SOLUTION OF LAPLACE'S EQUATION
in
INVERTED COORDINATE SYSTEMS

by

George Warner Swenson, Jr.

B.S., Michigan College of Mining and Technology

(1944)

Submitted in Partial Fulfillment of the

Requirements for the Degree of

MASTER OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

1948

Signature of Author . . . .
Department of Electrical Engineering

Certified by . . . .
Thesis Supervisor

. . . .
Chairman, Departmental Committee on Graduate Students

ACKNOWLEDGMENT

The author wishes to express his sincerest appreciation to Professor Parry Moon, of the Department of Electrical Engineering, who suggested the general topic of this thesis. His lectures in electrostatic field theory have provided the essential background for the investigation and his interest in the problem has prompted many helpful suggestions. In addition, the author is indebted to Dr. R. M. Redheffer, of the Department of Mathematics, without whose patient interpretation of his Doctorate thesis and frequent assistance in overcoming mathematical difficulties the present work could hardly have been accomplished.

CONTENTS

ABSTRACT

The treatment of problems in potential theory involves the solution of Laplace's equation subject to certain boundary conditions. This is accomplished by separating Laplace's equation, a partial differential equation, into three ordinary differential equations, whose combined solutions constitute a particular solution of the original equation. The complete solution of the problem consists of a Fourier series of particular solutions, whose coefficients are determined in such a way as to satisfy the boundary conditions.

The above procedure is effective only when the boundary surfaces of the problem coincide with the parametric surfaces of the coordinate system in which the problem is stated. This condition greatly limits the number of problems that can be solved, since the number of coordinate systems in which Laplace's equation can be solved by separation of variables is limited. Recently a number of new coordinate systems has been developed, each obtained by the inversion in a sphere of one of the conventional systems.

Inverted coordinate systems are useful for the solution of many problems whose boundaries are not compatible with the conventional coordinate systems. Two methods of attack on these problems are available. In the first, Laplace's equation is separated in the appropriate coordinate system and the solution of the problem is constructed in the

usual way, by the use of a Fourier series. This method has the disadvantage that not all of the particular solutions of Laplace's equation in inverted coordinates have been tabulated. The alternate method requires the inversion of the boundary conditions into one of the conventional systems, and the solution of the problem in that system, whereupon the solution is inverted back into the original system. This method has the disadvantage that simple boundary conditions are transformed by inversion into relatively complex ones. In the problems investigated, little advantage was observed in using one system instead of the other.

A table of the separated equations has been made, including all of the inverted coordinate systems. The Stäckel determinant has been obtained for each case. A number of graphs has been drawn, illustrating the physical appearances of parametric surfaces in inverted coordinates.

I

ORIENTATION

In 1782, while engaged in an investigation of the gravitational attraction of the earth, Pierre Simeon Laplace enunciated the relationship

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \qquad (1)$$

This equation, now usually written in the shorthand form

$$\nabla^2 \phi = 0,$$

was named for its discoverer, and has assumed great importance in applied mathematics, appearing as it does in discussions on mechanics, acoustics, hydrodynamics, heat, and electricity, in fact, in any science where the theory of potential is involved. The problem of solving this equation has naturally attracted the attention of a large number of scientific workers from the date of its introduction until the present time.

The most general solution of a partial differential equation, such as Laplace's equation, involves an arbitrary function or an infinite number of arbitrary constants. A particular solution of such an equation is a relation among the variables which satisfies the equation, but which, though included in it, is more restrictive than the general solution. If the general solution of a differential equation is to satisfy the conditions of a physical situation, enough must be known about the physical situation to specify the arbitrary function or all of the arbitrary constants. Very frequently the general solution of a partial differential equation cannot be obtained, but it is usually possible to obtain a number of particular solutions. In such cases, and Laplace's equation falls into this category, it is worthwhile to find a number of particular solutions and to combine them in such a way that they satisfy both the differential equation and the physical situation. The physical restrictions on the solution are called "boundary-values" and a problem of the type described here is called a "boundary-value problem".

Boundary-value problems involving Laplace's equation in three dimensions are ordinarily solved by the method of "separation of variables". This method requires that the partial differential equation be reduced to three ordinary differential equations, the solutions of which, when properly combined, constitute a particular solution of the partial equation. As an example[3] of this method, consider Laplace's equation in rectangular coordinates,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

Let $\phi = XYZ$, where $X = X(x)$, $Y = Y(y)$, and $Z = Z(z)$. Equation (1) then becomes

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2}.$$

The right-hand side of this equation involves z only and the left-hand side involves x and y only. In order for this condition to be true, it is apparent that each side must be equal to a constant. Thus

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{Z}\frac{d^2Z}{dz^2} = +\alpha_1.$$

$$\frac{d^2Z}{dz^2} + \alpha_1 Z = 0;$$

$$Z = \sin\sqrt{\alpha_1}\, z + \cos\sqrt{\alpha_1}\, z.$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \alpha_1 - \frac{1}{Y}\frac{d^2Y}{dy^2} = -\alpha_2,$$

$$\frac{d^2X}{dx^2} + \alpha_2 X = 0.$$

$$X = \sin\sqrt{\alpha_2}x + \cos\sqrt{\alpha_2}\, x.$$

$$\frac{d^2Y}{dy^2} - (\alpha_1 + \alpha_2)Y = 0$$

$$Y = \sinh\sqrt{\alpha_1 + \alpha_2}\, y + \cosh\sqrt{\alpha_1 + \alpha_2}\, y.$$

The particular solution to Laplace's equation is then

$$\begin{aligned}\phi &= XYZ \\ &= (\sin\sqrt{\alpha_1}\, z + \cos\sqrt{\alpha_1}\, z)(\sin\sqrt{\alpha_2}\, x + \cos\sqrt{\alpha_2}\, x) \\ &\quad (\sinh\sqrt{\alpha_1 + \alpha_2}\, y + \cosh\sqrt{\alpha_1 + \alpha_2}\, y).\end{aligned}$$

If the boundary conditions are specified on the surfaces of a rectangular parallelopiped, the potential on these boundaries can be expressed in terms of the particular solution for ϕ by using a Fourier series. Suppose, for example, that on the surface $z = d_1$ the potential is specified as a function of x and y, and that on the surfaces $x = 0$, $y = 0$, $z = 0$, $x = d_2$, $y = d_3$ the potential is zero. The form of each of the factors in the particular solution can be adjusted to make the potential go to zero on each of the prescribed boundaries. Because the term involving hyperbolic functions is constant when $z = d_1$, the specified potential function on that surface can be expressed as a double Fourier series[2] of trigonometric terms in the particular solution. Each term in the series has its coefficient determined by the boundary value, and since there are an infinite number of terms, in general, the series becomes a general solution of Laplace's equation. The boundary-value problem is thus completely and uniquely solved.

It is apparent that this method of solution is effective only when the boundaries are parametric surfaces. In rectangular coordinates, therefore, only the rectangular parallelopiped and its degenerate cases are acceptable as boundary surfaces. In order to handle other physical problems it is necessary to employ coordinate systems whose parametric surfaces correspond to the boundaries involved. Laplace's equation assumes a different form in each new coordinate system, and its solutions are not usually expressible in terms of elementary functions. Thus, a number of new functions has arisen in connection with boundary value problems. Numerical values of most of these functions have been tabulated for useful ranges of the independent variable.

In order for a coordinate system to be usable, it must have such properties that Laplace's equation, expressed in these coordinates, is separable into three ordinary differential equations. This circumstance imposes definite restrictions on the type of coordinate system that is admissable; among other requirements, the three families of parametric surfaces must be mutually orthogonal.

In 1934, in connection with an investigation of the Schroedinger wave-equation, L. P. Eisenhart tabulated the properties of eleven coordinate systems[4], including rectangular; circular, elliptic, and parabolic cylinder; spherical; prolate and oblate spheroidal; ellipsoidal; paraboloidal; conical; and parabolic coordinates. These systems are well known, and the solutions of Laplace's equation are tabulated for most of them. Laplace's equation also separates in toroidal coordinates. The twelve systems mentioned here are the only three-dimensional systems currently used in mathematical physics. In addition, there are a number of two-dimensional systems that can be used in physical problems where there is cylindrical symmetry.

In 1948, R. M. Redheffer showed in his doctorate thesis[10] that Laplace's equation also separates in the systems obtained when nine of the eleven Eisenhart systems, excluding elliptical and paraboloidal coordinates, are inverted in any sphere of any radius. He also tabulated the mathematical forms of three unnamed systems, which are as yet unidentified with geometrical configuration, and which may include the inversions of elliptical and paraboloidal coordinates. The twenty-four systems mentioned here were proved by Redheffer to be the only ones in which

Laplace's equation is separable by the methods employed in this thesis.

In 1845, Sir William Thompson (Lord Kelvin) pointed out the possibility of solving potential problems by inverting the boundary values in a sphere, to transform the problem from an arbitrary coordinate system into a familiar one. This technique is equivalent to the familiar "image method" of solving problems in electrostatics[7]. However, apparently no one has separated Laplace's equation in the inverted systems, or attempted to apply the method of separation of variables to problems whose boundaries coincide with parametric surfaces of the inverted systems.

It is the purpose of this thesis to separate Laplace's equation in the inversion of each of the eleven Eisenhart[4] coordinate systems, to illustrate the physical configurations of some of the systems which appear to have the greatest usefulness, and to compare the two methods of solving problems in these systems.

II

THE GEOMETRICAL PROPERTIES OF INVERTED SYSTEMS

A. The Process of Inversion

By inversion is meant the transformation by reciprocal radius-vectors; that is, the substitution for any set of points in space, another set obtained by drawing radii to the original points from a fixed origin, and measuring along these radii distances inversely proportional to their lengths.[14] Any desired constant of proportionality can be used, which then becomes the square of the radius of the "sphere of inversion". Each point on the inversion of a certain figure then lies on a line between the origin and the corresponding point on the original figure, with the radii so proportioned that the product of the radii of the original point and the inverted point is equal to the square of the radius of the sphere of inversion.

An inversion is a conformal transformation. This means that any angle between lines or surfaces in the original figure is preserved in the inverted figure. Elementary lengths, areas, and volumes are transformed by the inverse first, second, and third powers, respectively, of the ratios of the radii. Each straight line or plane in the given configuration is transformed into a circle or a sphere through the origin, respectively, and each circle or sphere in the given figure is transformed into a different circle or sphere. A point in the original figure lying outside of the sphere of inversion is transformed into one lying inside the sphere, and vice versa. It is apparent, then, that the inversion of a geometrical figure can be performed by geometrical means.

Mathematically, the process of inversion can be expressed by the formula[10]

$$\bar{r} = c + \frac{a^2}{|r - c|^2}(r - c),$$

where $\bar{r}$ is the radius vector from an arbitrary origin to the inverted point, r is the vector of the given point, c is the vector of the center of the sphere of inversion, and a is the scalar radius of the sphere.

The linear element, $\overline{ds}^2$, of an inverted coordinate system is given by[10]

$$\overline{ds}^2 = \frac{a^4}{|r - c|^4} ds^2,$$

where ds^2 is the linear element of the given system.

B. Practical Methods of Inversion

(1) Graphical Method

It is not feasible to represent exactly on a two-dimensional diagram the shape of an inverted figure; however, in some cases, the actual shape can be visualized with the aid of one or more such diagrams. For example, consider the inversion of a cylindrical object, shown in plan view in Figure 1. ABCD is the cylinder, while A'B'C'D' is its inversion. If the cylinder were infinitely long, the inverted figure would pass through the origin, as shown by the dotted lines. The cross-sectional area is infinitesimal

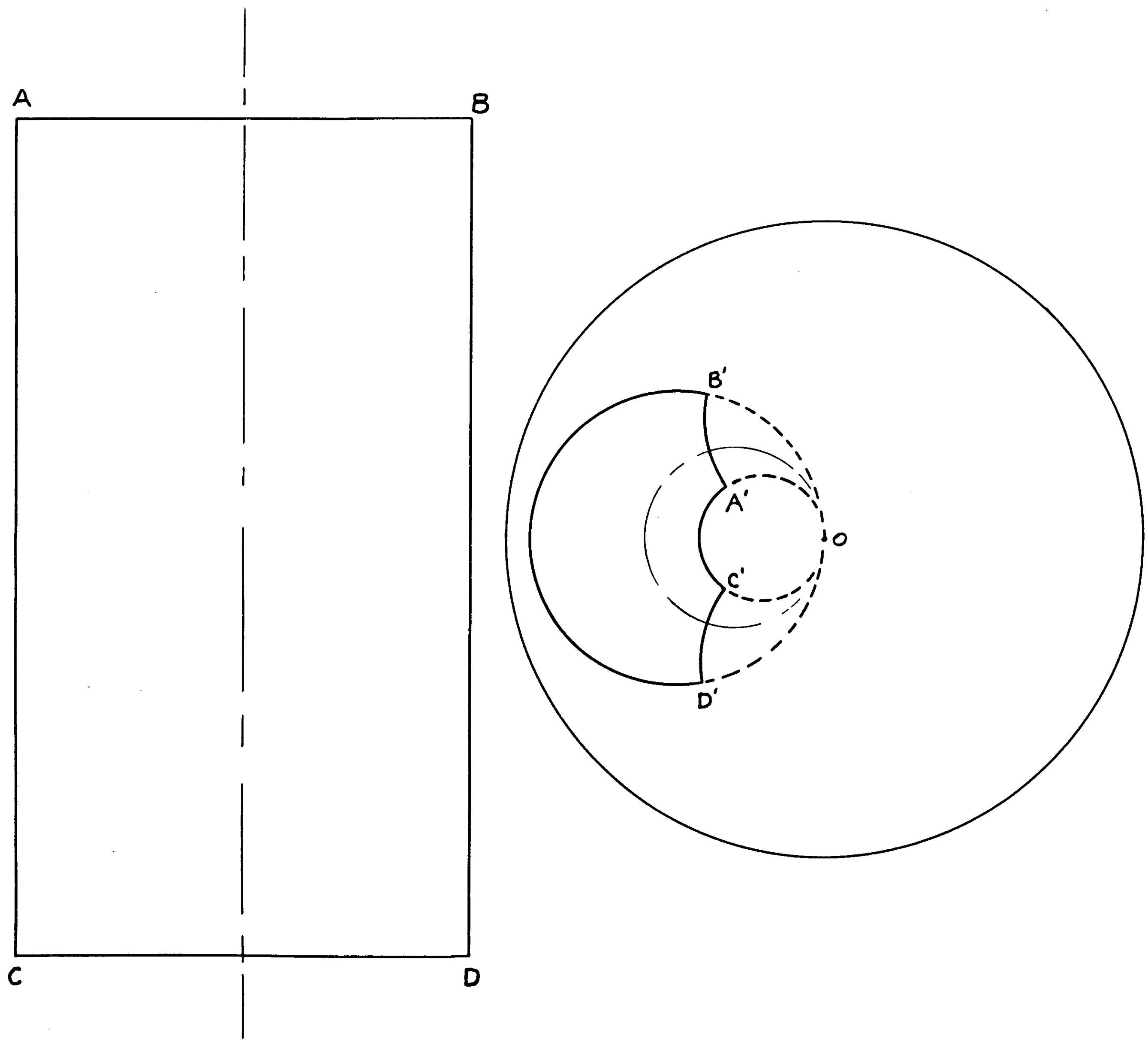


FIGURE 1. INVERSION OF A CYLINDRICAL OBJECT

at the center of the sphere of inversion, and varies to a maximum value at the point closest to the surface of the sphere. If, however, the cylinder is finite in length, its plane ends are transformed into spherical sectors. The cross-sectional shape of the inverted figure can be found by drawing the series of sectional views obtained by passing a family of planes, perpendicular to the paper, through the center of the sphere. Inverting the intersection of such a plane and the cylinder with respect to a circle, which is the intersection of the sphere and the plane, gives the cross-section of the inverted figure at its intersection with the plane. In the case of a circular cylinder, the intersection with a plane is an ellipse, the inversion of which is given in Figure 12.

(2) Optical Method

In working a physical boundary-value problem, one is usually obliged to choose a coordinate system whose parametric surfaces merely approximate the boundaries of the problem. In the case of inversions of the Eisenhart systems, the problem of choosing the best approximation may be difficult because of the infinite variety of shapes that can be obtained by varying the relative sizes and positions of the sphere of inversion and the figure to be inverted. In addition, it is often difficult to visualize the three-dimensional shape of an inverted object. As an experimental aid in determining the best coordinate system for a given problem, a spherical mirror can be used. Under certain conditions, the reflection of an object gives an indication of the shape of the inversion of the object in a similar sphere. It must be remembered, however, that such a reflection in a sphere is not an exact analogy to the inversion problem. Figure 2 illustrates the geometrical considerations involved.

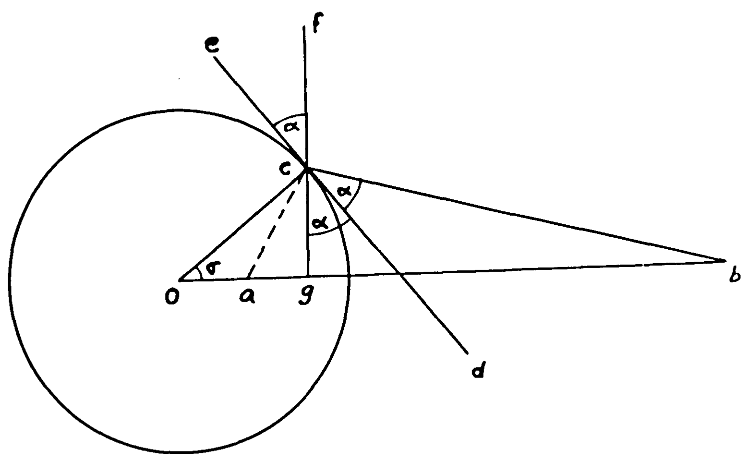


FIGURE 2.

Point b is a point whose inversion is to be found in the sphere with radius oc and center at O. Consider first the light-ray bc, which is reflected from the sphere at c.

To an external observer, the reflected ray cf appears to come from a virtual image at g. Now consider a point a on the line ob, so located that $\frac{oa}{oc} = \frac{oc}{ob}$, or $(oc)^2 = (oa)(ob)$. Point a is then the inversion of point b, and triangle oac is similar to triangle ocb. In order for the virtual image of point b to correspond to the inversion of point b, points a and g must coincide, and angle ogc must equal angle ocb.

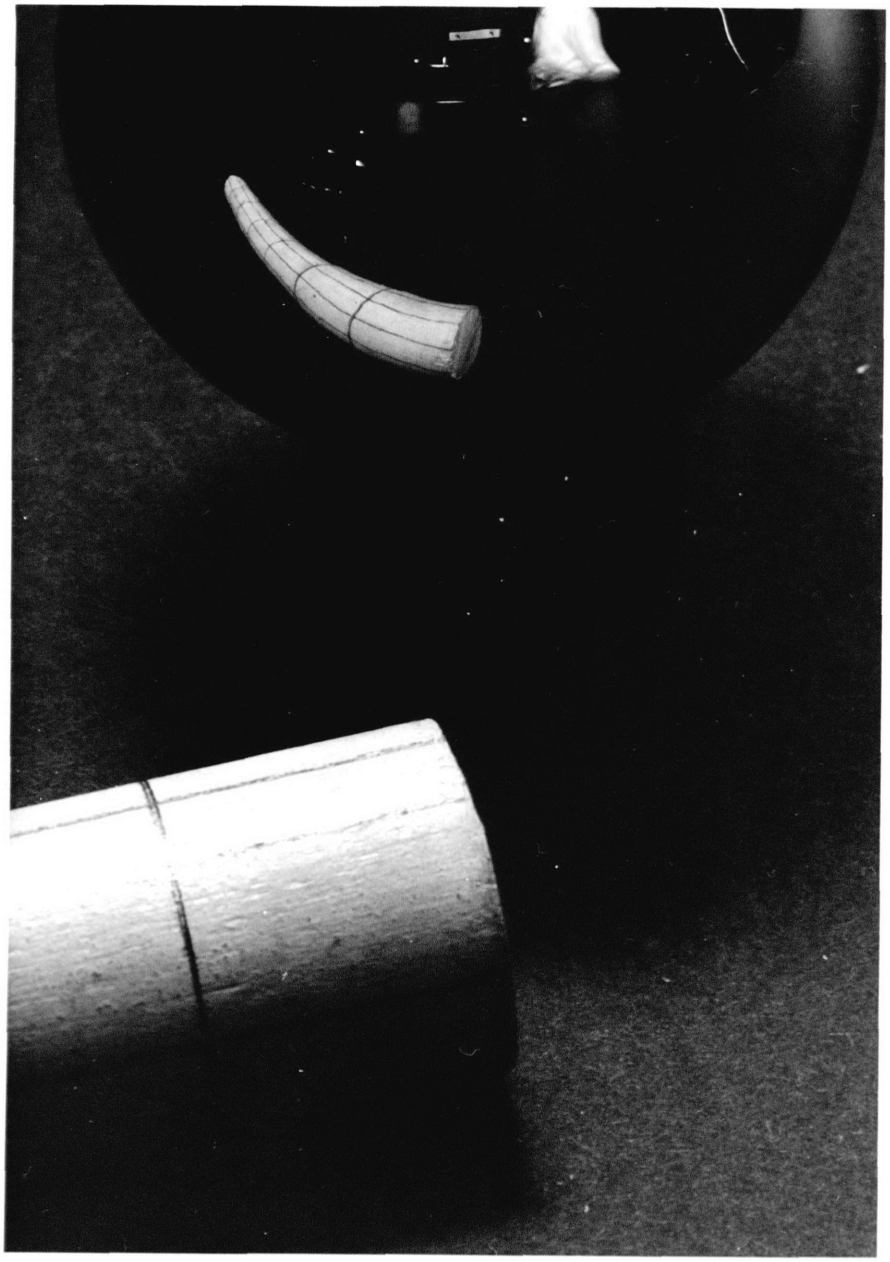
$$\angle\, ocd = \pi/2$$

$$\angle\, oc\ \ = \pi/2 + \alpha$$

$$\angle\, ogc = \pi - \sigma - \pi/2 + \alpha = \pi/2 + \alpha - \sigma.$$

$$\therefore \quad \angle\, ogc = \angle\, ocb - \sigma.$$

The maximum value of σ for a given position of b occurs when the ray cb is tangent to the sphere, and this limiting value of σ decreases as b moves closer to the sphere. Apparently, then, the virtual image g is close to the inversion a when b is close to the surface of the sphere. In other words, a spherical mirror is useful in visualizing the form of an inverted object providing the object is close to the sphere of inversion. Actually, the device is useful even though the geometrical analogy is far from being exact, because the general shape of the image is the same as that of the inversion. A number of photographs is shown in Figures 3 through 8, illustrating the optical analogy discussed here.

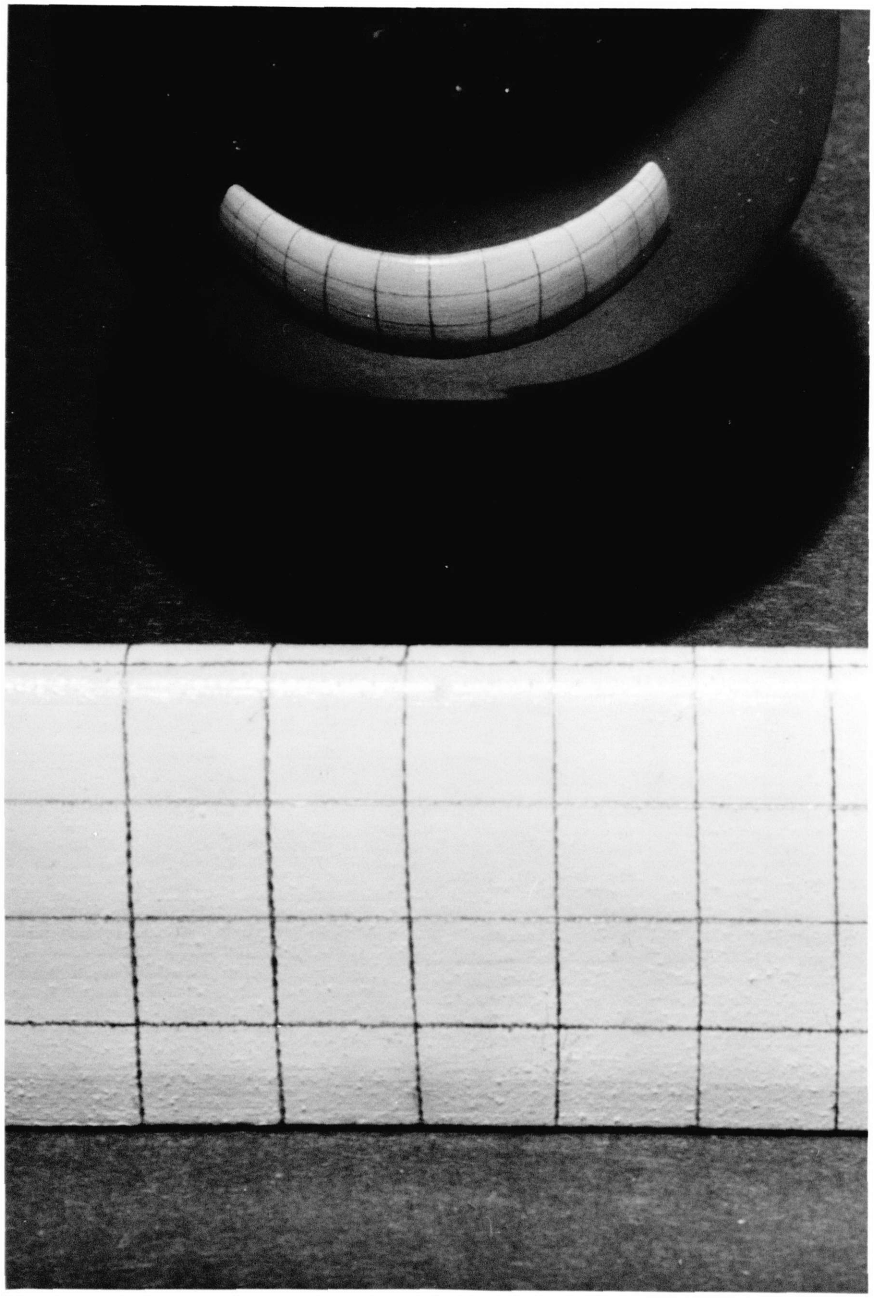


REFLECTION OF A CIRCULAR CYLINDER IN A CONVEX SPHERICAL MIRROR.

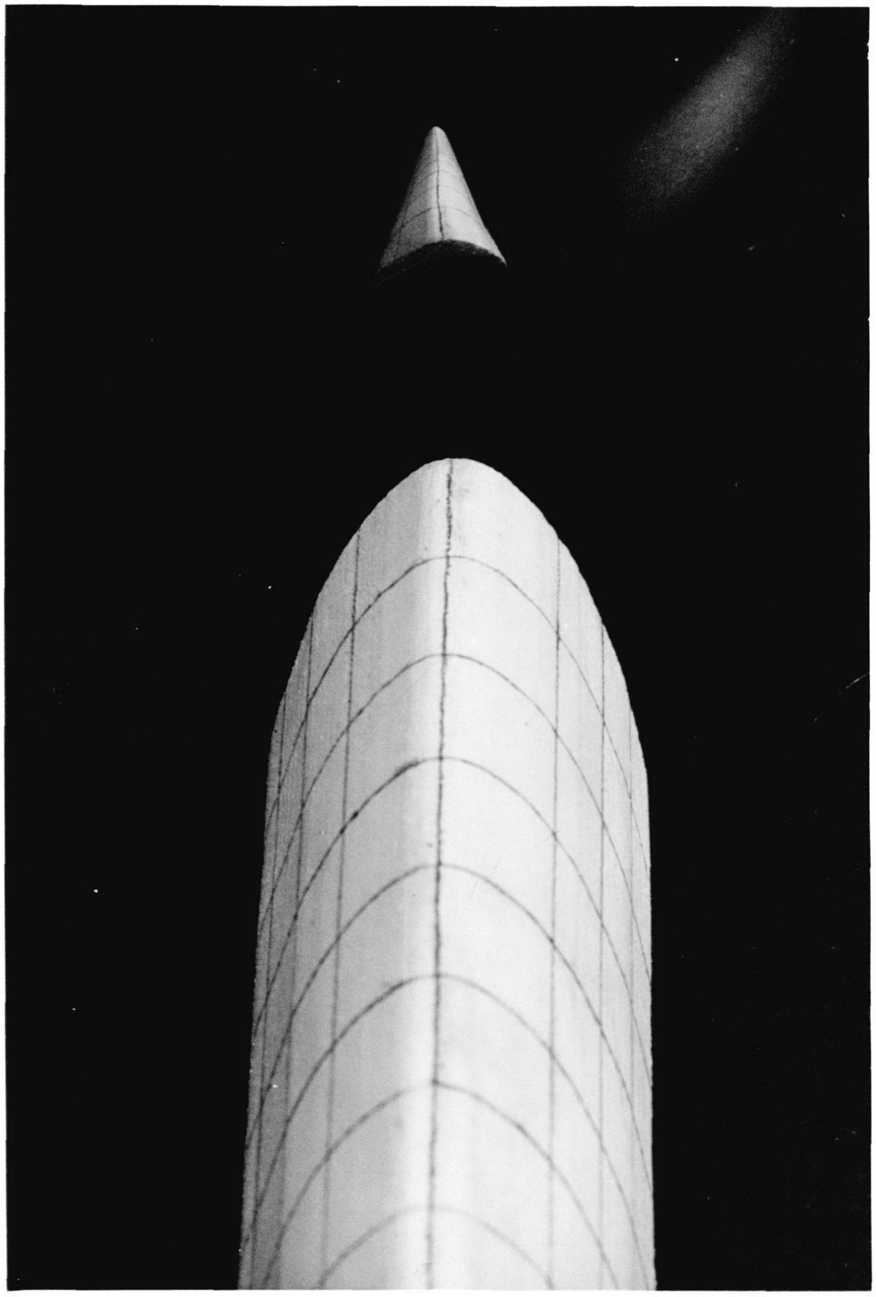
FIGURE 3.

REFLECTION OF A CIRCULAR CYLINDER IN A CONVEX SPHERICAL MIRROR.

FIGURE 4.

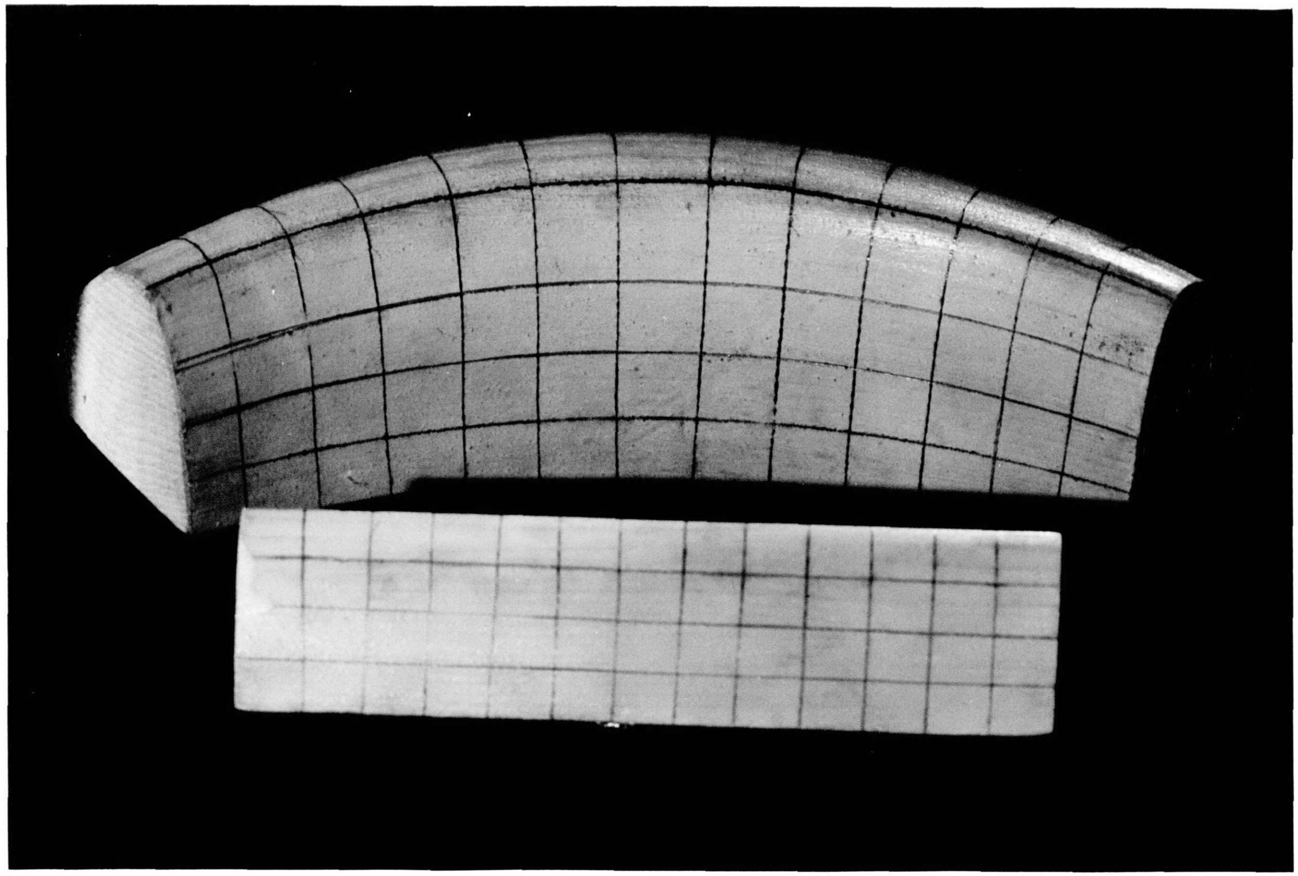


REFLECTION OF A PARABOLIC CYLINDER IN A CONVEX SPHERICAL MIRROR.
FIGURE 5.



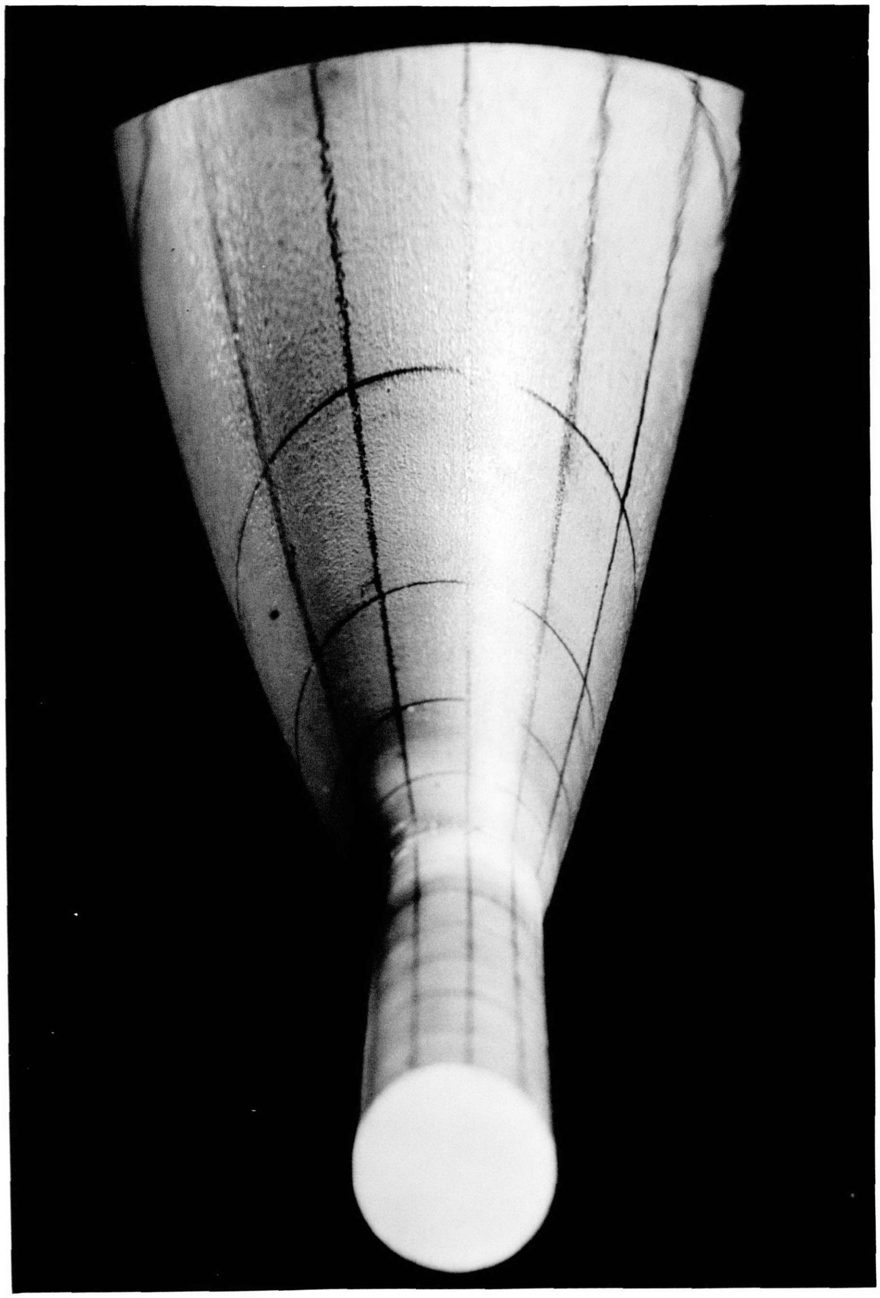
REFLECTION OF A PARABOLIC CYLINDER IN A CONVEX SPHERICAL MIRROR.

FIGURE 6.



REFLECTION OF A PARABOLIC CYLINDER IN A CONCAVE SPHERICAL MIRROR.

FIGURE 7.



REFLECTION OF A CIRCULAR CYLINDER IN A CONCAVE SPHERICAL MIRROR. AXIS OF CYLINDER NORMAL TO MIRROR SURFACE.

FIGURE 8.

C. The Inverted Coordinate Systems

Although it is not possible to represent accurately a three-dimensional coordinate system on a two-dimensional drawing, it is possible in certain cases to draw sectional views of the parametric surfaces which, when imagined to be rotated about suitable axes, are an adequate aid to accurate visualization of the system. Only inversions of those systems with rotational symmetry can be presented in this way, and only those inversions in which the symmetry is preserved. A number of coordinate systems capable of this type of representation are described in the following paragraphs.

(1) Inverse Rectangular Coordinates

Figure 9 is a sectional view in the XY plane of the inversion of rectangular coordinates in a sphere centered at the origin. (There is no reason to choose any other center with rectangular coordinates.) The circles u = constant and v = constant are inversions in the sphere of the lines x = constant and y = constant, respectively. If the family of circles designated by the parameter *u* is rotated about the *x* axis, and if the family designated by *v* is rotated about the *y* axis, and if a third family is visualized, designated by *w*, and is rotated about the *z* axis, a three dimensional system results, consisting of three families of spheres, all tangent to the origin.

(2) Inverse Spheroidal Coordinates

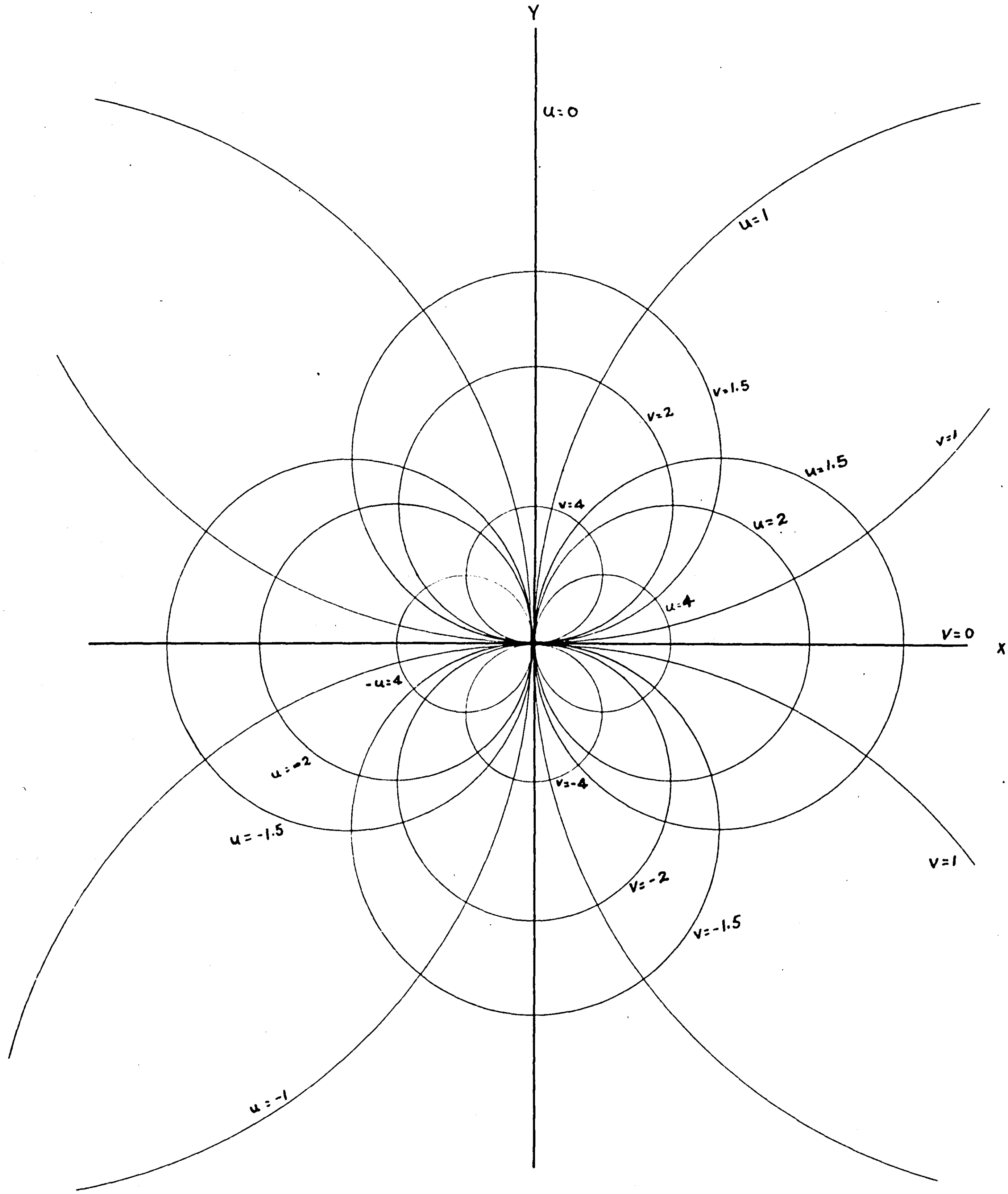
Figure 10 illustrates a two-dimensional elliptical coordinate system, from which may be generated three different three-dimensional systems. Translation of the figure in a direction perpendicular to the paper yields elliptic cylinder coordinates, which, in common with the other cylindrical systems posess inversions which are very difficult to represent diagramatically. Rotation of Figure 10 about the x axis yields the prolate-spheroidal coordinate system, and rotation about the y axis yields the oblate-spheroidal system.

Figure 11 represents the inversion of Figure 10 with respect to a circle centered at the origin. From this figure, then, can be visualized three special cases of inverted coordinate systems. Rotation of Figure 11 about the axis $u = \pi/2$ yields the inversion of the oblate spheroidal coordinates. Figure 11 as it stands represents any cross-section of the coordinate system resulting from an inversion of elliptic cylinder coordinates in a sphere whose center lies on the cylindrical axis.

Figure 12 is an inversion of Figure 10 with respect to a circle centered on the positive x axis. Apparently there is only one axis of symmetry in this diagram. If the figure is rotated about this axis, an inverse prolate-spheroidal coordinate system results, the center of the sphere of inversion being on the positive x axis. Similarly, Figure 13 can be rotated about its axis of symmetry ($u = \pi/2$) to produce an inverse oblate-spheroidal system whose center of inversion is on the positive y axis.

(3) Inverse Parabolic Coordinates

In Figure 14 is illustrated the two-dimensional parabolic coordinate-system. Rotation of this figure about the horizontal axis (z= 0) produces the three-dimensional parabolic coordinate system. Inversion of Figure 14 in any circle centered on the horizontal axis gives a figure which can be rotated about the same axis to produce a system of parabolic coordinates. For example, Figure 15 is an inversion with respect to a circle centered at the origin, and Figure 16 is an inversion with center on the positive y axis (horizontal axis).



INVERSION OF RECTANGULAR COORDINATES IN A CIRCLE
WITH CENTER AT THE ORIGIN.

FIGURE 9.

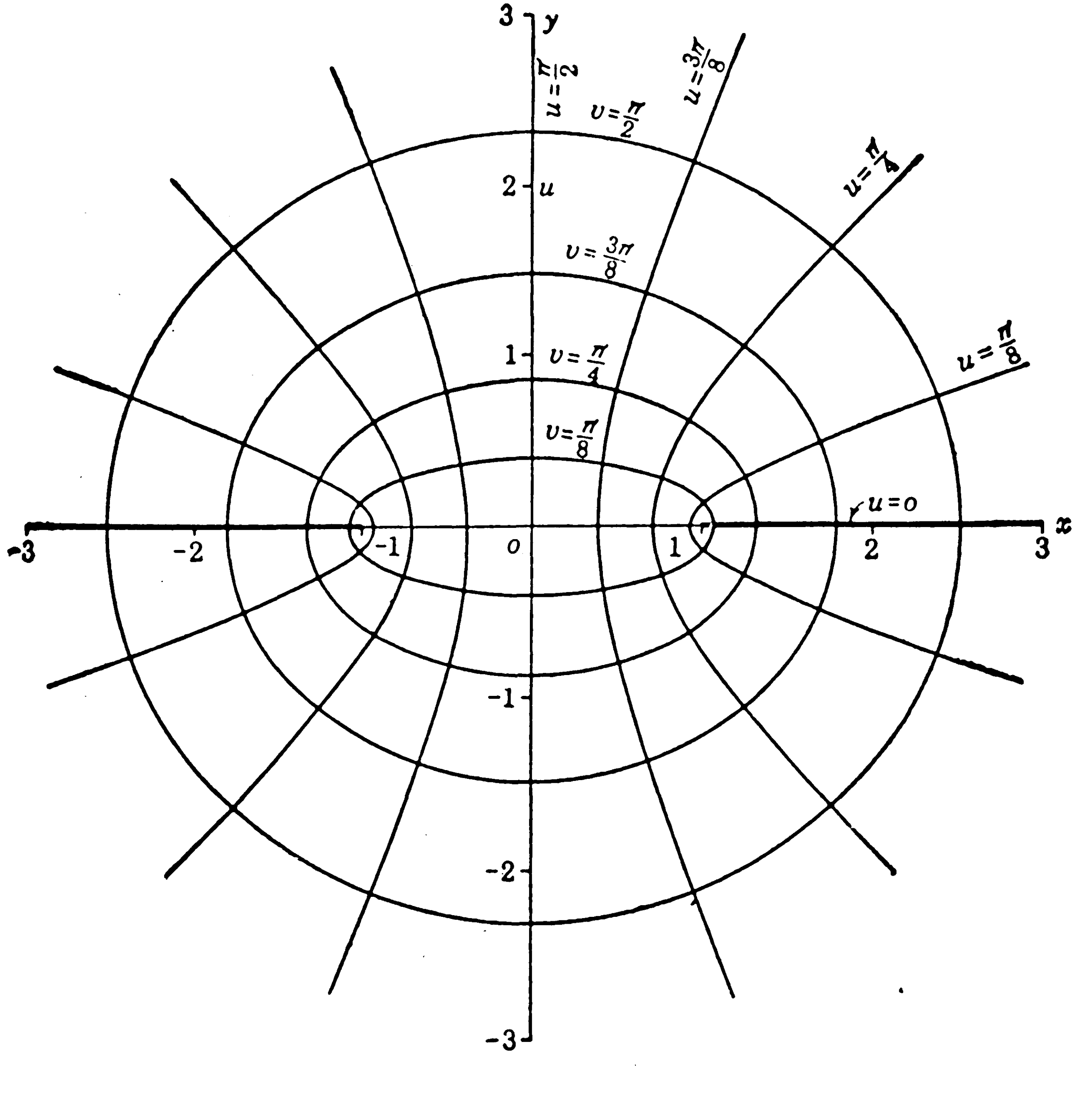
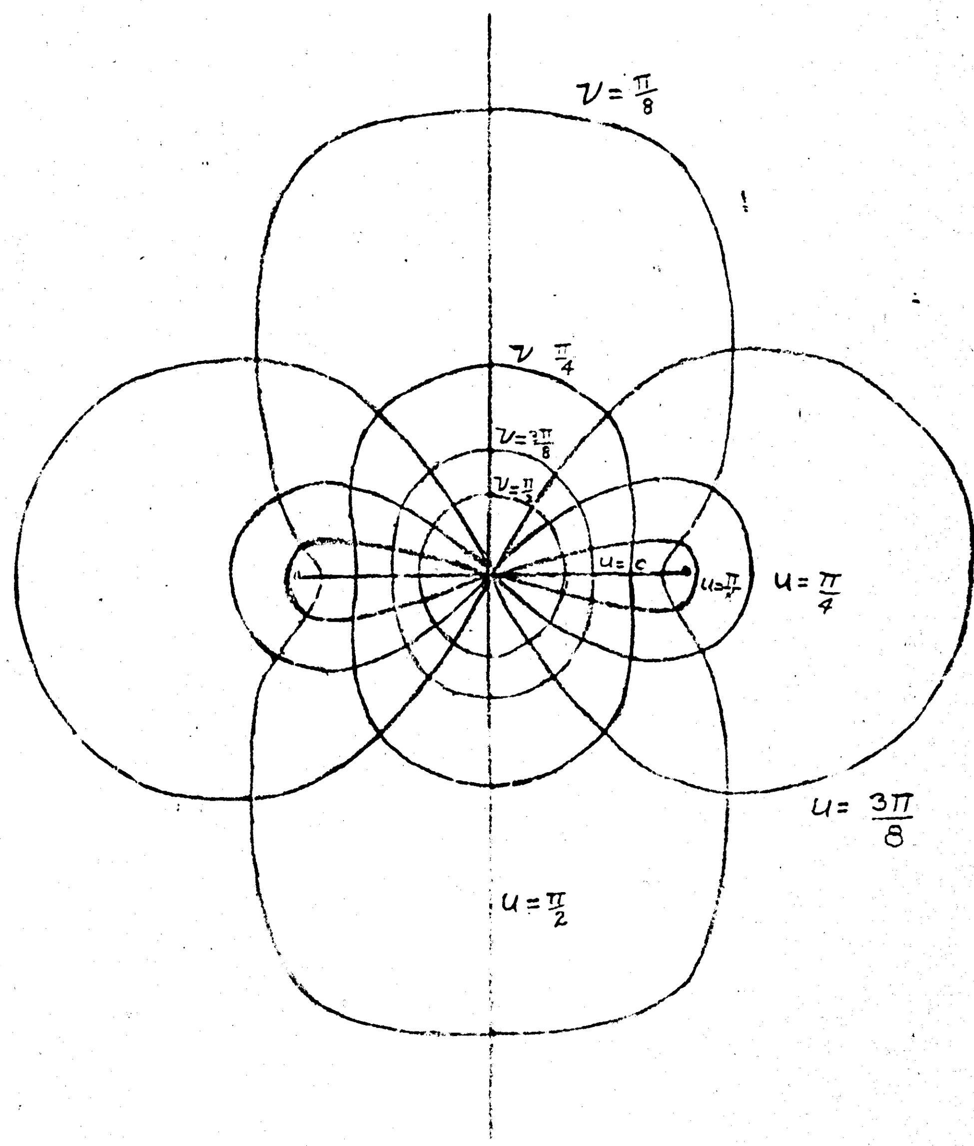


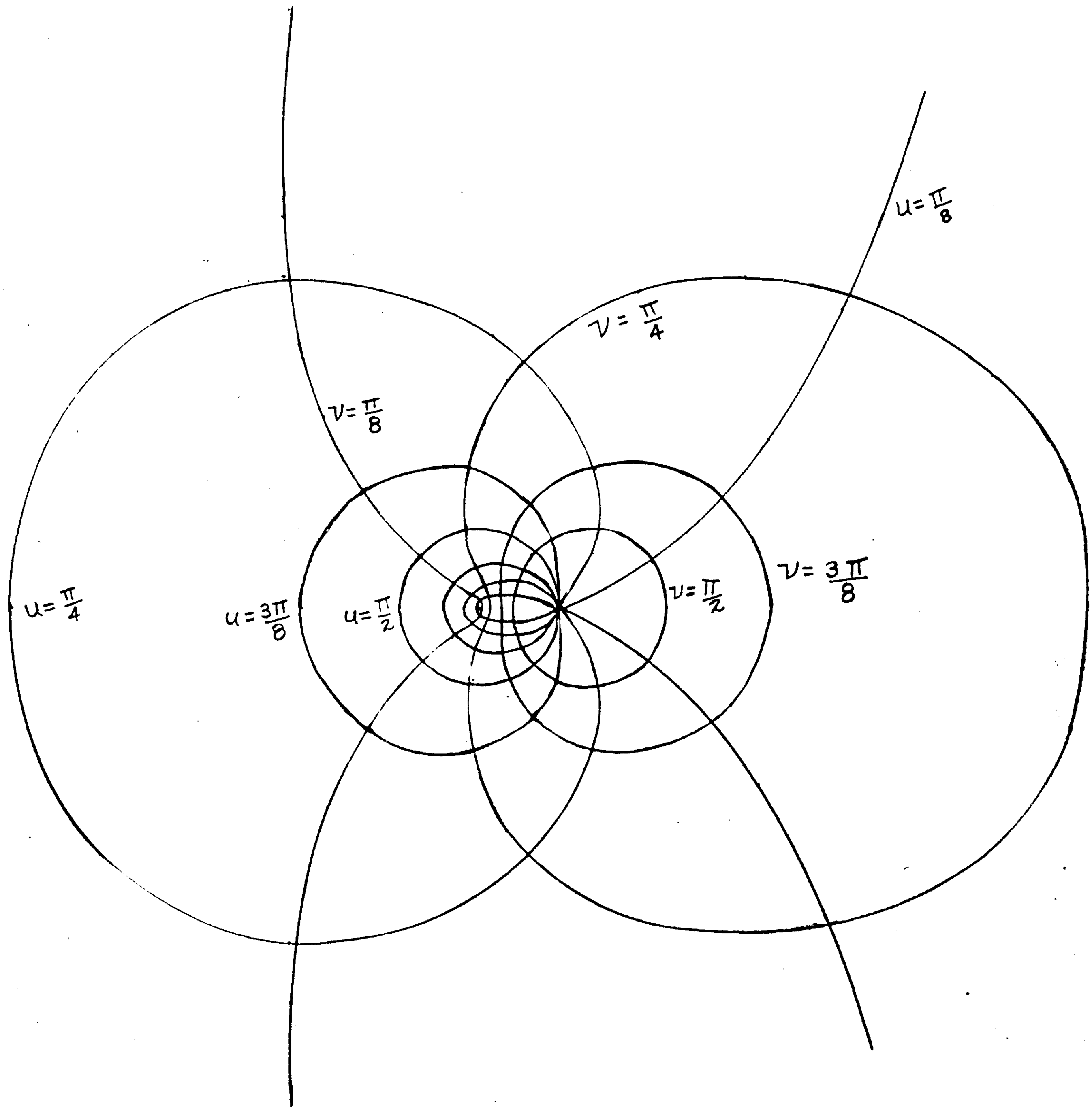
FIGURE 10. ELLIPTIC COORDINATES



INVERSION OF ELLIPTIC COORDINATES WITH RESPECT TO A CIRCLE WITH CENTER AT THE ORIGIN AND WITH RADIUS EQUAL TO THE FOCAL DISTANCE.

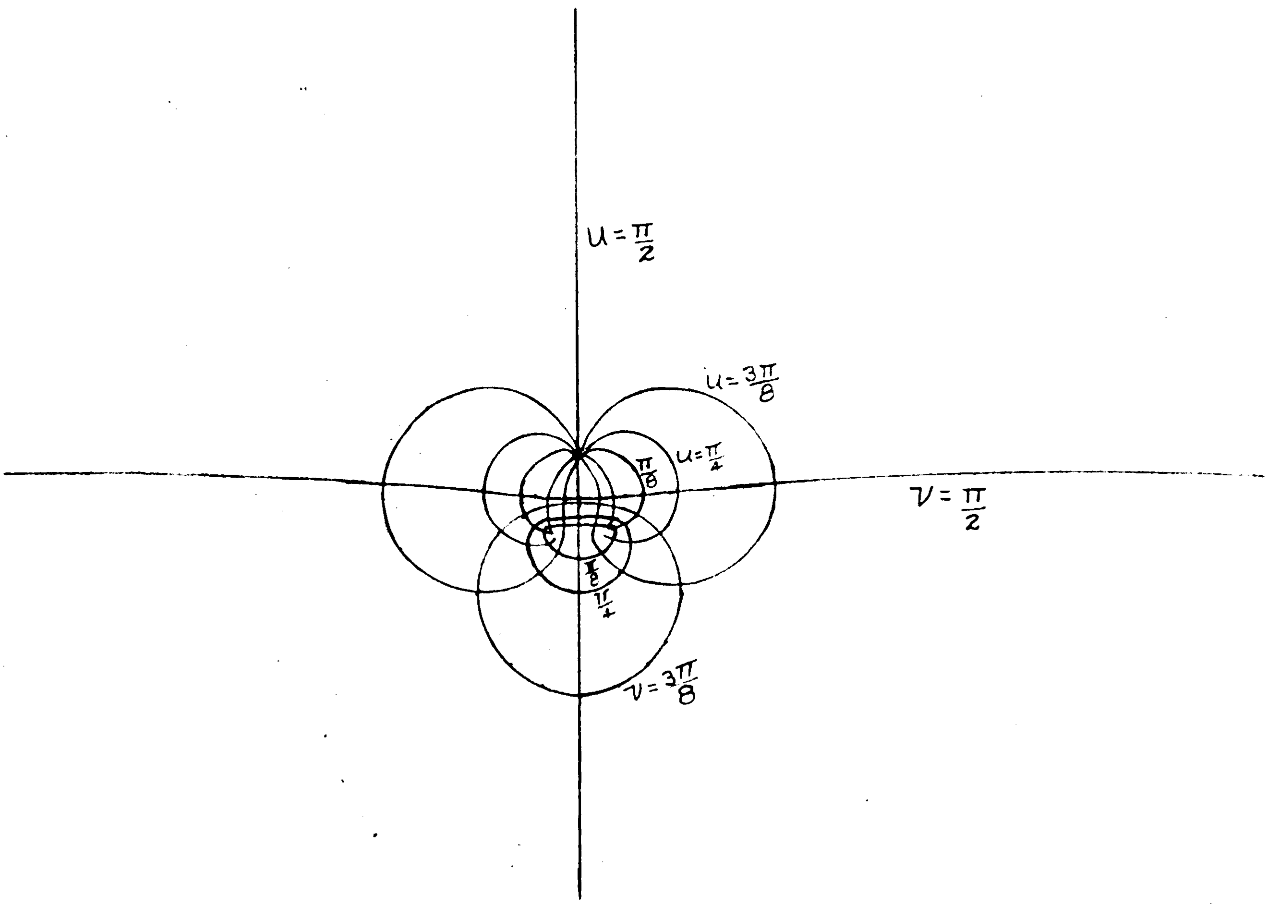
(Each curve labelled to correspond with its inversion.)

FIGURE 11.



INVERSION OF ELLIPTIC COORDINATES WITH RESPECT TO A CIRCLE WITH CENTER AT THE RIGHT-HAND FOCUS AND WITH RADIUS EQUAL TO THE FOCAL DISTANCE.

FIGURE 12.



INVERSION OF ELLIPTIC COORDINATES WITH RESPECT TO A CIRCLE WITH CENTER AT $(u=\frac{\pi}{2},\ v=\frac{\pi}{2})$ AND WITH RADIUS EQUAL TO THE FOCAL DISTANCE.

FIGURE 13.

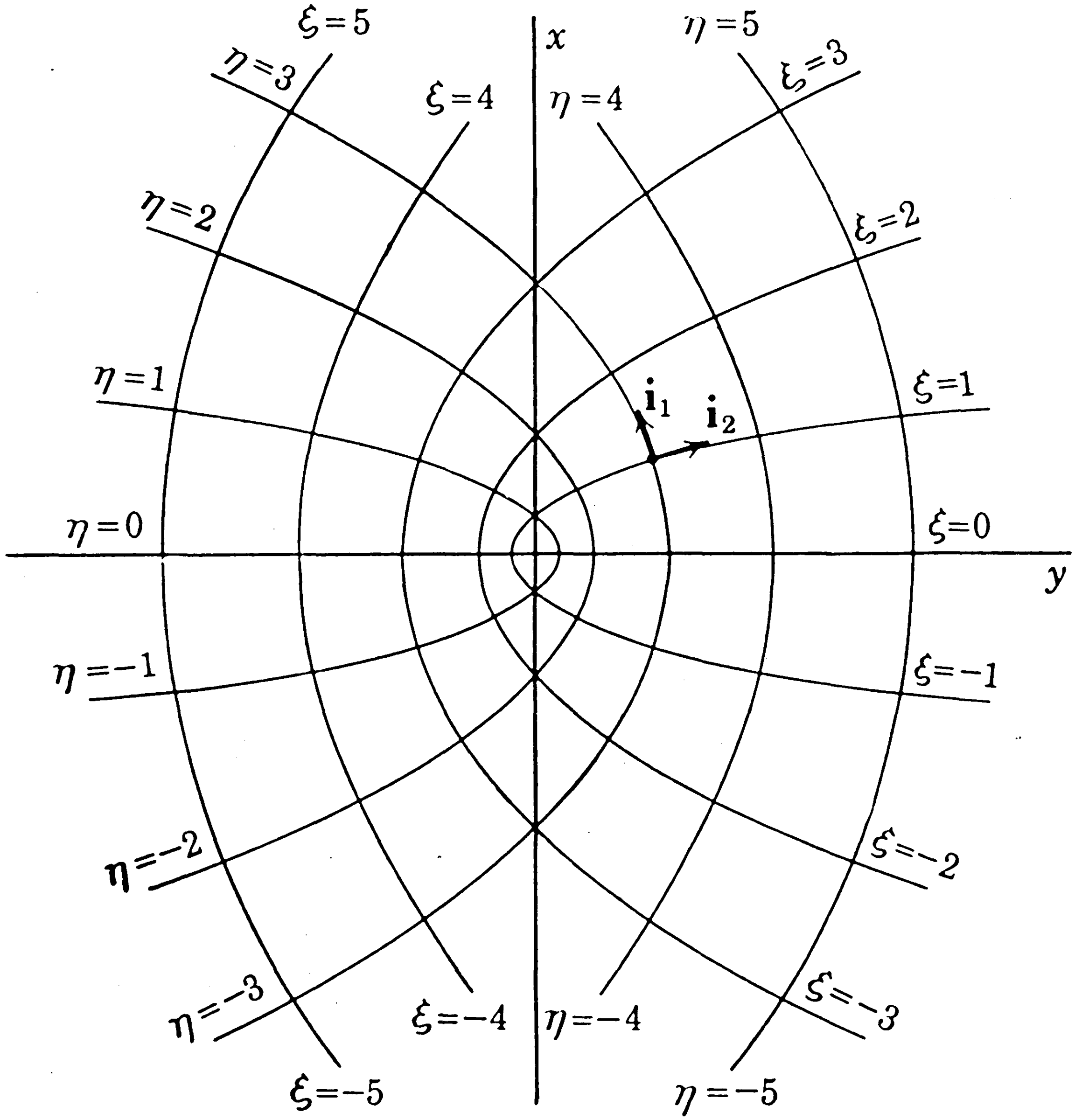
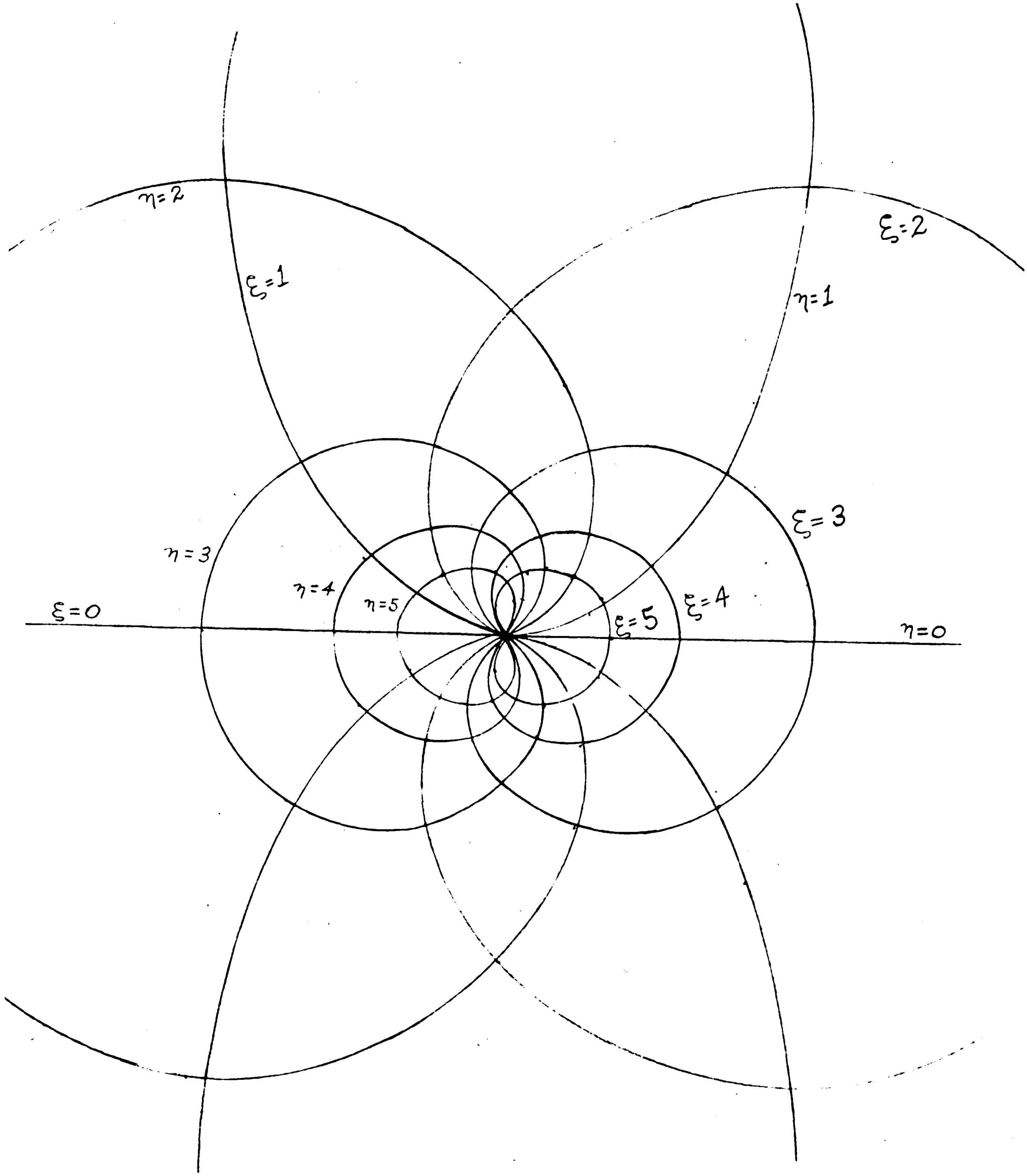
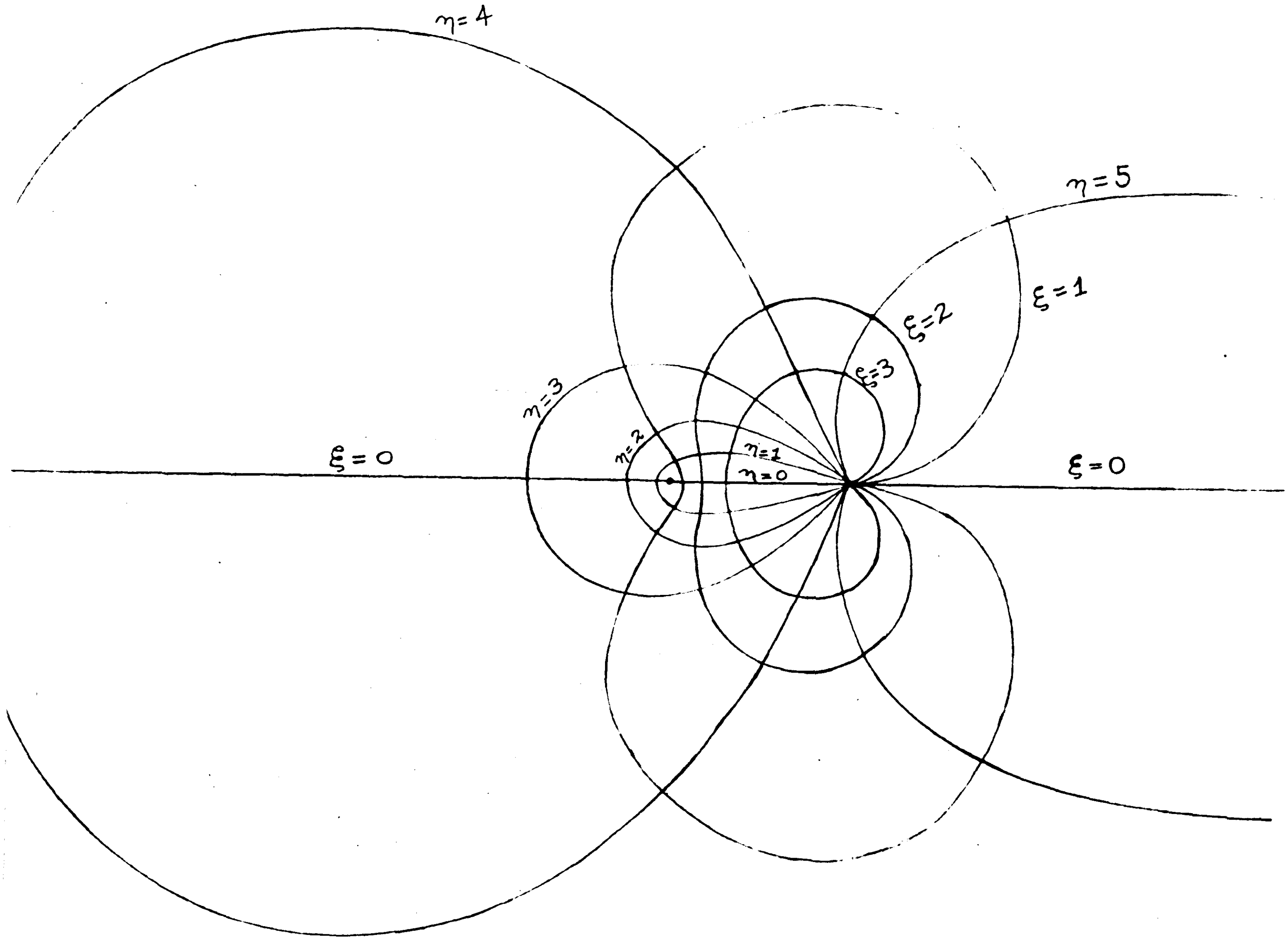


FIGURE 14. PARABOLIC COORDINATES



INVERSION OF PARABOLIC COORDINATES IN A CIRCLE WITH RADIUS EQUAL TO THE Y-INTERCEPT OF $\eta = 4$ AND WITH CENTER AT THE ORIGIN.

FIGURE 15.



INVERSION OF PARABOLIC COORDINATES IN A CIRCLE WITH RADIUS EQUAL TO THE Y-INTERCEPT OF $\eta = 4$ AND WITH CENTER AT THE Y-INTERCEPT OF $\eta = 4.3$.

FIGURE 16.

III

LAPLACE'S EQUATION IN INVERTED COORDINATE SYSTEMS

A. The Form of the Equation

As was mentioned in Chapter I, Laplace's equation assumes a different form in each different coordinate system. The general form, in any system is

$$\nabla^2\phi = \frac{1}{\sqrt{g}}\left\{\frac{\partial}{\partial u_1}\left(\frac{\sqrt{g}}{g_{11}}\frac{\partial\phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{\sqrt{g}}{g_{22}}\frac{\partial\phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{\sqrt{g}}{g_{33}}\frac{\partial\phi}{\partial u_3}\right)\right\} = 0,$$

where $\underline{u_1}$, $\underline{u_2}$, and $\underline{u_3}$ are the three independent variables, $\sqrt{g} = \sqrt{g_{11}\ g_{22}\ g_{33}}$, and g_{11}, g_{22}, and g_{33} are defined by

$$ds^2 = g_{11}\ du_1^2 + g_{22}\ du_2^2 + g_{33}\ du_3^2.$$

Hence, if the linear element ds^2 is known for the system, Laplace's equation can be written. The linear elements for the twelve conventional coordinate systems have been tabulated by a number of authors.[4,9,13,8] The linear elements for the inversions of the eleven Eisenhart systems are tabulated in part c and in reference 10.

B. Separation of Variables

In the eleven coordinate systems of Eisenhart it is possible to separate Laplace's equation by the straightforward method illustrated in Chapter I. This method assumes the potential equal to the product $U_1U_2U_3$, each U being a function of only one variable. This assumption does not produce separation, however, in toroidal coordinates or in any of the inverted systems. In these cases it is necessary to assume a potential of the form $\frac{U_1U_2U_3}{R}$, where R is a properly chosen function of all three independent variables. With this assumption it is possible to proceed in the usual way to obtain ordinary differential equations for U_1, U_2, and U_3. The method, however, involves a great deal of tedious manipulation, much of which can be avoided by a more general approach.

A more convenient method[8], originally due to P. Stäckel[12] and generalized by P. M. Morse[9], permits separation in all twenty-one of the systems in which Laplace's equation is known to be separable.

Without considering its derivation, this method is used as follows:

Define a determinant

$$S = \begin{vmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{vmatrix}, \qquad (1)$$

where the quantities in the first row are functions of u_1 only, those in the second row of u_2 only, and those in the third row of u_3 only.

Let $M_1 = \begin{vmatrix} \Phi_{22} & \Phi_{23} \\ \Phi_{32} & \Phi_{33} \end{vmatrix}$, $\quad M_2 = -\begin{vmatrix} \Phi_{12} & \Phi_{13} \\ \Phi_{32} & \Phi_{33} \end{vmatrix}$,

$$\text{and} \qquad M_3 = \begin{vmatrix} \Phi_{12} & \Phi_{13} \\ \Phi_{22} & \Phi_{23} \end{vmatrix}. \qquad (2)$$

Now determine the M's so that $g_{ii} = \frac{S}{M_i} x$, (3) where x is any convenient function of u_1, u_2, and u_3.

Let $\frac{\sqrt{g}}{S} = f_1 \, f_2 \, f_3 \, R^2 x$ (4)

where f_i is a function of u_i only and R is another convenient, arbitrary function of u_1, u_2, and u_3.

In separating Laplace's equation, first find g_{ii} from the linear element. Next, choose a value for x, and from equation (3) determine $\frac{S}{M_i}$. The three values of S/M_i determine the Stäckel determinant S, which can be written by inspection, subject to the restriction on Φ_{ij}. Having determined S, choose a value for R, and from equation (4), find f_1, f_2, and f_3. The separated equations are now given by

$$\frac{1}{f_i} \frac{d}{du_i} \left(f_i \frac{dU_i}{du_i}\right) + U_i \sum_{j=1}^{3} \Phi_{ij} \, \alpha_j = 0. \qquad (5)$$

U_i is a function of u_i only, and α_j is an arbitrary constant. The potential ϕ is equal to the function $\frac{U_1 U_2 U_3}{R}$.

The foregoing method is known as "R-separation", and may be applied to any coordinate system. In the eleven Eisenhart systems R is equal to unity, but in toroidal coordinates, and in all the inverted systems, R is a function of u_1, u_2, and u_3.

As an example of the use of the method, consider the inverse circular-cylinder system. The linear element, as found in reference 10, is

$$ds^2 = \frac{(du^2 + dv^2) + dw^2}{u^2\left(\frac{a^2+u^2+v^2}{2au} + \sin w\right)^2},$$ where a is the distance

from the axis of the cylinder to the center of inversion.

$$g_{11} = g_{22} = \frac{1}{u^2\left(\frac{a^2+u^2+v^2}{2au} + \sin w\right)^2}, \quad g_{33} = \frac{1}{\left(\frac{a^2+u^2+v^2}{2au} + \sin w\right)^2},$$

Let $x = \frac{1}{(\quad)^2}$.

$\frac{g_{33}}{x} = \frac{S}{M_3} = 1.$ Similarly $\frac{S}{M_1} = \frac{S}{M_2} = \frac{1}{u^2}.$

$$\text{Let } S = \begin{vmatrix} \frac{1}{u^2} & \ell & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{vmatrix}.$$

$M_3 = \frac{1}{u^2} = \ell n - m; \quad M_1 = 1, \quad M_2 = 1 = \ell.$

$\therefore \ \ell = 1, \quad n = 0, \quad m = -\frac{1}{u^2}.$

$$\text{Then } S = \begin{vmatrix} \frac{1}{u^2} & 1 & \frac{1}{u^2} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = \frac{1}{u^2}.$$

$$\frac{\sqrt{g}}{S} = \frac{1}{\left(\frac{a^2+u^2+v^2}{2au} + \sin w\right)^3} = f_1 f_2 f_3 R^2 x.$$

$$f_1 = f_2 = f_3 = 1; \quad R^2 = \frac{1}{\frac{a^2+u^2+v^2}{2au} + \sin w}.$$

Substituting into equation (5), where $u_1 = u$, $u_2 = v$, $u_3 = w$; the separated equations are:

$$\text{(i)} \quad \frac{d^2U}{du^2} + U \quad \frac{\alpha_1}{u^2} + \alpha_2 + \frac{\alpha_3}{u^2} = 0,$$

(ii) $\frac{d^2V}{dv^2} - a_2 V = 0,$

(iii) $\frac{d^2W}{dv^2} - a_3 W = 0.$

The solutions of the ordinary differential equations are:

(i) $U = \sqrt{u}\, Z_p(\sqrt{a_2}\, u)$, where Z_p is the solution of Bessel's Equation, and $p = \frac{1}{2}\sqrt{1-4(a_3+a_1)}$,

(ii) $V = \begin{matrix}\sinh\\ \cosh\end{matrix} \sqrt{a_2}\, v,$

(iii) $W = \begin{matrix}\sinh\\ \cosh\end{matrix} \sqrt{a_3}\, w.$

$$\phi = \frac{UVW}{R} .$$

It is to be expected that only two arbitrary constants will appear in the solution, indicating that a_1, a_2, and a_3 are interdependent. a_1 can be eliminated by substitution into the formula

$$a_1 = -\frac{x}{R} \sum_{i=1}^{3} \frac{1}{g_{ii} f_i} \frac{\partial}{\partial u_i}\left(f_i \frac{\partial R}{\partial u_i}\right), \qquad (6)$$

which yields a numerical value.

The greatest advantage of the foregoing method of separation is that it avoids a great deal of tedious differentiation and algebraic manipulation in determining the separated equations. On the other hand, in order to eliminate the unwanted separation constant, it is often necessary to perform almost the same amount of mathematical work as is necessary in the more straightforward separation method, which gives only the two irreducible separation constants. Nevertheless, the method of Stäckel is of great value in determining with the least possible effort the form of the separated equations and their solutions.

C. Tabulation of the Separated Equations for the Inverted Coordinate Systems

As an aid in using inverted coordinate systems in potential problems, the mathematical characteristics of each system have been tabulated, along with the ordinary differential equations resulting from the separation of Laplace's equation. The starting point in each case has been the linear element, ds^2, which has been obtained from reference 10. In cases where it was possible to recognize their solutions, the separated equations were solved. The method used in deriving the separated equations was that of section B; therefore three separation constants appear in each solution. In each case, α_1 must be eliminated by application of equation (6) of section B.

The Roman numeral following the name of each system corresponds to the designation of that system in reference 10.

(1) Inverse Rectangular Coordinates, X

$$ds^2 = \frac{2(du^2 + dv^2) + dw^2}{\left[2(u^2+v^2) + w^2\right]^2}$$

$$\frac{1}{R} = \sqrt{2(u^2+v^2) + w^2} \quad ,$$

$$S = \begin{vmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = 1$$

$$\phi = \frac{UVW}{R}$$

Separated Equations:

(i) $U'' + \alpha_1 U = 0$

(ii) $V'' - \alpha_2 V = 0$

(iii) $W'' - (\alpha_1 - \alpha_2)W = 0$

Note: The separation constants have been rearranged.

Solutions:

(i) $U = \frac{\sin}{\cos} \sqrt{\alpha_1}\, u$

(ii) $V = \frac{\sinh}{\cosh} \sqrt{\alpha_2}\, v$

(iii) $W = \frac{\sin}{\cos} \sqrt{\alpha_1 - \alpha_2}\, w$

Notes:

Parametric surfaces consist of three families of spheres, all tangent to the origin, with centers on the x, y, and z axes, respectively. See illustration problems in Chapter III.

The surfaces $u = u_1$, $v = v_1$, $w = w_1$, are inversions of the surfaces $x = x_1$, $y = y_1$, $z = z_1$, respectively.

(2) Inverse Circular-Cylinder Coordinates, XIII

$$ds^2 = \frac{\frac{1}{u^2}(du^2 + dv^2) + dw^2}{\left[\frac{a^2+u^2+v^2}{2au} + \sin w\right]^2}$$

$$R = \frac{1}{\left[\frac{a^2+u^2+v^2}{2au} + \sin w\right]^{1/2}}$$

$$S = \begin{vmatrix} \frac{1}{u^2} & 1 & \frac{1}{u^2} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = \frac{1}{u^2}$$

Separated Equations:

(i) $U'' + U\left(\alpha_1 + \alpha_3\right)\frac{1}{u^2} + \alpha_2 = 0$

(ii) $V'' - \alpha_2 V = 0$

(iii) $W'' - \alpha_3 W = 0$

Solutions:

(i) $U = \sqrt{u}\, Z_p(\sqrt{\alpha_2}\, u)$, where $p = \frac{1}{2}\sqrt{1-4(\alpha_3+\alpha_1)}$.

(ii) $V = \frac{\sinh}{\cosh} \sqrt{\alpha_2}\, v$

(iii) $W = \frac{\sinh}{\cosh} \sqrt{\alpha_3}\, w.$

Notes: See Figures 1, 11, and 12. The surfaces $u = u_1$, $v = v_1$, $w = w_1$ correspond to the surfaces $r = r_1$, $\theta = \theta_1$, $z = z_1$ in conventional circular-cylinder coordinates.

(3) Inverse Spherical Coordinates XI

$$ds^2 = \frac{\csc^2 u(du^2 + dv^2) + dw^2}{\left[\frac{a \cosh v + \cos u}{\sin u} + \sin w\right]^2}$$

$$R = \frac{1}{\sqrt{\frac{a \cosh v + \cos u}{\sin u} + \sin w}}$$

$$S = \begin{vmatrix} \csc^2 u & 1 & \csc^2 u \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = \csc^2 u$$

Separated Equations:

(i) $U'' + U\,(\alpha_1 + \alpha_2)\csc^2 u + \alpha_3 = 0$

(ii) $V'' - \alpha_2 V = 0$

(iii) $W'' - \alpha_3 W = 0$

Notes: To change the above notation to the more conventional form, let $v = \ln r$, $u = \theta$, $w = \Psi$.

(4) Inverse Prolate Spheroidal Coordinates. V

$$ds^2 = \frac{(\csc^2 u + \operatorname{csch}^2 v)(du^2 + dv^2) + dw^2}{\left[\frac{a^2 + b^2 + \sinh^2 u + \cos^2 v - 2b\cosh u \cos v + \sin w}{2a \sinh u \sin v}\right]^2}$$

$$R = \frac{1}{\sqrt{\left[\qquad\qquad\right]}}$$

$$S = \begin{vmatrix} \csc^2 u & 1 & \csc^2 u \\ \operatorname{csch}^2 v & -1 & \operatorname{csch}^2 v \\ 0 & 0 & -1 \end{vmatrix} = \csc^2 u + \operatorname{csch}^2 v$$

Separated Equations:

(i) $U'' + U(\alpha_1 + \alpha_3)\csc^2 u + \alpha_2 = 0$

(ii) $V'' + V(\alpha_1 + \alpha_3)\operatorname{csch}^2 v - \alpha_2 = 0$

(iii) $W'' - \alpha_3 W = 0$

Notes: See Figures 11 and 12, and article IIC(2). Above notation corresponds to the notation in Figures 10, 11, and 12. The focal distance is assumed to be unity in these formulas; an arbitrary value may be obtained by multiplying csc u and csch v by c, the focal distance. The constants a and b correspond respectively to $\sqrt{x_0^2 + y_0^2}$ and z_0, where (x_0, y_0, z_0) are the rectangular coordinates of the center of inversion.

(5) Inverse Oblate Spheroidal Coordinates, VI

$$ds^2 = \frac{(\sec^2 u - \operatorname{sech}^2 v)(du^2 + dv^2) + dw^2}{\left[\dfrac{a^2 + b^2 + \cosh^2 u + \sin^2 v - 2b \sinh u \sin v + \sin w}{2a \cosh u \cos v}\right]^2}$$

$$R = \frac{1}{\left[\qquad\qquad\right]^{1/2}}$$

$$S = \begin{vmatrix} \sec^2 u & 1 & \sec^2 u \\ \operatorname{sech}^2 v & -1 & \operatorname{sech}^2 v \\ 0 & 0 & -1 \end{vmatrix} = \sec^2 u + \operatorname{sech}^2 v$$

Separated Equations:

(i) $U'' + U\,(\alpha_1 + \alpha_3)\sec^2 u + \alpha_2 = 0$

(ii) $V'' + V\,(\alpha_1 + \alpha_3)\operatorname{sech}^2 v - \alpha_2 = 0$

(iii) $W'' - \alpha_3 W = 0$

Notes: See Figures 10, 13, and 14. The notation of the formulas corresponds to that of the figures. The focal distance in the formulas is assumed to be unity; an arbitrary value may be obtained by multiplying sec u and sech v, wherever they occur, by c, the focal distance. The constants a and b correspond respectively to $\sqrt{x_o^2 + y_o^2}$ and z_o, where (x_o, y_o, z_o) are the rectangular coordinates of the center of inversion.

(6) Inverse Parabolic Cylinder Coordinates, IX

$$ds^2 = \frac{(u^2 + v^2)(du^2 + dv^2) + dw^2}{\left[\frac{1}{4}(u^2 - v^2 - a)^2 + (uv - b)^2 + w^2\right]^2}$$

$$R = \frac{1}{\left[\frac{1}{4}(u^2 - v^2 - a)^2 + (uv - b)^2 + w^2\right]^{1/2}}$$

$$S = \begin{vmatrix} u^2 & 1 & u^2 \\ v^2 & -1 & v^2 \\ 0 & 0 & -1 \end{vmatrix} = u^2 + v^2$$

Separated Equations:

(i) $U'' + U\, u^2(\alpha_1 + \alpha_3) + \alpha_2 = 0$

(ii) $V'' + V\, v^2(\alpha_1 + \alpha_3) - \alpha_2 = 0$

(iii) $W'' - \alpha_3 W = 0.$

Notes: The notation of the formulas corresponds to that of Figure 14, if ξ, η, and z are substituted for u, v, and w, respectively.

(7) Inverse Elliptic Cylinder Coordinates, VIII

$$ds^2 = \frac{\frac{1}{2}(\cosh 2v - \cos 2u)(dv^2 + du^2) + dw^2}{\left[(\cosh v \cos u - a)^2 + (\sinh v \sin u - b)^2 + w^2\right]^2}$$

$$R = \frac{1}{\left[\qquad\right]^{1/2}}$$

$$S = \begin{vmatrix} \frac{1}{2}\cosh 2v & 1 & \frac{1}{2}\cosh 2v \\ \frac{1}{2}\cos 2u & -1 & \frac{1}{2}\cos 2u \\ 0 & 0 & -1 \end{vmatrix} = \frac{1}{2}(\cosh 2v + \cos 2u)$$

Separated Equations:

(i) $U'' + U\, \frac{1}{2}(\alpha_1 + \alpha_3)\cos 2u - \alpha_2 = 0$

(ii) $V'' + V\, \frac{1}{2}(\alpha_1 + \alpha_3)\cosh 2v + \alpha_2 = 0$

(iii) $W'' - \alpha_3 W = 0$

Notes: Above formulas assume unity focal distance, but may be generalized by multiplying cosh 2u and cos 2v by c^2, the square of the focal distance, wherever they appear. Notation above corresponds to that in Figure 10. The variable w is measured along the axis of the cylinder.

(8) Inverse Parabolic Coordinates, VII

$$ds^2 = \frac{\left(\frac{1}{u^2} + \frac{1}{v^2}\right)(du^2 + dv^2) + dw^2}{\left[\frac{(u^2+v^2)^2 + a^2 + b^2 + 2b(v^2 - u^2)}{4\,a\,u\,v} + \sin w\right]^2}$$

$$R = \frac{1}{\left[\qquad\qquad\right]^{1/2}}$$

$$S = \begin{vmatrix} \frac{1}{u^2} & 1 & \frac{1}{u^2} \\ \frac{1}{v^2} & -1 & \frac{1}{v^2} \\ 0 & 0 & -1 \end{vmatrix} = \frac{1}{u^2} + \frac{1}{v^2}$$

Separated Equations:

(i) $U'' + U \quad (\alpha_1 + \alpha_3)\frac{1}{u^2} + \alpha_2 \quad = 0$

(ii) $V'' + V \quad (\alpha_1 + \alpha_3)\frac{1}{v^2} - \alpha_2 \quad = 0$

(iii) $W'' - \alpha_3 W = 0.$

Solutions:

(i) $U = \sqrt{u}\, Z_p(\sqrt{\alpha_2}\, u)$, where $p = \frac{1}{2}\sqrt{1-4(\alpha_3+\alpha_1)}$

(ii) $V = \sqrt{v}\, Z_p(\sqrt{-\alpha_2}\, v)$

(iii) $W = \frac{\sinh}{\cosh} \sqrt{\alpha_3}\, w.$

Notes: See Figures 14, 15, and 16, and article IIC(3). In order for the notation of the formulas to correspond with that of the figures, ξ, η, and ψ must be substituted for u, v, and w, respectively.

(9) Inverse Conical Coordinates, I

$$ds^2 = \frac{k^2\,cn^2(k,u) + k'^2\,cn^2(k',v)\ (du^2+dv^2) + dw^2}{4\left[\dfrac{a\,dn(u,k)sn(v,k') + \ sn(u,k)dn(v,k') + c\,cn(u,k)cn(v,k')}{\sqrt{a^2+b^2+c^2}} + \dfrac{\cosh w}{4}\right]^2}$$

$$R = \frac{1}{\sqrt{2}\sqrt{[\qquad]}}$$

$$S = \begin{vmatrix} k^2cn^2(k,u) & 1 & k\,cn^2(k,u) \\ k'^2cn^2(k',v) & -1 & k'cn^2(k',v) \\ 0 & 0 & -1 \end{vmatrix} = k^2cn^2(k,u)+k'^2cn^2(k',v)$$

Separated Equations:

(i) $U'' + U\ (\alpha_1+\alpha_3)k\,cn^2(k,u)+\alpha_2 = 0$

(ii) $V'' + V\ (\alpha_1+\alpha_3)k'cn^2(k',v) - \alpha_2 = 0$

(iii) $W'' - \alpha_3 W = 0$

Note: a, b, and c are the rectangular coordinates of the center of inversion.

IV

THE APPLICATION OF THE INVERTED COORDINATE SYSTEMS TO BOUNDARY VALUE PROBLEMS

Problems in potential theory whose physical boundaries correspond with the parametric surfaces of the nine inverted coordinate systems can be attacked by either of two methods. The first method involves the separation of variables in the appropriate system, the solution of the separated equations, and the expansion of the potential in terms of these solutions. The alternate method involves inverting the given boundary conditions into one of the Eisenhart coordinate systems, solving the new problem in that system, and inverting the solution back into the original system. The choice of method depends upon which is more convenient for the problem at hand.

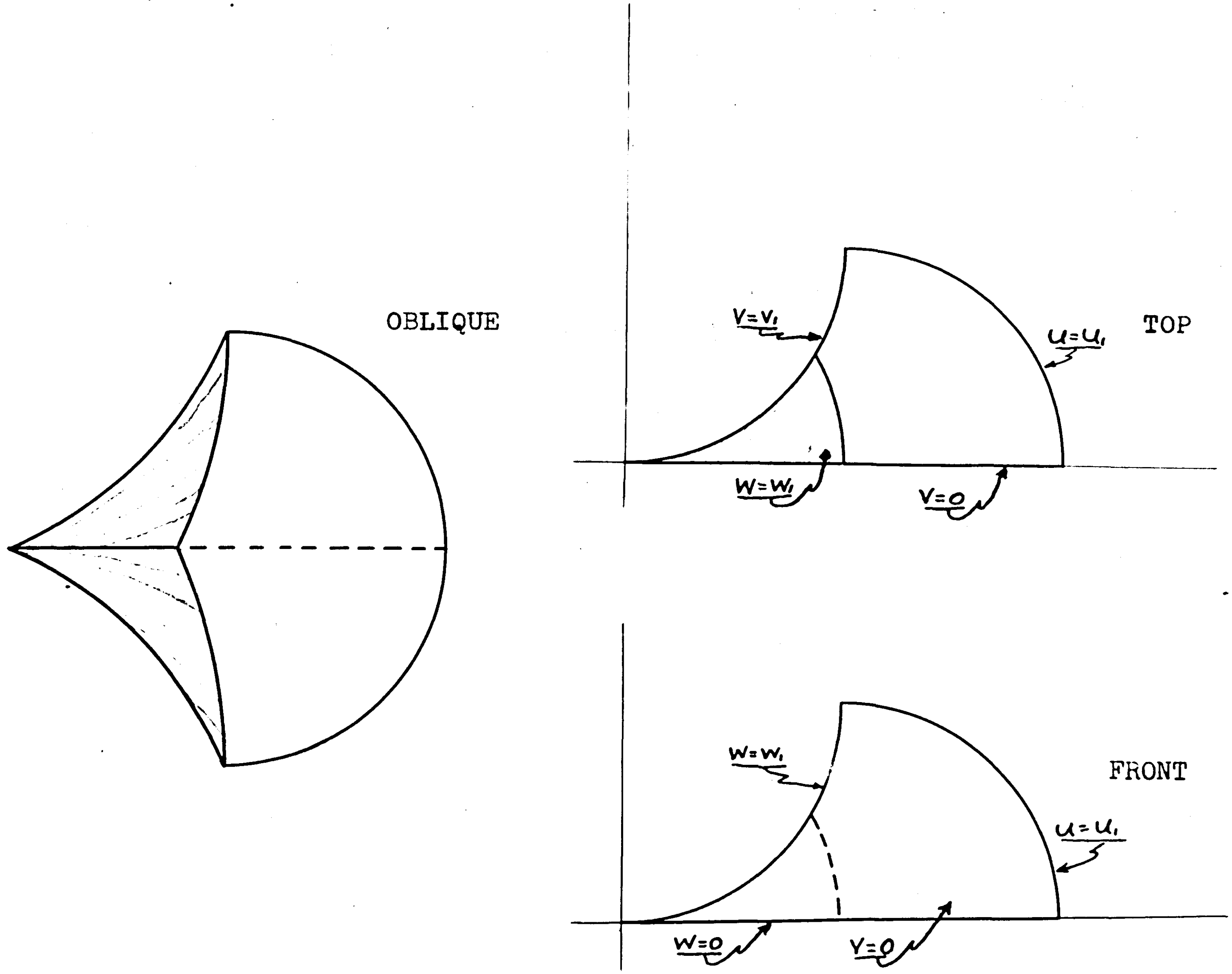
A. Solution by Separation of Variables

In applying the method of separation of variables it is necessary first to solve the separated differential equations, the separation process having been performed once and for all in part II. In general, it will be necessary to solve these equations in power series, and it is to be expected that new functions will arise thereby. Because of the orthogonality of the coordinate systems, it should be possible to expand any well-behaved, arbitrary function as a Fourier series of these new functions, so as to satisfy both Laplace's equation and the physical boundaries of the problem. In certain cases; for example, in inverse rectangular and in inverse circular-cylinder coordinates, the solutions of the separated equations are recognized as functions that have already been tabulated, and it may be that others of the nine inverted systems will also yield familiar solutions. In these cases, of course, the labor of solving a boundary-value problem is greatly reduced.

To illustrate the application of this method, two problems will be presented, both in the inverse rectangular coordinate system. The solutions will only be indicated formally, in order to avoid the labor of evaluating the final integrals.

(1) The first problem involves the potential field within a space bounded by spherical surfaces, as shown in Figure 17. As was mentioned in part II-B, the parametric surfaces of inverse rectangular coordinates consist of three families of spheres, all tangent to the origin, with centers along the x, y, and z axes, respectively. The present problem assumes that the sphere $u = u_1$ is intersected by the planes $v = 0$ and $w = 0$, and by the spheres $v = v_1$ and $w = w_1$. The boundary conditions on the potential ϕ are:

$$
\begin{aligned}
\text{when } u &= 0, & \phi &= 0; \\
u &= u_1, & \phi &= F(v,w); \\
v &= v_1, & \phi &= 0; \\
v &= 0, & \phi &= 0; \\
w &= w_1, & \phi &= 0; \\
w &= 0, & \phi &= 0.
\end{aligned}
$$



BOUNDARY SURFACES FOR SPHERICAL-SECTOR PROBLEM

FIGURE 17.

From article IIIC(1), after a slight manipulation of the separation constants, the particular solution for ϕ is

$$\frac{\phi}{\sqrt{u^2+v^2+w^2}} = \frac{\sinh}{\cosh} \sqrt{k_1^2 + k_2^2}\, u \, \frac{\sin}{\cos} k_1 v \, \frac{\sin}{\cos} k_2 w.$$

In order to satisfy the boundary conditions at all surfaces except $u = u_1$, it is necessary to restrict the particular solution to

$$\frac{\phi}{\sqrt{u^2+v^2+w^2}} = \frac{\sinh \ell u}{\sinh \ell u_1} \sin \frac{m\pi v}{v_1} \sin \frac{n\pi w}{w_1} ,$$

where m and n are integers and $\ell = \pi\sqrt{m^2 + n^2}$.

It is apparent that a double Fourier series must be used to represent the solution, since when $u = u_1$, ϕ is a function of both v and w. The solution

$$\phi = \sqrt{u^2+v^2+w^2} \sum_m \sum_n Amn \frac{\sinh \ell u}{\sinh \ell u_1} \sin \frac{m\pi v}{v_1} \sin \frac{n\pi w}{w_1} .$$

When $u = u_1$, $\quad \phi = F(v_1 w)$,

$$\therefore \frac{F(v,w)}{\sqrt{u^2+v^2+w^2}} = \sum_m \sum_n A_{mn} \sin \frac{m\pi v}{v_1} \sin \frac{n\pi w}{w_1} ,$$

$$A_{mn} = \frac{4}{v_1 w_1} \int_0^{v_1} \int_0^{w_1} \frac{F(v,w)}{\sqrt{u^2+v^2+w^2}} \sin \frac{m\pi v}{v_1} \sin \frac{n\pi w}{w_1} \, dv \, dw .$$

Upon evaluation of the integral for A_{mn}, the potential at any point within the bounded volume is determined.

(2) The second problem is to find the potential external to a charged, metal sphere which is very near to an infinite, grounded, conducting plane. The boundary surfaces can be represented in inverse-rectangular coordinates, as shown in Figure 18. The boundary conditions are:

$$\text{when } u = 0, \quad \phi = 0;$$
$$u = u_1, \quad \phi = F(v,w).$$

Here it is convenient to use an exponential form of the characteristic solution,

$$\frac{\phi}{\sqrt{u^2+v^2+w^2}} = A_\alpha B_\beta \frac{\sinh \ell u}{\sinh \ell u_1} e^{i\alpha p} e^{i\beta q}.$$

The constants A_α and B_β can be put into exponential form, whereupon the characteristic solution becomes

$$\frac{\phi}{\sqrt{u^2+v^2+w^2}} = \frac{\sinh \ell u}{\sinh \ell u_1} e^{i\,\alpha(v-p)} e^{i\,\beta(w-q)},$$

where $\ell = \sqrt{\alpha^2 + \beta^2}$ and $\underline{\alpha}$, $\underline{\beta}$, $\underline{p}$, and $\underline{q}$ are constant for any one characteristic solution.

The sphere is tangent to the plane at the point ($u = 0$, $v = \infty$, $w = \infty$), therefore the boundary value of the potential is specified over an infinite range of values

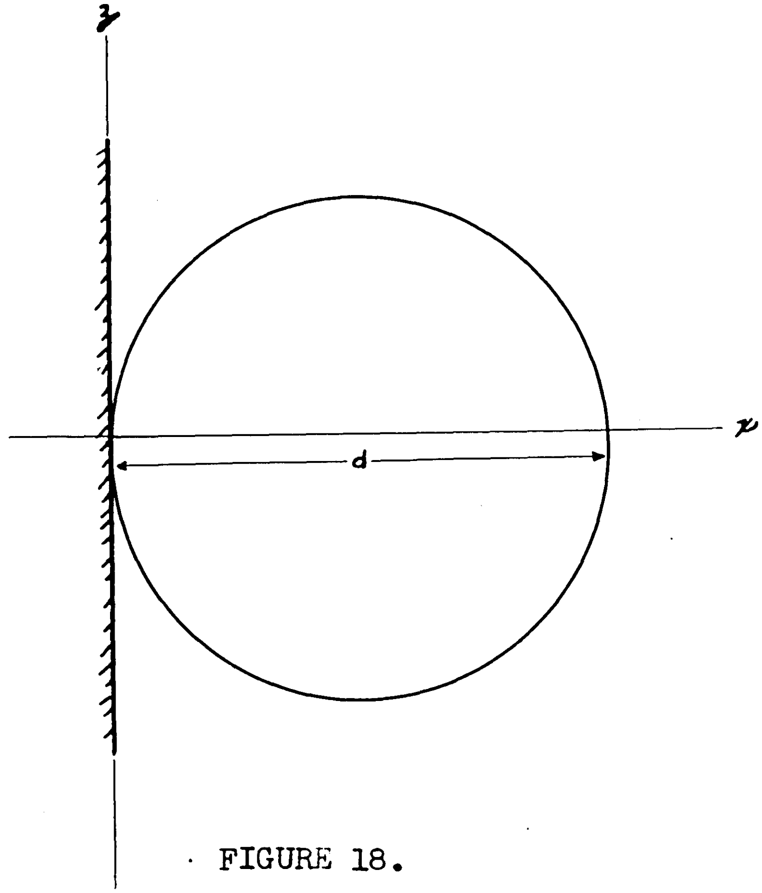


FIGURE 18.

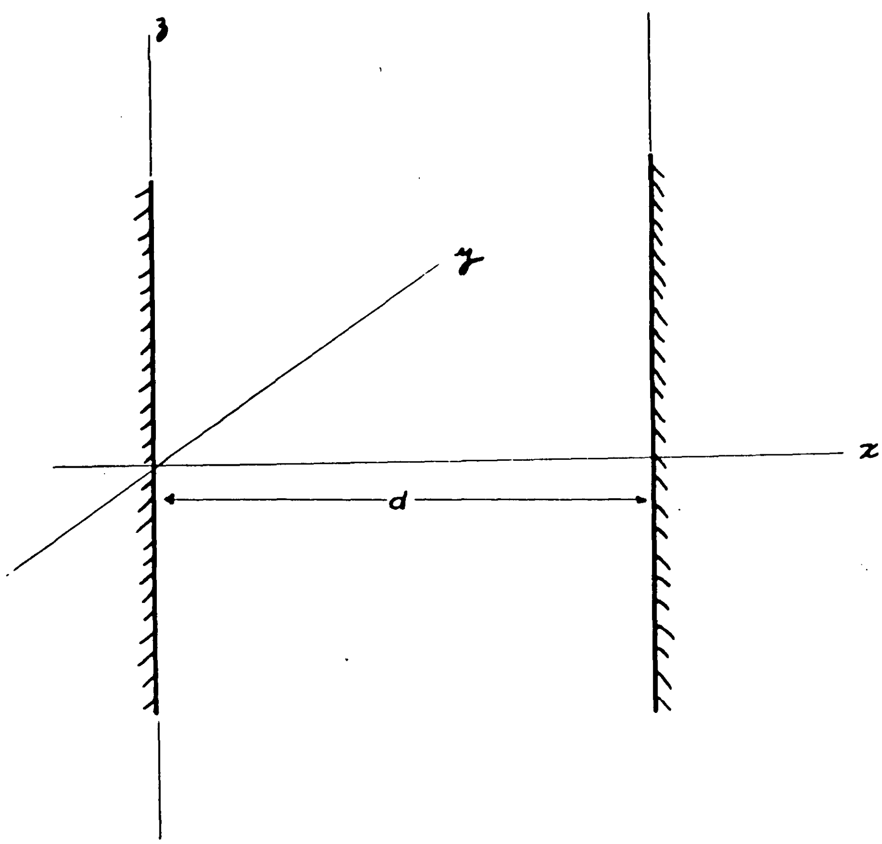


FIGURE 19.

for $\underline{v}$ and $\underline{w}$. The solution for the potential must therefore take the form of a double Fourier integral (See Appendix A). If $\underline{u}$ is zero, and if the characteristic solution is multiplied by

$$\frac{1}{4\pi^2} \frac{F(p,q)u_1}{\sqrt{u_1^2+p^2+q^2}}$$ and integrated four times,

with respect to α, β, p, and q, the result is

$$\frac{1}{4\pi^2} \iiiint_{-\infty}^{\infty} \frac{F(p,q)u_1}{u_1^2+p^2+q^2} e^{i\left[\alpha(v-p)+\beta(w-q)\right]} dp\, dq\, d\alpha\, d\beta.$$

This is the double Fourier integral representation of the function $\frac{F(v,w)}{\sqrt{u_1^2+v^2+w^2}}$ on the sphere $u = u_1$, where $F(v,w)$ is the potential on the surface of the sphere. The potential at any point is therefore

$$\phi = \frac{1}{4\pi^2} \iiiint_{-\infty}^{\infty} \frac{F(p,q)}{\sqrt{u_1^2+p^2+q^2}} \frac{\sinh \ell u}{\sinh \ell u_1} e^{i\left[\alpha(v-p)+\beta(w-q)\right]} dp\, dq\, d\alpha\, d\beta.$$

Suppose that $F(v,w)$ is a constant. It might be expected, from experience with the Eisenhart coordinate systems, that all the surfaces u = constant would be equipotentials, but such is clearly not the case. In fact, it can be shown that in none of the inverted systems can the potential be expressed as a function of only one variable. The general form of Laplace's equation is:

$$\frac{1}{\sqrt{g}} \sum_{i=1}^{3} \frac{\partial}{\partial u_i} \left(\frac{\sqrt{g}}{g_{ii}} \frac{\partial \phi}{\partial u_i} \right) = 0.$$

If ϕ depends only on u_1,

$$\frac{d}{du_1}\left(\frac{\sqrt{g}}{g_{11}} \frac{d\phi}{du_1}\right) = 0,$$

$$\frac{\sqrt{g}}{g_{11}} \frac{d\phi}{du_1} = K_1, \quad \text{where } K = f_1(u_2,u_3),$$

$$\frac{d\phi}{du_1} = f_2(u_1) = K_1 \frac{g_{11}}{\sqrt{g}},$$

$$\therefore \quad \frac{g_{11}}{\sqrt{g}} = f_2(u_1) f_1(u_2,u_3).$$

This result is a criterion for the expression of the potential as a function of one variable. Applying the criterion to the several coordinate systems shows that the eleven Eisenhart systems satisfy the requirement, and that the potential can therefore be expressed as a function of any one of the variables in these systems. In toroidal coordinates the condition is satisfied for only one of the variables, and in the nine inverted systems it is not satisfied for any of the variables.

B. Solution by Inversion of Boundary Conditions.

The alternate method of solving a potential problem, whose boundaries are parametric surfaces of one of the inverted coordinate systems, is to invert the boundary conditions into the corresponding Eisenhart system, in which the characteristic solution is well known. The problem is then completely solved in the familiar coordinates, and the solution inverted back into the original system. The boundary values can be stated either in terms of potentials or in terms of potential gradients. In the former case, the potential is inverted by the relation $\phi' = \frac{r}{a}\phi$,[14] where ϕ and ϕ' are the potentials of corresponding points in the original and inverted coordinates, respectively, and <u>r</u> is the distance from the origin of any point in the given configuration, and <u>a</u> is the radius of the sphere of inversion. In the latter case, the gradient is transformed by the relation

$$\phi' = \left(\frac{r}{a}\right)^3 \phi .$$

[14] The angle between the gradient and the boundary surface is preserved in the inversion.

This method of solving potential problems in three dimensions is analogous to the method of conformal transformations[11] which is applied to similar problems in two dimensions. Though these two-dimensional problems are customarily solved by conformal mapping, it is also possible to use the method of separation of variables, which may be more convenient in some cases. Appendix B compares the two methods of solution for a two-dimensional case.

As an illustration of the present method consider the problem of Figure 18, which has already been treated by separation of variables. It is assumed again that the conducting plane is at zero potential, and that the sphere is charged to an arbitrary potential $F(v,w)$. The given configuration is to be transformed by inversion into one compatible with rectangular coordinates, after which the problem can be solved in the conventional way. The center of inversion is placed at the point of tangency of the given sphere and the given plane, and the radius of the sphere of inversion is made equal to the diameter of the given sphere. The inverted configuration then becomes two infinite, parallel planes, as shown in Figure 19, with their separation equal to d, the diameter of the given sphere. The boundary potentials are transformed by the relation $\phi' = \frac{r\phi}{d}$, as mentioned previously. The potential of the left-hand plane is thus zero, but the potential of any point on the right-hand plane is transformed by a factor depending on the position of the corresponding point on the given sphere. Here it is most convenient to express the transformed potential in terms of rectangular coordinates. Since $rr' = d^2$, $\phi' = \frac{d}{r'}\phi$. But $r' = \sqrt{d^2+y^2+z^2}$, therefore

$$\phi' = \frac{F(y,z)d}{\sqrt{d^2 + y^2 + z^2}} \quad \text{at any point } (d,y,z)$$

on the right-hand plane. It is now necessary to solve this boundary value problem. The potential must be zero when x is zero; therefore, the characteristic solution is

$$A_\alpha B_\beta \frac{\sinh \ell x}{\sinh \ell d} e^{i\alpha y} e^{i\beta z}, \text{ where } \ell^2 = \alpha^2 + \beta^2.$$

Inasmuch as the boundary potentials are defined for an infinite range of values for y and z, it is evident that a double Fourier integral is again needed, in the same manner as in the previous treatment of the problem. Following the procedure of that treatment, one finds the potential in rectangular coordinates to be

$$\phi' = \frac{1}{4\pi^2} \iiiint_{-\infty}^{\infty} \frac{F(p,q)d}{\sqrt{d^2+p^2+q^2}} e^{i[\alpha(y-p)+\beta(z-q)]} dp\, dq\, d\alpha\, d\beta.$$

The potential can now be transformed back into the original coordinate system by the relation

$$\phi = \frac{d}{r}\,\phi'.$$

Upon comparison of the two methods of solving this problem, it is apparent that each presents about the same degree of difficulty and complexity. This appears to be generally true. Suppose that the function $F(u,v)$ were set equal to a constant in the problem. The boundary conditions are simple, but the nature of the inverted coordinate system is such as to require quadruple integration of a complicated function in obtaining a solution. On the other hand, if the problem is worked by transforming boundary conditions, a simple coordinate system is obtained, but the transformation changes the simple boundary conditions to a more complicated form, and the net result is the same.

V

CONCLUSIONS

It appears from the foregoing discussion that the inverted coordinate systems are potentially useful for the solution of boundary value problems. One of their advantageous features is the great variety of surface configurations that can be obtained by changing the location and radius of the sphere of inversion. The application of these coordinate systems consequently permits the treatment of a great many hitherto unvolved problems.

Because of the complicated nature of the inverted systems, it is not surprising that the solution of Laplace's equation is considerably more difficult here than in the more familiar systems. The complication arises partly from the fact that the potential must involve all three space-variables, and partly from the complex nature of the metric coefficients and the necessity of using the R-separation process. The necessary appearance of three independent variables in the expression for the potential makes it necessary in all cases to use double Fourier series or double Fourier integrals for fitting boundary conditions.

Adequate methods are available for finding the most desirable coordinate system to fit a given set of boundaries. It would be of value if more charts showing the shapes of the parametric surfaces should be made available, and if mirror surfaces should be developed which would present exact optical analogies of the process of inversion.

The separation of Laplace's equation has been performed for each of the nine inverted coordinate systems. In most cases, however, the rearrangement of the separation constants into more usable form is yet to be accomplished. Also, in several cases, the separated equations have yet to be solved. This is not a difficult problem, although a tedious one, since a simple trigonometric substitution will in every case permit the equation to be solved in a power series. If new functions are encountered in this process, it will be necessary to tabulate them before numerical examples can be solved.

In the cases in which the actual solution of problems has been investigated, it appears that the solution by separation of variables differs very little in complexity and difficulty from the solution by inversion of boundary conditions. This situation appears to obtain in all the inverted coordinate systems. The choice lies between a simple coordinate system with complex boundary conditions and a complex coordinate system with simple boundary conditions.

In conclusion, it appears that inverted coordinate systems constitute a useful tool in the solution of boundary value problems. The difficulty with which they are applies rules out their use as academic examples, but in cases where accurate solutions are needed, and where mechanical aids to analysis and computation are available, the systems should find valuable application.

APPENDIX A

THE DOUBLE FOURIER INTEGRAL

It is possible to arrive at the form of the Fourier integral in two variables by an extension of the process used in the case of one variable (See Reference 13, page 287).

The exponential form of the double Fourier Series is[2]:

$$f(x,y) = \sum_m \sum_n A_{mn}\, e^{i\, m \frac{x}{a}} e^{i\, m \frac{y}{b}} \quad ;$$

$$f(x,y) e^{-i\, p \frac{x}{a}} e^{-i\, q \frac{y}{b}} = \sum_m \sum_n A_{mn}\, e^{i(m-p)\frac{x}{a}} e^{i(n-q)\frac{y}{b}} \quad ;$$

$$\int_{-\pi a}^{\pi a} f(u,y) e^{-i(p\frac{u}{a} + q\frac{y}{b})} du = \sum_n A_{pn}\, e^{i(n-q)\frac{y}{b}} . 2\pi a \quad ;$$

$$\int_{-\pi b}^{\pi b} \int_{-\pi a}^{\pi a} f(u,v) e^{-i(p\frac{u}{a} + q\frac{v}{b})} du\, dv = A_{pq}(4\pi^2 ab) \quad ;$$

$$A_{pq} = \frac{1}{4\pi^2 ab} \int_{-\pi a}^{\pi a} \int_{-\pi b}^{\pi b} f(u,v) e^{-i(p\frac{u}{a} + q\frac{v}{b})} du\, dv \quad ;$$

$$f(x,y) = \frac{1}{4\pi^2 ab} \sum_{-N}^{N} \sum_{-M}^{M} \left\{ \int_{-\pi a}^{\pi a} \int_{-\pi b}^{\pi b} f(u,v) e^{-i(p\frac{u}{a} + q\frac{v}{b})} du\, dv \right\} \left[e^{i(m\frac{x}{a} + n\frac{y}{b})} \right] .$$

This is the double Fourier Series.

Let $\frac{m}{a} = K_m$, $\frac{n}{b} = K_n$, $\frac{1}{a} = \Delta K_m$, $\frac{1}{b} = \Delta K_n$.

$$f(x,y) = \frac{1}{4\pi^2} \sum_{-M}^{M} \sum_{-N}^{N} \left\{ \int_{-\pi a}^{\pi a} \int_{-\pi b}^{\pi b} f(u,v) e^{-i(K_m u + K_n v)} du\, dv \right\} \cdot \left[e^{i(K_m x + K_n y)} \Delta K_m \Delta K_n \right].$$

As a and b $\rightarrow \infty$, ΔK_m and $\Delta K_n \rightarrow dK_m$ and dK_n.

The above sum then approaches an integral which is

$$f(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u,v) e^{-i(K_m u + K_n v)} du\, dv \right\} \cdot \left[e^{i(K_m x + K_m y)}\, dK_m\, dK_n \right];$$

$$f(x,y) = \frac{1}{4\pi^2} \iiiint_{-\infty}^{\infty} f(u,v) e^{i\left[K_m(x-u) + K_n(y-v)\right]} du\, dv\, dK_m\, dK_n.$$

APPENDIX B

SOLUTION OF LAPLACE'S EQUATION IN A TWO-DIMENSIONAL COORDINATE SYSTEM

Consider the two-dimensional coordinate system represented by Figure 9. Such a system is suitable for the solution of Laplace's equation in a problem in which the boundary conditions are specified on two appropriate cylindrical surfaces, and in which there is no variation of potential in the direction perpendicular to the paper.

The x and y coordinates are defined in terms of U and v as:

$$x = \frac{2uv^2}{u^2+v^2} \quad , \quad y = \frac{2u^2v}{u^2+v^2},$$

from which the metric coefficients of the curvilinear system can be obtained.[8,13]

$$g_{11} = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 = 4v^4,$$

$$g_{22} = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 = 4u^4,$$

$$\sqrt{g} = \sqrt{g_{11}g_{22}} = 4u^2v^2.$$

Laplace's equation is

$$\frac{1}{4u^2v^2}\left[\frac{1}{v^2}\left(u^2\frac{\partial^2\phi}{\partial u^2} + 2u\frac{\partial\phi}{\partial u}\right) + \frac{1}{u^2}\left(v^2\frac{\partial^2\phi}{\partial v^2} + 2v\frac{\partial\phi}{\partial v}\right)\right] = 0.$$

To separate the equation, let $\phi = U(u)V(v)$.

Then $$\frac{1}{4u^2v^2}\left[\frac{1}{v^2}\left(u^2V\frac{d^2U}{du^2} + 2\,uV\frac{dU}{du}\right) + \frac{1}{u^2}\left(v^2U\frac{d^2V}{dv^2} + 2vU\frac{dV}{dv}\right)\right] = 0$$

or

$$\frac{u^4}{U}\frac{d^2U}{du^2} + \frac{2u^3}{U}\frac{dU}{du} = -\frac{v^4}{V}\frac{d^2V}{dv^2} - \frac{2v^3}{V}\frac{dV}{dv} = k.$$

The separated equations are therefore:

$$\text{(i)} \quad u^4 \frac{d^2U}{du^2} + 2u^3 \frac{dU}{du} - kU = 0,$$

$$\text{(ii)} \quad v^4 \frac{d^2V}{dv^2} + 2v^3 \frac{dV}{dv} + kV = 0.$$

These equations can be solved by assuming a solution in the form of a descending power series. Taking equation (i), let

$$U = K_0 + K_1 u^{-1} + K_2 u^{-2} + \dots + K_n u^{-n} + \dots \quad .$$

Substituting this series into (i), and equating like powers of u to zero, a recurrence formula for the coefficients is obtained,

$$K_{m+2} = \frac{K_m}{k\left[(m+2)(m+3) - 2(m+2)\right]},$$

whose form indicates that K_0 and K_1 are arbitrary. The solution can now be written:

$$U = A\left[1 + \frac{1}{2}\left(\frac{k}{v}\right)^2 + \frac{1}{24}\left(\frac{k}{u}\right)^4 + \dots\right]$$

$$+ B\left[\frac{k}{u} + \frac{1}{6}\left(\frac{k}{u}\right)^3 + \frac{1}{120}\left(\frac{k}{u}\right)^5 + \dots\right],$$

which is recognized to be equivalent to

$$U = A \cosh\frac{k}{u} + B \sinh\frac{k}{u}.$$

In a similar manner, the solution for equation (ii) is found to be

$$V = C \sin \frac{k}{v} + D \cos \frac{k}{v},$$

so that the solution for the potential has the form

$$\phi = \sum_n \frac{\sin}{\cos}\left(\frac{k}{v}\right) \frac{\sinh}{\cosh}\left(\frac{k}{u}\right).$$

It is apparent that the substitution of $\underline{x}$ for $\frac{1}{u}$ and $\underline{y}$ for $\frac{1}{v}$ in the separated equations would have put these equations into easily recognized forms, thus saving the labor of solving the equations. This fact suggests the solution of the boundary-value problem by the conformal transformation $w = \frac{1}{z}$, which transforms the coordinates of Figure 9 into rectangular coordinates. The boundary values also must be transformed, of course, and in this case the transformation is equivalent to the inversion of the boundary circles in a sphere centered at the origin. If the potentials on the boundary surfaces are specified, the transformation is accomplished by the method described in article IV B. If the potential gradients are specified, however, the transformation is given by the relation:

$$\phi' = \left(\frac{r}{a}\right) |\nabla\phi|,$$ [14],

where the notation corresponds to that of article IV B.

Problems in this coordinate system can thus be solved by either of the two methods of Section IV, the choice again depending upon which method produces the solution most easily.

BIBLIOGRAPHY

1. R.S. Burington and C.C. Torrance, "Higher Mathematics", 1939.

2. H.S. Carslaw and J.C.Jaeger, "Conduction of Heat in Solids", 1947.

3. R.V. Churchill, "Fourier Series and Boundary Value Problems", 1941

4. L.P. Eisenhart, "Separable Systems of Stäckel", Annals of Mathematics, 35:284, April, 1934.

5. P. Franklin, "Methods of Advanced Calculus", 1944.

6. E.W. Hobson, "Spherical and Ellipsoidal Harmonics", 1931.

7. J.H. Jeans, "Electricity and Magnetism", 3rd. ed., 1915.

8. Parry Moon, Class notes for course 6.586, Mass. Inst. of Tech.

9. P.M. Morse and H. Feshbach, "Notes on Theoretical Physics", Course 8.49, Mass. Inst. of Tech., 1945.

10. R.M. Redheffer, "Separation of Laplace's Equation", Ph.D. Thesis, Mass. Inst. of Tech., 1948.

11. Rothe, Ollendorf, and Pohlhausen, "Theory of Functions as Applied to Engineering Problems", 1933.

12. P. Stäckel, "Habilitationsschrift", 1891.

13. J.A. Stratton, "Electromagnetic Theory", 1941.

14. J.W. Thompson and P.G. Tait, "Treatise on Natural Philosophy", Vol. 2, 1890.