Dirac Operators and Monopoles with Singularities

by

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Abstract

This thesis consists of two parts. In the first part of the thesis, we prove an index theorem for Dirac operators of conic singularities with codimension 2. One immediate corollary is the generalized Rohklin congruence formula. The eta function for a twisted spin Dirac operator on a circle bundle over a even dimensional spin manifold is also derived along the way.

In the second part, we study the moduli space of monopoles with singularities along an embedded surface. We prove that when the base manifold is Kahler, there is a holomorphic description of the singular monopoles. The compactness for this case is also proved.

Thesis Supervisor: Tomasz Mrowka
Title: Professor of Mathematics

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Contents

Chapter 1

Introduction.

The work in this thesis is part of a project, in which we want to define Seiberg Witten Knot Floer Homology. More explicitly, this thesis studies the four dimensional analog of this question: given a triple (X, Σ, S), where X is a closed oriented four manifold, Σ is an embedded surface, and S is a $Spin^c$ structure on $X\backslash\Sigma$, restricted to the circle bundle P of Σ is torsion, can we define an appropriate monopole moduli space for this pair? The natural strategy is to study Seiberg Witten theory on the complement. Since here the base manifold is open, the asymptotic behavior of the monopoles might depend on the metric. And we choose to use the incomplete metric which is the restriction of a smooth metric on X. For the $Spin^c$ connection, we look at the ones which have nontrivial holonomy around the small circles linking the surface. In this thesis we compute the formal dimension of the moduli space defined under an appropriate Sobolev space, and prove that when the base manifold X is Kahler, the moduli space is compact with a holomorphic description.

The first part of the thesis is to find a proper space so that the Dirac operator is Fredholm and to compute the index. On an open manifold, the Fredholm property and index for Dirac operators certainly depends on the metric at the end of the manifold. If the metric is complete, it has been studied by [9], [15]. For incomplete metric, the situation is more sophisticated. For instance, the Dirac operators are no longer essentially selfadjoint and particular closed extensions have to be chosen. Some special cases of incomplete metrics have been carried out in the literature. If the metric has isolated conical point, i.e., $M = M_1 \cup_N U$, with $\partial M_1 = N$, $U = (0,1) \times N$, and $g|_U = dr^2 + r^2 g_N$ for some smooth metric g_N on N, the analysis has been initiated by Cheeger[7] who treated the case for signature operator. And for the Dirac operator, the index is calculated by Chou in [8]. Their analysis was extended to the more general case by Bruning and Seeley in [5], [4], [3]. To construct closed extensions such that the operator D is Fredholm, boundary conditions have to be imposed along the singular point. The following facts have been shown by the above authors. Let $\overline{D}$ denote the L^2 closure of $\{\phi \in L^2, D\phi \in L^2, \phi \in C^\infty\}$, then $\overline{D}$ is selfadjoint if and only if there is no eigenvalue of D_N such that $|\nu| < \frac{1}{2}$, (D_N is the associated self adjoint Dirac operator on N), which is also equivalent to that $\overline{D}$ is equal to the usual L^2_1 space. Thus the existence of the low eigenvalues makes the sections in $\overline{D}$ do not decay enough(or even have some singularities along the singular point) such that they do not sit inside L^2_1. The L^2 closed extensions of D are parametrized by the subspaces of $\oplus_{|s|<\frac{1}{2}} \ker(D_N - s)$. For any subspace $S \subset \oplus_{|s|<\frac{1}{2}} \ker(D_N - s)$ and its orthogonal complement $S^\perp$, the corresponding closed extensions D_S and $D_{S^\perp}$ are adjoint to each other. All these extensions are Fredholm and their indices can be calculated explicitly.

In this thesis, we consider another special case of incomplete metrics, for which the Dirac operator is singular along a codimension 2 submanifold. More specifically, let B be a codimension 2 submanifold of a closed even dimensional manifold M. S is a spinor bundle on $M \backslash B$, whose positive part S^+, when restricted to the tubular neighborhood of B, is the pull back of some spin bundle on B. (we will see later that this is equivalent to that S can not be extended to a spin bundle over M.) Denote the normal bundle of B by N, and the associated circle bundle by P. ω is a connection form of P. We are interested in the metric which is the restriction of a smooth metric on the entire M. Thus the Dirac operator D^+ looks like

$$\partial_r + \frac{1}{2r} + \frac{F}{r} + H,$$

where F is a family of first order differential operator on the fiber S^1 over B, H is essentially a family of first order differential operators along B. Because S is the pull back spin structure from B, it makes sense (which will be proved in Chapter 2) that F has eigenvalue 0 with the eigenspace being infinite dimensional, thus some APS type boundary conditions must be imposed along B to make D^+ Fredholm.

Define $\mathrm{Dom}(D^+_{\mathrm{max}}) = \{\phi \in L^2(S^+), D^+\phi \in L^2(S^-)\}$. Notice that $\mathrm{Dom}(D^+_{\mathrm{max}})$ is not the usual $L^2_1(S^+, M\backslash B)$, intuitively it is because that the Dirac operator D^+ has holonomy -1 along the small circles linking B, so it has some singularities along Σ. An important aspect of the behavior of the spinors in $\mathrm{Dom}(D^+_{\mathrm{max}})$ is given by the following lemma, which will be proved in Chapter2.

Lemma 1.0.1. *There is a bounded linear operator*

$$R : r^{\frac{1}{2}} Dom(D^+_{max}) \to L^2_{-\frac{1}{2}}(B; S_B)$$

The the range is $(L^2_{-\frac{1}{2}} \cap H^+) \oplus (L^2_{\frac{1}{2}} \cap H^-)$*. Here* r *is the distance function to* B*, which is extended to the entire* M *by* 1*,* $H^\pm$ *are the spectral subspaces associated to* D_B*.*

Just as the manifold with boundary case, we need to put some boundary conditions along B. Write P_0 for the projection

$$P_0 : L^2_{-\frac{1}{2}}(S_B) \to L^2_{-\frac{1}{2}}(S_B)$$

with image H^- and kernel H^+. This is just the APS boundary condition. We can also consider a slightly different boundary condition, given by a commensurate projection P, which is defined as in [12]:

Definition 1.0.2. Two projections P_1, $P_2 : L^2_{-\frac{1}{2}}(S_B) \to L^2_{-\frac{1}{2}}(S_B)$ are called commensurate if the difference

$$P_1 - P_2 : L^2_{-\frac{1}{2}}(S_B) \to L^2_{-\frac{1}{2}}(S_B)$$

is compact.

The following index theorem in Chapter 2.

Theorem 1.0.3. *Let M be a $2n$-dimensional closed manifold, B be a submanifold with codimension 2. Let S^+ be a spinor bundle on $M\backslash B$, which can not be extended to M as a spinor bundle, D^+ be the Dirac operator on S^+. $M\backslash B$ has the induced metric from M, r is the distance function near B as above. Let P be a projection commensurate with P_0, W^+ be the kernel of the composition map*

$$r^{\frac{1}{2}} Dom(D^+_{max}) \xrightarrow{R} L^2_{-\frac{1}{2}}(S_B) \xrightarrow{P} L^2_{-\frac{1}{2}}(S_B),$$

then on W^+, D^+ is Fredholm. Furthermore, if we take the projection to be P_0, the index of D^+ is

$$Ind(D^+) = \int_{M\backslash B} \hat{A}(M) + \int_B \hat{A}(B) \frac{1-\cosh\frac{e}{2}}{2\sinh\frac{e}{2}} - \frac{1}{2} dim Ker(D_B).$$

If we replace the above projection P by any projection P_1 such that $ker(P_1) \oplus Im(P_2) = L^2_{-\frac{1}{2}}(S_B)$ for some projection P_2 commensurate with P_0, D^+ is again Fredholm. In particular, if we project to the negative spinors, we have that D^+ is Fredholm and with index

$$d(D^+) = \int_{M\backslash B} \hat{A}(M) + \int_B \hat{A}(B) \frac{1-\cosh\frac{e}{2}}{2\sinh\frac{e}{2}} + \frac{1}{2} Ind Ker(D^+_B).$$

In the second part of this thesis we define a singular moduli space for the Seiberg Witten equations over $X\backslash\Sigma$ and a $Spin^c$ structure S, whose restriction to the circle bundle P(associated to the normal bundle N over Σ) is determined by a torsion line bundle. The $Spin^c$ extension of S to X is not unique, however, when coupled with a special $Spin^c$ connection $A_k + i\frac{1}{2}\omega$(which has trivial holonomy), $(S, A_k + i\frac{1}{2}\omega)$ extends uniquely to a $Spin^c$ pair (S_X, A_X) on X, such that $c_1(S^+_X)(\Sigma) = 2k + n$, $k \in \mathbb{Z}$ and $n = \Sigma \cdot \Sigma$ is the intersection number of Σ. Fix a reference connection $A_k + ic\omega$, A_k has holonomy $\exp(i2(c - \frac{1}{2})\pi)$ along the fibers. We consider the configuration space $\mathcal{C}_k$(detailed definition in Chapter 3.) with spinors singular along Σ and the connections having nontrivial holonomy. For the kahler case, we see that there is a holomorphic description of the singular monopoles. We will also show the compactness for Kahler case using a variational argument. More specifically, the following properties are proved in Chapter 3.

Theorem 1.0.4. *Let X be a closed Kahler surface, Σ be a holomorphically embedded curve. S is a $Spin^c$ structure on X, which restricted to the circle bundle of Σ is torsion. Then for any sequence $(A_i, \alpha_i) \in \mathcal{C}_k$ which are solutions to the Seiberg-Witten equations, there exists a subsequence, still denoted by (A_i, α_i), and a sequence of gauge transformation $g_i \in G_2^p$, such that $g_i \cdot (A_i, \alpha_i)$ converges in $\mathcal{C}_k$.*

Chapter 2

An Index Theorem for Dirac Operators with Conic Singularities.

In this chapter we will prove the index theorem stated in the introduction. Since local analysis on M is well understood, we first focus our attention to the tubular neighborhood of B. For the isolated conic situation, where the metric $g = dr^2 + r^2 g_P$ near the singular point, with P being the cross section, the eigenspaces for the Dirac operator D_P^r on P defined with respect to metrics $r^2 g_P$ remain the same for different r. For the metric we are working with, it only collapses along the fibers, i.e., the metric is dominated by the form $g = dr^2 + r^2\omega + \pi^* g_B$. Thus a priori the eigenspaces for D_P^r vary with r. Fortunately, as analyzed in Section 2.1.3., we can find another subspaces decomposition of $L^2(S_P)$, S_P is the induced spin bundle on P, which is independent of r, and allows us to use separation of variables to reduce the local analysis near B to the global analysis on P.

2.1 Dirac Operators on Circle bundles.

2.1.1 Review of Spin Geometry.

In this section, we collect some basic facts on spin manifolds and Dirac operators. For details the reader may refer to [14] or [2].

An orientable manifold X is called a spin manifold if its second Stiefel-Whitney class $w_2(X)$ is zero. Suppose X is equipped with a Riemannian metric, and let $P_{SO_n}(X)$ be the bundle of oriented orthonormal tangent frames. Let $Spin_n$ denote the spin group, which is the connected two-fold covering of SO_n for $n \geq 2$. A spin structure on X is a principal $Spin_n$ bundle $P_{Spin_n}(X)$ together with a $Spin_n$ equivariant map $p : P_{Spin_n} \to P_{SO_n}(X)$ which commutes with the projection onto X. The condition $w_2(X) = 0$ is equivalent to the existence of a spin structure. In fact using the Cech cohomology, we can easily see that the topological obstruction cocycle to the globalization of the local two-fold covering map

$$P_{Spin_n}(X) = Spin_n \times U \to P_{SO_n}(X)|_U = SO_n \times U,$$

where U is a small neighborhood, is exactly $w_2(X) \in H^2(X, \mathbb{Z}_2)$.

Let Cl_n denote the Clifford algebra of R^n with its standard inner product, and also denote the complexified Clifford algebra $Cl_n = Cl_n \otimes_R C$. The Spin representation of the Spin group $Spin_n$ is, by definition, the restriction of the algebra representation ρ of Cl_n to $Spin_n \subset Cl_n$. The Spin group $Spin_n$ has only one irreducible representation if n is odd and two irreducible representations $\triangle^{\pm}$ if n is even. These two irreducible representation $\triangle$ of Cl_{2k}:

$$\triangle : Cl_{2k} \to \mathrm{End}(C^{2^k}),$$

$\mathrm{End}(C^{2^k})$ is the group of endomorphisms of C^{2^k}.

When restricted to $Spin_{2k}$, $\triangle$ breaks into two irreducible representations $\triangle^{\pm}$ corresponding to the $(\pm)$ eigenspace of multiplication by the complex volume form $w = i^k e_1 e_2, \cdots, e_{2k}$, where $\{e_1, \cdots, e_{2k}\}$ is the orthonormal basis of R^{2k}.

Suppose now that X is a spin manifold of dimension n and $P_{Spin_n}(X) \to P_{SO_n}(X)$ is a spin structure on X. Then from the spin representation ρ of $Spin_n$, we can form the associated complex vector bundle

$$S(X) = P_{Spin_n}(X) \times_\rho V,$$

where V is the representation space of ρ. This is called the bundle of spinors. If $n = 2k$, then the above equation breaks into two pieces:

$$\begin{aligned} S(X) &= P_{Spin_n}(X) \times_{\triangle} V \\ &= P_{Spin_n}(X) \times_{\triangle^+} V^+ \oplus P_{Spin_n}(X) \times_{\triangle^-} V^- = S^+ \oplus S^- \end{aligned}$$

The section of $S(X)$ are called spinors, and the sections of $S^+(S^-)$ are called positive(negative) spinors. A local section $e = \{e_1, \cdots, e_n\}$ of $P_{SO_n}(X)$ can be lifted up to $P_{Spin_n}(X)$ and then embedded into P_{Spin_n} as a local section $\phi = \{\phi_1, \cdots, \phi_N\}$. This section ϕ is a local orthonormal basis of the bundle $S(X)$.

Let $Cl(X)$ denote the associated bundle of Clifford algebras. This is the bundle over X whose fiber at each point x is the Clifford algebra of the tangent space $T_x^*(X)$ with its given metric. This bundle carries a natural unitary connection ∇, induced from the principal SO_n bundle, and characterized by the condition that ∇ acts as a derivation on the algebra of sections $\Gamma(Cl(X))$, i.e., $\nabla(\alpha \cdot \beta) = (\nabla\alpha) \cdot \beta + \alpha \cdot (\nabla\beta)$ for all α, $\beta \in \Gamma(Cl(X))$, where $\cdot$ is the Clifford multiplication.

We can easily see that $S(X)$ is a bundle of modules over $Cl(X)$.

Lifting the Riemannian connection on $P_{SO_n}(X)$ to $P_{Spin_n}(X)$ via the Lie algebra isomorphism, we have an associated connection ∇ on $S(X)$ whose action on the spinor basis $\phi = \{\phi_1, \cdots, \phi_N\}$ can be described as follows. Let $e = \{e_1, \cdots, e_n\}$ be a local section of $P_{SO_n}(X)$ and ∇^T the Riemannian connection on the tangent bundle $T(X)$. Suppose that $\{\omega_{ij}\}$ are the one forms defined by

$$\nabla^T e_i = \sum_{j=1}^{n} \omega_{ij} e_j,$$

Then

$$\nabla^s \phi_i = \frac{1}{2} \sum_{i<j} \omega_{ij} e_i e_j \cdot \phi$$

It can be also be shown that ∇ acts as a derivation with respect to module multiplication, i.e.,

$$\nabla^s(\alpha \cdot \phi) = (\nabla\alpha) \cdot \phi + \alpha \cdot (\nabla\phi)$$

for all $\alpha \in \Gamma(Cl)$ and all $\phi \in \Gamma(s)$.

The Dirac operator $D : C^\infty(S) \to C^\infty(S)$ is defined by

$$D\phi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\phi,$$

where $\{e_1, \cdots, e_n\}$ is a local orthonormal basis on X and $\phi \in C^\infty(S)$. This is a first order elliptic differential operator with symbol $\sigma_\eta(D) = \eta\cdot$ for $\eta \in T^*(X)$.

We also include the famous Weitzenbock formula here:

Theorem 2.1.1. *(Lichnerowicz- Bochner-Weitzenbock Formula)*

$$D^2\phi = \nabla^*\nabla\phi + \frac{1}{4}s\phi,$$

where s is the scalar curvature of X.

The Dirac operator is a formally self adjoint elliptic differential operator of first order. If the manifold M is closed, then D has discrete real spectrum.

2.1.2 Spin structures on circle bundles.

In this section, we look at spin structures on a circle bundle P over a closed manifold B, $\dim B = 2m - 2$. A spin structure is called *extendable* if it can be extended to a spin structure on the disk bundle bounded by P, otherwise called *nonextendable*. Let $\zeta \in TP$ be the infinitesimal generator of the S^1 action on P, $i\omega \in i\Omega^1(P)$ is a connection 1-form on P such that $d\omega \in \pi^*\Omega^2(B)$. Let $P_{SO(2m-2)}(P)$ consists those frames on P having ζ as the first vector.

We will see that S is *nonextendable* is equivalent to that S is the pull back of a spin bundle over B. More precisely, pull back means the follows: let $\phi : P_{Spin}(B) \to P_{SO}(B)$ be a spin structure over B, then $\pi^*\phi : \pi^*P_{Spin}(N) \to \pi^*P_{SO}(N) = P_{SO}(B)$ is a $\Theta(2m-2) : Spin(2m-2) \to SO(2m-2)$ equivariant map. Extending the structure group to $Spin(2m-1)$ by

$$\tilde{\phi} : \pi^*P_{Spin}(B) \times_{Spin(2m-2)} Spin(2m-1) \to P_{SO(2m-2)} \times_{SO(2m-2)} SO(2m-1)$$

gives a spin structure on P, here $\tilde{\phi} := \pi^*\phi \times_{\Theta(2m-2)} \Theta(2m-1)$.

The S^1 action on P induces an S^1 action on $P_{SO(P)}$, the orthogonal bundle on P. For any spin structure $\pi : P_{Spin(P)} \to P_{SO(P)}$, it is called projectable if the action lifts. Otherwise it is called nonprojectable. See [1]. The relationship between projectable spin structures and spin structures on B is shown in [1], and we include it below.

Any projectable spin structure on P is the pull back of some spin structure on B: let $\tilde{\phi} : P_{Spin(P)} \to P_{SO(P)}$ be projectable. We can identify $P_{SO(B)}$ with $P_{SO(2m-2)}(P)/S^1$. Now $\tilde{\phi}^{-1}(P_{SO(2m-2)}/S^1$ is a $Spin$ bundle over $SO(B)$, and π induces a corresponding $Spin$ structure on B.

Conversely, any spin structure on B induces a projectable spin structure on P via pull back as above. The corresponding spinor bundle is just the pull back of the spinor bundle on B. Let $S_B = S_B^+ \oplus S_B^-$ be the spinor bundle on B, $S_P = \pi^* S_B$ be the corresponding spin bundle on P.

We conclude the discussion on projectable spin structures by remarking that projectable spin structures are exactly the nonextendable ones. First let us take a look at spin structures on the circle S^1. There are two different structures up to isomorphism, the trivial one $C_1 = S^1 \times \text{Spin}(1)$, and the nontrivial one $C_2 = ([0, 2\pi] \times \text{Spin}(1))/\sim$, where $\sim$ identifies 0 and 2π while interchanges the two elements of Spin(1). Let D be the disk with S^1 as its boundary. Since the disk is simply connected it can only have one spin structure. Thus only one spin structure on S^1 extends to the disk. The tangent vector to the boundary S^1 together with the outer unit normal vector forms a orthogonal frame which is a loop in the frame bundle on the disk, whose lift to the spin bundle does not close up. Thus the induced spin structure on the boundary S^1 is the nontrivial one. So the nontrivial spin structure on S^1 extends to the disk, and the trivial one does not extend.

Now given a projectable spin structure on P, from the previous discussion, it is the pull back of some spin structure. Hence when restricted to any fiber S^1, it induces a trivial spin structure, which does not extend to the corresponding disk. Thus the projectable spin structures do not extend to the disk bundles. On the other hand, if a spin structure S on P does not extend, then when restricted to the fiber, it does not extend neither. Thus restricted to any chart $U \times S^1 \subset P$, with $U \subset B$ being a small ball, S is isomorphic to the trivial spin structure, and the S^1 action lifts, so S is projectable.

2.1.3 η invariants of the perturbed Dirac operators on circle bundles.

In this section we consider the Dirac operator for the nonextendable spin bundle S on a circle bundle P over a spin manifold B. From the last section we know that S is the pull back of some spin bundle S_B over B, i.e., $S = \pi^* S_B$. We will calculate the eta function of the perturbed Dirac operator, which is going to be needed later for the index calculation. Recall that L is the line bundle associated to P.

Define the metric g_r on P by $g_r := r^2 \omega \otimes \omega + \pi^* g_B$, g_B is a Riemannian metric on B. Let $\mathcal{A}_r$ be the Dirac operator on S_P defined with respect to the metric g_r. The action of S^1 on $P_{so(2m-1)}(P)$ can be lifted to $P_{spin(2m-1)}(P)$, and it induces an isometric action on $L^2(S_P)$, where the inner product on $L^2(S_P)$ is defined with respect to the metric $g_1 = \omega \otimes \omega + \pi^* g_B$. Let L_ζ be the differential of the representation the Lie group S^1 on $L^2(S_P)$, we have the following decomposition:

$$L^2(S_P) = \oplus_{k \in \mathbb{Z}} V_k,$$

V_k is the eigenspace of L_ζ with eigenvalue ik, $k \in \mathbb{Z}$.

In section 4 of [1], it is proved that there is an isometry Q_k(which preserves the splitting of the bundle) from $L^2(S_B \otimes L^{-k})$ to V_k, such that

$$\mathcal{A}_r = \frac{1}{r} A_v + A_h - \frac{r}{4} \gamma(\frac{\zeta}{r}) \gamma(\pi^* d\omega).$$

with $A_v = \gamma(\frac{\zeta}{r}) L_\zeta$, $A_h \mid_{V_k} = Q_k \circ A_k \circ Q_k^{-1}$, A_k being the twisted Dirac operator on $L^2(S_B \otimes L^{-k})$, and γ is the Clifford multiplication.

In the rest of this section we will consider the perturbed Dirac operator defined by $A_r = \frac{1}{r} A_v + A_h$, whose spectrum is much easier to analyze, and we will calculate the eta function in this section. This perturbed operator also plays an essential role in the next section when we are trying to analyze the Dirac operator on $M\backslash B$, the main reason is that the perturbed term is of order r, which is not going to affect the index we want, but make the analysis much easier.

Now with respect to the bundle splitting $S_P = \pi^* S_B^+ \oplus \pi^* S_B^-$, the Clifford multiplication $\gamma(\frac{1}{r}\zeta)$ is given by

$$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

and the horizontal part A_h is

$$A_h = \begin{pmatrix} 0 & A_h^- \\ A_h^+ & 0 \end{pmatrix}$$

So $\gamma(\frac{\zeta}{r})$ anticommutes with A_h. Let $\phi_{k,a} = \alpha \oplus \beta \in L^2(S_P)$ be a unit norm common eigenspinor of L_ζ and A_h for eigenvalues ik and a respectively.

If $a \neq 0$, $k \neq 0$, then $\alpha \neq 0, \beta \neq 0$, and $\alpha \oplus 0$, $0 \oplus \beta$ are linearly independent and using them as a basis for the two dimensional space they span, A_r is represented by the matrix

$$A_{k,a} = \begin{pmatrix} k/r & a \\ a & -k/r \end{pmatrix}$$

If $a = 0$, let $V_{k,0}$ be the space spanned by the common eigenspinors of L_ζ and A_h for eigenvalues ik and 0 respectively, then for $\phi = \alpha \oplus \beta \in V_{k,0}$, α is the eigenvector of A_r with eigenvalue k which has multiplicity $dimKer(A_k|_{S_B^+\otimes L^{-k}})$, while β is eigenvector of A_r with eigenvalue $-k$ of multiplicity $dimKer(A_k|_{S_B^-\otimes L^{-k}})$.

If $k = 0$, V_0 is isometric to $L^2(S_B)$, and $A_r = A_v + A_h = D_B$ on V_0, $V_0 = \oplus V_{0,a}$, a runs through all the eigenvalues of D_B.

To sum up, $L^2(S_P)$ can be decomposed into direct sum of

$$
\begin{aligned}
L^2(S_P) &= \underset{k\neq 0, a\neq 0}{\oplus} (V_{k,a}) \underset{k\neq 0}{\oplus} V_{k,0} \oplus V_0 \\
&= H_1 \oplus H_2
\end{aligned} \tag{2.1.1}
$$

H_1 is the sum of the first two blocks, and H_2 is the third. Accordingly, A_r can be decomposed into

$$
A_r = \oplus A_{k,a} \oplus \frac{k}{r} \oplus D_B \tag{2.1.2}
$$

and $\frac{k}{r}$ has multiplicity $dim(Ker(A_k|_{S_B^+ \otimes L^k})) + dim(Ker(A_{-k}|_{S_B^- \otimes L^{-k}})$.

Remark: The Hilbert spaces H_1, H_2 are independent of r.

Due to the decomposition above, we can calculate the eta function $\eta_{A_r}(s)$ of A_r. By definition, $\eta_{A_r}(s) = \sum_{\lambda>0} \frac{m_\lambda - m_{-\lambda}}{\lambda^s}$, m_λ is the multiplicity of λ. On $V_{k,a}$, A_r has $\pm\sqrt{\frac{k^2}{r^2} + a^2}$ as eigenvalues with same multiplicities, so they have no contributions to $\eta_{A_r}(s)$. On V_0, since B is even dimensional, $\pm a$ has same multiplicity, so no contributions to the eta function. On $V_{k,0}$, we have

$$
m_k = dim(Ker(A_k|_{S_B^+ \otimes L^{-k}})) + dim(Ker(A_{-k}|_{S_B^- \otimes L^k})).
$$

$$
m_{-k} = dimKer(A_k|_{S_B^- \otimes L^{-k}}) + dim(Ker(A_{-k}|_{S_B^+ \otimes L^k})),
$$

so

$$
\eta_{A_r}(s) = \sum \frac{m_k - m_{-k}}{k^s} \tag{2.1.3}
$$

By Atiyah-Singer index theorem, $m_k - m_{-k} = -\int_B \hat{A}(B)(ch(L^k) - ch(L^{-k}))$. Define

$$
f_s(x) = -\sum_{k\geq 1} \frac{e^{kx} - e^{-kx}}{k^s} = -2\sum_{i\geq 1} \frac{\zeta(s-(2i-1))x^{2i-1}}{(2i-1)!}.
$$

Here $\zeta(s)$ is the Riemann-Zeta function. Noticing that $\zeta(-(2n-1)) = \frac{(-1)^{n-1}B_n}{2n}$, with B_n being the n-th Bernoulli number, and the Taylor expansion

$$
\frac{1}{2}\coth\frac{x}{2} = \sum_{i\geq 0} \frac{(-1)^{i-1}B_i}{(2i)!} x^{2i-1},
$$

we have that

$$f_s(x)\mid_{s=0}= -2(\frac{1}{2}\coth\frac{x}{2}-\frac{1}{x}). \qquad (2.1.4)$$

Hence

$$\eta_A(s)=\int_B \hat{A}(B)f_s(e).$$

and the eta invariant of A is

$$\eta_A(0)=-2\int_B \hat{A}(B)(\frac{1}{2}\coth\frac{e}{2}-\frac{1}{e}).$$

Here e is again the Euler class of the normal bundle on B. Since $\zeta(s)$ has the only simple pole at 1 with residue 1, $\eta_A(s)$ has simple poles only at $2,4,6,..$ with residue 1. When B is a Riemann surface, this eta function has been calculated in [18].

2.2 An Index Theorem for Dirac Operators on Complements of the Submanifolds.

In this chapter, we will prove our index theorem for the Dirac operators on the complements of the codimension 2 submanifolds.

2.2.1 The Dirac operator on the cone.

Let $C(P)=(0,1)\times P$ be the finite cone on P with metric $g=dr^2+g_r$, g_r is the metric on P defined in the last section. ∂_r denote the unit radial tangent vector field. Our orientation convention is that if $e_1,..,e_{2m-1}$ is an oriented orthonormal frame on P, then $\partial_r,e_1,..,e_{2m-1}$ will give an orientation on $C(P)$. Let $S^{\pm}_{C(P)}$ be the positive(negative) spinor bundle on $C(P)$, and let D^+ denote the Dirac operator on $S^+_{C(P)}$. $\gamma(\partial_r):S^+\to S^-$ gives an identification of S^+ and S^- as spinor bundles when restricted to P. In particular S^+ gives a spin bundle on P with Clifford multiplication defined as

$$\gamma(e_i)=\gamma(\partial_r)^{-1}\gamma_{C(P)}(e_i), i=1,..,2m-2.$$

The Dirac operator D^+ and $\mathcal{A}_r$ is related in the following way(cf[12]):

$$D^+\phi = \gamma(\partial_r)(\partial_r\phi - \frac{H}{2}\phi + \mathcal{A}_r\phi) \tag{2.2.1}$$

H is the mean curvature of $\{r\} \times P$ in $C(P)$, and can be computed as follows: let $\frac{\zeta}{r}, v_1, \cdots, v_{2m-1}$ be an oriented orthonormal frame on P, such that $v_i = \pi^* w_i$, $\{w_i\}$ is an oriented orthonormal frame on B. Obviously $[\partial_r, v_i] = 0$, and $[\partial_r, \frac{\zeta}{r}] = -\frac{\zeta}{r^2}$. Thus $< \nabla_{\frac{\zeta}{r}}\partial_r, \frac{\zeta}{r} >=< -\frac{\zeta}{r^2} + \nabla_{\partial_r}\frac{\zeta}{r}, \frac{\zeta}{r} >= -\frac{1}{r}$, and $< \nabla_{v_i}\partial_r, v_i >=< \nabla_{\partial_r} v_i, v_i >= 0$, we have $H = -\frac{1}{r}$.

So (2.2.1) becomes

$$D^+\phi = \gamma(\partial_r)(\partial_r\phi + \frac{1}{2r}\phi + \mathcal{A}_r\phi) \tag{2.2.2}$$

For any cross section ϕ in $S^+_{C(P)} |_{\{r\}\times P}$, we can extend it to a global section on $S^+_{C(P)}$ by parallel transport along radial geodesics, and still denote it by ϕ. So H_1, H_2 can be extended to $L^2(S^+_{C(P)})$, and

$$L^2(S^+_{C(P)}) = (H_1 \oplus H_2) \otimes L^2((0,1), rdr). \tag{2.2.3}$$

Define $\mathrm{Dom}(D^+_{\max}) = \{\phi \in L^2, D^+\phi \in L^2\}$.

In this section, we will prove the following:

Lemma 2.2.1. *There exists a bounded linear operator* $R : r^{\frac{1}{2}} Dom(D_{max}) \longrightarrow \pi^* L^2_{-\frac{1}{2}}(B; S_B)$. *The range is* $\pi^*(L^2_{-\frac{1}{2}} \cap H^+) \oplus \pi^*(L^2_{\frac{1}{2}} \cap H^-)$. *Here r is the distance function to B, which is extended to the entire M by 1, $H^{\pm}$ are the spectral subspaces associated to D_B.*

We first focus our attention to APS boundary condition on B. The show that the general case follows easily from this special one. We still use W^+ to denote the kernel of the following composition map:

$$r^{\frac{1}{2}} Dom(D^{\pm}_{max}) \xrightarrow{R} L^2_{-\frac{1}{2}}(S_B) \xrightarrow{P_0} L^2_{-\frac{1}{2}}(S_B).$$

Recall that P_0 is the projection map to the nonnegative eigenspaces of D_B. In this section, we prove the following:

Theorem 2.2.2. *With domain W^+, D^+ is Fredholm.*

Define $D_0^+\phi = \gamma(\partial_r)(\partial_r\phi + \frac{1}{2r}\phi + A_r\phi)$, which is just a perturbation of D^+ of order $O(r)$ near B. We will show later in this chapter that the Fredholm property of D_0^+ will guarantee the Fredholm property of D^+, and the two has the same index.

2.2.2 Construction of a boundary parametrix.

On $C(P) = (0,1) \times P$, any spin section can be written as the direct sum of three different parts as in (2.1.1),

$$\psi = \oplus\psi_{k,a} \oplus \psi_{k,0} \oplus \psi_{0,a} = \psi_1 \oplus \psi_2$$

we will construct the boundary parametrix according to the decomposition.

First, we want to solve $D_0^+\phi = \psi$, for $\phi, \psi \in L^2(S^+_{C(P)})$ then construct the boundary parametrix of ${D_0}^+$ based on the three components of ψ.

Case I:

Given $k \neq 0, a \neq 0$, $D^+_{k,a} = \partial_r + \frac{1}{2r} + A_{k,a}$. It has two fundamental solutions

$$\phi_1(r) = |a|^{\frac{1}{2}} \begin{pmatrix} K_{\nu_+}(|\, a \,|\, r)) \\ K_{\nu_-}(|\, a \,|\, r)) \end{pmatrix}, \quad \phi_2(r) = |a|^{\frac{1}{2}} \begin{pmatrix} -I_{\nu_+}(|\, a \,|\, r)) \\ I_{\nu_-}(|\, a \,|\, r)) \end{pmatrix}$$

with $\nu_\pm = |k \pm \frac{1}{2}|$, and K_ν, I_ν are generalized Bessel functions with order ν, and their asymptotic behavior has been studied extensively in the literature. The following two specific formulas for K_ν and I_ν

$$K_\nu(z) = \frac{\Gamma(\nu + \frac{1}{2})(\frac{1}{2}z)^\nu}{\sqrt{\pi}} \int_0^\infty \frac{\cos t dt}{(t^2 + z^2)^{\nu + \frac{1}{2}}}$$

$$I_\nu(z) = (\frac{1}{2}z)^\nu \sum_{k=0}^\infty \frac{(\frac{1}{4}z)^k}{k!\Gamma(\nu + k + 1)}.$$

give the estimates for $K_\nu(r)$ and $I_\nu(r)$ when r is small:

$$K_\nu(r) \le C\frac{1}{2}\Gamma(\nu)(\frac{1}{2}r)^{-\nu} \le Cr^{-\nu}$$
$$I_\nu(r) \le C(\frac{1}{2}r)^{\nu}/\Gamma(\nu+1) \le Cr^{\nu}$$

for some constant C independent of ν. the Wronskian for the fundamental system is

$$\begin{aligned} W(\phi_1(r), \phi_2(r)) = \det \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &= |a|(K_{\nu_-} I_{\nu_+} + K_{\nu_+} I_{\nu_-}) \\ = \frac{1}{r} \end{aligned} \tag{2.2.4}$$

With the given fundamental system $\phi_1(r)$, $\phi_2(r)$, the solution of the inhomogeneous equation $D_{k,a}\phi = \psi$ can be calculated by the method of variation of constants. Since $\phi \in L^2((0,1), rdr \otimes \mathbb{C}^2)$,

$$\phi_{k,a} = c\phi_2(r) + c_1(r)\phi_1(r) + c_2(r)\phi_2(r) \tag{2.2.5}$$

where c is constant, and

$$\begin{aligned} c_1(r) &= \int_0^r W(\phi_1(s), \phi_2(s)^{-1} W(\psi(s), \phi_2(s)))ds \\ &= \int_0^r W(\psi(s), \phi_2(s))sds \end{aligned} \tag{2.2.6}$$

$$\begin{aligned} c_2(r) &= -\int_r^1 W(\phi_1(s), \phi_2(s)^{-1} W(\phi_1(s), \psi(s)))ds \\ &= -\int_r^1 W(\phi_1, \psi)sds \end{aligned} \tag{2.2.7}$$

Define

$$Q_{k,a}\psi = c_1(r)\phi_1(r) + c_2(r)\phi_2(r).$$

Case II:

For $k \geq 1$, $D_{k,0}^{+} = \partial_r + \frac{1}{2r} + \frac{k}{r}$, define

$$Q_{k,0}\psi(r) = r^{-(k+\frac{1}{2})} \int_0^r s^{k+\frac{1}{2}}\psi(s)ds, \quad k \geq 1 \tag{2.2.8}$$

For $k \leq -1$, define

$$Q_{k,0}\psi(r) = -r^{-(k+\frac{1}{2})} \int_r^1 s^{k+\frac{1}{2}}\psi(s)ds, k \leq -1 \tag{2.2.9}$$

Case III: When $k = 0$, $T_{0,a}^{+} = \partial_r + \frac{1}{2r} + a$,

$$Q_{0,a}f = r^{-\frac{1}{2}} \int_0^r e^{a(s-r)} s^{\frac{1}{2}} f(s)ds \; a \geq 0 \tag{2.2.10}$$

$$Q_{0,a}f = -r^{-\frac{1}{2}} \int_r^1 e^{a(s-r)} s^{\frac{1}{2}} f(s)ds \; a < 0 \tag{2.2.11}$$

Formally, we write our boundary parametrix to be

$$Q = \oplus Q_{k,a} \oplus Q_{k,0} \oplus Q_{0,a}.$$

The following lemma makes sure that Q is well defined:

Lemma 2.2.3. *For $\psi \in L^2(S_{C(P)})$, there exists a constant C independent of k, a, such that*

1. $\| Q_{k,0}\psi_{k,0} \| (r) \leq (2k+2)^{-\frac{1}{2}} \| \psi_{k,0} \|_{L^2((0,r),rdr)}, \quad k \geq 1 \quad or \quad k \leq -1.$
2. $\| Q_{0,a}\psi_{0,a} \| (r) \leq C \| \psi_{0,a} \|_{L^2((0,1),rdr)}$.
3. $\| Q_{k,a}\psi_{k,a} \| (r) \leq C \| \psi_{k,a} \|_{L^2}$.

Proof. The first part follows from the Cauchy Schwartz inequality, so does the second part when $a = 0$. For part 3, when $|a| \leq 1$, by using the asymptotic behavior of $K_\nu(r)$ and $I_\nu(r)$ near r, Cauchy Schwartz inequality gives that $\| Q_{k,a}\psi_{k,a} \| (r) \leq |a|(2k+2)^{-\frac{1}{2}}(\| \psi_{k,a} \|_{L^2((0,1),rdr)}$

When $|a| \geq 1$, we can use the Hankel transform to get the desired bound:

Let $\psi = \begin{pmatrix} h \\ l \end{pmatrix} \in C_0^\infty((0,1), rdr \otimes C^2)$, $\phi = Q_{k,a}\psi = \begin{pmatrix} f \\ g \end{pmatrix}$. Since

$$(\partial_r + \frac{1}{2r} + \begin{pmatrix} k/r & a \\ a & -k/r \end{pmatrix})\phi = \psi \tag{2.2.12}$$

We have

$$(-\partial_r^2 + \frac{k^2 - k + \frac{1}{4}}{r^2} - \frac{1}{r}\partial_r)g + a^2 g = -(\partial_r - \frac{k + \frac{1}{2}}{r})h + al. \tag{2.2.13}$$

Now recall that for $g \in C_0^\infty(R_+)$, Hankel transform is defined by

$$H_\nu(g)(\lambda) = \int_0^\infty J_\nu(\lambda r) g(r) r dr,$$

where J_ν is the Bessel function of order ν and $\nu > -1$. Also we have the Plancherel equality:

$$\int_0^\infty \mid g(r) \mid^2 rdr = \int_0^\infty \mid H_\nu(g)(\lambda) \mid^2 \lambda d\lambda.$$

If we denote $\triangle_{\nu_\pm} = -\partial_r^2 + \frac{k^2 \pm k + \frac{1}{4}}{r^2} - \frac{1}{r}\partial_r$, then

$$H_{\nu_\pm}(\triangle_{\nu_\pm} g) = \lambda^2 H_{\nu_\pm}(g).$$

$$H_\nu(\partial_r - \frac{\nu+1}{r}) = \lambda H_{\nu+1}$$

From the above facts, and applying H_{ν_-} to (2.2.14), we get

$$\lambda^2 H_{\nu_-}(g) + a^2 H_{\nu_-}(g) = \lambda H_{\nu_+}(h) + a H_{\nu_-}(l)$$

$$(\lambda + ia) H_{\nu_-}(g) = \frac{\lambda H_{\nu_-}(h)}{\lambda + ia} + \frac{a H_{\nu_-}(l)}{\lambda + ia}.$$

So

$$(\lambda^2 + a^2) \mid g \mid^2_{L^2(R_+, rdr)} \leq \mid \tilde{h} \mid^2_{L^2(R_+, rdr)} + \mid l \mid^2_{L^2(R_+, rdr)}.$$

The same inequality holds for f. So

$$\| a\phi \|_{L^2} \leq \| \psi \|_{L^2(R_+, rdr)} .$$

and $\| \phi \|_{L^2} \leq \frac{1}{2a} \| \psi \|_{L^2}$.

The above argument also applies to the second part when $a \neq 0$, by letting $k = 0$ above and noting that $|a|$ has a lower bound now since a is the eigenvalue for the Dirac operator D_B, which is a self adjoint elliptic operator on closed manifold B.

□

With all these estimates at hand, we are ready to prove Theorem 2.2.2.

Proof of lemma 2.2.1: In the tubular neighborhood, write $\phi = \oplus\phi_{k,a} \oplus \phi_{k,0} \oplus \phi_{0,a} = \phi_1 \oplus \phi_2$ as usual. From Lemma 2.2.3 and Lemma 2.2.4, we see that $|r^{\frac{1}{2}}\phi_1|_{L^2(S_P, g_1)}$ goes to zero as r goes to 0. As for ϕ_2, we have $\psi(r) = r^{\frac{1}{2}}\phi_2 \in L^2((0,1), V_0)$, $\partial_r\psi(r) + D_B\psi(r) = r^{\frac{1}{2}}D_0^+\phi_2 \in L^2((0,1), V_0))$, so up to the isometry Q in Section 2.1.2, $\psi \in L_1^2(S_{B\times(0,1)})$, here $S_{B\times(0,1)} = \pi^* S_B$ is the pull back bundle over $B \times (0,1)$ from S_B, with $\pi : B \times (0,1) \to B$, and $\partial_r + D_B$ is an elliptic operator on $S_{B\times(0,1)}$. Now from the following lemma, we have the restriction map R with the corresponding range as claimed. Q. E. D.

Let X be a compact manifold with boundary ∂X. Let $D : C^\infty(E) \to C^\infty(E)$ be a first order elliptic operator. Near the boundary, D has the form

$$D = \partial_t + B,$$

here B is a first order elliptic self-adjoint operator on $E \mid \partial X$. And we write a collar neighborhood of ∂X as $(-1, 0] \times \partial X$. Define $D_{\text{max}} = \{\phi \in L^2 \mid D\phi \in L^2\}$, and let D_{min} denote the closure of smooth section which vanish on ∂X with respect to the L_1^2 norm. Let $H^\pm \subset L^2$ be the positive and nonnegative spectrum of B. Then we have the following lemma, which is due to Furutani and Booss.

Lemma 2.2.4. *The restriction of smooth sections extends to define a bounded linear* *operator*

$$R : D_{max} \to L^2_{-\frac{1}{2}}(Y; E\mid_Y).$$

The kernel of this map is D_{min} *and the range is* $L^2_{\frac{1}{2}} \cap H^- \oplus L^2_{-\frac{1}{2}} \cap H^+$.

Proof. Let D^* denote the adjoint of D. Then integration by part gives

$$\int_X < D\phi, \psi > - < \phi, D^*\psi >= \int_{\partial X} < \phi, \psi > .$$

Thus for any $\hat{\psi} \in L^2_{\frac{1}{2}}(\partial X, E \mid \partial X)$, by Sobolev restriction we have that there is a $\tilde{\psi} \in L^2_1(X, E)$ extending $\hat{\psi}$ and the extension can be chosen so as to define a bounded linear operator. Applying the equality to $\phi \in D_{\max}$ and $\tilde{\psi}$ we see that the left hand side of the equality is well defined and thus defines the right hand side. Thus the restriction of $\phi \in D_{\max}$ lies in $L^2_{-\frac{1}{2}}(\partial X, E \mid \partial X)$.

From APS we know that if $\phi \in D_{\max}$ and $R(\phi) \in H^-_{-\frac{1}{2}}$, then $R(\phi) \in H^-_{\frac{1}{2}}$.

If $\hat{\phi} \in H^+_{-\frac{1}{2}}$ then

$$\beta(t)e^{Bt}\hat{\phi} = \beta(t)\sum_{\lambda>0} e^{\lambda t}\hat{\phi}_\lambda \in D_{\max},$$

here we view the above a section on $(-1, 0] \times \partial X$ and β is a cutoff function which is 1 near $t = 0$ and zero near $t = -1$.

□

Remark 2.2.5. For any $\phi \in W^+$, $\phi = \phi_0 + r^{-\frac{1}{2}}e(\psi_B)$, such that $\phi_0 \in L^2_1(S^+, M\backslash B)$, $\psi_B \in L^2_{\frac{1}{2}}(S_B)$, and $e(\psi_B)$ is the L^2_1 extension of ψ_B to $M\backslash B$. This fact can be seen easily from the proof of Lemma 2.2.1: since we are projecting to the nonnegative eigenspaces, we know that $R(r^{\frac{1}{2}}\phi) \in H^-_{\frac{1}{2}}$ for any $\phi \in W^+$, so the extension map is well defined.

The estimates also give us the following

Lemma 2.2.6.

$$Q : C^\infty_0((0,1), H) \to C^\infty((0,1), H; P)$$

is a linear operator such that

$$D^+_0 Q = Id$$

$$QD_0^+\phi = \phi, \ for \ \phi \in C^\infty(R_+, H; P), \phi(r) = 0, \quad r \geq 1,$$

and there exists constant C such that

$$\| Q\psi \|_{L^2((0,1),rdr\otimes H)} \leq C \| \psi \|_{L^2((0,1),rdr\otimes H)} .$$

Proof. $D_0^+Q = Id$ follows from the construction of Q. Let $\phi = \oplus\phi_{k,a} \oplus \phi_{k,0} \oplus \phi_{0,a}$, $\psi = D_0^+\phi$. Then $\psi_{k,0} = (\partial_r + \frac{1}{2r} + \frac{k}{r})\phi_{k,0}$. If $k \leq -1$, from the construction, $(Q\psi_{k,0})(1) = 0$, on the other hand, $\phi_{k,0}(1) = 0$ from the assumption, thus $Q\psi_{k,0} = \phi_{k,0}$ since they both solve the same ODE. If $k \geq 1$,

$$Q\psi_{k,0} = r^{-(k+\frac{1}{2})} \int_0^r s^{k+\frac{1}{2}}\psi_{k,0}(s)ds = cr^{-(k+\frac{1}{2})} - r^{-(k+\frac{1}{2})} \int_r^1 s^{k+\frac{1}{2}}\psi_{k,0}(s)ds.$$

But $Q\psi_{k,0} \in L^2((0,1), rdr)$, so $c = 0$, thus $Q\psi_{k,0}(1) = 0$, and $Q\psi_{k,0} = \phi_{k,0}$. The same argument applies to $\phi_{k,a}$ and $\phi_{0,a}$, thus if $\phi(1) = 0$, $QD_0^+\phi = \phi$.

□

Let $0 < \epsilon \leq 1$, then for any bump function $0 \leq \chi \leq 1$, such that $\chi = 1$ for $0 < r < \frac{\epsilon}{2}$, and $\chi = 0$ for $r > \epsilon$, we have the following:

Lemma 2.2.7. *Let Q acting in $L^2((0,1), rdr)$, then there exists a constant such that*

$$\| \chi Q \| + \| Q\chi \| \leq C\epsilon.$$

Here ϵ, χ are as above.

Proof. From the lemma above, $\| Q \| \leq C$. Using Cauchy Schwartz inequality, $\| \chi Q \| \leq \| \chi \|_{L^2((0,1))} \| Q \| \leq C\epsilon$. Similarly for $Q\chi$. Here the norm means the L^2 operator norm. □

Proof of Theorem 2.2.2 To show that D^+ is Fredholm, we will use our boundary parametrix to construct a global right and left paremetrices. Choose $\phi, \overline{\phi} \in C_0^\infty(-\epsilon, \epsilon)$ such that $\phi = 1$ near 0 and $\overline{\phi} = 1$ near suppϕ, choose $\psi, \overline{\psi} \in C_0^\infty(X)$ such that $\phi + \psi = 1$ and $\overline{\psi} = 1$ in a neighborhood of suppψ. Let Q_i be an interior parametrix for D^+ with

$$D\overline{\psi}Q_i\psi = \psi + R_i \tag{2.2.14}$$

$$\overline{\psi}Q_i\psi D = \psi + L_i \tag{2.2.15}$$

with R_i, L_i compact smoothing operators in $L^2(S^+), L^2(S^-)$, respectively. Define

$$R := \overline{\phi}Q\phi + \overline{\psi}Q_i\psi \tag{2.2.16}$$

From Lemma 3.6, we know that

$$R : L^2(\mathcal{S}^-) \to Dom(D^+) \tag{2.2.17}$$

Now

$$DR = I + \overline{\phi}Q\phi + \frac{n}{2}\overline{\phi}rQ\phi + R_i \tag{2.2.18}$$

When the support of ϕ is sufficiently small, from Lemma2.2.4, we have

$$\| S = \overline{\phi}Q\phi + \frac{n}{2}\overline{\phi}rQ\phi \| < \frac{1}{2}$$

and we can write

$$DR = I + S_1 + R_i$$

since R_i is compact and

$$\| S_1 \| < \frac{1}{2}.$$

This implies

$$DR(I + S_1)^{-1} = I + R_i(I + S_1)^{-1},$$

Next we can see that with Lemma 2.2.7.,

$$\begin{aligned} RD &= \overline{\phi}Q\phi T + \psi + \frac{n}{2}\overline{\phi}rQ\phi + L_i \\ &= I + \overline{\phi}Q\frac{r}{2}\phi + \overline{\phi}Q\phi' + L_i \end{aligned} \tag{2.2.19}$$

and again, we have

$$\| S_2 = \overline{\phi} Q \frac{r}{2} \phi + \overline{\phi} Q \phi' \| < \frac{1}{2},$$

and

$$RD = I + S_2 + L_i$$

$$(I + S_2)^{-1} RD = I + (I + S_2)^{-1} L_i.$$

Thus we find operators A_R and A_L such that $DA_R = I + D_R$ and $A_L D = I + D_L$, D_R and D_L are compact operators. Notice that $A_L D A_R = A_R + D_L A_R$, on the other hand $A_L D A_R = A_L + A_L D_R$, we have $C = A_R - A_L = A_L D_R - A_R D_L$ is compact, $A_R D = (A_L + C) D = I + CD + D_R$. So we find a bounded operator $A_R : L^2 \to W^+$ such that $A_R D - I$ and $DA_R - I$ are compact respectively, which says D is Fredholm.

Q. E. D.

Lemma 2.2.8. *Let g be a bounded smooth function on $M \backslash B$, then*

$$\overline{D}^+ := D^+ + g$$

is a Fredhlom operator on W^+, and

$$Ind(\overline{D}^+) = Ind(D^+).$$

Proof. Define $D_t^+ = D^+ + tg$, $t \in [0, 1]$. Let χ be the bump function in Lemma 2.2.5., then $\| t\chi g Q \| + \| Q t \chi g \| \leq C\epsilon$ from the Cauchy Schwartz inequality. Thus we can still use Q to construct a boundary parametrix of D_t^+ to prove that $D_t{}^+$ is Fredholm in W^+, just as in Theorem 2.2.2. And $\text{Ind}(\overline{D}^+) = \text{Ind}(D_1^+) = \text{Ind}(D_0^+) = \text{Ind}(D^+)$ follows immediately.

□

Remark: we conclude this section by remarking that instead of L^2, D^+ is also Fredholm with the same index on the following domain: $\{\phi \in L^p, D^+\phi \in L^p, P(\phi) = 0\}$, for some $p > 2$. To see this, first observe that on this domain, D^+ has finite dimensional kernel since $p > 2$ and D^+ has finite kernel on W^+. Next, for a sequence

$D^+\phi \to \psi$ in L^p, we have $D^+\phi$ converges in L^2, thus there exists $\phi \in W^+$, such that $D^+\phi = \psi$. Since $\psi \in L^p$, we know that $\phi \in L^p_{loc}$. More over $\phi \in L^p(U)$, here U is the tubular neighborhood of Σ, from Remark 2.2.5., we can write $\phi = \phi_0 + r^{-\frac{1}{2}}e(\psi_B)$, $\phi_0 \in L^p$ since $\phi_0 \in L^2_1$, also $e(\psi_B) \in L^2_1(S^+, M\backslash B)$, from Sobolev multiplication we have $r^{-\frac{1}{2}}e(\psi_B) \in L^p$, thus $\phi \in L^p$, and D^+ has closed image. Finally, D^+ has finite dimensional cokernel: let $\psi \in L^q$ lie in the cokernel of D^+, with $\frac{1}{p} + \frac{1}{q} = 1$. From local regularity we have $\psi \in C^\infty(S^-)$, and $D^-\psi = 0$. By solving $D^-\psi = 0$ in U and from all the estimates above, we see that $\psi \in L^2(U)$, thus $\psi \in L^2$, so ψ lies in the cokernel of W^+. Thus D^+ is Fredholm on the above domain.

2.3 Index of $D^+_{M\backslash B}$.

The index of D^+ will be calculated in this section. We first consider the case when the metric near B is of the form $g_0 = dr^2 + r^2\omega\otimes\omega + \pi^* g_B$, then show that the Dirac operator associated with any general metric is just a bounded perturbation of the one with g_0, and the index remains equal from lemma 2.2.8. Also we will first focus on the APS boundary conditions, the generall case follows immediately.

Let

$$\triangle^+ = D^-D^+, \triangle^- = D^+D^-.$$

Since in the tubular neighborhood the operator is a combination of singular regular operator and product operator, from [5], we know that $(\triangle^\pm + \lambda)^{-m}$ are trace class for large m. And the nonzero eigenvalues of $\triangle^+$ and $\triangle^-$ coincide. Thus

$$tr(\triangle^+ + \lambda)^{-m} - tr(\triangle^- + \lambda)^{-m} = \lambda^{-m} ind D^+.$$

To get the expansion of $(\triangle^\pm + \lambda)^{-m}$, we will construct a parametrix for $(\triangle^\pm + \lambda)^{-m}$. As usual, we construct a boundary parametrix, and then patch it with the canonical interior one. Let $P_i^\pm$ be the interior pseudodifferential parametrix for $(\triangle^\pm + \lambda)^{-m}$.

In the tubular neighborhood $C_{0,1}(P)$, $\mathrm{tr}P_i^{\pm}$ has expansion

$$trP_i^{\pm}(r,r;x,x;\lambda)rdrdx = \sum_j p_j^{\pm}(r,x)\lambda^{-\frac{j}{2}}rdrdx,: \ r \in (0,1),: \ x \in P. \qquad (2.3.1)$$

Recall that we have the decomposition (2.1.1):

$$L^2(S_P) = \oplus(V_{k,a}) \oplus_{k\in\mathbb{Z}} V_{k,0} \oplus_{a\neq 0} (V_{0,a}),$$

and $dimKer(A_r) = dimV_{0,0} = dimKer(D_B)$.

On $V_{k,a} \otimes L^2((0,1), rdr)$,

$$\triangle^+ = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{4r^2} + \frac{1}{r^2}\begin{pmatrix} k^2 & k \\ k & k^2 \end{pmatrix} + a^2 \qquad (2.3.2)$$

After changing a basis, obviously we can write it as

$$\triangle^+ = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{4r^2} + \frac{1}{r^2}\begin{pmatrix} k^2+k & 0 \\ 0 & k^2-k \end{pmatrix} + a^2 \qquad (2.3.3)$$

Under the same basis,

$$\triangle^- = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{4r^2} + \frac{1}{r^2}\begin{pmatrix} k^2-k & 0 \\ 0 & k^2+k \end{pmatrix} + a^2 \qquad (2.3.4)$$

Both are with boundary condition $(r^{\frac{1}{2}}\phi)(r) = 0$. So on $V_{k,a}$, $tr(\triangle^+ + \lambda)^{-m} - tr(\triangle^- + \lambda)^{-m} = 0$.

Given $a \neq 0$, on $V_{0,\pm a} \otimes L^2((0,1), rdr)$, $\triangle^{\pm} = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{4r^2} + a^2$, with the same boundary condition

$$(r^{\frac{1}{2}}\phi_{0,a})(0) = 0, \quad a \geq 0 \qquad (2.3.5)$$

$$(r^{\frac{1}{2}}(\partial_r + \frac{1}{2r} + a)\phi_{0,a})(0) = 0, \quad a < 0. \qquad (2.3.6)$$

Thus we get $tr(\triangle^+ + \lambda)^{-m} - tr(\triangle^- + \lambda)^{-m} = 0$

On $\oplus_{k\in\mathbb{Z}\setminus\{0\}} V_{k,0}$,

$$\triangle^{\pm} = -\partial_r^2 + \frac{k^2 \pm k + \frac{1}{4}}{r^2} - \frac{1}{r}\partial_r.$$

with boundary conditions $(r^{\frac{1}{2}}\phi)(0) = 0$, this is just the conical operator considered in [5], where the following facts have been proved.

1. The kernels for $(\triangle^{\pm} + z^2)^{-1}$ with the above boundary conditions are

$$\oplus I_{\nu_{\pm}}(zr)K_{\nu_{\pm}}(zr)$$

Noting that

$$(\triangle^{\pm} + z^2)^{-m} = \frac{1}{(m-1)!}(-\frac{1}{2z}\partial_z)^{m-1}(\triangle^{\pm} + z^2)^{-1},$$

let $rz = \zeta$, define

$$\sigma^{+}(r,\zeta) = \frac{r^{2m-2}}{(m-1)!}(-\frac{1}{2\zeta}\partial_\zeta)^{m-1}\sum I_{\nu_+}(\zeta)K_{\nu_+}(\zeta) \qquad (2.3.7)$$

$$\sigma^{-}(r,\zeta) = \frac{r^{m-2}}{m-1}(-\frac{1}{2\zeta}\partial_\zeta)^{m-1}\sum I_{\nu_-}(\zeta)K_{\nu_-}(\zeta) \qquad (2.3.8)$$

2.

$$tr(\psi(\triangle^{\pm} + z^2)^{-m}\phi) = \int_0^\infty \phi(r)\sigma^{\pm}(r, rz)rdr.$$

ψ, ϕ are smooth functions, vanishing near $r = 0$, $\psi \equiv 1$ near suppϕ.

Let $\sigma^{\pm}(r,\zeta) \sim \sum_{j=1}^\infty \sigma_j^{\pm}(r)\zeta^{-j}$, $\zeta \to +\infty$. From the pointwise expansion of boundary parametrix on H_3 and the interior parametrix, we have

$$\sigma^{\pm}(1,\zeta) \sim \sum_j \int_P p_j^{\pm}(1,x)dx\zeta^{-j}.$$

Due to (4.6), (4.7), (4.8),

$$\sigma^{\pm}(r,\zeta) = r^{2m-2}\sigma^{\pm}(1,\zeta) \qquad (2.3.9)$$

$$\sigma_j^{\pm} = r^{2m-2}\int_P p_j^{\pm}(1,x)dx. \qquad (2.3.10)$$

3.

$$
\begin{aligned}
tr\phi(\triangle^{\pm}+z^2)^{-m} \sim & \sum_j \int_0^\infty \phi(r)\sigma_j^{\pm}(r)(rz)^{-j} r dr + \sum_j \Big(\int_X \phi_i p_j\Big) z^{-j} \\
& + \sum_{k=0}^{\infty} z^{-k-1} \int_0^\infty \frac{1}{k!}\zeta^k \sigma^{(k)}(0,\zeta) d\zeta \\
& + \sum_{k=0}^{\infty} z^{-k-1} a \log z \sigma_{k+1}^{(k)}(0)/k!
\end{aligned}
$$

To construct a parametrix for $(\triangle^{\pm}+\lambda)^{-m}$, we patch the interior parametrix $P_i^{\pm}$ with the boundary parametrix above. From [5], the remainder term does not affect the asymptotics of the trace. Thus we can use this parametrix to compute the expansion.

$$
\begin{aligned}
z^{-2m} ind(D^+) = & \sum_j \Big(\int_X \phi_i (p_j^+ - p_j^-)\Big) z^{-j} \\
& + \sum_j \int_0^\infty \phi(r)(\sigma_j^+(r)(rz)^{-j} - \sigma_j^-(r)(rz)^{-j}) r dr \\
& + \sum_{k=0}^{\infty} z^{-k-1} \int_0^\infty \frac{1}{k!}\zeta^k (\sigma_+^{(k)} - \sigma_+^{(k)})(0,\zeta)) d\zeta \\
& + \sum_{k=0}^{\infty} z^{-k-1} a \log z (\sigma_{+k+1}^{(k)}(0) - \sigma_{-k+1}^{(k)})(0)/k!
\end{aligned} \tag{2.3.11}
$$

So the terms in $z^{-2m}\log z$ coming from $\triangle^+$ and $\triangle^-$ must cancel:

$$
\int_N p_{2m}^+ dx = \int_N p_{2m}^- dx
$$

thus $\sigma_{2m}^+ = \sigma_{2m}^-$. Hence (4.11) becomes:

$$
\begin{aligned}
ind(D^+) = & \int_X \phi_i (p_+^{2m} - p_-^{2m})) \\
& + \int_0^\infty \frac{\zeta^{2m-2}}{(2m-2)!} (\sigma_+^{(2m-2)}(0,\zeta) - \sigma_-^{(2m-2)}(0,\zeta)) d\zeta
\end{aligned} \tag{2.3.12}
$$

Since $p_{2m}^+ - p_{2m}^-$ vanishes in the tubular neighborhood, we can drop ϕ_i in the first integral. Now it remains to compute the second integral above. Define

$$h_{\pm}(w) = \int_0^\infty \frac{\zeta^w}{(2m-1)!}\sigma_{\pm}^{(2m-1)}(0,\zeta)d\zeta.$$

It has been shown in [5] that the integral is just the analytic continuation of $h_+(w) - h_-(w)$, i. e., it is equal to $Res_0(h_+ - h_-)(2m-1)$, where $Res_k h(w_0)$ is the coefficient of $(w-w_0)^{-k}$ in the Laurent expansion of the meromorphic function $h(w)$.

Thus

$$\mathrm{Ind}(D^+) = \int_M \phi_i(p_+^{2m} - p_-^{2m}) + \mathrm{Res}_0(h_+ - h_-)(2m-1). \tag{2.3.13}$$

From [7],

$$h_{\pm}(w) = \frac{\Gamma(\frac{w+1}{2})\Gamma(m-1-\frac{w}{2})}{4\sqrt{\pi}\Gamma(m)}\sum_{b\neq 0}\frac{\Gamma(\nu_{\pm}+\frac{w+3}{2}-m)}{\Gamma(1+\nu_{\pm}-\frac{w+3}{2}+m)}.$$

Define ζ function of A_r as follows:

$$\zeta_{\pm}(z) = \sum_{b\neq 0} \mid b \pm \frac{1}{2} \mid^{-z}.$$

Let $z = (w+1-2m)/2$, then $h_{\pm}(w) = \frac{\Gamma(z+m)\Gamma(-z-\frac{1}{2})}{4\sqrt{\pi}(m-1)!}\sum_{b\neq 0}\frac{\Gamma(\nu_{\pm}+z+1)}{\Gamma(\nu_{\pm}-z)}$. This expression is analyzed in [7]:

$$\sum_{b\neq 0}\frac{\Gamma(\nu_{\pm}+z+1)}{\Gamma(\nu_{\pm}-z)} = \sum_{j=0}^{N} Q_j(z)\zeta_{\pm}(j-1-2z) + R(z),$$

$R(z)$ is analytic in $Re(z) < 1, R(0) = 0$. So if $\zeta_+(s) - \zeta_-(s)$ only has simple poles, we only need the linear parts of Q_j to get $Res_0(h_+ - h_-)(2m-1)$. From [7], we have

$$Q_j(z) = O(z^2), j \quad odd$$

$$Q_j(z) = -z\frac{2}{j}B_j, j \quad even > 0$$

B_j is the $j-$th Bernoulli number, we have $B_2 = \frac{1}{6}$ which we are going to use in our computations.

So $Res_0(h_+ - h_-)(2m-1)$ is the $0-$th residue at 0 of the difference function of the following two functions:

$$\frac{\Gamma(z+m)\Gamma(-z-\frac{1}{2})}{4\sqrt{\pi}(m-1)!}(\zeta_+(-1-2z) - \sum_{k=1}^{N/2} k^{-1}B_k z\zeta_+(2k-1-2z)),$$

$$\frac{\Gamma(z+m)\Gamma(-z-\frac{1}{2})}{4\sqrt{\pi}(m-1)!}(\zeta_-(-1-2z) - \sum_{k=1}^{N/2} k^{-1}B_k z\zeta_-(2k-1-2z)).$$

If f is analytic and g has a simple pole at 0, then $Res_0(fg)(0) = f(0)Res_0 g(0) + f^{'}(0)Res_1 g(0)$, from this and noting that $\frac{\Gamma^{'}(m)}{\Gamma(m)} = \sum_{j=1}^{m-1}\frac{1}{j} - \frac{1}{\gamma}$, γ is Euler's constant, and $\Gamma^{'}(-\frac{1}{2}) = -2\sqrt{\pi}$,

$$\begin{aligned} Res_0(h_+ - h_-)(2m-1) &= -\frac{1}{2}Res_0(\zeta_+ - \zeta_-)(-1) \\ &- \frac{1}{4}\sum_{k\geq 1}(-1)^k\frac{1}{k}B_k Res_1(\zeta_+ - \zeta_-)(2k-1) \\ &+ \frac{1}{2}(\frac{\Gamma^{'}(-\frac{1}{2})}{4\sqrt{\pi}} + \frac{1}{2}(\sum_{j=1}^{m-1}\frac{1}{j} - \gamma))Res_1(\zeta_+ - \zeta_-)(-1) \\ &- \frac{1}{2}dimV_{0,0} \end{aligned} \tag{2.3.14}$$

while $(\zeta_+ - \zeta_-)(z)$ is given by the following:

$$\begin{aligned}
(\zeta_+ - \zeta_-)(z) &= \sum_{b\neq 0}(\frac{1}{\mid b+\frac{1}{2}\mid^z} - \sum_{b\neq 0}\frac{1}{\mid b-\frac{1}{2}\mid^z}) \\
&= \sum_{b\neq 0}(\mid b\mid^{-z}((1+\frac{1}{2b})^{-z} - (1-\frac{1}{2b})^{-z}) \\
&= 2\sum_{b\neq 0}\mid b\mid^{-z}\sum_{k\geq 0}\binom{2k+1}{-z}(2b)^{-2k-1} \\
&= \sum_{k\geq 0}2^{-2k}\binom{2k+1}{-z}\sum_{b\neq 0}\mid b\mid^{-z-2k-1} signb \\
&= \sum_{k\geq 0}2^{-2k}\frac{(-z)(-z-1)\cdots(z-2k)}{(2k+1)!}\eta_{A_r}(z+2k+1) \\
&= \eta_{A_r}(z+1) + \sum_{k\geq 1}2^{-2k}\frac{(-t)(-t-1)\cdot\cdot(t-2k)}{(2k+1)!}\eta_{A_r}(z+2k+1)
\end{aligned}$$

here $\eta_{A_r}(z)$ is the η function of the operator A_r on N_r.

From (4.13), we have

$$Res_0(\zeta_+ - \zeta_-)(-1) = \eta_{A_r}(0) + \sum_{k\geq 1}2^{-2k}\frac{1}{2k(2k+1)}Res_1\eta_A(2k) \tag{2.3.15}$$

For $k \geq 1$,

$$Res_1(\zeta_+ - \zeta_-)(2k-1) = -\sum_{j\geq 0}2^{-2j}\binom{2k+2j-1}{2j+1-1}Res_1\eta_A(2j+2k) \tag{2.3.16}$$

Lastly we can get the explicit index formula by using the calculation for $\eta_A(s)$ in Section 2:

$$\begin{aligned}
Res_0(h_+ - h_-)(2m-1) = &-\frac{1}{2}\eta_A(s) - \frac{1}{2}\sum_{l\geq 1}2^{-2l}\frac{Res_1\eta_A(2l)}{2l(2l+1)} \\
&-\frac{1}{4}\sum_{l\geq 1}\sum_{i\geq 1, i+j=l}(-1)^i\frac{B_i}{i}(-1)2^{-2j}\binom{2l-1}{2j+1}Res_1\eta_A(2l) \\
&-\frac{1}{2}\dim V_{0,0}
\end{aligned} \tag{2.3.17}$$

And we can write the first three terms explicitly:

$$
\begin{aligned}
-\frac{1}{2}\eta_A &= -\frac{1}{2}(-2)\sum_{l\geq 1}\frac{\int_B \hat{A}(B)e^{2l-1}}{(2l-1)!}\zeta(-(2l-1)) \\
&= \sum_{l\geq 1}\frac{(-1)^{l-1}B_l \int_B \hat{A}(B)e^{2l-1}}{(2l)!} \\
&= \int_B \hat{A}(B)(\frac{1}{2}\coth\frac{e}{2} - \frac{1}{e})
\end{aligned}
$$

In the third equality above we used equation (2.1.4).

For the sum of the middle two terms in $Res_0(h_+ - h_-)(2m-1)$, we claim it is equal to $\int_B(-\frac{\coth\frac{e}{2}}{2\sinh\frac{e}{2}} + \frac{1}{e})$. To see this, note that

$$
\begin{aligned}
& -\frac{1}{4}\sum_{l\geq 1}\sum_{i+j=l,i\geq 1}\frac{(-1)^i B_i}{i}(-1)2^{-2j}\binom{2l-1}{2j+1}Res_1\eta_A(2l) \\
&= -\frac{1}{4}\sum_{l\geq 1}\sum_{i+j=1}\frac{(-1)^{i-1}B_i}{i}2^{-2j}\frac{(-2)\int_B \hat{A}(B)e^{2l-1}}{(2i-2)!(2j+1)!} \\
&= 2\sum_{l\geq 1}\sum_{i+j=l}\frac{(-1)^{i-1}B_i(2i-1)}{(2i)!}\frac{2^{-2j-1}\int_B \hat{A}(B)e^{2l-1}}{(2j+1)!}
\end{aligned}
$$

thus the sum of the middle two is

$$
2\int_B \hat{A}(B)\frac{d}{de}(\frac{1}{2}\coth\frac{e}{2})(\sinh\frac{e}{2}) = \int_B(-\frac{\cosh\frac{e}{2}}{2\sinh\frac{e}{2}} + \frac{1}{e}).
$$

Note $dimV_{0,0} = dimKer(D_B)$, we have

$$
Res_0(h+-h_-)(2m-1) = \int_B \hat{A}(B)\frac{1-\cosh\frac{e}{2}}{2\sinh\frac{e}{2}} - \frac{1}{2}\dim Ker(D_B).
$$

Combining this with (4.12), we have

$$
Ind(D^+) = \int_X \omega_{D^+} + \int_B \hat{A}(B)\frac{1-\cosh\frac{e}{2}}{2\sinh\frac{e}{2}} - \frac{1}{2}\dim Ker(D_B) \qquad (2.3.18)
$$

Since D^+ is the Dirac operator on a spin bundle, we know that $\omega_{D^+} = \hat{A}(X)$.

Thus

$$Ind(D^+) = \int_M \hat{A}(M) + \int_B \hat{A}(B)\frac{1-\cosh\frac{e}{2}}{2\sinh\frac{e}{2}} - \frac{1}{2}\dim Ker(D_B) \qquad (2.3.19)$$

Now suppose (M, g) is an arbitrary smooth Riemann metric on M, (p, rn) be a point near B, $p \in B$, n is a unit normal vector. Denote by A_n the shape operator of the submanifold B in the direction n. Then the metric g is of the following form near B, which is proved in [13]:

Lemma 2.3.1. *Let R denote the Riemann curvature $(0,4)$ tensor of (M, g). Then for u_1, $u_2 \in T_{(p,rn)}M$, we have*

$$\begin{aligned} exp^*g(u_1,u_2) &= g_0(u_1,u_2) - 2g(A_n\pi_*u_1, \pi_*u_2)r \\ &\quad + \{g(A_n\pi_*u_1, A_n\pi_*u_2) + R(\pi_*u_1, n, \pi_*u_2, n) + \frac{2}{3}R(\pi_*u_1, n, Ku_2, n) \\ &\quad \frac{2}{3}R(\pi_*u_2, n, Ku_1, n) + \frac{1}{3}R(Ku_1, n, Ku_2, n)\}r^2 + O(r^3) \end{aligned} \qquad (2.3.20)$$

Here exp is the exponential map, g_0 is the model metric we used in the previous sections, and K is the projection to the vertical tangent space.

Given a Riemannain metric g, the following Koszul formula is helpful if one wants to calculate the Levi-Civita connection:

Lemma 2.3.2. *Koszul Formula (cf [19])*

Let (M, g) be a Riemannian manifold, ∇ is the Levi-Civita connection, X, Y, Z are tangent vector fields, then

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X \cdot g(Y,Z) + Y \cdot g(Z,X) - Z \cdot g(X,Y) \\ &\quad + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) \end{aligned}$$

The Dirac operator associated with g is related to the Dirac operator D_0 associated with g_0 in the following way:

Theorem 2.3.3. *Let (M, g) be a smooth Riemannian manifold of dimension $2m$, B is an embedded submanifold with codimension 2. S is a Spin bundle over $M \backslash B$ which* *can not be extended to a spin bundle over M. Then near B, the Dirac operator D^+ can be written as*

$$D^+ = \partial_r + \frac{1}{2r} + \frac{1}{2}H + A_r + \eta(r).$$

H is the mean curvature of B, which we can view as a function on the unit circle bundle of B, and A_r is the perturbed operator introduced in Chapter 2. $\eta(r)$ is compactly supported in a small neighborhood of B, equal to $O(r)$ near B. Thus equivalently, we have that near B

$$D^+ = D_0^+ + \frac{1}{2}H + O(r),$$

D_0^+ is equal to $\partial_r + \frac{1}{2}r + A_r$.

Proof. Let $e_1, \cdots, e_{2m-2}$ be an oriented orthonormal frame for TB, and ζ is the infinitesimal of the s^1 action on N. First from the Koszul formula, it is straightforward to see that the contribution to the Levi-Civita connection ∇_M from the third and fourth part in the metric formula in Lemma 2.3.1 is $O(r)$, hence the same is true for D^+. Thus it suffices to consider metric $g_1 = g_0 - 2g(A_n\pi_*, \pi_*)r$. As shown in [13], at (p, rn), the Dirac operators D^+ and D_N are related by

$$D_1^+ = \partial_r + D_N - \frac{H_N}{2}.$$

Here D_N is the Dirac operator on N with metric induced from g_1, i.e., $g_N = r^2\omega \otimes \omega + \pi^* g_B - 2g(A_n\pi_*, \pi_*)$. H_N is the mean curvature of N at (p, rn).

For any fixed r, g_N is a small perturbation of the model metric we considered before, again from the Koszul formula, the Levi-Civita connection is perturbed by an $O(r)$ term, so is true for the Dirac operartor: $D_N^r = A_r + O(r)$.

At (p, rn), if we still use $e_1, \cdots, e_{2m-2}$ denote the pull back vectors from B, then under the metric g_1, $|e_i| = 1 + O(r)$, $|\zeta| = r$, and $\frac{e_i}{|e_i|}, \frac{\zeta}{r}, \partial_r$ is an orthonormal frame for TM. On the other hand, $H_N = \sum_{i=1}^{2m-2} g(\nabla_{e_i/|e_i|}n, e_i/|e_i|) + g(\nabla_{\zeta/r}n, \zeta/r)$. Since $g_1(\nabla_{\frac{\zeta}{r}}n, \frac{\zeta}{r}) = g_1(\nabla_n \frac{\zeta}{r} - \frac{\zeta}{r^2}, \frac{\zeta}{r}) = -\frac{1}{r}$, the last term in the above is $-\frac{1}{r}$. From the Koszul formula, $2g_1(\nabla_{e_i}n, e_i) = n \cdot g_1(e_i, e_i) = -2g_1(A_n e_i, e_i)$, the last equality follows from the definition of our metric g_1. So $H_N(p,rn) = -\frac{1}{r} - 2H(p,rn)$, thus $D_1^+ = \partial_r + \frac{1}{2r} + \frac{1}{2}H + D_N^r = \partial_r + \frac{1}{2r} + \mathcal{A}_r + \frac{1}{2}H = D_0 + \frac{1}{2}H$. $\square$

Now for any smooth metric g on M, the corresponding Dirac operator D is just a bounded perturbation of D_0: $D = D_0 + \frac{H}{2} + \eta(r)$, from Lemma 2.2.8., we know that the index of D will remain equal.

Proof of theorem 1.0.2. To prove that D^+ is Fredholm, notice that it suffices to prove

$$D^+ \oplus (P \circ R) : \mathrm{Dom}(D^+_{\max}) \to L^2(S^+) \oplus (\mathrm{Im}(P) \cap L^2_{-\frac{1}{2}}(S_B))$$

is Fredholm, for which the proof of proposition 17.2.5. in [12] applies here: first $PP_0 : \mathrm{Im}(P_0) \cap L^2_{-\frac{1}{2}} \to \mathrm{Im}(P) \cap L^2_{-\frac{1}{2}}$ is Fredholm, since $(P_0P)(PP_0) = P_0 + P_0(P - P_0)P_0$, so is a compact perturbation of the identity operator on $\mathrm{Im}(P_0)$, so PP_0 has a right inverse operator moduli compact operators, simliarly it also has a lefte inverse. Thus (D^+, PP_0) is Fredholm. At last, we can write $(D^+, P) = (D^+, PP_0) + (0, P(P - P_0))$, the first is Fredholm and the second is compact, so (D^+, P) is Fredholm. If we replace P by P_1 as in the theorem, following from the proof of proposition 17.2.6 in [12], D^+ is Fredholm.

As for the index, the first index formula has just been proved. For the second index formula, since the only difference in the boundary contribution is from $\ker(D_B)$, which now is projecting to $\ker(D_B^-)$ instead of to $\ker(D_B)$, thus the index formula follows immediately.

2.3.1 Weitzenbock formula.

In this section, we look at the Weitzenbock formula for spinors in the domains we considered before. For simplicity, we assume the metric g equal to $dr^2 + r^2\omega \otimes \omega + \pi^* g_B$ near B.

Let $M_\epsilon = \{p \in M \mid d(p,B) \geq \epsilon\}$, $\epsilon > 0$. For $\phi \in L^2(S^+), D^+\phi \in L^2(S^-)$, we have

$$\int_{M_\epsilon} \| D^+\phi \|^2 = \int_{M_\epsilon} \| \nabla\phi \|^2 + \int_{M_\epsilon} \frac{s}{4} \| \phi \|^2 + \int_{N_\epsilon} < \phi, D_\epsilon\phi > dg_\epsilon - \frac{1}{2\epsilon}\int_{N_\epsilon} \| \phi \|^2 \, dg_\epsilon \tag{2.3.21}$$

The last term is because with our metric, the boundary circle bundle has mean curvature $\frac{1}{\epsilon}$. As $\epsilon \to 0$, we will see that both $\int_{M_\epsilon} \| \nabla\phi \|^2$ and $\frac{1}{2\epsilon}\int_{N_\epsilon} \| \phi \|^2 \, dg_\epsilon$ diverge, but the divergent terms of this two cancel out, so the left hand side of the above equation remains finite.

For any $\phi \in L^2(S^+)$ such that $D^+\phi \in L^2(S^-)$, from the last section, $\phi = \phi_0 + r^{-\frac{1}{2}}\psi$, $\phi_0 \in L^2_1(S^+, M\backslash B)$, ψ is a section which in the tubular neighborhood is in $\pi^* L^2(S_B)$, and vanish in the interior. Thus $\int_{M_\epsilon} \| \nabla r^{-\frac{1}{2}}\psi \|^2$ ($\frac{1}{2\epsilon}\int_{N_\epsilon} \| \epsilon^{-\frac{1}{2}}\psi \|^2 \, dg_\epsilon$ respectively) is the term that renders $\int_{M_\epsilon} \| \nabla\phi \|^2$ ($\frac{1}{2\epsilon}\int_{N_\epsilon} \| \phi \|^2 \, dg_\epsilon$ respectively) divergent.

In the tubular neighborhood, the spin connection ∇ can be written as

$$\nabla = \nabla^1 + \nabla^2.$$

$$\nabla^1 = \partial_r \otimes dr + \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \otimes \omega \tag{2.3.22}$$

$$\begin{aligned}\nabla^2 = {} & \partial_{\varsigma_r} \otimes \eta_r + \partial_1 \otimes \eta^1 + \partial_2 \otimes \eta^2 - \frac{1}{2}k \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \otimes \eta^1 \\ & + \frac{1}{2}(-rn)\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \otimes \eta^2 + \frac{1}{2}rn \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \eta^1 \\ & + \frac{1}{2}rn \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \otimes \eta_r \end{aligned} \tag{2.3.23}$$

The divergent term of $\int_{M_\epsilon} \| \nabla\phi \|^2$ comes from $\int_{M_\epsilon} \| \nabla r^{-\frac{1}{2}}\psi \|^2$, which is

$$\begin{aligned}\int_{M_\epsilon} \| \nabla r^{-\frac{1}{2}}\psi \|^2 &= C + \int_{M_\epsilon\backslash M_a} \| \nabla^1 r^{-\frac{1}{2}}\psi \|^2 \\ &= C + \int_{M_\epsilon\backslash M_a} \frac{1}{4}r^{-3} \| \psi \|^2 + \| \frac{1}{2}r^{-\frac{1}{2}}\gamma(\frac{\zeta}{r}) \cdot \psi \otimes \omega \|^2 \\ &= C + \int_{M_\epsilon\backslash M_a} \frac{1}{4}r^{-3} \| \psi \|^2 + \frac{1}{4}r^{-3} \| \psi \|^2 \\ &= C + \int_\epsilon^a \frac{1}{2}r^{-2}dr \int_B \| \psi \|^2 \\ &= C - \frac{1}{2a}\int_B \| \psi \|^2 + \frac{1}{2\epsilon}\int_B \| \psi \|^2\end{aligned} \tag{2.3.24}$$

Here C is a constant which depends on ψ, and in the third equality above we used the fact that $\| \omega \|_{N_r} = \frac{1}{r}$.

At the same time, the divergent term of $\frac{1}{2\epsilon}\int_{N_\epsilon} \| \phi \|^2$ is

$$\frac{1}{2\epsilon}\int_{N_\epsilon} \| \epsilon^{-\frac{1}{2}}\psi \| \, dg_\epsilon = \frac{1}{2\epsilon}\int_B \| \psi \|^2 \, dg_B \tag{2.3.25}$$

Thus if we let

$$Q(\phi) = \lim_{\epsilon\to 0}\left(\int_{M_\epsilon} \| \nabla\phi \|^2 - \frac{1}{2\epsilon}\int_{N_\epsilon} \| \phi \|^2\right) \tag{2.3.26}$$

Also notice that $\lim_{\epsilon\to 0}\int_{N_\epsilon} < D_\epsilon\phi, \phi > = \int_B < D_B\psi, \psi >$, Weitzenbock formula for $\phi \in L^2(S^+), D^+\phi \in L^2(S^-)$ becomes

$$\int_{M\backslash B} \| \phi \|^2 = Q(\phi) + \int_{M\backslash B} \frac{s}{4} \| \phi \|^2 + \int_B < D_B\psi, \psi > \tag{2.3.27}$$

2.3.2 Index for *Spin^c* Case.

Let $\mathcal{S} \to M\backslash B$ be a *Spin^c* bundle, A_0 is a *Spin^c* connection such that restricted to the circle bundle P of B, it is $(\mathcal{S}\mid_P, A_0\mid_P) = \pi^*(S_B, A_B)$, where S_B is a *Spin^c* bundle over B, and A_B is a *Spin^c* connection on S_B. So A_0 has holonomy -1 along the circles linking B. Let ω be a connection form of P. Thus $(\mathcal{S}, A_0 + i\frac{1}{2}\omega)$ extends to a *Spin^c* pair $(\mathcal{S}_M, A_M)$, ω is a connection form on P. And $A_0 + i\frac{1}{2}\omega$ has holonomy zero along circles linking B.

Use A_0 as a reference connection, then all the $Spin^c$ connection can be characterized by $i\alpha \in \wedge^1(iT^*M\backslash B)$. Now suppose that when restricted to the tubular neighborhood , $i\alpha = ic\omega$, $c \in (0, \frac{1}{2})$ is constant. Let D_A^c be the Dirac operator associated with connection $A_c = A_0 + ic\omega$. In the tubular neighborhood U, as before, we have $L^2(S^+|_U) = L^2((0,1), L^2(S^+|_N) \otimes rdr)$. Since the $Spin^c$ bundle is a pull back of a $Spin^c$ bundle over B, the S^1 action on N again can be lifted to $\mathcal{S}$, and $L^2(S^+ |_N) = \oplus V_k$, where V_k is the eigenspace of the differential of the S^1 action, L_ζ.

The Dirac operator corresponding to A_c and $g |_N = r^2\omega \otimes \omega + \pi^* g_B$ is denoted by $\mathcal{A}_r^c$, and as in Chapter 2, can be written as

$$\mathcal{A}_r^c = \frac{1}{r}A_v + A_h - \frac{r}{4}\gamma(\frac{\zeta}{r})\gamma(\pi^* d\omega).$$

We also have the isometry $Q_k : V_k \to L^2(S_B \otimes L^{-k})$, such that when restricted to V_k, $A_h = Q_k^{-1} \circ A_k \circ Q_k$.

$A_r^c = \frac{1}{r}A_v + A_h$ is the twisted operator.

Let $\phi_{k,a} = \alpha \oplus \beta \in L^2(S_N)$ be a unit norm common eigenvector of L_ζ and A_h with eigenvalues ik and a respectively.

If $a \neq 0$, $k \neq 0$, then α, β span a two dimensional space, under this basis, A_r^c is represented by

$$A_{k,a} = \begin{pmatrix} -\frac{k+c}{r} & a \\ a & \frac{k+c}{r} \end{pmatrix}$$

If $a = 0$, $(\alpha, \beta) \in \ker(D_B)$, then α is the eigenvector of A_r, with eigenvalue $\frac{k+c}{r}$ and multiplicity $\dim\mathrm{Ker}(A_k |_{S_B} \otimes L^{-k})$, and β is the eigenvector of A_k with eigenvalue $-\frac{k+c}{r}$ and multiplicity $\dim\mathrm{Ker}(A_k |_{S_B^- \otimes L^{-k}})$.

If $k = 0$, up to the isometry, $A_r |_{V_0} = D_B$.

Then as in Chapter 2, the eta function $\eta_{A_1^0}$ of A_1^0 is again

$$\eta_{A^0}(s) = \int_B \hat{A}(B) f_s(e),$$

where $f_s(x) = -\sum_{k\geq 1} \frac{e^{kx} - e^{-kx}}{k^s} = -2\sum_{i\geq 1} \frac{\zeta(s-(2i-1))x^{2i-1}}{(2i-1)!}$.

For $0 < c < \frac{1}{2}$, the eta function $\eta_{A^c}(s)$ of A^c with holonomy $e^{i2\pi(c+\frac{1}{2})}$ is

$$\begin{aligned}\eta_{A^c}(s) &= \sum_{k\geq 0} \frac{\int_B \hat{A}(B)\mathrm{ch}(\det(S_B)L^{-k})}{(k+c)^s} + \sum_{k<0} \frac{\int_B \hat{A}(B)\mathrm{ch}((\det(S_B)L^{k}))}{(-k-c)^s} \\ &= \int_B \hat{A}(B) f_s(e)\end{aligned} \tag{2.3.28}$$

with

$$\begin{aligned}f_s(x) &= \sum_{i\geq 0}\sum_{j=0}^{i} c^{i-j}\zeta(s-j,c)\frac{x^i}{i!} \\ &+ \sum_{i\geq 0}\sum_{j=0}^{i} c^{i-j}\zeta(s-j,1-c)\frac{(-x)^i}{i!}\end{aligned}$$

Here $\zeta(s,a)$ is the Hurwitz Zeta function, given by

$$\zeta(s,a) = \sum_{k=0}^{+\infty} \frac{1}{(k+a)^s}.$$

$\zeta(s,a)$ defines a meromorphic function on the complex plane, with a simple pole at $s = 1$ of residue 1, and $\zeta(s,a)$ is continous in the second variable a.

Let

$$C^\infty(M\backslash B, \mathbb{S}^+; P) = \{\phi \in C^\infty \cap L^2, D^c_A\phi \in L^2\}. \tag{2.3.29}$$

Still let $D^c_{\max}$ denote the closure operator defined by above. When $c = 0$, define $Dom(D^k_A)$ is the kernel of the composition:

$$r^{\frac{1}{2}} Dom(D^c_{max}) \xrightarrow{R} L^2_{-\frac{1}{2}}(S_B \otimes L_B \otimes L^k) \xrightarrow{P_B} L^2_{-\frac{1}{2}}(S_B \otimes L_B \otimes L^k).$$

Here P_B is as usual the projection map to the negative spinors. When $c \neq 0$, for any $\phi \in D^c_{\max}$, the leading order term of ϕ is $\psi = c_1(r^{-\frac{1}{2}-c}\alpha \oplus r^{-\frac{1}{2}+c}\beta) + c_2(r^{\frac{1}{2}+c}\alpha' \oplus r^{\frac{1}{2}-c}\beta') + O(lnr)$, for some $\alpha, \beta, \alpha', \beta' \in L^2(S_P)$. Define

$$\mathrm{Dom}(D^c) = \{\phi \mid \lim_{r\to 0}|r^{\frac{1}{2}}\phi|_{L^2(S_N, g_1)} = 0\}.$$

Due to the behavior of $\phi \in D^c_{\max}$ for near B, we see that this boundary condition is exactly the same as the second boundary condition we considered before, i.e., projection to the negative spinors.

The main result in this section is the following theorem:

Theorem 2.3.4. *Let $S \to M\backslash B$ be a $Spin^c$ bundle, A_0 is a $Spin^c$ connection such that when restricted to the circle bundle P of B, $(S|_P, A_0|_P) = (\pi^* S_B, \pi^* A_B)$,$S_B$ is a $Spin^c$ bundle over B, A_B is a $Spin^c$ connection on S_B. If the connection in the tubular neighborhood is $A_0 + ic\omega$, then with the domain defined above, D^c_A is Fredholm and*

$$Ind(D^c_A) = \int_M \hat{A}(M) \wedge exp(\frac{i}{4\pi}(F_{A_M} - id\omega) - \int_B \hat{A}(B) \frac{1 - \cosh \frac{e}{2}}{2 \sinh \frac{e}{2}} + \frac{1}{2} Ind(D^+_B), \tag{2.3.30}$$

where F_M is the cuvature term of A_M. When M is four dimensional,

$$Ind(D^c_A) = \frac{\sigma(M) - c_1(S^+_M)^2}{8}.$$

where (S_M, A_M) is the extension of the pair $(S, A_0 + i\frac{1}{2}\omega)$.

Proof. Frehlomness can be proved in exactly the same way as before.

When $c = 0$, the index calculation is almost the same as the *Spin* case. The eta function of $A^r$is of the exact same form, which makes the boundary contribution the same. As for the interior form, notice that $F_X - id\omega$ is the curvature form of our $Spin^c$ connection.

When $c \neq 0$, the eta function $\eta_{A^c}(s)$ is given by (4.26), and since the Hurwitz function $\zeta(s, q)$ has simple pole at $s = 1$ with residue 1 and $\zeta(s, a)$ is continous in the second argument, we see that the boundary contribution to the index of D^c is continuous in c. But on the other hand, the index form of D^c also continously depends on c, they are all equal to $\int_M \hat{A}(M) \wedge \exp(\frac{i}{4\pi}(F_{A_m} - id\omega + i2cd\omega))$. Since $\mathrm{Ind}(D^c)$ is integer valued, we get that $\mathrm{Ind}(D^c) = \mathrm{Ind}(D^0)$. When $c = 0$, the boundary contribution is the same as before, the index formula follows immediately.

The formula for four dimensional case follows as below: Σ is *Spin*, we can write

$\mathcal{S}|_P = (\pi^* K_\Sigma^{-\frac{1}{2}} \oplus \pi^* K_\Sigma^{\frac{1}{2}}) \otimes H$, such that $(H, B) = \pi^*(H_\Sigma, B_\Sigma, A_\Sigma)$, H_Σ is a line bundle over Σ with degree k. And A_0 comes from the connection of the pull back *Spin* connection on $K_\Sigma^{-\frac{1}{2}} \oplus K_\Sigma^{\frac{1}{2}}$ and B_Σ. Thus $(\mathcal{S}, A_0 + i\frac{1}{2}\omega)$ extends to (S_X, A_X) such that $c_1(S_X^+)(\Sigma) = 2k + n$. Now from the general formula, $\text{Ind}(D^c) = \frac{\sigma(X) - (c_1(S_X^+) - d\omega)^2)}{8} - \frac{n}{8} - \frac{1}{2}k$, while $(c_1(S_X) - d\omega)^2 = c_1(S_X^+)^2 - 2c_1(S_X) \wedge d\omega + n$, and $c_1(S_X^+) \wedge d\omega = 2k + n$, we get

$$\text{Ind}(D^c) = \frac{\sigma(X) - c_1(S_X^+)}{8}.$$

□

Chapter 3

Moduli Space of Singular Monopoles.

3.1 Basic Seiberg Witten Theory.

Seiberg-Witten equations are first order elliptic equations. And the solutions are absolute minimum of a so called Seiberg-Witten functional. The Euler-Lagrange equations of the functional are second order equations, and Seiberg-Witten solutions also satisfy the Euler-Lagrange equations. One strategy to study the Seiberg-Witten solutions is to first solve the E-L equations, then to identify conditions under which certain solutions actually also solve the first order equations. The main source of reference is [10].

3.1.1 The monopole equations.

Let X be a compact connected oriented 4-manifold and fix a $Spin^c$ structure S on X. Such a $Spin^c$ structure always exists. As before, $\gamma : TX \to End(S)$ is the Clifford multiplication, and γ extends to an isomorphism of algebra bundles $C(TX) \to End(S)$. There is a natural splitting of S:

$$S^+ \oplus S^-$$

into the $\pm$ eigenspaces of $\gamma(e_1e_2e_3e_4)$ where e_1, e_2, e_3, e_4 is any positively oriented orthonormal frame of TX. Let L_S be the determinant line bundle of S^+ and S^-:

$$L_S = det(S^+) = det(S^-)$$

Denote by $\mathcal{A}$ the space of $Spin^c$ connection on S^+. For any given $A \in \mathcal{A}$, D_A is the associated Dirac operator.

The Seiberg-Witten monopole equations are a system of first order differential equations for a pair (A, Φ) where $A \in \mathcal{A}$ and $\Phi \in C^\infty(X, S^+)$. They read

$$D_A\Phi = 0, F_A^+ = \sigma^+((\Phi\Phi^*)_0). \tag{3.1.1}$$

Here the endomorphism $\Phi\Phi^* \in C^\infty(X, End(S^+))$ is defined by

$$\Phi\Phi^*\tau =< \Phi, \tau > \Phi$$

for $\tau \in C^\infty(X, S^+)$. Its traceless part is given by

$$(\Phi\Phi^*)_0\tau =< \Phi, \tau > \Phi - \frac{1}{2} \parallel \Phi \parallel^2 \tau.$$

Let $End_1(S^+)$ denote the bundle of traceless endomorphisms of S^+. The bundle isomorphism

$$\sigma^+ : End_0(S^+) \to \Omega^+T^*X \otimes \mathbb{C}$$

is the inverse of the map $\gamma : \Omega^+T^*X \to End_0(S^+)$ defined by Clifford multiplication. Recall that in the 4-dimensional case, γ identifies the imaginary self-dual 2-forms on X with the traceless Hermitian endomorphisms of S^+. Thus $\sigma^+((\Phi\Phi^*)_0)$ is an imaginary valued self-dual 2-form and so is F_A^+.

3.1.2 Space of Configurations.

In order to use these equations to produce a moduli space, out of which one can define Seiberg-Witten invariant of the $Spin^c$ structure, we need to put these equations in a nonlinear elliptic framework. The space of configurations is the space on which the equations define a function. It is the space of all pairs (A, Φ) where A is a $Spin^c$ connections and Φ is a spinor. For technical reasons, we need to work with Banach or Hilbert spaces. And usually people work with L_2^2 or L_1^p for $p > 4$. Here we choose the working space to be L_1^2, based on [10].

Let $L_1^2(S^+)$ be the space of sections of W^+ of L_1^2 class, and $\mathcal{A}_1^2$ be the connections on S^+ of L_1^2 class.

The Seiberg-Witten functional for a pair (A, Φ) is defined by

$$SW(A, \Phi) = \int_X (\| \nabla_A \Phi \|^2 + \| F_A^+ \|^2 + \frac{s}{4} \| \phi \|^2 + \frac{1}{8} \| \phi \|^4) dvol.$$

and s is the scalar curvature of (X, g).

Seiberg-Witten solutions are absolute minimum of this functional. The Euler-Lagrange equations of the Seiberg-Witten functional are

$$-\triangle_A \phi + \frac{s}{4}\phi + \frac{1}{4} \| \phi \|^2 \phi = 0, \tag{3.1.2}$$

$$d^* F_A^+ + \frac{1}{2}\text{Im} < \nabla_i \phi, \phi > e^i = 0. \tag{3.1.3}$$

Here $\triangle_A$ is the Laplacian, and $\nabla_i = \nabla_{e_i}$, $\{e_i\}$ is an orthonormal basis of TX.

The following maximum-principle property is proved by Kronheimer-Mrowka in [11]:

Theorem 3.1.1. *For a smooth solution (A, Φ) of the Seiberg-Witten equation,*

$$\| \phi(x) \| \leq max\{-s, 0\}.$$

It is not hard to see that the functional SW is well defined on $\mathcal{A}_1^2 \times L_1^2(S^+)$ and SW is smooth.

A suitable Lie group is required to serve as a gauge group. Let $\mathcal{G}_0 = \exp(iL_2^2(X, \mathcal{R}))$. Then $\mathcal{G}_0$ is a Lie group: obviously the quotient $Y = L_2^2/\sim$ is a Lie group with the usual addition of functions, $\sim$ is the equivalence relation in $L_2^2(X)$, $\phi_1 \sim \phi_2$ if and only if $\phi_1(x) - \phi_2(x) = 2\pi n$, for almost all $x \in X$, for some integer n. And $\mathcal{G}_0$ can be identified with Y by the exponential map. Hence $\mathcal{G}_0$ is a Lie group with the multiplication of functions.

Lemma 3.1.2. *Define* $\mathcal{G} = \bigcup g \cdot \mathcal{G}_0$. $\mathcal{G}$ *is a Lie group, here the union is over all components of* $C^\infty(X, s^1)$. $\mathcal{G}$ *acts smoothly on* $\mathcal{A}_1^2 \times L_1^2(S^+)$.

The proof of the following gauge fixing lemma can be found in [16].

Lemma 3.1.3. *For any pair* (A, ϕ), A_0 *a fixed smooth connection on* S^+, *there exists a gauge transformation* g, *such that* $\alpha = g \cdot A - A_0$ *satisfies* $d^*\alpha = 0$, *and*

$$\| \alpha \|_{L_1^2} \leq c_1 \| F_A^+ \|_{L^2} + c_2.$$

where c_1, c_2 *are constants.*

3.1.3 Regularity of solutions.

The crucial point in the proof of both the regularity and compactness of weak solutions is the L^∞ bound of the spinor ϕ. As long as this is guaranteed, the regularity can then be obtained through the standard elliptic bootstrap method.

Since now the configuration space is L_1^2, the usual maximal-principle argument does not apply to prove the L^∞ bound. And the authors in [10] use the method in [20] to prove the following:

Theorem 3.1.4. *Let* $(A, \phi) \in \mathcal{A}_1^2 \times L_1^2(S^+)$ *be a weak solution of* (5.1). *Then*

$$\| \phi \|_{L^\infty} \leq max\{-s(x), 0\}.$$

Proof. Let $s_0 = \min\{s(x) \mid x \in X\}$. If $s_0 \geq 0$, then since

$$\int \| \nabla_A \phi \|^2 + \frac{s}{4} \| \phi \|^2 + \frac{1}{4} \| \phi \|^4 = 0,$$

thus $\phi = 0$. Without loss of generality, assume $s_0 = -1$. Define test function η to be

$$\eta = \begin{cases} (\| \phi \| -1)\frac{\phi}{\|\phi\|}, & \text{for } \| \phi \| > 1, \\ 0, & \text{for } \| \phi \| \leq 1 . \end{cases} \tag{3.1.4}$$

and $\Omega = \{x \in X | \| \phi(x) \| > 1\}$.

Let $\nu = \frac{\phi}{\|\phi\|}$ for $\| \phi \| \geq 1$, $\chi(\Omega)$ be the characteristic function of Ω. By noticing that $\| \nu \| = 1$ and $\| \phi \| > 1$, we conclude

$$\nabla\eta = \chi(\Omega)((d \| \phi \|)\nu + (\| \phi \| -1)\nabla\nu) \in L^2,$$

since $d(\| \phi \|)\nu = \frac{<\nabla\phi,\phi>}{\|\phi\|}\nu =< \nabla\phi, \nu > \nu \in L^2$, and

$$\begin{aligned} \chi(\Omega)((\| \phi \| -1)\nabla\nu) &= \chi(\Omega)((\| \phi \| -1)\frac{\nabla\phi}{\| \phi \|} - \frac{1}{2}(\| \phi \| -1)\frac{< \nabla\phi, \phi > \phi}{\| \phi \|^3}) \\ &= \chi(\Omega)((1 - \frac{1}{\| \phi \|})\nabla\phi - \frac{1}{2}(1 - \frac{1}{\| \phi \|}) < \nabla\phi, \nu > \nu \in L^2). \end{aligned}$$

Now that (A, ϕ) is a solution of (5.1), we have

$$\begin{aligned} 0 &= \int_\Omega < \nabla_A\phi, \nabla_A\eta > + \frac{s}{4} < \phi, \eta > + \frac{1}{4} \| \phi \|^2 < \phi, \eta > \\ &= \int_\Omega < \nabla_A\phi, \nabla_A\eta > + \frac{1}{4}(\| \phi \|^2 + s)(\| \phi \| -1) \| \phi \| \\ &\geq \int_\Omega < \nabla_A\phi, \nabla_A\eta > + \frac{1}{4}(\| \phi \|^2 -1)(\| \phi \| -1) \| \phi \|, \end{aligned}$$

for $s \geq -1$. We show that the first term is also nonnegative: for

$$\begin{aligned} \int_\Omega < \nabla_A\phi, \nabla_A\eta > &= \int_\Omega < \nabla_A\phi, d(\| \phi \|)\nu > + < \nabla_A\phi, (\| \phi \| -1)\nabla_A\nu > \\ &=< d(\| \phi \|)\nu, d(\| \phi \|)\nu > + <\| \phi \| \nabla\nu, d(\| \phi \|) > \\ &+ < d(\| \phi \|)\nu, (\| \phi \| -1)\nabla_A\nu > + <\| \phi \| \nabla_A\nu, (\| \phi \| -1)\nabla_A\nu > \\ &=< d(\| \phi \|)\nu, d(\| \phi \|)\nu > + \| \phi \| (\| \phi \| -1) < \nabla_A\nu, \nabla_A\nu > \\ &\geq 0 \end{aligned} \tag{3.1.5}$$

The last equality in the above is due to the fact that $< \nu, \nabla_A \nu >= 0$ since ν is a unit normal vector. Thus Ω has measure zero, and $\| \phi \|_{L^\infty} \leq 1$.

□

With the L^∞ bound at hand, it is straightforward to prove the regularity property and the compactness theorem.

Theorem 3.1.5. *Let $(A, \phi) \in \mathcal{A}_1^2 \times L_1^2(S^+)$ be a solution of the Seiberg-Witten equation. Then there exists a gauge transformation $g \in \mathcal{G}$ such that $g(A, \phi) = (g(A), g^{-1}\phi)$ is smooth. Moreover, for any sequence (A_i, ϕ_i), there exists a subsequence(also denoted by (A_i, ϕ_i)), and g_i, such that $g_i \cdot (A_i, \phi_i)$ converges in C^∞ to a solution (A, ϕ).*

Proof. As before, we choose a base connection A_0. From the gauge fixing lemma, there exists a gauge transformation g, $\alpha = g \cdot A - A_0$, such that $d^*\alpha = 0$. Thus the Seiberg-Witten equation is of the form

$$\begin{cases} (D_{A_0}\phi = i\alpha \cdot \phi, \\ (d\alpha)^+ = \sigma(\phi) - F_{A_0}^+, \\ d^*\alpha = 0, \end{cases} \tag{3.1.6}$$

Since $\| \phi \|_{L^\infty} < C$, again from Lemma 3.1.3, α has uniformly bounded L_1^p norm for any $p > 1$. In particular, take $p > 4$, we see that it is bounded in C^0 by the Sobolev Embedding Theorem, and hence $\alpha \cdot \phi$ is bounded in L^p for any p, It follows that ϕ is bounded in L_1^p for any $p > 4$. Since L_1^p has Banach algebra structure, it follows from elliptic bootstrap that (α, ϕ) is bounded in L_k^p for any $k \geq 2$. Then sobolev embedding theorem says that (α, ϕ) is smooth.

For compactness property, notice that ϕ has a uniform L^∞ bound, from the exact argument above, for any sequence (A_i, ϕ_i), we have a uniform L_k^p bound for any integer $k \geq 1$ and $p > 4$, thus Rellich's Theorem produces a subsequence converges in L_k^p for all k. This proves the compactness.

□

3.2 Configuration space over the complement of the surface.

3.2.1 $Spin^c$ structures.

As before, X is a smooth, connected, oriented closed four manifold, Σ is a smooth, oriented, embedded surface in X. Denote by P the circle bundle of the normal bundle of Σ with degree n. ω is a connection form on P, such that $d\omega = \pi^* dvol_\Sigma$. The following lemma summarizes some relevant cohomological information about these spaces. The (co)homology groups are with integer coefficients.

Lemma 3.2.1. *Let X, P, Σ, $n \neq 0$ are as above.*

$$H^1(P) \cong H^1(\Sigma), \quad , H^2(P) \cong \mathbb{Z}_n \oplus H^1(\Sigma)$$

$$H^1(X \backslash \Sigma) \cong H^1(X), H^2(X \backslash \Sigma) \cong H^2(X)/\xi \oplus F,$$

where ξ is the Poincare dual of Σ and F is a subgroup of $H^1(\Sigma)$.

Proof. The cohomology groups of P comes from the Gysin exact sequence of $S^1 \hookrightarrow P \to \Sigma$. The Gysin exact sequence implies

$$H^2(P;\mathbb{Z}) \cong \pi^* H^2(\Sigma;\mathbb{Z}) \oplus H^1(\Sigma;\mathbb{Z}) \cong \mathbb{Z}_{|n|} \oplus \mathbb{Z}^{2g}.$$

As for the cohomology of $X \backslash \Sigma$, we can use the Poicare duality and the excision principle: $H^2(X \backslash \Sigma) \cong H_2(X \backslash \Sigma, P) \cong H_2(X, U)$, U is the tubular neighborhood of Σ. And $H_2(X, U)$ can be calculated from the exact sequence of the pair $H_2(X, U)$:

$$H_3(X) \to H_3(X, N) \to H_2(U) \to H_2(X) \to H_2(X, U) \to H_1(U).$$

Since $H_1(N)$ is a free abelian group, we have $H_2(X, U) \cong H_2(X)/\xi \oplus F$, F is some subgroup of $H_1(U)$. Similarly $H^1(X \backslash \Sigma) \cong H_3(X, U) \cong H^1(X)$.

□

Any line bundle H on P such that $c_1(H) = \hat{k} \in \mathbb{Z}_{|n|}$ can be obtained as a pull back of a line bundle H_Σ on Σ, $c_1(H_\Sigma) = k \in \mathbb{Z}$. Note that k is determined only modulo n. So this does not give a faithful correspondence between the line bundles on Σ and the torsion line bundles over P. However, if one equips the line bundle with a connection, one gets a faithful correspondence, as described in [17]:

Lemma 3.2.2. *([17]) There is a natural one-to-one correspondence between pairs bundles-with-connection over Σ and bundles- with-connection over P, whose curvature forms pull up from Σ and whose fiberwise holonomy is trivial.*

Let C be a flat connection on H with holonomy $\exp(2ik\pi/n)$. Set $C_k = C + (2ik\pi/n)\omega$, then C_k has trivial holonomy along the fibers and the curvature form of C_k is horizontal, $F_{C_k} = 2ik\pi d\omega$. Thus C_k is the pull back of a connection C_Σ on a line bundle H_Σ whose curvature satisfies

$$\pi^* F_{C_\Sigma} = 2\pi ik/nd\omega = 2ik\pi^* dvol_\Sigma.$$

Thus $c_1(H_\Sigma) = k$.

Now any $Spin^c$ bundle S_P over P with torsion determinant line bundle can be written as $S^+ = \pi^* K_\Sigma^{-\frac{1}{2}} \otimes H \oplus \pi^* K_\Sigma^{\frac{1}{2}} \otimes H$.

Suppose S is a $Spin^c$ structure on $X \backslash \Sigma$, whose restriction to P is determined by a torsion line bundle and it extends to a $Spin^c$ structure on X. The extension is not unique, and any two extensions differ by a power of the line bundle N, the normal bundle of Σ.(The pull back of N to P is trivial, so we can extend N to a line bundle over X, still denoted by N). This fact can be seen as follows: first changing the $Spin^c$ extension on X by a power of N does not change the induced $Spin^c$ structure on $X \backslash \Sigma$, since N is trivial on $X \backslash \Sigma$. On the other hand, suppose two $Spin^c$ structures on X differ by a line bundle L. If restricted to $X \backslash \Sigma$, they give the same $Spin^c$ structure, then L is trivial on $X \backslash \Sigma$, hence $c_1(L)$ lies in the kernel of the restriction map $H^2(X) \to H^2(X \backslash \Sigma)$, which is generated by the Poincare dual of Σ, thus $c_1(L) = mc_1(N)$, $L = N^m$.

Although the extension of a given $Spin^c$ structure S on $X \backslash \Sigma$ is not unique, when coupled with an extendable connection, from the above lemma, we do have a unique extension. More specifically, write $S^+|_P = (\pi^* K_\Sigma^{-\frac{1}{2}} \oplus \pi^* K_\Sigma^{\frac{1}{2}}) \otimes H$, let A_k a *Spin*^{c} connection on S^+, such that restricted to P, it is obtained by coupling the *Spin* connection on $\pi^* {K_\Sigma}^{-\frac{1}{2}} \oplus \pi^* {K_\Sigma}^{\frac{1}{2}}$ and a connection C_k on H. (Recall that $C_k = C + ik/n\omega$ is a connection with trivial holonomy on H.) And A_k has -1 holonomy since the *Spin* connection on $\pi^* K_\Sigma^{-\frac{1}{2}} \oplus \pi^* K_\Sigma^{\frac{1}{2}}$ does. From the above lemma, the pair (H, C_k) is a pull back of a unique pair (H_Σ, C_Σ) on Σ. Thus $(S^+, A_k + \frac{1}{2} i\omega)$ extends uniquely to a pair (S_X, A_X) on X. Let $\mathcal{L}$ denote the determinant line bundle of S_X, then $c_1(\mathcal{L})(\Sigma) = 2k + n$.

3.2.2 Configurations space of singular monopoles.

In order to be able to consider the Seiberg Witten equations over the complement of the surface, we need to choose an appropriate function space to study the equations. Recall that we chose to work with smooth metric g which can be extended to X. And the *Spin*^{c} structure S is the one we mentioned above, whose determinant line bundle is torsion restricted to P. Let $A_k + ic\omega$ be the reference connection. Thus $A_k + ic\omega$ has holonomy $\exp(2i(c - \frac{1}{2})\pi)$ along the circles. Notice that if we change c to $c + l$ for some integer l, then $A_k + il\omega + ic\omega$ has the same holonomy and $A_k + il\omega = A_{k+ln}$. Thus we only need to fix $c \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$. When $c = \frac{1}{2}$, the holonomy is trivial, and it reduces to the usual analysis on the compact manifold X.

Now for the spinors, we will use the space we studied in Chapter 2, i.e., $V^+ = \{\phi \in L^p(S^+), D_{A_k}\phi \in L^p(S^-), (r^{\frac{1}{2}}\phi)(0) \in S^+\}$ for $c = 0$, $V^+ = \{\phi \in L^p(S^+), D_{A_k}\phi \in L^p(S^-), (r^{\frac{1}{2}}\phi)(0) = 0\}$ for $c \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$.

For the connections, we use the following weighted sobolev spaces, with different weight of the spaces introduced in [11].

For some $p > 2$,

$$C_1^p = \{\alpha \in \Omega^1(X \backslash \Sigma) \mid \alpha \in L^p(i\Omega^1(X \backslash \Sigma)), \nabla_X \alpha \in r^{-1} L^p(i\Omega^+(X \backslash \Sigma))\}, \tag{3.2.1}$$

$$C_2^p = \{f \mid f \in rL^p, \nabla f \in L^p, \nabla^2 f \in r^{-1} L^p\}. \tag{3.2.2}$$

And $C_0^p = \{f \mid f \in r^{-1}L^p\}$. Here ∇_X is the induced Levi- Civita connection on 1-forms, and $\alpha \in r^c L^p$ means that $r^{-c}\alpha \in L^p$.

Let $\mathcal{A}_k = \{A_k + ic\omega + \alpha \mid \alpha \in C_1^p\}$. And the configuration space we choose to work with is $\mathcal{C}_k = V^+ \times \mathcal{A}_k$.

The above spaces are isometric to the following weighted Sobolev spaces introduced in [11]:

For nonnegative integer k and $p > 2$, W_k^p is the completion of the space of compactly supported smooth functions on $X \backslash \Sigma$ in the norm

$$\| f \|_{W_k^p} = \| \frac{1}{r^k} f \|_p + \| \frac{1}{r^{k-1}} \nabla f \|_p + \cdots + \| \nabla^k f \|_p .$$

Obviously $W_k^p \subset L_k^p$, and $W_0^p = L^p$. And the multiplication map $T : C_k^p \to W_k^p$, $T(f) = r^2 f$ is an isometry map.

In [11], the authors studied singular Yang-Mills equations along an embedded surface. There the connection near the surface is

$$A = a^0 + b$$

with a^0 being the diagonal part and b being the off diagonal part such that a^0 is in the ordinary L_1^p space and b is in W_1^p as above. The anti-self-dual equations can be written as

$$d^+ a^0 = -[b, b]^0$$
$$d^+ b = -[a^0, b]$$

Formally, the Seiberg Witten equations look simliar as the above equations, with a^0 being replaced by the *spin*^{c} connection and b being replaced by the spinor. The configuration spaces we chose here are actually the analogue of the configurations spaces for the singular connections, at least up to an isometry: for any spinor $\phi \in V^+$, since $\phi \sim O(r^{c-\frac{1}{2}})$, we can check that $r\phi \in W_1^p(S^+, X \backslash \Sigma)$ for appropriate constant $p > 2$, also for any $\alpha \in C_1^p$, we know that $r\alpha \in L^p$, and $\nabla(r\alpha) = dr \otimes \alpha + r\nabla\alpha \in L^p$, so $r\alpha \in L_1^p$. Thus up to the weight r, our configuration spaces are such that spinors lie in W_1^p and connections lie in L_1^p.

It is shown in [11] that usual embedding theorem also holds for W_k^p, and so does the elliptic theory: let

$$w_k^p = k - p/4,$$

Lemma 3.2.3. *([11].) If $k \geq l$ and $w_k^p \geq w_l^q$, then there is an inclusion $W_k^p \hookrightarrow W_l^q$. If both inequalities are strict, then the inclusion is compact.*

Lemma 3.2.4. *The usual statements of elliptic regularity also holds for S:*

$$S = (d^* + \frac{c_1}{r}) \oplus (d^+ + \frac{c_2}{r}) : \Omega^1(X) \to \Omega^0(X) \oplus \Omega^+(X).$$

Due to the isometry T, we have

Lemma 3.2.5. *The Sobolev embedding property also holds for C_k^p. And the elliptic regularity also holds for $d^* \oplus d^+$ acting on C_k^p.*

Define $\mathcal{A} = A_k + C_1^p$. The configuration space we choose to work with is $\mathcal{C} = V^+ \times \mathcal{A}$.

A gauge transformation $\sigma \in L^p_{2,loc}(X \backslash \Sigma, S^1)$ belongs to the gauge group based at Σ, $\mathcal{G}_\Sigma$, if there exists $f \in C_2^p$, so that $\sigma = \exp(f)$. Define $\mathcal{G} = \bigcup_g g \cdot \mathcal{G}_\Sigma$, where the union is over all the harmonic representatives of $H^1(X)$.

3.3 Deformation Complex.

Seiberg-Witten equations on the complement of the surface give rise to a well defined map

$$SW : V^+ \times \mathcal{A} \to L^p(S^-, X \backslash \Sigma) \times r^{-1} L^p(i\Omega^+).$$

The deformation complex $\mathcal{D}_{(A,\Psi)}$ of the solution (A, Ψ), taking into account of the action of $\mathcal{G}$, is

$$0 \to C_2^p \longrightarrow^{K_{(A,\Psi)}} V^+ \times \mathcal{A} \to^{T_{(A,\Psi)}SW} L^2(S^-) \times r^{-1} L^p(i\Omega^+(X \backslash \Sigma)) \to 0, \quad (3.3.1)$$

where $K_{(A,\psi)}(f) = (2df, -f\psi)$ is the infinitesimal gauge group action and

$$T_{(A,\psi)}SW(\alpha, \psi) = (d^+\alpha - 2Q(\Psi, \psi), D_A\psi + \alpha \cdot \Psi)$$

is the linearization of the Seiberg-Witten map at (A, Ψ). Here Q is the bilinear map associated to the quadratic map q in the Seiberg-Witten equations. The cohomology groups of this complex provide some local information about the based moduli space. The first observation is the following:

Lemma 3.3.1. *The zeroth cohomology group of the deformation complex is trivial.*

Proof. If $f \in C_2^p$ is in the kernel of $K_{(A,\Psi)}$, then $df = 0$. Thus f is constant and from our definition of G_2^2, f converges to 0 near Σ, it must be identically equal to zero. □

The first cohomology group of the deformation complex is called the *Zariski tangent space* of the moduli space and the second cohomology group is called the *obstruction space.* If (A, Ψ) is a regular point for the Seiberg-Witten map, then the obstruction space vanishes and the first cohomology of the complex is isomorphic to the tangent space of the moduli at $[A, \Psi]$.

Let (A, Ψ) be a configuration on $X \backslash \Sigma$. The adjoint $K^*_{(A,\Psi)}$ of the infinitesimal gauge group action $K_{(A,\Psi)}$ is defined with respect to the following inner products: for imaginary-valued forms α and β let

$$< \alpha, \beta >_r = \int_X \alpha \wedge *\beta r^{-2},$$

where $*$ is the complex anti-linear extension of the Hodge star operator.

Then $K^*_{(A,\Psi)}(\alpha, \psi) = 2r^2 d^* r^{-2} + 2i\mathrm{Im} < \Psi, \psi >$; we can drop the factor 2, thus obtaining the wrapped-up complex

$$F : \mathcal{A} \times V^+ \to L^p(i\Omega(X) \oplus i\Omega^+(X)) \oplus L^p(S^-),$$

$$F(\alpha, \psi) = (d^*\alpha - \frac{2}{r} + i\mathrm{Im} < \Psi, \psi >, d^+\alpha - 2Q(\Psi, \psi), D_A\psi + \alpha \cdot \Psi).$$

3.4 The equations over a Kahler manifold.

On a closed Kahler manifold (X, Ω), there is a canonical $Spin^c$ structure given by

$$S_0^+ = \Omega^0(X; C) \oplus \Omega^{0,2}(X; C),$$

$$S_0^- = \Omega^{0,1}(X; C).$$

Any other $Spin^c$ structure S differs from S_0 by tensoring with some complex line bundle L, and is given by

$$S^+ = \Omega^0(X; L) \oplus \Omega^{0,2}(X; L),$$

$$S^- = \Omega^{0,1}(X; L).$$

Furthermore, the determinant of S^+ is $K_X^{-1} \otimes L^2$.

Recall that (S, A_k) is a pair of $Spin^c$ bundle and connection over $X \backslash \Sigma$, such that $(S, A_k + i\frac{1}{2}\omega)$ extends uniquely to a pair of (S_X, A_X) on X. Write $S_X^+ = \Omega^0(X; L) \oplus \Omega^{0,2}(X; L)$. And $\mathcal{L}$ is the determinant line bundle of S_X, $c_1(\mathcal{L})(\Sigma) = 2k + n$. For any holomorphic connection B on $\mathcal{L}$, let B_1 be the induced holomorphic connection on L.

Let $(A, \phi) \in \mathcal{C}_k$ be a solution to the Seiberg-Witten equation. From our definition of the singular configuration space, $A = A_k + ic\omega + ia$, for some $a \in C_1^p$. Write $\phi = (\alpha, \beta) \in S^+ \oplus S^-$, α has leading order $r^{\frac{1}{2}-c}$ near Σ while β is of $O(1)$ near Σ. Due to the Kahler structure, we can write the Seiberg-Witten equations:

$$\begin{cases} \overline{\partial}_A \alpha + \overline{\partial}_A^* \beta = 0, \\ (F_A^+)^{1,1} = \frac{i}{4}(\| \alpha \|^2 - \| \beta \|^2)\omega, \\ F_A^{0,2} = \frac{\overline{\alpha}\beta}{2}. \end{cases} \tag{3.4.1}$$

Now the configuration space $\mathcal{C}_k$ is acted on by the complex gauge group which is defined to be $\mathcal{G}^C = \text{Map}(X \backslash \Sigma, C^*)$. A complex gauge transformation e^f acts on configurations by

$$A \to A + \partial \overline{f} - \overline{\partial} f$$

$$\alpha \to e^f \alpha$$

$$\beta \to e^{-\overline{f}} \beta$$

When f is pure imaginary this coincides with the usual action of $\mathcal{G}$. And it is easy to verify that the first and the third equation in (3.4.1) are preserved by this action.

Choose $e^f = r^{c-\frac{1}{2}}$, and $\alpha_1 = r^{c-\frac{1}{2}}\alpha$, $\beta_1 = r^{\frac{1}{2}-c}\beta$. Under this change, $B = A+(\frac{1}{2}-c)\partial \ln r - (\frac{1}{2}-c)\overline{\partial} \ln r$ is a nonsingular $Spin^c$ connection which can be extended to a $Spin^c$ connection on S_X.

Now the standard proof that Seiberg-Witten solutions on Kahler manifold have vanishing $F_A^{0,2}$ directly generalizes to our case: we have the harmonic spinor equation

$$\overline{\partial}_B \alpha + \overline{\partial}_B^* \beta = 0,$$

Apply $\overline{\partial}_B$ to this equation we get

$$\overline{\partial}_B \overline{\partial}_B \alpha_1 + \overline{\partial}_B \overline{\partial}_B^* \beta_1 = 0.$$

Of course, $\overline{\partial}_B \overline{\partial}_B \alpha = F_B^{0,2} \cdot \alpha_1$, but $F_A^{0,2} = F_B^{0,2} = \overline{\alpha}\beta = \overline{\alpha}_1 \beta_1$, we have

$$\| \alpha_1 \|^2 \beta_1 + \overline{\partial}_B \overline{\partial}_B^* \beta_1 = 0,$$

Due to the behavior of α_1 and β_1 near Σ(α_1 is of order $O(1)$, and β_1 is of order $O(r^{\frac{1}{2}-c})$), we have that $\int_P < \overline{\partial}^* \beta_1, \beta_1 > r d g_{N_1}$ goes to 0 when r goes to 0, here g_{N_1} is a fixed metric on N. Thus we are justified to take the L^2 inner product with β and get

$$\int_X \frac{1}{2} \| \alpha_1 \|^2 \| \beta_1 \|^2 \, dvol + \| \overline{\partial}_B^* \beta_1 \|_{L^2}^2 = 0,$$

Since each of these terms is nonnegative, it follows that they both vanish. And $\overline{\alpha}_1 \beta_1 = 0$. This means that $F_A^{0,2} = 0$, and hence A is a holomorphic connection, β_1 is an anti-holomorphic section, and α_1 is a holomorphic section. In particular, if X is connected, from unique continuation principle, either α_1 or β_1 vanishes identically on X.

Furthermore, since

$$(F_A^+)^{1,1} = (F_B^+)^{1,1} + i(c - \frac{1}{2})d\omega = \frac{i}{4}(\| \alpha \|^2 - \| \beta \|^2)\Omega.$$

Thus $(2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \cdot \Omega = \int_X (2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \wedge \Omega = \frac{1}{8\pi} \int_X (\| \beta_1 \|^2 - \| \alpha_1 \|^2) dvol$, here ξ is the Poincare dual of Σ.

Since at least one of α_1 and β_1 is zero, we see that if $(2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \cdot \Omega$ is nonnegative, then $\alpha_1 = 0$ and if $(2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \cdot \Omega$ is non-positive, then $\beta_1 = 0$. Without loss of generality, we assume that $(2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \cdot \Omega$ is nonpositive. Thus the Seiberg-Witten equations reduce to the following:

$$\begin{cases} \bar{\partial}_{B_1} \alpha_1 = 0, \\ (2F_{B_1^+} - F_{K_X^+})^{1,1} = r^{2c-1} \frac{i}{4} \| \alpha_1 \|^2 \omega - c d\omega, \\ F_{B_1}^{0,2} = 0. \end{cases} \tag{3.4.2}$$

To sum up, if $(2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \cdot \Omega \leq 0$, the Seiberg Witten equations reduce to the above singular vortex equation, (B_1, α_1) is an effective divisor on L.

3.5 Identification of two moduli spaces.

For compact Kahler manifold, there is a holomorphic description of the solutions to the Seiberg-Witten equations. More specifically, the following correspondence has been proved in [16]:

Theorem 3.5.1. *Let (X, ω) be a closed Kahler surface, S be a Spin^c bundle over X, $S^+ = (\Omega^0(X; L) \oplus \Omega^{0,2}(X; L))$. Let (A, ψ) be a solution to the Seiberg-Witten equations for S, with $\psi = (\alpha, \beta) \in \Omega^0(X; L) \oplus \Omega^{0,2}(X; L)$. Then if $\int_X (2c_1(L) - c_1(K_X)) \wedge \Omega \leq 0$, we have $\beta = 0$, A is a holomorphic connection on $\mathcal{L}$, α is a holomorphic section of L, with*

$$(F_A^+)^{1,1} = \frac{i}{4} |\alpha|^2 \omega.$$

if the $\int_X (2c_1(L) - c_1(K_X)) \wedge \Omega \geq 0$, we have $\alpha = 0$, A is a holomorphic connection *on $\mathcal{L}$, and β is an anti holomorphic section on L, with*

$$(F_A^+)^{1,1} = -\frac{i}{4}|\beta|^2\omega.$$

Conversely, suppose that $\int_X(2c_1(L) - c_1(K_X)) \wedge \Omega \leq 0$, A is a hermitian, holomorphic connection on $\mathcal{L}$, α is a nonzero holomorphic section of L, then there exists another hermitian structure h' on L such that for the connection A' which is hermitian with respect to h' and which defines the same holomorphic structure on L as A does we have

$$F_{A'}^{1,1} = \frac{i}{4}(|\alpha|_{h'})^2\omega$$

where $(|\alpha|_{h'})^2$ means the norm measured with respect to the hermitian structure on L_0 determined by h'.

Similar results hold for β when $\int_X(2c_1(L) - c_1(K_X)) \wedge \Omega \geq 0$.

In this section, we are going to prove an analogous identification theorem for the singular moduli space considered previously.

Let Σ be a holomorphically embedded curved in a closed Kahler manifold (X, Ω). Σ is a holomorphically embedded surface. (L, h_0) is hermitian line bundle over X, such that $L|_P$ is torsion, $c_1(L)(\Sigma) = k + g - 1$. B_k is the connection on L such that when restricted to P, B_k is a pull back connection from a connection on a line bundle L_Σ over Σ.

For some $p > 2$, define

$$\mathcal{B}_1^p = \overline{\partial}_{B_k} + \{a \in i\Omega^{0,1}(X), a \in \mathcal{C}_1^p\},$$

$$\mathcal{G}_2^{Cp} = \{e^g | g \in C_2^p \text{ is a complex valued function}\},$$

$$\mathcal{D}_1^{kp} = \{(\overline{\partial}_B, \alpha) \in \mathcal{B}_1^p \times L_1^p(L) | F_B^{0,2} = 0, \overline{\partial}_B\alpha = 0\},$$

where $L_1^p(L)$ is the usual Sobolev space. Also recall that $\mathcal{M}_k = \{(A, \phi) \in \mathcal{C}_k | SW(A, \phi) = 0\}$.

So $\mathcal{G}_2^{Cp}$ is the complex gauge group, and $\mathcal{D}_1^p$ is the collection of effective divisors.

$\mathcal{G}_2^{Cp}$ acts on $\mathcal{D}_1^p$ as follows: for any $e^g \in \mathcal{G}_2^{Cp}$, $(\overline{\partial}_B, \alpha) \in \mathcal{D}_1^p$,

$$\overline{\partial}_B \to e^g \overline{\partial}_B e^{-g},$$

$$\alpha \to e^g \alpha,$$

$$B \to B + \partial g - \overline{\partial} g.$$

here B is the connection compatible with $\overline{\partial}_B$ and h_0. From the definition the complex gauge group does form a group and the above action preserves $\mathcal{D}_1^p$.

Now we are ready to state the following correspondence theorem:

Theorem 3.5.2. *Let $S_X = S_0 \otimes L$ be a Spin^c bundle on a Kahler manifold (X, Ω), Σ be a holomorphically embedded surface, n is the intersection number of Σ. $\mathcal{L}$ is the determinant line bundle such that $c_1(\mathcal{L})(\Sigma) = 2k+n$. If $(2c_1(L)-c_1(K_X)+(c-\frac{1}{2})\xi)\cdot\Omega$ is nonpositive, then we have a one to one correspondence:*

$$i : \mathcal{M}_k/G_2^p \to \mathcal{D}_1^{kp}/\mathcal{G}_2^{pC}.$$

Proof. First from the adjunction formula, $c_1(\mathcal{L})(\Sigma) = 2k+n$ implies that $c_1(L)(\Sigma) = k + g - 1$. We have proved in Section 3.4 that if $(2c_1(L) - c_1(K_X) + (c - \frac{1}{2})\xi) \cdot \Omega$ is nonpositive, then any for solution $(A, \phi) \in \mathcal{C}_k$, $\phi = (\alpha, 0)$, after the complex gauge group transformation $e^f = r^{\frac{1}{2}-c}$, the pair $(B, \alpha_1)(B = A + (\frac{1}{2} - c)\partial\overline{f} - (\frac{1}{2} - c)\overline{\partial} f$, $\alpha_1 = r^{\frac{1}{2}-c}\alpha)$ satisfy equation (3.4.2), i.e., B defines a holomorphic connection on the determinant line bundle $\mathcal{L}$, let B also denote the induced holomorphic connection on L, $B - B_k \in C_1^p$, and α_1 is a holomorphic section of L, which give us the map $i : (A, \phi) \to (B, \alpha_1)$.

The following theorem tells us that for any $(\overline{\partial}_B, \alpha) \in \mathcal{D}_1^p$, there exists a unique $e^f \in \mathcal{G}_1^{Cp}$ such that after the gauge transform, the effective divisor also satisfies the curvature equation in (3.4.2), thus i is sujective and injective.

□

Theorem 3.5.3. *Let $S_X = S_0 \otimes L$ be a Spin^c bundle on X, Σ is a holomorphically* *embedded surface with genus g.* $c_1(L)(\Sigma) = k+g-1$. *If* $(2c_1(L)-c_1(K_X)+(c-\frac{1}{2})\xi)\cdot\Omega$ *is nonpositive, then for any pair* $(\overline{\partial_B}, \alpha) \in \mathcal{D}_1^p$, *there exists a* $f \in C_p^2$ *such that after the gauge transform* e^f, $(\overline{\partial}_{B'}, \alpha')$ *satisfies*

$$(2F_{B'} - F_{K_X})^{1,1} = \frac{1}{4} r^{2c-1} |\alpha_1|^2_{h'} - ic(d\omega)^{1,1}.$$

Proof. In order to solve

$$(2F_{B'} - F_{K_X})^{1,1} = \frac{1}{4} r^{2c-1} |\alpha_1|^2_{h'} - ic(d\omega)^{1,1}.$$

First observe that for the gauge transformation e^f, we can write the above as an equation for f:

$$-i \wedge_\Omega (2F_B - F_{K_X} + \partial f - \overline{\partial} f) = \frac{1}{4} r^{2c-1} |\alpha_1|^2 e^{2f} - ic(d\omega)^{1,1}.$$

Define $w = -4i \wedge_\Omega (2F_B - F_{K_X} - 4icd\omega)$, $\int w \geq 0$ from the assumption. And using the Kahler identity $-4i \wedge_\Omega \partial\overline{\partial} f = \triangle f$, the above equation becomes

$$\triangle f + r^{2c-1}|\alpha_1|^2 e^{2f} - w = 0, \tag{3.5.1}$$

Now follow the exact procedure in [6], we can construct a solution f. And the elliptic regularity in Section 3.2 will make sure that $f \in C_2^p$. First using the method in [6] to construct a sub-solution f_- and a super-solution f_+ for equation (3.5.1), i.e., functions f_- and f_+ such that $f_+ > f_-$ everywhere and

$$\triangle f_- + r^{2c-1}|\alpha_1|^2 e^{2f_-} - w < 0$$

$$\triangle f_+ + r^{2c-1}|\alpha_1|^2 e^{2f_+} - w > 0.$$

Then define a sequence of funcitons inductively by setting $f_0 = f_-$ and defining f_{i+1} to be the unique solution to

$$Lf_{i+1} = -r^{2c-1}|\alpha_1|^2 e^{2f_i} + w + kf_i$$

where $L = \Delta + k$ for a suitably defined nonnegative function k.

Then using f_i to construct a solution to equation (3.5.1).

Notice that $\Delta v = u$ has a solution whenever $\int u = 0$. If $u \in L^p$, then $v \in L^p_2$. To construct u_+, let v_1, v_2 be solutions to $\Delta v_1 = w - \overline{w}$ and $\Delta v_2 = c - r^{2c-1}|\alpha_1|^2$ with $c = \int r^{2c-1}|\alpha_1|^2$ and $\overline{w} = \int w$. Choose a to be constant large enough so that $ac > \overline{w}$ and then choose a constant b large enough so that $e^{v_1+av_2+b} - a > 0$ and $a - e^{-v_1-av_2-b} > 0$. Let $f_+ = v_1 + av_2 + b$ and it is easy to check that f_+ is a sup solution,

Simply define $f_- = v_1 - n$ with n large enough constant so that $f_- < f_+$ and $\overline{w} + r^{2c-1}|\alpha_1|^2 e^{v_1-n} < 0$, then f_- will be a sub solution.

Obviously $f_+, f_- \in L^p_2$.

Let $k = r^{2c-1}|\alpha_1|^2 e^{f_+}$.

Just as in [6], the sequence f_i defined recursively above satisfy $f_{i+1} \geq f_i$. Thus we have uniform upper and lower bounds on f_i, we get a uniform L^p_2 bounds on f_i from elliptic theory. Thus f_i has a subsequence converging uniformly to f in L^r_1 for some $r > 4$. In particular, f_i converge to f in C^0 norm.

From the definition of f_i, it is not hard to see that f is a weak solution of 3.5.1.: for any smooth function h,

$$\int_X \nabla f_{i+1} \nabla h + kf_{i+1}h = \int_X (\Delta f_{i+1} + kf_{i+1})h = \int_X (-r^{2c-1}|\alpha_1|^2 e^{2f_i} + w + kf_i)h$$

let i go to infinity, we have

$$\int_X \nabla f \nabla h + kfh = \int_X (r^{2c-1}|\alpha_1|^2 e^{2f} + w + kf)h.$$

Since f is a weak solution of 3.5.1, $f \in L^p_2$ from the elliptic theory.

To prove that $f \in C^p_2$, it suffices to prove $g = rf \in W^p_2$ by recalling the isometry we mentioned :

Since f is the solution to (3.5.1.), g is the ssolutionto the following equation

$$r\triangle(\frac{1}{r}g) + rr^{2c-1}|\alpha_1|^2 e^{2\frac{1}{r}g} - rw = 0 \tag{3.5.2}$$

Now that $\triangle = -4i \wedge_\omega \partial\overline{\partial}$, we have

$$-4i \wedge_\omega (r\partial\overline{\partial}(\frac{1}{r}g) + rr^{2c-1}|\alpha_1|^2 e^{2\frac{1}{r}g} - rw = 0,$$

alternatively, we can write

$$-4i \wedge_\omega (\partial(r\overline{\partial}(\frac{1}{r}g) - \frac{1}{r}\partial r \wedge r\overline{\partial}(\frac{1}{r}g))) = -rr^{2c-1}|\alpha_1|^2 e^{2\frac{1}{r}g} - rw = 0,$$

since $\frac{1}{r}g = f$ is ccontinuous the right hand side of the above equation is in L^p, and $r\overline{\partial}(\frac{1}{r}g)$ is also in L^p because $f \in L^p_2$, thus $r\overline{\partial}(\frac{1}{r}g)$ is in W^p_1, equivalently, $r\overline{\partial}(\frac{1}{r}g) = \overline{\partial}g - \frac{1}{r}g$ is in W^p_1, which shows that $g \in W^p_2$.

Uniqueness: if f_1, f_2 are two solutions, we have

$$\begin{aligned} 0 &\leq |d(f_1 - f_2)|^2 \\ &= \int_X < \triangle(f_1 - f_2), f_1 - f_2 > \\ &= \int_X -r^{-2(\frac{1}{2}-c)}|\alpha_1|^2(e^{f_1} - e^{f_2})(f_1 - f_2) \end{aligned}$$

and the last integral is nonpositive, thus $d(f_1 - f_2) = 0$, $f_1 - f_2 = c$ for some constant c. But c has to be 0 otherwise the integral above is strictly negative. Thus $f_1 = f_2$. □

3.6 Compactness of the moduli space for Kahler case.

In this section, we are going to prove that the moduli spaces we identified above is compact. The crucial point is to get a uniform L^∞ bound for the spinors. Notice that we are only working with L^p_1 space here, so the maximal principle does not apply. Fortunately the argument in Section 3.1 works here. When the base manifold is not

Kahler, we can not reduce the equation, and we still do not know whether the moduli is compact or not.

By using the argument introduced in Section 3.1.3., we prove that α_1 admits a uniform L^∞ bound.

Lemma 3.6.1. *Let X be a closed Kahler surface, Σ be a holomorphically embedded curve. S is a $Spin^c$ structure on $X\backslash\Sigma$, which restricted to the tubular neighborhood of Σ is torsion. If $(A,\phi) \in C_k$ is a solution to the Seiberg-Witten equation, then there exist a constant C independent of (A,ϕ), such that $\| r^{\frac{1}{2}-c}\phi \|_{L^\infty} < C$.*

Proof. First suppose that $(2c_1L) - c_1(K_X) + c\xi) \cdot \Omega$ is non-positive, thus $\beta = 0$, and $\phi = (\alpha, 0)$. From the above discussion, (B, α_1) satisfies equation (3.4.2), with $\alpha_1 = r^{\frac{1}{2}-c}\alpha$, and $B = A + (\frac{1}{2} - c)\partial lnr - (\frac{1}{2} - c)\overline{\partial} lnr$ is a $Spin^c$ connection being able to be extended to X. Thus it suffices to prove that α_1 has a uniform L^∞ bound. Notice that $\alpha_1 \in L^p_1$. The same method for Theorem 3.1.1 applies here:

Without loss of generality, we can assume that $s(x) \geq -1$, $s(x)$ is the scalar curvature. Again define the test function $\eta \in L^2_1$ to be

$$\eta = \begin{cases} (\| \alpha_1 \| - 1)\frac{\alpha_1}{\|\alpha_1\|}, & \text{for } \| \alpha_1 \| > 1, \\ 0, & \text{for } \| \alpha_1 \| \leq 1 . \end{cases} \tag{3.6.1}$$

and $\Omega = \{x \in X \mid \| \phi(x) \| > 1\}$.

Since (A_1, α) is a solution, we have

$$\begin{aligned} 0 &= \int_\Omega < \nabla_B \alpha_1, \nabla_B \eta > + \frac{s}{4} < \alpha_1, \eta > + \frac{1}{4} \| \phi \|^2 < r^{2c-1}\alpha_1, \eta > \\ &= \int_\Omega < \nabla_B \alpha, \nabla_B \alpha_1 > + \frac{1}{4}(\| \alpha_1 \|^2 + s)(r^{2c-1} \| \alpha_1 \| - 1) \| \alpha_1 \| \\ &\geq \int_\Omega < \nabla_B \phi, \nabla_A \eta > + \frac{1}{4}(\| \phi \|^2 - 1)(r^{2c-1} \| \phi \| - 1) \| \phi \|, \end{aligned}$$

for $s \geq -1$. The first term is nonnegative from the proof of Theorem 3.1.1. Thus Ω has measure zero, and $\| \alpha_1 \|_{L^\infty} \leq 1$.

□

Now we are ready to prove the following compactness theorem:

Theorem 3.6.2. *Let X be a closed Kahler surface, Σ be a holomorphically embedded curve. S is a $Spin^c$ structure on $X\backslash\Sigma$, which restricted to the tubular neighborhood of Σ is torsion. Then for any sequence $(A^i, \alpha^i) \in \mathcal{C}_k$ which are solutions to the Seiberg-Witten equations, there exists a subsequence, still denoted by (A^i, α^i), and a sequence of gauge transformation $g^i \in \mathcal{G}$, such that $g^i \cdot (A^i, \alpha^i)$ converges in $\mathcal{C}_k$.*

Proof. Let $B_k = A_k + ic\omega + \partial\overline{f} - \overline{\partial}f$ be the reference connection , with $e^f = r^{\frac{1}{2}-c}$. After the complex gauge transformation of e^f defined before, we have a sequence (a^i, α_1^i), such that $(B_k + a^i, \alpha^i)$ solve equation (3.4.2), with $a^i \in C_1^p$, and $\alpha_1^i \in L_1^2(S^+)$. And (A^i, α^i) converges in $\mathcal{C}$ if and only if (a^i, α_1^i) converges in $C_1^p \times L_1^2(S^+)$. From the usual gauge fixing theorem, for any sequence (B^i, α^i), we can find a sequence g^i of gauge transformations, such that $g^i \cdot B^i$, still denoted by B^i, satisfy $d^*a^i = 0$, and $|a^i|_{L^p} < C$ for some constant C. For convenience, we rewrite equation 3.4.2. here:

$$\begin{cases} \overline{\partial}_{B_k}\alpha_1^i + a^i \cdot \alpha_1^i = 0, \\ d^+a^i = F_{B_k} + r^{2c-1}\sigma(\alpha_1^i, \alpha_1^i), \\ d^*a^i = 0. \end{cases} \tag{3.6.2}$$

Since $d^+ \oplus d^*$ has the usual elliptic regularity on C_1^p, a^i has uniform C_1^p bound, which in turn, through the first equation, gives a uniform L_1^p bound on α_1^i.

To get the bound for higher derivatives, we use the argument in [10]:

α_1 is a weak solution to the following second order equation:

$$-\triangle_B\alpha_1 + \frac{s}{4}\alpha_1 + \frac{1}{4}r^{-2c} \parallel \alpha_1 \parallel^2 \alpha_1 = 0.$$

From the above equation, we get for $1 < r \leq 2$,

$$\parallel \triangle\alpha_1 \parallel_{L^r} \leq c(\parallel \triangle_B\alpha_1 \parallel_{L^r} + \parallel A \parallel_{L^r} + \parallel\parallel B \parallel\parallel \nabla_B\alpha_1 \parallel\parallel_{L^r} + \parallel\parallel B \parallel^2\parallel_{L^r}),$$

From the Holder inequality

$$\| \| \| B \| \nabla \alpha_1 \| \| \|_{L^r} \leq \| A \|_{L^{\frac{rq}{q-r}}} \| \nabla \alpha_1 \|_{L^q} .$$

Thus $\| \alpha_1^i \| L_2^r \leq C$.

From Rellich theorem, $L_2^r \to L_1^2$ is compact, thus there eexistsa subsequence α_1^i converging in L_1^2.

Now the right hand side of the second equation (3.4.4) has a uniform C_1^p bound: $\sigma(\alpha_1^i, \alpha_1^i)$ has uniform L_1^s bound for any $s < 4$, thus a uniform C_1^s bound, and $r^{2c-1} \in C_1^l$ for $l > 2$, thus from Holder inequality, the whole term is bounded in C_1^p. From Lemma 3.2.3, a^i has a uniform bound in C_2^p, and has a subsequence converging in C_1^p.

□

Remark: Notice that due to the identification theorem in the previous section, the compactness argument here also proves the compactness for the moduli space of the effective divisors $\mathcal{D}_1^p$.

Bibliography

[1] Bernd Ammann and Christian Bär, *The Dirac operator on nilmanifolds and collapsing circle bundles*, Ann. Global Anal. Geom. **16** (1998), no. 3, 221–253. MR MR1626659 (99h:58194)

[2] Nicole Berline, Ezra Getzler, and Michèle Vergne, *Heat kernels and Dirac operators*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 298, Springer-Verlag, Berlin, 1992. MR MR1215720 (94e:58130)

[3] J. Brüning and R. Seeley, *The expansion of the resolvent near a singular stratum of conical type*, J. Funct. Anal. **95** (1991), no. 2, 255–290. MR MR1092127 (93g:58146)

[4] Jochen Brüning and Robert Seeley, *The resolvent expansion for second order regular singular operators*, J. Funct. Anal. **73** (1987), no. 2, 369–429. MR MR899656 (88g:35151)

[5] ______, *An index theorem for first order regular singular operators*, Amer. J. Math. **110** (1988), no. 4, 659–714. MR MR955293 (89k:58271)

[6] James A. Bryan and Richard Wentworth, *The multi-monopole equations for Kähler surfaces*, Turkish J. Math. **20** (1996), no. 1, 119–128. MR MR1392667 (97i:58021)

[7] Jeff Cheeger, *On the spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. U.S.A. **76** (1979), no. 5, 2103–2106. MR MR530173 (80k:58098)

[8] Arthur Weichung Chou, *The Dirac operator on spaces with conical singularities and positive scalar curvatures*, Trans. Amer. Math. Soc. **289** (1985), no. 1, 1–40. MR MR779050 (86i:58124)

[9] Mikhael Gromov and H. Blaine Lawson, Jr., *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. (1983), no. 58, 83–196 (1984). MR MR720933 (85g:58082)

[10] Jürgen Jost, Xiaowei Peng, and Guofang Wang, *Variational aspects of the Seiberg-Witten functional*, Calc. Var. Partial Differential Equations **4** (1996), no. 3, 205–218. MR MR1386734 (97d:58055)

[11] P. B. Kronheimer and T. S. Mrowka, *Gauge theory for embedded surfaces. I*, Topology **32** (1993), no. 4, 773–826. MR MR1241873 (94k:57048)

[12] Peter Kronheimer and Tomasz Mrowka, *Monopoles and three manifolds*, To appear.

[13] M.-L. Labbi, *On two natural riemannian metrics on a tube*, Balkan Journal of Geometry and Its Applications **12** (2007), no. 2, 81–86.

[14] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn, *Spin geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989. MR MR1031992 (91g:53001)

[15] Robert B. Lockhart and Robert C. McOwen, *Elliptic differential operators on noncompact manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), no. 3, 409–447. MR MR837256 (87k:58266)

[16] John W. Morgan, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Mathematical Notes, vol. 44, Princeton University Press, Princeton, NJ, 1996. MR MR1367507 (97d:57042)

[17] Tomasz Mrowka, Peter Ozsváth, and Baozhen Yu, *Seiberg-Witten monopoles on Seifert fibered spaces*, Comm. Anal. Geom. **5** (1997), no. 4, 685–791. MR MR1611061 (98m:58017)

[18] Liviu I. Nicolaescu, *Eta invariants of Dirac operators on circle bundles over Riemann surfaces and virtual dimensions of finite energy Seiberg-Witten moduli spaces*, Israel J. Math. **114** (1999), 61–123. MR MR1738674 (2001h:58032)

[19] Peter Petersen, *Riemannian geometry*, second ed., Graduate Texts in Mathematics, vol. 171, Springer, New York, 2006. MR MR2243772 (2007a:53001)

[20] Clifford Henry Taubes, *On the equivalence of the first and second order equations for gauge theories*, Comm. Math. Phys. **75** (1980), no. 3, 207–227. MR MR581946 (83b:81098)