Simple Criteria to Prevent Sustained Oscillations in Nonlinear Fluid Flow Networks

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Simple Criteria to Prevent Sustained Oscillations in Nonlinear Fluid Flow Networks

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Abstract

Practically applicable criteria are found which guarantee that sustained oscillations and multiple equilibrium points cannot occur in nonlinear fluid flow networks. The method is particularly applicable to complex systems containing many pumps (or fans) whose pressure-rise characteristics do not decrease monotonically with increasing flow rate. The guarantees are shown to be remarkably robust to incomplete and/or inaccurate system modeling. The criteria primarily require that the non-monotone fluid resistors (pumps, fans or compressors) be *strictly passive with respect to their equilibrium operating points.* (This terminology is precisely defined and given a simple graphical interpretation in the text.) The analysis here makes novel use of graph theoretic methods originally developed for nonlinear electrical circuit applications. It particularly features the use of *Tellegen's Theorem* and the *Colored Arc Theorem.* It is notable that these methods allow strong conclusions to be drawn about a fluid network's dynamic behavior directly from the physical system model, without formulating or analyzing differential equations. Rigorous proofs of key results and numerous illustrative examples are given.

1 Nomenclature

M branch fluid momentum variable
$\mathbf{M}$ vector of branch fluid momentum variables
P branch pressure difference
$\mathbf{P}$ vector of branch pressure differences
Q branch volume flow rate
$\mathbf{Q}$ vector of branch volume flow rates
V branch volume
$\mathbf{V}$ vector of branch volumes

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t, τ time
n number of entries in a vector

subscripts

C fluid capacitor branches
I fluid inertia branches
k kth branch
R fluid resistor branches
S constant source branches

superscripts

$*$ distinguished operating point (typically equilibrium point)
T transpose of a vector

2 Introduction

The pressure-rise versus flow characteristic of turbomachine type pumps, fans and compressors are quite commonly non-monotonic, that is they have relative minima and maxima. This behavior often results from rotating stall. For example, the non-monotonic pressure-rise versus flow characteristic typical of an axial flow compressor is show in Figure 1. When

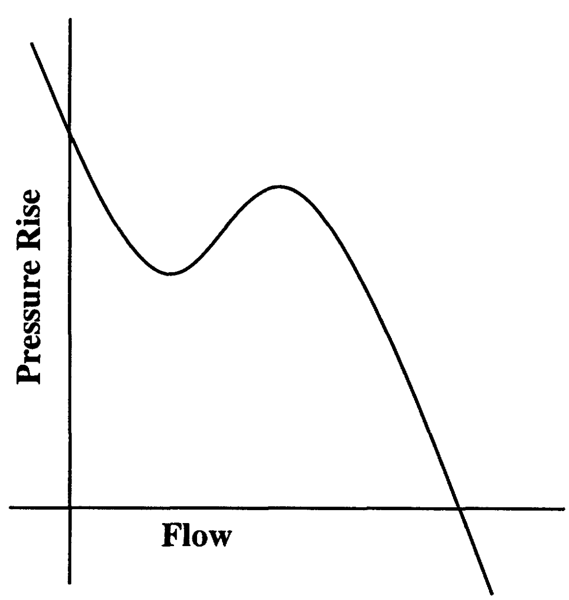

Figure 1: Non-monotone Pressure-Rise vs. Flow Characteristic Typical of an Axial Flow Compressor, Fan or Pump

such devices are connected to a system of conduits (ducts or pipes) to form a fluid flow network, it is possible that the resulting system will exhibit sustained, self-excited oscillations. Oscillations of this type can range in severity, with typical manifestations including: pulsation in building ventilation systems; unstable combustion and fatigue damage to ductwork in fossil fuel power plants; and the violent and destructive surge phenomena experienced in the compression systems of gas turbine engines and chemical process plants. (See for example [Greitzer 1981, Goldschmied, Wormley and Rowell 1980].)

The purpose of this paper is to show that for certain *safe* ranges of equilibrium operating points such undesirable oscillatory behavior cannot occur, even if the system is subjected to large external disturbances. These safe ranges of operation can easily be determined from inspection of the individual pressure-rise versus flow characteristics of the non-monotone pumping devices. By restricting the allowable set of system equilibrium points to the safe range thus established, the occurrence of these undesirable oscillations can be avoided. It is widely recognized in the practical design literature (for example [Jorgensen 1983] and [Stepanoff 1957]) that systems which include devices with non-monotone pressure-rise versus flow characteristics are subject to undesirable periodic behavior. Such instabilities have also been studied in detail for relatively simple systems with a single pumping device as for example in [Greitzer 1976]. In the research reported herein, mathematically precise and rigorous criteria are found which guarantee that oscillations cannot occur. These results provide a theoretical basis to support the empirical recommendations of the design literature and may be applied to system models with any number of pumping devices.

The notion of *safe* equilibrium operating points will be made precise by defining the concept of *passivity with respect to an operating point.* The key result is that fluid networks will generally not oscillate if all the fluid resistors are passive with respect to their operating points. Means for discriminating pathological cases where this result does not hold are also established. This result, which follows primarily from the constraints imposed by the network's structure, is robust with respect to incomplete modeling, does not depend upon linearity and is applicable to complex systems with many energy storage elements and non-monotone resistors. Since considerable intuition and insight can be gained by understanding why the claims made here are true, rigorous arguments will be included in the text for key illustrative points. Formal proofs have been relegated to the appendixes.

Finally, it is noted that the results presented here have their origins in the graph-theoretical methods of non-linear electrical circuit analysis. References will therefore be made to the appropriate literature, where many useful results are stated in great generality and proved for large classes of models. As such, the implications of this work are by no means limited to fluid flow networks. However, the specialization to the case of interest here allows us to do away with many (but unfortunately not all) intricate and often obscuring technical assumptions without sacrificing rigor. The basic aim here is to provide a *prescriptive approach* to avoiding real problems.

MOTIVATION

A simple example illustrates the basic problem features and motivates the subsequent developments. Consider the basic compression system, Figure 2(a), consisting of a fan or compressor which forces gas into a plenum volume which then discharges to atmosphere

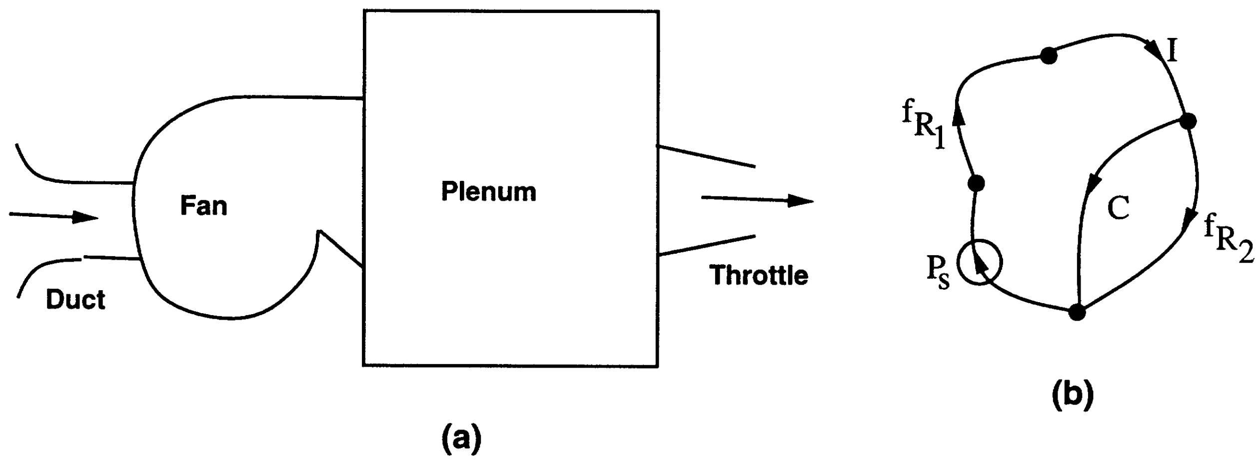

Figure 2: Basic Compression System

through a throttling valve. Such a configuration can be used to represent a gas turbine engine [Greitzer 1976] or the gas flow path through a forced draft boiler. A highly simplified linear graph model of this system is shown in Figure 2(b). The non-monotone pressure-rise versus flow characteristic of the fan is modeled by a series combination of a pressure source, P_{s} and a nonlinear resistor function, f_{R_1}, which gives pressure-drop as a function of flow. The pressure-drop through the throttle as a function of through flow is given by the strictly monotonically increasing fluid resistor function, f_{R_2}. The momentum of the fluid in the compressor duct, and compliance effects of the plenum are represented by the linear inertia, I and capacitor C respectively. It will be shown that the linearity of these elements is not essential to the analysis, they need only be invertible. For now this simplifies the problem description. This system is in equilibrium when the pressure-rise of the compressor is equal to the pressure-drop across the throttle. (Note: the pressure-rise of the compressor is found by subtracting the flow dependent pressure-drop across f_{R_1} from the pressure-rise of the constant source P_{s_1}) This condition is illustrated graphically in Figure 3(a) for two different throttle settings, denoted A and B. [1] Simulations of the system's initial condition response using a particular set of system parameter values, are compared for the equilibrium operating points A and B in the phase portraits shown in Figure 3(b) and Figure 3(c) respectively. In both cases, the system returns to equilibrium for sufficiently small perturbations. (The linearized system is stable.) However, for large initial perturbations, with the throttle set at A, the system trajectories tend to a large limit cycle. This type of undesirable sustained oscillation is not exhibited for any of the initial conditions which were tried when the throttle is set to B. These simulations show that the existence of oscillatory behavior, though dependent upon the equilibrium operating point, cannot be ascertained from a linear stability analysis alone. This theory rules out oscillations like those in Fig. 3b that are *not predicted* by linear

[1] The characteristic depicted here has two relative maxima which might appear atypical. Such a characteristic can result however from the series combination of two fans stages whose individual characteristics each have a only a single relative maximum.

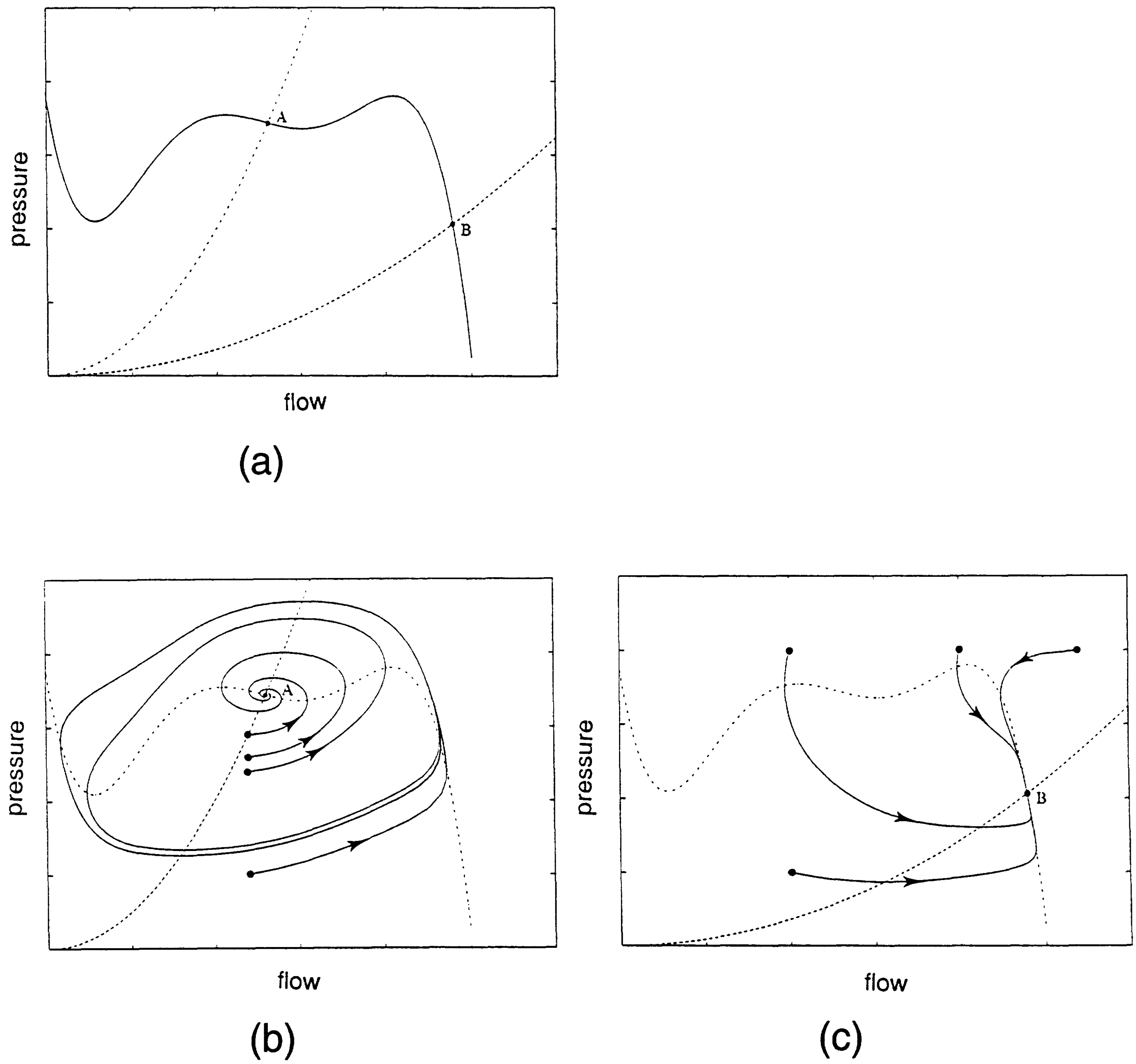

Figure 3: Dynamic behavior of basic compression system (a) Equilibrium operating points, (b) Phase portrait for throttle setting A, (c) Phase portrait for throttle setting B

methods.

Although simulation is useful for illustrative purposes, one cannot rely on simulations to rule out oscillation for three fundamental reasons. One can only simulate a finite number of i) initial conditions, ii) model parameter sets, and iii) model topologies. This theory provides a useful method for ruling out sustained oscillations where linear analysis and simulation are both inadequate.

3 Preliminaries

3.1 Linear Graph Models of Fluid Flow Networks

Familiarity with the basic techniques required to develop a lumped-element model of a fluid flow network and representing it with a linear graph or bond graph [Karnopp and Rosenberg 1983, Rowell and Wormley 1996, Shearer, Murphy and Richardson 1967] is assumed here.

Because terminology and notation in this field are not standardized, some basic definitions and nomenclature will be given to avoid ambiguity.

The entire development which follows will be in terms of linear graphs.[2] This choice seemed most natural because the results here are built on a graph theoretic foundation which directly applies to linear graph models.[3]

3.2 Branch Variables

A linear graph of a fluid flow network consists of a collection of points, called *nodes*, connected by directed (oriented) line segments, called *branches*. Associated with each branch, k, is a scalar *through variable*, Q_k, the volume flow rate, and a scalar *across variable*, P_k, the difference in pressure between the two nodes which the branch connects. An arrow is marked on each graph branch which indicates its orientation. The following sign convention will be used for all graph-theoretic arguments: Flow, Q_k, in the direction of the arrow is positive. The *pressure difference*, P_k, is defined as the pressure at the upstream node minus the pressure at the downstream node. (This convention, called *associated reference directions*, is standard in graph theory, and we strictly follow it in all proofs. But it is not traditional in fluid systems, and we don't use it in certain figures and parts of the text where confusion might arise. We use the explicit terms *pressure rise* (i.e., downstream pressure minus upstream pressure) and *pressure drop* (i.e., the opposite) whenever we deviate from associated reference directions.) Two other variables are also assigned to each branch, k, of the graph. These are the *fluid-momentum variable*[4] M_k, and *volume* V_k, defined respectively

[2] A *linear* graph can contain elements which have *nonlinear* constitutive relations. This term is an unfortunate historical misnomer.

[3] It is noted that systematic procedures exist for converting bond graphs to linear graphs [Ort and Martens 1974, Perelson and Oster 1976] and these may be employed if the original model is a bond graph.

[4] There does not seem to be a good, standard term for this quantity. Here it will be called the *fluid-momentum* because it plays the same role for a fluid-inertia element as momentum plays for a mechanical inertia. It is specifically not the fluid's momentum which for inviscid, incompressible flow in a constant area duct of cross section A, is given by, $\int_0^T P_k A \, dt$.

as the integrals over time of the branch pressure and flow rate:

$$M_k(t) = \int_0^t P_k(\tau)\, d\tau \tag{1}$$

$$V_k(t) = \int_0^t Q_k(\tau)\, d\tau. \tag{2}$$

By convention the same symbol is used for the branch variable name as for the function of time which assigns it values. Also, for brevity, P_k, the *branch pressure difference*, will simply be called the *branch pressure* and Q_k, the *branch volume flow rate*, will be called the *branch flow*. The branch variables for the network are given by the length n_b vectors, $\mathbf{Q}$, $\mathbf{P}$, $\mathbf{V}$, $\mathbf{M}$, whose kth components are the values of the branch variable on the kth branch of the graph.

3.3 Constitutive Relations

This paper is concerned with linear graphs constructed from four different types of *one-port*, *time invariant*, branch types. These branch types are distinguished based upon the particular pair of branch variables which are related by the branch's constitutive relation and are defined as follows:

Fluid-Capacitors: Let $\mathbf{V}_C$ and $\mathbf{P}_C$ be length n_C vectors whose components, V_{C_k} and P_{C_k} are the branch volumes and pressures respectively associated with the kth capacitor branch. The pressure for each capacitor is a function of the volume, $P_{C_k} = f_{C_k}(V_{C_k})$.

Fluid-Inertias: Similarly, for the n_I inertia branches, $\mathbf{M}_I$ and $\mathbf{Q}_I$ are defined to be length n_I vectors of fluid inertia branch fluid-momenta and flows respectively. The flow for each inertia is a function of the fluid-momentum, $Q_{I_k} = f_{I_k}(M_{I_k})$.

Fluid-Resistors: For the n_R resistor branches, $\mathbf{P}_R$ and $\mathbf{Q}_R$ are defined to be length n_R vectors of fluid-resistor branch pressures and flows respectively. Two types of fluid-resistor branches, *flow-controlled* and *pressure-controlled* are distinguished[5]. For a *flow-controlled* fluid-resistor the branch pressure, P_{R_k}, depends upon the branch flow. That is $P_{R_k} = f_{R_k}(Q_{R_k})$. In this case, P_{R_k} is referred to as the *dependent resistor variable* and Q_{R_k} is referred to as the *independent resistor variable.* For the *pressure-controlled* fluid-resistors the independent variable is the branch pressure, P_{R_k}, and the dependent variable is the branch flow, Q_{R_k}. That is $Q_{R_k} = f_{R_k}(P_{R_k})$.

Constant Sources: For the n_S constant source branches, $\mathbf{P}_S$ and $\mathbf{Q}_S$ are defined to be length n_S vectors of source branch pressures and flows respectively. Two types of source branches are distinguished: *pressure-sources* hold the branch pressure at a constant value and *flow-sources* maintain the branch flow at a constant value.

[5]The distinction between pressure-controlled and flow-controlled resistors becomes important for nonlinear resistors where, in contrast to the linear case, the functions, f_{R_k} are not necessarily invertible

3.4 Passivity

It will be useful to have available the following definition, adapted from [Chua and Green 1976], which makes precise the notion of a *safe* equilibrium operating pint, in terms of the *shape* of the resistor branch characteristics.

Definition 1 [Strictly passive with respect to an operating point] Let (P^*, Q^*) be a pair of points satisfying the constitutive relation of some resistor branch. If for any other pair of points, (P, Q) satisfying this resistor's constitutive relation, $(P - P^*)(Q - Q^*) > 0$, unless $P = P^*$ and $Q = Q^*$, then the resistor is said to be *strictly passive with respect to the operating point* (P^*, Q^*).

A resistor branch is strictly passive with respect to an operating point (P^*, Q^*), if the graph of its constitutive relation passes only through the open first and third quadrants of a set of axes whose origin is placed at (P^*, Q^*). This is illustrated in Figure 4. A resistor branch

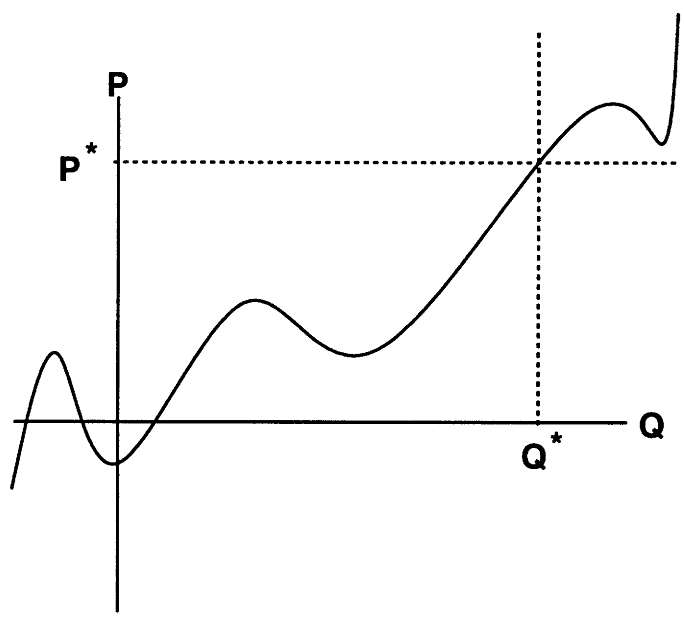

Figure 4: Resistor function which is strictly passive with respect to the operating point (P^*, Q^*)

which is passive with respect to the operating point $(0, 0)$ always absorbs physical power, which is perhaps the more conventional notion of passivity. In general however, a resistor branch may be passive with respect to a particular operating point and supply physical power to the network as in Figure 4. Finally, it is noted that a fluid-resistor with a *strictly increasing constitutive relation* [6] will be strictly passive with respect to every operating point.

[6] A scalar function, f, is said to be *strictly increasing* if $(f(y) - f(x))(y - x) > 0$ whenever, $y \neq x$

4 Undesirable Periodic Behavior

The possibility of a fluid flow network undergoing undesirable sustained oscillations (periodic behavior) will now be considered. To be more precise, let the vector functions of time, $\mathbf{P}(t)$ and $\mathbf{Q}(t)$ assign the time histories of the pressures and flows on all of the n_b network branches. At any instant of time, t, $\mathbf{P}(t)$ and $\mathbf{Q}(t)$ must satisfy the constraints imposed by continuity and compatibility for the network branches (see Appendix A). Also, at each time t, the components of $\mathbf{P}(t)$ and $\mathbf{Q}(t)$, along with those of $\mathbf{M}(t)$ and $\mathbf{V}(t)$ derived from them, must satisfy the constitutive relations of each graph branch. Any functions $\mathbf{P}(t)$ and $\mathbf{Q}(t)$ satisfying continuity, compatibility and the constitutive relations will be called a *network solution.* If for some real number, τ, $\mathbf{P}(t) = \mathbf{P}(t+\tau)$, and $\mathbf{Q}(t) = \mathbf{Q}(t+\tau)$, for all time, t, then the functions $\mathbf{P}(t)$ and $\mathbf{Q}(t)$ will be called a *periodic network solution.* By this definition, constant, i.e. equilibrium network solutions, are also periodic solutions. (The definition is satisfied for any value of τ.) When a distinction is required, a periodic solution which is not constant will be called a *nontrivial periodic solution.*

4.1 Ruling Out Periodic Behavior - A Particular Case

Consider the linear graph, Figure 5(b), which provides a highly simplified model of the *balanced draft furnace* shown schematically in Figure 5(a). A more realistic model might include additional elements to approximate better the distributed effects of the ductwork. This highly simplified model was chosen to provide a clear example and does not reflect any inherent limitation of the theory. In this linear graph model the fluid resistor R_1 represents the forced draft fan which pushes combustion air into the furnace volume, represented by the fluid capacitor C_1, from which the *induced draft fan*, represented by the fluid resistor R_2, removes combustion gases by sending them through the stack, which is represented by the fluid-inertia I_1. The constitutive relations for the graph branches are shown in Figure 5(c).

As indicated in Figure 5, the flow through the forced draft fan, f_{R_1}, increases monotonically as the difference between the upstream and downstream pressure increases. The induced draft fan, whose constitutive relation, f_{R_2}, is typical of an axial flow fan, does not increase monotonically with flow rate and in fact has a local minimum and maximum. The constitutive relations for the fluid-capacitor, f_{C_1}, which gives pressure as a function of the net volume which has entered the furnace is a continuous, increasing, invertible, function. In fact it may typically be modeled as linear but this is not required here. Similarly, f_{I_1} is a continuous, increasing, invertible function and is not necessarily linear. Suppose this network has a constant solution, denoted as $(\mathbf{P}^*, \mathbf{Q}^*)$, can it have any other periodic solution? This depends upon the nature of the resistor branch constitutive relations, as will now be demonstrated.

Suppose there were another periodic solution, $(\mathbf{P}(t), \mathbf{Q}(t))$. Since $\mathbf{P}(t)$ and $\mathbf{P}^*$ both must satisfy compatibility so must their difference, $(\mathbf{P}(t) - \mathbf{P}^*)$. Similarly, $(\mathbf{Q}(t) - \mathbf{Q}^*)$ must satisfy continuity. By Tellegen's Theorem (see Appendix A):

$$(\mathbf{Q}(t) - \mathbf{Q}^*)^T(\mathbf{P}(t) - \mathbf{P}^*) = 0 \tag{3}$$

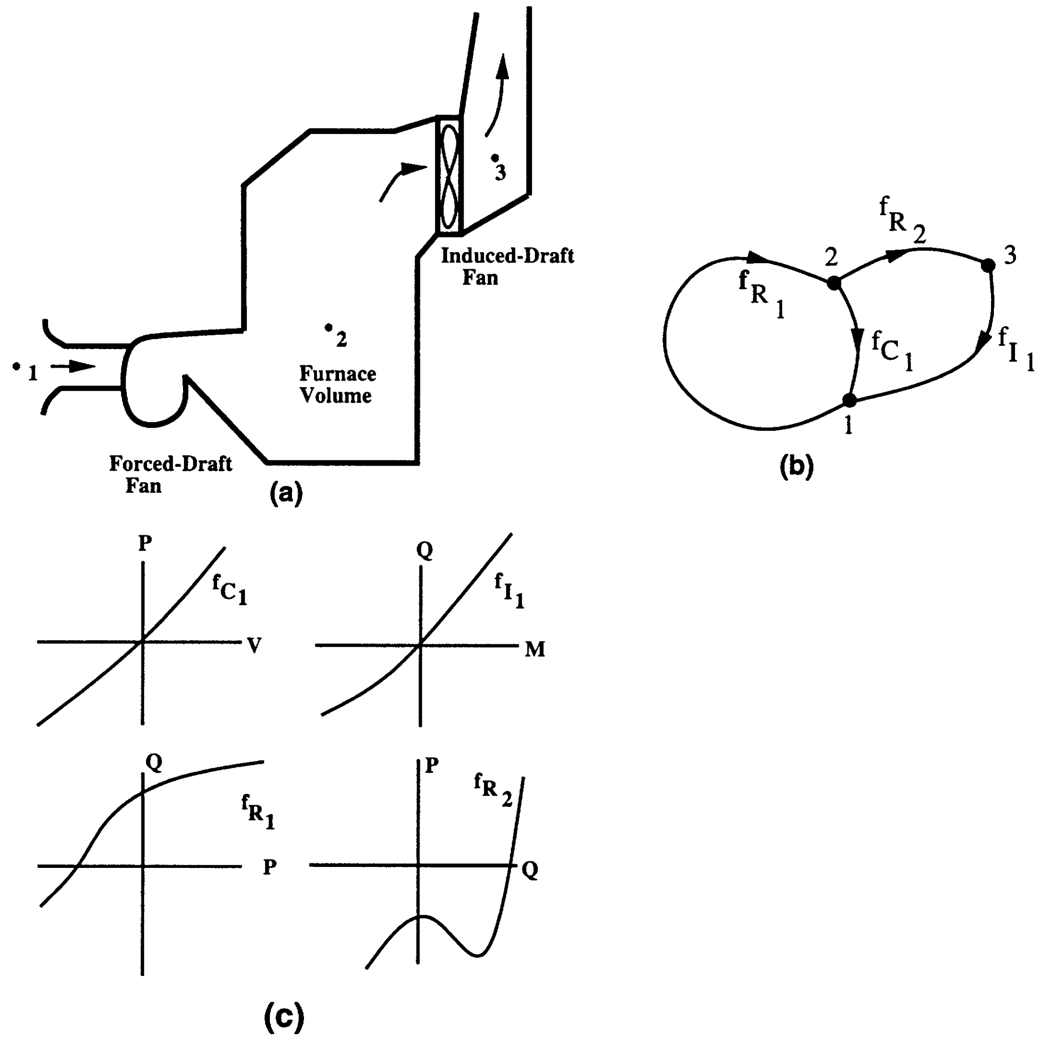

Figure 5: Balanced Draft Furnace System

Expanding in terms of the graph branches gives:

$$\begin{aligned} 0 &= \sum_{j=1}^{2}(Q_{R_j} - Q^*_{R_j})(P_{R_j} - P^*_{R_j}) + \\ &\quad (Q_{C_1} - Q^*_{C_1})(P_{C_1} - P^*_{C_1}) + \\ &\quad (Q_{I_1} - Q^*_{I_1})(P_{I_1} - P^*_{I_1}) \end{aligned} \tag{4}$$

Since $V^*_{C_1} = f_{C_1}{}^{-1}(P^*_{C_1})$ is a constant, $Q^*_{C_1} = dV^*_{C_1}/dt = 0$. Similarly, $P^*_{I_1} = 0$. (Note, this depends only upon f_{C_1} and f_{I_1} being invertible, not linear.) Dropping the terms which are identically zero and integrating Equation 4 over one period, τ, gives:

$$\begin{aligned} 0 &= \int_t^{t+\tau} \sum_{j=1}^{2}(Q_{R_j} - Q^*_{R_j})(P_{R_j} - P^*_{R_j})dt + \\ &\quad \int_t^{t+\tau} (Q_{C_1} P_{C_1})dt - \\ &\quad \int_t^{t+\tau} (Q_{C_1} P^*_{C_1})dt + \\ &\quad \int_t^{t+\tau} (Q_{I_1} P_{I_1})dt - \\ &\quad \int_t^{t+\tau} (Q^*_{I_1} P_{I_1})dt. \end{aligned} \tag{5}$$

However, as will now be shown, all but the first integral on the right hand side of Equation 5 vanish. Since the inertia branch flow is periodic $Q_{I_1}(t) = Q_{I_1}(t+\tau)$. Taking inverses, $f_{I_1}{}^{-1}(Q_{I_1}(t)) = f_{I_1}{}^{-1}(Q_{I_1}(t+\tau))$, which implies $M_{I_1}(t) = M_{I_1}(t+\tau)$. That is, the inertia branch fluid-momentum is periodic also. Similarly, the capacitor branch volume, V_{C_1} is also periodic. The periodicity of the inertia branch fluid-momentum and capacitor branch volume then imply that, $\int_t^{t+\tau}(Q^*_{I_1} P_{I_1})dt = 0$, and, $\int_t^{t+\tau}(Q_{C_1} P^*_{C_1})dt = 0$. Finally, $\int_t^{t+\tau}(Q_{I_1} P_{I_1})dt = 0$, and $\int_t^{t+\tau}(Q_{C_1} P_{C_1})dt = 0$, since I_1 and C_1 are non-dissipative elements. (This is developed more fully in Appendix B.)

Therefore, the following *conservation law* must be satisfied by any periodic solution:

$$0 = \int_t^{t+\tau} \left(\sum_{j=1}^{2}(Q_{R_j} - Q^*_{R_j})(P_{R_j} - P^*_{R_j})\right)dt \tag{6}$$

In fact, this conservation law holds for a large class of networks. The general statement of this fact, which will be called Duffin's Theorem,[7] along with its proof, is given in Appendix B. The general proof follows along the same basic line of reasoning which was just presented.

The following consequence of Duffin's Theorem shows its power. Consider again the linear graph model of Figure 5. Suppose, that both resistors are strictly passive with respect to the equilibrium operating point $(\mathbf{P}^*, \mathbf{Q}^*)$. Then it must be that $(\mathbf{P}^*, \mathbf{Q}^*)$, is the only constant solution for the network and there are no nontrivial periodic solutions. To see this, note that

[7] The theorem is so named because of its close similarity to a result presented by R. J. Duffin [Duffin 1955]. Note that the conserved quantity is not the physical energy absorbed by the resistors, though it has the same units.

each term in the sum forming the integrand in Equation 6 must always be non-negative and so Equation 6 can only be satisfied if each term in its integrand is identically zero. That is, $(Q_{R_j}(t) - Q^*_{R_j})(P_{R_j}(t) - P^*_{R_j}) = 0$ holds identically for $j = 1, 2$. But, by the assumption that both resistors are strictly passive with respect to the equilibrium operating point, this can only happen if for all time, t, both $Q_{R_j}(t) = Q^*_{R_j}$ and $P_{R_j}(t) = P^*_{R_j}$, for $j = 1, 2$.

Since the resistor operating points cannot vary with time, it is almost obvious that the capacitor and inertia branches must remain at equilibrium as well, i.e. : $Q_{C_1}(t) = Q^*_{C_1}$, $P_{C_1}(t) = P^*_{C_1}$, $Q_{I_1}(t) = Q^*_{I_1}$ and, $P_{I_1}(t) = P^*_{I_1}$. To show this rigorously, note that compatibility of branch pressures for the loop formed by the fluid-capacitor f_{C_1} and the fluid-resistor f_{R_1} requires that $P_{C_1}(t) = P^*_{C_1}$. Continuity for the flows leaving the cut-set formed by, f_{R_1}, f_{R_2}, and f_{C_1}, requires that $Q_{C_1}(t) = Q^*_{C_1}$. Similarly, compatibility for the loop formed by the inertia branch and the two resistor branches shows that $P_{I_1}(t) = P^*_{I_1}$. Applying continuity to the cut-set formed by the resistor branch f_{R_2} and the inertia branch f_{I_1} shows that $Q_{I_1}(t) = Q^*_{I_1}$. Therefore, $\mathbf{Q}(t) = \mathbf{Q}^*$ and, $\mathbf{P}(t) = \mathbf{P}^*$; the desired result.

4.2 General Theory

In the preceding example we showed that nontrivial periodic behavior is not possible if all the resistor branches are strictly passive with respect to an equilibrium operating point. The key feature of that example was eq. (6), which shows that the constant solution $(\mathbf{P}^*_R, \mathbf{Q}^*_R)$ is the only possible periodic behavior for the resistive subnetwork. This part of the result generalizes easily to essentially all practical networks for which the resistive subnetwork is passive with respect to its operating point. But to guarantee the network doesn't oscillate, we must also rule out the possibility that some subnetwork of inertias and capacitors is not damped by the resistors. The class of these potentially troublesome subnetworks is defined below.[8]

Definition 2 [undamped dynamic subnetwork][9] An undamped dynamic subnetwork is a collection u of inertia and capacitor branches such that every branch in u forms:

(i). a cutset consisting only of itself, other members of u and flow-sources, and

(ii). a loop consisting only of itself, other members of u and pressure-sources.

Example 1 In Figure 6, the branches marked I_1 and C_1 form an undamped dynamic subnetwork.

Proposition 1 will be useful in checking for an undamped dynamic subnetwork. The main result to be given is that fluid flow networks will not oscillate if all of the resistor branches are strictly passive with respect to their operating points, so long as the network contains no undamped dynamic subnetwork. This result is made precise by the following general theorem.

[8]We often will have need to specify collections of particular types of graph branches. To avoid awkward and undue formality this will be done throughout by listing the admissible branch types which can be in the collection with the understanding that any given collection need not include all of the admissible types.

[9]The term *topologically undamped dynamic subnetwork* is perhaps more apt as the network topology allows the subnetwork to be undamped but does not ensure that it is. For brevity, the shorter term, *undamped dynamic subnetwork* is used here.

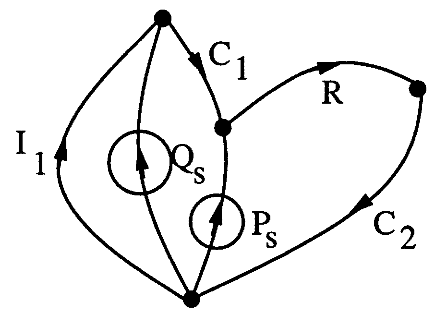

Figure 6: Network which Contains an Undamped Dynamic Subnetwork

Theorem 1 (Main Result) *A network $\mathcal{N}$ constructed exclusively of one-port fluid-resistors, fluid-capacitors, fluid-inertias and constant source branches, with the constant solution ($\mathbf{P}^*,\mathbf{Q}^*$), will have no non-trivial periodic solutions and no other constant solutions if the following conditions are satisfied by the branch constitutive relations and network topology.*

Constitutive Relations:

C-1 *Every resistor branch is strictly passive with respect to the steady operating point ($\mathbf{P}^*,\mathbf{Q}^*$)*

C-2 *Each capacitor and inertia branch constitutive relation is continuous and invertible*

Topology:

T-1 *There are no loops containing only pressure-source and inertia branches or cutsets containing only flow source and capacitor branches*

T-2 *There are no undamped dynamic subnetworks*

The proof, given in Appendix B, follows the same basic line of reasoning just illustrated for the previous particular case. For purposes of plant operation or design, assumption C-1 is the important one to ensure desirable plant behavior. The other assumptions are nearly always satisfied in practice and are included here solely to enable a rigorous proof.

Fluid-inertias, are typically modeled as linear [Karnopp and Rosenberg 1983]. Fluid-capacitors which are either linear or at least invertible are also typical. So assumption C-2 is commonly satisfied. A wide variety of reasonable models satisfy the topological conditions T-1 and T-2, as will be illustrated by subsequent examples. Thus failure to satisfy assumptions C-2, T-1 or T-2 is usually the result of erroneous or incomplete modeling.

Some key aspects of Theorem 1 will be illustrated by the following examples.

Example 2 Consider the linear graph of the simple series resistor, inertia, capacitor network shown in Figure 7. Suppose all the branches have linear constitutive relations and $R > 0$. Theorem 1 then shows that this system will have no nontrivial periodic solutions. Of course,

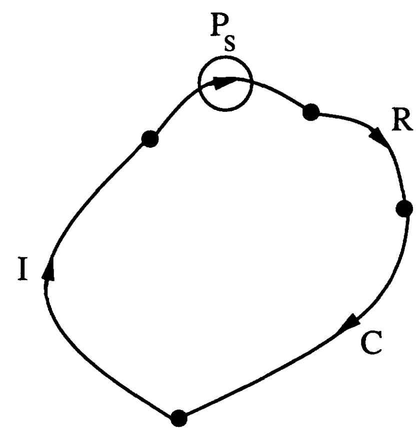

Figure 7: A simple system which can be shown not to oscillate

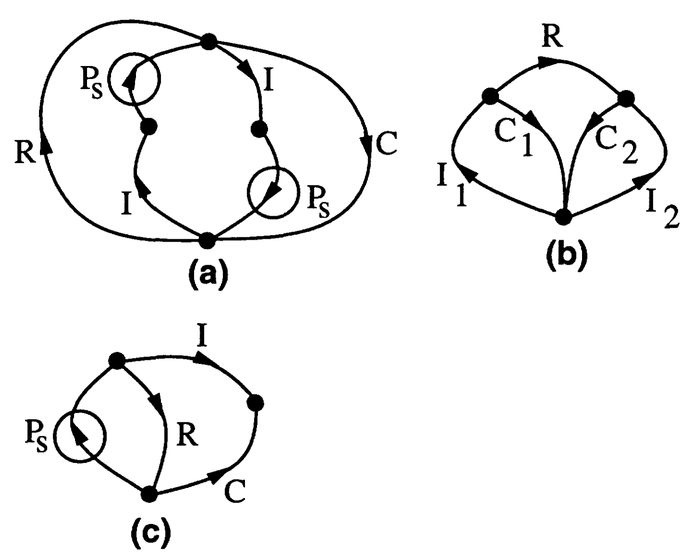

Figure 8: Networks where Theorem 1 is not applicable

for this simple case, this fact could also be established easily using a standard linear stability analysis. The power of the theorem is that this conclusion would remain unchanged even if the perturbations were large and the branch constitutive relations were nonlinear, provided the criteria of Theorem 1 were met.

Example 3 Three cases where the theorem is not applicable are shown in Figure 8. The network in Figure 8(a) has a loop consisting only of two identical pressure-sources and inertias which violates assumption T-1. For this network, any constant flow could occur around the pressure source loop and thus multiple solutions exist. Dual problems arise for cutsets containing only flow-sources and capacitors. The inertia and capacitor branches of the network shown in Figure 8(b) form an undamped dynamic subnetwork, violating condition T-2. If all the elements in this network were linear, and $I_1C_1 = I_2C_2$, this system could oscillate. This illustrates the need for condition T-2. The network shown in Figure 8(c) also has an undamped dynamic subnetwork. In this network, the parallel combination of the resistor and pressure source is equivalent to a pressure source alone. It is easy to see that the network could oscillate with any positive value of I and C.

The previous examples showed the importance of establishing that the network does not contain any undamped dynamic subnetworks. In those cases, it was fairly simple to identify the offending group of inertia and capacitor branches. The following result gives a useful means to check for undamped dynamic subnetworks in more complex systems.

Proposition 1 *An inertia or capacitor branch is* not *a member of an undamped dynamic subnetwork if:*

(i). the branch is in a loop formed exclusively with resistors, pressure sources and inertia and/or capacitor branches which are not *members of any undamped dynamic subnetwork or,*

(ii). the branch is in a cutset formed exclusively with resistors, flow sources and inertia and/or capacitor branches which are not *members of any undamped dynamic subnetwork.*

This proposition results from a straightforward application of the *colored arc corollary* given in Appendix A and the proof is omitted. It shows that any realistic network model with a resistance in series with each inertia and capacitance to represent dissipation cannot have an undamped dynamic subnetwork. The use of Proposition 1 is demonstrated by the following example.

Example 4 Consider the linear graph shown in Figure 9. The 'ladder network' of inertia and capacitor branches which occurs in this network is a typical representation of a long, lossless, duct. Repeated applications of Proposition 1 show that this network contains no undamped dynamic subnetwork, as will now be demonstrated. Let β be the (possibly empty) set of all branches which are in any undamped dynamic subnetwork. The first application of the proposition shows that C_1 is not in β because it forms a loop with R_1 and P_s. Further applications of the proposition show that branch I_1 is not in β because it forms a cutset with R_1 and C_1. (It was just shown that C_1 is not in β.) Branch C_2 is not a member of

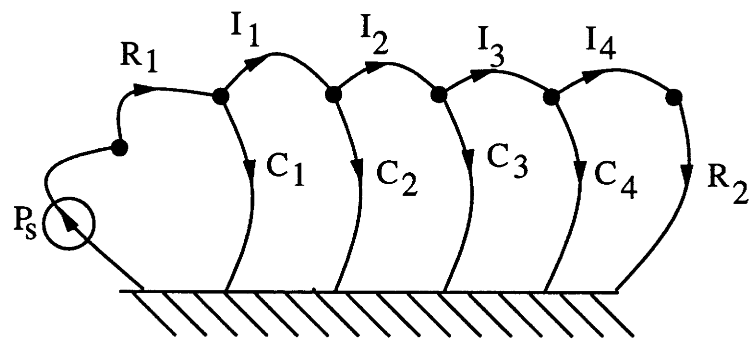

Figure 9: System containing ladder network representing lossless duct

β because it forms a loop with I_1 and C_1 (which were both just shown not to be in β). By continuing this process, eventually all the inertia and capacitor branches can be shown to not be elements of β. Thus, β is empty and the network contains no undamped dynamic subnetwork.

Although it is not always feasible, the simplest way to avoid undesirable oscillations is to insist that all the pumping devices used in the system have pressure-rise versus flow characteristics which strictly decrease with increasing flow (such devices can be modeled as a series combination of a pressure-source with a resistor whose pressure drop strictly increases with increasing flow). Since devices whose pressure-rise strictly decreases with increasing flow are strictly passive with respect to any operating point, as are the strictly increasing pressure-drop versus flow characteristics of duct and damper losses, the generally aperiodic behavior of such system follows from Theorem 1. This logic is formalized by the following corollary to Theorem 1, for which the proof is immediate, and hence omitted.

Corollary 1 *A fluid flow network $\mathcal{N}$ constructed exclusively of one-port fluid resistors, capacitors, inertias and constant source branches, with the constant solution, $(\mathbf{P}^*,\mathbf{Q}^*)$, has no nontrivial periodic solution and no other constant solution if the network satisfies conditions C-2, T-1, T-2 and every resistor branch constitutive relation is strictly increasing.*

The previous examples have deliberately been made simple and were frequently linear in order to illustrate particular points. To dispel any notion that applications of the theory which has been presented here are limited to such simple examples, this section will conclude with the following more elaborate example.

Example 5 A typical balanced draft furnace with multiple fans connected in parallel is shown schematically, along with a fairly detailed linear graph model in Figure 10. As indicated, the ducts have been represented by 'ladder networks' of inertia and capacitor branches. However, the exact number of 'rungs' in these ladders is not known. Suppose that the constitutive relations of all the network inertia and capacitor branches are unknown other than

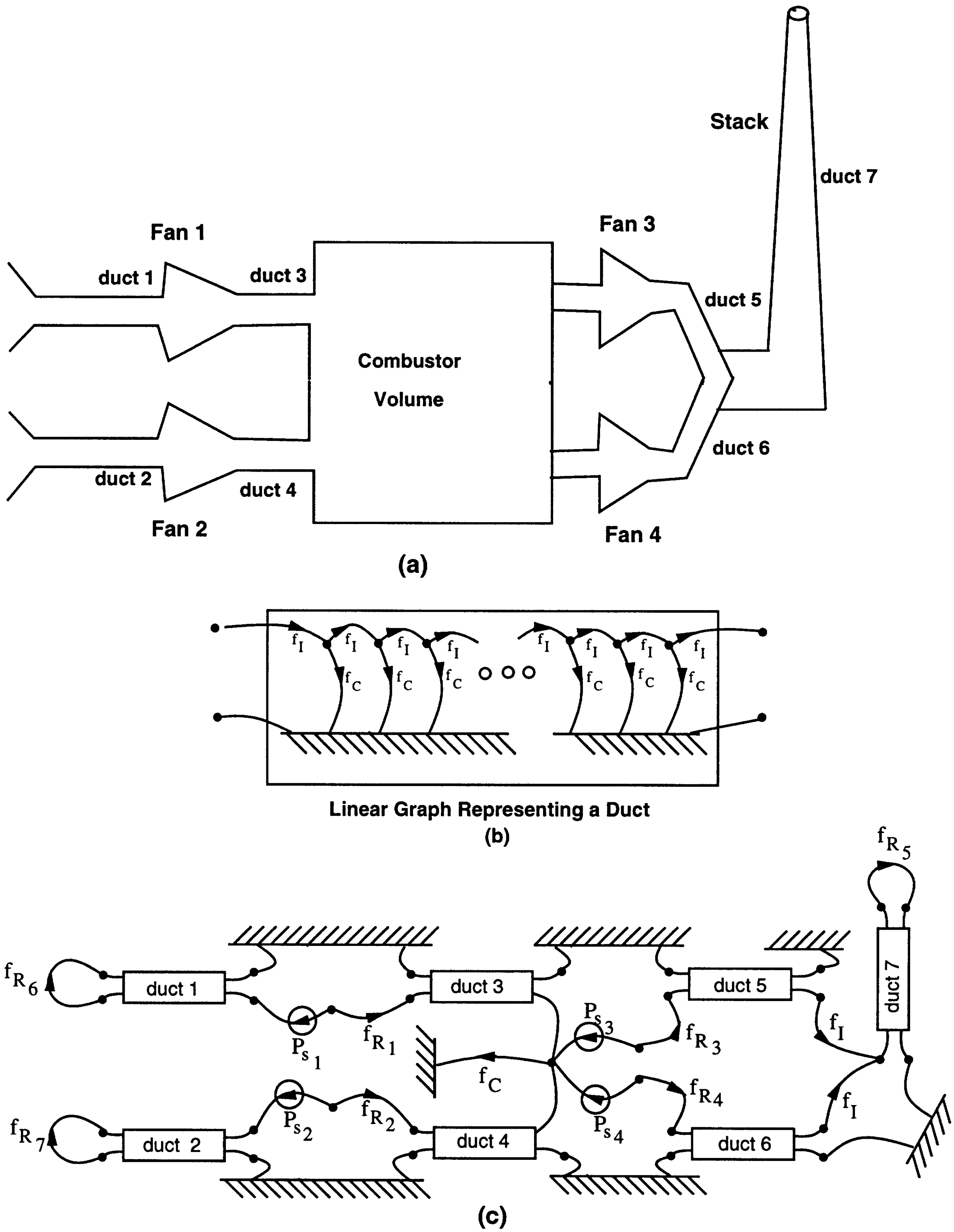

Figure 10: Balanced Draft Furnace with Parallel Fans

that they are continuous and invertible, so C-2 is satisfied. By inspection of the linear graph, T-1 is satisfied. Repeated application of Proposition 1 shows that T-2 is satisfied. So, the conclusion can be reached that the system has a unique steady solution and no nontrivial periodic solutions provided only that an operating point is chosen with respect to which all the fans and other resistors are strictly passive. Some typical fan characteristics are shown in Figure 11 along with the allowable *safe* range of operating points for which the requirement of strict passivity will be maintained. (We neglect the other safe region in the upper left portion, outside the normal operating range.) As has been previously noted, the strictly

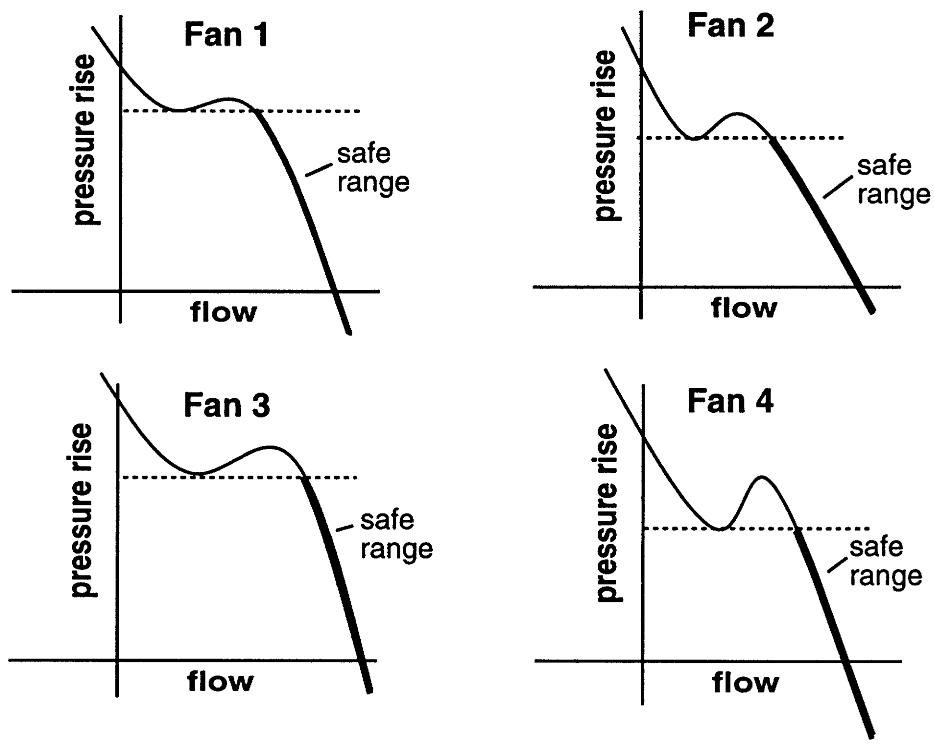

Figure 11: Fan Characteristics Showing the Safe Range Within the Typical Operating Region

increasing resistors typically used to model dampers and other losses are strictly passive with respect to any operating point. It is only the non-monotonic fan characteristics which give cause for concern.

It is remarkable that this simple analysis allows such a strong conclusion to be reached for such a complex and incompletely specified network.

5 Closing

This paper has addressed the problem of preventing sustained oscillations in fluid flow networks. The basic result is that for a large class of systems, oscillations can be prevented by ensuring that all of the network resistor branches are strictly passive with respect to their operating points. This is most simply accomplished at the design stage by utilizing only devices whose resistor characteristics are strictly increasing. If devices with non-monotone resistor characteristics must be utilized, then the allowable range of steady operating points must be restricted so that each resistor branch is strictly passive with respect to all allowed operating points. The criteria provided here are conservative, i.e., they are sufficient but by no means necessary for safe operation. Practical systems are operated successfully outside the safe regions presented. But in these cases additional safeguards may be required, since oscillations not predicted by linear analysis can arise outside the safe region, as shown in Fig. 3b. The criteria in this paper can be useful where nonoscillatory behavior must be guaranteed.

Exceptional cases can occur where oscillations involving only a portion of the network branches take place even when all the resistor branches are strictly passive with respect to their operating points. If the network contains no undamped dynamic subnetworks this is not possible and such pathological behavior will not occur. Proposition 1 provides a simple method for checking the topology of a system's linear graph to make sure that it contains no such undamped dynamic subnetworks, which can only arise from modelling errors in which dissipation associated with a fluid inertia or capacitance is neglected.

The generality of the results presented here and their robustness to incomplete and inexact modeling are remarkable. No detailed knowledge of the constitutive relations for the energy storage elements is required; it must only be assumed that they are continuous and invertible. Similarly, exact detail of the resistor characteristics need not be known; it is required only that the resistors be strictly passive with respect to their operating points. The conclusions of Theorem 1 will continue to hold as the model is altered by the addition or removal of any number of strictly increasing resistors, capacitors, and inertia branches, provided only that these do not alter the strict passivity of the non-monotone resistors with respect to their operating points or the basic topological assumptions. It is notable that these results have been derived without recourse to solving or even formulating any differential equations. This is indicative of the extent to which the behavior of this class of physical systems is dictated by the underlying structure imposed by the conditions of compatibility and continuity (Generalized Kirchoff Voltage and Current Laws). In closing, it is remarked that physical system modeling is often viewed merely as a means for obtaining differential equations which are then subjected to detailed analysis. Along with its immediate relevance to the particular problem of preventing fluid oscillations, the approach reported here also shows that in fact very general and practically applicable results can be obtained with great advantage directly from the physical system model (in this case the linear graph).

A Graph Theory

A.1 Terminology

Definition 3 [Linear Graph] *A linear graph $\mathcal{N}$ is a collection of points called* nodes *connected by directed line segments called* branches.

Definition 4 [Path] *[Wyatt 1992, Chua, Desoer and Kuh 1987] Given a graph $\mathcal{N}$, a path in $\mathcal{N}$ is an alternating sequence of nodes and branches, $n_{p_1}, b_{q_1}, n_{p_2}, b_{q_2}, \cdots, n_{p_{k-1}}, b_{q_{k-1}}, n_{p_k}$ such that*

(i). the first and last entries in the sequence are nodes, and

(ii). for each $j \in \{1, \ldots, k-1\}$, branch b_{q_j} terminates in nodes n_{p_j} and $n_{p_{j+1}}$.

A closed path *or* loop *is defined as one for which $n_{p_1} = n_{p_k}$.*

Definition 5 [Connected Graph] *[Wyatt 1992, Chua, Desoer and Kuh 1987] A linear graph $\mathcal{N}$ is* connected *if there is at least one path between any two nodes of $\mathcal{N}$*

Definition 6 [Cut-Set] *[Wyatt 1992, Chua, Desoer and Kuh 1987] Given a connected linear graph $\mathcal{N}$, a collection of branches C is a* cut-set *if and only if:*

(i). removing C from $\mathcal{N}$ disconnects $\mathcal{N}$

(ii). $\mathcal{N}$ minus the branches in C becomes connected upon replacing any one branch of C.

That is, C is a minimal separating set of branches.

A.2 Continuity and Compatibility

Continuity A set of branch volume flow rates, $\mathbf{Q}(t)$, is said to *satisfy continuity* for the linear graph $\mathcal{N}$ if and only if the sum of the volume flow rates flowing out of any cut-set in $\mathcal{N}$ is identically zero at all times t. (Continuity is called Kirchoff's Current Law in the electrical circuit literature)

Compatability A set of branch pressure differences, $\mathbf{P}(t)$, is said to *satisfy compatibility* for the linear graph $\mathcal{N}$ if and only if the sum of the branch pressure differences around any closed path of $\mathcal{N}$ is identically zero at all times t (Compatibility is called Kirchoff's Voltage Law in the electrical circuit literature)

A.3 Tellegen's Theorem

Theorem 2 (Tellegen) *Let $\mathbf{Q}$ be any vector of branch volume flow rates which satisfy continuity for the linear graph $\mathcal{N}$, and let $\mathbf{P}$ be any vector of branch pressure differences which satisfy compatibility for the graph $\mathcal{N}$. Then*

$$\mathbf{P}^T\mathbf{Q} = 0.$$

Proofs of Tellegen's Theorem along with considerably more general statements of it, can be found in [Penfield Jr., Spence and Duinker 1970, Wyatt 1992]. It is remarked that no assumptions are required on the constitutive relations for the branches of $\mathcal{N}$ and in particular they need not be linear functions or constant in time.

A.4 Colored Arc Corollary

The following corollary of the more general *Colored Arc Theorem*, [Vandewalle and Chua 1980] shows that any topological statement regarding the existence of a particular loop (respectively cutset) has a dual statement regarding the nonexistence of a particular cutset (respectively loop).

Corollary 2 (Colored Arc) *[Chua and Green 1976, Vandewalle and Chua 1980]*
Let b be a branch of a graph $\mathcal{N}$. Partition the remaining branches into two arbitrary sets, call them B and R. Then exactly one of the following statements is true:

(i). branch b forms a loop exclusively with branches in B, or

(ii). branch b forms a cutset exclusively with branches in R.

By cleverly choosing the two sets of branches, useful alternative descriptions of topological conditions can often be obtained, e.g. Lemma 2.

B Proofs

Theorem 3 (Duffin) *[Duffin 1955, Wyatt 1992] Let $\mathcal{N}$ be a linear graph constructed exclusively of one-port fluid-resistor, fluid-inertia, fluid-capacitor, and constant source branches, and require that:*

(i). for every fluid-capacitor branch k, the function $P_{C_k} = f_{C_k}(V_{C_k})$, defined over some interval of the real numbers, is continuous and invertible, and

(ii). for every fluid-inertia branch k, the function $Q_{I_k} = f_{I_k}(M_{I_k})$, defined over some interval of the real numbers, is continuous and invertible.

Suppose $\mathcal{N}$ has a constant solution, $(\mathbf{P}^*,\mathbf{Q}^*)$, for which the resistor branch pressures and flow are given by the vectors, $\mathbf{P}_R^*$, and $\mathbf{Q}_R^*$ and a periodic solution, $(\mathbf{P}(t),\mathbf{Q}(t))$, with period τ, for which the resistor branch pressures and flows are given by the vectors $\mathbf{P}_R(t)$, and $\mathbf{Q}_R(t)$. Then for any time, t_1,

$$\int_{t_1}^{t_1+\tau} (\mathbf{Q}_R(t) - \mathbf{Q}_R^*)^T(\mathbf{P}_R(t) - \mathbf{P}_R^*)dt = 0. \tag{7}$$

Proof: (Adapted and modified from [Duffin 1955]) Since $\mathbf{Q}(t)$ and $\mathbf{Q}^*$ both satisfy continuity for $\mathcal{N}$ so must their difference, $(\mathbf{Q}(t) - \mathbf{Q}^*)$. Similarly $(\mathbf{P}(t) - \mathbf{P}^*)$ satisfies compatibility for $\mathcal{N}$. So Tellegen's Theorem (Theorem 2) may be applied to get:

$$(\mathbf{Q}(t) - \mathbf{Q}^*)^T(\mathbf{P}(t) - \mathbf{P}^*) = 0. \tag{8}$$

This may be expanded in terms of the vectors of pressures and flows for the four types of branches in $\mathcal{N}$ as:

$$\begin{aligned} 0 \;=\; & (\mathbf{Q}_R(t) - \mathbf{Q}_R^*)^T(\mathbf{P}_R(t) - \mathbf{P}_R^*) + \\ & (\mathbf{Q}_I(t) - \mathbf{Q}_I^*)^T(\mathbf{P}_I(t) - \mathbf{P}_I^*) + \\ & (\mathbf{Q}_C(t) - \mathbf{Q}_C^*)^T(\mathbf{P}_C(t) - \mathbf{P}_C^*) + \\ & (\mathbf{Q}_S(t) - \mathbf{Q}_S^*)^T(\mathbf{P}_S(t) - \mathbf{P}_S^*) \end{aligned} \tag{9}$$

where the subscripts, R, I, C, S, denote the fluid-resistor, fluid-inertia, fluid-capacitor and constant source branches of $\mathcal{N}$. For any component, $Q^*_{I_k}$, of $\mathbf{Q}^*_I$, $M^*_{I_k} = f_{I_k}{}^{-1}(Q^*_{I_k})$ is a constant, which implies $P^*_{I_k} = dM^*_{I_k}/dt = 0$. Thus it can be concluded that $\mathbf{P}^*_I = 0$. By an argument dual to the one just given, it can also be shown that $\mathbf{Q}^*_C = 0$. For the constant source branches either $P_{S_k}(t) = P^*_{S_k}$ or $Q_{S_k}(t) = Q^*_{S_k}$. Dropping all the terms in Equation 9 which have just been shown to be zero, rearranging, and integrating over one period, τ, gives:

$$\begin{aligned} 0 \;=\; & \int_{t_1}^{t_1+\tau} (\mathbf{Q}_R(t) - \mathbf{Q}^*_R)^T(\mathbf{P}_R(t) - \mathbf{P}^*_R)dt + \\ & \int_{t_1}^{t_1+\tau} (\mathbf{Q}_I(t))^T(\mathbf{P}_I(t))dt + \int_{t_1}^{t_1+\tau} (\mathbf{Q}_C(t))^T(\mathbf{P}_C(t))dt - \\ & \int_{t_1}^{t_1+\tau} (\mathbf{Q}^*_I)^T(\mathbf{P}_I(t))dt - \int_{t_1}^{t_1+\tau} (\mathbf{Q}_C(t))^T(\mathbf{P}^*_C)dt \end{aligned} \qquad (10)$$

All but the first integral on the left hand side of Equation 10 will now be shown to vanish. Consider any inertia branch, call it branch k. The periodic branch pressure and flow are given by P_{I_k}, and Q_{I_k} respectively. Since the branch flow is periodic, $Q_{I_k}(t) = Q_{I_k}(t+\tau)$ Taking inverses, $f_{I_k}{}^{-1}(Q_{I_k}(t)) = f_{I_k}{}^{-1}(Q_{I_k}(t+\tau))$, or $M_{I_k}(t) = M_{I_k}(t+\tau)$. So the inertia branch fluid-momenta are also periodic. Define the stored energy:

$$E(M_{I_k}) = \int_0^{M_{I_k}} f_{I_k}(M_{I_k})dM_{I_k}.$$

By the chain rule,

$$\frac{dE(M_{I_k}(t))}{dt} = \frac{dE(M_{I_k})}{dM_{I_k}}\frac{dM_{I_k}(t)}{dt}.$$

Since $Q_{I_k}(t) = f_{I_k}(M_{I_k}) = dE(M_{I_k})/dM_{I_k}$, and, $P_{I_k} = dM_{I_k}(t)/dt$, the identity: $dE(M_{I_k}(t))/dt = Q_{I_k}(t)P_{I_k}(t)$ is obtained. Integrating both sides of this equation over one period, and using the fact that $M_{I_k}(t)$ periodic implies that $E(M_{I_k}(t))$ is periodic , shows that: $\int_{t_1}^{t_1+\tau} Q_{I_k}(t)P_{I_k}(t)dt = 0$. Since this is true for every inertia branch, the desired conclusion:

$$\int_{t_1}^{t_1+\tau} (\mathbf{Q}_I(t))^T(\mathbf{P}_I(t))dt = 0 \qquad (11)$$

is obtained.

A dual line of reasoning shows that:

$$\int_{t_1}^{t_1+\tau} (\mathbf{Q}_C(t))^T(\mathbf{P}_C(t))dt = 0 \qquad (12)$$

Since for any inertia branch, k, $\int_{t_1}^{t_1+\tau} P_{I_k}(t)dt = M_{I_k}(t_1+\tau) - M_{I_k}(t_1)$ and it was just demonstrated that $M_{I_k}(t)$ is periodic, it follows that: $\int_{t_1}^{t_1+\tau} P_{I_k}(t)dt = 0$. Since this is true for every fluid-inertia branch, the desired result is obtained:

$$\int_{t_1}^{t_1+\tau} (\mathbf{Q}^*_I)^T(\mathbf{P}_I(t))dt = 0 \qquad (13)$$

Following a dual line of reasoning for the capacitor branches gives:

$$\int_{t_1}^{t_1+\tau} (\mathbf{Q}_C(t))^T (\mathbf{P}_C^*) dt = 0 \tag{14}$$

Finally, combining equations 10-14 gives equation 7 as claimed.

Before proving the main result, Theorem 1, it will be useful to have available some facts which will be formally stated and proved in the form of the following lemmas and a theorem.

Lemma 1 *Let b be a capacitor branch whose constitutive law f_{C_k} is a continuous invertible function. Then the branch pressure P_{C_k} is constant if and only if the branch flow is identically zero. Similarly, for an inertia branch with an invertible constitutive law f_{I_k}, the branch flow is constant if and only if the branch pressure is identically zero.*

The proof for a capacitor is immediate from the one-to-one relation between pressure and volume, and the proof for an inertia follows a dual line of reasoning. Details are omitted.

Lemma 2 *Let $\mathcal{N}$ be a network constructed exclusively of one-port resistor, capacitor, inertia and constant source branches. Then*

(i). $\mathcal{N}$ contains no cutset consisting exclusively of capacitor and flow-source branches if and only if every capacitor and every flow-source branch in $\mathcal{N}$ forms a loop exclusively with resistor, inertia and pressure-source branches.

(ii). $\mathcal{N}$ contains no loop consisting only of inertia and pressure-source branches if and only if every inertia and every pressure source branch forms a cutset exclusively with resistor, capacitor and flow-source branches.

proof: This is a direct consequence of the Colored Arc Corollary, with the sets B and R of the corollary chosen as follows:

(i). Let the set $\hat{R}$ be the set of all the capacitor and flow-source branches in $\mathcal{N}$, and B be the remaining branches. Given any capacitor or flow-source branch, b_k, let R be the set $\hat{R} - \{b_k\}$.

(ii). Let the set $\hat{B}$ be the set of all the inertia and pressure source branches in $\mathcal{N}$ and R be the remaining branches. Given any inertia or pressure-source branch, b_k, let B be the set $\hat{B} - \{b_k\}$.

Theorem 4 *A network $\mathcal{N}$ constructed exclusively of one-port resistors, capacitors, inertias and constant source branches, with the constant solution $(\mathbf{P}^*, \mathbf{Q}^*)$, will have no other constant solution if the following conditions are met:*

(i). Every resistor branch is strictly passive with respect to the steady operating point, $(\mathbf{P}^*, \mathbf{Q}^*)$.

(ii). Each capacitor and inductor branch constitutive relation is a continuous and invertible real valued function defined over the entire real line.

(iii). There are no loops in $\mathcal{N}$ containing only pressure-source and inertia branches, and no cutsets containing only flow source and capacitor branches

proof: Suppose there were another steady solution, $(\mathbf{P}, \mathbf{Q})$. By Duffin's Theorem (Theorem 3), and assumption (i), $P_{R_k}(t) = P^*_{R_k}$ and $Q_{R_k}(t) = Q^*_{R_k}$ for every resistor branch. (Note, a constant solution is periodic with arbitrary period τ, thus Duffin's Theorem is applicable.) By Lemma 1, for any steady solution the capacitor branch flows and inertia branch pressures must be identically zero. Thus, $Q_{C_k}(t) = Q^*_{C_k}$ and $P_{I_k}(t) = P^*_{I_k}$ for all the capacitor and inertia branches respectively. The flows through the constant flow-source branches and pressures across the constant pressure-source branches are by definition constant and the same as those of the steady solution $(\mathbf{P}^*, \mathbf{Q}^*)$. It remains to show that the capacitor branch pressures, inertia branch flows, flow-source pressures, and pressure-source flows are the same as those in $(\mathbf{P}^*, \mathbf{Q}^*)$. By condition (iii) and Lemma 2, every capacitor branch b_{C_k} is in a loop exclusively with resistor, inertia and pressure-source branches. Since it has already been shown that the pressure across all these branch types is the same as for the steady solution $(\mathbf{P}^*, \mathbf{Q}^*)$, compatibility requires that $P_{C_k}(t) = P^*_{C_k}$ for this capacitor branch. Similar considerations hold for all the other capacitor branches. A dual line of reasoning shows, $Q_{I_k}(t) = Q^*_{I_k}$ for all the inertia branches, $P_{S_k}(t) = P^*_{S_k}$ for all the flow-source branches, and $Q_{S_k}(t) = Q^*_{S_k}$ for all the pressure-source branches. So it has been shown that $(\mathbf{P}, \mathbf{Q}) = (\mathbf{P}^*, \mathbf{Q}^*)$, which completes the proof of the theorem.

Lemma 3 *Let $\mathcal{N}$ be a network constructed exclusively of one-port resistor, capacitor, inertia and constant source branches. Let δ be any set of inertia and capacitor branches in $\mathcal{N}$ such that no element in δ forms a loop exclusively with resistor and pressure-source branches, or a cutset exclusively with resistor and flow-source branches.*

Then the following statements are equivalent:

(i). $\mathcal{N}$ contains no undamped dynamic subnetwork.

(ii). Every such set δ contains at least one branch which forms a loop exclusively with resistors, pressure-sources, inertias and capacitors not in δ or a cutset exclusively with resistors, flow-sources, inertias and capacitors not in δ.

proof:

$(i) \Rightarrow (ii)$ Choose any such set δ. By assumption $\mathcal{N}$ contains no undamped dynamic subnetwork. So some branch of δ either: 1) forms no loop exclusively with other branches of δ and pressure-source branches or, 2) forms no cutset exclusively with other branches of δ and flow-source branches. By the Colored Arc Corollary this implies either: 1) Some branch in δ forms a cutset exclusively with resistor, flow-source, inertia and capacitor branches not in δ or, 2) Some branch in δ forms a loop exclusively with resistors, pressure-sources, inertias and capacitors not in δ. Thus condition (ii) is satisfied.

$(ii) \Rightarrow (i)$ Choose any collection α of inertia and capacitor branches. If some element of α forms a loop exclusively with resistors and pressures-sources, then, by the Colored Arc Corollary it can not form a cutset exclusively with inertias, capacitors and flow-sources. So α is not an undamped dynamic subnetwork. Similarly we are done if some element of α forms a cutset exclusively with resistors and flow sources. Otherwise, by assumption (ii) and the Colored Arc Corollary, some branch b of α either: 1) forms no cutset exclusively with flow-sources and members of α or 2) forms no loop exclusively with pressure-sources and branches of α. In either case α is not an undamped dynamic subnetwork, which completes the proof.

Proof of Theorem 1 The uniqueness of the constant solution is immediate because $\mathcal{N}$ satisfies all the conditions of Theorem 4. Suppose that there is some other periodic solution, $(\mathbf{P}(t), \mathbf{Q}(t))$. Strict passivity of the resistors with respect to the constant solution $(\mathbf{P}^*, \mathbf{Q}^*)$ along with Duffin's Theorem implies $P_{R_k}(t) = P^*_{R_k}$ and $Q_{R_k}(t) = Q^*_{R_k}$ for all resistor branches. By definition, the pressures across constant pressure-source branches and the flows through constant flow-source branches are the same as those in the constant solution. Consider any capacitor which is in a loop exclusively with pressure-source and resistor branches. Since the resistor and pressure-source branch pressures are the same as in the steady solution, compatibility for the loop implies that the capacitor branch pressure is also the same as for the steady solution. By Lemma 1, the capacitor branch flow must be identically zero as in the steady solution. Dual considerations show that $Q_{I_k}(t) = Q^*_{I_k}$ and $P_{I_k}(t) = P^*_{I_k}$ for any inertia branch which is in a cutset exclusively with flow sources and resistor branches. Let δ_1 be the finite set containing the n remaining inertia and capacitor branches which form no loops exclusively with pressure-sources and resistors or cutsets exclusively with flow-sources and resistors. The assumption that $\mathcal{N}$ contains no undamped dynamic subnetwork, along with Lemma 3, shows that δ_1 contains a branch b_1 which either forms a loop exclusively with pressure-sources, resistors, inertias, and capacitors not in δ_1 or a cutset exclusively with flow sources, resistors, inertias and capacitors not in δ_1. In the former case compatibility for the loop implies that the pressure for this branch is the same as in the steady solution. In the latter case continuity for the cutset shows that the flow is the same as in the steady solution. In either case, Lemma 1 shows that the other branch variable (flow and pressure respectively) must also be constant. Now form a set δ_2 by removing branch b_1 from δ_1. The previous arguments just given for set δ_1 go through for set δ_2 showing that for some branch b_2 in δ_2 the pressure and flow is constant. This process can be repeated a finite number of times until all of the remaining inertia and capacitor branches are shown to have constant pressure and flow. It only remains to show that the flows through the pressure-source branches and the pressures across the flow-source branches are constant. The assumption that there are no loops formed exclusively by pressure source and inertia branches, along with Lemma 2, shows that every pressure source is in a cutset containing no other pressure source. But the flow through all branches other than pressure sources have already been shown to be constant. So, continuity for these cutsets implies that the flow through the pressure-source branches is constant. Dual considerations show that the pressure across the flow source branches is constant. It has now been shown that $(\mathbf{P}(t), \mathbf{Q}(t))$ is also a constant solution. Thus there is no nontrivial periodic solution. This completes the proof.

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