PAIR PRODUCTION OF S-WAVE PI MESONS

by

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ABSTRACT

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A theory for pair production of s-wave π-mesons is constructed along the lines of the Chew-Low-Wick formalism. A bilinear s-wave interaction of the form $\lambda_0/\mu \; \underline{\phi}\cdot\underline{\phi} + \lambda/\mu^2 \; \underline{\tau}\cdot\underline{\phi}\times\underline{\pi}$ as used by Drell, Friedman and Zachariasen is added to the p-wave interaction $\sqrt{4\pi}\; f/\mu \; \vec{\sigma}\cdot\vec{\nabla} \; \underline{\tau}\cdot\underline{\phi}$ as used by Chew and Low. It is shown that if the s-wave interaction is limited to the λ_0 term (scalar pair theory) the cross section for pair production of s-waves vanishes.

Using both the λ_0 and λ terms the meso-production cross section near threshold (total energy of produced mesons ≤ 350 Mev) per unit energy of one of the produced mesons is determined to be of the order of millimicrobarns/Mev; for photoproduction the corresponding number is ~ 100-1000 times smaller.

Cutkosky and Zachariasen have calculated the cross section for photoproduction of an s-wave and a p-wave meson and obtained a number ~ 1000 times larger than the corresponding number here calculated. Their results are in agreement with the fragmentary experimental data available.

It is concluded that if there is no meson - meson interaction s-wave pair production may be neglected except possibly at the very threshold.

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TABLE OF CONTENTS

I. INTRODUCTION

In 1935 Yukawa[1] suggested that nuclear forces are due to the exchange between nucleons of quanta of the nuclear force field. These quanta have since been identified with π-mesons. The discovery of π-mesons in 1947 by Lattes, Occhialini and Powell[2] admirably fulfilled the qualitative predictions of Yukawa's theory. Thus, the π-meson has the correct mass to give rise to the observed short range of nuclear forces. Also, the π-meson interacts strongly with nucleons. Quantitatively, however, the Yukawa theory and experiment seemed to disagree in every aspect.

This failure of Yukawa's theory can be ascribed to the methods of calculation. The most powerful tool in field theoretical calculations is perturbation theory in one form or another. If the coupling between the sources of the field and its quanta is weak, perturbation theory may be expected to give answers in reasonable agreement with experiment. This is the case in quantum electro-dynamics: electrons, the sources of the field and photons, the field quanta are coupled weakly, the strength of the interaction being characterized by the fine structure constant. The fine structure constant is equal to 1/137 and is therefore small in comparison with 1 . The situation is quite different in meson theory since the coupling between nucleons and mesons is strong.

Recently Chew and Low [3] proposed a method of calculation, which, unlike perturbation theory, can be applied to strongly coupled systems. The Yukawa theory, as used by Chew and Low, gave numerical answers in excellent agreement with experiment. Chew and Low have concentrated on treating p-wave mesons since it is a consequence of the pseudoscalar nature of the π-meson that it will interact singly with the nucleon only when it is in a p-state.

Thus the Chew and Low theory was incomplete since it disregarded s-wave mesons, whose interaction with nucleons, although weaker than that of p-wave mesons, is nevertheless present. Drell, Friedman and Zachariasen [4] have recently extended the work of Chew and Low to include interactions with s-wave mesons.

By considering scattering of p-wave and s-wave mesons respectively, Chew and Low and Drell, Friedman and Zachariasen fix the values of the parameters in the theory, i.e. the values of the coupling constants and the cut off energy. The theory can then be applied to other processes. The process of photoproduction of mesons is treated by Chew and Low, their work being extended to include interactions of s-wave mesons by Drell, Friedman, and Zachariasen . Another process of interest is that of pair production of mesons - either by a meson (inelastic scattering) or by a photon (photo pair production). Photo pair production has been treated by Cutkosky and Zachariasen [5], not including interactions of s-wave mesons.

The purpose of this work is to consider pair production of mesons (by either mesons or photons), including the interactions between s-wave mesons and nucleons. The reasons for considering such a process are mainly the two following : In the first place it should serve as an additional test of the Drell, Friedman and Zachariasen theory. In the second place, if there exists a meson-meson interaction, as has been recently conjectured [6], it should certainly play a role in a process where two s-wave mesons are simultaneously present. Should the experimental data disagree with our results a possible explanation could be a meson-meson interaction since no mechanism for such an interaction has been incorporated into our theory.

II. THE HAMILTONIAN

Our starting point is the Chew-Low theory as extended by Drell, Friedman and Zachariasen to include interactions between s-wave mesons and nucleons. The Hamiltonian for the system of mesons and nucleons (not including electromagnetic interactions) can be written as a sum of two parts

$$H = H_0 + H' \qquad (1)$$

where H_0 is the sum of the Hamiltonians of the nucleon alone and the meson field alone and H' is the interaction Hamiltonian. In the static theory the energy of the physical nucleon is a constant which remains unchanged in any interaction and we propose to eliminate it from the Hamiltonian by redefining the zero of the energy scale. Thus taking the energy of the system of one physical nucleon and no free mesons to be zero, we have for H_0:

$$H_0 = \frac{1}{2} \int \left\{ \underset{\rightharpoonup}{\pi}(\vec{r}) \cdot \underset{\rightharpoonup}{\pi}(\vec{r}) + \vec{\nabla}\underset{\rightharpoonup}{\phi}(\vec{r}) \cdot \vec{\nabla}\underset{\rightharpoonup}{\phi}(\vec{r}) + \mu^2 \underset{\rightharpoonup}{\phi}(\vec{r}) \cdot \underset{\rightharpoonup}{\phi}(\vec{r}) \right\} d\vec{r} \qquad (2)$$

We are using throughout the system of units in which $\hbar = c = 1$. An arrow under a symbol indicates a vector in the isospin space. The components π_α, ϕ_α, with $\alpha = 1,2,3$, of the vectors $\underset{\rightharpoonup}{\pi}$, $\underset{\rightharpoonup}{\phi}$ are the canonically conjugate momenta and amplitudes of the charge symmetric meson field, and μ is the meson's rest mass.

In the Chew-Low theory H' is assumed to be

$$H' = \sqrt{4\pi}\, f^\circ/_\mu \int u(r)(\vec{\sigma}\cdot\vec{\nabla})(\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}(\vec{r}))\, d\vec{r} \qquad (3)$$

where f° is the nonrenormalized (nonrationalized) meson-nucleon coupling constant, $\vec{\sigma}$ and $\underset{\sim}{\tau}$ are the usual Pauli spin matrices operating respectively in space and isospin space, and $u(r)$ is a density function which is supposed to simulate the space distribution of the nucleon density. It is taken to be spherically symmetric : $u(\vec{r}) = u(r)$. As will be shown later, this type of an interaction Hamiltonian permits only p-wave mesons to interact with the nucleons.

Drell, Friedman and Zachariasen have taken instead of eq. (3) the following expression

$$H' = \lambda_\circ^\circ/_\mu \left(\int u(r)\,\underset{\sim}{\phi}(\vec{r})\, d\vec{r}\right)\cdot\left(\int u(r')\,\underset{\sim}{\phi}(\vec{r}')\, d\vec{r}'\right)$$

$$-\lambda^\circ/_{\mu^2}\, \underset{\sim}{\tau}\cdot\left(\int u(r)\,\underset{\sim}{\pi}(\vec{r})\, d\vec{r}\right) x \left(\int u(r')\,\underset{\sim}{\phi}(\vec{r}')\, d\vec{r}'\right) \qquad (4)$$

where the second term is to be understood as a triple vector product in isospin space. Here $\lambda_\circ^\circ$, λ° are two new (nonrenormalized) coupling constants. As will be shown later this type of an interaction Hamiltonian permits only s-wave mesons to interact with the nucleons.

Expressions (3) and (4) may be justified as follows by an appeal to field theory: Since the meson field is a pseudo-scalar field the simplest form that H' would assume in a relativistic field theoretical treatment is

$$H' = ic \int \psi(\vec{r})\, \gamma_5 \underline{\tau}\cdot\underline{\phi}(\vec{r})\, \psi(\vec{r})\, d\vec{r} \tag{5}$$

where c is some coupling constant, ψ and $\bar{\psi} = \psi^* \gamma_4$ are the quantized nucleon field amplitudes, $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ where γ_ν are the usual Dirac matrices. As shown by Drell and Henley (7) H' as given by eq.(5) may be brought by a canonical transformation to a form which lends itself better to a nonrelativistic interpretation.

As a result of this canonical transformation eq.(5) may be written as a sum of a number of terms among which are

$$c' \int \psi^*(\vec{r}) (\vec{\sigma}\cdot\vec{\nabla}) (\underline{\tau}\cdot \underline{\phi}(\vec{r}))\, \psi(\vec{r})\, d\vec{r} \tag{6}$$

$$c'' \int \psi^*(\vec{r})\, \underline{\phi}(\vec{r})\cdot\underline{\phi}(\vec{r})\, \psi(\vec{r})\, d\vec{r} \tag{7}$$

$$c''' \int \psi^*(\vec{r})\, \underline{\pi}(\vec{r})\cdot\underline{\tau} \times \underline{\phi}(\vec{r})\, \psi(\vec{r})\, d\vec{r} \tag{8}$$

If now the nucleon density $\psi^*(\vec{r})\,\psi(\vec{r})$ is replaced by the density function $u(r)$ we see that (6) transforms into (3) and (7) and (8) transform into (4) provided we make the additional assumption of separability. We introduce the separability requirement because if we should take expressions (7) and (8) as they stand, with $\psi^*(\vec{r})\,\psi(\vec{r})$ replaced by u(r), then these terms would in addition to giving us an interaction for s-wave mesons also lead to an interaction for p-wave mesons and all higher partial waves. We wish to extract from expressions (7) and (8) only the s-wave part. It should be pointed out that if $u(r) \to \delta(\vec{r})$ (which is the space dependence of $\psi^*(\vec{r})\,\psi(\vec{r})$ for point particles) then the s-wave parts of expressions (7) and (8) are the same as the expressions obtained under the separability assumption.

Of course, field theory makes definite predictions about the relations between the coupling constants c and c', c'', c''' . Therefore, presumably similar relations should exist between f°, $\lambda^\circ_\circ$ and λ° . However in our final expressions only the renormalized quantities f, $\lambda_\circ$ and λ appear and since they are renormalized according to different prescriptions no comparison is possible. Rather, in the work of Chew and Low f, and in the work of Drell, Friedman and Zachariasen $\lambda_\circ$ and λ , are treated as adjustable parameters. We shall use the values of these constants as determined by Chew and Low and Drell, Friedman and Zachariasen and so there will be no further adjustable parameters in our theory.

In the first part of this work we are concerned with computing the transition amplitude for inelastic meson scattering (meso-production), i.e. we take as the initial state of the system a nucleon and one meson, and as the final state a nucleon and two mesons. We wish to calculate eventually the cross section for this process near threshold, where phase space considerations will favor production of a pair of s-wave mesons. Then the total angular momentum and parity of the final state are $\frac{1}{2}^+$ and it follows that the meson in the initial state must be a p-wave meson. Thus in our work we will need both equations (3) and (4) for H'. Also it is clear that it will be convenient to expand the meson field amplitudes in spherical waves rather than in the more conventional plane waves. Furthermore it is convenient to replace ϕ_α, $\alpha = 1,2,3$ by ϕ_q, $q = -1, 0, +1$ where

$$\phi_{+1} = -\frac{\phi_1 + i\phi_2}{\sqrt{2}}, \quad \phi_0 = \phi_3, \quad \phi_{-1} = \frac{\phi_1 - i\phi_2}{\sqrt{2}} \tag{9}$$

the peculiar phases being chosen for reasons that will become apparent later. The details of the expansion in spherical waves and of the replacement (9) are worked out in Appendix A. The result is that we can now write the Hamiltonian in terms of creation and annihilation operators for mesons of definite energy, charge and angular momentum as follows:

$$H = H_\circ + H'$$

$$H_\circ = \sum_p \omega_p \sum_q (-)^q \Big\{ a_q^\dagger(p) a_{-q}(p) + \sum_{m=-1}^{+1} (-)^m a_q^\dagger(pm) a_{-q}(p-m)$$
$$+ \sum_{t=2}^{\infty} \sum_{m=-t}^{+t} (-)^m a_q^\dagger(ptm) a_{-q}(pt-m) \Big\} \tag{10}$$

$$H' = f^{\circ}/\mu \sum_p N/\sqrt{3}\; pv(p)/\sqrt{2\omega_p} \sum_{q,m=-1}^{+1} (-)^q (-)^m \tau_q \sigma_m \left[a_{-q}(p-m) + a^{\dagger}_{-q}(p-m)\right]$$

$$+ \lambda_o^{\circ}/\mu \sum_{pp'} N^2/4\pi\; v(p)v(p')/\sqrt{4\omega_p\omega_{p'}} \sum_q (-)^q \qquad (11)$$

$$\left[a_q(p)a_{-q}(p') + a_q(p)a^{\dagger}_{-q}(p') + a^{\dagger}_q(p)a_{-q}(p') + a^{\dagger}_q(p)a^{\dagger}_{-q}(p')\right]$$

$$+ \lambda^{\circ}/\mu^2 \sum_{pp'} N^2/4\pi\; v(p)v(p')/\sqrt{4\omega_p\omega_{p'}} \left\{ (\omega_p - \omega_{p'}) \sum_q \tau_q \left(a_r(p)a_s(p')\right.\right.$$

$$\left.\left. - a^{\dagger}_s(p')a^{\dagger}_r(p)\right) + (\omega_p + \omega_{p'}) \sum_q \tau_q \left(a_r(p)a^{\dagger}_s(p') - a_s(p')a^{\dagger}_r(p)\right)\right\}$$

(q, r, s = cyclic permutations of -1, 0, +1)

The commutation relations for the a's are

$$\left[a_q(ptm),\; a^{\dagger}_{q'}(p't'm')\right] = (-)^{q+m}\, \delta_{q,-q'}\, \delta_{m,-m'}\, \delta_{t,t'}\, \delta_{p,p'} \qquad (12)$$

all other commutators vanishing.

The meaning of the various symbols is as follows :

$a^{\dagger}_q(ptm)$ = creation operator for a meson of charge = qe, energy = $\omega_p = \sqrt{p^2+\mu^2}$; square of angular momentum = t(t+1), and z-component of angular momentum = m .

$a_q(ptm)$ = annihilation operator for a meson of charge = -qe, energy = $\omega_p = \sqrt{p^2+\mu^2}$; square of angular momentum = t(t+1) , and z-component of angular momentum = - m

(Note the minus signs in these definitions.)

$a_q(p)$ is a shorthand notation for $a_q(p00)$
$a_q(pm)$ is a shorthand notation for $a_q(p1m)$

$$\sigma_{\pm1} = \mp \frac{\sigma_1 \pm i\sigma_2}{\sqrt{2}} , \quad \sigma_0 = \sigma_3 ; \qquad \tau_{\pm1} = \mp \frac{\tau_1 \pm i\tau_2}{\sqrt{2}} , \quad \tau_0 = \tau_3$$

where $\sigma_{1,2,3}$ and $\tau_{1,2,3}$ are the standard Pauli matrices in spin and isospin space respectively.

N = normalization factor; it is related to the density of states in such a manner that the transition from summing over a discrete set of magnitudes of the momentum to integrating over a continuous set is accomplished by

$$N^2 \sum_p \rightarrow 2/\pi \int p^2 dp$$

Finally $$v(p) = \int u(r)\, e^{\pm i\vec{p}\cdot\vec{r}}\, d\vec{r}$$

Note that $a_q(ptm)$ and $a^\dagger_q(ptm)$ are <u>not</u> each others hermitian conjugates but rather $a_q(ptm)$ and $(-)^{q+m}\, a^\dagger_{-q}(pt{-m})$ are each others hermitian conjugates. The reason for this peculiar notation is that now $a_q(ptm)$ (as well as $a^\dagger_q(ptm)$) behaves as the m-component of an irreducible tensor of rank t under rotations in space; and as the q-component of an irreducible tensor of rank 1 under rotations in isospace. We note that the relation between $a_q(ptm)$ and its hermitian conjugate is simply an extension of the relation : " $Y_{l,m}(\Omega)$ and $(-)^m\, Y_{l,-m}(\Omega)$ are hermitian conjugates" , to nonhermitian quantities. Here $Y_{l,m}(\Omega)$ is a spherical harmonic - an example of an irreducible tensor.

It is now seen that in the expression for H', eq. (11), the term proportional to f° contains creation and annihilation operators for p-wave mesons only and the terms proportional to $\lambda^{\circ}_{\circ}$ and λ° contain operators for s-wave mesons only. Thus this form of H' leads to an interaction between p-wave mesons (f° term) and nucleons and s-wave mesons ($\lambda^{\circ}_{\circ}$ and λ° terms) and nucleons as stated without proof previously.

Having H' in a form convenient for calculations we turn our attention to the expression for the transition amplitude for meso-production of a pair of s-wave mesons. The starting point is the scattering matrix - we wish to calculate the following element of the scattering matrix:

$$\langle BRS^{-} \mid AL^{+} \rangle \qquad (13)$$

The superscripts + and - are used to denote that the corresponding states are scattering eigenstates defined by the boundary condition at infinity of only outgoing or incoming waves respectively. We use the symbols A, B for nucleons, R, S, L for mesons. Each symbol is to be understood as a shorthand notation for the following aggregate of quantum numbers:

$$
\text{A:} \quad \begin{cases} A = \text{the isospin} \\ \alpha = \text{z-component of the isospin} \\ A' = \text{angular momentum (spin)} \\ \alpha' = \text{z-component of angular momentum} \end{cases} \tag{14}
$$

$$
\text{R:} \quad \begin{cases} R = \text{the isospin} \\ \rho = \text{z-component of the isospin} \\ R' = \text{angular momentum (orbital)} \\ \rho' = \text{z-component of angular momentum} \\ r = \text{magnitude of linear momentum} \end{cases} \tag{15}
$$

We require expression (13) for the special case of mesons R and S being s-waves and meson L being a p-wave. The states $\langle BRS^{-}|$ and $|AL^{+}\rangle$ are eigenstates of the total Hamiltonian and therefore

$$
|AL^{+}\rangle \neq a_{\lambda}^{\dagger}(\ell\lambda')|A\rangle
$$

We determine $|AL^{+}\rangle$ as follows: consider operating on $a_{\lambda}^{\dagger}(\ell\lambda')|A\rangle$ with H:

$$
\begin{aligned}
H\, a_{\lambda}^{\dagger}(\ell\lambda')|A\rangle &= [H, a_{\lambda}^{\dagger}(\ell\lambda')]|A\rangle \\
&= \{ [H_0, a_{\lambda}^{\dagger}(\ell\lambda')] + [H', a_{\lambda}^{\dagger}(\ell\lambda')] \} |A\rangle \\
&= \{ \omega_{\ell}\, a_{\lambda}^{\dagger}(\ell\lambda') + [H', a_{\lambda}^{\dagger}(\ell\lambda')] \} |A\rangle
\end{aligned}
$$

or

$$(H - \omega_\ell) a^\dagger_\lambda(\ell\lambda') \,|\, A\rangle = [H', a^\dagger_\lambda(\ell\lambda')] \,|\, A\rangle \tag{16}$$

which may be formally inverted to read

$$a^\dagger_\lambda(\ell\lambda') \,|\, A\rangle = |\,\chi^\pm\rangle + \frac{1}{H - \omega_\ell \mp i\epsilon} [H', a^\dagger_\lambda(\ell\lambda')] \,|\, A\rangle \tag{17}$$

The $\pm i\epsilon$ is supplied to define the manner in which the singular operator $(H - \omega_\ell)^{-1}$ is to be treated. The limit $\epsilon \to 0$ is understood. Equation (17) is equivalent to eq. (16) if $(H - \omega_\ell) |\,\chi^\pm\rangle = 0$ or

$$H \,|\,\chi^\pm\rangle = \omega_\ell \,|\,\chi^\pm\rangle \tag{18}$$

because then when we operate on eq.(17) from the left with $(H - \omega_\ell)$ we reproduce eq.(16). But from eq.(18) it is clear that $|\,\chi^\pm\rangle$ is an eigenstate of H to the eigenvalue ω_ℓ, hence it is $|\,AL^\pm\rangle$.

Thus

$$|\,AL^+\rangle = a^\dagger_\lambda(\ell\lambda') \,|\, A\rangle - \frac{1}{H - \omega_\ell - i\epsilon} [H', a^\dagger_\lambda(\ell\lambda')] \,|\, A\rangle \tag{19}$$

where the manner in which the pole of $(H - \omega_\ell)^{-1}$ is treated is precisely the required one to lead to an eigenstate with the + superscript[8].

In exactly the same fashion one can show that

$$|\,BRS^-\rangle = \left\{ a^\dagger_\rho(r)\, a^\dagger_\sigma(s) - \frac{1}{H - \omega_r - \omega_s + i\epsilon} [H', a^\dagger_\rho(r)\, a^\dagger_\sigma(s)] \right\} |\,B\rangle \tag{20}$$

An equivalent way of expressing $|BRS^-\rangle$ which will be useful later is

$$|BRS^-\rangle = \left\{ a_\rho^\dagger(r) - \frac{1}{H-\omega_r-\omega_s+i\epsilon}\left[H', a_\rho^\dagger(r)\right]\right\}|BS^-\rangle \qquad (20')$$

In deriving above results our starting point was an expression of the form: creation operator acting on a state vector. We derive here for future reference the result of considering an annihilation operator acting on a state vector. Let $|X^\pm\rangle$ describe a scattering eigenstate consisting of a nucleon and any number of mesons. Then

$$H\alpha(K)|X^\pm\rangle = \left\{[H,\alpha(K)] + \alpha(K)H\right\}|X^\pm\rangle$$

$$= (E_X - \omega_k)\alpha(K)|X^\pm\rangle + [H', \alpha(K)]\,|X^\pm\rangle$$

where $\alpha(K)$ is the annihilation operator for a meson whose properties are specified by K and where E_X is the energy of the state $|X^\pm\rangle$. Then

$$(H - E_X + \omega_k)\alpha(K)\,|X^\pm\rangle = [H', \alpha(K)]\,|X^\pm\rangle \qquad (21)$$

which is inverted to read

$$\alpha(K)\,|X^\pm\rangle = |Y^\pm\rangle + \frac{1}{H-E_X+\omega_k\mp i\epsilon}\,[H', \alpha(K)]\,|X^\pm\rangle \qquad (22)$$

Eq.(22) is equivalent to eq.(21) provided $|Y^\pm\rangle$ satisfies the homogeneous equation

$$(H - E_X + \omega_k)\,|Y^\pm\rangle = 0$$

or

$$H\left|Y^{\pm}\right\rangle = (E_x - \omega_k)\left|Y^{\pm}\right\rangle \tag{23}$$

From eq.(23) and the boundary conditions on eq.(22) it follows that $\left|Y^{\pm}\right\rangle$ is a state obtained from $\left|X^{\pm}\right\rangle$ by removing from the latter the meson of type K . If the state $\left|X^{\pm}\right\rangle$ didn't contain a meson of type K then $\left|Y^{\pm}\right\rangle = 0$. Thus in particular if $\left|X^{\pm}\right\rangle$ is taken as the physical nucleon state we have $E_x = 0$, $\left|Y^{\pm}\right\rangle = 0$ and

$$a(K)\left|A\right\rangle = \frac{1}{H + \omega_k}\left[H', a(K)\right]\left|A\right\rangle \tag{24}$$

We now have most of the relations necessary to derive the equations for the transition amplitude but before doing so we digress for a moment.

III. SCALAR PAIR THEORY

It will have been noted that our expression for H' contains two types of terms leading to interactions of s-wave mesons, the $\lambda^{\circ}_{\circ}$ and λ° terms. It has been shown in the work of Drell, Friedman and Zachariasen that both of these terms are necessary to account for the experimental s-wave phase shifts and therefore we will use both terms. However, we would like to show first that the assumption $\lambda^{\circ} = 0$ leads to zero transition amplitude for the process that we are considering.

Substituting eq.(20) into eq.(13) gives

$$\begin{aligned}\langle BRS^{-} | AL^{+} \rangle &= \langle \{ a^{+}_{\rho}(r)\, a^{+}_{\sigma}(s) - \frac{1}{H - \omega_r - \omega_s + i\epsilon} [H', a^{+}_{\rho}(r) a^{+}_{\sigma}(s)] \} B | AL^{+} \rangle \\ &= \langle B | \{ (-)^{\rho+\sigma} a_{-\rho}(r)\, a_{-\sigma}(s) + (-)^{\rho+\sigma} [H', a_{-\rho}(r) a_{-\sigma}(s)] \frac{1}{H - \omega_r - \omega_s - i\epsilon} \} AL^{+} \rangle\end{aligned} \qquad (25)$$

where we have used the fact that H and H' are hermitian and that the hermitian conjugates of $a^{+}_{\rho}(r)$, $a^{+}_{\sigma}(s)$ and $i\epsilon$ are respectively $(-)^{\rho} a_{-\rho}(r)$, $(-)^{\sigma} a_{-\sigma}(s)$ and $-i\epsilon$. Using the same approach that gave us eq.(22) we prove that

$$a_{-\rho}(r)\, a_{-\sigma}(s) | AL^{+} \rangle = \frac{1}{H + \omega_r + \omega_s - \omega_\ell - i\epsilon} [H', a_{-\rho}(r)\, a_{-\sigma}(s)] | AL^{+} \rangle \qquad (26)$$

Introducing eq.(26) into eq.(25) then gives

$$\langle BRS^{-} | AL^{+} \rangle = (-)^{\rho+\sigma} \langle B | \frac{1}{H + \omega_r + \omega_s - \omega_\ell - i\epsilon} [H', a_{-\rho}(r) a_{-\sigma}(s)] + [H', a_{-\rho}(r) a_{-\sigma}(s)] \frac{1}{H - \omega_\ell - \omega_s - i\epsilon} | AL^{+} \rangle$$

$$= (-)^{\rho+\sigma} \langle B | [H', a_{-\rho}(r)\, a_{-\sigma}(s)]\, AL^+ \rangle \left(\frac{1}{\omega_r + \omega_s - \omega_l - i\epsilon} + \frac{1}{\omega_l - \omega_r - \omega_s - i\epsilon} \right)$$

$$= 2\pi i \;\delta(\omega_l - \omega_r - \omega_s)\; (-)^{\rho+\sigma} \;\langle B | [H', a_{-\rho}(r)\, a_{-\sigma}(s)]\, AL^+ \rangle \tag{27}$$

and $\langle B | [H', a_{-\rho}(r) a_{-\sigma}(s)] AL^+ \rangle = \langle B | [a_{-\rho}(r), [H', a_{-\sigma}(s)]] + [H', a_{-\rho}(r)]\, a_{-\sigma}(s) + [H', a_{-\sigma}(s)]\, a_{-\rho}(r) | AL^+ \rangle$

$$= \langle B | [a_{-\rho}(r), [H', a_{-\sigma}(s)]] + [H', a_{-\rho}(r)] \frac{1}{H + \omega_s - \omega_l - i\epsilon} [H', a_{-\sigma}(s)] + [H', a_{-\sigma}(s)] \frac{1}{H + \omega_r - \omega_l - i\epsilon} [H', a_{-\rho}(r)] | AL^+ \rangle$$

$$= \langle B | [a_{-\rho}(r), [H', a_{-\sigma}(s)]]\, AL^+ \rangle + \sum_{N^+} \langle B | [H', a_{-\rho}(r)] N^+ \rangle \langle N^+ | [H', a_{-\sigma}(s)]\, AL^+ \rangle (E_{N^+} + \omega_s - \omega_l - i\epsilon)^{-1}$$

$$+ \sum_{N^+} \langle B | [H', a_{-\sigma}(s)] N^+ \rangle \langle N^+ | [H', a_{-\rho}(r)]\, AL^+ \rangle (E_{N^+} + \omega_r - \omega_l - i\epsilon)^{-1} \tag{28}$$

where we have inserted a complete set of eigenstates of the Hamiltonian with the $+$ convention denoted by N^+ .

In the special case when H' contains only the terms proportional to f° and $\lambda_\circ^\circ$ we have from eqs.(11) and (12):

$$[H', a_{-\sigma}(s)] = -\lambda_\circ^\circ/\mu \;\; N^2/4\pi \;\; v(s)/\sqrt{2\omega_s} \sum_p 2v(p)/\sqrt{2\omega_p} \left\{ a_{-\sigma}(p) + a^\dagger_{-\sigma}(p) \right\}$$

$$= -U_s \sum_p U_p \left\{ a_{-\sigma}(p) + a^\dagger_{-\sigma}(p) \right\} \tag{29}$$

where

$$U_x = \left\{ \frac{\lambda_\circ^\circ}{\mu} N^2 \frac{v^2(x)}{4\pi\omega_x} \right\}^{\frac{1}{2}} \tag{30}$$

Similarly

$$[H', a^\dagger_{-\sigma}(s)] = U_s \sum_p U_p \left\{ a_{-\sigma}(p) + a^\dagger_{-\sigma}(p) \right\} \tag{31}$$

Now consider $\langle M^+ | [H', a_{-\sigma}(s)] | N^+ \rangle$ where $|M^+\rangle$ and

$|N^+\rangle$ are any eigenstates of the system with the + convention. Using eq.(29) we have

$$\langle M^+|[H', a_{-\sigma}(s)]|N^+\rangle = - U_s \sum_p U_p \langle M^+| a_{-\sigma}(p) + a^\dagger_{-\sigma}(p)|N^+\rangle \quad (32)$$

But from eq.(22) we have

$$a_{-\sigma}(p)|N^+\rangle = |(N-1_\sigma)^+\rangle \delta^N_{\sigma p} + \frac{1}{H+\omega_p - E_N - i\epsilon}[H', a_{-\sigma}(p)]|N^+\rangle \quad (33)$$

where the symbol $|(N-1_\sigma)^+\rangle \delta^N_{\sigma p}$ is supposed to represent an eigenstate obtained from the state $|N^+\rangle$ by removing an s-wave meson of momentum p and charge σe - if the state $|N^+\rangle$ didn't contain such a meson then $|(N-1_\sigma)^+\rangle \delta^N_{\sigma p} = 0$.

On the other hand in analogy to eqs.(19) and (20) we have

$$a^\dagger_{-\sigma}(p)|N^+\rangle = |(N+1_{-\sigma})^+\rangle + \frac{1}{H-\omega_p - E_N - i\epsilon}[H', a^\dagger_{-\sigma}(p)]|N^+\rangle \quad (34)$$

where $|(N + 1_{-\sigma})^+\rangle$ is supposed to represent an eigenstate containing in addition to what was in $|N^+\rangle$ an s-wave meson of momentum p and charge $(-\sigma e)$. In both eqs. (33) and (34) E_N stands for the energy of the state $|N^+\rangle$.

Introducing eqs. (33) and (34) into eq. (32) leads to

$$\langle M^+|[H', a_{-\sigma}(s)]N^+\rangle = - U_s \sum_p U_p \Big\{ \langle M^+|(N-1_\sigma)^+\rangle \delta^N_{\sigma p} + \langle M^+|(N+1_{-\sigma})^+\rangle$$
$$+ \langle M^+|(H+\omega_p - E_N - i\epsilon)^{-1}[H', a_{-\sigma}(p)] + (H-\omega_p - E_N - i\epsilon)^{-1}[H', a^\dagger_{-\sigma}(p)]|N^+\rangle\Big\}$$

$$= - U_s \sum_p U_p \Big\{ \langle M^+ | (N-1_\sigma)^+ \rangle \delta^N_{\sigma p} + \langle M^+ | (N+1_{-\sigma})^+ \rangle$$

$$+ \frac{\langle M^+ | [H', a_{-\sigma}(p)] | N^+ \rangle}{E_M - E_N - i\epsilon + \omega_p} + \frac{\langle M^+ | [H', a^\dagger_{-\sigma}(p)] | N^+ \rangle}{E_M - E_N - i\epsilon - \omega_p} \Big\} \qquad (35)$$

It follows from eq.(35) that $\frac{1}{U_s} \langle M^+ | [H', a_{-\sigma}(s)] N^+ \rangle$ is independent of the momentum s and therefore

$$\frac{1}{U_p} \langle M^+ | [H', a_{-\sigma}(p)] N^+ \rangle = -\frac{1}{U_p} \langle M^+ | [H', a^\dagger_{-\sigma}(p)] N^+ \rangle = \frac{1}{U_s} \langle M^+ | [H', a_{-\sigma}(s)] N^+ \rangle \qquad (36)$$

Hence eq.(35) becomes

$$\frac{1}{U_s} \langle M^+ | [H', a_{-\sigma}(s)] | N^+ \rangle = - \sum_p U_p \left(\langle M^+ | (N-1_\sigma)^+ \rangle \delta^N_{\sigma p} + \langle M^+ | (N+1_{-\sigma})^+ \rangle \right)$$

$$- \frac{1}{U_s} \langle M^+ | [H', a_{-\sigma}(s)] | N^+ \rangle \sum_p U_p^2 \cdot$$

$$\left\{ 1/(E_M - E_N - i\epsilon + \omega_p) - 1/(E_M - E_N - i\epsilon - \omega_p) \right\} \qquad (37)$$

or

$$\langle M^+ | [H', a_{-\sigma}(s)] N^+ \rangle = - \frac{U_s \sum_p U_p \left\{ \langle M^+ | (N-1_\sigma)^+ \rangle \delta^N_{\sigma p} + \langle M^+ | (N+1_{-\sigma})^+ \rangle \right\}}{1 - 2 \sum_p U_p^2 \, \omega_p / \left\{ (E_M - E_N - i\epsilon)^2 - \omega_p^2 \right\}} \qquad (38)$$

Disregarding the pathological case when the value of λ_o^o is such as to make the denominator in eq.(38) equal to zero, it follows that $\langle M^+ | [H', a_{-\sigma}(s)] | N^+ \rangle = 0$ unless the states $|M^+\rangle$ and $|N^+\rangle$ differ by one s-wave meson of charge σe, but are otherwise identical. In particular the right hand side of eq.(38) vanishes if states $|M^+\rangle$ and $|N^+\rangle$ differ by one p-wave meson. We now return to eq.(28) and see that as a consequence of eq.(38) the summation over N^+ vanishes. This is because, of the two factors $\langle B|[H', a_{\rho}(r)]|N^+\rangle$ and $\langle N^+|[H', a_{-\sigma}(s)]|AL^+\rangle$ one or the other must always vanish in accordance with eq. (38) since $|AL^+\rangle$ contains one p-wave meson and $|B\rangle$ does not. Hence eq.(28) becomes

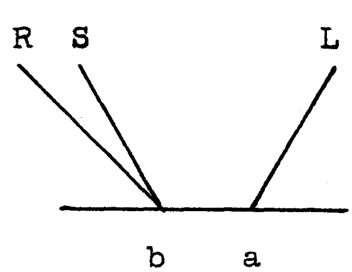
$$\langle B|[H', a_{-\rho}(r)a_{-\sigma}(s)]|AL^+\rangle = \langle B|[a_{-\rho}(r), [H', a_{-\sigma}(s)]]|AL^+\rangle$$

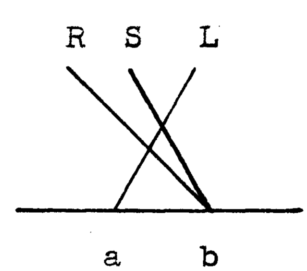
But from eq.(29) and eq.(12) we have for the double commutator:

$$[a_{-\rho}(r), [H', a_{-\sigma}(s)]] = -(-)^{\rho}\,\delta_{\rho,-\sigma}\,U_s\,U_r$$

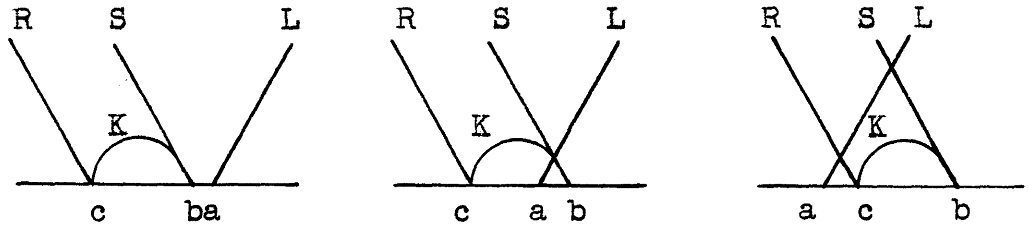
This is a c-number which can be pulled out and what remains is $\langle B|AL^+\rangle = 0$ due to orthogonality of these two eigenstates. This completes the proof that $\langle BRS^-|AL^+\rangle = 0$ if the s-wave interaction is assumed to be due only to the scalar pair term.

We observe that above result could be anticipated if the process in question were pictured in terms of a series of diagrams grouped together according to the number of vertices involved. We have the following two-vertex diagrams

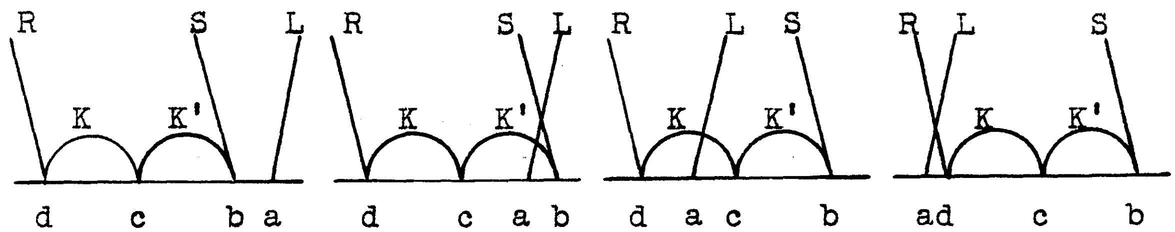




the following three-vertex diagrams



the following four-vertex diagrams



etc. In above we have pictured a family of 2-vertex diagrams, another family can be obtained by interchanging mesons R and S . Similarly we have pictured a family of 3-vertex diagrams, another family can be obtained by interchanging vertices b and c (i.e. mesons R and S) . In the same fashion other families of 4-vertex diagrams can be obtained by interchanging vertices b and c , b and d , and c and d .

The diagrams are to be read from right to left. The horizontal line represents the nucleon, the other lines are meson lines. At the vertex a the p-wave meson L is absorbed. Hence the operator in question, as far as the nucleon is concerned, is $\sigma\tau$. At all the other vertices s-wave mesons (R, S, K, K') are scattered, or created or annihilated in pairs. Hence the operator in question, as far as the nucleon is concerned, is simply unity (scalar pair theory !) . Therefore all the vertices commute and it does not matter whether we write ab or ba . Hence the only difference between contributions from diagrams with the same number of vertices is in the energy denominators.

Thus the contribution from the 2-vertex diagrams pictured is

$$ab\left\{1/(-\omega_\ell) + 1/(\omega_\ell)\right\} = 0 \;;$$

that from the 3-vertex diagrams pictured is

$$\sum_k abc\left\{1/(-\omega_\ell)(\omega_s+\omega_k-\omega_\ell) + 1/(\omega_s+\omega_k)(\omega_s+\omega_k-\omega_\ell) + 1/(\omega_s+\omega_k)(\omega_\ell)\right\}=0;$$

that from the 4-vertex diagrams pictured is

$$\sum_{kk'} abcd\left\{1/(-\omega_\ell)(\omega_s+\omega_{k'}-\omega_\ell)(\omega_s+\omega_k-\omega_\ell) + \right.$$

$$1/(\omega_s+\omega_{k'})(\omega_s+\omega_{k'}-\omega_\ell)(\omega_s+\omega_k-\omega_\ell) + 1/(\omega_s+\omega_{k'})(\omega_s+\omega_k)(\omega_s+\omega_k-\omega_\ell)$$

$$\left. + 1/(\omega_s+\omega_{k'})(\omega_s+\omega_k)(\omega_\ell)\right\} = 0 \;; \text{ etc.}$$

The crucial point in this "proof" by diagrams is the commutability of vertices, valid in the scalar pair theory. We note that if not all diagrams with a given number of vertices are considered the mutual cancellation will not take place and a nonvanishing contribution will be obtained. This is what happens in approximation methods such as the Tamm - Dancoff(9), leading to erroneous results.

IV. MESO-PRODUCTION

We again start with expression (13) for the relevant matrix element of the scattering matrix. In section III we proceeded by replacing $\langle BRS^{-}|$ by eq.(20) -- this approach was dictated by some special features of the scalar pair theory. In the present case we proceed in the orthodox way of replacing $|AL^{+}\rangle$ by eq.(19) to obtain

$$\langle BRS^{-}|AL^{+}\rangle = \langle BRS^{-}|\; a^{+}_{\lambda}(\ell\lambda') - (H-\omega_{\ell}-i\epsilon)^{-1}[H', a^{+}_{\lambda}(\ell\lambda')]\,|A\rangle$$

$$= (-)^{\lambda+\lambda'}\langle a_{-\lambda}(\ell-\lambda')BRS^{-}|A\rangle - \langle BRS^{-}|(H-\omega_{\ell}-i\epsilon)^{-1}[H', a^{+}_{\lambda}(\ell\lambda')]A\rangle$$

$$= \langle \frac{(-)^{\lambda+\lambda'}}{H-\omega_{r}-\omega_{s}+\omega_{\ell}+i\epsilon}[H', a_{-\lambda}(\ell-\lambda')]\,BRS^{-}|A\rangle - \langle BRS^{-}|\frac{1}{H-\omega_{\ell}-i\epsilon}[H', a^{+}_{\lambda}(\ell\lambda')]\,A\rangle$$

$$= -\langle BRS^{-}|[H', a^{+}_{\lambda}(\ell\lambda')]\,(H-\omega_{r}-\omega_{s}+\omega_{\ell}-i\epsilon)^{-1} + (H-\omega_{\ell}-i\epsilon)^{-1}[H', a^{+}_{\lambda}(\ell\lambda')]|A\rangle$$

$$= -2\pi i\,\delta(\omega_{\ell}-\omega_{r}-\omega_{s})\,\langle BRS^{-}|[H', a^{+}_{\lambda}(\ell\lambda')]\,A\rangle$$

$$\equiv -2\pi i\,\delta(\omega_{\ell}-\omega_{r}-\omega_{s})\,\langle BRS|F_{s}(r,\ell)|AL\rangle \qquad (39)$$

where the last line defines the function $F_{s}(r,\ell)$. We recognize that the coefficient of $-2\pi i\,\delta(\omega_{\ell}-\omega_{r}-\omega_{s})$ is just the required matrix element of the transition amplitude in view of the relation between the scattering matrix and the transition matrix

$$S_{ab} = \delta_{ab} - 2\pi i\,\delta(\omega_{a}-\omega_{b})\,T_{ab} \qquad (40)$$

We are dealing with an off diagonal matrix element of S and therefore the δ_{ab} term does not appear.

Since the cross section for any process is directly expressible in terms of the square of the absolute value of T_{ab} we concentrate on evaluating that quantity.

Using the form of $|BRS^{-}\rangle$ given by eq.(20') we have

$$\langle BRS|F_s(r,\ell)|AL\rangle \equiv \langle BRS^{-}|[H', a_\lambda^{+}(\ell\lambda)]A\rangle$$

$$= \langle BS^{-}|\{(-)^{\rho}a_{-\rho}(r) + [H', (-)^{\rho}a_{-\rho}(r)](H-\omega_r-\omega_s-i\epsilon)^{-1}\}[H', a_\lambda^{+}(\ell\lambda')]A\rangle$$

$$= \langle BS^{-}|[H', (-)^{\rho}a_{-\rho}(r)](H-\omega_r-\omega_s-i\epsilon)^{-1}[H', a_\lambda^{+}(\ell\lambda')]$$

$$+ [H', a_\lambda^{+}(\ell\lambda')](H+\omega_r)^{-1}[H', (-)^{\rho}a_{-\rho}(r)]|A\rangle \qquad (41)$$

where we make use of eq.(24) and the fact that $[H', a_\lambda^{+}(\ell\lambda')]$ commutes with $a_{-\rho}(r)$ since meson L is a p-wave and meson R an s-wave.

Equation (41) is the basic equation of this problem: however it is not very useful unless certain approximations are made. Consider a complete set $|X^{-}\rangle$ of eigenstates of the total Hamiltonian with the "-" convention. If there are no bound states then the states $|X^{-}\rangle$ are of the following kind: physical nucleon; physical nucleon and one incident meson plus incoming scattered waves; physical nucleon and two incident mesons plus incoming scattered waves, etc.. Thus we may write eq.(41) by introducing such a complete set in a form similar to eq. (28):

$$\langle BRS | F_s(r,\ell) | AL \rangle =$$

$$\sum_X \Big\{ \langle BS^- | [H'; (-)^\rho a_{-\rho}(r)] X^- \rangle \langle X^- | [H'; a_\lambda^+(\ell\lambda')] A \rangle / (E_X - \omega_r - \omega_s - i\epsilon)$$

$$+ \langle BS^- | [H'; a_\lambda^+(\ell\lambda')] X^- \rangle \langle X^- | [H'; (-)^\rho a_{-\rho}(r)] A \rangle / (E_X + \omega_r) \Big\} \qquad (42)$$

where E_X stands for the energy of the state $|X^-\rangle$. We propose to approximate eq.(42) by making the so-called "one meson approximation". In the work of Chew and Low and Drell, Friedman and Zachariasen this meant omitting all states $|X^-\rangle$ containing more than one incident meson. In our case, as will be seen presently, the procedure is slightly different - nevertheless the spirit of the "one meson approximation" is maintained. It is not clear whether this approximation is valid. It might be argued that whereas the one meson approximation may be valid in the work of Chew and Low and Drell, Friedman and Zachariasen, it cannot be valid here since we are interested in states such as $|BRS^-\rangle$, which is a two meson state. This is not the case - contributions from states such as $|BRS^-\rangle$ are included in our calculation provided only one of the two mesons in $|BRS^-\rangle$ is rescattered. Thus effectively what is neglected corresponds to terms quadratic in the amplitudes involving two meson states. It is just this modification that makes our "one meson approximation" look different from that of Chew and Low and Drell, Friedman and Zachariasen. We also neglect entirely contributions from three or more meson states, hoping as Chew and Low and Drell, Friedman and Zachariasen do, that at low energies this does not falsify our results appreciably.

Let us consider successively the various terms that are obtained by assuming $|X^-\rangle$ to be the physical nucleon state, physical nucleon and one meson state, and finally physical nucleon and two mesons state.

1. $|X^-\rangle = |C\rangle$ = physical nucleon state

The first term in eq.(42) - to be referred to as the direct term from now on - then gives in the numerator

$$\langle BS^-|[H', (-)^{\rho} a_{-\rho}(r)]C\rangle\langle C|[H', a^{+}_{\lambda}(\ell\lambda')]A\rangle \qquad (43)$$

By investigating the matrix element of the scattering matrix corresponding to creation of a pair of s-wave mesons we show that

$$\langle BS^-|[H', (-)^{\rho} a_{-\rho}(r)]C\rangle \equiv -\langle BSR|N_r(s)|C\rangle \qquad (44)$$

corresponds to the matrix element of the transition amplitude for this process:

$$\langle BSR^-|C\rangle = \langle\{a^{+}_{\rho}(r) - (H-\omega_r-\omega_s+i\epsilon)^{-1}[H', a^{+}_{\rho}(r)]\}BS^-|C\rangle$$

$$= \langle BS^-|(H+\omega_r-i\epsilon)^{-1}[H', (-)^{\rho} a_{-\rho}(r)] + [H', (-)^{\rho} a_{-\rho}(r)]\cdot (H-\omega_r-\omega_s-i\epsilon)^{-1}|C\rangle$$

$$= \langle BS^- | [H', (-)^\rho a_{-\rho}(r)] C \rangle \left(\frac{1}{\omega_r + \omega_s - i\epsilon} - \frac{1}{\omega_r + \omega_s + i\epsilon} \right)$$

$$= - 2\pi i \; \delta(\omega_r + \omega_s) \langle BSR | N_r(s) | C \rangle$$

On the other hand

$$\langle C | [H', a_\lambda^+(\ell\lambda')] A \rangle \equiv \langle C | \vartheta(\ell) | AL \rangle \qquad (45)$$

is the matrix element of the transition amplitude for absorption of a p-wave meson:

$$\langle C | AL^{\pm} \rangle = \langle C | \{ a_\lambda^+(\ell\lambda') - (H - \omega_\ell - i\epsilon)^{-1} [H', a_\lambda^+(\ell\lambda')] \} A \rangle$$

$$= - \langle C | [H', a_\lambda^+(\ell\lambda')] (H + \omega_\ell - i\epsilon)^{-1} + (H - \omega_\ell - i\epsilon)^{-1} [H', a_\lambda^+(\ell\lambda')] | A \rangle$$

$$= - \langle C | [H', a_\lambda^+(\ell\lambda')] A \rangle \left(\frac{1}{\omega_\ell - i\epsilon} - \frac{1}{\omega_\ell + i\epsilon} \right)$$

$$= - 2\pi i \; \delta(\omega_\ell) \langle C | \vartheta(\ell) | AL \rangle$$

We now look at the second term in eq.(42) - to be referred to as the crossing term from now on. It gives in the numerator

$$\langle BS^- | [H', a_\lambda^+(\ell\lambda')] C \rangle \langle C | [H', (-)^\rho a_{-\rho}(r)] A \rangle \qquad (46)$$

Both terms in the product (46) vanish on the basis of arguments involving parity conservation. In simplest terms the argument runs as follows: The state $\langle BS^- |$ contains a physical nucleon and one s-wave meson -

hence its parity is odd relative to the state $|C\rangle$ = a physical nucleon; on the other hand the operator $[H', a^{\dagger}_{\lambda}(\ell\lambda')]$ has even parity since meson L is a p-wave. Hence $\langle BS^{-}|[H', a^{\dagger}_{\lambda}(\ell\lambda')]C\rangle = 0$. Similarly $\langle C|[H', (-)^{\varrho} a_{-\varrho}(r)]A\rangle = 0$ since states $\langle C|$ and $|A\rangle$ have the same parity and the operator $[H', (-)^{\varrho} a_{-\varrho}(r)]$ has odd parity (meson R being an s-wave.)

2. $|X^{-}\rangle = |CK^{-}\rangle$ = physical nucleon and one meson. The direct term in eq.(42) gives in the numerator

$$\langle BS^{-}|[H', (-)^{\varrho} a_{-\varrho}(r)]CK^{-}\rangle \langle CK^{-}|[H', a^{\dagger}_{\lambda}(\ell\lambda')]A\rangle \qquad (47)$$

The meson K is either an s-wave or a p-wave or a higher partial wave. If it is a partial wave higher than a p-wave then the state $|CK^{-}\rangle$ is orthogonal to the result of operating with $[H', a^{\dagger}_{\lambda}(\ell\lambda')]$ on $|A\rangle$ as well as to the result of operating with $[H', (-)^{\varrho} a_{-\varrho}(r)]^{\dagger}$ on $|BS^{-}\rangle$ since H' doesn't contain operators for mesons in angular momentum states higher than the p- state. On the other hand, if meson K is an s-wave then both terms in expression (47) vanish when parity conservation is taken into account (see preceeding paragraph) . There remains the case when meson K is a p-wave. In this case $\langle CK^{-}|[H', a^{\dagger}_{\lambda}(\ell\lambda')]A\rangle$ corresponds to the matrix element of the transition amplitude for scattering of a p-wave meson :

$$\langle CK^- | AL^+ \rangle = \langle CK^- | \{ a^+_\lambda(\ell\lambda') - \frac{1}{H-\omega_\ell - i\epsilon} [H', a^+_\lambda(\ell\lambda')] \} A \rangle$$

$$= \langle C | A \rangle \delta_{L,K} -$$

$$\langle CK^- | [H', a^+_\lambda(\ell\lambda')] \frac{1}{H+\omega_\ell - \omega_k - i\epsilon} + \frac{1}{H-\omega_\ell - i\epsilon} [H', a^+_\lambda(\ell\lambda')] | A \rangle$$

$$= \langle C | A \rangle \delta_{L,K} - \langle CK^- | [H', a^+_\lambda(\ell\lambda')] A \rangle \left(\frac{1}{\omega_\ell - \omega_k - i\epsilon} + \frac{1}{\omega_k - \omega_\ell - i\epsilon} \right)$$

$$= \langle C | A \rangle \delta_{L,K} - 2\pi i \delta(\omega_\ell - \omega_k) \langle CK^- | [H', a^+_\lambda(\ell\lambda')] A \rangle$$

where $\delta_{L,K}$ is one if mesons K and L have all quantum numbers the same, zero otherwise. The term $\langle C | A \rangle \delta_{L,K}$ corresponds thus to the δ_{ab} in eq.(40).

At this stage we propose to make one more approximation: It is known experimentally (and the Chew and Low work reproduces this result theoretically) that the scattering of p-wave mesons is extremely weak in all but the 3/2 3/2 state. This is the state in which the p-wave meson and the nucleon are coupled together in such a way as to produce an eigenstate of the total angular momentum J to the eigenvalue 3/2 and total isotopic spin T to the eigenvalue 3/2 . However in our work the final state is one of the total angular momentum eigenvalue = 1/2 (a single nucleon and two s-wave mesons.) . From angular momentum conservation we see that the initial state must also be one in which the eigenvalue of the total angular momentum is 1/2 . Therefore the scattering of a p-wave represented by $\langle CK^- | [H', a_\lambda^+(\ell\lambda')] A\rangle$ can never occur in the resonant 3/2 3/2 state. Consistent with the experimental data we assume that the scattering of p-waves in any but the 3/2 3/2 state vanishes. Hence expression (47) vanishes.

The crossing term gives in the numerator

$$\langle BS^- | [H', a_\lambda^+(\ell\lambda')] CK^-\rangle \langle CK^- | [H', (-)^\rho a_{-\rho}(r)] A\rangle \qquad (48)$$

Again above vanishes due to orthogonality of eigenstates of the Hamiltonian if meson K is a partial wave higher than p-wave; and it vanishes due to parity conservation if meson K is a p-wave. If meson K is an s-wave we have

$$\begin{aligned}\langle BS^- | [H', a_\lambda^+(\ell\lambda')] CK^-\rangle &= \langle B | [H', a_\lambda^+(\ell\lambda)] C\rangle \delta_{K,S} \\ &- \langle BS^- | [H; a_\kappa^+(k)] \frac{1}{H+\omega_k-\omega_s-i\epsilon} [H', a_\lambda^+(\ell\lambda')] \\ &+ [H; a_\lambda^+(\ell\lambda')] \frac{1}{H-\omega_k+i\epsilon} [H', a_\kappa^+(k)] | C\rangle \\ &= \langle BS | E_s(k,\ell) | CKL\rangle + \langle B | \theta(\ell) | CL\rangle \delta_{K,S}\end{aligned} \qquad (49)$$

where use was made of eq.(45) and the definition of the function E is

$$\begin{aligned}\langle BS | E_s(k,\ell) | CKL\rangle \equiv &- \langle BS^- | [H; a_\lambda^+(\ell\lambda')] \frac{1}{H-\omega_k+i\epsilon} [H; a_\kappa^+(k)] \\ &+ [H; a_\kappa^+(k)] \frac{1}{H+\omega_k-\omega_s-i\epsilon} [H; a_\lambda^+(\ell\lambda)] | C\rangle\end{aligned} \qquad (50)$$

The expression $\langle BS^- | [H', a^+_\lambda(\ell\lambda')] CK^- \rangle$ can not be easily related to the scattering matrix in view of the fact that the states $\langle BS^- |$ and $| CK^- \rangle$ are both with the minus convention. If we had instead $| CK^+ \rangle$ then $\langle BS^- | [H', a^+_\lambda(\ell\lambda')] CK^+ \rangle$ would be the matrix element of the transition amplitude corresponding to the element $\langle BS^- | CKL^+ \rangle$ of the scattering matrix. $\langle BS^- | CKL^+ \rangle$ represents the process of a p-wave being absorbed (meson L) while simultaneously an s-wave is scattered (meson K goes into meson S) . This suggests the following interpretation of the two terms in eq.(49) : The first term $\langle B | \vartheta(\ell) | CL \rangle \delta_{K,S}$ represents the absorption of the meson L in the presence of an s-wave meson $K = S$ which undergoes no interaction; the second term $\langle BS | E_s(k,\ell) | CKL \rangle$ corresponding to the absorption of meson L simultaneously with the s-wave meson being scattered from the state K to S . Note that even if $K = S$ both terms contribute, the frist term representing no interaction between the s-wave and the nucleon, the second term containing the effects of the scattering of an s-wave from one state into the same state.

The second term in the product (48) has already been considered in connection with expression (43).

3. $|X^-\rangle = |CKP^-\rangle$ = physical nucleon and two mesons. The direct term in eq.(42) gives in the numerator:

$$\langle BS^- | [H', (-)^\rho a_{-\rho}(r)] CKP^- \rangle \langle CKP^- | [H', a^+_\lambda(\ell\lambda')] A \rangle \quad (51)$$

From orthogonality of eigenstates and parity conservation arguments we conclude that above may be different from zero if either both mesons K and P are s-waves or both are p-waves. Consider the first term $\langle BS^- | [H', (-)^\rho a_{-\rho}(r)] CKP^- \rangle$. Again there are difficulties in relating above directly to the transition amplitude for a process, due to the "-" convention in both states. As in the preceeding paragraph we consider $\langle BS^- | [H', (-)^\rho a_{-\rho}(r)] CKP^+ \rangle$. This, by the now familiar proof, is easily shown to be the matrix element of the transition amplitude corresponding to the element $\langle BSR^- | CKP^+ \rangle$ of the scattering matrix. $\langle BSR^- | CKP^+ \rangle$ represents the scattering of two mesons from the states K and P into the states S and R . It is at this point that we define our "one meson approximation". Namely we keep from $\langle BSR^- | CKP^+ \rangle$ only those contributions which correspond to the scattering of one meson only, the scattering taking place in the presence of the second meson which undergoes no change. Since both mesons S and R are s-waves there will be such a contribution only provided one of the mesons K and P is also an s-wave, but then the other must be too (see comment after expression (51)). What was said above is expressed mathematically as follows:

$$\langle BS^- | [H', (-)^{\rho} a_{-\rho}(r)] \, CKP^- \rangle \tag{52}$$

$$= \langle BS^- | [H', (-)^{\rho} a_{-\rho}(r)] \Big\{ a^{+}_{\kappa}(k) - \frac{1}{H-\omega_k-\omega_p+i\epsilon} [H', a^{+}_{\kappa}(k)] \Big\} CP^- \rangle$$

$$= - \langle BS^- | [a^{+}_{\kappa}(k), [H', (-)^{\rho} a_{-\rho}(r)]] + [H', (-)^{\rho} a_{-\rho}(r)] \frac{1}{H-\omega_k-\omega_p+i\epsilon} [H', a^{+}_{\kappa}(k)]$$

$$+ [H', a^{+}_{\kappa}(k)] \frac{1}{H+\omega_k-\omega_s-i\epsilon} [H', (-)^{\rho} a_{-\rho}(r)] | CP^- \rangle + \langle B | [H', (-)^{\rho} a_{-\rho}(r)] \, CP^- \rangle \delta_{S,K}$$

and the approximation consists in taking from the above only $\langle B | [H', (-)^{\rho} a_{-\rho}(r)] \, CP^- \rangle \delta_{S,K}$ and ignoring the rest. That this coincides with what was said above is proved as follows : The factor $\delta_{S,K}$ indicates - as required - that one of the mesons undergoes no change; the remaining factor is the negative complex conjugate of the element of the transition amplitude corresponding to the scattering of an s-wave because

$$\langle CP^- | BR^+ \rangle = \langle CP^- | \Big\{ a^{+}_{\rho}(r) - \frac{1}{H-\omega_r-i\epsilon} [H', a^{+}_{\rho}(r)] \Big\} B \rangle$$

$$= \langle C | B \rangle \delta_{R,P} - \langle CP^- | [H', a^{+}_{\rho}(r)] \frac{1}{H+\omega_r-\omega_p-i\epsilon} + \frac{1}{H-\omega_r-i\epsilon} [H', a^{+}_{\rho}(r)] | B \rangle$$

$$= \langle C | B \rangle \delta_{R,P} - \langle CP^- | [H', a^{+}_{\rho}(r)] \, B \rangle \left(\frac{1}{\omega_r-\omega_p-i\epsilon} + \frac{1}{\omega_p-\omega_r-i\epsilon} \right)$$

$$= \langle C | B \rangle \delta_{R,P} - 2\pi i \, \delta(\omega_r-\omega_p) \langle CP^- | [H', a^{+}_{\rho}(r)] \, B \rangle$$

Thus

$$\langle CP^- | [H', a_\rho^+(r)] B \rangle \equiv \langle CP | M_r(p) | BR \rangle \qquad (53)$$

is the required matrix element of the transition amplitude and

$$\langle B | [H', (-)^\rho a_{-\rho}(r)] CP^- \rangle = - \langle CP | M_r(p) | BR \rangle^* \qquad (54)$$

The other term in expression (51) is nothing else but the function F defined in eq.(39) - the actual quantity that we wish to find:

$$\langle CKP^- | [H', a_\lambda^+(\ell\lambda')] A \rangle = \langle CKP | F_p(k,\ell) | AL \rangle \qquad (55)$$

(this is a consequence of the fact that unless mesons K and P are taken to be s-waves the other term in eq.(51) contributes a zero) .

Finally we must consider the crossing term which gives in the numerator in the present case

$$\langle BS^- | [H', a_\lambda^+(\ell\lambda')] CKP^- \rangle \langle CKP^- | [H', (-)^\rho a_{-\rho}(r)] A \rangle \quad (56)$$

The orthogonality and parity arguments allow expression (56) to be different from zero if either meson K is an s-wave and meson P a p-wave, or vice versa. Assuming that it is P which is the s-wave, the one meson approximation leads to the replacement of $\langle BS^- | [H', a_\lambda^+(\ell\lambda')] CKP^- \rangle$ by $\langle B | [H', a_\lambda^+(\ell\lambda')] CK^- \rangle \delta_{P,S}$. $\langle B | [H', a_\lambda^+(\ell\lambda')] CK^- \rangle$ corresponds to the absorption of a pair of p-wave mesons. As a consequence of the

identity $[H', a^{\dagger}_{\lambda}(\ell\lambda')] = -[H', a_{\lambda}(\ell\lambda')]$ which follows by examination of eq.(11), the absorption of two p-waves is directly expressible in terms of scattering of p-waves. But as in expression (47) the scattering occurs in an eigenstate of J to the eigenvalue 1/2 , because the state $\langle CKP^{-}|$ is a state of $J = 1/2$. That conclusion follows from investigating $\langle CKP^{-}|[H', (-)^{\rho} a_{-\rho}(r)]A\rangle$ and noting that meson R is an s-wave, therefore the total angular momentum of $\langle CKP^{-}|$ is the same as that of $|A\rangle$.
Thus we conclude that expression (56) vanishes.

Collecting our results we have as an approximate version of eq.(42) :

$$\langle BRS|F_s(r,\ell)|AL\rangle = \sum_C \frac{\langle BSR|N_r(s)|C\rangle\langle C|\theta(\ell)|AL\rangle}{\omega_s + \omega_r}$$

$$- \sum_C \frac{\langle B|\theta(\ell)|CL\rangle\langle CSR|N_r(s)|A\rangle}{\omega_s + \omega_r}$$

$$- \sum_{CK} \frac{\langle CK|M_r(k)|BR\rangle^{*}\langle CKS|F_s(k,\ell)|AL\rangle}{\omega_k - \omega_r - i\epsilon}$$

$$- \sum_{CK} \frac{\langle BS|E_s(k,\ell)|CKL\rangle\langle CKR|N_r(k)|A\rangle}{\omega_k + \omega_r} \quad (57)$$

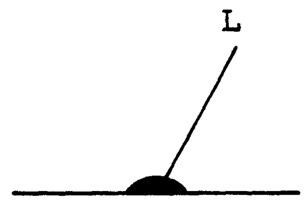
Eq. (57) is an integral equation for the function F. Of the quantities appearing in it, the function ϑ can be determined from the work of Chew and Low, the functions M and N from the work of Drell, Friedman and Zachariasen. The function E, however, is unknown. This is similar to the situation in the work of Drell, Friedman and Zachariasen: they set out to calculate the function M and find that they need the function N. In the work of Chew and Low such a situation does not arise. This is a consequence of the fact that H' is linear in meson operators in the work of Chew and Low, whereas it contains bilinear terms in our work. From the definition of the function E by eq. (50) it is seen, by comparison with eq. (41) which defines the function F, that an integral equation for E may be derived in precisely the same manner as the integral equation for F. Subject to the same approximations we obtain

$$\langle BS|E_s(r,\ell)|ARL\rangle = \sum_C \frac{\langle BS|M_r(s)|CR\rangle\langle C|\vartheta(\ell)|AL\rangle}{\omega_s-\omega_r+i\epsilon} - \sum_C \frac{\langle B|\vartheta(\ell)|CL\rangle\langle CS|M_r(s)|AR\rangle}{\omega_s-\omega_r+i\epsilon} - \sum_{CK} \frac{\langle BS|E_s(k,\ell)|CKL\rangle\langle CK|M_r(k)|AR\rangle}{\omega_k-\omega_r+i\epsilon} - \sum_{CK} \frac{\langle CKR|N_r(k)|B\rangle^*\langle CKS|F_s(k,\ell)|AL\rangle}{\omega_k+\omega_r} \tag{58}$$

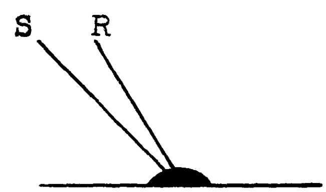
Equations (57) and (58) form a system of two coupled linear integral equations for the functions F and E which must be solved.

In terms of diagrams the various functions ϑ,N,M,F and E can be pictured as follows :

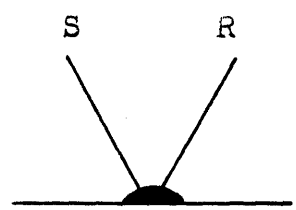
$\vartheta(\ell)$:



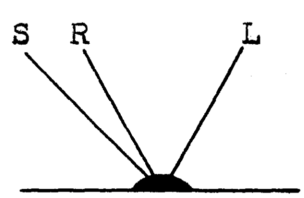
$N_r(s)$:



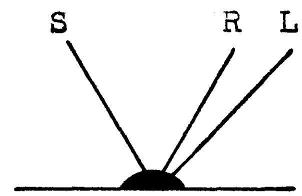
$M_r(s)$:



$F_s(r,\ell)$:

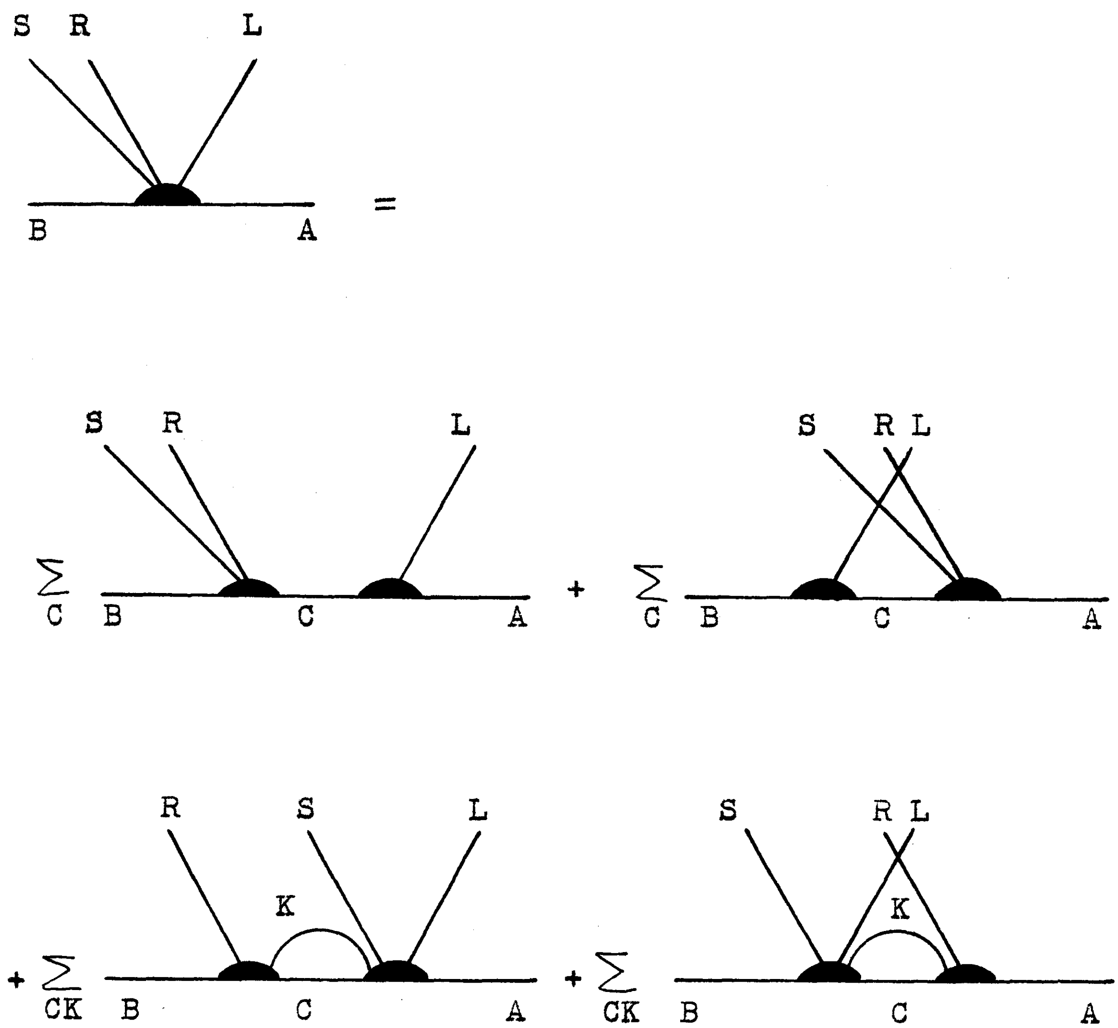


$E_s(r,\ell)$:

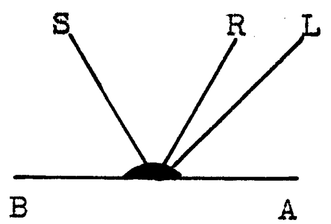


where the blob indicates the complete physical interaction.

We can also write eqs.(57) and (58) in terms of diagrams :



and



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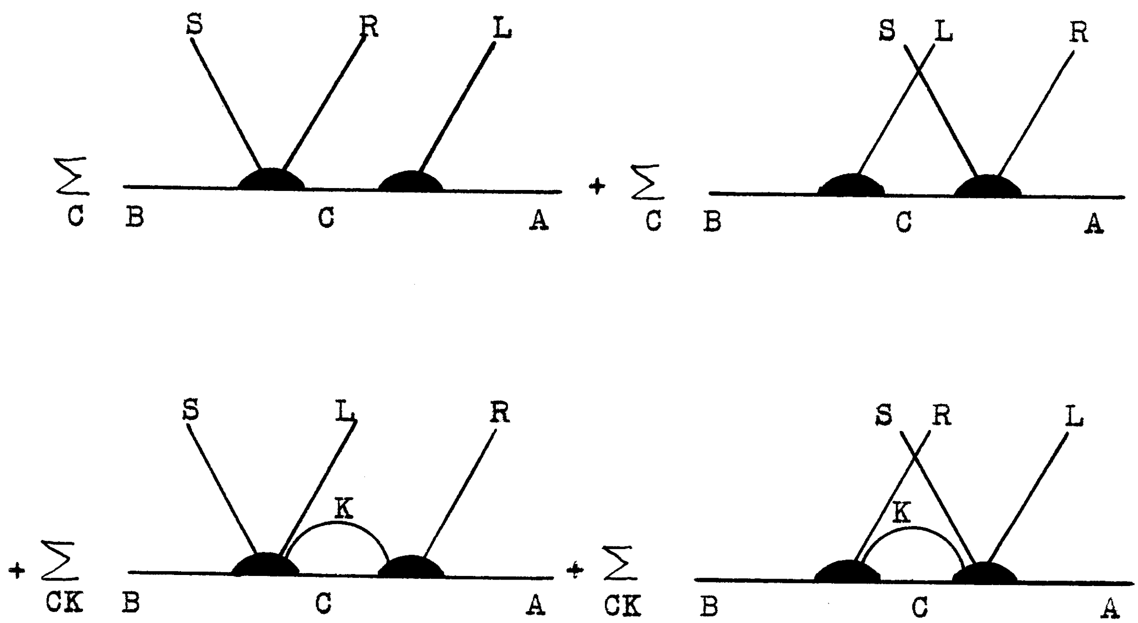
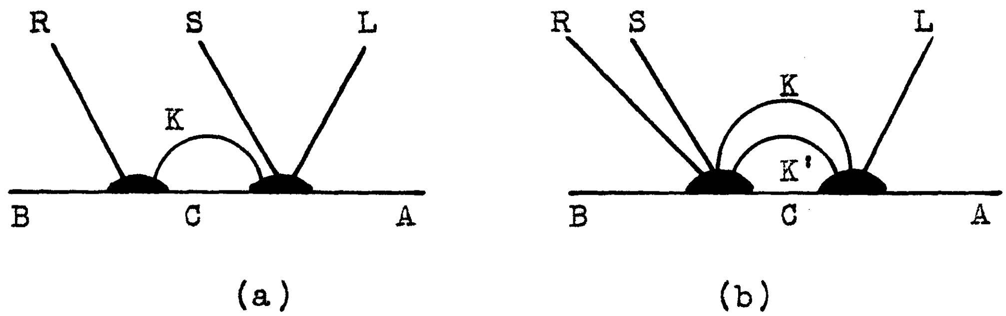


Finally we note that, in terms of diagrams, our one meson approximation as applied to two-meson states means : keep diagrams like (a) below but disregard diagrams like (b) below. In diagram (a) we have a two-meson state between the vertices, but only one of the mesons (K) is rescattered at the second vertex. In diagram (b) on the other hand both mesons K and K', forming the two-meson state between vertices, are rescattered at the second vertex.



So far we have said nothing about the magnetic quantum numbers. Thus for example $\langle BRS|F_s(r,\ell)|AL\rangle$ is the transition amplitude for a process in which the nucleon in the initial state is specified by the magnetic quantum numbers α, α'; the meson in the initial state is specified by the magnetic quantum numbers λ, λ' ; the nucleon in the final state is specified by β, β' ; and the mesons in the final state are specified by ρ, ρ' and σ, σ' . Our Hamiltonian has been so constructed as to be a scalar under rotations in both space and isospace - therefore the transition amplitudes corresponding to various choices of magnetic quantum numbers are not all unrelated. That is to say, if the initial state of the system is constructed to be an eigenstate of the total angular momentum J and the total isotopic spin T, then the only difference between choosing such a state with one set of magnetic quantum numbers as compared with another set will be a numerical factor expressing the different geometry of the two sets of magnetic quantum numbers. It thus is clear that it would be advantageous to eliminate from our equations this dependence on the geometry and consider only what remains and expresses the physics of the problem .

The elimination of the magnetic quantum numbers is carried out in Appendix B and we quote here the result:

$$F_s(r,\ell) = \Lambda' \frac{N_r^{\frac{1}{2}}(s) - N_r^{\frac{3}{2}}(s)}{\omega_r + \omega_s} \theta(\ell) - \sum_k \left\{ \frac{M_r^*(k) F_s(k,\ell)}{\omega_k - \omega_r - i\epsilon} + \Delta' \frac{N_r(k)\, E_s(k,\ell)}{\omega_k + \omega_r} \right\} \quad (59)$$

$$E_s(r,\ell) = \Delta' \Lambda' \frac{M_r^{\frac{1}{2}}(s) - M_r^{\frac{3}{2}}(s)}{\omega_s - \omega_r + i\epsilon} \theta(\ell) - \sum_k \left\{ \frac{M_r(k)\, E_s(k,\ell)}{\omega_k - \omega_r + i\epsilon} + \Delta' \frac{N_r^*(k)\, F_s(k,\ell)}{\omega_k + \omega_r} \right\} \quad (60)$$

The meaning of the various symbols in above equations is as explained in Appendix B .

As stated previously, the functions θ, M and N are known. From the definitions (45), (B-6') and (B-15) we have

$$\langle c | [H', a_\lambda^\dagger(\ell\lambda')] A \rangle = \langle c | \theta(\ell) | AL \rangle$$

$$= -2(-)^{\gamma+\gamma'} V(\tfrac{1}{2}1\tfrac{1}{2};\alpha\lambda-\gamma)\, V(\tfrac{1}{2}1\tfrac{1}{2};\alpha'\lambda'-\gamma')\, \theta(\ell) \quad (61)$$

On the other hand from eqs. (11, 12) we have

$$[H', a_\lambda^\dagger(\ell\lambda')] = N\, f^{\circ}/\mu\, (6\omega_\ell)^{-\frac{1}{2}}\, v(\ell)\, \ell\, \tau_\lambda\, \sigma_{\lambda'} \quad (62)$$

and, using Racah's[10] definition of a reduced matrix element,

$$\langle C | f^{\circ} \tau_{\lambda} \sigma_{\lambda'} | A \rangle =$$

$$- (-)^{\gamma+\gamma'} \; V(\tfrac{1}{2} 1 \tfrac{1}{2}; \alpha \lambda - \gamma) \; V(\tfrac{1}{2} 1 \tfrac{1}{2}; \alpha' \lambda' - \gamma') \; \langle \tfrac{1}{2} \| f^{\circ} \tau \sigma \| \tfrac{1}{2} \rangle$$

Hence we conclude that

$$\theta(\ell) = \frac{N}{2\mu\sqrt{3}} (2\omega_{\ell})^{-\frac{1}{2}} \, v(\ell) \, \ell \, \langle \tfrac{1}{2} \| f^{\circ} \tau \sigma \| \tfrac{1}{2} \rangle \tag{63}$$

where $\langle \tfrac{1}{2} \| f^{\circ} \tau \sigma \| \tfrac{1}{2} \rangle$ is the reduced matrix element of $f^{\circ} \tau \sigma$ taken between physical nucleon states. But this last quantity is simply a multiple, say f/f°, of the same matrix element taken between <u>bare</u> nucleon states.

$$\langle \tfrac{1}{2} \| f^{\circ} \tau \sigma \| \tfrac{1}{2} \rangle = \langle \tfrac{1}{2} \| f \, \tau \sigma \| \tfrac{1}{2} \rangle_{\text{bare}} = 6 \, f \tag{64}$$

and so

$$\theta(\ell) = \frac{N \sqrt{3} \, f}{\mu \sqrt{2\omega_{\ell}}} \; v(\ell) \; \ell \tag{65}$$

From the work of Chew and Low $f^2 = .08$.

The functions M and N are not as easily determined. They are obtained as solutions of a set of coupled nonlinear integral equations. We now derive briefly these equations: From the definition (53) we have

$$\langle BS | M_r(s) | AR \rangle \equiv \langle BS^- | [H', a^{+}_{\rho}(r)] A \rangle$$

$$= \langle \{ a^{+}_{\sigma}(s) - \frac{1}{H - \omega_s + i\epsilon} [H', a^{+}_{\sigma}(s)] \} B | [H', a^{+}_{\rho}(r)] A \rangle$$

$$= \langle B | [(-)^{\sigma} a_{-\sigma}(s), [H', a_{\rho}^{\dagger}(r)]] | A \rangle + \langle B | [H', (-)^{\sigma} a_{-\sigma}(s)] \frac{1}{H - \omega_s - i\epsilon} [H', a_{\rho}^{\dagger}(r)]$$

$$+ [H', a_{\rho}^{\dagger}(r)] \frac{1}{H + \omega_s} [H', (-)^{\sigma} a_{-\sigma}(s)] | A \rangle \qquad (66)$$

which reduces, in the one meson approximation, to

$$\langle BS | M_r(s) | AR \rangle = \langle B | [(-)^{\sigma} a_{-\sigma}(s), [H', a_{\rho}^{\dagger}(r)]] A \rangle$$

$$- \sum_{CK} \frac{\langle CK | M_s(k) | BS \rangle^* \langle CK | M_r(k) | AR \rangle}{\omega_k - \omega_s - i\epsilon}$$

$$- \sum_{CK} \frac{\langle CKR | N_r(k) | B \rangle^* \langle CKS | N_s(k) | A \rangle}{\omega_k + \omega_s} \qquad (67)$$

Similarly, starting with the definition (44) we have in the one meson approximation

$$\langle BSR | N_r(s) | A \rangle = - \langle B | [(-)^{\sigma} a_{-\sigma}(s), [H', (-)^{\rho} a_{-\rho}(r)]] A \rangle$$

$$- \sum_{CK} \frac{\langle CK | M_s(k) | BS \rangle^* \langle CKR | N_r(k) | A \rangle}{\omega_k - \omega_s - i\epsilon}$$

$$- \sum_{CK} \frac{\langle CK | M_r(k) | BR \rangle^* \langle CKS | N_s(k) | A \rangle}{\omega_k + \omega_s} \qquad (68)$$

Eliminating the magnetic quantum numbers in a manner analogous to that described in Appendix B we get

$$\begin{pmatrix} M_r^{\frac{1}{2}}(s) \\ \\ M_r^{\frac{3}{2}}(s) \end{pmatrix} = \frac{N^2\ v(s)\ v(r)}{4\pi\ \mu \sqrt{\omega_s \omega_r}} \left[\begin{pmatrix} 1 \\ \\ 1 \end{pmatrix} \lambda_\circ + \begin{pmatrix} -1 \\ \\ \frac{1}{2} \end{pmatrix} \lambda \frac{\omega_s + \omega_r}{\mu} \right]$$

$$- \sum_K \frac{1}{\omega_k - \omega_s - i\epsilon} \begin{pmatrix} M_s^{\frac{1}{2}}(k) & 0 \\ \\ 0 & M_s^{\frac{3}{2}}(k) \end{pmatrix}^* \begin{pmatrix} M_r^{\frac{1}{2}}(k) \\ \\ M_r^{\frac{3}{2}}(k) \end{pmatrix}$$

$$- \sum_K \frac{1}{\omega_k + \omega_s} \begin{pmatrix} -\frac{1}{3} & \frac{4}{3} \\ \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} N_r^{\frac{1}{2}}(k) & 0 \\ \\ 0 & N_r^{\frac{3}{2}}(k) \end{pmatrix}^* \begin{pmatrix} N_s^{\frac{1}{2}}(k) \\ \\ N_s^{\frac{3}{2}}(k) \end{pmatrix} \qquad (69)$$

and

$$\begin{pmatrix} N_r^{\frac{1}{2}}(s) \\ \\ N_r^{\frac{3}{2}}(s) \end{pmatrix} = \frac{N^2\ v(s)\ v(r)}{4\pi\ \mu \sqrt{\omega_s \omega_r}} \left[\begin{pmatrix} 1 \\ \\ 1 \end{pmatrix} \lambda_\circ + \begin{pmatrix} -1 \\ \\ \frac{1}{2} \end{pmatrix} \lambda \frac{\omega_s - \omega_r}{\mu} \right]$$

$$- \sum_K \frac{1}{\omega_k - \omega_s - i\epsilon} \begin{pmatrix} M_s^{\frac{1}{2}}(k) & 0 \\ \\ 0 & M_s^{\frac{3}{2}}(k) \end{pmatrix}^* \begin{pmatrix} N_r^{\frac{1}{2}}(k) \\ \\ N_r^{\frac{3}{2}}(k) \end{pmatrix}$$

$$- \sum_K \frac{1}{\omega_k + \omega_s} \begin{pmatrix} -\frac{1}{3} & \frac{4}{3} \\ \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} M_r^{\frac{1}{2}}(k) & 0 \\ \\ 0 & M_r^{\frac{3}{2}}(k) \end{pmatrix}^* \begin{pmatrix} N_s^{\frac{1}{2}}(k) \\ \\ N_s^{\frac{3}{2}}(k) \end{pmatrix} \qquad (70)$$

Here $\lambda_\circ$ and λ are the renormalized coupling constants defined in a manner analogous to eq. (64) :

$$\left\langle \tfrac{1}{2} \left\| \lambda_0^\circ \right\| \tfrac{1}{2} \right\rangle \equiv \left\langle \tfrac{1}{2} \left\| \lambda_0 \right\| \tfrac{1}{2} \right\rangle_{\text{bare}} \tag{71}$$

$$\left\langle \tfrac{1}{2} \left\| \lambda^\circ \tau \right\| \tfrac{1}{2} \right\rangle \equiv \left\langle \tfrac{1}{2} \left\| \lambda \tau \right\| \tfrac{1}{2} \right\rangle_{\text{bare}} \tag{72}$$

(It follows from definition (71) that $\lambda_0^\circ = \lambda_0$.)

Equations (69) and (70) were solved approximately by Drell, Friedman and Zachariasen . They find that M and N may be taken to be real and of the form

$$\begin{pmatrix} M_r^{\frac{1}{2}}(s) \\ \\ M_r^{\frac{3}{2}}(s) \end{pmatrix} = \frac{N^2 v(s) v(r)}{4\mu\sqrt{\omega_s\omega_r}} \left\{ 2c_0 \begin{pmatrix} 1 \\ \\ 1 \end{pmatrix} + \left[c_1 \frac{\omega_r + \omega_s}{\mu} + c_2 \frac{\omega_r - \omega_s}{\mu} \right] \begin{pmatrix} -1 \\ \\ \frac{1}{2} \end{pmatrix} \right\} \tag{73}$$

$$\begin{pmatrix} N_r^{\frac{1}{2}}(s) \\ \\ N_r^{\frac{3}{2}}(s) \end{pmatrix} = \frac{N^2 v(s) v(r)}{4\mu\sqrt{\omega_s\omega_r}} \left\{ 2c_0 \begin{pmatrix} 1 \\ \\ 1 \end{pmatrix} - \left[c_1 \frac{\omega_r - \omega_s}{\mu} + c_2 \frac{\omega_r + \omega_s}{\mu} \right] \begin{pmatrix} -1 \\ \\ \frac{1}{2} \end{pmatrix} \right\} \tag{74}$$

$$\text{with} \quad c_0 = .04 \; : \; c_1 = .14 \; : \; c_2 \leq .01 \tag{75}$$

Introducing eqs. (73), (74) into eq. (59) and (60) gives

$$F_s(r,\ell) = \frac{3N^2 \; v(s) \; v(r)}{8\mu^2 \; \sqrt{\omega_r\omega_s}} \; \theta(\ell) \left(c_1 \frac{\omega_r - \omega_s}{\omega_r + \omega_s} + c_2 \right) \Lambda'$$

$$- \frac{N^2\ v(r)}{4\mu\sqrt{\omega_r}} \sum_k \frac{v(k)}{\sqrt{\omega_k}} \left\{ 2c_0 + \left[c_1 \frac{\omega_r+\omega_k}{\mu} + c_2 \frac{\omega_r-\omega_k}{\mu} \right] \Gamma \right\} \frac{F_s(k,\ell)}{\omega_k-\omega_r}$$

$$- \frac{N^2\ v(r)}{4\mu\sqrt{\omega_r}} \sum_k \frac{v(k)}{\sqrt{\omega_k}} \left\{ 2c_0\Delta' - \left[c_1 \frac{\omega_r-\omega_k}{\mu} + c_2 \frac{\omega_r+\omega_k}{\mu} \right] \Delta'\Gamma \right\} \frac{E_s(k,\ell)}{\omega_k+\omega_r} \tag{76}$$

and

$$E_s(r,\ell) = \frac{3N^2\ v(s)\ v(r)}{8\mu^2\ \sqrt{\omega_r\omega_s}}\ \theta(\ell) \left(c_1 \frac{\omega_r+\omega_s}{\omega_r-\omega_s} + c_2 \right) \Delta'\Lambda'$$

$$- \frac{N^2 v(r)}{4\mu\sqrt{\omega_r}} \sum_k \frac{v(k)}{\sqrt{\omega_k}} \left\{ 2c_0 + \left[c_1 \frac{\omega_r+\omega_k}{\mu} + c_2 \frac{\omega_r-\omega_k}{\mu} \right] \Gamma \right\} \frac{E_s(k,\ell)}{\omega_k-\omega_r}$$

$$- \frac{N^2 v(r)}{4\mu\sqrt{\omega_r}} \sum_k \frac{v(k)}{\sqrt{\omega_k}} \left\{ 2c_0\Delta' - \left[c_1 \frac{\omega_r-\omega_k}{\mu} + c_2 \frac{\omega_r+\omega_k}{\mu} \right] \Delta'\Gamma \right\} \frac{F_s(k,\ell)}{\omega_k+\omega_r} \tag{77}$$

where Γ is a 4x4 diagonal matrix :

$$\Gamma = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{78}$$

and we assume that the functions F and E may be approximated by real functions. Integrals over singular quantities are to be interpreted as Cauchy's principal value integrals.

V. PHOTOPRODUCTION

We first generalize our Hamiltonian to include the effects of electro-magnetic fields. The Hamiltonian in the absence of electro-magnetic fields was taken to be

$$H = H_0 + H_1 + H_2 + H_3$$

$$H_0 = \frac{1}{2}\sum_q (-)^q \int \{\pi_q(\vec{r})\pi_{-q}(\vec{r}) + \vec{\nabla}\phi_q(\vec{r})\cdot\vec{\nabla}\phi_{-q}(\vec{r}) + \mu^2\phi_q(\vec{r})\phi_{-q}(\vec{r})\} d\vec{r} \quad (79)$$

$$H_1 = \sqrt{4\pi}\, f^0/\mu \sum_q (-)^q \int u(r)\tau_{-q}\vec{\sigma}\cdot\vec{\nabla}\phi_q(\vec{r})\, d\vec{r} \quad (80)$$

$$H_2 = \lambda_0^0/\mu \sum_q (-)^q \int u(r)\,\phi_q(\vec{r})\, d\vec{r} \int u(r')\,\phi_{-q}(\vec{r}')\, d\vec{r}' \quad (81)$$

$$H_3 = i\lambda^0/\mu^2 \sum_q \tau_q \Big\{ \int u(r)\,\pi_p(\vec{r})\, d\vec{r} \int u(r')\,\phi_s(\vec{r}')\, d\vec{r}'$$

$$- \int u(r)\,\pi_s(\vec{r})\, d\vec{r} \int u(r')\,\phi_p(\vec{r}')\, d\vec{r}' \Big\} \quad (82)$$

where q, p, s = cyclic permutations of -1, 0, +1.

In the presence of electro-magnetic fields above H goes over into $H(\vec{A})$. $H(\vec{A})$ must have a structure which is gauge invariant. This means that the following equation must be satisfied:(11)

$$e^{iD}\, H(\vec{A})\, e^{-iD} = H(\vec{A} + \vec{\nabla} G) \quad (83)$$

where $D = \int d\vec{r}\ G(\vec{r})\rho(\vec{r})$ (84)

where $\rho(\vec{r})$ is the charge density of the system given in this case by

$$\rho(\vec{r}) = \rho_{\pi}(\vec{r}) + (1 + \tau_{o})/2 \quad e\,\delta(\vec{r})$$

$$= ie\left\{\pi_{+1}(\vec{r})\phi_{-1}(\vec{r}) - \pi_{-1}(\vec{r})\phi_{+1}(\vec{r})\right\} + (1 + \tau_{o})/2 \quad e\,\delta(\vec{r}) \tag{85}$$

where ρ_{π} is the charge density of the mesons and $(1 + \tau_{o})/2 \quad e\,\delta(\vec{r})$ is the charge density of the nucleon assumed to be located at the origin of the coordinate system.

The function $G(\vec{r})$ is a scalar gauge function - any electro-magnetic field operator which commutes with $\vec{A}$.

If it were not for the source function $u(r)$ the transition $H \rightarrow H(\vec{A})$ could be accomplished by the standard prescription:

$$\vec{\nabla}\phi_{-q}(\vec{r}) \rightarrow \left(\vec{\nabla} + qie\vec{A}\right)\phi_{-q}(\vec{r}) \tag{86}$$

It can be easily verified that with the substitution (86) the terms in H denoted by H_2 and H_3 fail to satisfy eq. (83) . (Note that the substitution (86) leaves H_2 and H_3 unchanged). We demonstrate this for H_2 :

$$e^{iD} H_s e^{-iD} =$$

$$\lambda_0^0/\mu \sum_q (-)^q \int d\vec{r} \int d\vec{r}\,' \, u(r)\, u(r') \, e^{iD} \phi_q(\vec{r})\phi_{-q}(\vec{r}\,') \, e^{-iD} \tag{87}$$

Let us define $F(x) \equiv e^{ixD} \phi_q(\vec{r})\phi_{-q}(\vec{r}\,') \, e^{-ixD}$: then

$$dF/dx = ie^{ixD}\left[D, \phi_q(\vec{r})\phi_{-q}(\vec{r}\,')\right] e^{-ixD}$$

$$= - e \, e^{ixD} \int d\vec{r}\,''\left[\pi_{+1}(\vec{r}\,'')\phi_{-1}(\vec{r}\,'') - \pi_{-1}(\vec{r}\,'')\phi_{+1}(\vec{r}\,''),\right.$$

$$\left.\phi_q(\vec{r})\phi_{-q}(\vec{r}\,')\right] G(\vec{r}\,'') \, e^{-ixD}$$

$$= - ieq \, e^{ixD} \int d\vec{r}\,''\left\{\phi_q(\vec{r})\phi_{-q}(\vec{r}\,'')\,\delta(\vec{r}\,'' - \vec{r}\,')\right.$$

$$\left. -\phi_{-q}(\vec{r}\,')\phi_q(\vec{r}\,'')\,\delta(\vec{r}\,'' - \vec{r})\right\} G(\vec{r}\,'') \, e^{-ixD}$$

$$= - ieq \, e^{ixD} \phi_q(\vec{r})\phi_{-q}(\vec{r}\,') \left\{G(\vec{r}\,') - G(\vec{r})\right\} e^{-ixD}$$

$$= - ieq \left\{G(\vec{r}\,') - G(\vec{r})\right\} F(x)$$

Integrating $dF/dx = - ieq\left\{G(\vec{r}\,') - G(\vec{r})\right\} F(x)$ between 0 and 1 we get $\ln\left\{F(1)/F(0)\right\} = -ieq\left\{G(\vec{r}\,') - G(\vec{r})\right\}$ or

$$F(1) = e^{iD} \phi_q(\vec{r})\phi_{-q}(\vec{r}\,') \, e^{-iD} = F(0) \, e^{-ieq\left\{G(\vec{r}\,') - G(\vec{r})\right\}}$$

$$= \phi_q(\vec{r})\phi_{-q}(\vec{r}\,') \, e^{-ieq\left\{G(\vec{r}\,') - G(\vec{r})\right\}}$$

and thus

$$e^{iD} H_s e^{-iD} = \qquad (88)$$

$$\lambda_0^\circ/\mu \sum_q (-)^q \int d\vec{r} \int d\vec{r}' \; u(r)\, u(r') \{e^{ieqG(\vec{r})} \phi_q(\vec{r})\}\{e^{-ieqG(\vec{r}')} \phi_{-q}(\vec{r}')\}$$

By similarly investigating H_s we conclude that under a gauge transformation ϕ_q, π_q and τ_q change as follows

$$\phi_q(\vec{r}) \rightarrow e^{ieq\, G(\vec{r})} \phi_q(\vec{r})$$

$$\pi_q(\vec{r}) \rightarrow e^{ieq\, G(\vec{r})} \pi_q(\vec{r}) \qquad (89)$$

$$\tau_q \rightarrow e^{ieq\, G(0)} \tau_q$$

(For the nucleon located not at the origin but at $\vec{r}_\circ$: $\tau_q \rightarrow e^{ieq\, G(\vec{r}_\circ)} \tau_q$.) This then suggests that the Hamiltonian will be properly gauge invariant if it is obtained (in addition to substitution (86)) by modifying ϕ_q, π_q, τ_q in such a way as to produce the factors appearing in (89), when $\vec{A} \rightarrow \vec{A} + \vec{\nabla} G$.

Now consider the expression

$$-ieq/4\pi \int \frac{\vec{\nabla}' \cdot \vec{A}(\vec{r}')}{|\vec{r}-\vec{r}'|} d\vec{r}' \qquad (90)$$

Under a gauge transformation $\vec{A} \to \vec{A} + \vec{\nabla} G$ expression (90) becomes

$$-\frac{ieq}{4\pi}\int \frac{\vec{\nabla}'\cdot\vec{A}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}' \longrightarrow -\frac{ieq}{4\pi}\int \frac{\vec{\nabla}'\cdot\vec{A}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}' - \frac{ieq}{4\pi}\int \frac{\nabla^2 G(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}'$$

$$= -\frac{ieq}{4\pi}\int \frac{\vec{\nabla}'\cdot\vec{A}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}' + ieq\,G(\vec{r}) \qquad (91)$$

provided G vanishes at infinity.

Thus we propose to introduce the electro-magnetic field into the theory by taking, instead of substitution (86), the following :

$$\phi_q(\vec{r}) \longrightarrow \exp\left[-ieq/4\pi \int \frac{\vec{\nabla}'\cdot\vec{A}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}'\right] \phi_q(\vec{r})$$

$$\pi_q(\vec{r}) \longrightarrow \exp\left[-ieq/4\pi \int \frac{\vec{\nabla}'\cdot\vec{A}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}'\right] \pi_q(\vec{r})$$

$$\tau_q \longrightarrow \exp\left[-ieq/4\pi \int \frac{\vec{\nabla}'\cdot\vec{A}(\vec{r}')}{|\vec{r}'|}\,d\vec{r}'\right] \tau_q$$

$$\vec{\nabla}\phi_q(\vec{r}) \longrightarrow \exp\left[-ieq/4\pi \int \frac{\vec{\nabla}'\cdot\vec{A}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}'\right] \left(\vec{\nabla} + ieq\vec{A}(\vec{r})\right) \phi_q(\vec{r}) \qquad (92)$$

and one easily verifies that the $H(\vec{A})$ so obtained satisfies eq. (83) and hence is gauge invariant [1]. We observe that

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[1] It is well known that the manner in which an extended source theory is made gauge invariant is not unique. For a general treatment see reference (12). We note that if the theory is

every term in H picks up two exponential factors with opposite signs of q, hence the terms in the exponent are always of the form

$$\int \vec{\nabla}\cdot\vec{A}(\vec{r}) \left[1/|\vec{r} - \vec{r}\,'| - 1/|\vec{r} - \vec{r}\,''| \right] d\vec{r}$$

with $\vec{r}\,'$ a meson coordinate and $\vec{r}\,''$ also a meson coordinate or the nucleon coordinate. Therefore the restriction that G must vanish as $\vec{r} \to \infty$ required in eq. (91) may be relaxed to $\vec{\nabla}G$ vanishing as $\vec{r} \to \infty$ thus allowing G to be a constant. Since $\vec{\nabla}G = \vec{A} - \vec{A}\,'$, where $\vec{A}$, $\vec{A}\,'$ are two vector potentials differing by a gauge transformation, we see that our theory is gauge invariant provided we only allow gauge transformations such that the different vector potentials have the same behaviour at infinity. We feel that this is sufficiently general.

The advantage of this formulation is that, now that the theory is gauge invariant, we may choose a particular gauge to work in and we take the gauge defined by $\vec{\nabla}\cdot\vec{A} = 0$. Then all the exponential factors reduce to unity and the Hamiltonian $H(\vec{A})$ is :

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[1] cont.

made gauge invariant in a manner different from ours additional currents appear ; however at low energies their contribution is negligible.

$$H_0(\vec{A}) = H_0(\vec{A}=0) + \tfrac{1}{2}\sum_q (-)^q ieq \int \vec{A}(\vec{r})\cdot\left[\phi_q(\vec{r})\vec{\nabla}\phi_{-q}(\vec{r}) - \vec{\nabla}\phi_q(\vec{r})\phi_{-q}(\vec{r})\right]d\vec{r}$$

$$H_1(\vec{A}) = H_1(\vec{A}=0) + \sqrt{4\pi}\, f^\circ/\mu \sum_q (-)^q ieq \int u(r)\tau_{-q}\vec{\sigma}\cdot\vec{A}(\vec{r})\,\phi_q(\vec{r})\, d\vec{r}$$

$$H_2(\vec{A}) + H_3(\vec{A}) = H_2(\vec{A} = 0) + H_3(\vec{A} = 0) \qquad (93)$$

Thus

$$H(\vec{A}) = H + H''$$

where $H = H(\vec{A} = 0)$ is the same as that used in meso-production, and

$$H'' = \sum_q (-)^q ieq \Big\{ \int \phi_q(\vec{r})\, \vec{A}(\vec{r})\cdot\vec{\nabla}\phi_{-q}(\vec{r})\, d\vec{r}$$

$$+ \sqrt{4\pi}\, f^\circ/\mu \int u(r)\tau_{-q}\vec{\sigma}\cdot\vec{A}(\vec{r})\,\phi_q(\vec{r})\, d\vec{r} \Big\} \qquad (94)$$

We are interested in the transition amplitude corresponding to the absorption of a photon and creation of two s-wave mesons. The matrix element for the absorption of a photon of type k by a nucleon described by spin, isospin = A leading to two mesons described by spin, isospin R and S and a nucleon described by spin, isospin B , is given by

$$\langle BRS^- | \, H''_k \, | A \rangle \qquad (95)$$

where H''_k is the matrix element of H'' taken between states of the radiation field of one photon of type k and no photons. (We are treating the electro-magnetic effects in perturbation theory.)

In expression (95) the state $|A\rangle$ and the state $\langle BRS^-|$ are both states of even parity and total angular momentum $\frac{1}{2}$ because R, S are s-waves. Therefore expression (95) will give non-vanishing contributions provided H''_k has even parity and angular dependence corresponding to a tensor of rank 0 or 1. It is therefore advantageous to expand the vector potential $\vec{A}(\vec{r})$ in H'' in spherical waves since then H''_k will be simply equal to the coefficient of the annihilation operator for a photon characterized by an even parity and angular momentum 0 or 1, and linear momentum $= k$. In the expansion of $\vec{A}(\vec{r})$ angular momentum $= 0$ does not exist and so the only term to contribute is the one generally associated with magnetic dipole transitions. (We may do this because H'' is a scalar and therefore the rotation and reflection properties of H''_k are entirely determined by the rotation and reflection properties of the photon annihilation operator.)

The solutions of the vector equation

$$\nabla^2 \vec{A} + k^2 \vec{A} = 0$$

representing spherical transverse vector waves regular at the origin are[13]

$$\vec{M}_{\ell m}(\vec{r}) = \vec{\nabla} \times \left[\vec{r} Y_{\ell m} (\Omega_r) \, j_\ell(kr)\right]$$

and

$$\vec{N}_{\ell m}(\vec{r}) = 1/k\ \vec{\nabla} \times \vec{M}_{\ell m}(\vec{r}) \tag{96}$$

Hence in the usual manner we may expand $\vec{A}$ as follows

$$\vec{A}(\vec{r}) = \sum_k N/\sqrt{2k} \sum_{\ell m} (-)^m \Big\{ \vec{M}_{\ell -m}(\vec{r}) \big[c(k\ell m) + c^{\dagger}(k\ell m)\big] + \vec{N}_{\ell -m}(\vec{r}) \big[d(k\ell m) + d^{\dagger}(k\ell m)\big] \Big\} \tag{97}$$

where N is a normalization constant related to the inverse square root of the volume in which $\vec{A}$ is quantized, and the sum over ℓ starts at $\ell = 1$, the sum over m goes for each value of ℓ from $-\ell$ to $+\ell$. Since the parity of $Y_{\ell m}$ is $(-)^{\ell}$ it follows that the parity of $\vec{M}_{\ell m}$ is $(-)^{\ell}$ and that of $\vec{N}_{\ell m}$ is $-(-)^{\ell}$. Since the photon has intrinsic spin one (we are dealing with a vector (polar) field) it has intrinsic odd parity and so it follows that the parity of $c(k\ell m)$, $c^{\dagger}(k\ell m)$ is $-(-)^{\ell}$ whereas the parity of $d(k\ell m)$, $d^{\dagger}(k\ell m)$ is $(-)^{\ell}$. Thus we need the coefficient of $c(k1m)$ and H''_k is obtained from H'' by replacing $\vec{A}(\vec{r})$ by

$$N/\sqrt{2k}\ \vec{M}_{1m_k}(\vec{r}) = N/\sqrt{2k}\ \vec{\nabla} \times \big[\vec{r}\ Y_{1m_k}(\Omega_r)\ j_1(kr)\big] \tag{98}$$

This is because the state of the radiation field corresponding to one photon of even parity, angular momentum = 1, angular momentum z-component = m_k and linear momentum = k is given by

$$\big| c^{\dagger}(k1m_k)\Psi_o \big\rangle \tag{99}$$

(where $|\Psi_\circ\rangle$ is the vacuum state of the radiation field) and because

$$\left|(-)^{m_k} c(kl{-}m_k)\, c^\dagger(klm_k)\, \Psi_\circ\right\rangle = |\Psi_\circ\rangle \qquad (100)$$

The expression (98) can be written more explicitly as

$$N/\sqrt{2k}\, \vec{M}_{1m_k}(\vec{r}) = N/\sqrt{2k}\, j_1(kr)\, \sqrt{3/8\pi}\, \cdot$$
$$\Big\{ \left[- i\cos\vartheta\, \vec{u}_x + \cos\vartheta\, \vec{u}_y + i\sin\vartheta\, e^{i\varphi}\, \vec{u}_z\right] \delta_{m_k,1}$$
$$+ \left[- \sin\vartheta\, \sin\varphi\, \vec{u}_x + \sin\vartheta\, \cos\varphi\, \vec{u}_y\right] \sqrt{2}\, \delta_{m_k,0} \qquad (101)$$
$$+ \left[- i\cos\vartheta\, \vec{u}_x - \cos\vartheta\, \vec{u}_y + i\sin\vartheta\, e^{-i\varphi}\vec{u}_z\right] \delta_{m_k,-1}$$

Thus

$$H''_k = \sum_q (-)^q\, ieq\, N/\sqrt{2k} \Big\{ \int \phi_q(\vec{r})\, \vec{M}_{1m_k}(\vec{r}) \cdot \vec{\nabla} \phi_{-q}(\vec{r})\, d\vec{r}$$
$$+ \sqrt{4\pi}\, f^\circ/\mu \int u(r)\, \vec{\sigma} \cdot \vec{M}_{1m_k}(\vec{r})\, \tau_{-q}\, \phi_q(\vec{r})\, d\vec{r} \Big\} \qquad (102)$$

We next show that if we expand $\phi_q(\vec{r})$ (wherever it appears in eq. (102)) in spherical waves then the only contribution to

$$\langle BRS^- |\, H''_k\, |\, A \rangle$$

comes from the p-waves.

Consider the state $|A\rangle$: it represents a "physical" nucleon and in view of the structure of our interaction Hamiltonian if we should expand $|A\rangle$ in "bare" states it would contain amplitudes for only the following type of "bare" states :

$$\begin{aligned} &\text{one nucleon + any number of p-wave mesons} \\ &\qquad\qquad + \text{any even number of s-wave mesons} \end{aligned} \tag{103}$$

Since the state $|BRS^-\rangle$ differs from $|A\rangle$ by the presence of a pair of s-waves, in an expansion of the type considered above, amplitudes for the same type of bare states - and no other - will appear.

Thus H_k'' must contain only such meson operators, which , when operating on $|A\rangle$ produce a state that contains again only amplitudes for "bare" states of type (103) or else we obtain a vanishing contribution.

Consider first the term in eq. (102) linear in ϕ_q : In the spherical wave expansion

$$\phi_q(\vec{r}) = N \sum_{ptm} (2\omega_p)^{-\frac{1}{2}} (-)^m j_t(pr)\, Y_{t-m}(\Omega_r) \left[a_q(ptm) + a_q^{\dagger}(ptm)\right]$$

the term corresponding to $t = 0$ when acting on $|A\rangle$ will produce a state containing amplitudes for an odd number of s-wave mesons, hence not of the type (103) . The terms corresponding to $t \geq 2$ will produce states containing amplitudes for a meson of angular momentum $t \geq 2$, hence again not of type (103) . Only for $t = 1$ do we obtain a nonvanishing result.

Now consider the term in eq. (102) bilinear in ϕ_q: After introducing the spherical wave expansion we will have terms containing products of two operators - one for a meson of angular momentum t and one for a meson of angular momentum t' .

a) $t = t' = 0$

In this case the angular dependence of the integral in eq.(102) is entirely that due to $\vec{M}_{1,m_k}(\vec{r})$ and , as is clear from eq. (101) , the angular integration gives 0 .

b) $t = 0,\ t' \neq 0$ or vice versa

The result of operating with such a term on $|A\rangle$ is a state with amplitudes for an odd number of s-wave mesons, hence not of type (103) .

c) $t \geq 2,\ t' \neq t$ or vice versa

The result of operating with such a term on $|A\rangle$ is a state with amplitudes for a meson of angular momentum $t \geq 2$, hence not of type (103) .

d) $t = t' \geq 2$

1) $m \neq - m'$

The meson created by the operator corresponding to $t(t')$ cannot be annihilated by the operator corresponding to $t'(t)$ because $m \neq - m'$ and the result of operating with this term on $|A\rangle$ is a state containing amplitudes for two mesons of angular momentum $t = t' \geq 2$, hence not of type (103).

2) $m = - m'$

Now it is possible (provided the energies are the same) to have one of the meson operators create the meson (t,m) and the other annihilate it and therefore the result of operating with this term on $|A\rangle$ would be a state containing amplitudes of type (103). However, in this case the φ-dependence of the integral in eq. (102) is entirely due to $\vec{M}_{1,m_k}(\vec{r})$ and then it is clear from eq. (101) that all φ-dependent terms will vanish upon φ-integration. Then for ϑ-integration we have (see eq. (101))

$$\int d(\cos\vartheta) \cos\vartheta \, Y_{tm} \, Y_{t-m}$$

We can write $Y_{tm} \, Y_{t-m} = \sum_{t''=0}^{t} c_{t''} \, Y_{2t''0}$ where $c_{t''}$ is some coefficient. Hence the ϑ-integral becomes

$$\sqrt{4\pi/3} \sum_{t''=0}^{t} c_{t''} \int_{-1}^{1} d(\cos\vartheta)\, Y_{2t''0}\, Y_{10} =$$

$$\sqrt{4\pi/3} \sum_{t''=0}^{t} c_{t''}\, \delta_{2t'',1} = 0$$

e) Finally we will have a term $t = t' = 1$ which gives a non zero contribution.

Thus eq. (102) can be written as follows :

$$H''_k = \sum_q (-)^q\, ieq\, N/\sqrt{2k} \Big\{ \sqrt{4\pi}\, f^{\circ}/\mu\, N \sum_{pm} (2\omega_p)^{-\frac{1}{2}} (-)^m \tau_{-q} \cdot$$

$$\left[a_q(pm) + a_q^{\dagger}(pm)\right] \int u(r)\, j_1(pr)\, Y_{1-m}(\Omega_r)\vec{\sigma}\cdot\vec{M}_{1m_k}(\vec{r})\, d\vec{r}$$

$$+ N^2 \sum_{\substack{pm\\p'm'}} (4\omega_p\omega_{p'})^{-\frac{1}{2}} (-)^{m+m'} \left[a_q(pm) + a_q^{\dagger}(pm)\right]\left[a_{-q}(p'm') + a_{-q}^{\dagger}(p'm')\right] \cdot$$

$$\int j_1(pr)\, Y_{1-m}(\Omega_r)\, \vec{M}_{1m_k}(\vec{r})\cdot\vec{\nabla}\left[j_1(p'r)\, Y_{1-m'}(\Omega_r)\right] d\vec{r} \Big\} \qquad (104)$$

The explicit structure of H''_k as given by eq. (104) will not be made use of except for the following features :

1) H''_k contains operators for p-wave mesons only

2) H''_k as far as rotations in space are concerned behaves like the m_k-component of an irreducible tensor of rank 1 (this is a simple consequence of the fact that H'' is a scalar and H''_k was obtained by extracting from H'' a quantity behaving like the $-m_k$-component of a vector)

3) H''_k as far as rotations in isospace are concerned behaves like the 0-component of a tensor of rank 1. This is a consequence of the fact that H''_k would be an isotopic scalar were it not for the factor q in "ieq". Since $q\,\phi_q = t_0\,\phi_q$ where t_0 is the zeroth component of the meson isospin operator which is a vector, our conclusion follows.

Keeping above features in mind we now compare

$$\langle BRS^- | H''_k | A\rangle$$

and

$$\langle BRS^- | [H', a^\dagger_0(k,m_k)]\ A\rangle$$

Recalling the notation of the sections on meso-production we observe that $a^\dagger_0(k,m_k)$ is the creation operator for a neutral meson with linear momentum = k, angular momentum = 1, angular momentum z-component = m_k.

We see that $[H', a^\dagger_0(k,m_k)]$ contains only operators for p-wave mesons - just like H''_k. The behaviour under rotations in space and isospace is also the same as that of H''_k. On the other hand H''_k contains as a factor $1/\sqrt{2k}$ where $[H', a^\dagger_0(k,m_k)]$ has $1/\sqrt{2\omega_k}$, due to this difference in normalization of a photon and a meson. Lastly we observe that whereas the matrix elements of $[H', a^\dagger_0(k,m_k)]$ are proportional to the p-wave coupling constant the corresponding factor for H''_k should be

$-\, ge/2M$ where g is the anomalous nuclear gyromagnetic ratio. Since H''_k is an isotopic vector, in our theory g is equal and opposite for the proton and neutron and may be taken as the average of the experimental values $g = (g_p - g_n)/2$. (Chew and Low use the complete static moments ; we feel that it is more correct to use the anomalous parts of the moments only. The theory is not accurate enough to decide this question in any case.)

Thus we conclude that

$$\langle BRS^- | H''_k | A \rangle = - \frac{(g_p - g_n)/2 \;\; e/2M}{\sqrt{4\pi} \; f/\mu \; \sqrt{k/\omega_k}} \langle BRS^- | [H', a^\dagger_\bullet(km_k)] A \rangle \qquad (105)$$

(here f is the renormalized but not rationalized quantity - according to Chew and Low $f^2 = 0.08$)

Therefore once the meso-production equations are solved eq. (105) gives us immediately the transition amplitude for photoproduction.

VI. CROSS SECTIONS ; CALCULATION AND DISCUSSION

Our next task is to solve the meso-production equations and compute cross sections. Since the dependence of $F_s(r,\ell)$ on the momentum ℓ is of a trivial nature we eliminate it from our equations. Also, it is more convenient for purposes of calculations to rationalize the matrices Δ' and Λ' . We define two new functions $H_s(r)$ and $G_s(r)$ by

$$F_s(r,\ell) = \frac{N^2\ v(s)\ v(r)}{12\mu^2\ \sqrt{\omega_r\omega_s}}\ \theta(\ell)\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \sqrt{2} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \sqrt{5}\end{pmatrix} H_s(r) \qquad (106)$$

$$E_s(r,\ell) = \frac{N^2\ v(s)\ v(r)}{12\mu^2\ \sqrt{\omega_r\omega_s}}\ \theta(\ell)\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \sqrt{2} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \sqrt{5}\end{pmatrix} G_s(r) \qquad (107)$$

Then instead of eqs. (76) and (77) for $F_s(r,\ell)$ and $E_s(r,\ell)$ we must solve the following equations for $H_s(r)$ and $G_s(r)$:

$$H_s(r) = \left(c_1 \frac{\omega_r - \omega_s}{\omega_r + \omega_s} + c_2\right)\Lambda$$

$$- \frac{N^2}{4\mu}\sum_k \frac{v^2(k)}{\omega_k}\left\{\left[2c_0 + \left(c_1\frac{\omega_r + \omega_k}{\mu} + c_2\frac{\omega_r - \omega_k}{\mu}\right)\Gamma\right]\frac{H_s(k)}{\omega_k - \omega_r}\right.$$

$$+ \Delta \left[2c_0 + \left(c_1 \frac{\omega_k - \omega_r}{\mu} - c_2 \frac{\omega_k + \omega_r}{\mu} \right) \Gamma \right] \frac{G_s(k)}{\omega_k + \omega_r} \Bigg\} \qquad (108)$$

and

$$G_s(r) = \left(c_1 \frac{\omega_r + \omega_s}{\omega_r - \omega_s} + c_2 \right) \Delta \Lambda$$

$$- \frac{N^2}{4\mu} \sum_k \frac{v^2(k)}{\omega_k} \Bigg\{ \left[2c_0 + \left(c_1 \frac{\omega_k + \omega_r}{\mu} - c_2 \frac{\omega_k - \omega_r}{\mu} \right) \Gamma \right] \frac{G_s(k)}{\omega_k - \omega_r}$$

$$+ \Delta \left[2c_0 + \left(c_1 \frac{\omega_k - \omega_r}{\mu} - c_2 \frac{\omega_k + \omega_r}{\mu} \right) \Gamma \right] \frac{H_s(k)}{\omega_k + \omega_r} \Bigg\} \qquad (109)$$

where

$$\Lambda = -\begin{pmatrix} 4 \\ 2 \\ 1 \\ -1 \end{pmatrix} ; \ \Delta = \frac{1}{9} \begin{pmatrix} 1 & 4 & 8 & 20 \\ 2 & 8 & -2 & -5 \\ 4 & -2 & 5 & -10 \\ 2 & -1 & -2 & 4 \end{pmatrix} ; \ \Delta\Lambda = -\begin{pmatrix} 0 \\ 3 \\ 3 \\ 0 \end{pmatrix} \qquad (110)$$

and $\Delta\Delta = 1$

We are interested in pair production near threshold and therefore the quantity $(\omega_r - \omega_s)/(\omega_r + \omega_s)$ is small compared with unity. Thus the first term in the equation for $H_s(r)$ is small and the first term in the equation for $G_s(r)$ is large with the consequence that

$$G_s(r) \cong \frac{\omega_r + \omega_s}{\omega_r - \omega_s} \; c_1 \Delta \Lambda \tag{111}$$

is an excellent approximation for the function $G_s(r)$ but the corresponding assumption for the function $H_s(r)$ results in large errors.

Using the identity

$$\frac{1}{\omega_k \pm \omega_r} = \frac{1}{\omega_k \pm \omega_s} \left(1 \mp \frac{\omega_r - \omega_s}{\omega_k \pm \omega_r}\right) \tag{112}$$

we rewrite eq. (108) as

$$\begin{aligned} H_s(r) = {} & \left(c_1 \frac{\omega_r - \omega_s}{\omega_r + \omega_s} + c_2\right) \Lambda + \overline{H}_s + \Delta \overline{G}_s \\ & - \frac{\omega_r - \omega_s}{\pi\mu} \int k \, d\omega_k \left[\frac{c_0 + c_1 \Gamma \omega_k/\mu}{(\omega_k - \omega_s)(\omega_k - \omega_r)} H_s(k) \right. \\ & \left. - \Delta \frac{c_0 + c_1 \Gamma \omega_k/\mu}{(\omega_k + \omega_s)(\omega_k + \omega_r)} G_s(k) \right] \end{aligned} \tag{113}$$

where

$$\overline{H}_s = - \frac{1}{2\pi\mu} \int k \, d\omega_k \left\{ 2c_0 + \left(c_1 \frac{\omega_k + \omega_s}{\mu} - c_2 \frac{\omega_k - \omega_s}{\mu} \right) \Gamma \right\} \frac{H_s(k)}{\omega_k - \omega_s} \tag{114}$$

$$\overline{G}_s = - \frac{1}{2\pi\mu} \int k \, d\omega_k \left\{ 2c_0 + \left(c_1 \frac{\omega_k - \omega_s}{\mu} - c_2 \frac{\omega_k + \omega_s}{\mu} \right) \Gamma \right\} \frac{G_s(k)}{\omega_k + \omega_s} \tag{115}$$

and we have replaced summation over k by integration according to the relation

$$N^2 \sum_k \rightarrow 2/\pi \int k^2 \, d\omega_k = 2/\pi \int k \, \omega_k \, d\omega_k$$

(see Appendix A) and omitted $v^2(k)$ which is assumed to be equal to 1 up to some cut-off energy ω_{max} and equal to 0 for $\omega_k > \omega_{max}$.

We now assume that the function $H_s(r)$ can be approximated by taking

$$H_s(r) \cong \left(c_1 \frac{\omega_r - \omega_s}{\omega_r + \omega_s} + c_2\right)\Lambda + \overline{H}_s + \Delta\overline{G}_s \tag{116}$$

Eq. (116) is equivalent to taking in eq. (108) $\omega_r = \omega_s$ everywhere but in the first term ; therefore the error is proportional to $(\omega_r - \omega_s)$ as is clear from eq. (113). Thus for $(\omega_r - \omega_s)$ small this is a good approximation.

Using eqs. (111) and (116) in eqs. (114) and (115) we obtain $(\overline{H}_s + \Delta\overline{G}_s)$ from the following equation

$$\left\{1 + \frac{1}{2\pi\mu}\int \frac{k \, d\omega_k}{\omega_k - \omega_s}\left[2c_0 + \left(c_1 \frac{\omega_k + \omega_s}{\mu} - c_2 \frac{\omega_k - \omega_s}{\mu}\right)\Gamma\right]\right\}(\overline{H}_s + \Delta\overline{G}_s) =$$

$$- \frac{1}{2\pi\mu}\int k \, d\omega_k \left\{\frac{c_1^2 - c_2^2}{\mu}\Gamma\Lambda + \frac{c_1(c_1 - c_2)}{\mu}\Delta\Gamma\Delta\Lambda + \frac{2c_1}{\omega_k + \omega_s}\left(c_0\Lambda + \frac{c_2\omega_s}{\mu}\Gamma\Lambda\right)\right.$$

$$+ \frac{2}{\omega_k - \omega_s} \left[c_0 (c_1 + c_2) \Lambda + \frac{c_1 c_2 \omega_s}{\mu} (\Gamma\Lambda - \Delta\Gamma\Delta\Lambda) \right] \Bigg\} \qquad (117)$$

All the integrals in eq. (117) are easily evaluated. We take $\omega_{max} = 4.5\,\mu$, the same as Drell, Friedman and Zachariasen . The error caused by assuming that $H_s(r)$ and $G_s(r)$ are given by eqs. (116) and (111) instead of eqs. (108) and (109) is estimated by taking $H_s(r)$ and $G_s(r)$ as given by eqs. (116) and (111) , introducing these into eqs. (108) and (109) and comparing the new values of $H_s(r)$ and $G_s(r)$ so obtained with the old ones.

We find that $H_s(r)$ is a weak function of ω_r and ω_s and is well approximated in the small range of energies of interest by a constant , namely the value of $H_s(r)$ as obtained from eq. (116) with $\omega_r = \omega_s = (\omega_r + \omega_s)/2$. For $\omega_r + \omega_s = 2.5\,\mu$ the error (estimated in the manner outlined above) is less than 25% ; the error is smaller for smaller values of $\omega_r + \omega_s$. On the other hand the auxiliary function $G_s(r)$ is given by eq. (111) with an error of less than 1% .

We give below $H_s(r)$ for two choices of $\omega_r + \omega_s$; it is seen that within the narrow range of energies considered $H_s(r)$ varies very slowly.

$$\omega_r + \omega_s = \qquad 2.5\,\mu \qquad\qquad 2.25\,\mu$$

$$H_s(r) = \begin{pmatrix} -.241 \\ .060 \\ -.123 \\ -.007 \end{pmatrix} \qquad \begin{pmatrix} -.223 \\ .062 \\ -.114 \\ -.006 \end{pmatrix} \tag{118}$$

In order to calculate cross sections it is necessary to form actual physical states from the eigenstates of the total isospin so far considered. The following reactions are possible:

$$\begin{aligned}
p + \pi^+ &\rightarrow p + \pi^+ + \pi^\circ \\
&\rightarrow n + \pi^+ + \pi^+ \\[1ex]
p + \pi^\circ &\rightarrow p + \pi^\circ + \pi^\circ \\
&\rightarrow p + \pi^+ + \pi^- \\
&\rightarrow n + \pi^+ + \pi^\circ \\[1ex]
p + \pi^- &\rightarrow p + \pi^- + \pi^\circ \\
&\rightarrow n + \pi^- + \pi^+ \\
&\rightarrow n + \pi^\circ + \pi^\circ
\end{aligned} \tag{119}$$

In above we have assumed that the nucleon in the initial state is a proton - another 8 reactions are possible with the nucleon in the initial state being a neutron. If we replace the π° meson in the initial state by a photon we obtain the 6 possible photoproduction reactions. The transition amplitudes for all these 22 reactions can be obtained from $H_s(r)$

by forming the appropriate linear combinations of the four rows of the matrix $H_s(r)$. We demonstrate the procedure by calculating the 3 cases involving a proton and π° meson in the initial state (we choose this example since it will permit us to calculate later cross sections for both meso- and photo-production ; how to calculate the other cases will be obvious from this example.)

The initial state of the system is $|p\pi^\circ\rangle$ which in terms of isospin states can be written as

$$|p\pi^\circ\rangle = \frac{1}{\sqrt{3}}\left|\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{3}{2}\right\rangle \tag{120}$$

The three possible final states of the system are $|p\pi^\circ\pi^\circ\rangle$, $|p\pi^-\pi^+\rangle$ and $|n\pi^+\pi^\circ\rangle$ which in terms of isospin states can be written as

$$|p\pi^\circ\pi^\circ\rangle = \frac{1}{3}\left|\frac{1}{2}(\tfrac{1}{2})\right\rangle - \frac{\sqrt{2}}{3}\left|\frac{1}{2}(\tfrac{3}{2})\right\rangle + \frac{\sqrt{2}}{3}\left|\frac{3}{2}(\tfrac{1}{2})\right\rangle + \frac{1}{3}\sqrt{\frac{2}{5}}\left|\frac{3}{2}(\tfrac{3}{2})\right\rangle + \sqrt{\frac{2}{5}}\left|\frac{5}{2}(\tfrac{3}{2})\right\rangle \tag{121}$$

$$|p\pi^-\pi^+\rangle = -\frac{2}{3}\left|\frac{1}{2}(\tfrac{1}{2})\right\rangle + \frac{1}{3\sqrt{2}}\left|\frac{1}{2}(\tfrac{3}{2})\right\rangle + \frac{\sqrt{2}}{3}\left|\frac{3}{2}(\tfrac{1}{2})\right\rangle - \frac{2}{3}\sqrt{\frac{2}{5}}\left|\frac{3}{2}(\tfrac{3}{2})\right\rangle + \frac{1}{\sqrt{10}}\left|\frac{5}{2}(\tfrac{3}{2})\right\rangle \tag{122}$$

$$|n\pi^+\pi^\circ\rangle = -\frac{\sqrt{2}}{3}\left|\frac{1}{2}(\tfrac{1}{2})\right\rangle - \frac{1}{3}\left|\frac{1}{2}(\tfrac{3}{2})\right\rangle - \frac{2}{3}\left|\frac{3}{2}(\tfrac{1}{2})\right\rangle + \frac{1}{3\sqrt{5}}\left|\frac{3}{2}(\tfrac{3}{2})\right\rangle + \frac{1}{\sqrt{5}}\left|\frac{5}{2}(\tfrac{3}{2})\right\rangle \tag{123}$$

In eqs. (120 - 123) we omit the symbol for the z-component of the isospin and the quantity in brackets refers to the isospin of the subsystem. The subsystem is formed with the meson whose symbol appears next to the nucleon on the left hand side of the equations. Identifying the meson used to form the subsystem with the meson R , identifying the other meson with meson S and identifying the meson in the initial state with meson L , we have from eqs. (120 - 123) the following expressions for the transition amplitudes :

$$F^{\circ\circ} \equiv F^{p\pi^{\circ} \to p\pi^{\circ}\pi^{\circ}} = \tag{124}$$

$$\frac{1}{3\sqrt{3}} F_S^{\frac{1}{2}\frac{1}{2}}(r,\ell) - \frac{1}{3}\sqrt{\frac{2}{3}}\, F_S^{\frac{1}{2}\frac{3}{2}}(r,\ell) + \frac{2}{3\sqrt{3}} F_S^{\frac{3}{2}\frac{1}{2}}(r,\ell) + \frac{2}{3\sqrt{15}} F_S^{\frac{3}{2}\frac{3}{2}}(r,\ell)$$

$$F^{-+} \equiv F^{p\pi^{\circ} \to p\pi^{-}\pi^{+}} = \tag{125}$$

$$- \frac{2}{3\sqrt{3}} F_S^{\frac{1}{2}\frac{1}{2}}(r,\ell) + \frac{1}{3\sqrt{6}} F_S^{\frac{1}{2}\frac{3}{2}}(r,\ell) + \frac{2}{3\sqrt{3}} F_S^{\frac{3}{2}\frac{1}{2}}(r,\ell) - \frac{4}{3\sqrt{15}} F_S^{\frac{3}{2}\frac{3}{2}}(r,\ell)$$

$$F^{+\circ} \equiv F^{p\pi^{\circ} \to n\pi^{+}\pi^{\circ}} = \tag{126}$$

$$- \frac{1}{3}\sqrt{\frac{2}{3}}\, F_S^{\frac{1}{2}\frac{1}{2}}(r,\ell) - \frac{1}{3\sqrt{3}} F_S^{\frac{1}{2}\frac{3}{2}}(r,\ell) - \frac{2}{3}\sqrt{\frac{2}{3}}\, F_S^{\frac{3}{2}\frac{1}{2}}(r,\ell) + \frac{1}{3}\sqrt{\frac{2}{15}}\, F_S^{\frac{3}{2}\frac{3}{2}}(r,\ell)$$

The transition amplitudes as given by eqs. (124 - 126) would be the appropriate ones if the experimental set up for measuring cross sections were such as to detect states of definite angular momentum. The normal experimental techniques are however such as to measure mesons of definite linear, and not angular, momentum. Therefore we transform at this stage from spherical to plane waves .

Consider

$$\langle \tfrac{1}{2}\, \beta'\, \vec{r}\, \vec{s} \,|\, M \,|\, \tfrac{1}{2}\, \alpha' \vec{\ell} \rangle \tag{127}$$

This represents the matrix element of some quantity M taken between the following states (we supress here the isospin dependence) :

Initial state : nucleon described by a spin = $\frac{1}{2}$, z-component of the spin = α' and a meson described by a plane wave characterized by the momentum $\vec{\ell}$.

Final state : nucleon described by a spin = $\frac{1}{2}$, z-component of the spin = β' and two mesons described by plane waves characterized by the momenta $\vec{r}$ and $\vec{s}$.

Using the standard methods of transformation theory we now write

$$\langle \tfrac{1}{2}\beta'\vec{r}\vec{s} \,|M|\, \tfrac{1}{2}\alpha'\vec{\ell}\rangle =$$

$$\sum_{R'\rho'S'\sigma'J'j'J''j''L'\lambda'} (\tfrac{1}{2}\beta'\vec{r}\vec{s} \,|\, \tfrac{1}{2}\beta' R'\rho' S'\sigma') \, (\tfrac{1}{2}\beta' R'\rho' S'\sigma' \,|\, J'j')$$

$$\langle J'j' \,|\, M \,|\, J''j''\rangle \, (J''j'' |\, \tfrac{1}{2}\alpha' L'\lambda') \, (\tfrac{1}{2}\alpha' L'\lambda' \,|\, \tfrac{1}{2}\alpha'\vec{\ell}) \tag{128}$$

Eq. (128) provides the means for expressing a matrix element taken between states of definite angular momentum

- $\langle J'j' | M | J''j'' \rangle$ - in terms of the matrix element taken between states of definite linear momentum - $\langle \frac{1}{2}\beta'\vec{r}\vec{s} | M | \frac{1}{2}\alpha'\vec{\ell} \rangle$; the second and fourth terms on the right hand side of eq. (128) are simply vector addition coefficients, the first and last terms are inner products of plane and spherical waves.

For the processes that we have been considering $\langle J'j' | M | J''j'' \rangle$ is different from zero provided $J' = J'' = \frac{1}{2}$ and $j' = j''$. Then parity conservation together with angular momentum conservation require $R' = S' = 0$, $\rho' = \sigma' = 0$, $L' = 1$. Thus eq. (128) simplifies to

$$\langle \tfrac{1}{2}\beta'\vec{r}\vec{s} | M | \tfrac{1}{2}\alpha'\vec{\ell} \rangle = \sum_{j'\lambda'} (\tfrac{1}{2}\beta'\vec{r}\vec{s} | \tfrac{1}{2}\beta' 0000)\,(\tfrac{1}{2}\beta' 0000 | \tfrac{1}{2}j')$$
$$\langle \tfrac{1}{2}j' | M | \tfrac{1}{2}j' \rangle\,(\tfrac{1}{2}j' | \tfrac{1}{2}\alpha' 1\lambda')\,(\tfrac{1}{2}\alpha' 1\lambda' | \tfrac{1}{2}\alpha'\vec{\ell}) \qquad (129)$$

Here $\langle \frac{1}{2}j' | M | \frac{1}{2}j' \rangle$ is independent of j' and equal to F as given by eqs. (124 - 126), i.e. $\langle \frac{1}{2}j' | M^{oo} | \frac{1}{2}j' \rangle = F^{oo}$ etc. . The vector addition coefficients are given by

$$(\tfrac{1}{2}\beta' 0000 | \tfrac{1}{2}j') = \delta_{\beta', j'} \qquad (130)$$

and

$$(\tfrac{1}{2}j' | \tfrac{1}{2}\alpha' 1\lambda') = \delta_{j', \alpha'+\lambda'}\,(-)^{\alpha'-\frac{1}{2}}\sqrt{\frac{1+|j'-\alpha'|}{3}} \qquad (131)$$

The inner products of plane and spherical waves are given by (see Appendix A):

$$(\tfrac{1}{2}\beta'\vec{r}\vec{s} \,|\, \tfrac{1}{2}\beta' 0000) = 4\pi/N^2V \qquad (132)$$

and

$$(\tfrac{1}{2}\alpha' 1\lambda' \,|\, \tfrac{1}{2}\alpha'\vec{\ell}) = 4\pi/N\sqrt{V} \quad (-i)(-)^{\lambda'} \; Y_{1-\lambda'}(\Omega_\ell) \qquad (133)$$

Thus eq. (129) becomes

$$\langle \tfrac{1}{2}\beta'\vec{r}\vec{s} \,|M|\, \tfrac{1}{2}\alpha'\vec{\ell} \rangle =$$

$$(-i)(-)^{\beta'-\frac{1}{2}} \; \frac{\sqrt{1+|\beta'-\alpha'|}}{\sqrt{3}} \; \frac{(4\pi)^2}{N^3V^{3/2}} \; Y_{1(\alpha'-\beta')}(\Omega_\ell) \quad F \qquad (134)$$

Combining eqs. (134), (124 - 126), (106) and (65) we have

$$\langle \tfrac{1}{2}\beta'\vec{r}\vec{s} \,|M^{oo}|\, \tfrac{1}{2}\alpha'\vec{\ell} \rangle = X\, H^{oo} \qquad (135)$$

$$\langle \tfrac{1}{2}\beta'\vec{r}\vec{s} \,|M^{-+}|\, \tfrac{1}{2}\alpha'\vec{\ell} \rangle = X\, H^{-+} \qquad (136)$$

$$\langle \tfrac{1}{2}\beta'\vec{r}\vec{s} \,|M^{+o}|\, \tfrac{1}{2}\alpha'\vec{\ell} \rangle = X\, H^{+o} \qquad (137)$$

where

$$H^{oo} = \tfrac{1}{3}\left[H_s^{\frac{1}{2}\frac{1}{2}}(r) - 2\, H_s^{\frac{1}{2}\frac{3}{2}}(r) + 2\, H_s^{\frac{3}{2}\frac{1}{2}}(r) + 2\, H_s^{\frac{3}{2}\frac{3}{2}}(r)\right] \qquad (135')$$

$$H^{-+} = \tfrac{1}{3}\left[-2\, H_s^{\frac{1}{2}\frac{1}{2}}(r) + H_s^{\frac{1}{2}\frac{3}{2}}(r) + 2\, H_s^{\frac{3}{2}\frac{1}{2}}(r) - 4\, H_s^{\frac{3}{2}\frac{3}{2}}(r)\right] \qquad (136')$$

$$H^{+o} = -\tfrac{\sqrt{2}}{3}\left[H_s^{\frac{1}{2}\frac{1}{2}}(r) + H_s^{\frac{1}{2}\frac{3}{2}}(r) + 2\, H_s^{\frac{3}{2}\frac{1}{2}}(r) - H_s^{\frac{3}{2}\frac{3}{2}}(r)\right] \qquad (137')$$

and where

$$X = (-i)(-)^{\beta'-\frac{1}{2}} \; \frac{\sqrt{1+|\beta'-\alpha'|}}{\sqrt{3}} \; \frac{(4\pi)^3}{\mu^3V^{3/2}} \; \frac{f}{12}$$

$$v(s)\; v(r)\; v(\ell)/\sqrt{2\omega_s\omega_r\omega_\ell} \;\; \ell \; Y_{1(\alpha'-\beta')}(\Omega_\ell) \qquad (138)$$

Both values of α' are equally likely and we average over them; also we must sum over β' . Using a bar to denote the averaging we have

$$\sum_{\beta'} \overline{X} = \frac{(-i)}{2} \frac{(4\pi)^2}{\mu^3 V^{3/2}} \frac{f}{12} \frac{v(s)\ v(r)\ v(\ell)}{\sqrt{2\omega_s \omega_r \omega_\ell}}\ \ell \left[\sqrt{\tfrac{1}{3}}\ Y_{10}(\Omega_\ell) - \sqrt{\tfrac{2}{3}}\ Y_{11}(\Omega_\ell)\right]$$

and

$$\left|\sum_{\beta'} \overline{X}\right|^2 = \frac{1}{4} \frac{(4\pi)^3}{\mu^6 V^3} \frac{f^2}{12^2} \frac{v^2(s)v^2(r)v^2(\ell)}{2\omega_s \omega_r \omega_\ell}\ \ell^2 \left[1 + 2\sin\vartheta_\ell \cos\vartheta_\ell \cos\varphi_\ell\right]$$

$$\longrightarrow \frac{(4\pi)^3}{\mu^6 V^3} \frac{f^2}{12^2} \frac{\ell^2}{8\omega_s \omega_r \omega_\ell} \qquad (139)$$

where we have averaged over φ_ℓ and omitted the form factors $v^2(s)v^2(r)v^2(\ell)$ to obtain the last line.

The cross section is given by multiplying (139) by

$$2\pi\ |H|^2 \ \ d^2n/dE\ \ \mu V/\ell \qquad (140)$$

where H is given by eqs. (135' - 137'), where $\ell/\mu V$ is the flux of the incident mesons and d^2n/dE is the density of final states per unit energy interval. This last quantity is obtained as follows : the number of states available to a meson of energy ω_r lying within the interval $d\omega_r$, within the solid angle $d\Omega_r$ when the mesons are quantized within a cube of volume V is

$$V/(2\pi)^3 \quad r\,\omega_r\, d\omega_r\, d\Omega_r \tag{141}$$

For two mesons the number of states is a product of two factors each of the form (141) . Now the total energy available to the two mesons is

$$E = \omega_r + \omega_s$$

and therefore $\left(\dfrac{\partial \omega_s}{\partial E}\right)_{\omega_r = \text{const}} dE = dE$. Thus the required density of final states per unit energy interval is

$$\frac{d^4 n}{dE} = V^2/(2\pi)^6 \quad r\, s\, \omega_r\, \omega_s\, d\omega_r\, d\Omega_r\, d\Omega_s$$

or

$$\frac{d^2 n}{dE} = 4V^2/(2\pi)^4 \quad r\, s\, \omega_r\, \omega_s\, d\omega_r \tag{142}$$

where we have integrated over $d\Omega_r$ and $d\Omega_s$ (nothing in the cross section depends on these angles so the integration simply gives $(4\pi)^2$).

Thus finally we have for the cross section per unit energy of the meson R :

$$\begin{aligned} d\sigma/d\omega_r &= (f/6)^2\, H^2\, (\ell r s)/(\omega_\ell\, \mu^5) \\ &= (f/6)^2\, H^2/\mu^3\, \ell/\omega_\ell \sqrt{\left(\frac{\omega_r^2}{\mu^2}-1\right)\left[\frac{(E-\omega_r)^2}{\mu^2}-1\right]} \end{aligned} \tag{143}$$

From eq.(143) we immediately obtain the cross section for the corresponding photoproduction process by multiplying by

$$\left(\frac{(g_p - g_n)/2 \quad e/2M}{\sqrt{4\pi} \quad f/\mu \quad \sqrt{\ell/\omega_\ell}}\right)^2 \frac{\ell}{\mu} \tag{144}$$

where ℓ/μ arises from the different forms of the expressions for the flux of mesons as compared with photons and the other factor follows from eq. (105) .

The value of E (= energy available to the two produced mesons) will be taken equal to ω_ℓ , this corresponds to neglecting any recoil of the nucleon .

As is seen from eq. (143) the ω_r-dependence of $d\sigma/d\omega_r$ is due entirely to the term under the square root sign (this is a result of having approximated $H_r(s)$ by a constant in view of its weak dependence on ω_r , ω_s) . This term goes to zero at the two limits $\omega_r = \mu$ and $E - \omega_r = \omega_s = \mu$, and reaches its peak value of $\left[(E/2\mu)^2 - 1\right]^2$ at $\omega_r = \omega_s = E/2$. Using eq. (118) to obtain numerical values for H^2 and replacing the term under the square root sign by its peak value we obtain as the maximum value of $d\sigma/d\omega_r$ in millimicrobarns (10^{-33} cm^2) per Mev the following

$d\sigma/d\omega_r$ \ E	2.5μ	2.25μ
$p\pi^0 \rightarrow p\pi^0\pi^0$	4.7	2.0
$p\pi^0 \rightarrow p\pi^-\pi^+$	1.4	.54
$p\pi^0 \rightarrow n\pi^+\pi^0$	4.5	1.7

(145)

The corresponding photoproduction cross sections are obtained by multiplying eq. (145) by the factor (144) :

$d\sigma/d\omega_r$ \ E	2.5μ	2.25μ
$p\gamma \rightarrow p\pi^0\pi^0$	20×10^{-3}	7.8×10^{-3}
$p\gamma \rightarrow p\pi^-\pi^+$	6.1×10^{-3}	2.1×10^{-3}
$p\gamma \rightarrow n\pi^+\pi^0$	20×10^{-3}	6.6×10^{-3}

(146)

There are as yet no data available with which to compare the numbers (145) , (146) . Some experimental data exist for higher energies, in particular for the reaction $p + \gamma \rightarrow p + \pi^- + \pi^+$ [14] . Cutkosky and Zachariasen[5] obtain a good fit to these data by using the Chew and Low theory (no interactions for s-waves) and assuming that one of the mesons is produced in an S-state and one in a P-state . Cutkosky and Zachariasen obtain for $p\gamma \rightarrow p\pi^-\pi^+$ 250 millimicrobarns/Mev at $E = 2.5\mu$ and for $p\gamma \to n\pi^+\pi^0$ 50 millimicrobarns/ Mev at $E = 2.5\mu$. Our corresponding numbers are roughly 10,000 times smaller for $p\gamma \to p\pi^-\pi^+$ and 1,000 times smaller for $p\gamma \to n\pi^+\pi^0$. One reason why our numbers are so much smaller is that in order to produce two s-wave mesons the photon must be absorbed by the nucleon, whereas an s- and a p-wave meson can be produced by having one of the mesons absorb the photon. Thus, even if the s-wave and p-wave interactions were equally strong our process would lead to cross sections smaller by a factor $(M/\mu)^2 \approx 50$. The fact that the cross sections turn out to be much smaller must be blamed on the weakness of s-wave interactions as compared with p-wave interactions. Although phase space does favor the production of 2 s-waves over an s-wave and a p-wave, it does not compensate sufficiently the difference in strength of the interactions, except possibly at an energy of a few electron-volts above threshold.

APPENDIX A

The free meson field amplitudes satisfy the Klein - Gordon equation and therefore may be expanded in terms of a complete set of solutions of the Klein - Gordon equation such as plane waves or spherical waves. The standard expansion is in terms of the orthonormal set

$$1/\sqrt{V} \quad e^{i\vec{k}\cdot\vec{x}} \tag{A-1}$$

where the plane waves are normalized in a cube of volume V . Periodic boundary conditions at the faces of the cube lead to discrete values for the momentum $\vec{k}$. The number of eigen-states in a volume element $d\vec{k}$ of momentum space is then

$$V/(2\pi)^3 \quad d\vec{k} \tag{A-2}$$

The expansion of the field amplitudes in terms of the set (A-1) has the familiar form

$$\phi_q(\vec{x}) = \sum_{\vec{k}} 1/\sqrt{2V\omega_k}\left[\alpha_q(\vec{k})e^{i\vec{k}\cdot\vec{x}} + \alpha_q^{\dagger}(\vec{k})e^{-i\vec{k}\cdot\vec{x}}\right]; \quad q = -1,0,+1. \tag{A-3}$$

Similarly we have for the canonically conjugate momenta

$$\pi_q(\vec{x}) = (-i)\sum_{\vec{k}} \sqrt{\omega_k/2V}\left[\alpha_q(k)e^{i\vec{k}\cdot\vec{x}} - \alpha_q^{\dagger}(\vec{k})e^{-i\vec{k}\cdot\vec{x}}\right]; \quad q = -1,0,+1. \tag{A-4}$$

Here $\omega_k = \sqrt{k^2 + \mu^2}$ = energy of a meson of momentum k .

Whereas the space dependence of $\phi_q(\vec{x})$ is in the $e^{\pm i\vec{k}\cdot\vec{x}}$, the operator properties of $\phi_q(\vec{x})$ are in $a_q(\vec{k})$, $a_q^{+}(\vec{k})$. As a consequence of the definition (9) $\phi_q(\vec{x})$ describes mesons of definite charge : the operator $a_q(k)$ annihilates a meson of charge = -qe , the operator $a_q^{+}(k)$ creates a meson of charge = + qe. Thus ϕ_q acting on an eigenstate of the charge operator leads to another eigenstate to an eigenvalue increased by the amount qe .

Introducing the expansions (A-3,4) into eqs. (2,3,4) yields

$$H_\circ = \sum_{k} \omega_k \sum_{q} (-)^q \, a_q^{+}(\vec{k})\, a_{-q}(\vec{k}) \qquad \text{(A-5)}$$

$$H' = f^\circ/\mu \sum_{k} \sqrt{2\pi/V\omega_k}\; v(k)\; \vec{\sigma}\cdot i\vec{k} \sum_{q} (-)^q \tau_q \left[a_{-q}(\vec{k}) - a_{-q}^{+}(\vec{k})\right]$$

$$+ \lambda_\circ^\circ/2\mu V \sum_{\vec{k},\vec{k}'} v(k)v(k')/\sqrt{\omega_k\omega_{k'}} \;\cdot$$

$$\sum_{q}(-)^q \left[a_q(\vec{k})a_{-q}(\vec{k}') + a_q(\vec{k})a_{-q}^{+}(\vec{k}') + a_q^{+}(\vec{k})a_{-q}(\vec{k}') + a_q^{+}(\vec{k})a_{-q}^{+}(\vec{k}')\right]$$

$$+ \lambda^\circ/2\mu^2 V \sum_{\vec{k},\vec{k}'} v(k)v(k')/\sqrt{\omega_k\omega_{k'}} \;\cdot$$

$$\left[(\omega_k - \omega_{k'}) \sum_{q} \tau_q \left(a_r(\vec{k})a_s(\vec{k}') - a_s^{+}(\vec{k}')a_r^{+}(\vec{k})\right)\right.$$

$$\left. + (\omega_k + \omega_{k'}) \sum_{q} \tau_q \left(a_r(\vec{k})a_s^{+}(\vec{k}') - a_s(\vec{k}')a_r^{+}(\vec{k})\right)\right] \qquad \text{(A-6)}$$

(q, r, s = cyclic permutations of -1, 0, +1)

In eq. (A-6) $v(k)$ is the Fourier transform of the source function :

$$v(k) = \int u(x)\, e^{\pm i\vec{k}\cdot\vec{x}}\, d\vec{x} \quad ; \tag{A-7}$$

τ_q , $q = -1, 0, +1$, is related to τ_α , $\alpha = 1, 2, 3$, in the same way as ϕ_q is related to ϕ_α (eq. (9)) , the τ_α's being the standard Pauli spin matrices acting in isospin space.

Now consider the following orthonormal set

$$N\, j_\ell(px)\, Y_{\ell m}\,(\Omega_x) \tag{A-8}$$

where $j_\ell(px)$ is a spherical Bessel function and $Y_{\ell m}\,(\Omega_x)$ is a spherical harmonic and N is a normalization constant. These spherical waves also represent solutions of the Klein - Gordon equation and we may therefore expand the field amplitudes in terms of (A-8) . Quantization now must be performed in a spherical volume of radius $R = 2p^2/N^2$. Discrete values of p result from imposing the boundary condition that $j_\ell(px)$ vanish for $x = R$. A spherical Bessel function $j_\ell(px)$ approaches asymptotically for large values of its argument $1/px \cos\left[px - (\ell + 1)\pi/2\right]$ (15), hence the boundary condition is $pR = \pi\ell/2$. The number of eigenstates in a volume element dp of momentum space is then

$$R/\pi \;\; dp = 2/\pi N^2 \;\; p^2 \; dp \tag{A-9}$$

The expansion of the field amplitudes in terms of the set

(A-8) has the form

$$\phi_q(\vec{x}) = \sum_{p\ell m} N/\sqrt{2\omega_p}\ (-)^m\ j_\ell(px)\ Y_{\ell -m}(\Omega_x)\ \left[a_q(p\ell m) + a_q^\dagger(p\ell m)\right] \quad \text{(A-10)}$$

Whereas the operators $a_q(\vec{k})$, $a_q^\dagger(\vec{k})$ served to annihilate and create mesons whose space dependence was that of a plane wave, the operators in (A-10) annihilate and create mesons with a space dependence corresponding to spherical waves. Thus $a_q(p\ell m)$ annihilates a meson of momentum magnitude = p, angular momentum magnitude = $\sqrt{\ell(\ell+1)}$, and z-component of angular momentum = - m ; $a_q^\dagger(p\ell m)$ creates a similar meson except with z-component of angular momentum = + m . The choice of phases of these operators is such that they behave as irreducible tensors in the sense of Racah(10) under rotations in both space and isospace. Under rotations in isospace $\phi_q(\vec{x})$ behaves as the q-component of a tensor of rank 1 since the π-mesons are members of an isotopic triplet ; in the expansion (A-10) this property is preserved by having $a_q(p\ell m)$, $a_q^\dagger(p\ell m)$, behave as the q-component of a tensor of rank 1 under such rotations. Under rotations in space $\phi_q(\vec{x})$ behaves as a scalar since π-mesons are pseudoscalars ; in the expansion (A-10) this property is preserved by having $a_q(p\ell m)$, $a_q^\dagger(p\ell m)$ behave as the m-component of a tensor of rank ℓ under such rotations - hence the scalar product

$$\sum_m (-)^m\ Y_{\ell,-m} \left[a_q(p\ell m) + a_q^\dagger(p\ell m)\right] \quad \text{(A-11)}$$

behaves as a scalar under such rotations . The pseudoscalar nature of the π-meson manifests itself in the fact that $\alpha_q(p\ell m)$, $\alpha_q^\dagger(p\ell m)$ behave as pseudotensors under inversions in space . Lastly, our choice of phases implies that the hermitian conjugate of $\alpha_q(p\ell m)$ is $(-)^m \alpha_q^\dagger(p\ell m)$. We note that in this notation the only nonvanishing commutator is

$$\left[\alpha_q(p\ell m),\ \alpha^\dagger_{q'}(p'\ell'm')\right] = (-)^q(-)^m\,\delta_{q,-q'}\,\delta_{m,-m'}\,\delta_{\ell,\ell'}\,\delta_{p,p'} \tag{A-12}$$

Comparing (A-10) and (A-3) we see that the annihilation and creation operators in the two expansions are related by

$$\sum_{\vec k} 1/\sqrt{2V\omega_k}\;\; a_q(\vec k)\, e^{i\vec k\cdot\vec x} = \sum_{p\ell m} N/\sqrt{2\omega_p}\;(-)^m\, j_\ell(px)\, Y_{\ell -m}(\Omega_x)\, a_q(p\ell m) \tag{A-13}$$

$$\sum_{\vec k} 1/\sqrt{2V\omega_k}\;\; a^\dagger_q(\vec k)\, e^{-i\vec k\cdot\vec x} = \sum_{p\ell m} N/\sqrt{2\omega_p}\;(-)^m\, j_\ell(px)\, Y_{\ell -m}(\Omega_x)\, a^\dagger_q(p\ell m) \tag{A-14}$$

Now

$$\int e^{i(\vec k-\vec k')\cdot\vec x}\, d\vec x = V\,\delta_{\vec k,\vec k'} \tag{A-15}$$

and

$$\int e^{-i\vec k'\cdot\vec x}\, j_\ell(px)\, Y_{\ell -m}(\Omega_x)\, d\vec x$$

$$= 4\pi \sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'} (-i)^{\ell'} \int j_{\ell'}(k'x)\, j_\ell(px)\, x^2\, dx \cdot$$

$$Y_{\ell'm'}(\Omega_{k'}) \int Y^*_{\ell'm'}(\Omega)\, Y_{\ell -m}(\Omega)\; d\Omega$$

$$= 4\pi\, (-i)^\ell\, N^{-2}\, Y_{\ell -m}(\Omega_{k'})\, \delta_{k',p} \tag{A-16}$$

because

$$\int Y^*_{\ell' m'}(\Omega)\; Y_{\ell -m}(\Omega)\; d\Omega = \delta_{\ell',\ell}\;\delta_{m',-m} \tag{A-17}$$

and

$$\int j_\ell(k'x)\; j_\ell(px)\; x^2\; dx = N^{-2}\;\delta_{k',p} \tag{A-18}$$

Hence if we multiply both sides of eqs. (A-13,14) by $e^{-i\vec{k}'\cdot\vec{x}}$ and integrate over $\vec{x}$ using eqs. (A-15,16) we obtain

$$a_q(\vec{k}) = \sum_{p\ell m} 4\pi/N\sqrt{V}\;(-i)^\ell(-)^m\; Y_{\ell -m}(\Omega_k)\; a_q(p\ell m)\;\delta_{k,p} \tag{A-19}$$

$$a^\dagger_q(\vec{k}) = \sum_{p\ell m} 4\pi/N\sqrt{V}\;(i)^\ell(-)^m\; Y_{\ell -m}(\Omega_k)\; a^\dagger_q(p\ell m)\;\delta_{k,p} \tag{A-20}$$

We now want to introduce (A-19,20) into (A-5,6) and perform the required summations over $\vec{k}$ and $\vec{k}'$. As a consequence of (A-2) we see that the summation over a discrete spectrum can be replaced by integration over a continuous spectrum according to

$$\sum_{\vec{k}} \rightarrow V/(2\pi)^3 \int d\vec{k} = V/(2\pi)^3 \int k^2\; dk \int d\Omega_k \tag{A-21}$$

On the other hand from (A-9) we have

$$\sum_{k} \rightarrow 2/\pi N^2 \int k^2\; dk \tag{A-22}$$

hence we conclude

$$\sum_{\vec{k}} \rightarrow VN^2/(4\pi)^2 \sum_{k} \int d\Omega_k \tag{A-23}$$

Using eqs. (A-19, 20, 23) in eq. (A-5) we obtain

$$H_\circ = VN^2/(4\pi)^2 \sum_k \int d\Omega_k \, \omega_k \sum_q (-)^q$$

$$\cdot \sum_{p\ell m} 4\pi/N\sqrt{V} \, (-)^m (i)^\ell \, Y_{\ell -m}(\Omega_k) \, a_q^\dagger(p\ell m) \, \delta_{k,p}$$

$$\cdot \sum_{p'\ell' m'} 4\pi/N\sqrt{V} \, (-)^{m'} (-i)^{\ell'} \, Y_{\ell' -m'}(\Omega_k) \, a_{-q}(p'\ell' m') \, \delta_{k,p'}$$

$$= \sum_{p\ell m} \omega_p \sum_q (-)^q \, (-)^m \, a_q^\dagger(p\ell m) \, a_{-q}(p\ell -m) \qquad \text{(A-24)}$$

since

$$\int d\Omega_k \, Y_{\ell -m}(\Omega_k) \, Y_{\ell' -m'}(\Omega_k) = (-)^m \, \delta_{\ell,\ell'} \, \delta_{m,-m'} \qquad \text{(A-25)}$$

Eq. (A-24) is equivalent to eq. (10) in the text.

Next we use eqs. (A-19, 20, 23) in eq. (A-6) . For the term proportional to f° we find :

$$f^\circ/\mu \sum_k \int d\Omega_k \, N/\sqrt{8\pi\omega_k} \, v(k) \, \vec{\sigma}\cdot i\vec{k} \, \sum_q (-)^q \tau_q \qquad \text{(A-26)}$$

$$\cdot \sum_{p\ell m} (-)^m \, (-i)^\ell \, Y_{\ell -m}(\Omega_k) \, \delta_{k,p} \left[a_{-q}(p\ell m) - (-)^\ell a_{-q}^\dagger(p\ell m) \right]$$

$$= f^\circ/\mu \sum_p N/\sqrt{3} \, pv(p)/\sqrt{2\omega_p} \sum_{q,m} (-)^q \, (-)^m \tau_q \sigma_m \left[a_{-q}(p1-m) + a_{-q}^\dagger(p1-m) \right]$$

where σ_m, $m = -1, 0, +1$, is related to σ_α, $\alpha = 1, 2, 3$, in the same way as ϕ_q is related to ϕ_α (eq. 9), the σ_α's being the standard Pauli matrices . Thus this term is responsible for the interaction with the nucleon of mesons in the P-state only.

For the term proportional to λ_o^o we find :

$$\lambda_o^o/2\mu \sum_q (-)^q \sum_{kk'} \int\int d\Omega_k\, d\Omega_{k'}\; v(k)\; v(k')/\sqrt{\omega_k \omega_{k'}}$$

$$\cdot \sum_{\substack{p\ell m \\ p'\ell'm'}} N^2/(4\pi)^2\; (-)^{m+m'}\; (-1)^{\ell+\ell'}\; Y_{\ell -m}(\Omega_k)\; Y_{\ell' -m'}(\Omega_{k'})\; \delta_{k,p}\; \delta_{k',p'}$$

$$\cdot \Big[a_q(p\ell m)\, a_{-q}(p'\ell'm') + (-)^{\ell'} a_q(p\ell m)\, a^{\dagger}_{-q}(p'\ell'm') +$$

$$(-)^{\ell} a^{\dagger}_q(p\ell m)\, a_{-q}(p'\ell'm') + (-)^{\ell+\ell'} a^{\dagger}_q(p\ell m)\, a_{-q}(p'\ell'm') \Big]$$

$$= \lambda_o^o/\mu \sum_{pp'} N^2/4\pi \quad v(p)\; v(p')/\sqrt{4\omega_p \omega_{p'}} \;\; \sum_q (-)^q \qquad \text{(A-27)}$$

$$\cdot \Big[a_q(p) a_{-q}(p') + a_q(p) a^{\dagger}_{-q}(p') + a^{\dagger}_q(p) a_{-q}(p') + a^{\dagger}_q(p) a^{\dagger}_{-q}(p') \Big]$$

where $a_q(p) \equiv a_q(p00)$.

Finally for the term proportional to λ^o we find in an analogous manner

$$\lambda^\circ/\mu^2 \sum_{pp'} N^2/4\pi \quad v(p)\, v(p')/\sqrt{4\omega_p \omega_{p'}}$$

$$\cdot\Big\{(\omega_p - \omega_{p'}) \sum_q \tau_q \left[a_r(p)\, a_s(p') - a_s^\dagger(p')\, a_r^\dagger(p)\right]$$

$$+(\omega_p + \omega_{p'}) \sum_q \tau_q \left[a_r(p)\, a_s^\dagger(p') - a_s(p') a_r^\dagger(p)\right]\Big\} \qquad \text{(A-28)}$$

(q, r, s = cyclic permutations of -1, 0, +1)

Thus (A-27, 28) is responsible for the interaction with the nucleon of mesons in the S-state only.

The sum of (A-26), (A-27) and (A-28) is equivalent to eq. (11) in the text.

APPENDIX B

We wish to eliminate from eqs. (57) and (58) the dependence on the magnetic quantum numbers since they describe merely the geometry and not the physics of the problem. It was just for the purpose of eliminating the magnetic quantum numbers that the quantities F, E, M, N and ϑ were introduced because they are scalars.

We present the proof that $M_r(s)$ is a scalar - the proof is the same for F, E, N and ϑ. Consider the defining equation (53) :

$$\langle C\gamma S\sigma | M_r(s) | B\beta R\rho \rangle = \langle C\gamma S\sigma | [H', a^{+}_{\rho}(r)] | B\beta \rangle \tag{53}$$

where we write out in detail all the quantum numbers in isospin space but ignore the quantum numbers referring to the angular momentum - we shall carry out the proof for isospace only, the method is the same for regular space. In this particular case $C = B = \frac{1}{2}$ and $S = R = 1$ but we carry out the proof in general, independent of numerical values. The left hand side of eq. (53) may be written as

$$\langle C\gamma S\sigma | M_r(s) | B\beta R\rho \rangle$$

$$= \sum_{T\tau T'\tau'} (CS\gamma\sigma | CST\tau)\ (BR\beta\rho | BRT'\tau')\ \langle T\tau | M_r(s) | T'\tau' \rangle$$

$$= \sum_{T\tau T'\tau'} (-)^{T+\tau+T'+\tau'} \sqrt{2T+1}\,\sqrt{2T'+1}\;\; V(CST\,;\gamma\sigma-\tau)$$

$$V(BRT'\,;\beta\rho-\tau')\;\langle T\tau|M_r(s)|T'\tau'\rangle \qquad \text{(B-1)}$$

where $(CS\gamma\sigma|CST\tau)$ is the standard vector addition coefficient as defined by Condon and Shortley[16] and where $V(CST;\gamma\sigma-\tau)$ is again the vector addition coefficient but with phase and normalization as defined by Racah[10] .

On the other hand the right hand side of eq. (53) may be written as follows : we observe that H' is a scalar and therefore the tensor properties of the commutator $[H', \overset{+}{a}_\rho(r)]$ are the same as those of $\overset{+}{a}_\rho(r)$. But the latter has been so defined as to behave as the ρ-component of a tensor of rank R $(=1)$ under rotations in isospace. Therefore, as shown by Racah

$$\langle C\gamma S\sigma|[H', \overset{+}{a}_\rho(r)]|B\beta\rangle$$

$$= \sum_{T\tau} (-)^{T+\tau} \sqrt{2T+1}\;\; V(CST;\gamma\sigma-\tau)\;\;\langle T\tau|[H', \overset{+}{a}_\rho(r)]|B\beta\rangle$$

$$= \sum_{T\tau} (-)^{2R} \sqrt{2T+1}\;\; V(CST;\gamma\sigma-\tau)\; V(BRT;\beta\rho-\tau)\;\langle T\|[H',\overset{+}{a}(r)]\|B\rangle \qquad \text{(B-2)}$$

where $\langle T\|[H',\overset{+}{a}(r)]\|B\rangle$ is the reduced matrix element of $\langle T\tau|[H', \overset{+}{a}_\rho(r)]|B\beta\rangle$ as defined by Racah - a quantity independent of all magnetic quantum numbers . Comparing (B-1) and (B-2) we conclude that

$$\langle T\tau | M_r(s) | T'\tau' \rangle = \delta_{T,T'} \, \delta_{\tau,\tau'} \, (2T+1)^{-\frac{1}{2}} (-)^{2R} \langle T \| [H', \overset{+}{a}(r)] \| B \rangle \tag{B-3}$$

Thus $M_r(s)$ connects only states with the same total isotopic spins and the same z-components ; also $\langle T\tau | M_r(s) | T\tau \rangle$ is independent of all magnetic quantum numbers. Hence $M_r(s)$ is a scalar.

Having established that F, E, M, N and ϑ are scalars we proceed to the elimination of the magnetic quantum numbers and deal first with the angular momentum magnetic quantum numbers. Suppressing the isospin dependence and writing out explicitly the angular momentum dependence we find that the various states in eqs. (57), (58) become :

$$|A\rangle = |AR\rangle = |\tfrac{1}{2}\alpha'\rangle$$

$$|B\rangle = |BR\rangle = |BS\rangle = |BRS\rangle = |BSR\rangle = |\tfrac{1}{2}\beta'\rangle$$

$$|C\rangle = |CR\rangle = |CS\rangle = |CK\rangle = |CSR\rangle = |CKS\rangle = |CKR\rangle = |\tfrac{1}{2}\gamma'\rangle$$

$$|AL\rangle = |ARL\rangle = |\tfrac{1}{2}\alpha' 1 \lambda'\rangle = \sum_{Jj} (-)^{J+j} \sqrt{2J+1} \, V(\tfrac{1}{2}1J; \alpha' \lambda' -j) \, |Jj\rangle$$

$$|CL\rangle = |CKL\rangle = |\tfrac{1}{2}\gamma' 1 \lambda'\rangle = \sum_{Jj} (-)^{J+j} \sqrt{2J+1} \, V(\tfrac{1}{2}1J; \gamma' \lambda' -j) \, |Jj\rangle \tag{B-4}$$

Using (B-4) we have for eq. (57)

$$\sum_{Jj} (-)^{J+j} \sqrt{2J+1} \, V(\tfrac{1}{2}1J; \alpha' \lambda' -j) \, \langle \tfrac{1}{2}\beta' | F_s(r,\ell) | Jj \rangle$$

$$= \sum_{\gamma' J j} \frac{(-)^{J+j}\sqrt{2J+1}\ V(\tfrac{1}{2}1J;\alpha'\lambda'-j)\ \langle \tfrac{1}{2}\beta'|N_r(s)|\tfrac{1}{2}\gamma'\rangle\langle\tfrac{1}{2}\gamma'|\vartheta(\ell)|Jj\rangle}{\omega_r+\omega_s}$$

$$- \sum_{\gamma' J j} \frac{(-)^{J+j}\sqrt{2J+1}\ V(\tfrac{1}{2}1J;\gamma'\lambda'-j)\ \langle \tfrac{1}{2}\beta'|\vartheta(\ell)|Jj\rangle\langle\tfrac{1}{2}\gamma'|N_r(s)|\tfrac{1}{2}\alpha'\rangle}{\omega_r+\omega_s}$$

$$- \sum_{k\gamma' J j} \frac{(-)^{J+j}\sqrt{2J+1}\ V(\tfrac{1}{2}1J;\alpha'\lambda'-j)\ \langle \tfrac{1}{2}\gamma'|M_r(k)|\tfrac{1}{2}\beta'\rangle^*\langle\tfrac{1}{2}\gamma'|F_s(k,\ell)|Jj\rangle}{\omega_k-\omega_r-i\epsilon}$$

$$- \sum_{k\gamma' J j} \frac{(-)^{J+j}\sqrt{2J+1}\ V(\tfrac{1}{2}1J;\gamma'\lambda'-j)\ \langle \tfrac{1}{2}\beta'|E_s(k,\ell)|Jj\rangle\langle\tfrac{1}{2}\gamma'|N_r(k)|\tfrac{1}{2}\alpha'\rangle}{\omega_k+\omega_r} \qquad \text{(B-5)}$$

Taking into consideration that F, E, M, N and ϑ are scalars, eq. (B-5) reduces to

$$(-)^{\frac{1}{2}+\beta'}\sqrt{2}\ V(\tfrac{1}{2}1\tfrac{1}{2};\alpha'\lambda'-\beta')\ \langle\tfrac{1}{2}\beta'|F_s(r,\ell)|\tfrac{1}{2}\beta'\rangle$$

$$= (-)^{\frac{1}{2}+\beta'}\sqrt{2}\ V(\tfrac{1}{2}1\tfrac{1}{2};\alpha'\lambda'-\beta')$$

$$\cdot\Big\{\langle\tfrac{1}{2}\beta'|N_r(s)|\tfrac{1}{2}\beta'\rangle\langle\tfrac{1}{2}\beta'|\vartheta(\ell)|\tfrac{1}{2}\beta'\rangle/(\omega_r+\omega_s)$$

$$-\langle\tfrac{1}{2}\beta'|\vartheta(\ell)|\tfrac{1}{2}\beta'\rangle\langle\tfrac{1}{2}\alpha'|N_r(s)|\tfrac{1}{2}\alpha'\rangle/(\omega_r+\omega_s)$$

$$-\sum_k\langle\tfrac{1}{2}\beta'|M_r(k)|\tfrac{1}{2}\beta'\rangle^*\langle\tfrac{1}{2}\beta'|F_s(k,\ell)|\tfrac{1}{2}\beta'\rangle/(\omega_k-\omega_r-i\epsilon)$$

$$-\sum_k\langle\tfrac{1}{2}\beta'|E_s(k,\ell)|\tfrac{1}{2}\beta'\rangle\langle\tfrac{1}{2}\alpha'|N_r(k)|\tfrac{1}{2}\alpha'\rangle/(\omega_k+\omega_r)\Big\} \qquad \text{(B-6)}$$

All the matrix elements in eq. (B-6) are independent of magnetic quantum numbers.

Abbreviating

$$\langle \tfrac{1}{2}x' | Y | \tfrac{1}{2}x' \rangle \quad \text{by } Y \tag{B-6'}$$

we have

$$F_s(r,\ell) = \left[N_r(s)\,\vartheta(\ell) - \vartheta(\ell)\,N_r(s)\right]/(\omega_r + \omega_s)$$

$$- \sum_k \left\{ \frac{M_r(k)^* \;F_s(k,\ell)}{\omega_k - \omega_r - i\epsilon} + \frac{E_s(k,\ell)\;N_r(k)}{\omega_k + \omega_r} \right\} \tag{B-7}$$

Eq. (B-7) is of the required form, i.e. it is eq. (57) with the angular momentum quantum numbers eliminated.

In the same fashion we obtain from eq. (58)

$$E_s(r,\ell) = \left[M_r(s)\,\theta(\ell) - \theta(\ell)\,M_r(s)\right]/(\omega_s - \omega_r + i\epsilon)$$

$$- \sum_k \left\{ \frac{E_s(k,\ell)\;M_r(k)}{\omega_k - \omega_r + i\epsilon} + \frac{N_r(k)^*\;F_s(k,\ell)}{\omega_k + \omega_r} \right\} \tag{B-8}$$

Next we must dispose of the isospin magnetic quantum numbers. Reintroducing the suppressed isospin dependence, eq. (B-7) reads

$$\langle \tfrac{1}{2}\beta 1\rho 1\sigma | F_s(r,\ell) | \tfrac{1}{2}\alpha 1\lambda \rangle$$

$$= (\omega_r + \omega_s)^{-1} \sum_\gamma \Big\{ \langle \tfrac{1}{2}\beta 1\sigma 1\rho | N_r(s) | \tfrac{1}{2}\gamma \rangle \langle \tfrac{1}{2}\gamma | \vartheta(\ell) | \tfrac{1}{2}\alpha 1\lambda \rangle$$

$$- \langle \tfrac{1}{2}\beta | \vartheta(\ell) | \tfrac{1}{2}\gamma 1\lambda \rangle \langle \tfrac{1}{2}\gamma 1\sigma 1\rho | N_r(s) | \tfrac{1}{2}\alpha \rangle \Big\}$$

$$- \sum_{k\gamma\kappa} \Big\{ (\omega_k - \omega_r - i\epsilon)^{-1} \langle \tfrac{1}{2}\gamma 1\kappa | M_r(k) | \tfrac{1}{2}\beta 1 \rho \rangle^* \langle \tfrac{1}{2}\gamma 1\kappa 1\sigma | F_s(k,\ell) | \tfrac{1}{2}\alpha 1 \lambda \rangle$$

$$+ (\omega_k + \omega_r)^{-1} \langle \tfrac{1}{2}\beta 1\sigma | E_s(k,\ell) | \tfrac{1}{2}\gamma 1\kappa 1\lambda \rangle \langle \tfrac{1}{2}\gamma 1\kappa 1\rho | N_r(k) | \tfrac{1}{2}\alpha \rangle \Big\} \qquad \text{(B-9)}$$

and eq. (B-8) changes similarly.

To specify completely states consisting of a nucleon and two mesons it is not enough to specify the total isospin and its z-component. We must in addition specify the isospin and its z-component of the subsystem consisting of the nucleon and one of the two mesons. We introduce the convention of forming this subsystem always with the meson whose symbol appears next to the symbol for the nucleon in the expression for the state vector of the system. Thus for the state $\langle \tfrac{1}{2}\beta 1\rho 1\sigma |$ we form the subsystem with meson R ; for the state $\langle \tfrac{1}{2}\beta 1\sigma 1\rho |$ we form the subsystem with meson S ; etc. . Then the various state vectors appearing in eq. (B-9) become

$$\begin{aligned} | \tfrac{1}{2}\beta 1\rho 1\sigma \rangle &= \sum_{T'\tau'} (-)^{T'+\tau'} \sqrt{2T'+1}\, V(\tfrac{1}{2}1T'; \beta\rho - \tau') \quad | T'\tau' 1\sigma \rangle \\ &= \sum_{T'\tau'} (-)^{T'+\tau'} \sqrt{2T'+1}\, V(\tfrac{1}{2}1T'; \beta\rho - \tau') \\ &\cdot \sum_{T\tau} (-)^{T+\tau} \sqrt{2T+1}\, V(T'1T; \tau'\sigma - \tau) \quad | T\tau(T'\tau') \rangle \end{aligned} \qquad \text{(B-10)}$$

$$| \tfrac{1}{2}\alpha 1 \lambda \rangle = \sum_{T\tau} (-)^{T+\tau} \sqrt{2T+1}\, V(\tfrac{1}{2}1T; \alpha\lambda - \tau) \quad | T\tau \rangle \qquad \text{(B-11)}$$

etc. . The symbol $| T\tau(T'\tau') \rangle$ denotes a state of the system of nucleon and two mesons with isotopic spin T (z-component τ)

formed from a subsystem of nucleon and one meson with isotopic spin T' (z-component τ').

Making use of eqs. (B-10, 11) eq. (B-9) becomes :

$$\sum_{T\tau T'\tau'} (-)^{T'+\tau'} (2T+1)\sqrt{2T'+1}\; V(\tfrac{1}{2}1T';\beta\rho-\tau')\; V(T'1T;\tau'\sigma-\tau)$$

$$\cdot V(\tfrac{1}{2}1T;\alpha\lambda-\tau)\,\langle T\tau(T'\tau')|F_s(r,\ell)|T\tau\rangle$$

$$= \sum_{\gamma} (\omega_r+\omega_s)^{-1}\Big\{\sum_{T'\tau'}\langle\tfrac{1}{2}\gamma(T'\tau')|N_r(s)|\tfrac{1}{2}\gamma\rangle\langle\tfrac{1}{2}\gamma|\theta(\ell)|\tfrac{1}{2}\gamma\rangle$$

$$\cdot(-)^{T'+\tau'}\, 2\sqrt{2T'+1}\; V(\tfrac{1}{2}1T';\beta\sigma-\tau')\; V(T'1\tfrac{1}{2};\tau'\rho-\gamma)\; V(\tfrac{1}{2}1\tfrac{1}{2};\alpha\lambda-\gamma)$$

$$-\sum_{T'\tau'}(-)^{1+\alpha+\beta}\langle\tfrac{1}{2}\beta|\theta(\ell)|\tfrac{1}{2}\beta\rangle\langle\tfrac{1}{2}\alpha(T'\tau')|N_r(s)|\tfrac{1}{2}\alpha\rangle$$

$$\cdot(-)^{T'+\tau'}\, 2\sqrt{2T'+1}\; V(\tfrac{1}{2}1T';\gamma\sigma-\tau')\; V(T'1\tfrac{1}{2};\tau'\rho-\alpha)\; V(\tfrac{1}{2}1\tfrac{1}{2};\gamma\lambda-\beta)\Big\}$$

$$-\sum_{\gamma\kappa k}(\omega_k-\omega_r-i\epsilon)^{-1}\sum_{T''\tau''T'\tau'T\tau}(-)^{T'+\tau'}(2T+1)(2T''+1)\sqrt{2T'+1}$$

$$\cdot\langle T''\tau''|M_r(k)|T''\tau''\rangle^{*}\langle T\tau(T'\tau')|F_s(k,\ell)|T\tau\rangle\; V(\tfrac{1}{2}1T'';\gamma\kappa-\tau'')$$

$$\cdot V(\tfrac{1}{2}1T'';\beta\rho-\tau'')\; V(\tfrac{1}{2}1T';\gamma\kappa-\tau')\; V(T'1T;\tau'\sigma-\tau)\; V(\tfrac{1}{2}1T;\alpha\lambda-\tau)$$

$$-\sum_{\gamma\kappa k}(\omega_k+\omega_r)^{-1}\sum_{T''\tau''T'\tau'T\tau}(-)^{T''+\tau''+T'+\tau'+\frac{1}{2}+\alpha}(2T+1)\sqrt{(2T'+1)(4T''+2)}$$

$$\langle T\tau|E_s(k,\ell)|T\tau(T'\tau')\rangle\langle\tfrac{1}{2}\alpha(T''\tau'')|N_r(k)|\tfrac{1}{2}\alpha\rangle\; V(\tfrac{1}{2}1T;\beta\sigma-\tau)$$

$$V(\tfrac{1}{2}1T';\gamma\kappa-\tau')\; V(T'1T;\tau'\lambda-\tau)\; V(\tfrac{1}{2}1T'';\gamma\kappa-\tau'')\; V(T''1\tfrac{1}{2};\tau''\rho-\alpha) \qquad \text{(B-12)}$$

where we have made use of the fact that F, E, M, N and ϑ are scalars. All the matrix elements in eq. (B-12) are independent of magnetic quantum numbers and we abbreviate them as follows:

$$\langle T\tau(T'\tau')|F_s(r,\ell)|T\tau\rangle \equiv F_s^{TT'}(r,\ell) \qquad \text{(B-13)}$$

$$\langle \tfrac{1}{2}\gamma(T'\tau')|N_r(s)|\tfrac{1}{2}\gamma\rangle = (-)^{T'-\frac{1}{2}}\sqrt{(2T'+1)/2}\; N_r^{T'}(s) \qquad \text{(B-14)}$$

$$\langle \tfrac{1}{2}\gamma|\vartheta(\ell)|\tfrac{1}{2}\gamma\rangle \equiv \Theta(\ell) \qquad \text{(B-15)}$$

$$\langle T''\tau''|M_r(k)|T''\tau''\rangle \equiv M_r^{T''}(k) \qquad \text{(B-16)}$$

$$\langle T\tau|E_s(k,\ell)|T\tau(T'\tau')\rangle \equiv E_s^{TT'}(k,\ell) \qquad \text{(B-17)}$$

etc.. (The normalization in definition (B-14) is chosen for later convenience). The meaning of the superscripts is as follows:

Where two superscripts appear the first indicates the total isospin, the second indicates the isospin of the subsystem formed with the meson whose symbol appears first inside the bracket. Thus $F_s^{TT'}(r,\ell)$ indicates the transition amplitude between two states of total isospin T , with the two-meson state formed by coupling meson R to the nucleon to obtain an isospin T' and then coupling on meson S to obtain the isospin T. $E_s^{TT'}(k,\ell)$ indicates the transition amplitude between two states of total isospin T, with the two-meson state formed by coupling meson K to the nucleon to obtain an isospin T' and then coupling on meson L to obtain the isospin T.

Where one superscript appears it indicates either the total isospin of the system as in $M_r^{T''}(k)$ (here meson R and initial nucleon, as well as meson K and final nucleon, couple together to give isospin T"), or the isospin of the subsystem as in $N_r^{T'}(s)$ (here meson S couples with the nucleon to give isospin T' and then meson R is coupled on to give always the isospin $\frac{1}{2}$).

The only dependence on magnetic quantum numbers remaining in eq. (B-12) is in the V-coefficients and is now eliminated by making use of various symmetry and orthogonality properties of the V-coefficients[10].

The V-coefficients have the following well known symmetry properties:

$$V(abc;\alpha\beta\gamma) = (-)^{a+b-c} V(bac;\beta\alpha\gamma) = (-)^{a+b+c} V(acb;\alpha\gamma\beta)$$

$$= (-)^{a-b+c} V(cba;\gamma\beta\alpha) = (-)^{2b} V(cab;\gamma\alpha\beta)$$

$$= (-)^{2c} V(bca;\beta\gamma\alpha) = (-)^{a+b+c} V(abc;-\alpha-\beta-\gamma) \quad \text{(B-18)}$$

They also satisfy the following orthogonality relation:

$$\sum_{\alpha\beta} V(abc;\alpha\beta\gamma)\ V(abc';\alpha\beta\gamma') = \delta_{c,c'}\ \delta_{\gamma,\gamma'}\ /(2c+1) \qquad \text{(B-19)}$$

Products of three V-coefficients may be expressed in terms of the Racah coefficient W by the following relation:

$$\sum_{\alpha\beta\varphi} (-)^{f+\varphi}\ V(abe;\alpha\beta-\epsilon)\ V(afc;-\alpha\varphi\gamma)\ V(fbd;\varphi\beta-\delta) =$$

$$(-)^{b+c-a-d+e+\epsilon}\ W(aefd;bc)\ V(edc;-\epsilon\delta\gamma) \qquad \text{(B-20)}$$

Finally, if eqs. (B-19, 20) are not immediately applicable it is sometimes useful to recouple some angular momenta by using the relation:

$$\sum_{\epsilon}(-)^{e+\epsilon}\ V(abe;\alpha\beta-\epsilon)\ V(edc;\epsilon\delta-\gamma) =$$

$$\sum_{f\varphi}(-)^{f+\varphi}\ (2f+1)\ V(afc;\alpha\varphi-\gamma)\ V(bdf;\beta\delta-\varphi)\ W(abcd;ef) \qquad \text{(B-21)}$$

Using eqs. (B-19) and (B-21) we rewrite eq. (B-12) as follows:

$$\sum_{T\tau T'\tau'} (-)^{T'+\tau'}\ (2T+1)\sqrt{2T'+1}\ F_s^{TT'}(r,\ell)\ V(\tfrac{1}{2}1T';\beta\rho\tau')\ V(T'1T;\tau'\sigma-\tau)\ V(\tfrac{1}{2}1T;\alpha\lambda-\tau)$$

$$= \sum_{\gamma}(\omega_r+\omega_s)^{-1}\Big\{\sum_{T\tau}(-)^{2T+\tau-\frac{1}{2}}\ \sqrt{2}\ (2T+1)\ N_r^T(s)\ \theta(\ell)$$

$$V(\tfrac{1}{2}1T;\beta\sigma-\tau)\ V(T1\tfrac{1}{2};\tau\rho-\gamma)\ V(\tfrac{1}{2}1\tfrac{1}{2};\alpha\lambda-\gamma)$$

$$- \sum_{T\tau T'} (-)^{T+\tau+T'+\frac{1}{2}+\alpha+\beta} \sqrt{2}(2T+1)(2T'+1)\, \vartheta(\ell)\, N_r^{T'}(s)\, W(\tfrac{1}{2}1\tfrac{1}{2}1;T'T)$$

$$\cdot V(\tfrac{1}{2}T\tfrac{1}{2};\gamma\tau-\alpha)\, V(11T;\sigma\rho-\tau)\, V(\tfrac{1}{2}1\tfrac{1}{2};\gamma\lambda-\beta) \Big\}$$

$$- \sum_k (\omega_k-\omega_r-i\epsilon)^{-1} \sum_{T\tau T'\tau'} (-)^{T'+\tau'} (2T+1)\sqrt{2T'+1}\, M_r^{T'*}(k)\, F_s^{TT'}(k,\ell)$$

$$\cdot V(\tfrac{1}{2}1T';\beta\rho-\tau')\, V(T'1T;\tau'\sigma-\tau)\, V(\tfrac{1}{2}1T;\alpha\lambda-\tau)$$

$$- \sum_k (\omega_k+\omega_r)^{-1} \sum_{T\tau T'\tau'} (-)^{T'+\alpha} (2T+1)\sqrt{2T'+1}\, E_s^{TT'}(k,\ell)\, N_r^{T'}(k)$$

$$\cdot V(\tfrac{1}{2}1T;\beta\sigma-\tau)\, V(T'1T;\tau'\lambda-\tau)\, V(T'1\tfrac{1}{2};\tau'\rho-\alpha) \qquad \text{(B-22)}$$

Next we multiply both sides of eq. (B-22) by $V(\tfrac{1}{2}1X;\alpha\lambda-x)\, V(\tfrac{1}{2}1X';\beta\rho-x')$ and sum over α, λ, β, and ρ. Using the various identities (B-18) through (B-21) we obtain:

$$(-)^{X'+x'} V(X'1X;x'\sigma-x)\, (2X'+1)^{-\frac{1}{2}}\, F_s^{XX'}(r,\ell) = -\, (-)^{X'+x'} V(X'1X;x'\sigma-x)$$

$$\cdot \Big\{ (\omega_r+\omega_s)^{-1}\, \delta_{X,\frac{1}{2}} \sum_T (-)^{T-\frac{1}{2}} (2T+1)/\sqrt{2}\, N_r^{T}(s)\, \vartheta(\ell)\, W(T\tfrac{1}{2}\tfrac{1}{2}X';11)$$

$$+ (-)^{X+X'} (\omega_r+\omega_s)^{-1} \sum_{TT'} (-)^{T+T'-\frac{1}{2}} \sqrt{2}(2T+1)(2T'+1)$$

$$\cdot \vartheta(\ell)\, N_r^{T'}(s)\, W(\tfrac{1}{2}1\tfrac{1}{2}1;T'T)\, W(\tfrac{1}{2}X\tfrac{1}{2}\tfrac{1}{2};1T)\, W(TX1X';\tfrac{1}{2}1)$$

$$+ \sum_k (\omega_k-\omega_r-i\epsilon)^{-1} (2X'+1)^{-\frac{1}{2}}\, M_r^{X'*}(k)\, F_s^{XX'}(k,\ell)$$

$$+ (-)^{2X} \sum_{k} (\omega_k+\omega_r)^{-1} \sum_{TT'} (-)^{2T'} (2T+1)\sqrt{2T'+1}\; E_s^{TT'}(k,\ell)\; N_r^{T'}(k)$$

$$\cdot W(\tfrac{1}{2}XT'T;11)\; W(TX\tfrac{1}{2}X';11) \qquad \text{(B-23)}$$

or, with an obvious relabelling of dummy indices:

$$F_s^{TT'}(r,\ell) = -\delta_{T,\frac{1}{2}} \sum_{T''} \frac{N_r^{T''}(s)\;\theta(\ell)}{\omega_r+\omega_s} (-)^{T''-\frac{1}{2}}(2T''+1)\sqrt{(2T'+1)/2}\; W(T''\tfrac{1}{2}\tfrac{1}{2}T';11)$$

$$- (-)^{T+T'} \sum_{T''T'''} \frac{\theta(\ell)\; N_r^{T'''}(s)}{\omega_r+\omega_s} (-)^{T''+T'''-\frac{1}{2}}(2T''+1)(2T'''+1)\sqrt{4T'+2}$$

$$\cdot W(\tfrac{1}{2}1\tfrac{1}{2}1;T'''T'')\; W(\tfrac{1}{2}T\tfrac{1}{2}\tfrac{1}{2};1T''')\; W(T''T1T';\tfrac{1}{2}1)$$

$$- \sum_{kT''T'''} \frac{E_s^{T''T'''}(k,\ell)\; N_r^{T'''}(k)}{\omega_k+\omega_r} (2T''+1)\sqrt{(2T'+1)(2T''+1)}\; W(\tfrac{1}{2}TT'''T'';11)\; W(T''T\tfrac{1}{2}T';11)$$

$$- \sum_{k} M_r^{T'}(k)^{*}\; F_s^{TT'}(k,\ell)\;/(\omega_k-\omega_r-i\epsilon) \qquad \text{(B-24)}$$

Eq. (B-24) is of the desired form, all magnetic quantum numbers having been eliminated.

In much the same fashion the isospin magnetic quantum numbers are eliminated from eq. (B-7). The result is

$$E_s^{TT'}(r,\ell) =$$

$$(\omega_s-\omega_r+i\epsilon)^{-1}(-)^{T+\frac{1}{2}}\sqrt{4T'+2}\left[M_r^{T}(s)\;\theta(\ell) - \theta(\ell)\;M_r^{T'}(s)\right] W(\tfrac{1}{2}T\tfrac{1}{2}T';11)$$

$$- \sum_{k} M_r^{T'}(k)\;\; E_s^{TT'}(k,\ell)\;/(\omega_k-\omega_r+i\epsilon)$$

$$- \sum_{kT''T'''} (\omega_k+\omega_r)^{-1} \overset{T'''\,*}{N_r}(k) \overset{T''T'''}{F_s}(k,\ell)\ (2T''+1)$$

$$\sqrt{(2T'''+1)(2T'+1)}\ W(\tfrac{1}{2}TT'''T'';11)\ W(T''T\tfrac{1}{2}T';11) \qquad \text{(B-25)}$$

In eqs. (B-24, 25) T and T' can take on the values $\frac{1}{2}$ and $\frac{3}{2}$, for other values the W-coefficients vanish. Thus eqs. (B-24, 25) represent four equations each, corresponding to the four possible distinct combinations of T and T'. Each set of four equations may be conveniently summarized by using matrix notation:

$$F_s(r,\ell) = \Lambda' \frac{N_r^{\frac{1}{2}}(s) - N_r^{\frac{3}{2}}(s)}{\omega_r+\omega_s}\,\theta(\ell)$$

$$- \sum_k \left\{ \frac{M_r^*(k)\ F_s(k,\ell)}{\omega_k-\omega_r-i\epsilon} + \Delta' \frac{N_r(k)\ E_s(k,\ell)}{\omega_k+\omega_r} \right\} \qquad \text{(B-26)}$$

$$E_s(r,\ell) = \Delta'\Lambda' \frac{M_r^{\frac{1}{2}}(s) - M_r^{\frac{3}{2}}(s)}{\omega_s-\omega_r+i\epsilon}\,\theta(\ell)$$

$$- \sum_k \left\{ \frac{M_r(k)\ E_s(k,\ell)}{\omega_k-\omega_r+i\epsilon} + \Delta' \frac{N_r^*(k)\ F_s(k,\ell)}{\omega_k+\omega_r} \right\} \qquad \text{(B-27)}$$

Here $F_s(r,\ell)$ and $E_s(r,\ell)$ are 4-row, 1-column matrices:

$$F = \begin{pmatrix} F^{\frac{1}{2}\frac{1}{2}} \\ F^{\frac{1}{2}\frac{3}{2}} \\ F^{\frac{3}{2}\frac{1}{2}} \\ F^{\frac{3}{2}\frac{3}{2}} \end{pmatrix} \; ; \qquad E = \begin{pmatrix} E^{\frac{1}{2}\frac{1}{2}} \\ E^{\frac{1}{2}\frac{3}{2}} \\ E^{\frac{3}{2}\frac{1}{2}} \\ E^{\frac{3}{2}\frac{3}{2}} \end{pmatrix} \; ; \qquad \text{(B-28)}$$

$M_r(k)$ and $N_r(k)$ are 4x4 diagonal matrices:

$$M = \begin{pmatrix} M^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & M^{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & M^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & M^{\frac{3}{2}} \end{pmatrix} \; ; \qquad N = \begin{pmatrix} N^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & N^{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & N^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & N^{\frac{3}{2}} \end{pmatrix} \; ; \qquad \text{(B-29)}$$

and finally Λ' and Δ' are numerical matrices:

$$\Lambda' = -\frac{2}{9} \begin{pmatrix} 4 \\ 2\sqrt{2} \\ 1 \\ -\sqrt{5} \end{pmatrix} \; ; \quad \Delta' = \frac{1}{9} \begin{pmatrix} 1 & 2\sqrt{2} & 8 & 4\sqrt{5} \\ 2\sqrt{2} & 8 & -2\sqrt{2} & -\sqrt{10} \\ 4 & -\sqrt{2} & 5 & -2\sqrt{5} \\ 2\sqrt{5} & -\sqrt{\frac{5}{2}} & -2\sqrt{5} & 4 \end{pmatrix} \qquad \text{(B-30)}$$

We note that the crossing matrix Δ' has the property $\Delta'\Delta' = 1$. Eqs. (B-26) and (B-27) are identical to eqs. (59) and (60) in the text.

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BIOGRAPHY

The author was born in Kraków, Poland, on the 25th of April 1930. He attended elementary school in that city until the outbreak of World War II in September 1939. In 1940 the author was deported into the Soviet Union and spent one year in a Labor Camp in the Ural Mountains. From 1941 to 1946 he was in Djambul, Kazachstan, U.S.S.R., and completed his elementary and high school education in that town. In 1946 he returned to Kraków, Poland, where he took an examination to have his schooling in the U.S.S.R. recognized. In 1947 in Stockholm, Sweden, and in 1948-9 in São Paulo, Brazil, he was employed as a radio-technician.

In September 1949 the author was admitted to the undergraduate school at the Massachusetts Institute of Technology, was on the Dean's List every term, and received the degree of S.B. in Physics in June 1953. From 1953 to 1956 he was a half-time Research Assistant in the Department of Physics, Massachusetts Institute of Technology, while pursuing his graduate studies in Physics.

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