

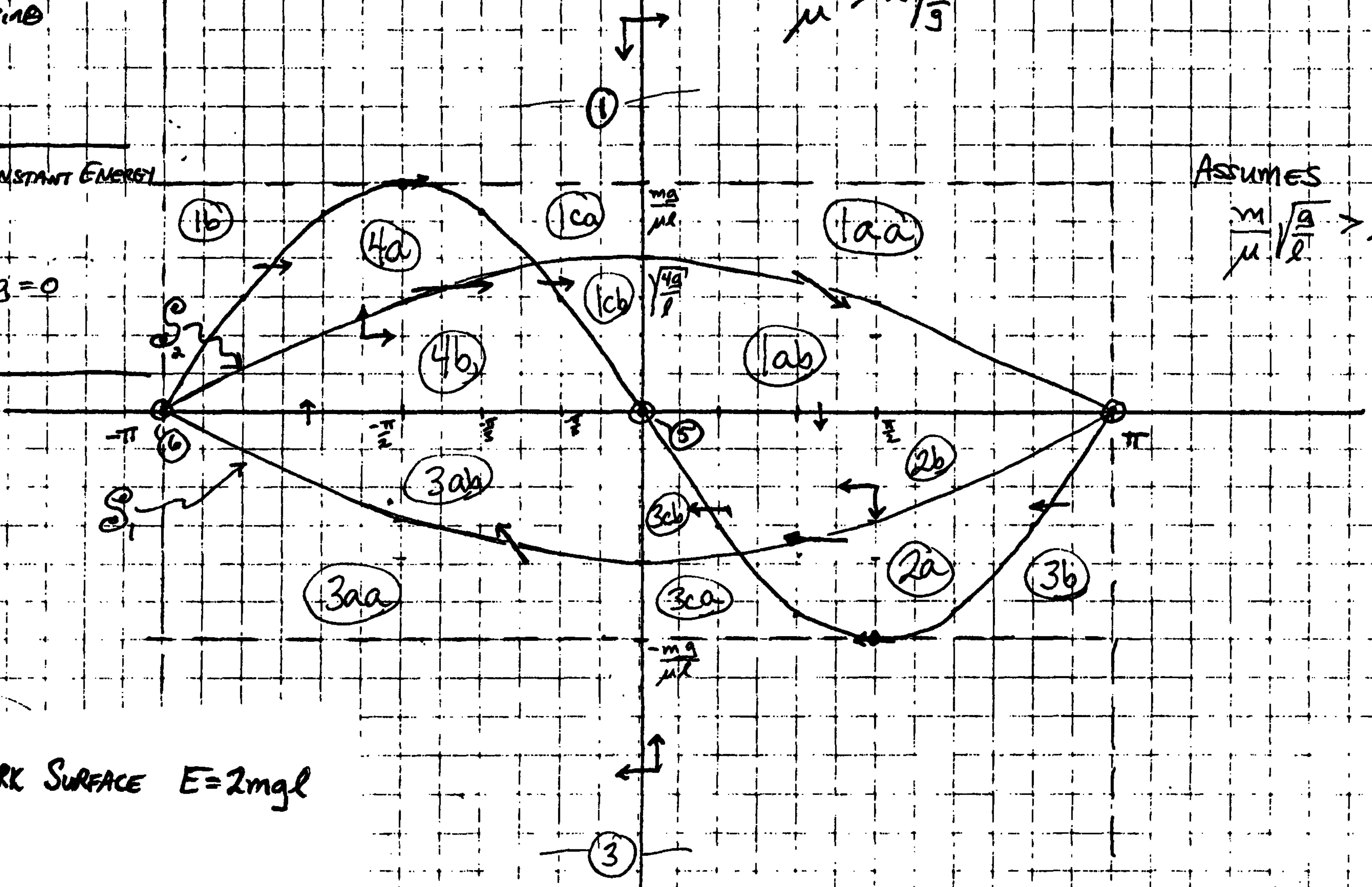
SURFACE OF CONSTANT ENERGY

E = 2mlg

$\gamma'^2 - g\cos\theta - g = 0$

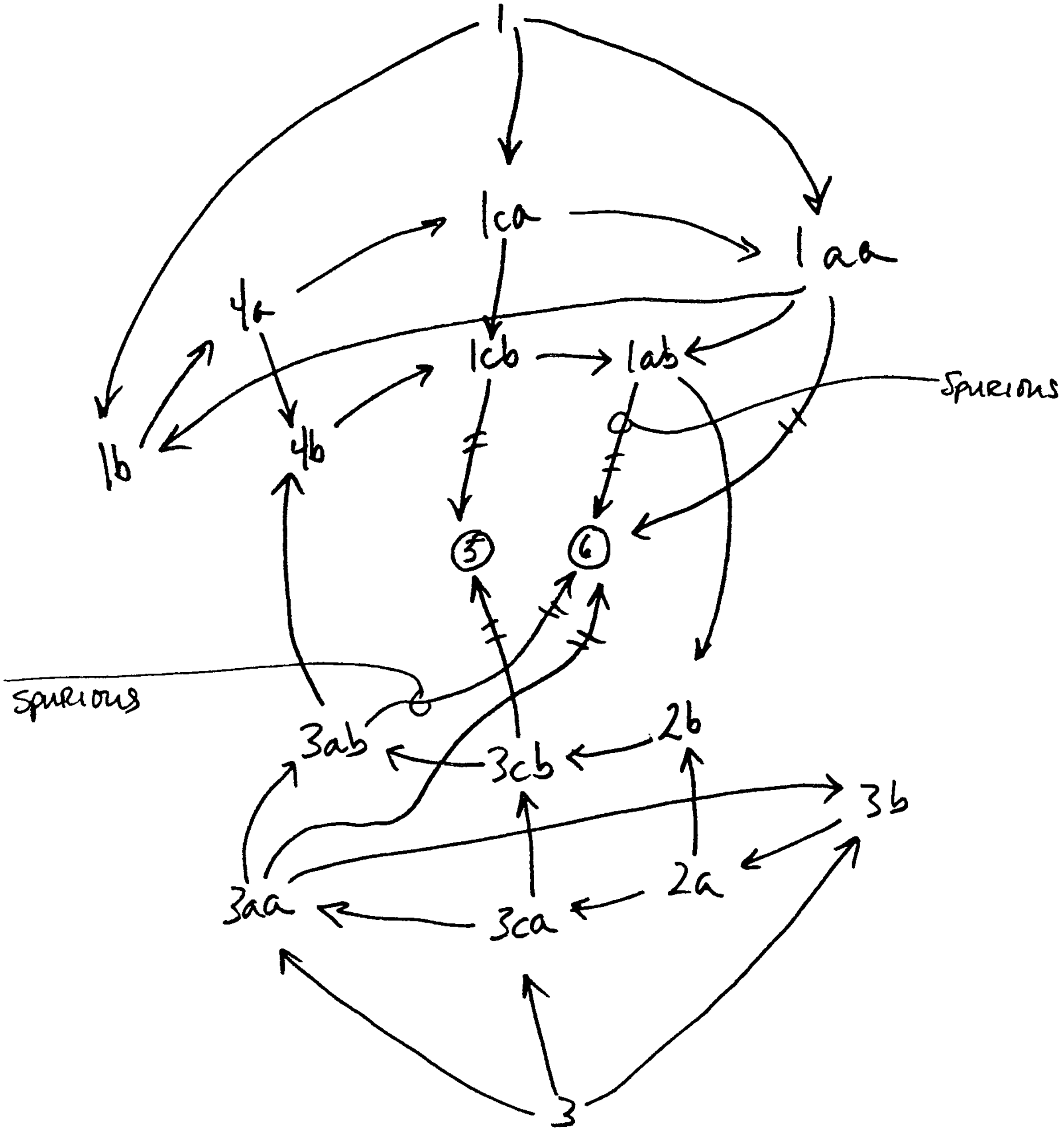
Assumes

$\frac{m}{\mu}\sqrt{\frac{g}{\ell}} > 2$



S°: LANDMARK SURFACE $E=2mg\ell$

PENDULUM: LOW FRICTION REGIME



$l = 2, \mu = m = g = 1$

$$\boxed{\frac{\mu}{n}\theta' + \frac{g}{l}\sin\theta = 0}$$

$$\theta' = -\frac{mg}{\mu l}\sin\theta$$

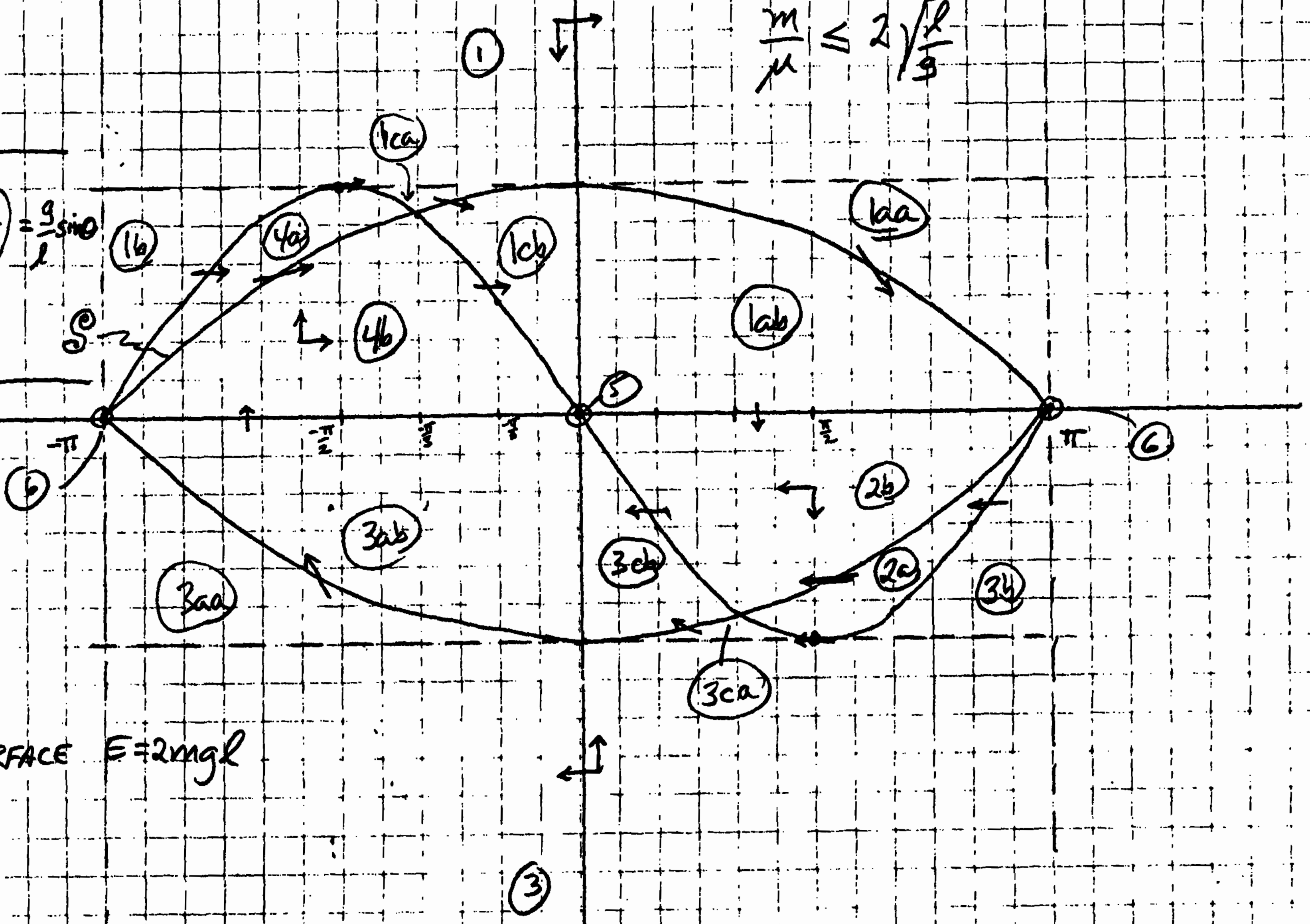
$$\theta' = -\frac{1}{2}\sin\theta$$

$$\theta'\left(\frac{mg}{\mu l}\cos\theta - \frac{\mu}{m}\right) = \frac{g}{l}\sin\theta$$

$$' = \frac{\sin\theta}{\cos\theta - 2}$$

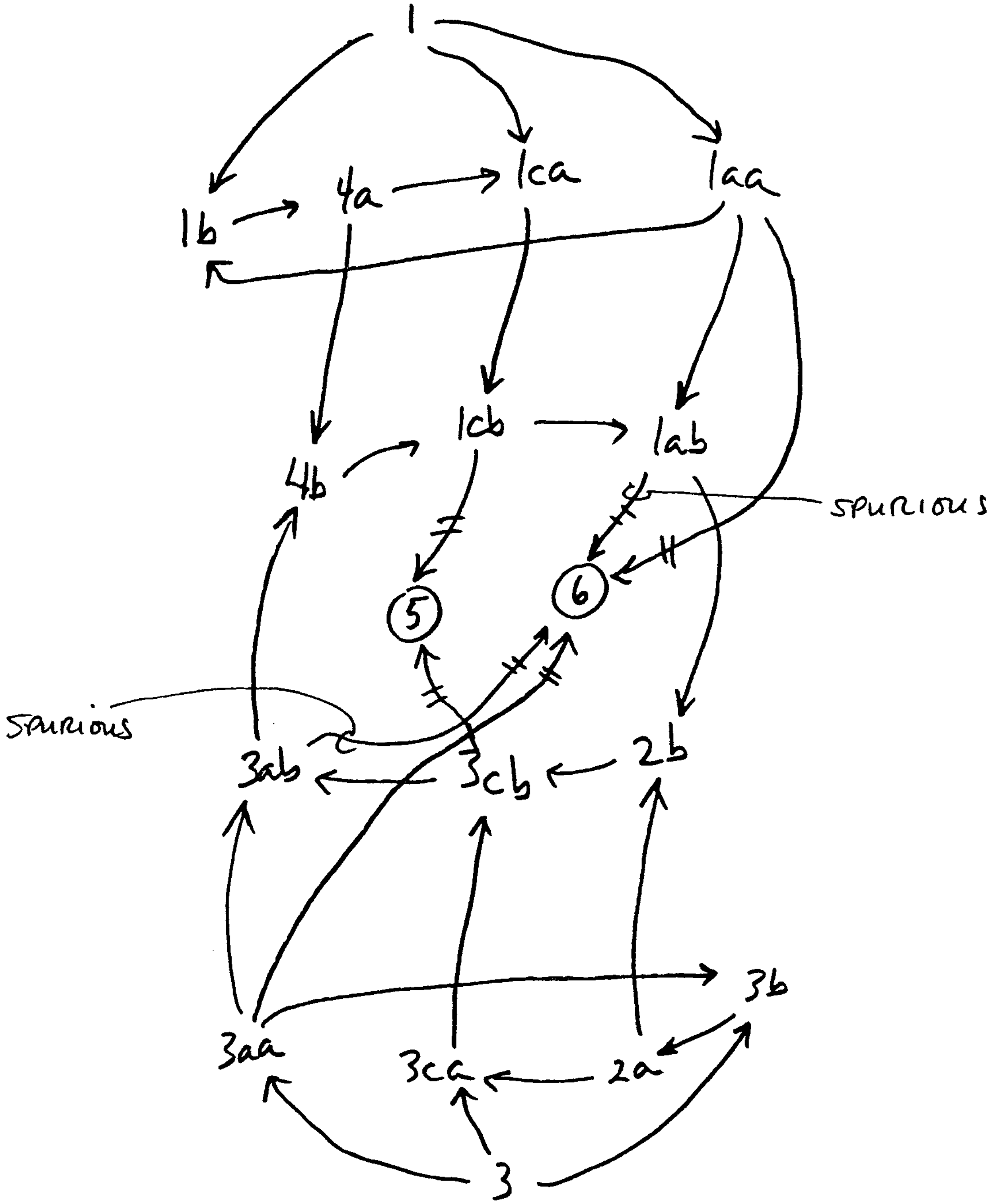
PENDULUM: HIGH FRICTION REGIME

$$\frac{m}{\mu} \le 2\sqrt{\frac{l}{g}}$$



∴ LANDMARK SURFACE $E = 2mgl$

PENDULUM: HIGH FRICTION REGIME

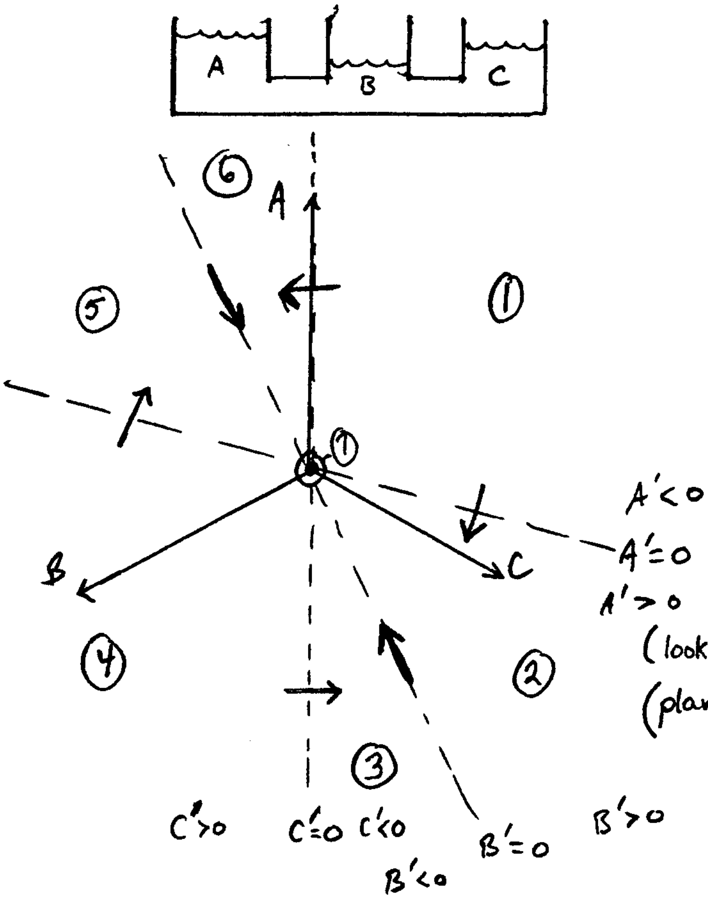


7. 3 COUPLED TANKS

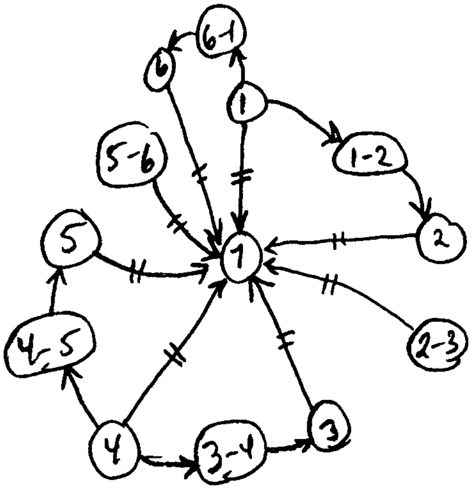
$A' = B - A$

$B' = A - 2B + C$

$C' = B - C$



(looking toward origin from (1,1,1))
(planes intersect in line A=B=C)



EXAMPLE 8. HOD COUNTEREXAMPLE

$$B = X^2 - A$$
$$A'' = -4$$
$$X'' = -2$$

INITIAL CONDITIONS
OF INTEREST:

$X(1) = 9$
$X'(1) = -2$
$A'(1) = -4$
$A(1) = -2$

REGION BOUNDARIES OF INTEREST:

$x' = 0$
$A' = 0$
$\boxed{B' = 0}$

$B' = 2xx' - A'$ So $B' = 0$ SURFACE SATISFIES $A' = 2xx'$

normal vector to $B' = 0$:

$$\underset{\sim}{n} = \hat{x} + \left(\frac{x}{x'}\right)\widehat{x'} \quad \text{(points from } B' > 0 \text{ to } B' < 0\text{)}$$

phase tangent:

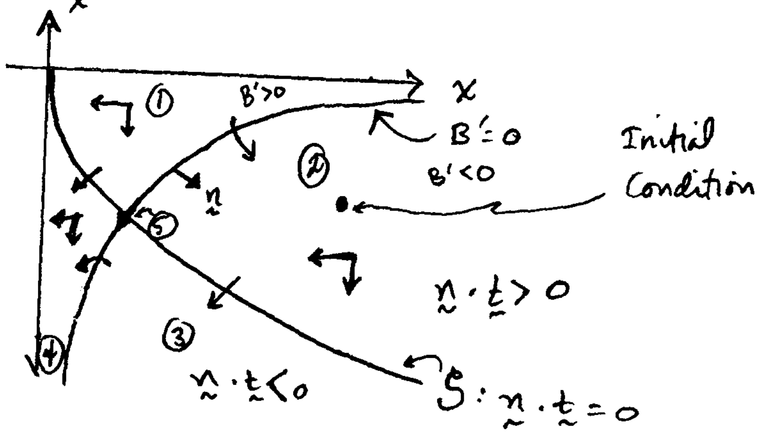
$$\underset{\sim}{t} = x'\hat{x} - 2\widehat{x'} + A'\hat{A} - 4\widehat{A'}$$

$$\underset{\sim}{n} \cdot \underset{\sim}{t} = x' - \frac{2x}{x'}$$

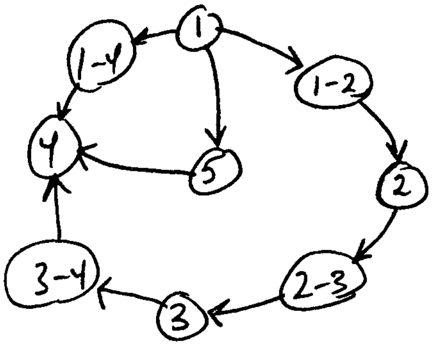
THIS SUGGESTS A NEW "LANDMARK SURFACE": $x' - \frac{2x}{x'} = 0$

CALL IT $\mathcal{S}$.

PHASE SPACE
SLICE: $A' = K < 0$
($\hat{A}$ COMPLETELY
SUPPRESSED)



EXAMPLE 8 (CONT)



PARTIAL (COMPRESSED)
QUALITATIVE STATE DIAGRAM

INITIAL CONDITION IS IN ②, SO CORRECT BEHAVIOR IS B' INCREASES THROUGH 0.

QSIM (WITHOUT HOD) PREDICTS

(1) B' INCREASES THROUGH 0; OR
(2) B' INCREASES TO 0 AND STOPS THERE; OR
(3) B' INCREASES TO 0, STOPS, THEN DECREASES.

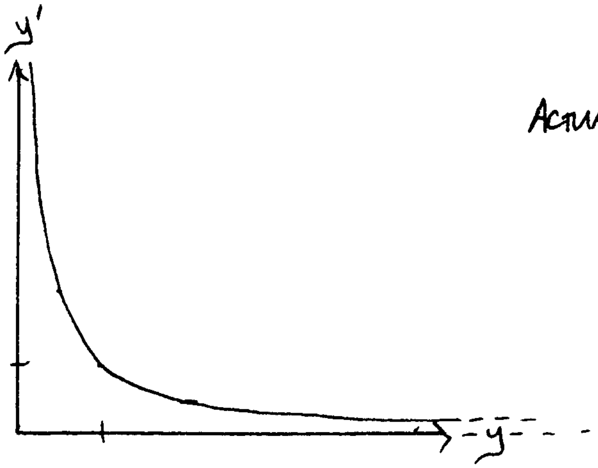
QSIM (WITH HOD) PREDICTS

(3) B' INCREASES TO 0, STOPS, THEN DECREASES.

(THIS IS INCORRECT)

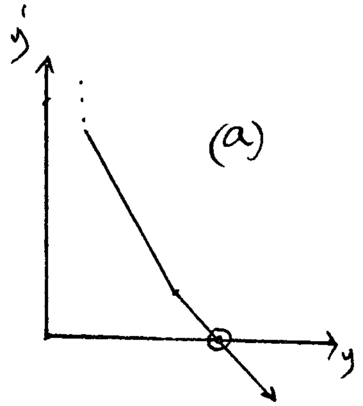
EXAMPLE 9. A NON-LINEARIZABLE EQUATION

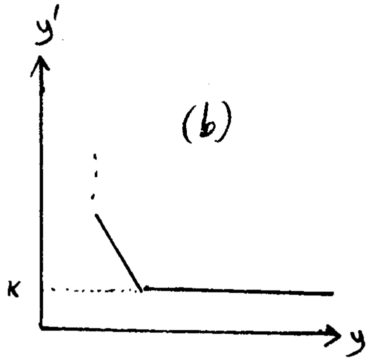
$$y' = \frac{1}{y} \quad , \quad y > 0$$



ACTUAL BEHAVIOR:

y' APPROACHES ZERO,
y GOES TO INFINITY

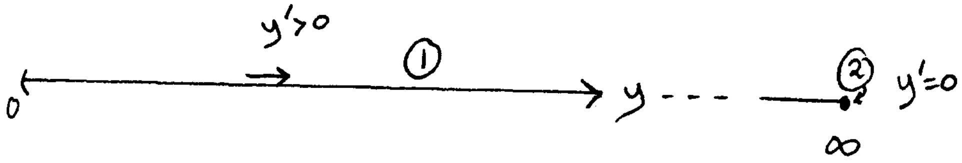




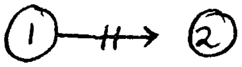
CANDIDATE LINEARIZATIONS :

(a) y' APPROACHES ZERO, y APPROACHES FINITE VALUE
(b) y' FAILS TO APPROACH ZERO

EXAMPLE 9 (CONT)



PHASE SPACE



QUALITATIVE STATE DIAGRAM PRODUCED BY PHASE SPACE GEOMETRY APPROACH