Exercise 4.4
If the first plant wants to produce $y$, its cost will be:

$$C_1(y) = K(w_1, w_2, a)y = \min_{x_1 x_2 \geq y} w_1 x_1 + w_2 x_2$$

where $K(w_1, w_2, a) = a^{-a}(1 - a)^{a-1}w_1^{1-a}w_2^{1-a}$.

The cost of the second plant is symmetric. To allocate the production between its two plants, the firm solves:

$$C(y) = \min_{y' + y'' \geq y} C_1(y') + C_2(y'') = \min_{y' + y'' \geq y} K(w_1, w_2, a)y' + K(w_1, w_2, b)y''$$

Note that both the constraint and the cost function are linear in $y'$ and $y''$, hence the solution is a corner solution not an interior one.

if $K(w_1, w_2, a) < K(w_1, w_2, b) \Rightarrow y' = y$ and $y'' = 0$, $C(y) = C_1(y)$
if $K(w_1, w_2, a) > K(w_1, w_2, b) \Rightarrow y' = 0$ and $y'' = y$, $C(y) = C_2(y)$
if $K(w_1, w_2, a) = K(w_1, w_2, b) \Rightarrow y' \in [0, y]$ and $y'' = y - y'$, $C(y) = C_1(y) = C_2(y)$

Exercise 4.6
The firm solves:

$$\min \quad c_1(y_1) + c_2(y_2)$$
$$y_1 + y_2 \geq y$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$

Set up the Lagrangian:

$$L = c_1(y_1) + c_2(y_2) - \lambda(y_1 + y_2 - y) - \mu_1 y_1 - \mu_2 y_2$$

where $\lambda$, $\mu_1$ and $\mu_2$ are non-negative. The Kuhn-Tucker conditions are:

$$\frac{2}{\sqrt{y_1}} - \lambda - \mu_1 = 0$$
\[
\frac{1}{\sqrt{y_2}} - \lambda - \mu_2 = 0
\]
\[
\lambda(y_1 + y_2 - y) = 0
\]
\[
\mu_1 y_1 = 0
\]
\[
\mu_2 y_2 = 0
\]

- Interior solution: both \( \mu_i \) are equal to 0. This implies that

\[
\lambda = \frac{2}{\sqrt{y_1}} = \frac{1}{\sqrt{y_2}} > 0
\]

\( \lambda > 0 \) implies that \( y_1 + y_2 = y \). Solving for \( y_1 \), you find:

\[
y_1 = \frac{y}{5}, \quad y_2 = \frac{4y}{5} \quad \text{and} \quad c(y) = 2\sqrt{5}\sqrt{y}
\]

- Corner solution: \( \mu_1 > 0 \) and \( \mu_2 = 0 \).

\[
y_1 = 0, \quad y_2 = y \quad \text{and} \quad c(y) = 2\sqrt{y}
\]

As the corner solution is better than the interior one (which indeed is a maximum and not a minimum), the firm will produce using solely its plant #2. Note that it is no use of looking at the other corner solution as \( c_1(y) = 2c_2(y) \), if the firm decides to use a single plant it will use only the second one. Note that it is not enough to have \( c_1(y) = 2c_2(y) \) to conclude that the firm produces using on plant only. (you can check that with \( c_1(y) = 2y^2 \) and \( c_1(y) = y^2 \)).

**Exercise 4.8**

The cost function is:

\[
c(w_1, w_2, y) = 2w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} y^{\frac{1}{2}}
\]

\[
c(1, 1, y) = 4 \Rightarrow y = 4
\]

**Exercise 5.4**

Start by drawing the production function, then you can read the solution on the graph. See figure 1

\[
\begin{align*}
&\text{if } \frac{w_1}{w_2} > 2, \quad x_1 = 0, \quad x_2 = y, \quad c = w_2y \\
&\text{if } \frac{w_1}{w_2} = 2, \quad x_1 \in [0, \frac{y}{2}], \quad x_2 = y - 2x_1, \quad c = w_2y = \frac{w_1}{2}y \\
&\text{if } 2 > \frac{w_1}{w_2} > \frac{1}{2}, \quad x_1 = \frac{y}{3}, \quad x_2 = \frac{y}{3}, \quad c = (w_1 + w_2)\frac{y}{3} \\
&\text{if } \frac{w_1}{w_2} = \frac{1}{2}, \quad x_1 \in \left[\frac{y}{3}, y\right], \quad x_2 = \frac{y-x_1}{2}, \quad c = w_1y = \frac{w_2}{2}y \\
&\text{if } \frac{1}{2} > \frac{w_1}{w_2}, \quad x_1 = y, \quad x_2 = 0, \quad c = w_1y
\end{align*}
\]
Exercise 5.6

• $x_1$ has to be homogeneous of degree 0 $\Rightarrow a = \frac{1}{2}$.

• $x_2$ has to be homogeneous of degree 0 $\Rightarrow c = -\frac{1}{2}$.

• The substitution matrix has to be symmetric $\frac{\partial x_1}{\partial w_2} = \frac{\partial x_2}{\partial w_1} \Rightarrow b = 3$.

Exercise 5.8

If the firm produces, what is the best it could do?

\[
\frac{p^2}{4} - 1 = \max_y py - (y^2 + 1)
\]

indeed FOC: $y = \frac{p}{2}$ and SOC: $-2 < 0$.
But the firm has also the option of not producing which leads to zero profit.

• if $p > 2$, $y = \frac{p}{2}$ and $\pi = \frac{p^2}{4} - 1$

• if $p = 2$, $y = \frac{p}{2}$ or $y = 0$ and $\pi = 0$

• if $p < 2$, $y = 0$ and $\pi = 0$
Exercise 5.12
(a) see figure 2
(b) Let’s start by showing a general result for an interior solution when the production function is CRtS. (easily checked on a graph)

\[ x(w, ty) = tx(w, y) \]

The demand functions are characterized by:

\[ f(x(w, y)) = y \]
\[ \frac{\partial f}{\partial x_i}(x(w, y)) = \frac{w_i}{w_j} \]

\[ \frac{\partial f}{\partial x_j}(x(w, y)) = \frac{w_i}{w_j} \]

\( f \) is CRtS means it is homogeneous of degree 1, which implies that its partial derivatives are homogeneous of degree 0.

\[ f(tx(w, y)) = tf(x(w, y)) = ty \]
\[ \frac{\partial f}{\partial x_i}(tx(w, y)) = \frac{\partial f}{\partial x_j}(tx(w, y)) = \frac{w_i}{w_j} \]

therefore \( x(w, ty) = tx(w, y) \). Differentiate w.r.t. \( t \) and set \( t = 1 \) to get

\[ \frac{\partial x_j}{\partial y}(w, y) = \frac{x_i(w, y)}{y} \geq 0 \]. No factor can be inferior.
(c) Assume $\frac{\partial^2 c}{\partial w_i \partial w_j} < 0$ i.e. the marginal cost decreases when the price $w_i$ increases. Use Shephard’s lemma $x_i = \frac{\partial c}{\partial w_i}$ to conclude that $\frac{\partial x_i}{\partial y} < 0$, $x_i$ is inferior.

Exercise 5.16

1. $c$ is not homogeneous of degree 1 as $c(tw, y) = t^{1.5}c(w, y)$.

2. - $c$ is homogeneous of degree 1 as $c(tw, y) = tc(w, y)$.
   - $c$ is non decreasing as $\frac{\partial c}{\partial w_i} = y(1 + 0.5\sqrt{\frac{w_j}{w_i}}) > 0$.
   - $c$ is concave as $c(tw + (1 - t)w', y) \geq tc(w, y) + (1 - t)c(w', y)$ $\iff$ $\sqrt{(tw_1 + (1 - t)w_2)(tw'_1 + (1 - t)w'_2)} \geq t\sqrt{w_1w_2} + (1 - t)\sqrt{w'_1w'_2} \iff \frac{1 - t}{\sqrt{w_1w_2} + \sqrt{w'_1w'_2}} \geq 0$.

   Note: as it was suggested in class, checking that the Hessian matrix is negative semi-definite is also valid.

   - $c$ is continuous.

The idea for going from the cost function back to the production function is to use the demand functions to eliminate the prices and find a relation linking the output and the inputs. The demand functions are $x_i = y(1 + 0.5\sqrt{\frac{w_j}{w_i}})$. Extract $\frac{w_i}{w_2}$ from the demand functions in order to eliminate the $w$s and get a relation between $y$, $x_1$ and $x_2$ only. Then rearrange this relation to get the production function.

\[
\sqrt{\frac{w_1}{w_2}} = \frac{2(x_2 - 1)}{y} = \left(\frac{2}{y} - 1\right)^{-1}
\]
\[
\Rightarrow y = \frac{2}{3}(x_1 + x_2) - \sqrt{\frac{4}{9}(x_1 + x_2)^2 - \frac{4}{3}x_1x_2}
\]

You initially get two candidates for the production function which are the two roots. Knowing that it must be concave in $x_i$ allows to select one of them.

3. $c$ is not homogeneous

4. $c$ is not monotonic as $\frac{\partial c}{\partial w_i} = y \left(1 - \frac{1}{2}\sqrt{\frac{w_j}{w_i}}\right) < 0$ when $4w_i < w_j$.

5. $c$ is not increasing in $y$. 

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