Answer keys for problem set 6

Exercise 7.2
Call $X^1$ and $X^2$ the total amounts of good 1 and 2. Consider two consumers with the same Leontief utility function

$$u_i(x_i) = \min\left(\frac{x^1_i}{X^1}, \frac{x^2_i}{X^2}\right)$$

For any endowment $(\omega_1, \omega_2)$, any price $p$ (different from $(0, 0)$) is part of a Walrasian equilibrium. Indeed consider the allocation

$$x^i_j = \frac{p \cdot \omega_i}{p \cdot X}$$

$(p, x)$ is a Walrasian equilibrium.

Exercise 7.4
Consumer A has a Cobb-Douglas utility function, his demand functions are

$$x^1_A = a \frac{m_A}{p_1} = a \frac{p_2}{p_1}$$

$$x^2_A = (1 - a) \frac{m_A}{p_2} = 1 - a$$
The demand functions of consumer B are
\[ x_B^1 = x_B^2 = \frac{m_B}{p_1 + p_2} = \frac{p_1}{p_1 + p_2} \]

By Walras’s law, if one market is cleared the other one will also be cleared. Check for example that the market for good 2 has to clear:
\[ x_A^2 + x_B^2 = 1 \iff \frac{p_1}{p_2} = \frac{a}{1-a} \]

In equilibrium the prices will be such that \( \frac{p_1}{p_2} = \frac{a}{1-a} \) and the allocation will be \( x_A^1 = x_A^2 = 1 - a \) and \( x_B^1 = x_B^2 = a \).

**Exercise 7.6**
Those agents have Cobb-Douglas utility functions
\begin{align*}
    u_1 & = x_1^a x_2^{1-a} \\
    u_2 & = x_1^b x_2^{1-b}
\end{align*}

Hence their demand functions are
\begin{align*}
    x_1^1 & = a \frac{p_1 + p_2}{p_1} \\
    x_1^2 & = (1-a) \frac{p_1 + p_2}{p_2} \\
    x_2^1 & = b \frac{p_1 + p_2}{p_1} \\
    x_2^2 & = (1-b) \frac{p_1 + p_2}{p_2}
\end{align*}

The market for good 1 has to clear
\[ x_1^1 + x_2^1 = 2 \iff \frac{p_1}{p_2} = \frac{a + b}{2 - a - b} \]

**Exercise 7.8**
Let an allocation be defined by \((x_A^1, x_B^1)\) where A gets \((x_A^1, x_B^2)\) and B gets \((1-x_A^1, 2-x_B^2)\).

The set of strongly Pareto efficient allocations is \(S = \{(1,0), (0,2)\}\).

The set of weakly Pareto efficient allocations is \(W = \{(x_A^1, 0) \text{ s.t. } x_A^1 \in [0,1]\} \cup \{(x_A^1, 2) \text{ s.t. } x_A^1 \in [0,1]\} \cup \{(x_A^1, 1) \text{ s.t. } x_A^1 \in [0,1]\} \cup \{(1, x_A^2) \text{ s.t. } x_A^2 \in [0,1]\} \cup \{(0, x_A^2) \text{ s.t. } x_A^2 \in [1,2]\}\).
Exercise 7.10

The demonstration is identically the same as the one on p321, until the equality

$$\left[ \sum_{i=1}^{k} \max(0, z_i(p^*)) \right] p^*.z(p^*) = \sum_{i=1}^{k} z_i(p^*) \max(0, z_i(p^*))$$

The LHS is non-positive as \( \max(0, z_i(p^*)) \geq 0 \) and \( p^*.z(p^*) \leq 0 \). The RHS is non-negative as \( z_i(p^*) \max(0, z_i(p^*)) \) is either 0 or \( z_i(p^*)^2 \). Therefore both the LHS and the RHS must be equal to 0.

$$\sum_{i=1}^{k} z_i(p^*) \max(0, z_i(p^*)) = 0$$

And the end of the proof remains the same.