Problem 7.7. *Three-dimensional motion of closed strings and cusps.*

In this problem we show that instantaneous cusps moving with the speed of light appear generically in the free motion of closed strings in three dimensions. For this purpose we consider the motion described by

$$\vec{X}(t,\sigma) = \frac{1}{2}\Big(\vec{F}(u) + \vec{G}(v)\Big)\,, \quad \text{with} \quad u = ct + \sigma\,,\ v = ct - \sigma\,. \tag{1}$$

Here $\sigma \sim \sigma + \sigma_1$, where $\sigma_1 = E/T_0$ and E the energy of the string. In this solution of the wave equation we have introduced two independent vector functions $\vec{F}$ and $\vec{G}$.

(a) Show that the reparameterization conditions and the periodicity conditions required by the identification $\sigma \sim \sigma + \sigma_1$ imply that for arbitrary u and v,

$$|\vec{F}'(u)|^2 = |\vec{G}'(v)|^2 = 1\,, \quad \vec{F}(u+\sigma_1) - \vec{F}(u) = \vec{G}(v) - \vec{G}(v-\sigma_1)\,. \tag{2}$$

Explain why the last equality above implies that

$$\vec{F}'(u+\sigma_1) = \vec{F}'(u) \quad \text{and} \quad \vec{G}'(v+\sigma_1) = \vec{G}'(v)\,. \tag{3}$$

Equations (2) and (3) show that $\vec{F}'(u)$ and $\vec{G}'(v)$ can be described as two independent closed parameterized paths on the surface of a unit two-sphere.

(b) Let us now assume that the paths intersect at $u = u_0$ and $v = v_0$

$$\vec{F}'(u_0) = \vec{G}'(v_0)\,. \tag{4}$$

The quantities u_0 and v_0 define a time t_0 and position σ_0. Verify that at $t = t_0$, the point $\sigma = \sigma_0$ on the string moves with the speed of light. What is the direction of the motion? Under general conditions this point turns out to be the tip of a cusp. Note that arbitrary paths on the surface of a two-sphere generically intersect.

(c) We choose a coordinate system so that the cusp generated by (4) appears at the origin: $\vec{F}(u_0) + \vec{G}(v_0) = \vec{0}$. Use the Taylor expansions of $\vec{F}(u)$ and $\vec{G}(v)$ around u_0 and v_0 to prove that for σ near σ_0,

$$\vec{X}(t_0,\sigma) = \vec{T}\,(\sigma-\sigma_0)^2 + \vec{R}\,(\sigma-\sigma_0)^3 + \dots\,, \tag{5}$$

where the vectors $\vec{T}$ and $\vec{R}$ are given by

$$\vec{T} = \frac{1}{4}\big(\vec{F}''(u_0) + \vec{G}''(v_0)\big)\,, \quad \vec{R} = \frac{1}{12}\big(\vec{F}'''(u_0) - \vec{G}'''(v_0)\big)\,. \tag{6}$$

Assume that the intersection of the paths on the two-sphere indicated in equation (4) is regular: the paths are not parallel at the intersection and neither $\vec{F}''(u_0)$ nor $\vec{G}''(v_0)$ vanishes. Explain why $\vec{T}$ is non-zero and orthogonal to $\vec{F}'(u_0)$.

(d) One can use equation (5) to show that the cusp opens up along the direction of the vector $\vec{T}$ and is contained in the plane spanned by $\vec{T}$ and $\vec{R}$. For this, align the positive y axis along $\vec{T}$, position the x axis so that $\vec{R}$ lies on the (x,y) plane, and demonstrate that near the cusp $y \sim x^{2/3}$. In what plane does the velocity of the cusp lie?

(e) Consider the functions $\vec{F}(u)$ and $\vec{G}(v)$ given by

$$\vec{F}(u) = \frac{\sigma_1}{2\pi}\Big(\sin\frac{2\pi u}{\sigma_1}\,, -\cos\frac{2\pi u}{\sigma_1},\, 0\Big)\,, \quad \vec{G}(v) = \frac{\sigma_1}{4\pi}\Big(\sin\frac{4\pi v}{\sigma_1}\,, 0\,, -\cos\frac{4\pi v}{\sigma_1}\,\Big)\,. \tag{7}$$

Verify that the conditions in (2) and (3) are satisfied. For the cusp at $t = \sigma = 0$ give its direction, the plane it lies on, and its velocity. The motion of the closed string has period σ_1/c. How many cusps are formed during a period?