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THE FUNDAMENTAL EEL EQUATIONS

by

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ABSTRACT. Details of the kinematics, statics, and dynamics of a particularly simple form of locomotory system are developed to demonstrate the importance of understanding the behavior of the mechanical system interposed between the commands to the actuators and the generation of displacements in manipulation and locomotion systems, both natural and artificial.

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INTRODUCTION

When one studies manipulation and locomotion in artificial or biological systems one often forgets that something comes between the actuator and the displacement, between the command to move and the actual motion. One may be tempted further to ignore the interactions between parts of the mechanism. It is therefore of the utmost importance to have a clear picture of the kinematics, statics, and dynamics of the mechanical system that lies between actuator and motion. For articulated linkages this is quite hard, but some important work has been done in this area [1].

In this paper, we explore a particularly simple system capable of propelling itself through a fluid by means of waves travelling along its length, in a direction opposite to the desired motion. The continuous nature of such a system allows one to apply differential equation methods; and a complete solution for propulsion forces, actuator torques, and body accelerations is developed.

The gulf between actuator inputs and displacements of body segments will become apparent, since the one lags behind the other by $\pi/2$ in phase. The interaction of the body segments will also be seen to be of importance, since it is only indirectly, through the interaction of different segments, that the propulsive forces are generated from the actuator torques.

As an added bonus we discover that this mode of locomotion is remarkably efficient. We also show that an optimum amplitude to wavelength ratio exists that minimizes power expenditure. This ratio depends on the drag-lift ratio and is independent of velocity. In steady motion the rearward slippage of the waveform relative to the fluid balances the drag due to the forward motion of the body. The results have additional application in the design of fish-like vehicles, in understanding the locomotion of a wide variety of species, and possibly in the design of elephant trunk-like manipulator devices.

The fundamental eel equations are:

$$A\rho\ddot{y} \approx \frac{dF}{dx} - f_t$$

and

$$I\rho\ddot{\theta} \approx -\frac{d\tau}{dx} + F$$

Here F is the shear force in a body cross-section, while τ is the torque transmitted across such a body cross-section. The rest of the notation will be explained later. The analysis starts with an assumed travelling wave of body displacement. Initially the mass of the body is ignored and only actuator torques required to overcome the forces generated by motion through the fluid are considered. The effect of the masses and inertias of body segments is introduced subsequently.

SUMMARY

The following will be shown:

Propulsion by this means is very efficient. Not much more power is required than that needed to push a stick of equal dimensions and shape through the fluid.

The velocity at which the travelling wave propagates rearward along the body is a bit larger than the velocity at which the body moves forward through the fluid.

In steady motion, the force generated by the rearward slip of the waveform relative to the fluid balances the drag force due to the forward motion of the body through the fluid.

It is advantageous for the wavelength of the travelling wave to be a sub-multiple of the length of the body. This ensures steady motion.

The internally generated torque function needed to support this motion is also a travelling wave and lags the displacement waveform by $\pi/2$.

There is a value of amplitude to wavelength ratio that minimizes power. It does not depend on velocity and is proportional to the fourth root of the drag-lift ratio.

The relative slip-rate for minimum power approximately equals the square root of the drag-lift ratio.

Small bodies in viscous fluids need large amplitude to wavelength ratios, compared to large bodies in fluids of low viscosity, which can move most efficiently with relatively small amplitude to wavelength ratios.

A second component of internally generated torque is required to accelerate body segment masses during the motion. This torque is in phase with the body displacement and can be generated by passive elastic means.

A good method for controlling forward velocity is to vary only the rearward velocity of the travelling wave. The wave length is kept equal to some submultiple of the body length, while the amplitude-wavelength ratio is kept at the optimal value for least power.

Equivalently, one varies the frequency of undulation to control forward velocity, keeping amplitude and wavelength fixed.

WAVEFORM OF THE UNDULATION

Consider a travelling wave of body displacement as in Figure 1. Let Y be the amplitude of this wave, ω its angular frequency and u the velocity at which the wave propagates backwards along the body. If we let x and y be the coordinates of points in the body measured in a system fixed in the body and moving with it, we have

$$y = Y \cos\{\omega(t + x/u)\}$$

x lies between -L and 0, where L is the length of the body. It is convenient to use the abbreviation $\phi = \omega(t + x/u)$.

$$y = Y \cos \phi$$

Clearly the wavelength $\lambda = 2\pi(u/\omega)$. We will assume that the amplitude is much smaller than the wavelength in order to make analysis feasible. Later we may discuss what happens when this condition is violated. A convenient dimensionless parameter is $\alpha = Y(\omega/u)$. From the previous assumption it follows that α is small.

DESCRIPTION IN TERMS OF MOTION RELATIVE TO FLUID

Let x' and y' be coordinates measured in a system fixed in the fluid, but aligned with the coordinate system moving with the body. If the body is moving with velocity v in the x' direction we have

$$x' = x + vt$$

and

$$y' = y$$

For forward propulsion we will find that u is a bit larger than v. Expressing the waveform in terms of the new coordinate system we get,

$$y' = Y \cos\{\omega[(1 - v/u)t + x'/u]\}$$

so $\omega(1-v/u)$ is the angular frequency as observed at a fixed point in the fluid and $(u-v)$ is the velocity at which points of fixed phase slip rearward with respect to the fluid.

INCLINATION, CURVATURE, AND RATE OF FLEXURE OF BODY SEGMENTS

The inclination of a body segment as shown in Figure 2 can be found by differentiation.

$$\tan\theta = \frac{dy}{dx} = -\alpha \sin\phi$$

Since α is small we will be able to use the approximations $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$ when needed. Next we calculate curvature as the rate of change of inclination along the body.

$$\frac{d\theta}{dx} \simeq -(\omega/u)(\alpha \cos\phi)$$

Later we will need the rate of change of curvature with time in order to calculate power.

$$\frac{d}{dt}\left(\frac{d\theta}{dx}\right) \approx \omega(\omega/u)(\alpha \sin \phi)$$

MOVEMENT OF BODY SEGMENTS RELATIVE TO FLUID

The two components of motion relative to the coordinate system fixed in the fluid are:

$$\frac{dx'}{dt} = v \quad \text{and} \quad \frac{dy'}{dt} = -\alpha u \sin \phi$$

Let us call these v_x and v_y. Next we decompose the velocity into components along and across the body segment as in Figure 3. Let the longitudinal and transverse components be v_l and v_t respectively.

$$v_l = v_x \cos \theta + v_y \sin \theta$$

$$v_t = -v_x \sin \theta + v_y \cos \theta$$

Or,

$$v_l = [v + (\alpha \sin \phi)^2 u] \cos \theta$$

$$v_t = -(u - v)(\alpha \sin \phi) \cos \theta$$

FORCES GENERATED IN RESPONSE TO MOVEMENT OF BODY SEGMENTS

Motion along the axis of a body segment is not greatly impeded, while motion in a transverse direction generates a large force. One can think of these as "drag" and "lift" components of the force generated by motion through the fluid. Let the force generated per unit body length have a longitudinal component f_l and a transverse component f_t,

$$f_l = dv_l \quad \text{and} \quad f_t = lv_t$$

where d is a "drag" factor while l is a "lift" factor. Normally d will be much smaller than l. These quantities will depend on the shape, area, and surface properties of the skin covering the body as well as the properties of the fluid, such as its density and viscosity. The above analysis assumes movement is slow enough to guarantee laminar flow.

FORCES GENERATED IN DIRECTION OF MOTION AND ACROSS IT

At this point we will return to the coordinate system moving with the body as in Figure 4. Let the components of force produced per unit length in the negative x and y directions be f_x and f_y respectively, then

$$f_x = f_l \cos\theta - f_t \sin\theta$$

$$f_y = f_l \sin\theta + f_t \cos\theta$$

Or,

$$f_x = \{d[v + (\alpha \sin\phi)^2 u] - l(u - v)(\alpha \sin\phi)^2\}\cos^2\theta$$

$$f_y = \{d[v + (\alpha \sin\phi)^2 u] - l(u - v)\}(\alpha \sin\phi)\cos^2\theta$$

Simplifying,

$$f_x = \{dv + [du - l(u - v)](\alpha \sin\phi)^2\}\cos^2\theta$$

$$f_y = \{[dv - l(u - v)] + du(\alpha \sin\phi)^2\}(\alpha \sin\phi)\cos^2\theta$$

We will use $\cos\theta \approx 1$ to simplify further calculations.

AVERAGE FORCES OVER ONE CYCLE OR ONE WAVELENGTH

If we use the relation $\sin^2\theta = \frac{1}{2}(1 + \sin 2\theta)$, we can easily obtain the average forces in the x and y directions per unit length.

$$\bar{f}_x = dv + \tfrac{1}{2}\alpha^2[du - l(u - v)] \quad \text{and} \quad \bar{f}_y = 0$$

There is a clear advantage in choosing a wavelength that is a sub-multiple of the length of the body. If the length of the body is an integer multiple of the wavelength, the force in the x-direction is constant with respect to time and the forces in the y-direction cancel out at all times as well. This can be shown by integrating the expressions for f_x and f_y with respect to x from -L to 0. If the wavelength does not divide evenly the body length, there are sideway oscillations of the body and periodic oscillations in forward velocity. This effect is of less importance if the body is many wavelengths long.

BALANCE OF FORCES DURING STEADY MOTION

During steady motion, the acceleration is zero and so the overall force must be zero.

$$dv + \tfrac{1}{2}\alpha^2[du - 1(u - v)] = 0$$

Or,

$$u = \frac{1 + (2/\alpha^2)d}{1 - d} v = [1 + \frac{d}{1 - d} (1 + \frac{2}{\alpha^2})]v$$

This confirms that u will be greater than v. One can think of the forces produced by the rearward slip of the waveform relative to the fluid as having to balance the drag forces due to forward motion of the body. If α is not too small and d is much smaller than 1, it is clear that the slip-rate need not be very big, that is, $(u - v)$ can be small relative to v.

$$\frac{u - v}{v} = \frac{d}{1 - d} (1 + \frac{2}{\alpha^2})$$

RELATIVE SLIP-RATE

Evidently the minimum relative slip-rate is fixed by the drag-lift ratio. That is, if α becomes large,

$$\frac{u - v}{v} \rightarrow \frac{d}{1 - d}$$

One comes within a factor of two of this minimal value, for $\alpha = \sqrt{2}$.
Since $Y/\lambda = \alpha/2\pi$, this corresponds to an amplitude of about a quarter of the wavelength. Since $\cos^2\theta = 1/(1 + \alpha^2\sin^2\theta)$, one easily can see that this also

corresponds to a maximum inclination of 55°. One should be cautious when using these formulae for values of α larger than this, since we have ignored various deleterious effects of large angles of body segment inclination, such as body shortening. Fortunately, we show later that the minimum power required for propulsion tends to occur for smaller values of α anyway.

If l is much larger than d and α is small one can further approximate the relative slip-rate

$$\frac{u-v}{v} \approx \frac{d}{l}\frac{2}{\alpha^2} = \frac{2d}{l}\left(\frac{u}{\omega}\right)^2\frac{1}{y^2} = \frac{1}{2\pi^2}\;\frac{d}{l}\left(\frac{\lambda}{y}\right)^2$$

The relative slip-rate is approximately proportional to the drag-lift ratio and the square of the wavelength-amplitude ratio.

GENERATION OF REQUIRED INTERNAL TORQUES

A short section of the body can be modelled discretely as in Figure 5. The pin-jointed links in the center represent the flexible, but incompressible spine. Each joint of course can transmit longitudinal and transverse forces, but no torques. The "muscles" attached to the rigid plates generate the torques required to drive the undulating motion. This model may not correspond to a practical way of doing things, but captures the basic idea. In particular, with fixed wavelength, one can come up with arrangements that make more efficient use of a smaller number of muscles that extend over longer segments of the body.

This discrete model can now be related to a slice of a continuous model shown in Figure 6. If we let r be the distance from the body center-line to the points of attachment of the "muscles", then $\tau(x)$ corresponds to $(f_1 - f_2)r$, while $\tau(x + \delta)$ corresponds to $(f_3 - f_4)r$. We prefer to work with the continuous model since it is mathematically more tractable.

BALANCING DRAG AND LIFT FORCES WITH INTERNAL TORQUES

Initially we will assume that the body is massless and no forces or torques are required to accelerate body segments. Later we calculate separately the additional torques required to accelerate the body segments. Consider a short segment of our continuous model. We find forces δf_l and δf_t in the longitudinal and transverse direction respectively.

Balance of forces: $F(x + \delta) - F(x) - \delta f_t = 0$

Balance of torques: $\tau(x + \delta) - \tau(x) - (\delta/2)[F(x + \delta) + F(x)] = 0$

That is: $\frac{dF}{dx} = f_t$ and $\frac{d\tau}{dx} = F$

Here we have once again assumed small inclinations of body segments.

INTERNAL TORQUES REQUIRED TO DRIVE UNDULATING BODY MOTION

Now $f_t = l v_t = -l(u - v)(\alpha \sin \phi)$,

so $F = l(u - v)(u/\omega)(\alpha \cos \phi)$ and $\tau = l(u - v)(u/\omega)^2(\alpha \sin \phi)$.

Here we have ignored the end conditions, namely that the torque has to drop to zero at both ends of the body. This is easily taken care of by adding a linear term of the form $(ax + b)$. If $\tau = T \sin \{\omega(t + x/u)\}$,

$$b = -T \sin \omega t$$

$$a = (T/L)\ [\sin\{\omega(t - L/u)\} - \sin \omega t]$$

The term a becomes 0 if the wavelength divides the body-length evenly. We will ignore this additional torque term for now.

POWER REQUIRED TO SUPPORT THIS MOTION

Work is force times distance or torque times angle. Power then can be calculated from the product of torque times the rate of change of angle with time. Applying this to our model, we have, per unit body length:

$$p = \tau\frac{d}{dt}\left(\frac{d\theta}{dx}\right)$$

$$p = l(u - v)(u/\omega)^2(\alpha \sin \phi)(\omega/u)\omega(\alpha \sin \phi)$$

$$p = l(u - v)u(\alpha \sin \phi)^2$$

Averaged over one cycle or one wavelength this becomes:

$$\bar{p} = (\alpha^2/2)l(u - v)u$$

Now for steady forward motion we found $(u - v) \approx (d/l)(2/\alpha^2)v$. So,

$$\bar{p} \approx duv$$

This remarkable result shows that the power required to propel this body is little more than the power required to push a stick of the same dimensions through the fluid at the same speed. This latter quantity is of course dv^2.

CONDITIONS FOR MINIMUM POWER

Let us calculate the power more precisely using

$$\bar{p} = (\alpha^2/2)l(u - v)u$$

$$u = \frac{1 + (2/\alpha^2)d}{1 - d} v$$

and

$$u - v = \frac{d}{1 - d} [1 + (2/\alpha^2)]v$$

Then

$$\bar{p} = \frac{ld}{(1 - d)} [1 + (\alpha^2/2)][1 + (2/\alpha^2)d]v^2$$

This is minimal for $(\alpha^2/2) = \sqrt{d/l}$, that is, $Y/\lambda = (1/\sqrt{2}\pi)\sqrt{\sqrt{d/l}}$. This suggests that α typically will be a lot less than 1 since d will usually be small compared to 1. The maximum is very broad and a range of amplitude to wavelength ratios will produce near minimum power comsumption. We also see that smaller bodies in hgh viscosity fluids require larger amplitude to wavelength ratios for least power compared to larger bodies in low viscosity fluids. The minimal power is,

$$\bar{p}_{min} = \frac{dv^2}{(1 - \sqrt{d/l})^2} = du^2$$

This occurs for a relative slip-rate of

$$\frac{u - v}{v} = \frac{\sqrt{d/l}}{1 - \sqrt{d/l}}$$

$$\frac{u - v}{u} = \sqrt{d/l}$$

So the optimal slip-rate as far as power consumption is concerned is approximately the square root of the drag-lift ratio.

PHASE RELATIONSHIPS OF VARIOUS WAVEFORMS

The body displacement is $y = Y \cdot \cos \phi$, while the driving torque generated by the actuators is $\tau = (u - v)(u/\omega)^2(\alpha \sin \phi)$. The torque waveform thus lags $\pi/2$ on the displacement. This illustrates the indirect nature of the generation of propulsive force and the importance of interactions of body segments to transmit the motion. This is illustrated in Figure 7.

The curvature equals $-(\omega/u)(\alpha \cos \phi)$ and is thus exactly π out of phase with the displacement. The propulsive force and the power used by the actuators vary as $(\alpha \sin \phi)^2$ and thus fluctuate at twice the frequency of the body displacements and torques, being maximal at the zero crossings of the displacement waveform.

MASS OF BODY SEGMENTS

Forces are required to accelerate the finite mass of the body segments. This is an important factor limiting the amplitude and speed of propagation of the travelling wave. Since the internal torques required will have a different phase relationship to the body displacement than the torques required for propulsion, we might expect that no average power will be needed to support this torque and that it could therefore be generated by elastic means.

Let a cross section through the body as in Figure 8 have area A, then the mass of a thin slice will be $A\rho\delta$. Here ρ is the density of the body, while δ is the thickness of the slice.

TORQUES REQUIRED TO ACCELERATE BODY PARTS

Referring to the body segment of Figure 6 and ignoring now the forces and torques related to propulsion we have:

Balance of forces: $(A\rho\delta)\ddot{y} = F(x + \delta) - F(x)$

Balance of torques: $0 = -\tau(x + \delta) + \tau(x) + (\delta/2)[F(x + \delta) + F(x)]$

That is: $A\rho\ddot{y} = \frac{dF}{dx}$ and $0 = -\frac{d\tau}{dx} + F$

Now $\ddot{y} = -Y\omega^2 \cos\phi$

Hence $F = -A\rho Y\omega u \sin\phi$

$$\frac{d\tau}{dx} = -\rho Y(\omega/u)(Au^2)\sin\phi$$

So $\tau = \rho Y(Au^2)\cos\phi$

Here once again we have ignored integration constants reflecting the conditions on F and τ at the ends of the body.

POWER NEEDED TO SUPPORT ACCELERATION OF BODY PARTS

Since this torque component is out of phase with the rate of change of curvature we expect no average power.

$$\rho = \tau\frac{d}{dt}\left(\frac{d\theta}{dx}\right) = AY^2\rho\omega^2\sin\phi\cos\phi$$

Since $\sin\phi\cos\phi = \frac{1}{2}\sin 2\phi$, the average is indeed zero. Energy does have to be pumped in and out of this system however and this limits maximum amplitude, travelling wave velocity and angular frequency of oscillation.

This component of torque is exactly π out of phase with curvature. It is thus possible to generate it with passive springs. The necessary spring constant per unit length is:

$$\tau/(\frac{d\theta}{dx}) = \rho(u/\omega)^2 \, Au^2$$

While this is independent of amplitude, as expected, it does vary with angular frequency and travelling wave velocity. In effect, the body has to stiffen up for higher speeds.

PHASE RELATIONSHIPS BETWEEN TORQUE AND DISPLACEMENT WAVEFORMS

If the torque required to accelerate body segment masses is generated by actuators and not passive elastic means, it will introduce a component that is in phase with the displacement waveform. The overall torque waveform now can lag on displacement anywhere from 0 to $\pi/2$, depending on the relative magnitudes of the two components. If the body is light, moving slowly in a viscous fluid for example, the lag will be near $\pi/2$, since most of the actuator torque will be needed to exert forces on the fluid. On the other hand if the body is moving rapidly through a fluid of low viscosity, the phase lag will be near 0, since most of the torque balances body segment accelerations.

If the body cross-section varies along the length of the body, one can be faced with a situation where the phase-shift between torque and displacement waveforms varies along the body. This has been observed in a natural system [2].

EFFECT OF INERTIA OF BODY SEGMENTS

In addition to lateral accelerations of body segments we also have to consider their angular acceleration. We will show that the torque required for this is small and in phase with the torque required to linearly accelerate body segments. Again consider a cross-section of the body as in Figure 8 having movement about the y-axis. The moment of inertia of a thin slice of thickness δ and density ρ will be $I\rho\delta$. The equations become, ignoring the other components now:

$$0 = \frac{dF}{dx} \qquad \text{and} \qquad I\rho\ddot{\theta} = -\frac{d\tau}{dx} + F$$

since $$\ddot{\theta} = Y(\omega/u)\omega^2 \sin\phi$$ we get:

$$\tau = \rho Y(I\omega^2)\cos\phi$$

This is clearly in phase with the torque required for lateral acceleration of body segments and can be accounted for by adding $I\omega^2$ to the Au^2 term appearing in the equations for that torque, the power, as well as the equation for the required spring constant.

RELATIVE IMPORTANCE OF INERTIAL AND MASS TERMS

The relative importance of the inertial and mass terms depends on the magnitude of $I\omega^2$ compared to Au^2. Since $\lambda = 2\pi(u/\omega)$, we see that the mass component dominates if

$$I/A < (\lambda/2\pi)^2$$

Roughly, if the wavelength is much larger than the radius of gyration, we can ignore the inertial term.

FUNDAMENTAL EEL EQUATIONS -- SUMMARY

If we consider all contributions to the balance of forces and torques we get the equations:

$$A\rho\ddot{y} \approx \frac{dF}{dx} - ft$$

$$I\rho\ddot{\theta} \approx -\frac{d\tau}{dx} + F$$

We then assumed a travelling wave displacement of the form

$$y = Y\cos\phi \qquad \text{where} \qquad \phi = \omega(t + x/u)$$

This produces an average propulsive force.

$$\overline{F}x = dv + \tfrac{1}{2}Y^2(\omega/u)^2[du - \ell(u - v)]$$

The lateral force on the body is:

$$f_t = -\ell(u - v)Y(\omega/v)\sin\phi$$

Finally we found three torque components that must be generated internally:

$$\tau_1 = \ell(u - v)Y(u/\omega)\sin\phi$$

$$\tau_2 = \rho YAu^2\cos\phi$$

$$\tau_3 = \rho YI\omega^2\cos\phi$$

DISCUSSION

We have developed details of the mechanical aspects of a form of swimming locomotion. This discussion applies perhaps most directly to animals such as sea-serpents and moray eels, but also to some extent to fish and various kinds of worms and microscopic organisms. The vertical undulations of the wings of rays and mantas are clearly also included.

John Purbrick has proposed another form of swimming locomotion utilizing undulating waves of body cross-section instead of side-way displacement. One might call this exterior peristalsis. We conjecture that this may be an equally efficient method, but depends on compressibility of body segments. Observations suggest that sea worms utilize this mode of locomotion. These creatures are essentially fluid-filled tubes with musculature arranged to allow contraction of selected body-segments. Waves of contraction propagate rearward along their bodies.

An interesting question is the preference for drop-shapes and ancilliary fins and other appendages amongst certain species. Are these forms of advantage for anatomical or fluid-mechanical reasons? In fact, what importance is there to varying body cross-section and amount of musculature along the length of the bodies of some animals? Do such characteristics help or hinder the locomotory process?

Several other issues remain to be explored. We have treated the end-conditions on lateral force and torque somewhat cavalierely for example. Neither have we discussed the longitudinal force in the "spine" or alternate ways of arranging the musculature. We have not explored the relative efficiency of of short bursts of high frequency undulations followed by free coasting versus that of steady forward motion. It is possible of course that such maneuvers are adopted by certain animals to avoid predators instead of savings in energy expenditure. We have concentrated largely on the case of steady forward motion, ignoring acceleration, deceleration, and turning or changing the direction of motion.

Some readers may have noticed a direct analogy between the generation of propulsive force here as a result of rearward slippage of the body undulations relative to the fluid with the generation of torque in an induction motor as a result of slippage of the rotor relative to the rotating magnetic field. Many such issues remain to be explored. Perhaps the most interesting is the possibility of building a mechanical "eel", with the potential for very efficient rapid propulsion with relatively little generation of noise.

REFERENCES

1. Paul, Richard. "Modelling, Trajectory Calculation and Servoing of a Computer Controlled Arm". Stanford Artificial Intelligence Memo 177, November 1972.

2. Grillner, S. and Kashin, S. "On the Generation and Performance of Swimming in Fishes". International Conference on Neural Control of Locomotion, Temple University, September 1975.

3. Kashin, S. M., Feldman, A. G., and Orlovsky, G. N. "Locomotion of Fish Evoked by Electrical Stimulation of the Brain". Brain Research, 82, 1974, 41-47.

4. Gray, J. "The Movement of the Spermatozoa of the Bull". J. Exp. Biol., 35, 1958, 96-108.

5. Bainbridge, Richard. "The Speed of Swimming of Fish as Related to Size and to the Frequency and Amplitude of the Tail Beat". J. Exp. Biol., 35, 1958, 109-133.

6. Gray, J. "The Propulsive Mechanism of the Whiting". J. Exp. Biol., 10, 1933, 386-390.

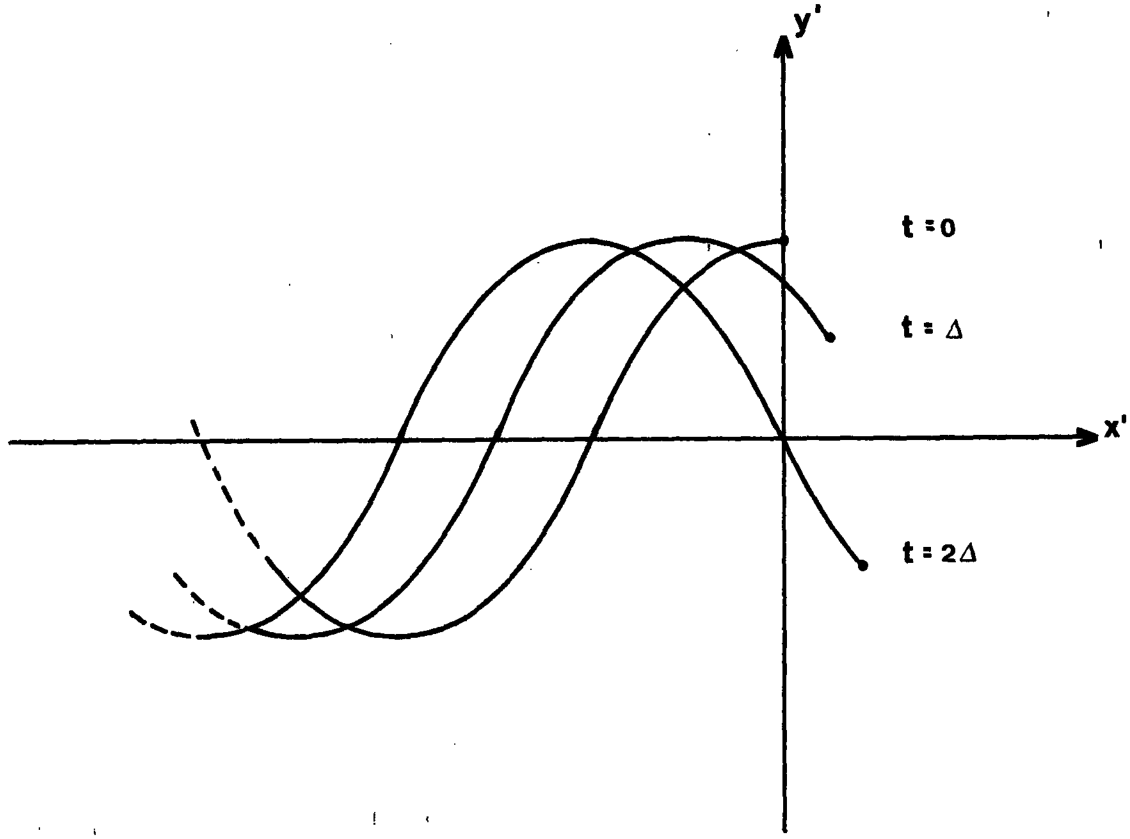

FIGURE 1. Travelling wave of body motion. The amplitude of the wave is Y, the length of the body is L. The wave, of angular frequency ω, travels backward along the body at velocity u, while the body is propelled forward at velocity v relative to the fluid. u is somewhat larger than v.

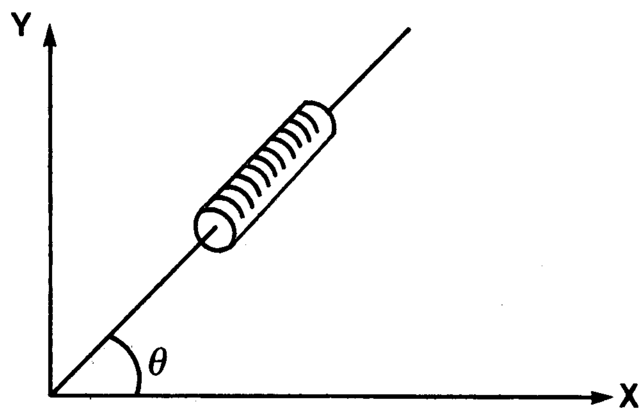

FIGURE 2. Inclination of Body Segment.

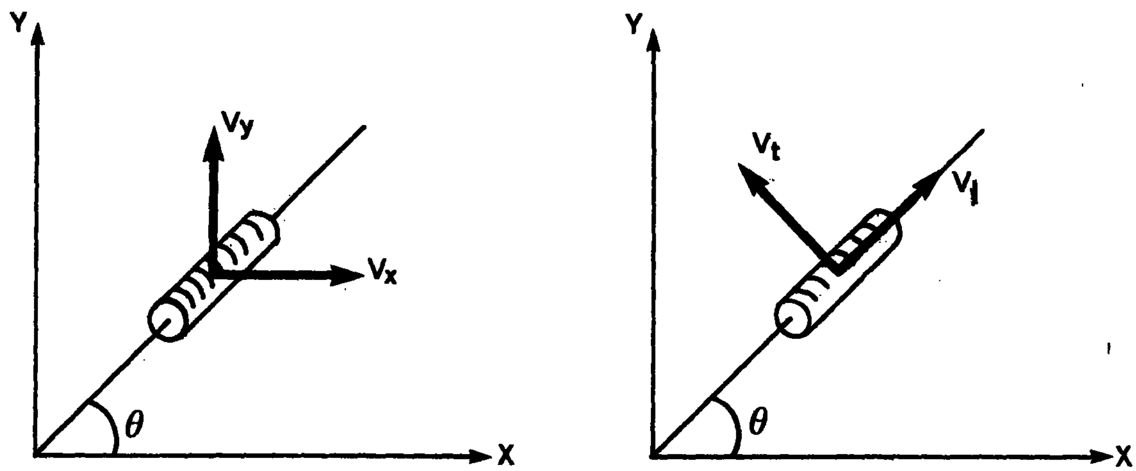

FIGURE 3. Determination of velocity along direction of body segment and velocity at right angles to body segment.

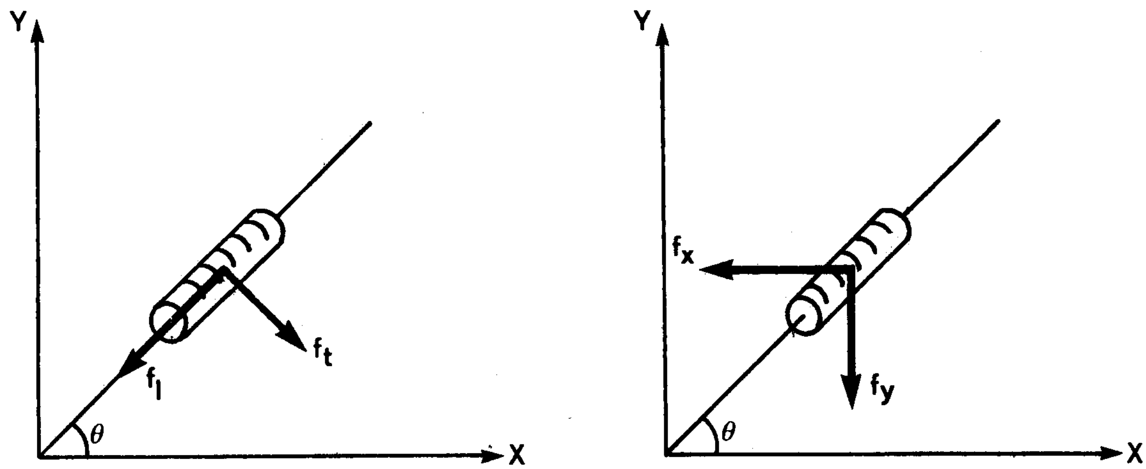

FIGURE 4. Determination of forces per unit length in direction of coordinate axes.

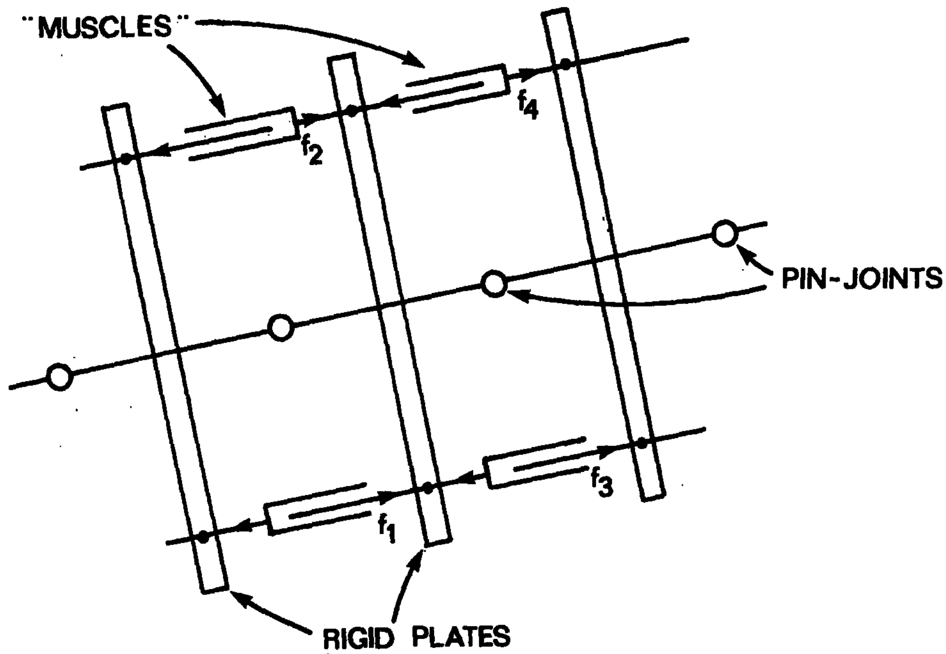

FIGURE 5. Discrete Model of Body Section. The "muscles" generate torques around the pin-jointed "spine". The rigid plates are 2r long and the separation between successive pin-joints is δ.

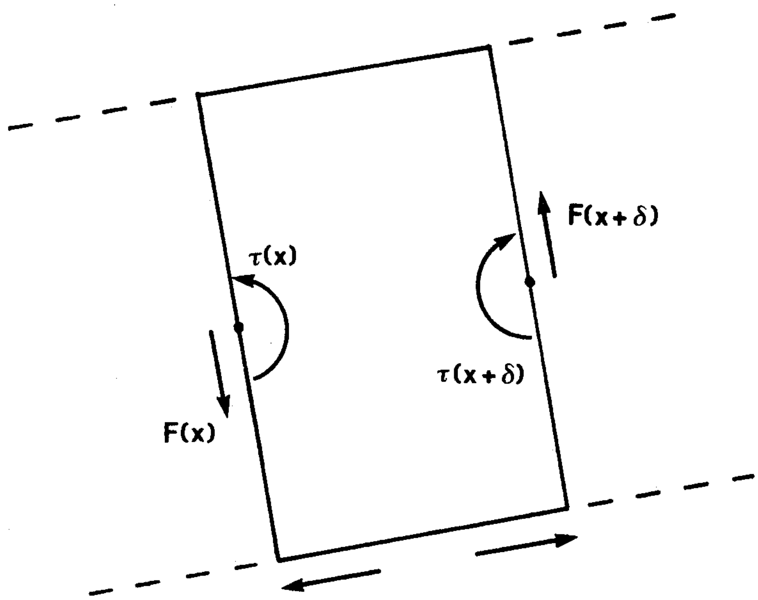

FIGURE 6. Slice of Continuous Model of Body. F is the shear force across the body, while τ is the torque.

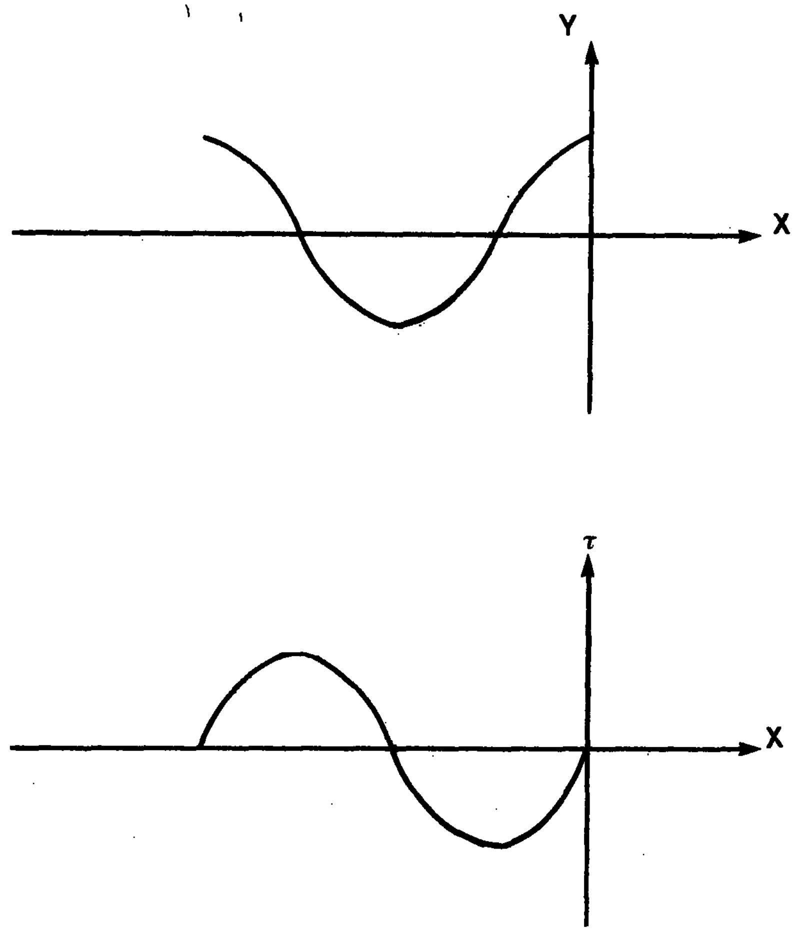

FIGURE 7. Phase relationship between driving torque and displacement for the massless body model.

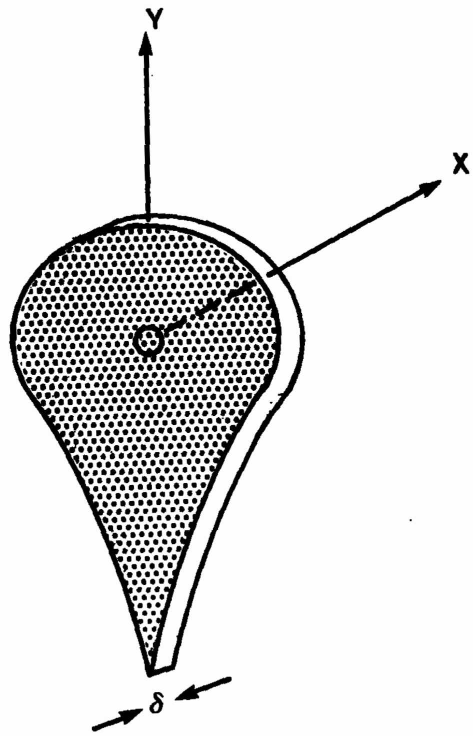

FIGURE 8. Thin slice of body for mass and inertia calculation.