THE PLASTIC BENDING OF CURVED BARS

by

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Professor Joseph S. Newell
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge, Massachusetts

Dear Professor Newell:

I hereby submit this thesis entitled, "THE PLASTIC BENDING OF CURVED BARS," in partial fulfillment of the requirements for the degree of Master of Science in Mechanical Engineering from the Massachusetts Institute of Technology.

Respectfully submitted,

Charles O. Smith

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NOMENCLATURE

a - a constant, see Case I

A - cross-sectional area of bar, square inches

b - width of bar, inches

c - a constant, see Case I

C_1, C_2 - constants of integration

E - modulus of elasticity, pounds per square inch

h - depth of bar, inches

I - moment of inertia of cross-section, $(\text{inches})^4$

k - a constant, used for integrating, Cases II and III

M - bending moment at any point, inch pounds, considered positive when decreasing the curvature of the bar

M_c - bending moment for complete yielding, inch pounds

M_i - bending moment for initial yielding, inch pounds

M_s - bending moment for secondary yielding, inch pounds

P - applied load, pounds

R_i - inner radius of curved bar, inches

R_o - outer radius of curved bar, inches

S - stress at any point, pounds per square inch

S_o - yield stress, pounds per square inch

w - deflection, inches

Nomenclature

y - distance of a point from the neutral axis, inches, measured positive upward

y_1 - distance from neutral axis to bottom of bar, inches

y_2 - distance from neutral axis to top of bar, inches

y' - distance from neutral axis to edge of plastic region, measured downward, inches

y'' - distance from neutral axis to edge of plastic region, measured upward, inches

$\bar{y}$ - vertical distance between neutral axis and horizontal axis through the centroid, inches

z - width of bar at any point, inches

ε - unit strain at any point, inches per inch

ε_1 - unit strain on bottom of bar, inches per inch

ε_2 - unit strain on top of bar, inches per inch

ε_o - unit strain at yield stress, inches per inch

ρ - radius of curvature at any point, inches

ρ_o - initial radius of curvature, inches

θ - angle of position, measured from fixed end, degrees or radians

θ_1 - angle to which plastic region extends on bottom of bar - degrees or radians

θ_2 - same to top of bar

Nomenclature

Numbers in parentheses refer to bibliography.

ASSUMPTIONS

1. Transverse cross-sections of the bar originally plane and normal to the center line of the bar remain so after bending.

2. The material of which the bar is composed is homogeneous and isotropic.

I N T R O D U C T I O N

The general field of the plastic behavior of materials is of great importance since many common fabricating processes are entirely dependent upon the plastic properties of materials. The bending of curved bars, in the plastic as well as in the elastic range, is of great importance since many such elements are used in structural engineering and electrical machinery, while others serve as essential machine parts. Consequently, it is desired to know of the behavior of such elements and to be able to predict the stress conditions and the deflection of such curved bars under various loading conditions.

P U R P O S E

It is the purpose of this thesis to develop analytical expressions for stress distribution and for deflection of a curved bar, which is built in at one end, of a length sufficient to form an arc of 90°, and loaded by a single load at the free end, this load being applied in the plane of curvature. (See Figure 1 for the geometry of the curved bar.)

PREVIOUS CONTRIBUTIONS TO THE SUBJECT

Several contributions have been made to knowledge of the general field of plastic behavior of materials by various individuals. A great deal of the work of formulating the laws of plastic behavior has been done by NADAI (2, 5, 6, 7).

Murphy and Timoshenko (4, 11) have written textbooks on the advanced theory of the mechanics of materials. Both of these texts contain expressions for stress distribution and deflection of a curved bar loaded in the plane of curvature, but these expressions apply only within the elastic range and for a stress-strain curve in which stress is directly proportional to strain. WINSLOW and EDMONDS (12) did considerable work on comparison of experimental results with theoretical predictions but, again, this work was only within the elastic range. HOGAN (3) developed expressions for design purposes for stress distribution, deflection angle of slope, and angle of twist in circular cantilever beams loaded normal to the plane of curvature under various load conditions; but these expressions also hold only for the elastic region.

NADAI (7) has developed expressions for determining the extent of the plastic region, the stress distribution, and deflection of a bar subjected to plastic bending using an idealized stress-strain curve and a curve accounting for the effect of work hardening (somewhat idealized), but these expressions apply only to a straight beam, not to an

PREVIOUS CONTRIBUTIONS TO THE SUBJECT

initially curved bar.

So far as the author could determine, there have been no publications of work on the plastic bending of a curved bar.

PROCEDURE

Three separate cases were considered. The case of a bar with rectangular cross-section and an idealized stress-strain curve, the case of a bar with rectangular cross-section and an arbitrary stress-strain curve of the form $S = S_0 \left(\frac{2\epsilon}{\epsilon_0}\right)^{\frac{1}{2}}$, and the case of the bar with triangular cross-section and an arbitrary stress-strain curve of the form $S = S_0 \left(\frac{2\epsilon}{\epsilon_0}\right)^{\frac{1}{2}}$; these being hereafter referred to as Case I, Case II, and Case III, respectively.

The first consideration was to determine a general method and then apply it to each of these cases separately. This is necessary since both the cross-section of the bar and the stress-strain curve will affect the results obtained. In fact, the results and their form are entirely dependent on these two factors.

The method developed is simple in idea but very likely to be unwieldy in execution. The method followed is:

(1) The use of the equilibrium condition, $\int S\,dA = 0$, to determine the position of the neutral axis.

(2) The use of the equilibrium condition, $M = \int S y\,dA$, to determine the bending moment in terms of the geometry of the cross-section and the stress-strain curve.

(3) The combination of the bending moment expression just derived and the stress-strain expression to obtain an expression for stress distribution in terms of the bending moment and the geometry of the cross-section.

PROCEDURE

(4) The determination of the expression, $\frac{\rho_0 - \rho}{\rho}$, from the bending moment expression derived in Step 2, and the substitution of this term in the differential equation

$$\frac{d^2w}{d\theta^2} + w = \rho_0\left(\frac{\rho_0 - \rho}{\rho}\right) \qquad (10)$$

This will give expressions for the stress distribution and deflection in general terms of M for the particular cross-section and stress-strain curve being considered. These expressions must then be used to determine the stress distribution equation and the deflection equation for the geometry of the bar and the particular type of loading.

For the geometry and the loading considered in this work, the steps to be followed are:

(5) The substitution of $-P\rho_0\cos\theta$ for M in the stress distribution relation derived in Step 3.

(6) The substitution of $-P\rho_0\cos\theta$ for M in the differential equation derived in Step 4, and the solution of the differential equation for the deflection in terms of P and $\cos\theta$.

For each of the three cases, the extent of the plastic region, the stress distribution on the plane of maximum stress, and the deflection curve was determined for five different loads, varying from that required for initial yielding to that required for complete yielding (Case I) or for extensive yielding (Cases I and II). The maximum deflection versus load relation was determined for each of the three cases.

S U M M A R Y   A N D   D I S C U S S I O N

Case I

For the geometry of the curved bar and the type of loading considered (Figure 1), the stress distribution is given by:

For the completely elastic beam,

$$S = \frac{M}{A\bar{y}}\left(\frac{y}{e_o + y}\right)$$

For the partially plastic-partially elastic beam,

$$S = -So \quad \text{for } y'' < y < y_2$$

$$S = -So\left(\frac{e_o - y'}{y'}\right)\left(\frac{y}{e_o + y}\right) \quad \text{for } y' < y < y'' \quad \text{(elastic portion)}$$

$$S = So \text{ for } y' < y < y_1$$

$$M = \frac{S_o b e_o y'^2}{3(e_o - 2y')} - \frac{S_o b h^2}{4} = -P e_o \cos\theta$$

It is possible to combine the expression for stress distribution and bending moment to eliminate y' and obtain a new expression for stress distribution in the elastic portion. This expression is:

$$S = -S_o\left(\sqrt{1 + \frac{4e_o^2/3h^2}{1 - \frac{4Pe_o\cos\theta}{S_o b h^2}}}\right)\left(\frac{y}{e_o + y}\right)$$

For the geometry of the curved bar and the type of loading considered (Figure 1), the deflection equation is:

Summary and Discussion - Case I

For the completely elastic beam,

$$w = -\frac{P e_0^3 \theta \sin\theta}{2EI}$$

For the partially plastic-partially elastic beam,

for the partially plastic portion

$$\frac{d^2w}{d\theta^2} + w = -e_0 \varepsilon_0 \sqrt{1 + \frac{4 e_0^2/3h^2}{1 - \frac{4Pe_0\cos\theta}{S_0 b h^2}}}$$

for the completely elastic portion

$$w' = C_1 \sin\theta + C_2 \cos\theta - \frac{P e_0^3 \theta \sin\theta}{2EI}$$

Several attempts were made to find an analytical solution to the differential equation given above. Unfortunately all attempts met with no success. A solution may be made by numerical integration by the method of successive approximations (1, 8, 9). This was done making use of the boundary conditions that

$$\theta = 0, \quad w = 0, \quad \frac{dw}{d\theta} = 0$$

At the point where the partially plastic-partially elastic and the completely elastic portions of the beam meet, the constants of integration, C_1 and C_2, may be evaluated from the boundary conditions that

$$w = w', \quad \frac{dw}{d\theta} = \frac{dw'}{d\theta}$$

Summary and Discussion - Case I

A question might well be raised as to the validity of the expressions derived for this case since the position of the neutral axis does not remain fixed but shifts from its position at initial yielding to coincide with the horizontal axis through the centroid at complete yielding. In this particular case, with the depth of the beam small in comparison with the radius of curvature, this neutral axis shift is very small and therefore its effect is negligible. For example, in the numerical case for which results were calculated, this neutral axis shift was 0.002" in an original radius of curvature of 10.248".

Case II

For the geometry of the curved bar and the type of loading considered (Figure 1), the stress distribution is given by:

$$S = \frac{\mp P e_0 \cos\theta \left(\frac{\pm y}{e_0 + y}\right)^{\frac{1}{2}}}{b\left[\frac{3e_0^2}{4}\left(\sin^{-1}\sqrt{\frac{y_1}{e_0}} + \sinh^{-1}\sqrt{\frac{y_2}{e_0}}\right) + \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i}\right) - \frac{3e_0}{4}\left(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i}\right)\right]}$$

For the geometry of the curved bar and the type of loading considered (Figure 1), the deflection equation is:

Summary and Discussion - Case II

$$W = \frac{P^2 e_o^3 \mathcal{E}_o \left(\frac{2\cos\theta}{3} + \frac{\cos 2\theta}{3} - 1 \right)}{4 S_o^2 b^2 \left[\frac{3e_o^2}{4}\left(\sin^{-1}\sqrt{\frac{y_1}{e_o}} + \sinh^{-1}\sqrt{\frac{y_2}{e_o}} \right) + \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i} \right) - \frac{3e_o}{4}\left(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i} \right) \right]^2}$$

These expressions for stress distribution and deflection apply to both the completely elastic and the partially plastic-partially elastic cases. The reason for this is that the stress-strain curve applies to both the elastic and plastic regions.

Case III

For the geometry of the curved bar and the type of loading considered (Figure 1), the stress distribution is given by:

$$S = \frac{\mp P e_o \cos\theta \, h \left(\frac{\pm y}{e_o + y} \right)^{\frac{1}{2}}}{b \left[\left(\frac{3e_o^2 y_2}{4} + \frac{5e_o^3}{8} \right)\left(\sin^{-1}\sqrt{\frac{y_1}{e_o}} + \sinh^{-1}\sqrt{\frac{y_2}{e_o}} \right) - \frac{1}{3}\left(\sqrt{y_2^5}\sqrt{R_o} + \sqrt{y_1^5}\sqrt{R_i} \right) + \left(\frac{y_2}{2} + \frac{5e_o}{12} \right)\left(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i} \right) - \left(\frac{3e_o y_2}{4} + \frac{5e_o^2}{8} \right)\left(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i} \right) \right]}$$

Summary and Discussion - Case III

For the geometry of the curved bar and the type of loading considered (Figure 1), the deflection equation is:

$$W = \frac{P^2 e_0^3 h^2 \varepsilon_0 \left(\frac{2\cos\theta}{3} + \frac{\cos 2\theta}{3} - 1 \right)}{4 S_0^2 b^2 \left[\left(\frac{3e_0^2 y_2}{4} + \frac{5e_0^3}{8} \right)\left(\sin^{-1}\sqrt{\frac{y_1}{e_0}} + \sinh^{-1}\sqrt{\frac{y_2}{e_0}} \right) + \left(\frac{y_2}{2} + \frac{5e_0}{12} \right)\left(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i} \right) - \frac{1}{3}\left(\sqrt{y_2^5}\sqrt{R_o} + \sqrt{y_1^5}\sqrt{R_i} \right) - \left(\frac{3e_0 y_2}{4} + \frac{5e_0^2}{8} \right)\left(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i} \right) \right]^2}$$

These expressions for stress distribution and deflection apply to both the completely elastic and the partially plastic-partially elastic cases. The reason for this is that the stress-strain curve applies to both the elastic and plastic regions.

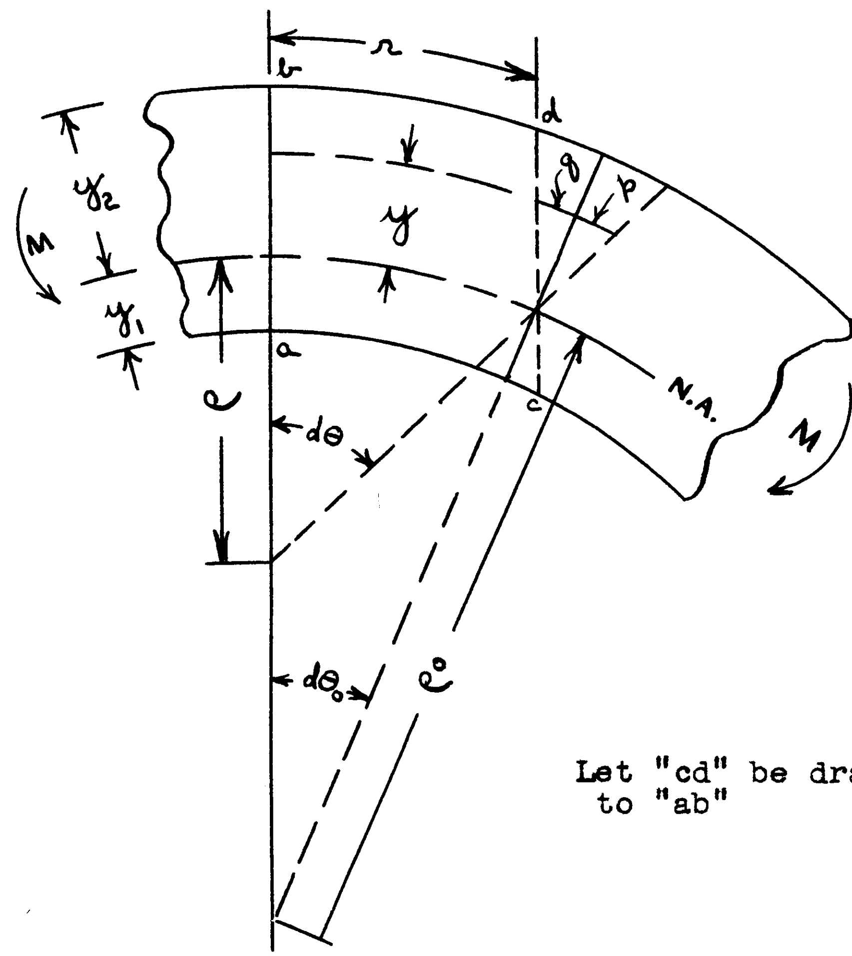
SUGGESTIONS FOR FUTURE INVESTIGATION

Since this thesis covers only a very narrow range of possibilities, future investigation could well be done in this field by using different cross-sections (circular, trapezoidal), various other stress-strain relations, other boundary conditions (arcs greater or smaller than 90°, bars built in at both ends, bars simply supported), and other conditions of loading (constant moment, uniformly distributed load, uniformly varying load, more than one concentrated load, loading normal to the plane of curvature).

An investigation which should prove interesting would be to determine the analytical expression for stress as a function of strain for a particular material, derive the stress distribution and deflection equations as was done in this thesis, and compare the predicted results with those obtained experimentally for a curved bar of the same material having the same boundary and loading conditions. This could be done with a number of materials and for a variety of combinations of cross-sections, boundary conditions and loading conditions.

GENERAL PRELIMINARY INVESTIGATION

Consider the general case of a curved bar in bending (elastic or plastic)



Let "cd" be drawn parallel to "ab"

From the similarity of triangles,

$$\frac{q}{r} = \frac{y}{e_0} \qquad \text{AND} \qquad \frac{p+q}{r} = \frac{y}{e}$$

$$\frac{p}{r} + \frac{q}{r} = \frac{y}{e}$$

$$\frac{p}{r} = \frac{y}{e} - \frac{q}{r} = \frac{y}{e} - \frac{y}{e_0} = y\left(\frac{1}{e} - \frac{1}{e_0}\right)$$

$$\frac{q+r}{r} = \frac{y}{e_0} + 1 = y\left(\frac{1}{e_0} + \frac{1}{y}\right)$$

$$\varepsilon = \frac{p}{q+r} = \frac{\frac{p}{r}}{\frac{q+r}{r}} = \frac{y\left(\frac{1}{e}-\frac{1}{e_o}\right)}{y\left(\frac{1}{e_o}+\frac{1}{y}\right)}$$

$$\varepsilon = \frac{\frac{1}{e}-\frac{1}{e_o}}{\frac{1}{e_o}+\frac{1}{y}}$$

$$\varepsilon = \left(\frac{e_o-e}{e}\right)\left(\frac{y}{e_o+y}\right) \qquad \text{EQUATION 1}$$

It should be noted that for $e_o = \infty$, $\varepsilon = \frac{y}{e}$, which is the case of a straight beam.

FROM EQ. 1

$$\frac{1}{e_o}+\frac{1}{y} = \frac{\frac{1}{e}-\frac{1}{e_o}}{\varepsilon}$$

$$\frac{1}{y} = \frac{\frac{1}{e}-\frac{1}{e_o}}{\varepsilon} - \frac{1}{e_o}$$

$$\frac{1}{y} = \frac{e_o-e}{e e_o \varepsilon} - \frac{1}{e_o}$$

$$\frac{1}{y} = \frac{e_o-e-e\varepsilon}{e e_o \varepsilon}$$

$$y = \frac{e e_o \varepsilon}{e_o - e(1+\varepsilon)} \qquad \text{EQUATION 2}$$

$$dy = \frac{e\, e_o (e_o - e)}{\left[e_o - e(1+\varepsilon)\right]^2}\, d\varepsilon$$

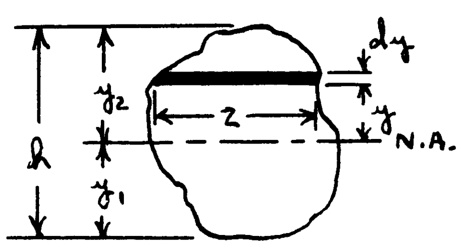
$$y_1 = \frac{e\, e_o\, \varepsilon_1}{e_o - e(1 - \varepsilon_1)} \qquad \text{EQUATION 3}$$

$$y_2 = \frac{e\, e_o\, \varepsilon_2}{e_o - e(1 + \varepsilon_2)} \qquad \text{EQUATION 4}$$

Where $-\varepsilon_1$ & ε_2 are the unit strains at $-y_1$ & y_2 respectively.

$$y_1 + y_2 = h \qquad \text{EQUATION 5}$$

Consider the cross-section



$$z = f(y)$$

$$dA = z\, dy$$

$$dA = f(y)\, dy$$

LET $S = f_2(\varepsilon)$

For equilibrium $\int S\,dA = 0$ over the cross-section of the beam.

$$\int S\,dA = 0$$

$$\int_{-y_1}^{y_2} f_2(\varepsilon)\, f(y)\, dy = 0$$

BUT FROM EQ. 2

$$y = \frac{e e_o \varepsilon}{e_o - e(1+\varepsilon)} \qquad \therefore\ f(y) = f_1(\varepsilon, e)$$

$$\int S\,dA = \int_{-\varepsilon_1}^{\varepsilon_2} f_2(\varepsilon) f_1(\varepsilon, e) \frac{e e_o (e_o - e)}{\left[e_o - e(1+\varepsilon)\right]^2}\, d\varepsilon = 0 \qquad \text{EQUATION 6}$$

$$M = \int S y\, dA$$

$$M = \int_{-y_1}^{y_2} f_2(\varepsilon)\, y\, f(y)\, dy$$

$$M = \int_{-\varepsilon_1}^{\varepsilon_2} f_2(\varepsilon) \frac{e e_o \varepsilon}{\left[e_o - e(1+\varepsilon)\right]} f_1(\varepsilon, e) \frac{e e_o (e_o - e)}{\left[e_o - e(1+\varepsilon)\right]^2}\, d\varepsilon$$

$$M = \int_{-\varepsilon_1}^{\varepsilon_2} e^2 e_o^2 (e_o - e) \frac{f_2(\varepsilon) f_1(\varepsilon, e)\, \varepsilon}{\left[e_o - e(1+\varepsilon)\right]^3}\, d\varepsilon \qquad \text{EQUATION 7}$$

BUT $M = f(\theta)$

$$\int_{-\varepsilon_1}^{\varepsilon_2} e^2 e_o^2 (e_o - e) \frac{f_2(\varepsilon) f_1(\varepsilon, e)\, \varepsilon}{\left[e_o - e(1+\varepsilon)\right]^3}\, d\varepsilon = f(\theta) \qquad \text{EQUATION 8}$$

There are six variables:

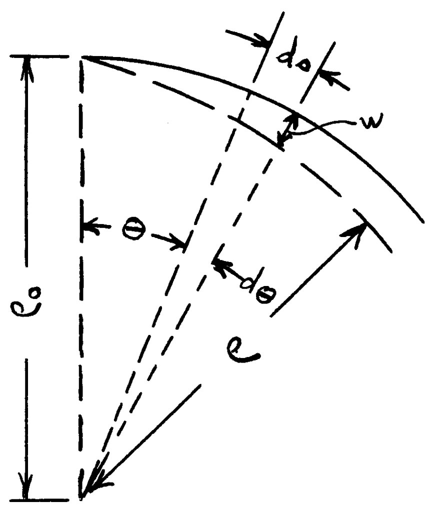
$$\gamma_1 , \gamma_2 , \varepsilon_1 , \varepsilon_2 , \rho , \theta$$

There are five equations:

3, 4, 5, 6, 8

Therefore, $\rho = f_2(\theta)$ can be found. EQUATION 9

Deflection Equation



It has been shown (10) that the solution of the following differential equation will yield the deflection equation.

$$\frac{d^2 w}{ds^2} + \frac{w}{\rho_0^2} = \frac{1}{\rho} - \frac{1}{\rho_0}$$

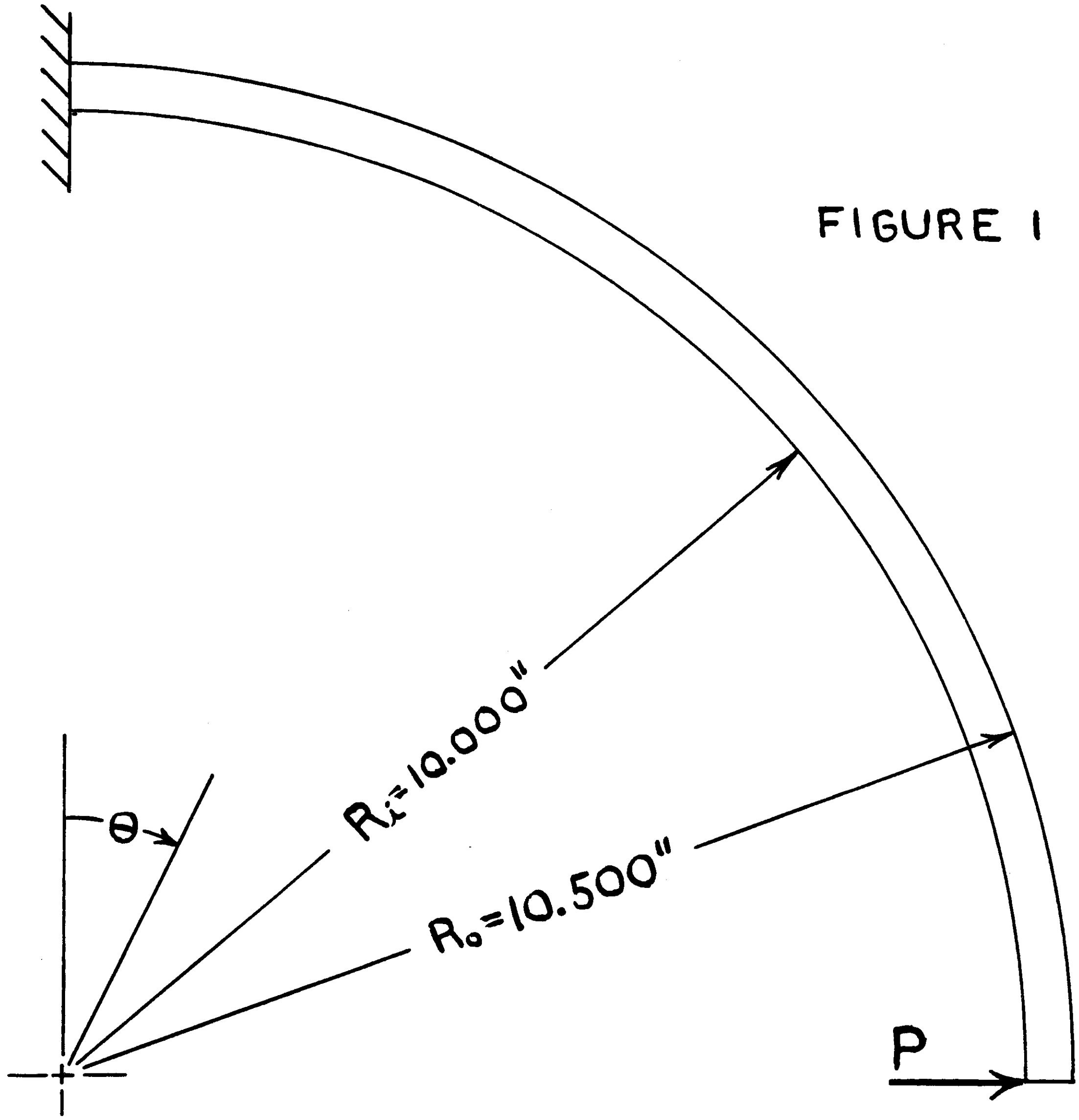
$$ds = \rho_0 \, d\theta$$

$$ds^2 = \rho_0^2 \, d\theta^2$$

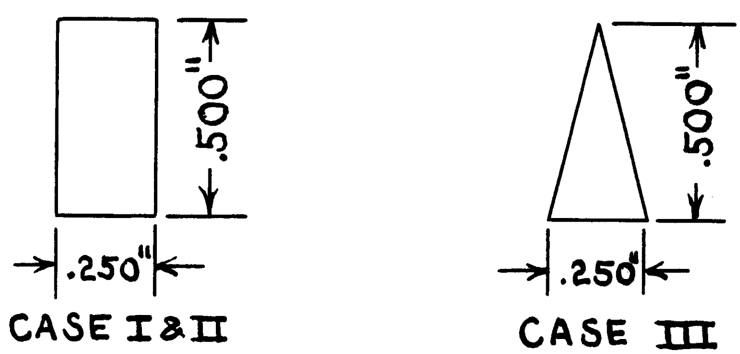
$$\frac{d^2 w}{d\theta^2} + w = \rho_0^2 \left(\frac{1}{\rho} - \frac{1}{\rho_0} \right) = \rho_0 \left(\frac{\rho_0 - \rho}{\rho} \right) \quad \text{EQUATION 10}$$

Substituting Equation 9 in Equation 10, and solving for $w = f_3(\theta)$ will yield deflection equation of the bar.

FIGURE I



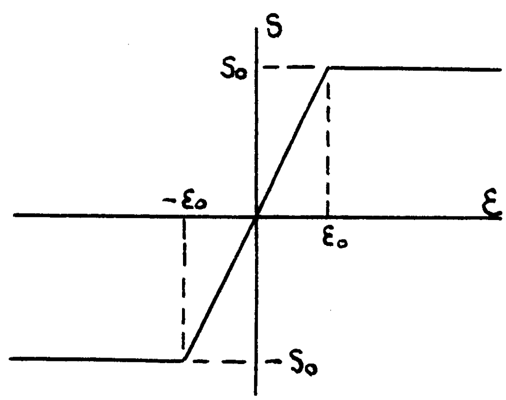
GEOMETRY OF THE CURVED BAR



CROSS-SECTION OF THE BAR

Case I

Consider a curved bar of rectangular cross-section
with depth of beam small
in comparison with the radius of curvature
and an idealized stress-strain curve such as



For the Completely Elastic Condition (4)

$$\rho_o = \frac{h}{\ln \frac{R_o}{R_i}} \qquad \text{Equation 11}$$

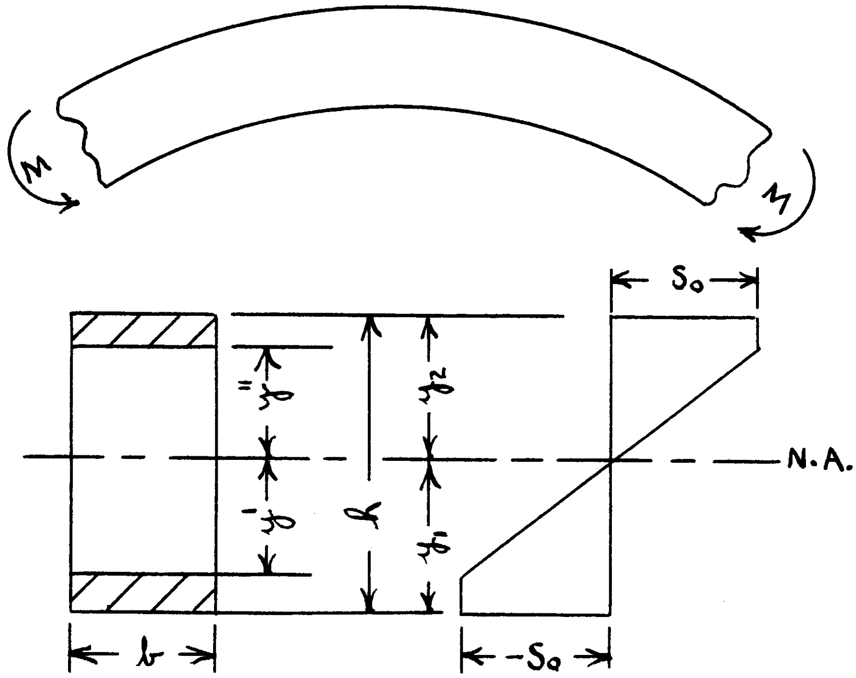
$$S = \frac{M}{A\bar{y}} \left(\frac{y}{\rho_o + y} \right) \qquad \text{Equation 12}$$

$$\frac{1}{\rho} - \frac{1}{\rho_o} = \frac{M}{EI}$$

$$\frac{d^2 w}{d\theta^2} + w = \rho_o^2 \frac{M}{EI} \qquad \text{Equation 13}$$

If the idealized stress-strain curve is assumed, it is obvious that beyond the yield point the neutral axis will shift toward the centroid of the cross-section until at the point of complete yielding, the neutral axis will coincide with the horizontal axis passing through the centroid.

Partial Plastic Yielding



Considering the elastic stress equation (Equation 12) assume $S = \frac{ay}{c+y}$ in the elastic portion of the bar, in which "a" and "c" are constants for the particular section.

For the plastic portion, $S = \pm S_0$

$$S_0 = \frac{ay''}{c+y''} \qquad - S_0 = \frac{a(-y')}{c-y'}$$

DIVIDING

$$-\frac{S_0}{S_0} = \frac{\frac{ay''}{c+y''}}{\frac{-ay'}{c-y'}} = \frac{y''(c-y')}{y'(c+y'')}$$

$$y'(c+y'') = y''(c-y')$$

$$cy' + y'y'' = cy'' - y'y''$$

$$cy' - cy'' = -2y'y''$$

$$c(y'-y'') = -2y'y''$$

$$c = \frac{-2y'y''}{y'-y''} = \frac{2y'y''}{y''-y'}$$

$$c+y'' = \frac{2y'y''}{y''-y'} + y'' = \frac{y''(y''+y')}{y''-y'}$$

$$\frac{y''}{c+y''} = \frac{y''-y'}{y''+y'}$$

$$S_o = \frac{a y''}{c+y''} = a\left(\frac{y''-y'}{y''+y'}\right)$$

$$a = S_o \left(\frac{y''+y'}{y''-y'}\right)$$

$$\varepsilon = \left(\frac{e_o-e}{e}\right)\left(\frac{y}{e_o+y}\right)$$

$$\varepsilon_o = \left(\frac{e_o-e}{e}\right)\left(\frac{y''}{e_o+y''}\right) \qquad -\varepsilon_o = \left(\frac{e_o-e}{e}\right)\left(\frac{-y'}{e_o-y'}\right)$$

DIVIDING

$$-\frac{\varepsilon_o}{\varepsilon_o} = \frac{\left(\frac{e_o-e}{e}\right)\left(\frac{y''}{e_o+y''}\right)}{\left(\frac{e_o-e}{e}\right)\left(\frac{-y'}{e_o-y'}\right)} = -\left(\frac{y'}{e_o-y'}\right)\left(\frac{e_o+y''}{y''}\right)$$

$$y''(e_0 - y') = y'(e_0 + y'')$$

$$e_0 y'' - y'y'' = e_0 y' + y'y''$$

$$e_0 (y'' - y') = 2y'y''$$

$$e_0 = \frac{2y'y''}{y''-y'} \qquad \text{BUT} \quad c = \frac{2y'y''}{y''-y'}$$

$$\therefore \; c = e_0$$

$$2y'y'' = e_0 y'' - e_0 y'$$

$$y''(2y' - e_0) = -e_0 y'$$

$$y'' = \frac{e_0 y'}{e_0 - 2y'}$$

$$2y'y'' + e_0 y' = e_0 y''$$

$$y'(2y'' + e_0) = e_0 y''$$

$$y' = \frac{e_0 y''}{e_0 + 2y''}$$

$$y'' + y' = \frac{e_0 y'}{e_0 - 2y'} + y' = \frac{2y'(e_0 - y')}{e_0 - 2y'}$$

$$y'' - y' = \frac{e_0 y'}{e_0 - 2y'} - y' = \frac{2y'^2}{e_0 - 2y'}$$

$$a = S_o \left(\frac{y'' + y'}{y'' - y'} \right)$$

$$a = S_o \left(\frac{\dfrac{2y'(e_o - y')}{e_o - 2y'}}{\dfrac{2y'^2}{e_o - 2y'}} \right)$$

$$a = \frac{S_o (e_o - y')}{y'} = \frac{S_o (e_o + y'')}{y''}$$

$$S = S_o \left(\frac{e_o - y'}{y'} \right) \left(\frac{y}{e_o + y} \right) \quad \text{EQUATION 14}$$

$$\int S\, dA = 0$$

$$\int_{-y_1}^{-y'} -S_0 b\, dy + \int_{-y'}^{y''} \frac{ay}{c+y} b\, dy + \int_{y''}^{y_2} S_0 b\, dy = 0$$

$$-yS_0 b\Big]_{-y_1}^{-y'} + ab\Big[c+y - c \ln c+y\Big]_{-y'}^{y''} + S_0 b\, y\Big]_{y''}^{y_2} = 0$$

$$-S_0(-y'+y_1) + a\left[c+y'' - c + y' - c\ln\frac{c+y''}{c-y'}\right] + S_0(y_2 - y'') = 0$$

$$a\left[y''+y' - c\ln\frac{c+y''}{c-y'}\right] = S_0\left[y_1 - y' - (y_2 - y'')\right]$$

$$c - y' = \frac{y'(y''+y')}{y''-y'}$$

$$c + y'' = \frac{y''(y''+y')}{y''-y'}$$

OR

$$\frac{c+y''}{c-y'} = \frac{y''}{y'} = \frac{\dfrac{e_0 y'}{e_0 - 2y'}}{y'} = \frac{e_0}{e_0 - 2y'}$$

$$a\left[y''+y' - c\ln\frac{e_0}{e_0-2y'}\right] = S_0\left[y_1 - y_2 - y' + y''\right]$$

$$y_2 = h - y_1 \qquad\qquad y'' = \frac{e_0 y'}{e_0 - 2y'}$$

$$c = e_0 \qquad\qquad a = S_0\left(\frac{e_0 - y'}{y'}\right)$$

$$S_o\left(\frac{e_o - y'}{y'}\right)\left[y' + \frac{e_o y'}{e_o - 2y'} - e_o \ln \frac{e_o}{e_o - 2y'}\right] = S_o\left[y_1 - (h - y_1) - y' + \frac{e_o y'}{e_o - 2y'}\right]$$

$$\left(\frac{e_o - y'}{y'}\right)\left[\frac{y'(e_o - 2y') + e_o y'}{e_o - 2y'} - e_o \ln \frac{e_o}{e_o - 2y'}\right] = 2y_1 - h + \frac{(-y')(e_o - 2y') + e_o y'}{e_o - 2y'}$$

$$\frac{e_o(e_o - y')}{y'} \ln \frac{e_o}{e_o - 2y'} = h - 2y_1 - \frac{2y'^2}{e_o - 2y'} + \frac{2(e_o - y')^2}{e_o - 2y'}$$

$$\frac{e_o(e_o - y')}{y'} \ln \frac{e_o}{e_o - 2y'} = h - 2y_1 + 2e_o$$

$$h - 2y_1 = \frac{e_o(e_o - y')}{y'} \ln \frac{e_o}{e_o - 2y'} - 2e_o$$

$$y_1 = -\frac{1}{2}\left[\frac{e_o(e_o - y')}{y'} \ln \frac{e_o}{e_o - 2y'} - (2e_o + h)\right] \quad \text{EQUATION 15}$$

This expression is used to determine the position of the neutral axis. A trial and error solution is recommended since the expression is not readily solvable by algebraic means. Care should be taken to insure accuracy since the solution involves the small difference of two comparatively large numbers.

$$M = \int S_y \, dA$$

$$M = \int_{-y_1}^{-y'} -S_0 b \, y \, dy + \int_{-y'}^{y''} \frac{a y}{c+y} \, y b \, dy + \int_{y''}^{y_2} S_0 \, y \, b \, dy$$

$$M = -S_0 b \int_{-y_1}^{-y'} y \, dy + ab \int_{-y'}^{y''} \frac{y^2 dy}{c+y} + S_0 b \int_{y''}^{y_2} y \, dy$$

$$M = -\frac{S_0 b y^2}{2} \Big]_{-y_1}^{-y'} + ab \Big[\frac{1}{2}(c+y)^2 - 2c(c+y) + c^2 \ln (c+y) \Big]_{-y'}^{y''} + \frac{S_0 b y^2}{2} \Big]_{y''}^{y_2}$$

$$M = ab \left[\frac{y''^2 - y'^2}{2} - c(y''+y') + c^2 \ln \frac{c+y''}{c-y'} \right] + \frac{S_0 b}{2} \left[y_1^2 + y_2^2 - (y'^2 + y''^2) \right]$$

$$a = \frac{S_0 (e_0 + y'')}{y''} \qquad\qquad c = e_0$$

$$y_2 = h - y_1 \qquad\qquad \frac{c+y''}{c-y'} = \frac{e_0 + 2y''}{e_0}$$

$$M = \frac{S_0 (e_0 + y'')}{y''} b \left[\frac{y''^2 - y'^2}{2} - e_0 (y'' + y') + e_0^2 \ln \frac{e_0 + 2y''}{e_0} \right]$$

$$+ \frac{S_0 b}{2} \left[y_1^2 + (h - y_1)^2 - (y'^2 + y''^2) \right]$$

$$M = \frac{S_0 b}{2} \left(\frac{e_0 + y''}{y''} \right) \left(y''^2 - y'^2 \right) - S_0 b e_0 \left(\frac{e_0 + y''}{y''} \right) (y'' + y') + \frac{S_0 b h}{2} (h - 2y_1)$$

$$+ S_0 b e_0^2 \left(\frac{e_0 + y''}{y''} \right) \ln \frac{e_0 + 2y''}{y''} + S_0 b y_1^2 - \frac{S_0 b}{2} \left(y'^2 + y''^2 \right)$$

$$M = \frac{2 S_0 b y''^2 (e_0 + y'')^2}{(e_0 + 2y'')^2} - \frac{2 S_0 b e_0 (e_0 + y'')^2}{e_0 + 2y''} - \frac{S_0 b y''^4}{(e_0 + 2y'')^2}$$

$$+ S_0 b e_0^2 \left(\frac{e_0 + y''}{y''}\right) \ln \frac{e_0 + 2y''}{e_0} + \frac{S_0 b h e_0}{2} \left(\frac{e_0 + y''}{y''}\right) \ln \frac{e_0 + 2y''}{e_0}$$

$$- S_0 b h e_0 + \frac{S_0 b}{4} (2e_0 + h)^2 - \frac{S_0 b}{2} (2e_0 + h) e_0 \left(\frac{e_0 + y''}{y''}\right) \ln \frac{e_0 + 2y''}{e_0}$$

$$+ \frac{S_0 b e_0^2}{4} \left(\frac{e_0 + y''}{y''}\right)^2 \left[\ln \left(\frac{e_0 + 2y''}{e_0}\right)\right]^2 - \frac{S_0 b y''^2 (e_0 + y'')^2}{(e_0 + 2y'')^2}$$

$$M = -\frac{S_0 b y''^2}{(e_0 + 2y'')^2} \left[-(e_0 + y'')^2 + y''^2\right] + \frac{S_0 b e_0^2}{4} \left(\frac{e_0 + y''}{y''}\right)^2 \left[\ln \left(\frac{e_0 + 2y''}{e_0}\right)\right]^2$$

$$- \frac{2 S_0 b e_0 (e_0 + y'')^2}{e_0 + 2y''} - \frac{S_0 b}{4} \left[4 h e_0 - (2e_0 + h)^2\right]$$

$$M = -S_0 b e_0^2 + \frac{S_0 b h^2}{4} - \frac{S_0 b e_0 y''^2}{e_0 + 2y''} + \frac{S_0 b e_0^2}{4} \left(\frac{e_0 + y''}{y''}\right)^2 \left[\ln \left(\frac{e_0 + 2y''}{e_0}\right)\right]^2$$

LET $\dfrac{e_0 + 2y''}{e_0} = x$

EXPANSION

$$\ln x = 2 \left[\frac{x-1}{x+1} + \frac{1}{3} \left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5} \left(\frac{x-1}{x+1}\right)^5 + \cdots\right]$$

$$\frac{x-1}{x+1} = \frac{y''}{e_0 + y''} << 1$$

$$\frac{S_0 b e_0^2}{4} \left(\frac{e_0 + y''}{y''}\right)^2 \left[\ln \left(\frac{e_0 + 2y''}{e_0}\right)\right]^2 \simeq \frac{S_0 b e_0^2}{4} \left(\frac{e_0 + y''}{y''}\right)^2 4 \left[\left(\frac{y''}{e_0 + y''}\right)^2 + \frac{2}{3} \left(\frac{y''}{e_0 + y''}\right)^4\right]$$

NEGLECTING FURTHER TERMS

$$= S_o b e_o^2 + \frac{2}{3} S_o b e_o^2 \left(\frac{y''}{e_o + y''}\right)^2$$

$$= S_o b e_o^2 + \frac{2}{3} \frac{S_o b e_o^2 y''^2}{e_o^2 + 2e_o y'' + y''^2}$$

NEGLECT y''^2 — $y''^2 <<< e_o^2 + 2e_o y''$

$$M = \frac{S_o b h^2}{4} - \frac{S_o b e_o y''^2}{3(e_o + 2y'')} \qquad \text{EQUATION 16a}$$

OR

$$M = \frac{S_o b h^2}{4} - \frac{S_o b e_o y'^2}{3(e_o - 2y')} \qquad \text{EQUATION 16b}$$

$$M = \frac{S_0 b h^2}{4} - \frac{S_0 b e_0 y'^2}{3(e_0 - 2y')}$$

For Complete Yielding $y' = 0$

$$M_c = \frac{S_0 b h^2}{4}$$

Let $q = \frac{1}{y'}$

$$M = \frac{S_0 b h^2}{4} - \frac{S_0 b e_0}{3(e_0 q^2 - 2q)}$$

$$\frac{4M}{S_0 b h^2} - 1 = \frac{-4 S_0 b e_0}{3 S_0 b h^2 (e_0 q^2 - 2q)}$$

$$\left(\frac{M}{M_c} - 1\right) = -\frac{4 e_0}{3h^2 (e_0 q^2 - 2q)}$$

$$\left(\frac{M}{M_c} - 1\right) e_0 q^2 - 2q\left(\frac{M}{M_c} - 1\right) + \frac{4 e_0}{3h^2} = 0$$

$$q = \frac{2\left(\frac{M}{M_c} - 1\right) \pm \sqrt{4\left(\frac{M}{M_c} - 1\right)^2 - 4\left(\frac{M}{M_c} - 1\right) e_0 \left(\frac{4 e_0}{3h^2}\right)}}{2 e_0 \left(\frac{M}{M_c} - 1\right)}$$

$$q = \frac{1}{y'} = \frac{1}{e_0} + \frac{1}{e_0}\sqrt{1 - \frac{4 e_0^2/3h^2}{\frac{M}{M_c} - 1}}$$

$$\frac{1}{y'} - \frac{1}{e_0} = \frac{1}{e_0}\sqrt{1 - \frac{4 e_0^2/3h^2}{\frac{M}{M_c} - 1}}$$

$$\frac{e_0 - y'}{e_0 y'} = \frac{1}{e_0}\sqrt{1 - \frac{4 e_0^2/3h^2}{\frac{M}{M_c} - 1}}$$

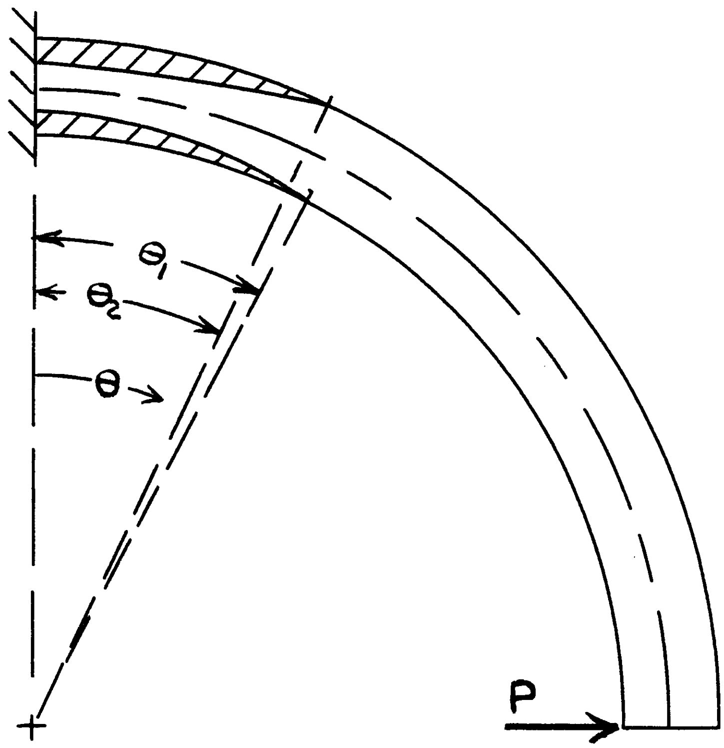
$$\frac{e_o - y'}{y'} = \frac{1}{\varepsilon_o}\left(\frac{e_o - e}{e}\right)$$

$$\frac{e_o - e}{e} = \varepsilon_o \sqrt{1 - \frac{4e_o^2/3h^2}{\frac{M}{M_c} - 1}}$$

SUBSTITUTING IN EQUATION 10

$$\frac{d^2 w}{d\theta^2} + w = e_o \varepsilon_o \sqrt{1 + \frac{4e_o^2/3h^2}{1 - \frac{M}{M_c}}} \qquad \text{EQUATION 17}$$

Calculations for the Specific Case (Figure 1)



$$M = -P \rho_o \cos\theta$$

$b = 0.250"$
$h = 0.500"$
$R_o = 10.500"$
$R_i = 10.000"$

Initial Yielding

From Equation 11

$$\rho_o = \frac{.500}{\ln \frac{10.500}{10.000}} = 10.248"$$

From Equation 12 and using y = y' = y_1

$$S_o = \frac{M_i}{.250 \times .500 \times .002}\left(\frac{-.248}{10.000}\right)$$

$$M_i = -0.001008\ S_o \quad \text{IN.LB.}$$

$$P = 0.000984\ S_o \quad \text{LB}$$

Secondary Yielding

From Equation 15

$y_1 = -0.248"$ or $\rho_o = 10.248"$

From Equation 16a using $y" = y_2 = 0.252"$

$M_s = -0.01058\ S_o$ in.-lb

$P = 0.001032\ S_o$ lb

Complete Yielding

From inspection of the equilibrium condition,

$\rho_o = 10.250"$

From Equation 16a or 16b using $y' = y" = 0$

$M_c = -0.01562\ S_o$ in.-lb

$P = 0.001524\ S_o$ lb

Stress Distribution

For any value of applied load up to that required for initial yielding ($P = 0.000984\ S_o$) and for the elastic portion (completely so) of the curved bar, the stress distribution is given by Equation 12.

$$S = \frac{M}{A\bar{y}}\left(\frac{y}{\rho_o + y}\right)$$

For larger values of applied load up to that required for complete yielding, in any section which has partially yielded, the stress is given in the elastic portion by Equation 14, $S = -S_o\left(\frac{\rho_o - y'}{y'}\right)\left(\frac{y}{\rho_o + y}\right)$, and in the plastic portion by S_o for $-y$ and by $-S_o$ for $+y$.

Deflection Equations

$0 < M < M_i$

EQUATION 13 APPLIES

$$\frac{d^2w}{d\theta^2} + w = \rho_0^2 \frac{M}{EI}$$

BUT $M = -P\rho_0 \cos\theta$

$$\frac{d^2w}{d\theta^2} + w = -\frac{P\rho_0^3 \cos\theta}{EI}$$

COMPLEMENTARY SOLUTION

$$w = C_1 \sin\theta + C_2 \cos\theta$$

PARTICULAR SOLUTION

$$w = -\frac{P\rho_0^3}{2EI} \theta \sin\theta$$

GENERAL SOLUTION

$$W = C_1 \sin\theta + C_2 \cos\theta - \frac{P\rho_0^3}{2EI} \theta \sin\theta$$

$$\theta = 0, \quad w = 0, \quad \frac{dw}{d\theta} = 0$$

EVALUATING C_1 & C_2

$$C_1 = C_2 = 0$$

$$W = -\frac{P\rho_0^3}{2EI} \theta \sin\theta \qquad \text{EQUATION 18}$$

$M_i < M < M_c$

$0 < \theta < \theta_1$ Equation 17 Applies

$$\frac{d^2 w}{d\theta^2} + w = -\rho_o \mathcal{E}_o \sqrt{1 + \frac{4\rho_o^2/3h^2}{1 - \frac{4P\rho_o \cos\theta}{S_o b h^2}}} \qquad \text{Equation 17a}$$

$\theta_1 < \theta < \frac{\pi}{2}$ Equation 13 Applies

$$\frac{d^2 w'}{d\theta^2} + w' = -\frac{P\rho_o^3 \cos\theta}{EI} \qquad \text{Equation 13a}$$

Strictly, Equation 17 applies only $0 < \theta < \theta_2$, and Equation 13 applies only $\theta_1 < \theta < \frac{\pi}{2}$, leaving the small range $\theta_2 < \theta < \theta_1$. For this range there would be a new expression. However, since for the case under consideration, ie, a thin bar with large curvature, the difference between Mi and Ms is very small and consequently the difference between θ_1 and θ_2 is very small. This small difference can be neglected for Equation 17 will extend over this region with sufficient accuracy.

Several attempts were made to find an analytical solution to Equation 17a. Unfortunately, no success was met with. Accordingly, the equation was solved by numerical integration using the method of successive approximations (1, 8, 9).

The general solution for Equation 13a is

$$w' = C_1 \sin\theta + C_2 \cos\theta - \frac{P \rho_0^3 \theta \sin\theta}{2EI}$$

$$\theta = 0 \,, \quad w = 0, \quad \frac{dw}{d\theta} = 0$$

$$\theta = \theta_1 \,, \quad w = w' \,, \quad \frac{dw}{d\theta} = \frac{dw'}{d\theta}$$

The four constants of integration may be determined from the boundary conditions as stated above. Thus the complete deflection curve may be determined; by numerical integration from 0 to θ_1 and by substitution in Equation 13a (after evaluation of the integration constants) from θ_1 to $\frac{\pi}{2}$.

CASE I

INITIAL YIELDING

($y' = -0.248''$)

$M = -0.01008\ S_o$ in lbs

$P = 0.000984\ S_o$ lbs

$\rho_o = 10.248$ in

Extent of Plastic Region

There is no plastic region.

Case I - Initial Yielding

Stress Distribution on Plane of Maximum Stress

Y inches	S psi
0.252	-0.968 So
0.200	-0.772 So
0.150	-0.582 So
0.100	-0.390 So
0.050	-0.196 So
0	0
-0.050	0.198 So
-0.100	0.397 So
-0.150	0.599 So
-0.200	0.803 So
-0.248	1.000 So

Deflection Curve

θ radians	w inches
0	0
0.100	0.203 ε_o
0.200	0.808 ε_o
0.300	1.802 ε_o
0.400	3.166 ε_o
0.500	4.873 ε_o
0.600	6.886 ε_o
0.700	9.166 ε_o
0.800	11.66 ε_o
0.900	14.33 ε_o
1.000	17.10 ε_o
1.100	19.93 ε_o
1.200	22.74 ε_o
1.300	25.46 ε_o
1.400	28.04 ε_o
1.500	30.41 ε_o
1.5708	31.93 ε_o

CASE I

YIELDING TO ONE QUARTER OF DEPTH OF BEAM

($y' = -0.186"$)

M = -0.01267 So in lbs

P = 0.001236 So lbs

ρ_o = 10.248 in

Extent of Plastic Region

y' inches	y" inches	θ degrees
-0.186	0.191	0
-0.190	0.197	9°24'
-0.200	0.208	16°28'
-0.220	0.230	25°46'
-0.230	0.241	29°45'
	0.252	33°23'
-0.248		37°17'

Case I - Yielding to One-quarter of Depth of Beam

Stress Distribution on Plane of Maximum Stress

y inches	S psi
0.252	-1.000 So
0.200	-1.000 So
0.191	-1.000 So
0.150	-0.780 So
0.100	-0.523 So
0.050	-0.263 So
0	0
-0.050	0.265 So
-0.100	0.533 So
-0.150	0.804 So
-0.186	1.000 So
-0.200	1.000 So
-0.248	1.000 So

Deflection Curve

θ radians	w inches
0	0
0.100	4.818 ε_0
0.200	19.13 ε_0
0.300	42.50 ε_0
0.400	74.29 ε_0
0.500	113.7 ε_0
0.600	159.7 ε_0
0.700	210.7 ε_0
0.800	260.8 ε_0
0.900	308.6 ε_0
1.000	353.6 ε_0
1.100	395.4 ε_0
1.200	433.5 ε_0
1.300	467.4 ε_0
1.400	496.8 ε_0
1.500	521.3 ε_0
1.5708	535.6 ε_0

CASE I

YIELDING TO ONE-HALF OF DEPTH OF BEAM

($y' = -0.124''$)

$M = -0.01431$ So in lbs

$P = 0.001396$ So lbs

$\varrho_o = 10.249$ in

Extent of Plastic Region

y'	y"	θ-
inches	inches	degrees
-0.124	0.127	0
-0.132	0.136	9°6'
-0.150	0.154	16°56'
-0.175	0.181	24°54'
-0.200	0.208	31°53'
-0.225	0.235	38°26'
	0.251	42°19'
-0.249		45°13'

Case I - Yielding to One-half of Depth of Beam

Stress Distribution on Plane of Maximum Stress		Deflection Curve	
y	S	θ	w
inches	psi	radians	inches
0.251	-1.000 So	0	0
0.200	-1.000 So	0.100	7.206 ε_o
0.150	-1.000 So	0.200	28.39 ε_o
0.127	-1.000 So	0.300	62.35 ε_o
0.100	-0.789 So	0.400	107.6 ε_o
0.075	-0.593 So	0.500	162.4 ε_o
0.050	-0.396 So	0.600	225.0 ε_o
0.025	-0.199 So	0.700	293.9 ε_o
0	0	0.800	367.6 ε_o
-0.025	0.200 So	0.900	440.7 ε_o
-0.050	0.400 So	1.000	509.7 ε_o
-0.075	0.602 So	1.100	574.0 ε_o
-0.100	0.805 So	1.200	632.8 ε_o
-0.124	1.000 So	1.300	685.5 ε_o
-0.150	1.000 So	1.400	731.5 ε_o
-0.200	1.000 So	1.500	765.4 ε_o
-0.249	1.000 So	1.5708	793.2 ε_o

CASE I

YEILDING TO THREE-QUARTERS OF DEPTH OF BEAM

(y = -0.062")

M = -0.01530 So in lbs

P = 0.001493 So lbs

ρ_o = 10.250 in

Extent of Plastic Region

y'	y"	θ
inches	inches	degrees
-0.062	0.063	0
-0.100	0.102	14°59'
-0.150	0.154	26°31'
-0.200	0.208	37°22'
	0.250	46°15'
-0.250		48°47'

Case I - Yielding to Three-quarters of Depth of Beam

Stress Distribution on Plane of Maximum Stress

y inches	S psi
0.250	-1.000 So
0.200	-1.000 So
0.150	-1.000 So
0.100	-1.000 So
0.063	-1.000 So
0.050	-0.797 So
0.025	-0.400 So
0	0
-0.025	0.401 So
-0.050	0.805 So
-0.062	1.000 So
-0.100	1.000 So
-0.150	1.000 So
-0.200	1.000 So
-0.250	1.000 So

Deflection Curve

θ radians	w inches
0	0
0.100	14.26 ε_0
0.200	54.32 ε_0
0.300	114.8 ε_0
0.400	190.8 ε_0
0.500	278.4 ε_0
0.600	374.4 ε_0
0.700	476.3 ε_0
0.800	581.8 ε_0
0.900	688.2 ε_0
1.000	788.9 ε_0
1.100	882.0 ε_0
1.200	966.7 ε_0
1.300	1042 ε_0
1.400	1107 ε_0
1.500	1161 ε_0
1.5708	1192 ε_0

CASE I

COMPLETE YIELDING

$(y' = y'' = 0)$

$M = -0.01562\ S_o$ in lb

$P = 0.001524\ S_o$ lb

$\rho_o = 10.250$ in

Extent of Plastic Region

y' inches	y'' inches	θ degrees
0	∞	0
-0.050	0.051	$9^{\circ}14'$
-0.100	0.102	$18^{\circ}54'$
-0.150	0.154	$28^{\circ}48'$
-0.200	0.208	$38^{\circ}53'$
	0.250	$47^{\circ}22'$
-0.250		$49^{\circ}49'$

Case I - Complete Yielding

Stress Distribution on Plane of Maximum Stress

y	S
inches	psi
0.250	-1.000 So
0.200	-1.000 So
0.150	-1.000 So
0.100	-1.000 So
0.050	-1.000 So
0	-1.000 So
0	1.000 So
-0.050	1.000 So
-0.100	1.000 So
-0.150	1.000 So
-0.200	1.000 So
-0.250	1.000 So

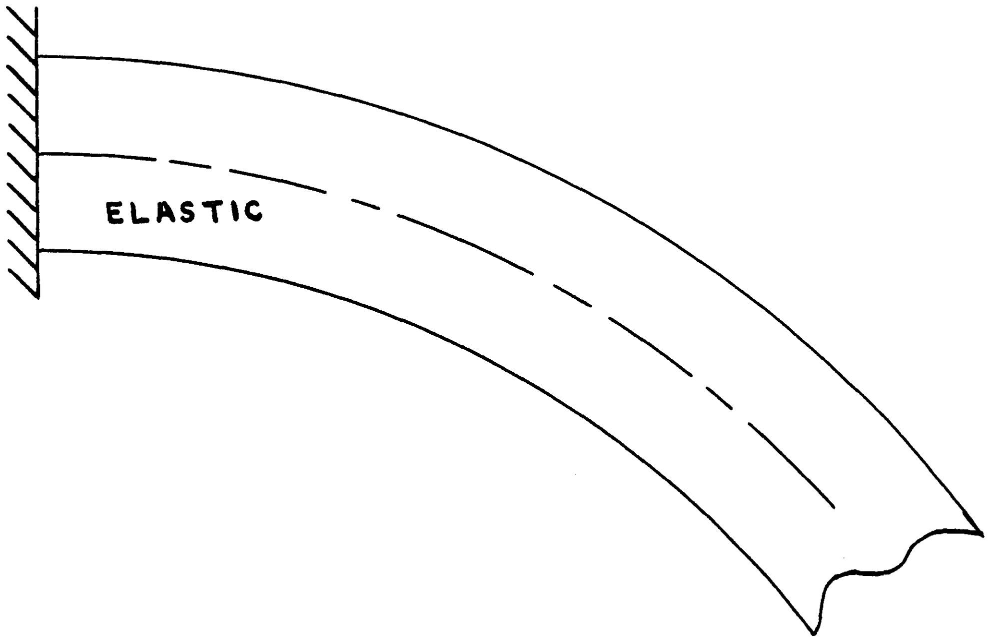
Deflection Curve

θ	w
radians	inches
0	0
any other value	∞

Case I

Maximum Deflection

P lbs	Wmax inches
0	0
0.000492 So	15.96 ε_o
0.000984 So	31.93 ε_o
0.001236 So	535.6 ε_o
0.001396 So	793.2 ε_o
0.001493 So	1192 ε_o
0.001524 So	∞



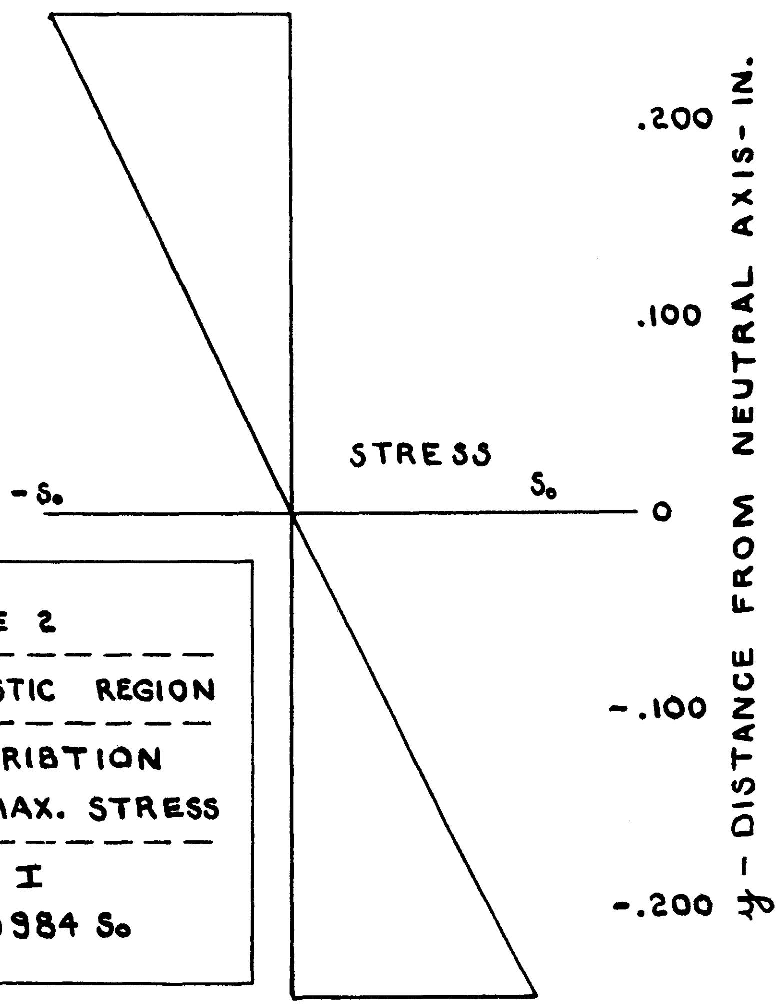


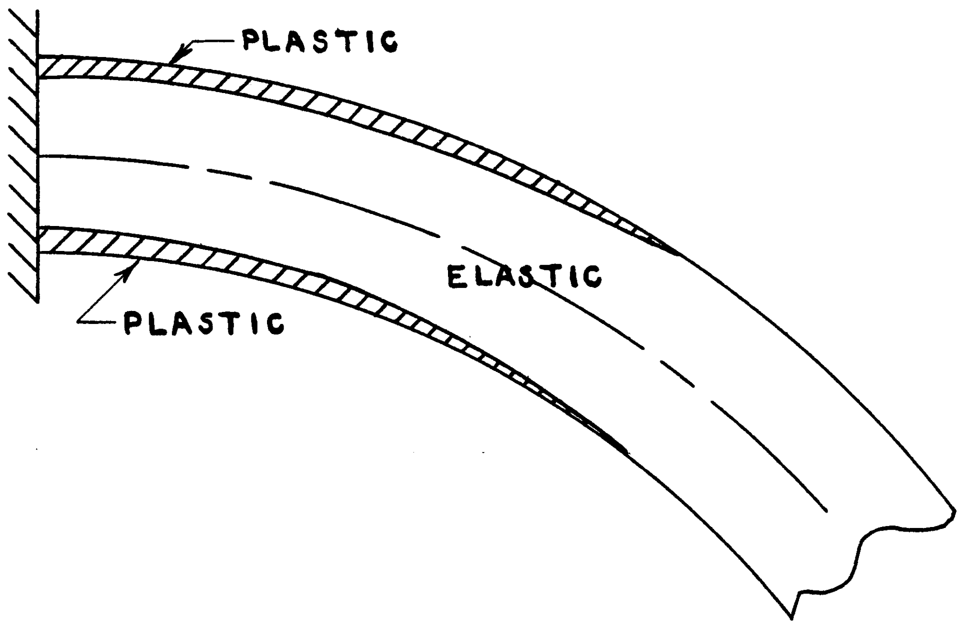
FIGURE 2

EXTENT OF PLASTIC REGION

STRESS DISTRIBTION
ON PLANE OF MAX. STRESS

CASE I

$P = 0.000984\ S_0$



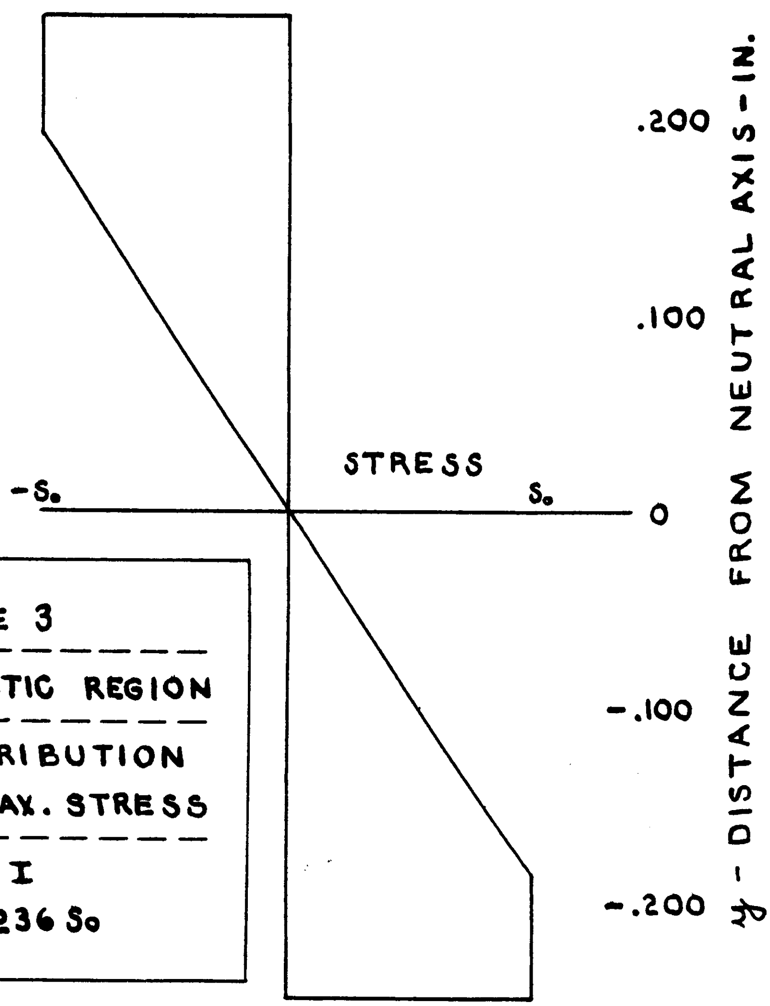


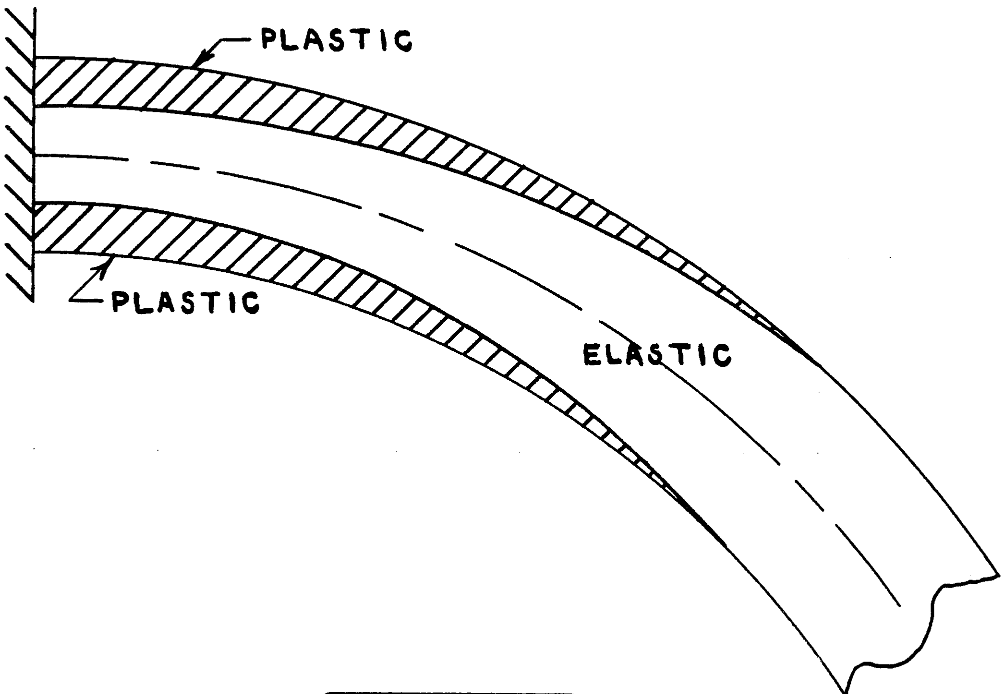
FIGURE 3

EXTENT OF PLASTIC REGION

STRESS DISTRIBUTION ON PLANE OF MAX. STRESS

CASE I

$P = 0.001236\, S_0$



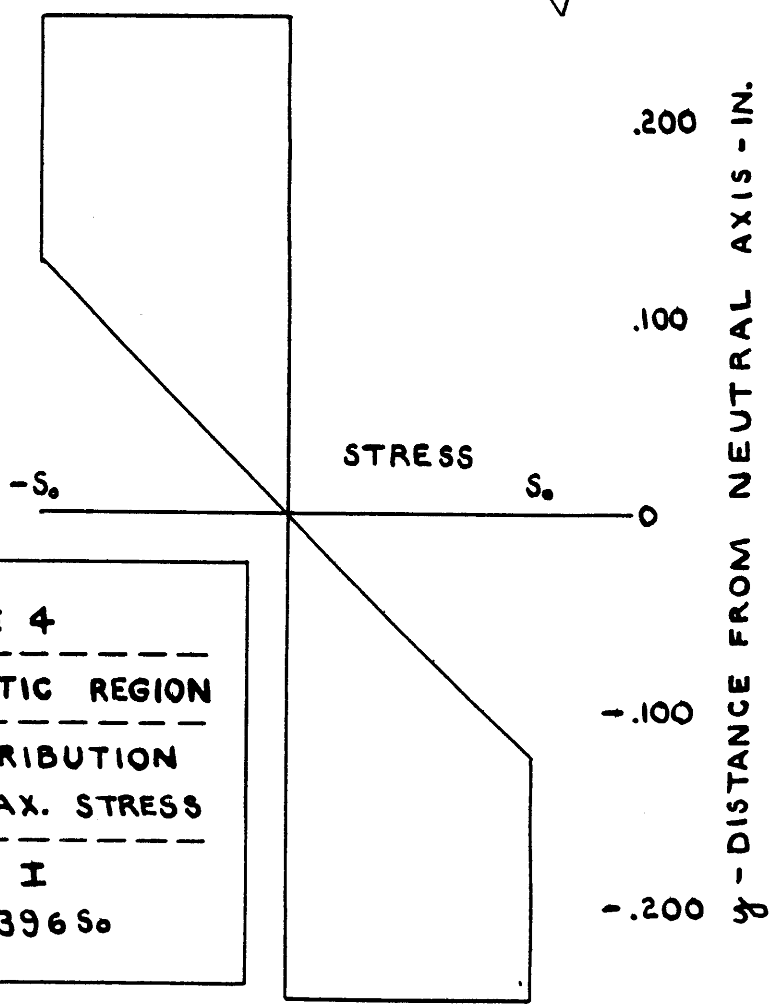
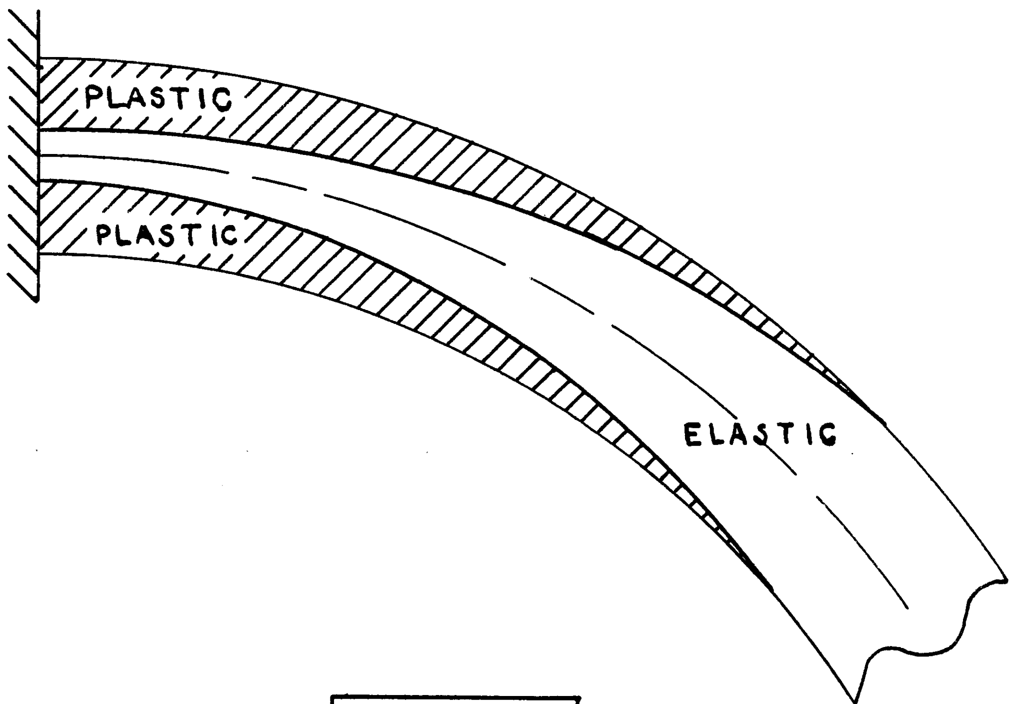


FIGURE 4

EXTENT OF PLASTIC REGION

STRESS DISTRIBUTION
ON PLANE OF MAX. STRESS

CASE I
$P = 0.001396 S_0$



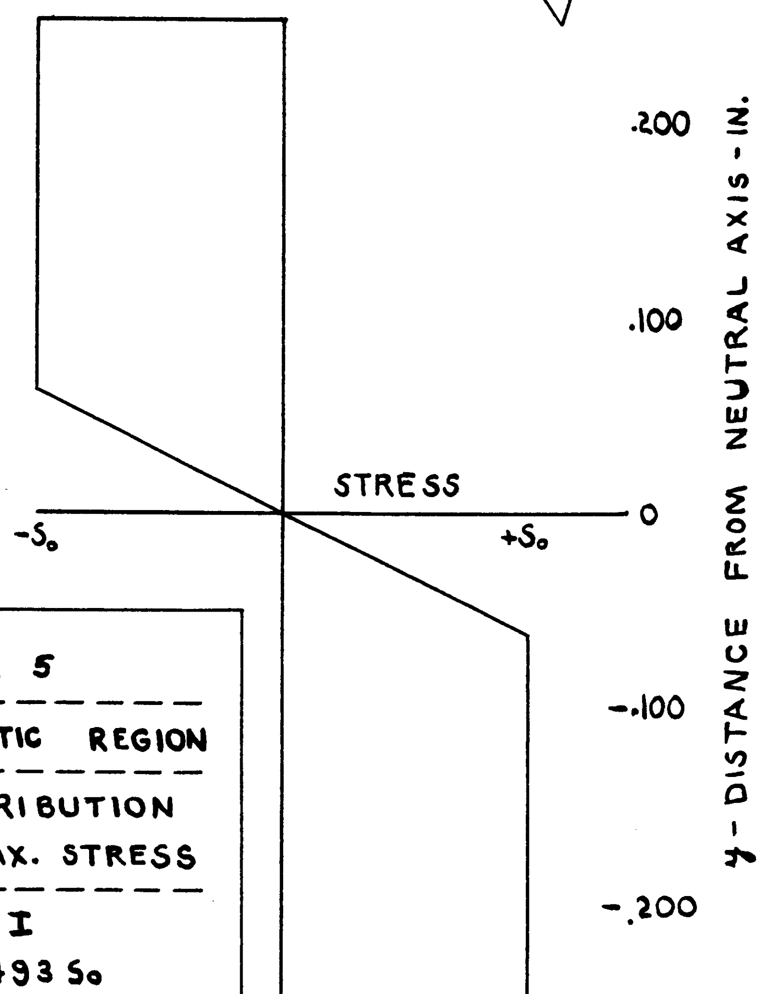


FIGURE 5

EXTENT OF PLASTIC REGION

STRESS DISTRIBUTION ON PLANE OF MAX. STRESS

CASE I

$P = 0.001493\, S_o$

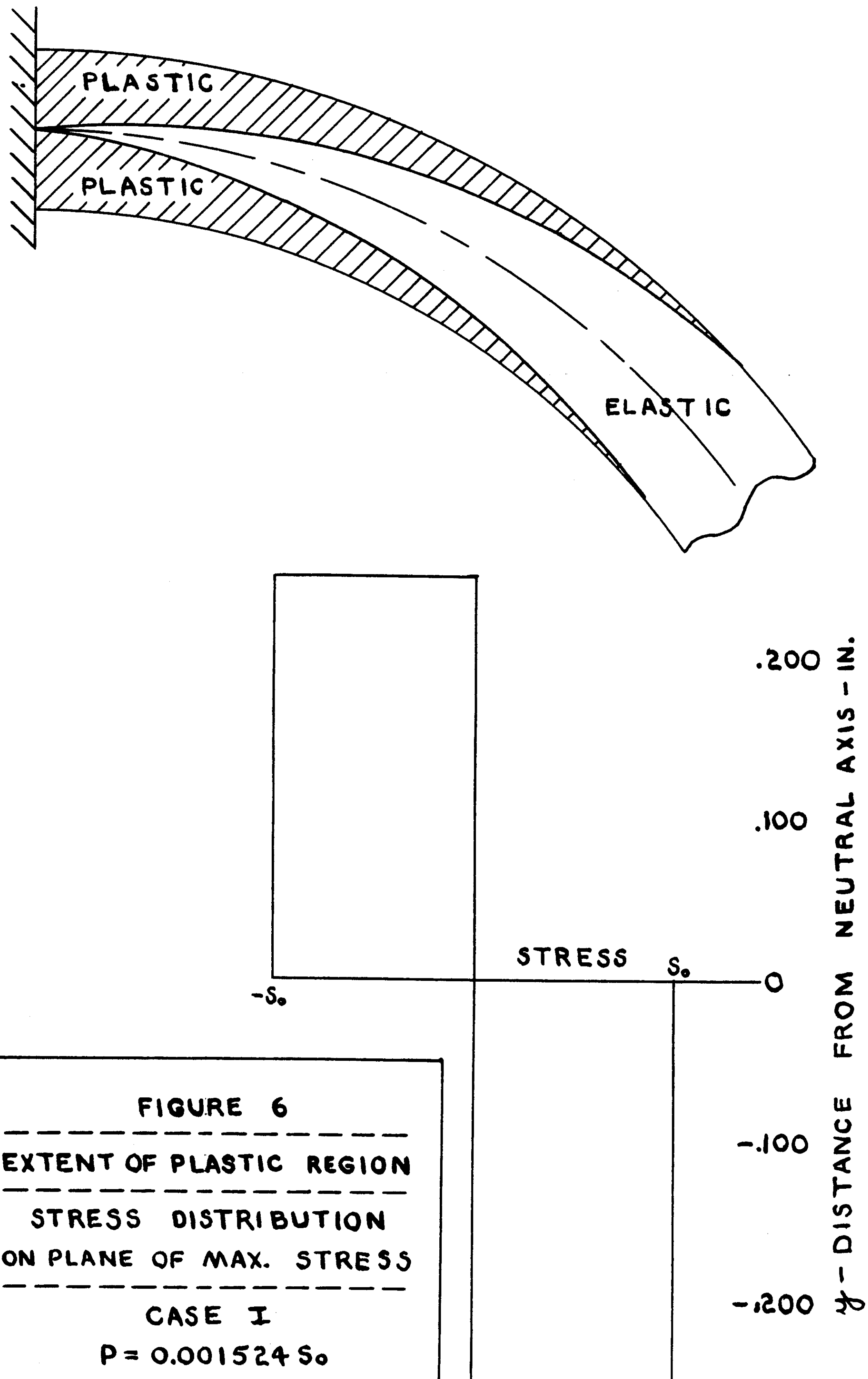


FIGURE 6

EXTENT OF PLASTIC REGION

STRESS DISTRIBUTION
ON PLANE OF MAX. STRESS

CASE I
$P = 0.001524\, S_o$

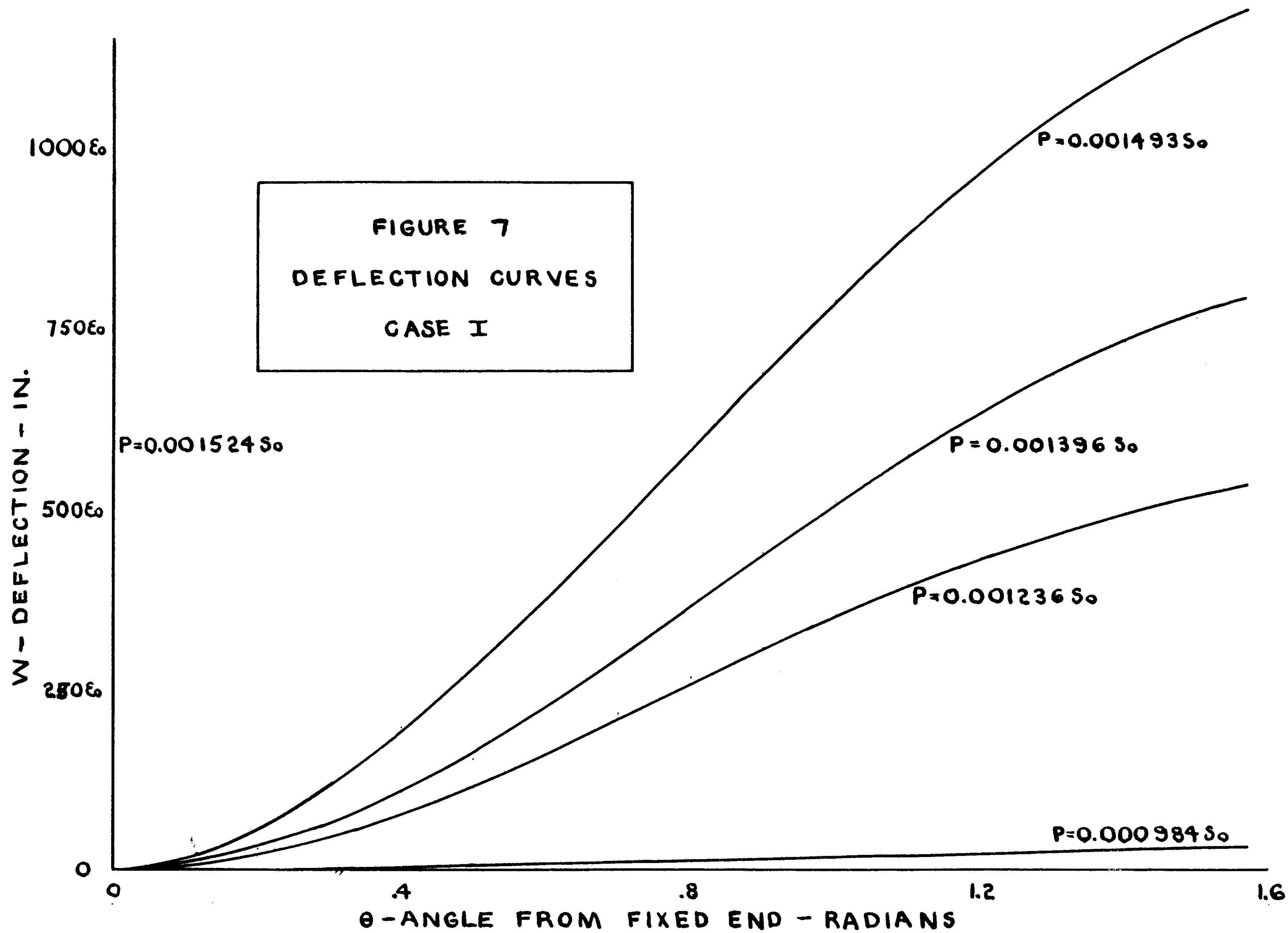


FIGURE 7
DEFLECTION CURVES
CASE I
P=0.001493 So
P=0.001396 So
P=0.001236 So
P=0.000984 So
P=0.001524 So
1000Eo
750Eo
500Eo
250Eo
0
W - DEFLECTION - IN.
0
.4
.8
1.2
1.6
θ - ANGLE FROM FIXED END - RADIANS

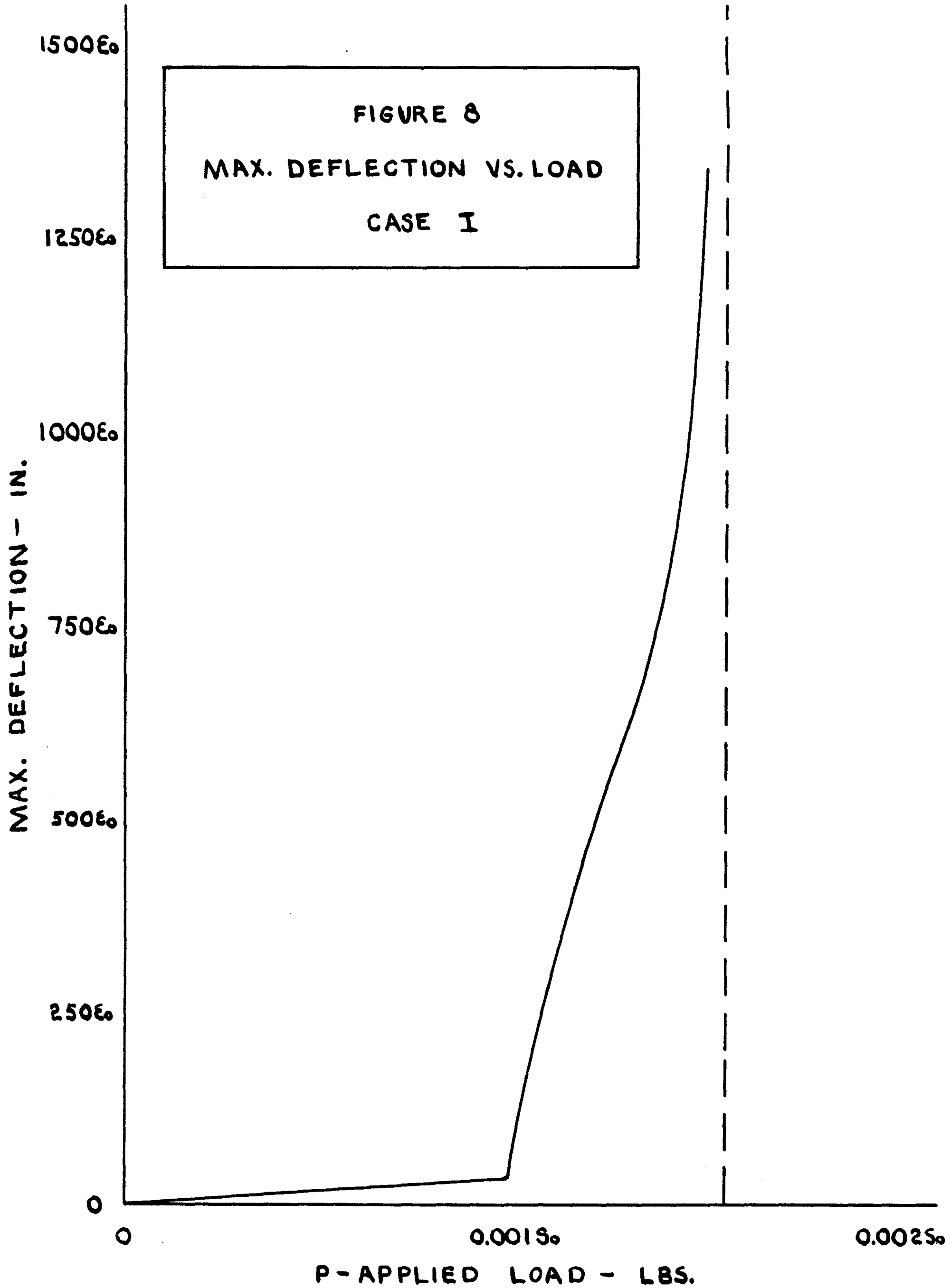
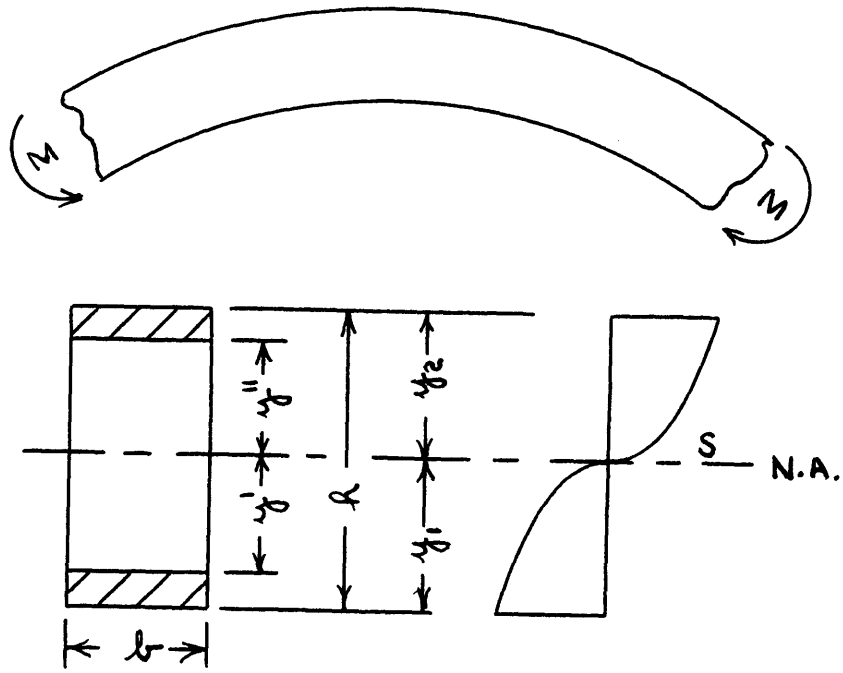


FIGURE 8
MAX. DEFLECTION VS. LOAD
CASE I
1500E₀
1250E₀
1000E₀
750E₀
500E₀
250E₀
0
MAX. DEFLECTION - IN.
0
0.0015₀
0.0025₀
P - APPLIED LOAD - LBS.

Case II

Consider a curved bar of rectangular cross-section
with depth of beam small
in comparison with the radius of curvature
and a stress-strain curve of the form

$$S = S_0 \left(\frac{2\varepsilon}{\varepsilon_0}\right)^{\frac{1}{2}}$$



$$S = S_0 \left(\frac{2\varepsilon}{\varepsilon_0} \right)^{\frac{1}{2}} \quad \text{For } +\varepsilon$$

$$S = -S_0 \left(\frac{-2\varepsilon}{\varepsilon_0} \right)^{\frac{1}{2}} \quad \text{For } -\varepsilon$$

$$\varepsilon = \left(\frac{\rho_0 - \rho}{\rho} \right) \left(\frac{y}{\rho_0 + y} \right)$$

$$S = S_0 \left[\left(\frac{2}{\varepsilon_0} \right) \left(\frac{\rho_0 - \rho}{\rho} \right) \left(\frac{y}{\rho_0 + y} \right) \right]^{\frac{1}{2}} \quad \text{For } +y$$

$$S = -S_0 \left[\left(\frac{2}{\varepsilon_0} \right) \left(\frac{\rho_0 - \rho}{\rho} \right) \left(\frac{-y}{\rho_0 + y} \right) \right]^{\frac{1}{2}} \quad \text{For } -y$$

$$\text{Let } k = S_0 \left[\left(\frac{2}{\varepsilon_0} \right) \left(\frac{\rho_0 - \rho}{\rho} \right) \right]^{\frac{1}{2}}$$

$$\int S\,dA = 0$$

$$\int_{y_1}^{0} -kb\left(\frac{-y}{e_o+y}\right)^{\frac{1}{2}} dy + \int_{0}^{y_2} kb\left(\frac{y}{e_o+y}\right)^{\frac{1}{2}} dy = 0$$

$$-\int_{y_1}^{0}\left(\frac{-y}{e_o+y}\right)^{\frac{1}{2}} dy + \int_{0}^{y_2}\left(\frac{y}{e_o+y}\right)^{\frac{1}{2}} dy = 0$$

$$-\left[(-y)^{\frac{1}{2}}(e_o+y)^{\frac{1}{2}} - e_o \sin^{-1}\frac{(-y)^{\frac{1}{2}}}{(e_o)^{\frac{1}{2}}}\right]_{y_1}^{0} + \left[(y)^{\frac{1}{2}}(e_o+y)^{\frac{1}{2}} - e_o \sinh^{-1}\frac{(y)^{\frac{1}{2}}}{(e_o)^{\frac{1}{2}}}\right]_{0}^{y_2} = 0$$

$$\sqrt{-y_1}\sqrt{e_o+y_1} - e_o \sin^{-1}\sqrt{\frac{-y_1}{e_o}} + \sqrt{y_2}\sqrt{e_o+y_2} - e_o \sinh^{-1}\sqrt{\frac{y_2}{e_o}} = 0$$

$$\sqrt{-y_1}\sqrt{e_o+y_1} + \sqrt{y_2}\sqrt{e_o+y_2} = e_o\left[\sin^{-1}\sqrt{\frac{-y_1}{e_o}} + \sinh^{-1}\sqrt{\frac{y_2}{e_o}}\right]$$

$$e_o = \frac{\sqrt{-y_1}\sqrt{e_o+y_1} + \sqrt{y_2}\sqrt{e_o+y_2}}{\sin^{-1}\sqrt{\frac{-y_1}{e_o}} + \sinh^{-1}\sqrt{\frac{y_2}{e_o}}} \qquad \text{EQUATION 19}$$

Using absolute values, Equation 19 becomes

$$e_o = \frac{\sqrt{y_1}\sqrt{R_i} + \sqrt{y_2}\sqrt{R_o}}{\sin^{-1}\sqrt{\frac{y_1}{e_o}} + \sinh^{-1}\sqrt{\frac{y_2}{e_o}}} \qquad \text{EQUATION 19a}$$

This expression is used to determine the position of the neutral axis. A trial and error solution is recommended since the expression is not readily solvable by algebraic means.

$$M = \int S_y \, dA$$

$$M = \int_{y_1}^{0} -k\left(\frac{-y}{e_0+y}\right)^{\frac{1}{2}} y\, b\, dy + \int_{0}^{y_2} k\left(\frac{y}{e_0+y}\right)^{\frac{1}{2}} y\, b\, dy$$

$$M = kb\left[-\int_{y_1}^{0} y\left(\frac{-y}{e_0+y}\right)^{\frac{1}{2}} dy + \int_{0}^{y_2} y\left(\frac{y}{e_0+y}\right)^{\frac{1}{2}} dy\right]$$

$$M = kb\left(-\left[-\frac{(-y)^{\frac{3}{2}}(e_0+y)^{\frac{1}{2}}}{2} - \frac{3e_0}{4}(-y)^{\frac{1}{2}}(e_0+y)^{\frac{1}{2}} + \frac{3e_0^2}{4}\sin^{-1}\frac{(-y)^{\frac{1}{2}}}{(e_0)^{\frac{1}{2}}}\right]_{y_1}^{0} + \left[\frac{(y)^{\frac{3}{2}}(e_0+y)^{\frac{1}{2}}}{2} - \frac{3e_0}{4}(y)^{\frac{1}{2}}(e_0+y)^{\frac{1}{2}} + \frac{3e_0^2}{4}\sinh^{-1}\frac{(y)^{\frac{1}{2}}}{(e_0)^{\frac{1}{2}}}\right]_{0}^{y_2}\right)$$

$$M = kb\left[-\frac{\sqrt{-y_1^3}\sqrt{e_0+y_1}}{2} - \frac{3e_0}{4}\sqrt{-y_1}\sqrt{e_0+y_1} + \frac{3e_0^2}{4}\sin^{-1}\sqrt{\frac{-y_1}{e_0}} + \sqrt{y_2^3}\sqrt{e_0+y_2} - \frac{3e_0}{4}\sqrt{y_2}\sqrt{e_0+y_2} + \frac{3e_0^2}{4}\sinh^{-1}\sqrt{\frac{y_2}{e_0}}\right]$$

$$M = kb\left[\frac{3e_0^2}{4}\left(\sin^{-1}\sqrt{\frac{-y_1}{e_0}} + \sinh^{-1}\sqrt{\frac{y_2}{e_0}}\right) + \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{e_0+y_2} - \sqrt{-y_1^3}\sqrt{e_0+y_1}\right) - \frac{3e_0}{4}\left(\sqrt{y_2}\sqrt{e_0+y_2} + \sqrt{-y_1}\sqrt{e_0+y_1}\right)\right]$$

$$M = S_0 b\left[\left(\frac{2}{\varepsilon_0}\right)\left(\frac{e_0-e}{e}\right)\right]^{\frac{1}{2}}\left[\frac{3e_0^2}{4}\left(\sin^{-1}\sqrt{\frac{-y_1}{e_0}} + \sinh^{-1}\sqrt{\frac{y_2}{e_0}}\right) + \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{e_0+y_2} - \sqrt{-y_1^3}\sqrt{e_0+y_1}\right) - \frac{3e_0}{4}\left(\sqrt{y_2}\sqrt{e_0+y_2} + \sqrt{-y_1}\sqrt{e_0+y_1}\right)\right]$$

EQUATION 20

Using absolute values, Equation 20 becomes

$$M = S_o b\left[\left(\frac{2}{\varepsilon_o}\right)\left(\frac{e_o - e}{e}\right)\right]^{\frac{1}{2}}\left[\frac{3e_o^2}{4}\left(\sin^{-1}\sqrt{\frac{y_1}{e_o}} + \sinh^{-1}\sqrt{\frac{y_2}{e_o}}\right) + \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i}\right) - \frac{3e_o}{4}\left(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i}\right)\right]$$

EQUATION 20a

Care must be taken to use a high degree of accuracy when evaluating the expression in Equation 20 (or 20a) contained in the brackets since it involves the very small difference of two comparatively large numbers.

$$S = \pm S_o\left[\left(\frac{2}{\varepsilon_o}\right)\left(\frac{e_o - e}{e}\right)\left(\frac{\pm y}{e_o + y}\right)\right]^{\frac{1}{2}}$$

$$S = \frac{\pm M \left(\frac{\pm y}{e_o + y}\right)^{\frac{1}{2}}}{b\left[\frac{3e_o^2}{4}\left(\sin^{-1}\sqrt{\frac{y_1}{e_o}} + \sinh^{-1}\sqrt{\frac{y_2}{e_o}}\right) + \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i}\right) - \frac{3e_o}{4}\left(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i}\right)\right]}$$

EQUATION 21

When using Equation 21, the + signs apply to + values of y, the - signs to - values of y.

FROM EQUATION 20a

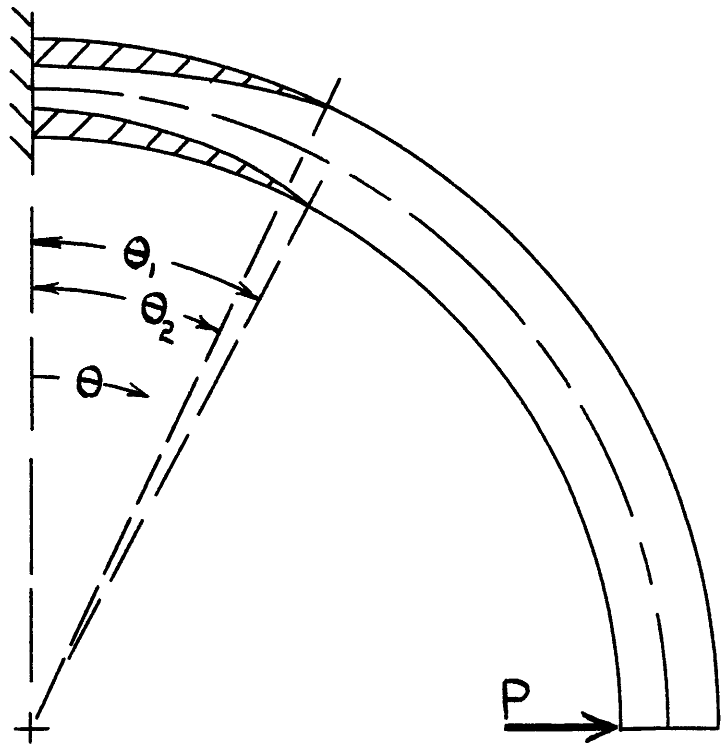
$$\frac{\rho_o - \rho}{\rho} = \frac{M^2 \mathcal{E}_o}{2S_o^2 b^2\left[\frac{3\rho_o^2}{4}\left(\sin^{-1}\sqrt{\frac{y_1}{\rho_o}} + \sinh^{-1}\sqrt{\frac{y_2}{\rho_o}}\right) + \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i}\right) - \frac{3\rho_o}{4}\left(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i}\right)\right]^2}$$

SUBSTITUTING IN EQUATION 10

$$\frac{d^2 w}{d\theta^2} + w = \frac{\rho_o M^2 \mathcal{E}_o}{2S_o^2 b^2\left[\frac{3\rho_o^2}{4}\left(\sin^{-1}\sqrt{\frac{y_1}{\rho_o}} + \sinh^{-1}\sqrt{\frac{y_2}{\rho_o}}\right) + \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i}\right) - \frac{3\rho_o}{4}\left(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i}\right)\right]^2}$$

EQUATION 22

Calculations for the Specific Case (Figure 1)



$$M = -P\rho_0 \cos\theta$$

b	= 0.250"
h	= 0.500"
Ro	= 10.500"
Ri	= 10.000"

A trial and error solution of Equation 19a yields the result that

$$\rho_0 = 10.249"$$

By the substitution of $M = -P\rho_0 \cos\theta$ in Equation 21

$$S = \frac{\mp P\rho_0 \cos\theta \left(\frac{\pm y}{\rho_0 + y}\right)^{\frac{1}{2}}}{b\left[\frac{3\rho_0^2}{4}\left(\sin^{-1}\sqrt{\frac{y_1}{\rho_0}} + \sinh^{-1}\sqrt{\frac{y_2}{\rho_0}}\right) + \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i}\right) - \frac{3\rho_0}{4}\left(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i}\right)\right]}$$

Equation 21a

Evaluating

$$\left[\frac{3e_0^2}{4}\left(\sin^{-1}\sqrt{\frac{y_1}{c_0}}+\sinh^{-1}\sqrt{\frac{y_2}{c_0}}\right)+\frac{1}{2}\left(\sqrt{y_2^3}\sqrt{R_o}-\sqrt{y_1^3}\sqrt{R_i}\right)-\frac{3e_0}{4}\left(\sqrt{y_2}\sqrt{R_o}+\sqrt{y_1}\sqrt{R_i}\right)\right]=0.00789$$

Initial Yielding

From Equation 21a, $S = So$, $y = y' = y_1$

$Mi = -0.01250\ So$ in-lb

$P = 0.001220\ So$ lb

Secondary Yielding

From Equation 21a, $S = -So$, $y = y'' = y_2$

$Ms = -0.01276\ So$ in-lb

$P = 0.001245\ So$ lb

Complete Yielding

From Equation 21a, $S = So$, $y = y' = y'' = 0$

$Mc = -\infty$ in-lb

$P = \infty$ lb

Stress Distribution

The stress distribution at any angle θ from 0 to $\frac{\pi}{2}$ for any load P from 0 to ∞ is given by Equation 21a. This applies to both the elastic and plastic regions since the stress-strain curve also applies to both the elastic and plastic regions.

Deflection Equation

Substitution of $M = -P\rho_0 \cos\theta$ and $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$ in Equation 22 yields

$$\frac{d^2 w}{d\theta^2} + w = \frac{P^2 \rho_0^3 \, \varepsilon_0 \left(\cos 2\theta + 1\right)}{4 S_0^2 b^2 \left[\frac{3e_0^2}{4}\left(\sin^{-1}\sqrt{\frac{y_1}{c_0}} + \sinh^{-1}\sqrt{\frac{y_2}{c_0}}\right) + \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{R_0} - \sqrt{y_1^3}\sqrt{R_i}\right) - \frac{3e_0}{4}\left(\sqrt{y_2}\sqrt{R_0} + \sqrt{y_1}\sqrt{R_i}\right)\right]^2}$$

Complementary Solution

$$w = C_1 \sin\theta + C_2 \cos\theta$$

Particular Solution

$$w = \frac{-P^2 \rho_0^3 \, \varepsilon_0 \left(\frac{\cos 2\theta}{3} - 1\right)}{4 S_0^2 b^2 \left[\frac{3e_0^2}{4}\left(\sin^{-1}\sqrt{\frac{y_1}{c_0}} + \sinh^{-1}\sqrt{\frac{y_2}{e_0}}\right) + \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{R_0} - \sqrt{y_1^3}\sqrt{R_i}\right) - \frac{3e_0}{4}\left(\sqrt{y_2}\sqrt{R_0} + \sqrt{y_1}\sqrt{R_i}\right)\right]^2}$$

General Solution

$$w = C_1 \sin\theta + C_2 \cos\theta - \frac{P^2 \rho_0^3 \, \varepsilon_0 \left(\frac{\cos 2\theta}{3} - 1\right)}{4 S_0^2 b^2 \left[\frac{3e_0^2}{4}\left(\sin^{-1}\sqrt{\frac{y_1}{c_0}} + \sinh^{-1}\sqrt{\frac{y_2}{c_0}}\right) + \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{R_0} - \sqrt{y_1^3}\sqrt{R_i}\right) - \frac{3e_0}{4}\left(\sqrt{y_2}\sqrt{R_0} + \sqrt{y_1}\sqrt{R_i}\right)\right]^2}$$

Boundary Conditions

$$\theta = 0, \quad w = 0, \quad \frac{dw}{d\theta} = 0$$

Evaluation of C_1 and C_2

$$C_1 = 0$$

$$C_2 = -\frac{P^2 \rho_0^3 \, \varepsilon_0}{6 S_0^2 b^2 \left[\frac{3e_0^2}{4}\left(\sin^{-1}\sqrt{\frac{y_1}{e_0}} + \sinh^{-1}\sqrt{\frac{y_2}{c_0}}\right) + \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{R_0} - \sqrt{y_1^3}\sqrt{R_i}\right) - \frac{3e_0}{4}\left(\sqrt{y_2}\sqrt{R_0} + \sqrt{y_1}\sqrt{R_i}\right)\right]^2}$$

$$w = \frac{-P^2 \rho_0^3 \, \varepsilon_0 \left(\frac{2\cos\theta}{3} + \frac{\cos 2\theta}{3} - 1\right)}{4 S_0^2 b^2 \left[\frac{3e_0^2}{4}\left(\sin^{-1}\sqrt{\frac{y_1}{c_0}} + \sinh^{-1}\sqrt{\frac{y_2}{c_0}}\right) + \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{R_0} - \sqrt{y_1^3}\sqrt{R_i}\right) - \frac{3e_0}{4}\left(\sqrt{y_2}\sqrt{R_0} + \sqrt{y_1}\sqrt{R_i}\right)\right]^2}$$

Equation 23

This deflection equation (Equation 23) applies for all values of P from 0 to ∞, ie, it applies over both the elastic and plastic regions. The sign of the deflection must be the same as that of the bending moment before the moment term is squared and substituted in Equation 22.

CASE II

INITIAL YIELDING

$(y' = -0.249)$

$M = -0.01250$ So in lbs

$P = 0.001220$ So lbs

$e_o = 10.249$ in

Extent of Plastic Region

There is no plastic region.

Case II - Initial Yielding

Stress Distribution on Plane of Maximum Stress

y inches	S psi
0.251	-0.980 So
0.200	-0.877 So
0.150	-0.761 So
0.100	-0.623 So
0.050	-0.442 So
0.040	-0.395 So
0.030	-0.342 So
0.020	-0.280 So
0.010	-0.198 So
0	0
-0.010	0.198 So
-0.020	0.280 So
-0.030	0.343 So
-0.040	0.397 So
-0.050	0.444 So
-0.100	0.629 So
-0.150	0.772 So
-0.200	0.894 So
-0.249	1.000 So

Deflection Curve

θ radians	w inches
0	0
0.100	1.027 ε_o
0.200	4.075 ε_o
0.300	9.055 ε_o
0.400	15.85 ε_o
0.500	25.84 ε_o
0.600	33.85 ε_o
0.700	44.60 ε_o
0.800	56.11 ε_o
0.900	68.00 ε_o
1.000	80.11 ε_o
1.100	91.97 ε_o
1.200	103.3 ε_o
1.300	113.9 ε_o
1.400	123.6 ε_o
1.500	132.0 ε_o
1.5708	137.2 ε_o

CASE II

YIELDING TO ONE-FIFTH OF DEPTH OF BEAM

($y' = -0.200$)

$M = -0.01398$ So in lb

$P = 0.001364$ So lb

$\rho_o = 10.249$ in

Extent of Plastic Region

y' inches	y'' inches	θ degrees
-0.200	0.208	0
-0.209	0.218	12° 6'
-0.215	0.224	15°28'
-0.230	0.241	21°24'
	0.251	24° 9'
-0.249		26°37'

Case II - Yielding to One-fifth of Depth of Beam

Stress Distribution on Plane of Maximum Stress		Deflection Curve	
y	S	θ	w
inches	psi	radians	inches
0.251	-1.096 So	0	0
0.200	-0.981 So	0.100	1.285 ε_o
0.150	-0.851 So	0.200	5.098 ε_o
0.100	-0.697 So	0.300	11.33 ε_o
0.050	-0.494 So	0.400	19.83 ε_o
0.040	-0.442 So	0.500	32.33 ε_o
0.030	-0.383 So	0.600	42.35 ε_o
0.020	-0.313 So	0.700	55.80 ε_o
0.010	-0.221 So	0.800	70.20 ε_o
0	0	0.900	85.07 ε_o
-0.010	0.222 So	1.000	100.2 ε_o
-0.020	0.313 So	1.100	115.1 ε_o
-0.030	0.384 So	1.200	129.3 ε_o
-0.040	0.444 So	1.300	142.6 ε_o
-0.050	0.496 So	1.400	154.6 ε_o
-0.100	0.704 So	1.500	165.2 ε_o
-0.150	0.864 So	1.5708	171.7 ε_o
-0.200	1.000 So		
-0.249	1.118 So		

CASE II

YIELDING TO TWO-FIFTHS OF DEPTH OF BEAM

($y' = -0.150''$)

$M = -0.01618$ So in lb

$P = 0.001579$ So lb

$\rho_o = 10.249$ in

Extent of Plastic Region

y' inches	y" inches	θ degrees
-0.150	0.154	0
-0.155	0.160	10°26'
-0.164	0.169	17° 7'
-0.180	0.187	24°17'
-0.209	0.218	32°22'
-0.230	0.241	36°27'
	0.251	37°59'
-0.249		39°26'

Case II - Yielding to Two-fifths of Depth of Beam

Stress Distribution on Plane of Maximum Stress

y inches	S psi
0.251	-1.269 So
0.200	-1.135 So
0.150	-0.985 So
0.100	-0.807 So
0.050	-0.572 So
0.040	-0.512 So
0.030	-0.443 So
0.020	-0.362 So
0.010	-0.256 So
0	0
-0.010	0.256 So
-0.020	0.363 So
-0.030	0.445 So
-0.040	0.514 So
-0.050	0.574 So
-0.100	0.814 So
-0.150	1.000 So
-0.200	1.158 So
-0.249	1.295 So

Deflection Curve

θ radians	w inches
0	0
0.100	1.722 ε_o
0.200	6.832 ε_o
0.300	15.18 ε_o
0.400	26.57 ε_o
0.500	43.32 ε_o
0.600	56.75 ε_o
0.700	74.78 ε_o
0.800	94.07 ε_o
0.900	114.0 ε_o
1.000	134.3 ε_o
1.100	154.2 ε_o
1.200	173.2 ε_o
1.300	191.0 ε_o
1.400	207.2 ε_o
1.500	221.3 ε_o
1.5708	230.0 ε_o

CASE II

YIELDING TO THREE-FIFTHS OF DEPTH OF BEAM

(y' = -0.100")

M = -0.01987 So in lb

P = 0.001939 So lb

ρ_o = 10.249 in

Extent of Plastic Region

y'	y"	θ
inches	inches	degrees
-0.100	0.102	0
-0.110	0.112	17°38'
-0.125	0.128	26°27'
-0.150	0.155	35°28'
-0.200	0.210	45°17'
	0.251	50° 4'
-0.249		51° 1'

Case II - Yielding to Three-fifths of Depth of Beam

Stress Distribution on Plane of Maximum Stress

y	S
inches	psi
0.251	-1.558 So
0.200	-1.394 So
0.150	-1.210 So
0.100	-0.990 So
0.050	-0.702 So
0.040	-0.628 So
0.030	-0.544 So
0.020	-0.445 So
0.010	-0.314 So
0	0
-0.010	0.315 So
-0.020	0.445 So
-0.030	0.546 So
-0.040	0.631 So
-0.050	0.705 So
-0.100	1.000 So
-0.150	1.228 So
-0.200	1.421 So
-0.249	1.590 So

Deflection Curve

θ	w
radians	inches
0	0
0.100	2.595 ε_o
0.200	10.28 ε_o
0.300	22.88 ε_o
0.400	40.06 ε_o
0.500	65.31 ε_o
0.600	85.55 ε_o
0.700	112.7 ε_o
0.800	141.8 ε_o
0.900	171.8 ε_o
1.000	202.4 ε_o
1.100	232.4 ε_o
1.200	261.2 ε_o
1.300	288.0 ε_o
1.400	312.3 ε_o
1.500	326.0 ε_o
1.5708	346.7 ε_o

CASE II

YIELDING TO FOUR-FIFTHS OF DEPTH OF BEAM

($y' = -0.050''$)

$M = -0.02817$ So in lb

$P = 0.002749$ So lb

$\rho_o = 10.249$ in

Extent of Plastic Region

y'	y''	θ
inches	inches	degrees
-0.050	0.051	0
-0.053	0.054	13°48'
-0.058	0.059	21°56'
-0.066	0.067	29°35'
-0.075	0.076	35°22'
-0.100	0.102	45° 8'
-0.150	0.155	54°56'
-0.200	0.210	60°15'
	0.251	63° 4'
-0.249		63°40'

Case II - Yielding to Four-fifths of Depth of Beam

Stress Distribution on Plane of Maximum Stress

y inches	S psi
0.251	-2.208 So
0.200	-1.976 So
0.150	-1.715 So
0.100	-1.404 So
0.050	-0.995 So
0.040	-0.891 So
0.030	-0.772 So
0.020	-0.630 So
0.010	-0.446 So
0	0
-0.010	0.446 So
-0.020	0.632 So
-0.030	0.774 So
-0.040	0.894 So
-0.050	1.000 So
-0.100	1.418 So
-0.150	1.741 So
-0.200	2.015 So
-0.249	2.254 So

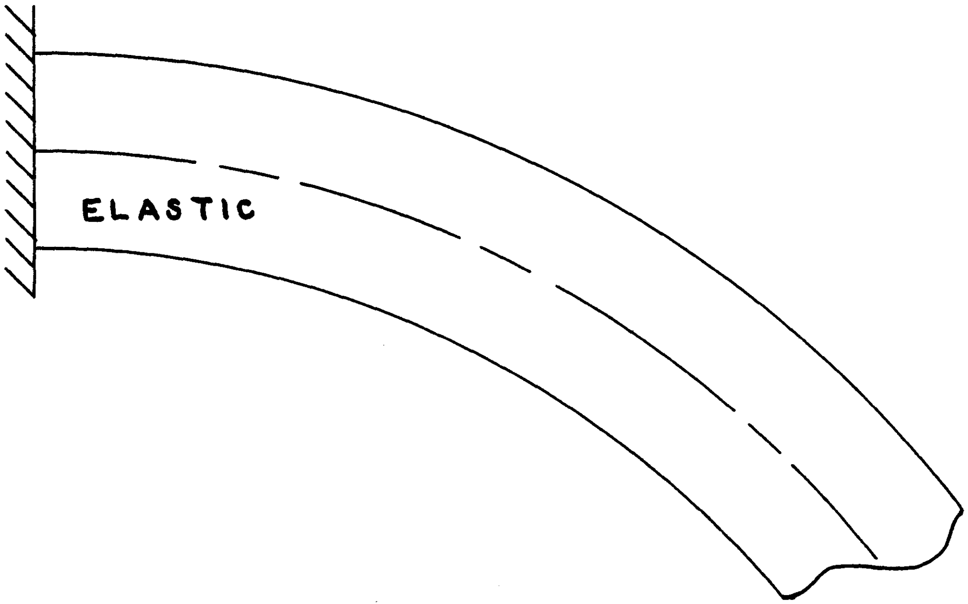
Deflection Curve

θ radians	w inches
0	0
0.100	5.216 ε_0
0.200	20.70 ε_0
0.300	45.99 ε_0
0.400	80.51 ε_0
0.500	131.3 ε_0
0.600	172.0 ε_0
0.700	226.5 ε_0
0.800	285.0 ε_0
0.900	345.4 ε_0
1.000	406.9 ε_0
1.100	467.1 ε_0
1.200	524.9 ε_0
1.300	578.8 ε_0
1.400	627.6 ε_0
1.500	670.5 ε_0
1.5708	696.9 ε_0

Case II

Maximum Deflection

P lbs	Wmax inches
0	0
0.0002 So	3.689 ε_o
0.0004 So	14.76 ε_o
0.0006 So	33.20 ε_o
0.0008 So	58.29 ε_o
0.0010 So	92.23 ε_o
0.0012 So	132.8 ε_o
0.0014 So	180.4 ε_o
0.0016 So	236.1 ε_o
0.0018 So	298.8 ε_o
0.0020 So	368.9 ε_o
0.0022 So	446.4 ε_o
0.0024 So	531.2 ε_o
0.0026 So	623.5 ε_o
0.0028 So	723.1 ε_o
0.0030 So	830.1 ε_o



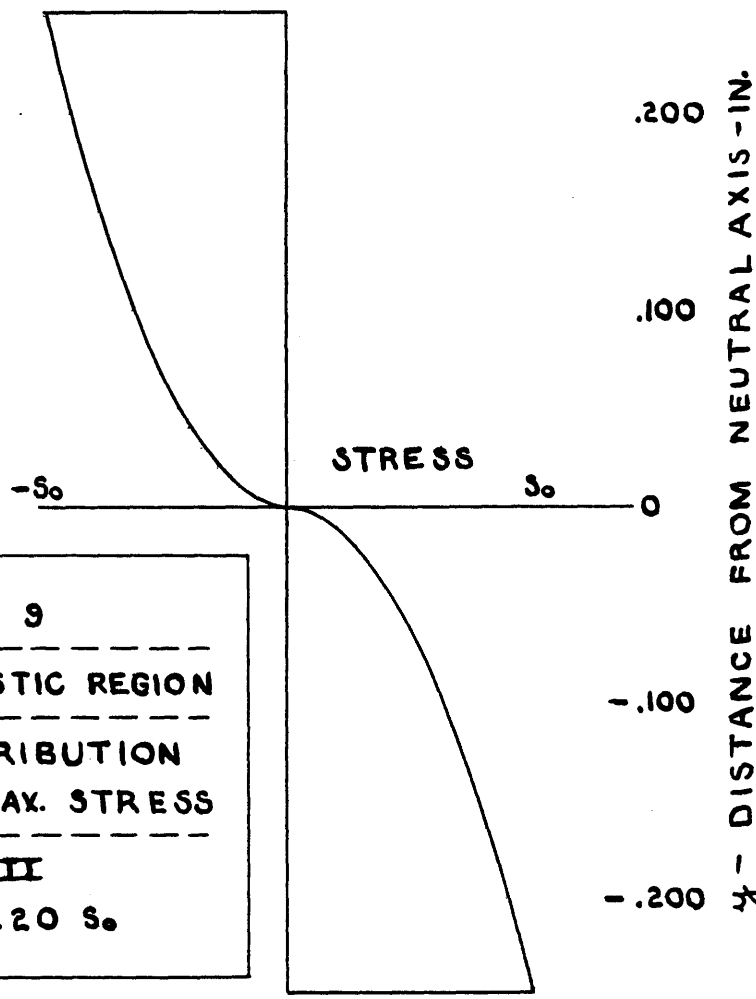
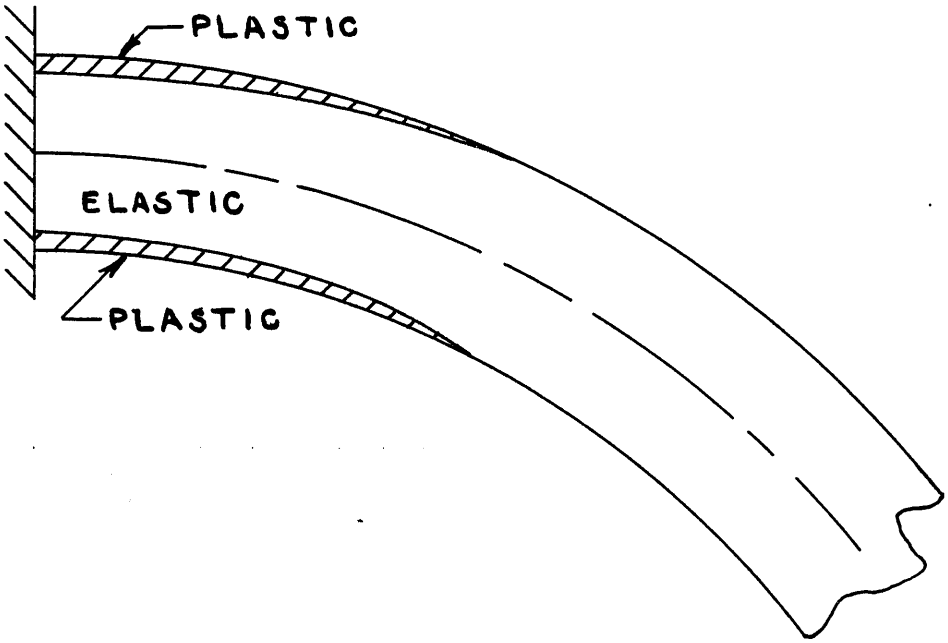


FIGURE 9

EXTENT OF PLASTIC REGION

STRESS DISTRIBUTION
ON PLANE OF MAX. STRESS

CASE II
P = 0.001220 S_o



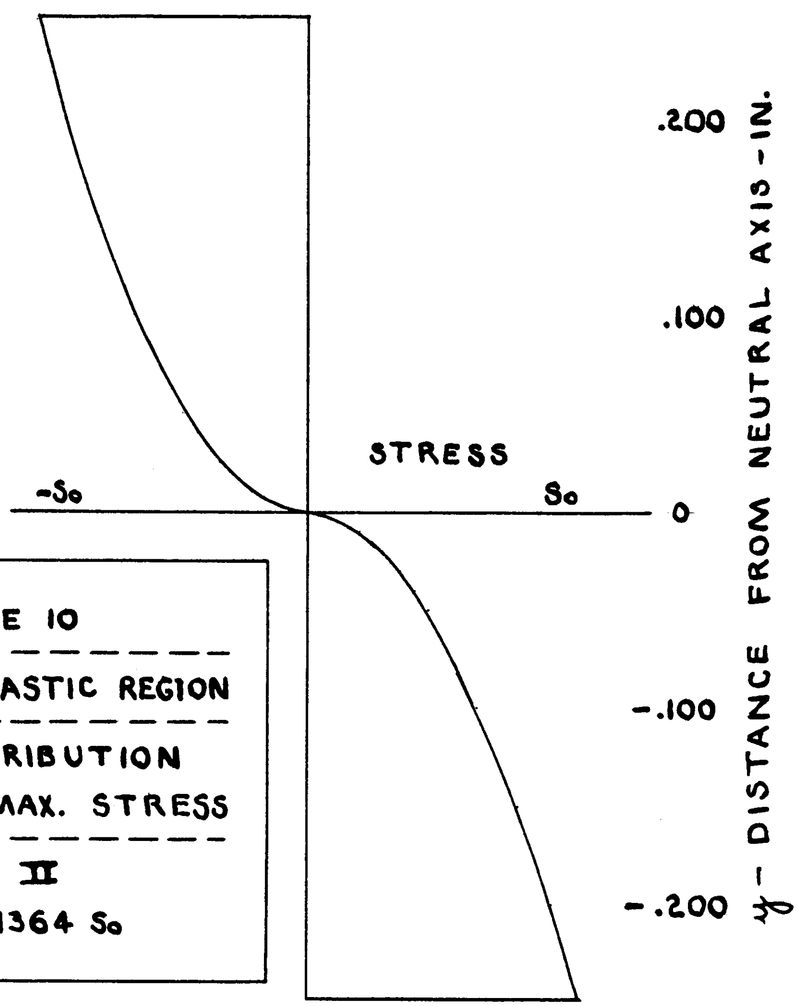


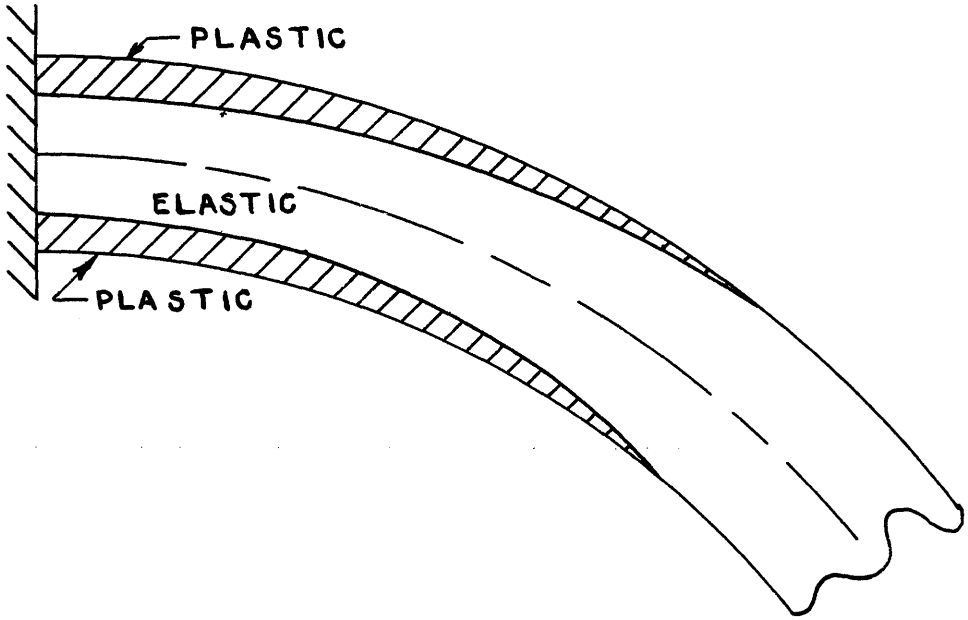
FIGURE 10

EXTENT OF PLASTIC REGION

STRESS DISTRIBUTION
ON PLANE OF MAX. STRESS

CASE II

P = 0.001364 So



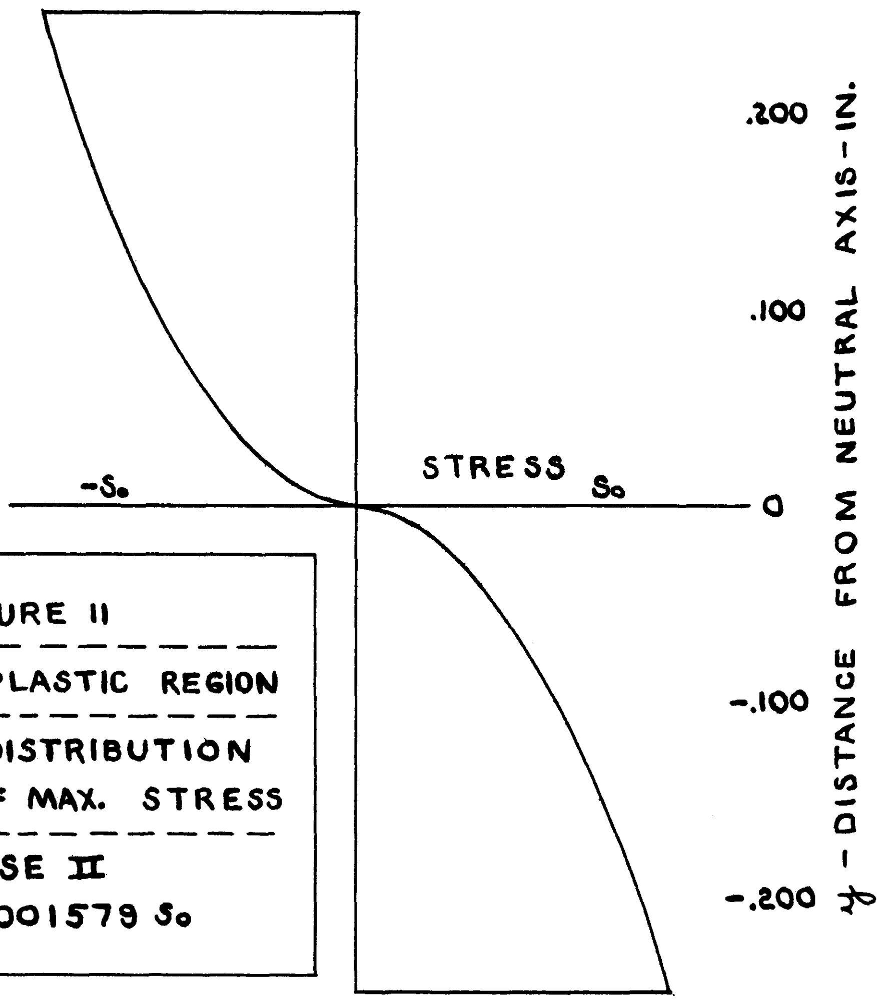


FIGURE 11

EXTENT OF PLASTIC REGION

STRESS DISTRIBUTION ON PLANE OF MAX. STRESS

CASE II

$P = 0.001579\ S_o$

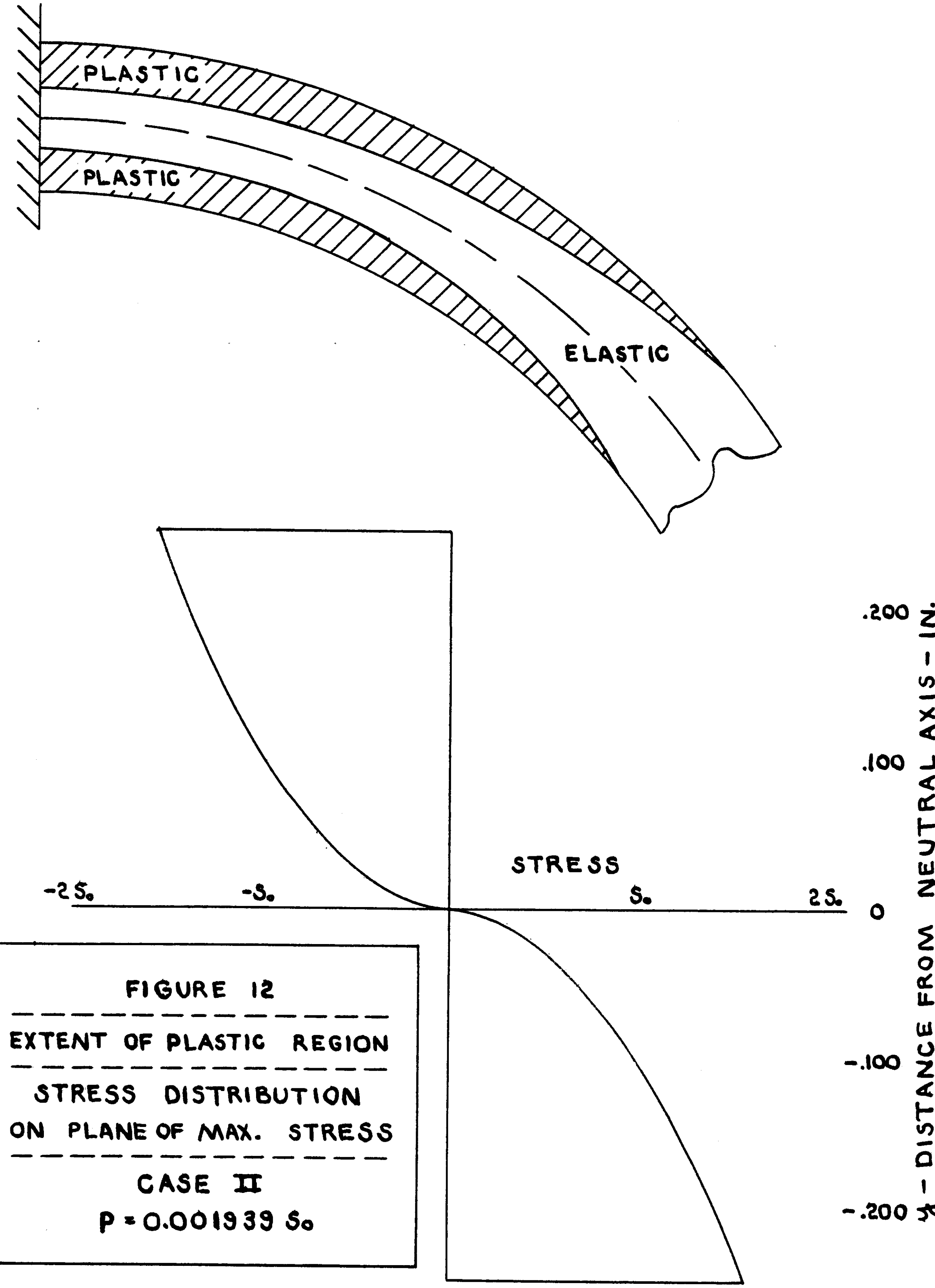


FIGURE 12

EXTENT OF PLASTIC REGION

STRESS DISTRIBUTION
ON PLANE OF MAX. STRESS

CASE II
$P = 0.001939\ S_o$

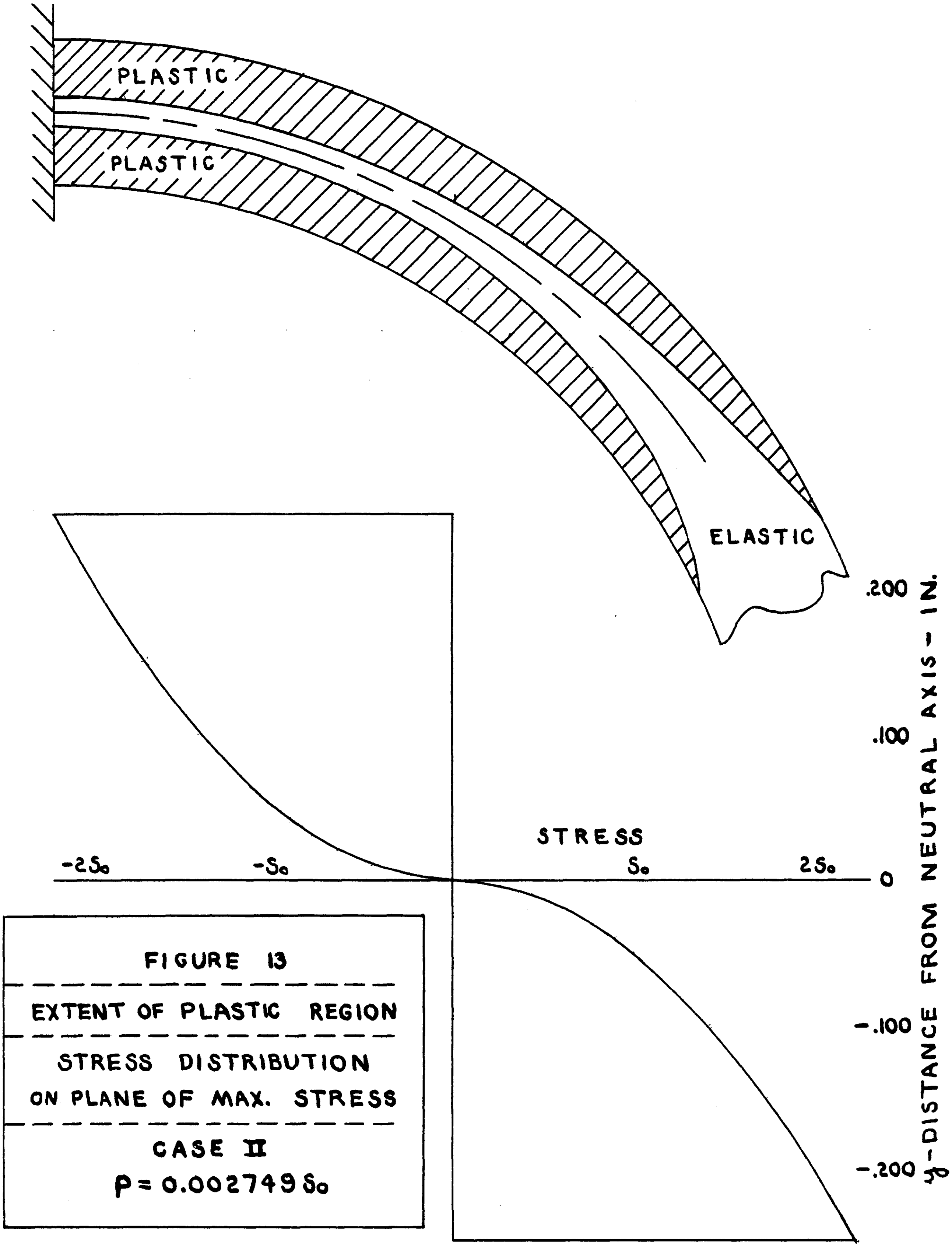


FIGURE 13

EXTENT OF PLASTIC REGION

STRESS DISTRIBUTION ON PLANE OF MAX. STRESS

CASE II

$P = 0.002749 S_0$

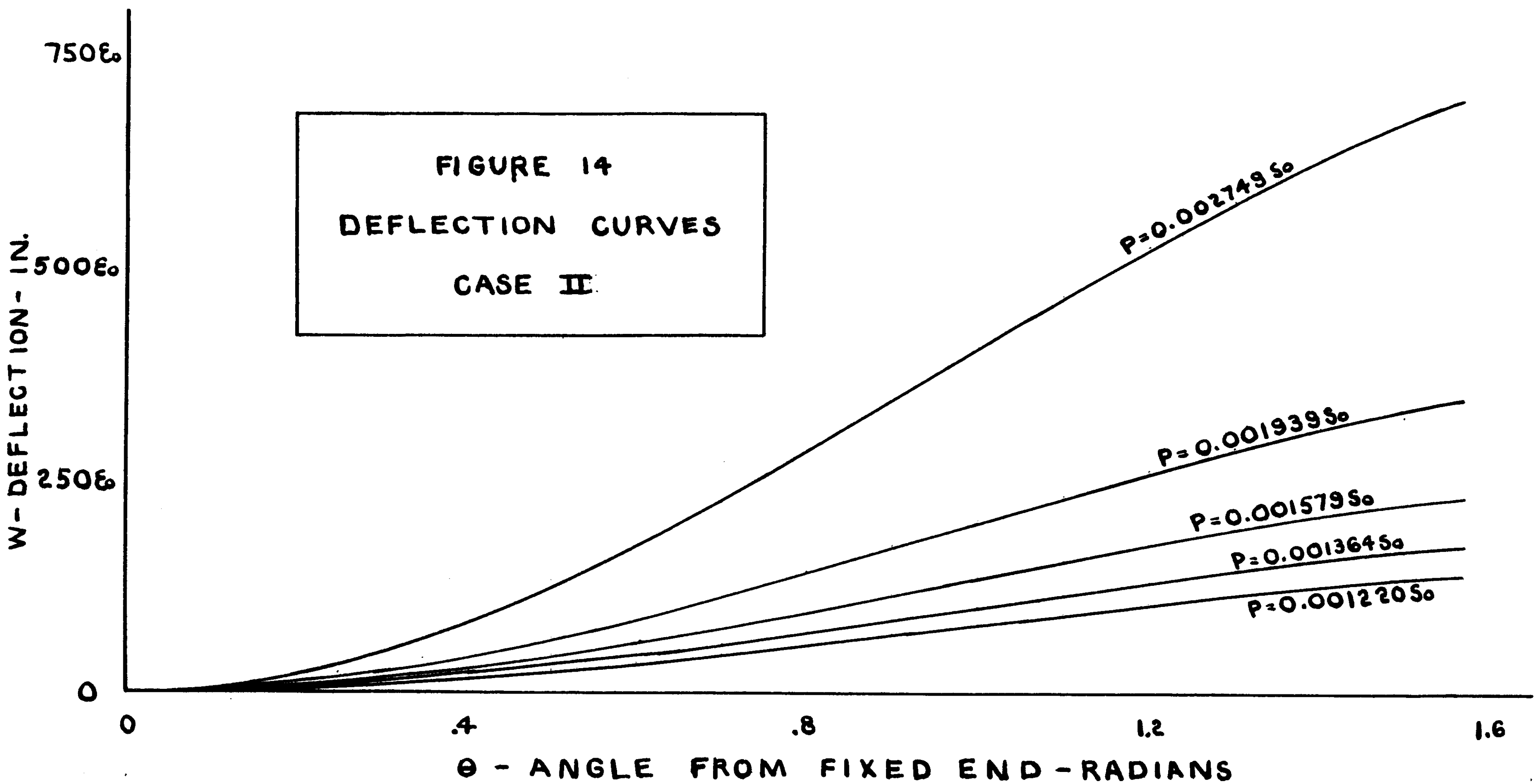


FIGURE 14
DEFLECTION CURVES
CASE II

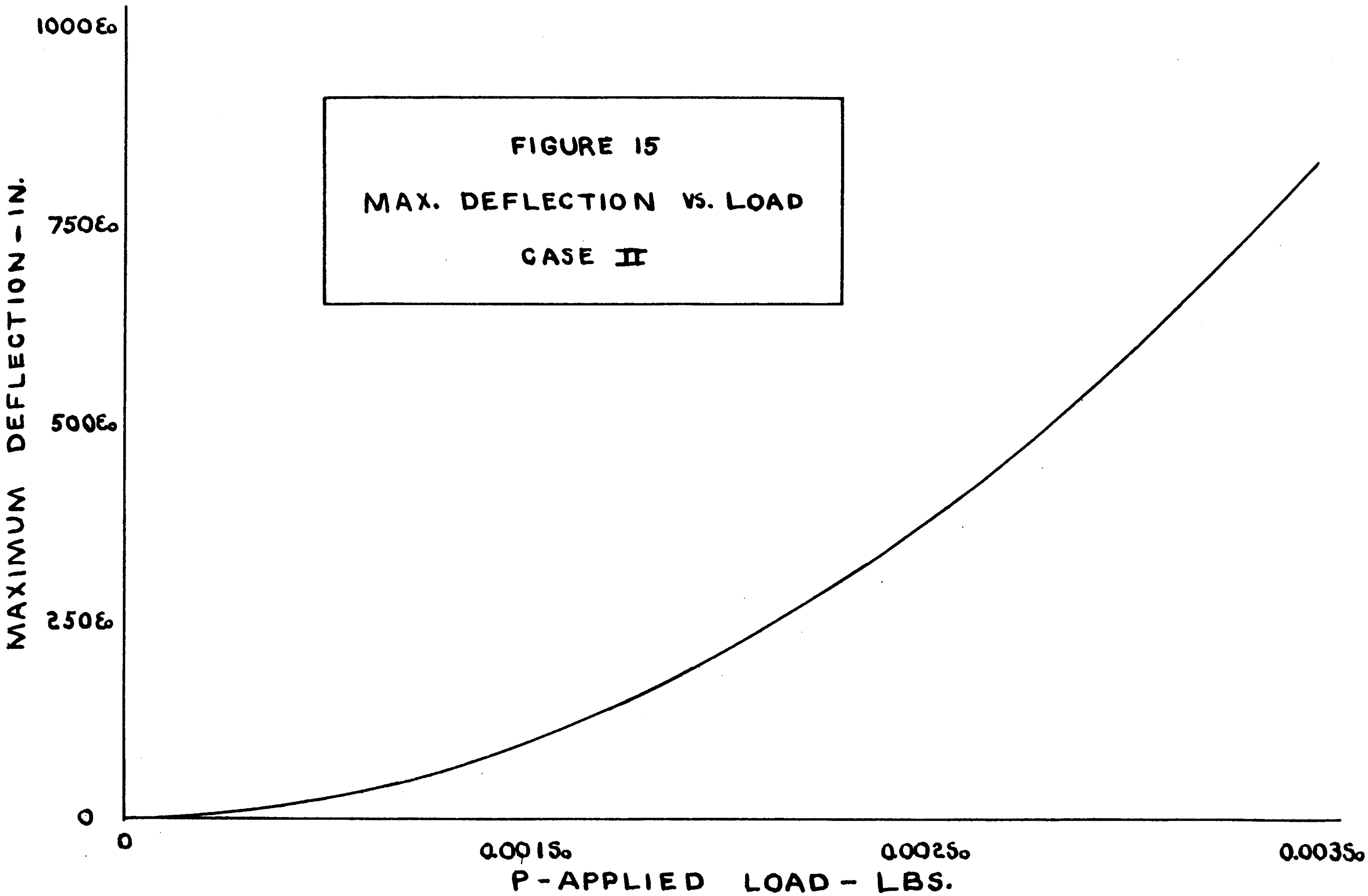


FIGURE 15

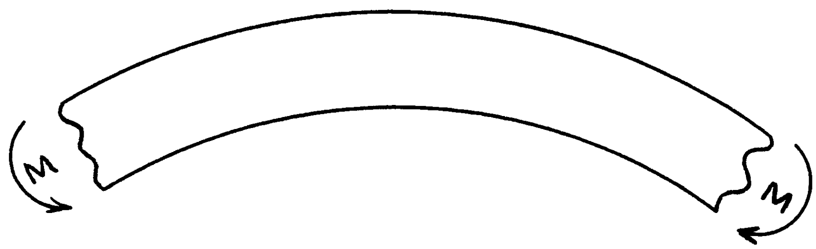
MAX. DEFLECTION VS. LOAD

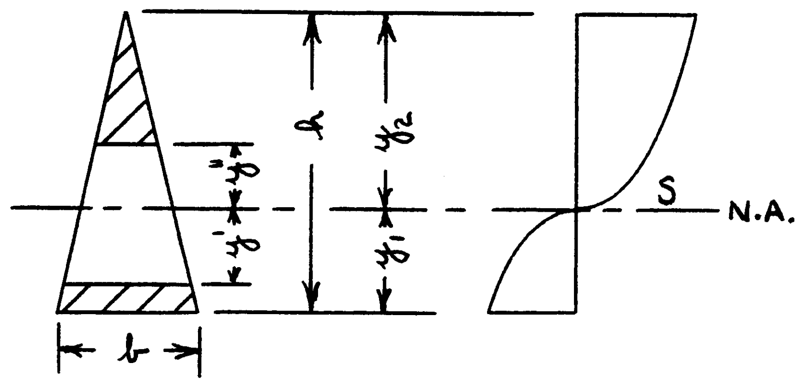
CASE II

Case III

Consider a curved bar of triangular cross-section
with depth of beam small
in comparison with the radius of curvature
and a stress-strain curve of the form

$$S = S_0\left(\frac{2\varepsilon}{\varepsilon_0}\right)^{\frac{1}{2}}$$





$$S = S_o \left(\frac{2\varepsilon}{\varepsilon_o}\right)^{\frac{1}{2}} \quad \text{For } +\varepsilon$$

$$S = -S_o \left(\frac{-2\varepsilon}{\varepsilon_o}\right)^{\frac{1}{2}} \quad \text{For } -\varepsilon$$

$$\varepsilon = \left(\frac{\rho_o - \rho}{\rho}\right)\left(\frac{y}{\rho_o + y}\right)$$

$$S = S_o \left[\left(\frac{2}{\varepsilon_o}\right)\left(\frac{\rho_o - \rho}{\rho}\right)\left(\frac{y}{\rho_o + y}\right)\right]^{\frac{1}{2}} \quad \text{For } +y$$

$$S = -S_o \left[\left(\frac{2}{\varepsilon_o}\right)\left(\frac{\rho_o - \rho}{\rho}\right)\left(\frac{-y}{\rho_o + y}\right)\right]^{\frac{1}{2}} \quad \text{For } -y$$

$$\text{Let } k = S_o \left[\left(\frac{2}{\varepsilon_o}\right)\left(\frac{\rho_o - \rho}{\rho}\right)\right]^{\frac{1}{2}}$$

$$\int S\,dA = 0 \qquad dA = Z\,dy \qquad dA = \frac{b}{h}(y_2 - y)\,dy$$

$$\int_{y_1}^{0} -k\left(\frac{-y}{e_0+y}\right)^{\frac{1}{2}}\frac{b}{h}(y_2-y)\,dy + \int_{0}^{y_2} k\left(\frac{y}{e_0+y}\right)^{\frac{1}{2}}\frac{b}{h}(y_2-y)\,dy = 0$$

$$-\frac{kb}{h}\left[\int_{y_1}^{0} y_2\left(\frac{-y}{e_0+y}\right)^{\frac{1}{2}}dy - \int_{y_1}^{0} y\left(\frac{-y}{e_0+y}\right)^{\frac{1}{2}}dy\right]$$

$$+\frac{kb}{h}\left[\int_{0}^{y_2} y_2\left(\frac{y}{e_0+y}\right)^{\frac{1}{2}}dy - \int_{0}^{y_2} y\left(\frac{y}{e_0+y}\right)^{\frac{1}{2}}dy\right] = 0$$

$$\left[y_2\int_{y_1}^{0}\left(\frac{-y}{e_0+y}\right)^{\frac{1}{2}}dy - \int_{y_1}^{0} y\left(\frac{-y}{e_0+y}\right)^{\frac{1}{2}}dy\right] =$$

$$\left[y_2\int_{0}^{y_2}\left(\frac{y}{e_0+y}\right)^{\frac{1}{2}}dy - \int_{0}^{y_2} y\left(\frac{y}{e_0+y}\right)^{\frac{1}{2}}dy\right]$$

$$y_2\left[\sqrt{-y}\sqrt{e_0+y} - e_0\sin^{-1}\sqrt{\frac{-y}{e_0}}\right]_{y_1}^{0}$$

$$-\left[-\frac{\sqrt{-y}^{3}\sqrt{e_0+y}}{2} + \frac{3e_0}{4}\left(-\sqrt{-y}\sqrt{e_0+y} + e_0\sinh^{-1}\sqrt{\frac{-y}{e_0}}\right)\right]_{y_1}^{0} =$$

$$y_2\left[\sqrt{y}\sqrt{e_0+y} - e_0\sinh^{-1}\sqrt{\frac{y}{e_0}}\right]_{0}^{y_2}$$

$$-\left[\frac{\sqrt{y}^{3}\sqrt{e_0+y}}{2} - \frac{3e_0}{4}\left(\sqrt{y}\sqrt{e_0+y} - e_0\sinh^{-1}\sqrt{\frac{y}{e_0}}\right)\right]_{0}^{y_2}$$

$$-y_2\left[\sqrt{-y_1}\sqrt{e_0+y_1} - e_0 \sin^{-1}\sqrt{\frac{-y_1}{e_0}}\right]$$

$$+\left[-\frac{\sqrt{-y_1^3}\sqrt{e_0+y_1}}{2} + \frac{3e_0}{4}\left(-\sqrt{-y_1}\sqrt{e_0+y_1} + e_0 \sin^{-1}\sqrt{\frac{-y_1}{e_0}}\right)\right] =$$

$$y_2\left[\sqrt{y_2}\sqrt{e_0+y_2} - e_0 \sinh^{-1}\sqrt{\frac{y_2}{e_0}}\right]$$

$$-\left[\frac{\sqrt{y_2^3}\sqrt{e_0+y_2}}{2} - \frac{3e_0}{4}\left(\sqrt{y_2}\sqrt{e_0+y_2} - e_0 \sinh^{-1}\sqrt{\frac{y_2}{e_0}}\right)\right]$$

$$\left(y_2 e_0 + \frac{3}{4}e_0^2\right)\left(\sin^{-1}\sqrt{\frac{-y_1}{e_0}} + \sinh^{-1}\sqrt{\frac{y_2}{e_0}}\right) =$$

$$\left(y_2 + \frac{3}{4}e_0\right)\left(\sqrt{-y_1}\sqrt{e_0+y_1} + \sqrt{y_2}\sqrt{e_0+y_2}\right) - \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{e_0+y_2} - \sqrt{-y_1^3}\sqrt{e_0+y_1}\right)$$

$$e_0 = \frac{\left(y_2+\frac{3}{4}e_0\right)\left(\sqrt{-y_1}\sqrt{e_0+y_1} + \sqrt{y_2}\sqrt{e_0+y_2}\right) - \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{e_0+y_2} - \sqrt{-y_1^3}\sqrt{e_0+y_1}\right)}{\left(y_2+\frac{3}{4}e_0\right)\left(\sin^{-1}\sqrt{\frac{-y_1}{e_0}} + \sinh^{-1}\sqrt{\frac{y_2}{e_0}}\right)}$$

EQUATION 24

Using absolute values, Equation 24 becomes

$$e_0 = \frac{\left(y_2+\frac{3}{4}e_0\right)\left(\sqrt{y_1}\sqrt{R_i} + \sqrt{y_2}\sqrt{R_0}\right) - \frac{1}{2}\left(\sqrt{y_2^3}\sqrt{R_0} - \sqrt{y_1^3}\sqrt{R_i}\right)}{\left(y_2+\frac{3}{4}e_0\right)\left(\sin^{-1}\sqrt{\frac{y_1}{e_0}} + \sinh^{-1}\sqrt{\frac{y_2}{e_0}}\right)}$$

EQUATION 24a

This expression is used to determine the position of the neutral axis. A trial and error solution is recommended since the expression is not readily solvable by algebraic means.

$$M = \int S_y \, dA$$

$$M = \int_{y_1}^{0} -k\left(\frac{-y}{e_0+y}\right)^{\frac{1}{2}} y \frac{b}{h}(y_2 - y)\,dy + \int_{0}^{y_2} k\left(\frac{y}{e_0+y}\right)^{\frac{1}{2}} y \frac{b}{h}(y_2 - y)\,dy$$

$$M = \frac{kb}{h}\left[-y_2 \int_{y_1}^{0} y\left(\frac{-y}{e_0+y}\right)^{\frac{1}{2}} dy + \int_{y_1}^{0} y^2 \left(\frac{-y}{e_0+y}\right)^{\frac{1}{2}} dy \right.$$

$$\left. + y_2 \int_{0}^{y_2} y\left(\frac{y}{e_0+y}\right)^{\frac{1}{2}} dy - \int_{0}^{y_2} y^2 \left(\frac{y}{e_0+y}\right)^{\frac{1}{2}} dy \right]$$

$$M = \frac{kb}{h}\left[-y_2 \left\{ \frac{-\sqrt{-y^3}\sqrt{e_0+y}}{2} - \frac{3e_0}{4}\sqrt{-y}\sqrt{e_0+y} + \frac{3e_0^2}{4}\sin^{-1}\sqrt{\frac{-y}{e_0}} \right\}_{y_1}^{0} \right.$$

$$- \left\{ -\frac{\sqrt{-y^5}\sqrt{e_0+y}}{3} - \frac{5e_0\sqrt{-y^3}\sqrt{e_0+y}}{12} - \frac{5e_0^2\sqrt{-y}\sqrt{e_0+y}}{8} + \frac{5e_0^3}{8}\sin^{-1}\sqrt{\frac{-y}{e_0}} \right\}_{y_1}^{0}$$

$$+ y_2 \left\{ \frac{\sqrt{y^3}\sqrt{e_0+y}}{2} - \frac{3e_0}{4}\sqrt{y}\sqrt{e_0+y} + \frac{3e_0^2}{4}\sinh^{-1}\sqrt{\frac{y}{e_0}} \right\}_{0}^{y_2}$$

$$\left. - \left\{ \frac{\sqrt{y^5}\sqrt{e_0+y}}{3} - \frac{5e_0\sqrt{y^3}\sqrt{e_0+y}}{12} + \frac{5e_0^2\sqrt{y}\sqrt{e_0+y}}{8} - \frac{5e_0^3}{8}\sinh^{-1}\sqrt{\frac{y}{e_0}} \right\}_{0}^{y_2} \right]$$

$$M = \frac{kb}{h}\left[-\frac{y_2\sqrt{-y_1^3}\sqrt{e_0+y_1}}{2} - \frac{3e_0 y_2}{4}\sqrt{-y_1}\sqrt{e_0+y_1} \right.$$

$$+ \frac{3e_0^2 y_2}{4}\sin^{-1}\sqrt{\frac{-y_1}{e_0}} - \frac{\sqrt{-y_1^5}\sqrt{e_0+y_1}}{3}$$

$$- \frac{5e_0\sqrt{-y_1^3}\sqrt{e_0+y_1}}{12} - \frac{5e_0^2\sqrt{-y_1}\sqrt{e_0+y_1}}{8}$$

$$+ \frac{5e_0^3}{8}\sin^{-1}\sqrt{\frac{-y_1}{e_0}} + \frac{y_2\sqrt{y_2^3}\sqrt{e_0+y_2}}{2}$$

$$- \frac{3e_0 y_2}{4}\sqrt{y_2}\sqrt{e_0+y_2} + \frac{3e_0^2 y_2}{4}\sinh^{-1}\sqrt{\frac{y_2}{e_0}}$$

$$- \frac{\sqrt{y_2^5}\sqrt{e_0+y_2}}{3} + \frac{5e_0\sqrt{y_2^3}\sqrt{e_0+y_2}}{12}$$

$$\left. - \frac{5e_0^2\sqrt{y_2}\sqrt{e_0+y_2}}{8} + \frac{5e_0^3}{8}\sinh^{-1}\sqrt{\frac{y_2}{e_0}} \right].$$

$$M = S_0\left[\left(\frac{2}{E_0}\right)\left(\frac{e_0-e}{e}\right)\right]^{\frac{1}{2}}\frac{b}{h}\left[\left(\frac{3e_0^2 y_2}{4} + \frac{5e_0^3}{8}\right)\left(\sin^{-1}\sqrt{\frac{-y_1}{e_0}} + \sinh^{-1}\sqrt{\frac{y_2}{e_0}}\right)\right.$$

$$+ \left(\frac{y_2}{2} + \frac{5e_0}{12}\right)\left(\sqrt{y_2^3}\sqrt{e_0+y_2} - \sqrt{-y_1^3}\sqrt{e_0+y_1}\right)$$

$$- \left(\frac{3e_0 y_2}{4} + \frac{5e_0^2}{8}\right)\left(\sqrt{y_2}\sqrt{e_0+y_2} + \sqrt{-y_1}\sqrt{e_0+y_1}\right)$$

$$\left. - \frac{1}{3}\left(\sqrt{y_2^5}\sqrt{e_0+y_2} + \sqrt{-y_1^5}\sqrt{e_0+y_1}\right)\right]$$

EQUATION 25

Using absolute values, Equation 25 becomes

$$M = S_0\left[\left(\frac{2}{E_0}\right)\left(\frac{c_0-l}{c}\right)\right]^{\frac{1}{2}}\frac{b}{h}\left[\left(\frac{3c_0^2 y_2}{4}+\frac{5c_0^3}{8}\right)\left(\sin^{-1}\sqrt{\frac{y_1}{c_0}}+\sinh^{-1}\sqrt{\frac{y_2}{c_0}}\right)\right.$$
$$+\left(\frac{y_2}{2}+\frac{5c_0}{12}\right)\left(\sqrt{y_2^3}\sqrt{R_o}-\sqrt{y_1^3}\sqrt{R_i}\right)-\frac{1}{3}\left(\sqrt{y_2^5}\sqrt{R_o}+\sqrt{y_1^5}\sqrt{R_i}\right)$$
$$\left.-\left(\frac{3c_0 y_2}{4}+\frac{5c_0^2}{8}\right)\left(\sqrt{y_2}\sqrt{R_o}+\sqrt{y_1}\sqrt{R_i}\right)\right]$$

EQUATION 25a

Care must be taken to use a high degree of accuracy when evaluating the expression in Equation 25 (or 25a) contained in the brackets since it involves the very small difference of two comparatively large numbers.

$$S = \pm S_0\left[\left(\frac{2}{E_0}\right)\left(\frac{c_0-l}{c}\right)\left(\frac{\pm y}{c_0+y}\right)\right]^{\frac{1}{2}}$$

$$S = \frac{\pm M h\left(\frac{\pm y}{c_0+y}\right)^{\frac{1}{2}}}{b\left[\left(\frac{3c_0^2 y_2}{4}+\frac{5c_0^3}{8}\right)\left(\sin^{-1}\sqrt{\frac{y_1}{c_0}}+\sinh^{-1}\sqrt{\frac{y_2}{c_0}}\right)-\frac{1}{3}\left(\sqrt{y_2^5}\sqrt{R_o}+\sqrt{y_1^5}\sqrt{R_i}\right)+\left(\frac{y_2}{2}+\frac{5c_0}{12}\right)\left(\sqrt{y_2^3}\sqrt{R_o}-\sqrt{y_1^3}\sqrt{R_i}\right)-\left(\frac{3c_0 y_2}{4}+\frac{5c_0^2}{8}\right)\left(\sqrt{y_2}\sqrt{R_o}+\sqrt{y_1}\sqrt{R_i}\right)\right]}$$

EQUATION 26

When using Equation 26, the signs apply to values of y, the - signs to - values of y.

From Equation 25a

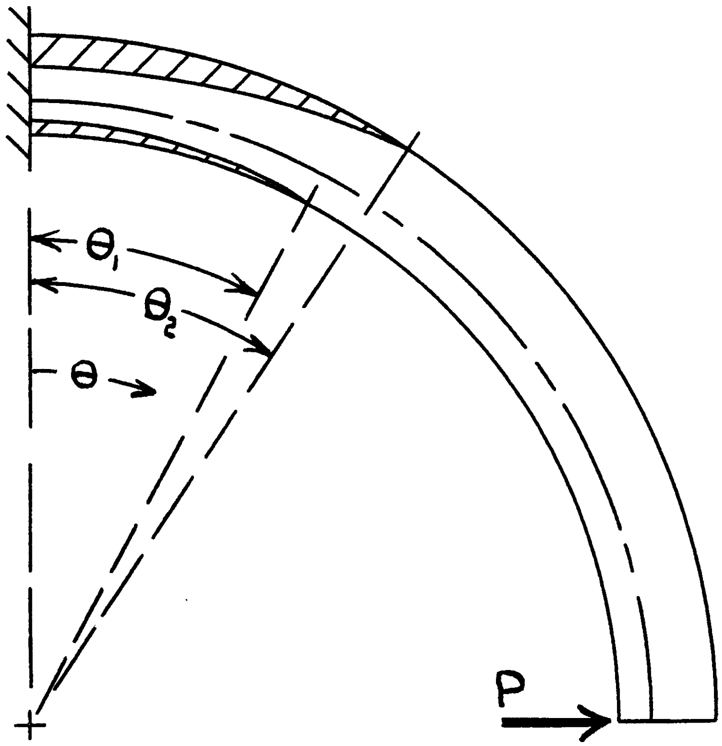
$$\frac{e_o - e}{e} = \frac{M^2 h^2 \varepsilon_o}{2 S_o^2 b^2 \left[\left(\frac{3 e_o^2 y_2}{4} + \frac{5 e_o^3}{8} \right) \left(\sin^{-1} \sqrt{\frac{y_1}{e_o}} + \sinh^{-1} \sqrt{\frac{y_2}{e_o}} \right) + \left(\frac{y_2}{2} + \frac{5 e_o}{12} \right) \left(\sqrt{y_2^3} \sqrt{R_o} - \sqrt{y_1^3} \sqrt{R_i} \right) - \frac{1}{3} \left(\sqrt{y_2^5} \sqrt{R_o} + \sqrt{y_1^5} \sqrt{R_i} \right) - \left(\frac{3 e_o y_2}{4} + \frac{5 e_o^2}{8} \right) \left(\sqrt{y_2} \sqrt{R_o} + \sqrt{y_1} \sqrt{R_i} \right) \right]^2}$$

Substituting in Equation 10

$$\frac{d^2 w}{d\theta^2} + w = \frac{e_o M^2 h^2 \varepsilon_o}{2 S_o^2 b^2 \left[\left(\frac{3 e_o^2 y_2}{4} + \frac{5 e_o^3}{8} \right) \left(\sin^{-1} \sqrt{\frac{y_1}{e_o}} + \sinh^{-1} \sqrt{\frac{y_2}{e_o}} \right) + \left(\frac{y_2}{2} + \frac{5 e_o}{12} \right) \left(\sqrt{y_2^3} \sqrt{R_o} - \sqrt{y_1^3} \sqrt{R_i} \right) - \frac{1}{3} \left(\sqrt{y_2^5} \sqrt{R_o} + \sqrt{y_1^5} \sqrt{R_i} \right) - \left(\frac{3 e_o y_2}{4} + \frac{5 e_o^2}{8} \right) \left(\sqrt{y_2} \sqrt{R_o} + \sqrt{y_1} \sqrt{R_i} \right) \right]^2}$$

EQUATION 27

Calculations for the Specific Case (Figure 1)



$$M = -P\rho_o \cos\theta$$

A trial and error solution of Equation 24a yields the result that

$$\rho_o = 10.165''$$

b = 0.025"

h = 0.500"

Ro = 10.500"

Ri = 10.000"

By the substitution of $M = -P\rho_o\cos\theta$ in Equation 26

$$S = \frac{\mp P\rho_o \cos\theta \, h \left(\frac{\pm y}{e_o + y}\right)^{\frac{1}{2}}}{b\left[\left(\frac{3e_o^2 y_2}{4} + \frac{5e_o^3}{8}\right)\left(\sin^{-1}\sqrt{\frac{y_1}{e_o}} + \sinh^{-1}\sqrt{\frac{y_2}{e_o}}\right) - \frac{1}{3}\left(\sqrt{y_2^5}\sqrt{R_o} + \sqrt{y_1^5}\sqrt{R_i}\right) + \left(\frac{y_2}{2} + \frac{5e_o}{12}\right)\left(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i}\right) - \left(\frac{3e_o y_2}{4} + \frac{5e_o^2}{8}\right)\left(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1} R_i\right)\right]}$$

EQUATION 26a

Evaluating

$$\Bigg[\left(\frac{3e_o^2 y_2}{4} + \frac{5e_o^3}{8}\right)\left(\sin^{-1}\sqrt{\frac{y_1}{e_o}} + \sinh^{-1}\sqrt{\frac{y_2}{e_o}}\right) - \frac{1}{3}\left(\sqrt{y_2^5}\sqrt{R_o} + \sqrt{y_1^5}\sqrt{R_i}\right)$$

$$+\left(\frac{y_2}{2} + \frac{5e_o}{12}\right)\left(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i}\right) - \left(\frac{3e_o y_2}{4} + \frac{5e_o^2}{8}\right)\left(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i}\right)\Bigg] = 0.00400$$

Initial Yielding

From Equation 26a, $S = -So$, $y = y'' = y_2$

$Mi = -\ 0.01120\ So$ in-lb

$P = 0.001102\ So$ lb

Secondary Yielding

From Equation 26a, $S = So$, $y = y' = y_1$

$Ms = -0.01557\ So$ in-lb

$P = 0.001532\ So$ lb

Complete Yielding

From Equation 26a, $S = So$, $y = y' = y'' = 0$

$Mc = -\ \infty$ in-lb

$P = \infty$ lb

Stress Distribution

The stress distribution at any angle θ from 0 to $\frac{\pi}{2}$ for any load P from 0 to ∞ is given by Equation 26a. This applies to both the elastic and plastic regions since the stress-strain curve also applies to both the elastic and plastic regions.

Deflection Equation

Substitution of $M = -P\rho_o \cos\theta$ and $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$ in Equation 27 yields

$$\frac{d^2w}{d\theta^2} + w = \frac{P^2 e_o^3 \varepsilon_o h^2 (\cos 2\theta + 1)}{4 S_o^2 b^2 \Big[\Big(\frac{3e_o^2 y_2}{4} + \frac{5e_o^3}{8} \Big) \Big(\sin^{-1}\sqrt{\frac{y_1}{e_o}} + \sinh^{-1}\sqrt{\frac{y_2}{e_o}} \Big) + \Big(\frac{y_2}{2} + \frac{5e_o}{12} \Big) \Big(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i} \Big) - \frac{1}{3}\Big(\sqrt{y_2^5}\sqrt{R_o} + \sqrt{y_1^5}\sqrt{R_i} \Big) - \Big(\frac{3e_o y_2}{4} + \frac{5e_o^2}{8} \Big) \Big(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i} \Big) \Big]^2}$$

Complementary Solution

$$W = C_1 \sin\theta + C_2 \cos\theta$$

Particular Solution

$$W = \frac{-P^2 e_o^3 h^2 \varepsilon_o \left(\frac{\cos 2\theta}{3} - 1 \right)}{4 S_o^2 b^2 \Big[\Big(\frac{3e_o^2 y_2}{4} + \frac{5e_o^3}{8} \Big) \Big(\sin^{-1}\sqrt{\frac{y_1}{e_o}} + \sinh^{-1}\sqrt{\frac{y_2}{e_o}} \Big) + \Big(\frac{y_2}{2} + \frac{5e_o}{12} \Big) \Big(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i} \Big) - \frac{1}{3}\Big(\sqrt{y_2^5}\sqrt{R_o} + \sqrt{y_1^5}\sqrt{R_i} \Big) - \Big(\frac{3e_o y_2}{4} + \frac{5e_o^2}{8} \Big) \Big(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i} \Big) \Big]^2}$$

General Solution

$$W = C_1 \sin\theta + C_2 \cos\theta - \frac{p^2 e_o^3 h^2 \varepsilon_o \left(\frac{\cos 2\theta}{3} - 1\right)}{4 S_o^2 b^2 \left[\left(\frac{3e_o^2 y_2}{4} + \frac{5e_o^3}{8}\right)\left(\sin^{-1}\sqrt{\frac{y_1}{e_o}} + \sinh^{-1}\sqrt{\frac{y_2}{e_o}}\right) + \left(\frac{y_2}{2} + \frac{5e_o}{12}\right)\left(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i}\right) - \frac{1}{3}\left(\sqrt{y_2^5}\sqrt{R_o} + \sqrt{y_1^5}\sqrt{R_i}\right) - \left(\frac{3e_o y_2}{4} + \frac{5e_o^2}{8}\right)\left(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i}\right)\right]^2}$$

Boundary Conditions

$$\theta = 0, \quad W = 0, \quad \frac{dW}{d\theta} = 0$$

Evaluation of C_1 and C_2

$$C_1 = 0$$

$$C_2 = \frac{-p^2 e_o^3 h^2 \varepsilon_o}{6 S_o^2 b^2 \left[\left(\frac{3e_o^2 y_2}{4} + \frac{5e_o^3}{8}\right)\left(\sin^{-1}\sqrt{\frac{y_1}{e_o}} + \sinh^{-1}\sqrt{\frac{y_2}{e_o}}\right) + \left(\frac{y_2}{2} + \frac{5e_o}{12}\right)\left(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i}\right) - \frac{1}{3}\left(\sqrt{y_2^5}\sqrt{R_o} + \sqrt{y_1^5}\sqrt{R_i}\right) - \left(\frac{3e_o y_2}{4} + \frac{5e_o^2}{8}\right)\left(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i}\right)\right]^2}$$

$$W = \frac{-P^2 e_0^3 h^2 \varepsilon_0 \left(\frac{2\cos\theta}{3} + \frac{\cos 2\theta}{3} - 1 \right)}{4 S_0^2 b^2 \Big[\left(\frac{3e_0^2 y_2}{4} + \frac{5e_0^3}{8} \right)\left(\sin^{-1}\sqrt{\frac{y_1}{e_0}} + \sinh^{-1}\sqrt{\frac{y_2}{e_0}} \right) + \left(\frac{y_2}{2} + \frac{5e_0}{12} \right)\left(\sqrt{y_2^3}\sqrt{R_o} - \sqrt{y_1^3}\sqrt{R_i} \right) - \frac{1}{3}\left(\sqrt{y_2^5}\sqrt{R_o} + \sqrt{y_1^5}\sqrt{R_i} \right) - \left(\frac{3e_0 y_2}{4} + \frac{5e_0^2}{8} \right)\left(\sqrt{y_2}\sqrt{R_o} + \sqrt{y_1}\sqrt{R_i} \right) \Big]^2}$$

Equation 28

This deflection equation (Equation 28) applies for all values of P from 0 to ∞ , ie, it applies over both the elastic and plastic regions. The sign of the deflection must be the same as that of the bending moment before the moment term is squared and substituted in Equation 27.

CASE III

INITIAL YIELDING

($y'' = 0.335''$)

$M = -0.01120\ S_o$ in lb

$P = 0.001102\ S_o$ lb

$\rho_o = 10.165$ in

Extent of Plastic Region

There is no plastic region.

Case III - Initial Yielding

Stress Distribution on Plane of Maximum Stress

y inches	S psi
0.335	-1.000 So
0.300	-0.948 So
0.250	-0.867 So
0.200	-0.778 So
0.150	-0.675 So
0.100	-0.553 So
0.050	-0.392 So
0.040	-0.351 So
0.030	-0.304 So
0.020	-0.248 So
0.010	-0.176 So
0	0
-0.010	0.176 So
-0.020	0.249 So
-0.030	0.305 So
-0.040	0.352 So
-0.050	0.394 So
-0.100	0.558 So
-0.150	0.685 So
-0.165	0.719 So

Deflection Curve

θ radians	w inches
0	0
0.100	0.795 ε_0
0.200	3.154 ε_0
0.300	7.010 ε_0
0.400	12.24 ε_0
0.500	19.55 ε_0
0.600	26.20 ε_0
0.700	34.52 ε_0
0.800	43.43 ε_0
0.900	52.63 ε_0
1.000	62.01 ε_0
1.100	71.19 ε_0
1.200	79.99 ε_0
1.300	88.20 ε_0
1.400	95.64 ε_0
1.500	102.2 ε_0
1.5708	106.2 ε_0

CASE III

YIELDING TO ONE SIXTH OF DEPTH OF BEAM (ON TOP)

($y'' = 0.255''$)

$M = -0.01278$ So in lb

$P = 0.001258$ So lb

$\rho_o = 10.165$ in

Extent of Plastic Region

y'' inches	θ degrees
0.255	0
0.260	$7^\circ 52'$
0.265	$11^\circ 4'$
0.275	$15^\circ 27'$
0.300	$22^\circ 29'$
0.335	$28^\circ 51'$

Case III - Yielding to One-sixth of Depth of Beam

Stress Distribution on Plane of Maximum Stress

y inches	S psi
0.335	-1.142 So
0.300	-1.082 So
0.250	-0.990 So
0.200	-0.888 So
0.150	-0.771 So
0.100	-0.631 So
0.050	-0.447 So
0.040	-0.400 So
0.030	-0.347 So
0.020	-0.283 So
0.010	-0.200 So
0	0
-0.010	0.201 So
-0.020	0.284 So
-0.030	0.348 So
-0.040	0.402 So
-0.050	0.449 So
-0.100	0.637 So
-0.150	0.782 So
-0.165	0.821 So

Deflection Curve

θ radians	w inches
0	0
0.100	1.036 ε_o
0.200	4.112 ε_o
0.300	9.138 ε_o
0.400	15.96 ε_o
0.500	25.43 ε_o
0.600	34.16 ε_o
0.700	45.01 ε_o
0.800	56.62 ε_o
0.900	68.62 ε_o
1.000	80.84 ε_o
1.100	92.81 ε_o
1.200	104.3 ε_o
1.300	115.0 ε_o
1.400	124.7 ε_o
1.500	133.2 ε_o
1.5708	138.5 ε_o

CASE III

SECONDARY YIELDING

($y' = -0.165''$, $y'' = 0.171''$)

$M = -0.01557$ So in lb

$P = 0.001532$ So lb

$\rho_0 = 10.165$ in

Extent of Plastic Region

y'' inches	θ degrees
0.171	0
0.175	9° 7'
0.185	16° 6'
0.200	22°22'
0.250	34° 0'
0.300	40° 39'
0.335	44° 1'

Case III - Secondary Yielding

Stress Distribution on Plane of Maximum Stress

y inches	S psi
0.335	-1.391 So
0.300	-1.318 So
0.250	-1.206 So
0.200	-1.081 So
0.150	-0.939 So
0.100	-0.786 So
0.050	-0.545 So
0.040	-0.487 So
0.030	-0.422 So
0.020	-0.345 So
0.010	-0.244 So
0	0
-0.010	0.244 So
-0.020	0.346 So
-0.030	0.424 So
-0.040	0.489 So
-0.050	0.547 So
-0.100	0.794 So
-0.150	0.953 So
-0.165	1.000 So

Deflection Curve

θ radians	w inches
0	0
0.100	1.537 ε_o
0.200	6.099 ε_o
0.300	13.55 ε_o
0.400	23.68 ε_o
0.500	37.71 ε_o
0.600	50.67 ε_o
0.700	66.76 ε_o
0.800	83.98 ε_o
0.900	101.8 ε_o
1.000	119.9 ε_o
1.100	137.7 ε_o
1.200	154.7 ε_o
1.300	170.5 ε_o
1.400	184.9 ε_o
1.500	197.6 ε_o
1.5708	205.4 ε_o

CASE III

YIELDING TO ONE-HALF OF DEPTH OF BEAM (ON TOP)

($y'' = 0.110''$)

$M = -0.01933$ So in lb

$P = 0.001902$ So lb

$\rho_o = 10.165$ in

Extent of Plastic Region

y' inches	y'' inches	θ degrees
-0.108	0.110	0
-0.113	0.115	11°59'
-0.122	0.125	20° 9'
-0.146	0.150	30°54'
-0.165		36°21'
	0.175	37°19'
	0.200	41°51'
	0.250	48° 6'
	0.300	52°20'
	0.335	54°36'

Case III - Yielding to One-half of Depth of Beam (On Top)

Stress Distribution on Plane of Maximum Stress

y	S
inches	psi
0.335	-1.726 So
0.300	-1.636 So
0.250	-1.497 So
0.200	-1.343 So
0.150	-1.165 So
0.100	-0.932 So
0.050	-0.676 So
0.040	-0.605 So
0.030	-0.524 So
0.020	-0.428 So
0.010	-0.303 So
0	0
-0.010	0.303 So
-0.020	0.429 So
-0.030	0.526 So
-0.040	0.607 So
-0.050	0.680 So
-0.100	0.963 So
-0.150	1.183 So
-0.165	1.241 So

Deflection Curve

θ	w
radians	inches
0	0
0.100	2.369 ε_0
0.200	9.400 ε_0
0.300	20.89 ε_0
0.400	36.49 ε_0
0.500	58.12 ε_0
0.600	78.09 ε_0
0.700	102.9 ε_0
0.800	129.4 ε_0
0.900	156.9 ε_0
1.000	184.8 ε_0
1.100	212.2 ε_0
1.200	238.4 ε_0
1.300	262.9 ε_0
1.400	285.0 ε_0
1.500	304.5 ε_0
1.5708	316.5 ε_0

CASE III

YIELDING TO SIX-TENTHS OF DEPTH OF BEAM (ON TOP)

($y'' = 0.050''$)

$M = -0.02859$ So in lb

$P = 0.002812$ So lb

$\rho_o = 10.165$ in

Extent of Plastic Region

y' inches	y'' inches	θ degrees
-0.050	0.050	0
-0.054	0.055	17°30'
-0.064	0.065	28°38'
-0.074	0.075	35°10'
-0.098	0.100	44°52'
-0.146	0.150	54°32'
-0.165		57° 0'
	0.200	59°46'
	0.250	63° 9'
	0.300	65°36'
	0.335	66°56'

Case III - Yielding to Six-tenths of Depth of Beam (On Top)

Stress Distribution on Plane of Maximum Stress

y inches	S psi
0.335	-2.553 So
0.300	-2.420 So
0.250	-2.214 So
0.200	-1.986 So
0.150	-1.724 So
0.100	-1.411 So
0.050	-1.000 So
0.040	-0.895 So
0.030	-0.775 So
0.020	-0.633 So
0.010	-0.448 So
0	0
-0.010	0.449 So
-0.020	0.635 So
-0.030	0.778 So
-0.040	0.898 So
-0.050	1.005 So
-0.100	1.425 So
-0.150	1.749 So
-0.165	1.836 So

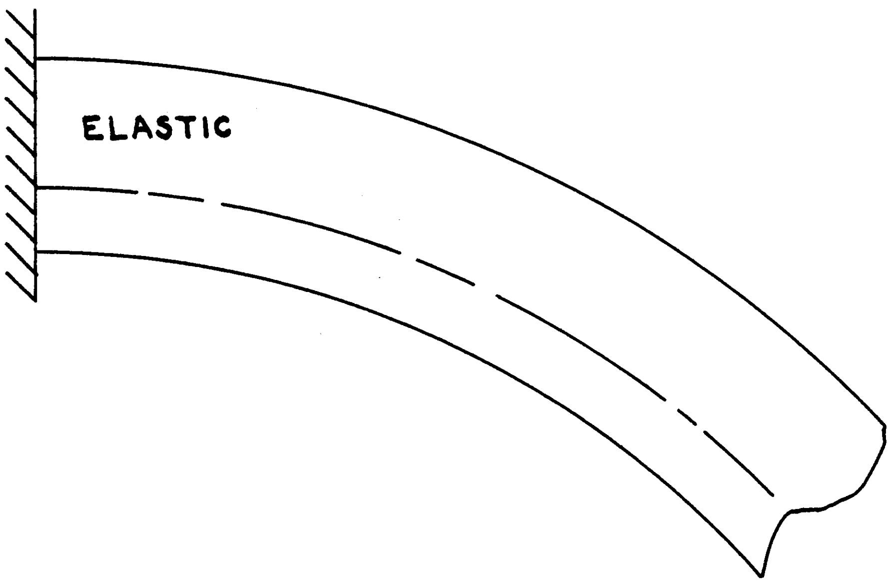
Deflection Curve

θ radians	w inches
0	0
0.100	5.181 ε_o
0.200	20.56 ε_o
0.300	45.69 ε_o
0.400	79.81 ε_o
0.500	127.1 ε_o
0.600	170.8 ε_o
0.700	225.0 ε_o
0.800	283.1 ε_o
0.900	343.1 ε_o
1.000	404.2 ε_o
1.100	464.0 ε_o
1.200	521.4 ε_o
1.300	574.9 ε_o
1.400	623.4 ε_o
1.500	666.1 ε_o
1.5708	692.3 ε_o

Case III

Maximum Deflection

P lbs	Wmax inches
0	0
.0002 So	3.501 ε_o
.0004 So	14.00 ε_o
.0006 So	31.51 ε_o
.0008 So	55.32 ε_o
.0010 So	87.53 ε_o
.0012 So	126.0 ε_o
.0014 So	171.6 ε_o
.0016 So	224.1 ε_o
.0018 So	283.6 ε_o
.0020 So	350.1 ε_o
.0022 So	423.6 ε_o
.0024 So	504.2 ε_o
.0026 So	591.7 ε_o
.0028 So	686.2 ε_o
.0030 So	787.8 ε_o



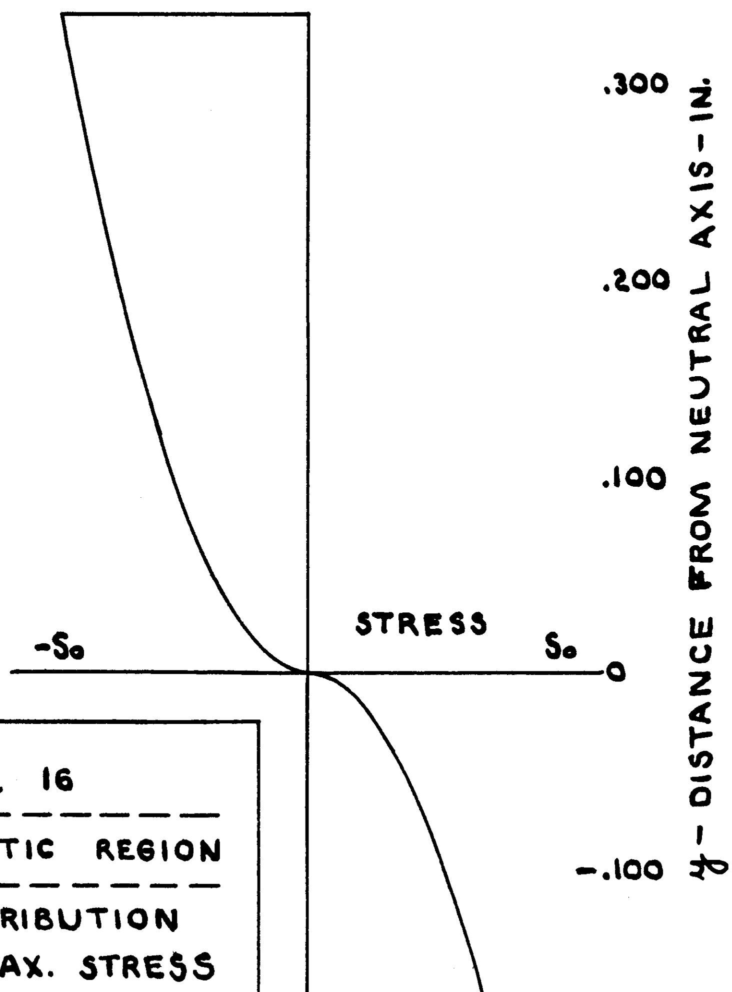
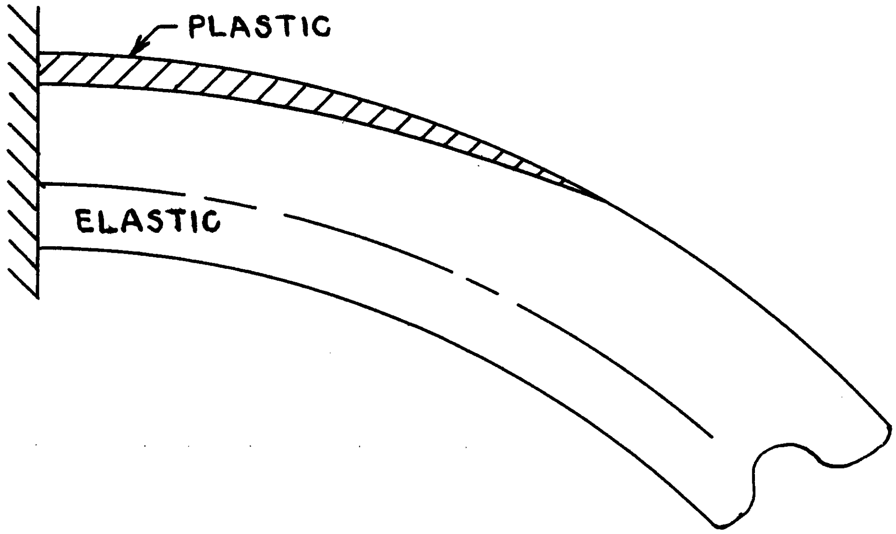


FIGURE 16

EXTENT OF PLASTIC REGION

STRESS DISTRIBUTION
ON PLANE OF MAX. STRESS

CASE III
$P = 0.001102\ S_o$



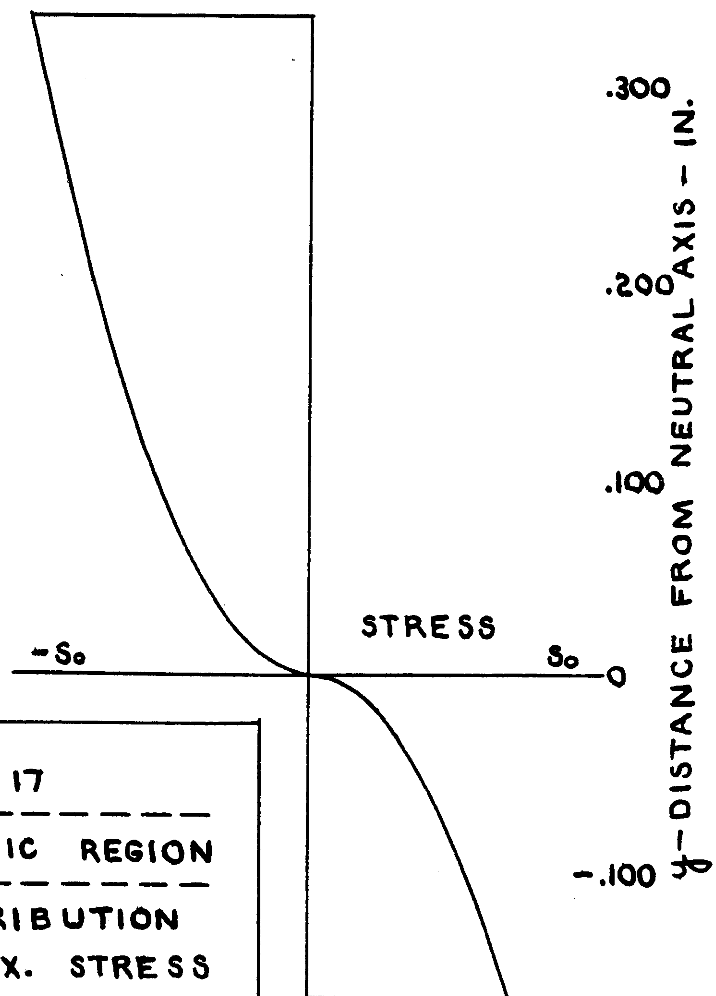


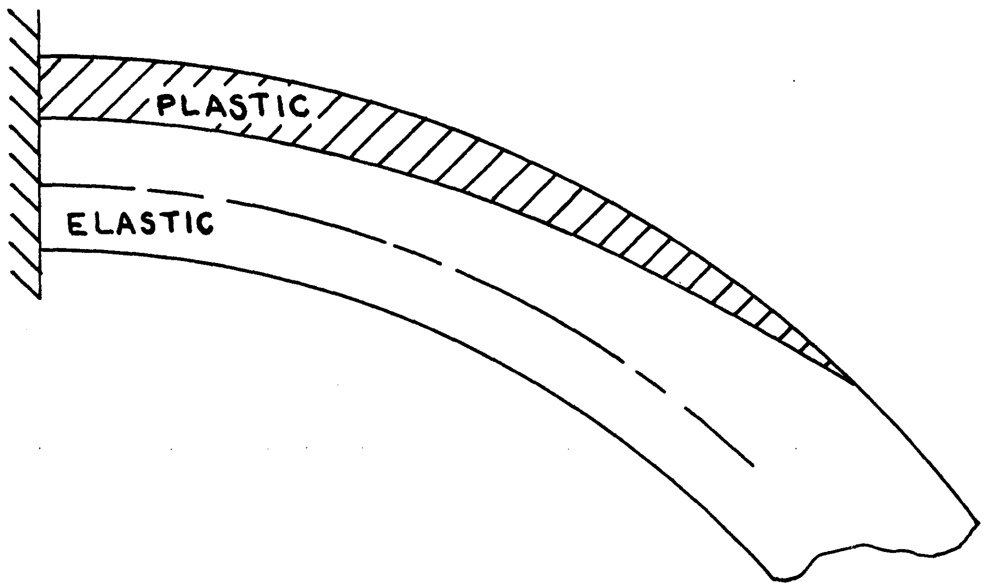
FIGURE 17

EXTENT OF PLASTIC REGION

STRESS DISTRIBUTION ON PLANE OF MAX. STRESS

CASE III

$P = 0.001258\, S_0$



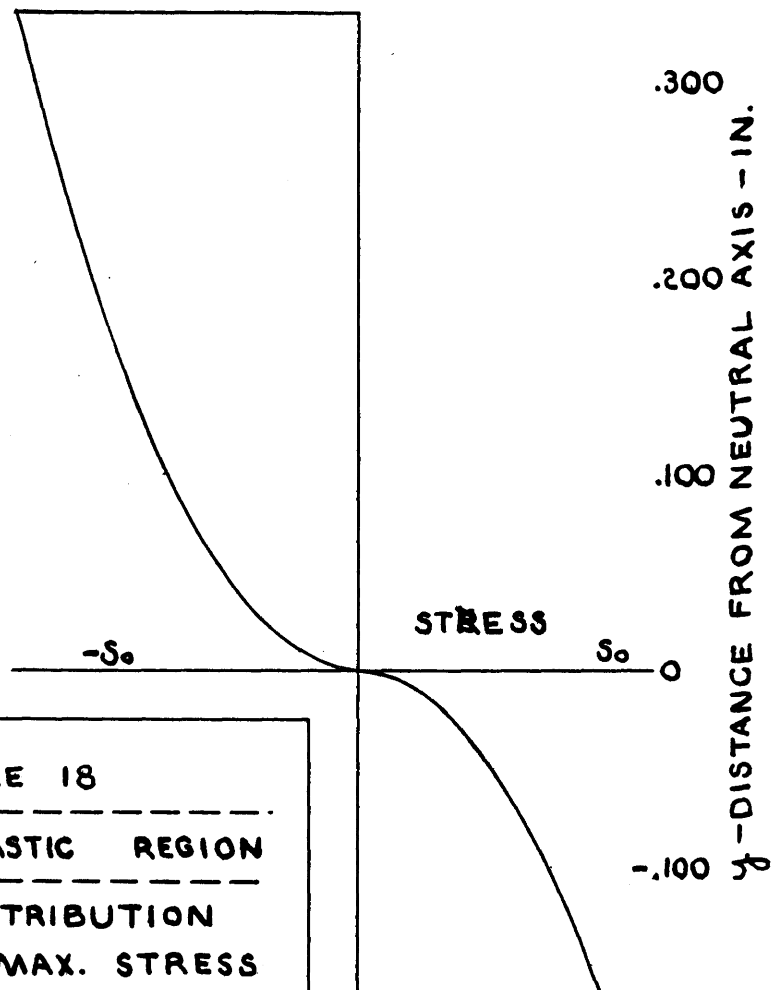


FIGURE 18

EXTENT OF PLASTIC REGION

STRESS DISTRIBUTION
ON PLANE OF MAX. STRESS

CASE III
$P = 0.001532\ S_o$

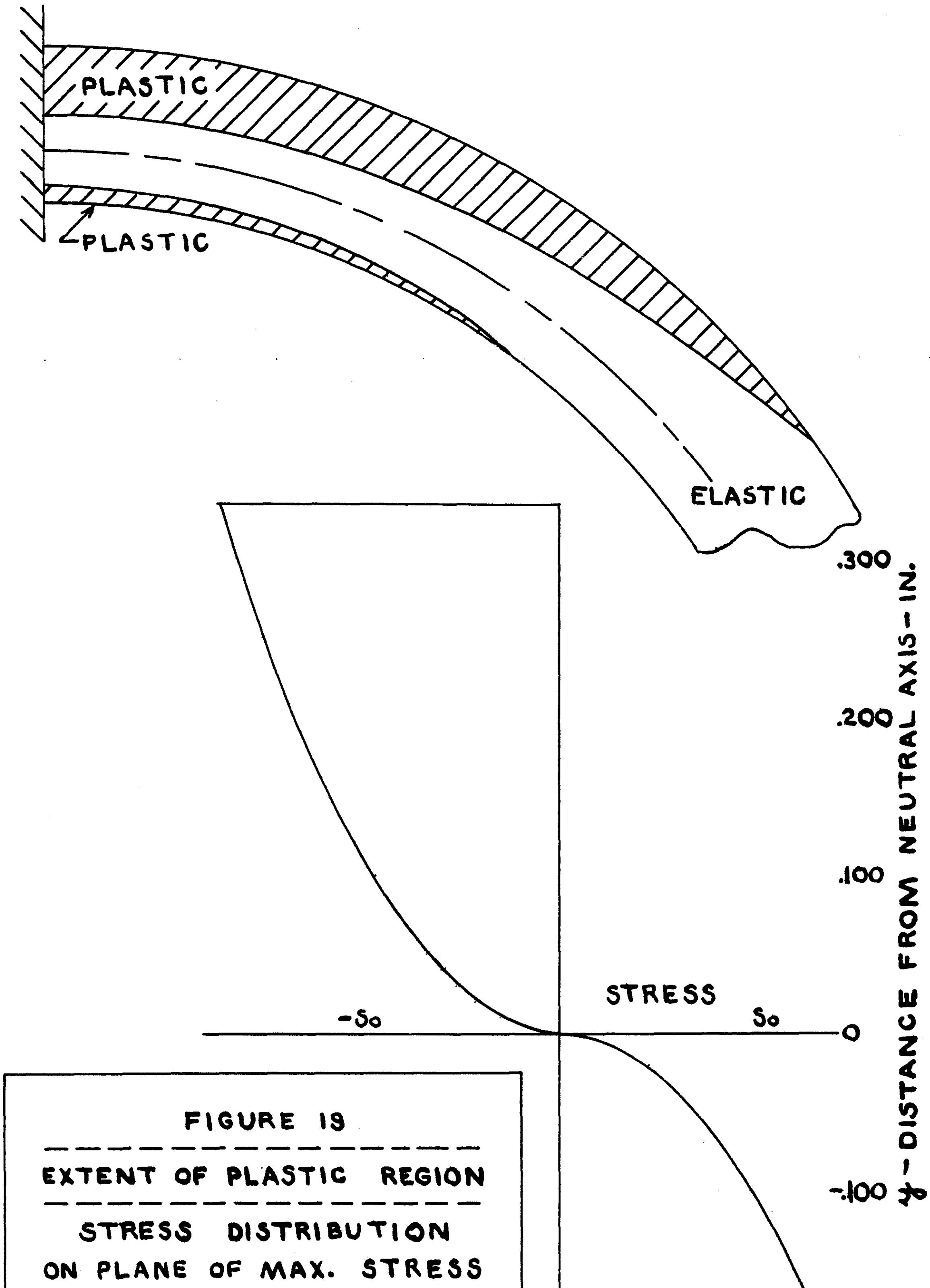


FIGURE 19

EXTENT OF PLASTIC REGION

STRESS DISTRIBUTION
ON PLANE OF MAX. STRESS

CASE III
P = 0.001902 So

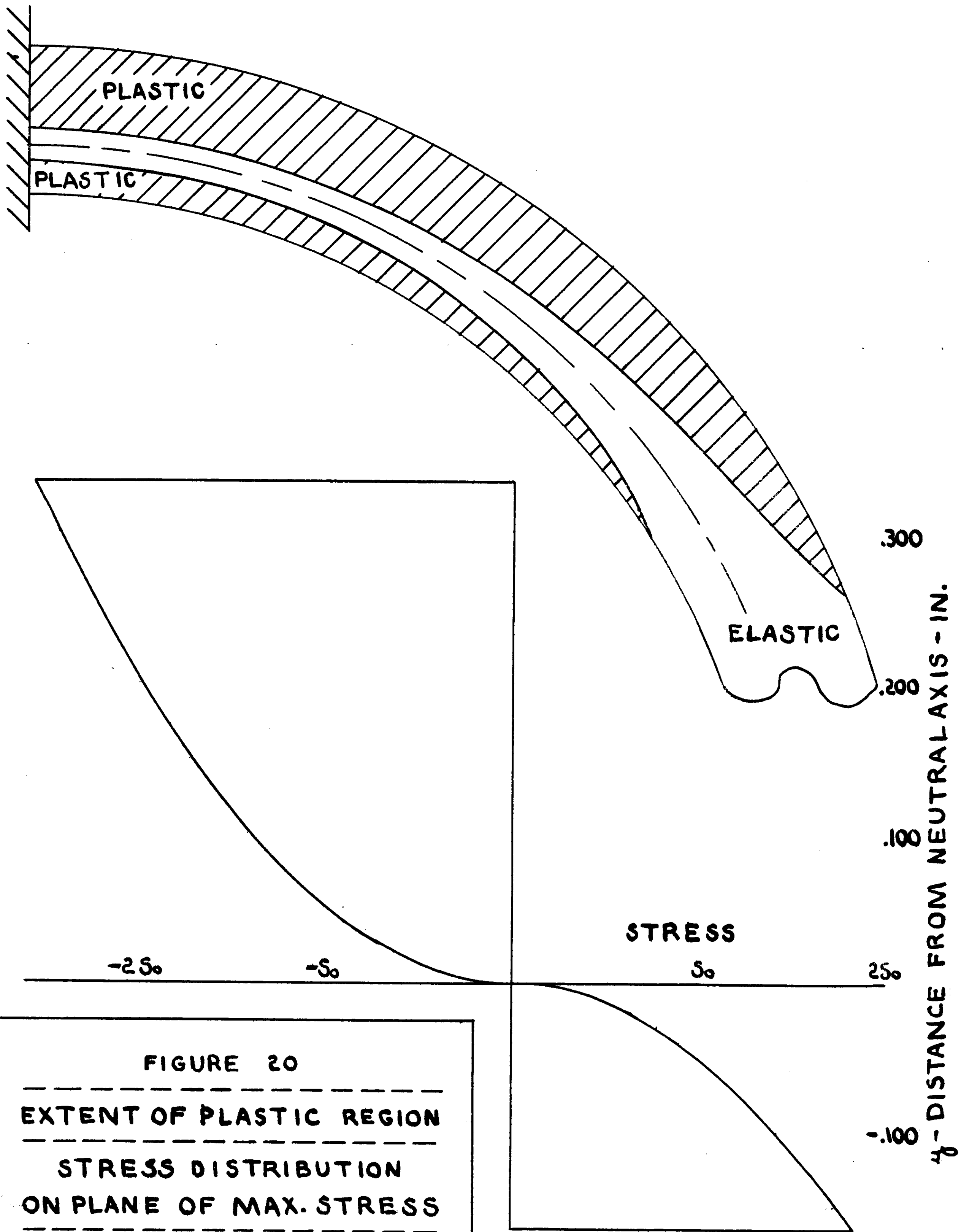


FIGURE 20

EXTENT OF PLASTIC REGION

STRESS DISTRIBUTION
ON PLANE OF MAX. STRESS

CASE III

$P = 0.002812\ S_0$

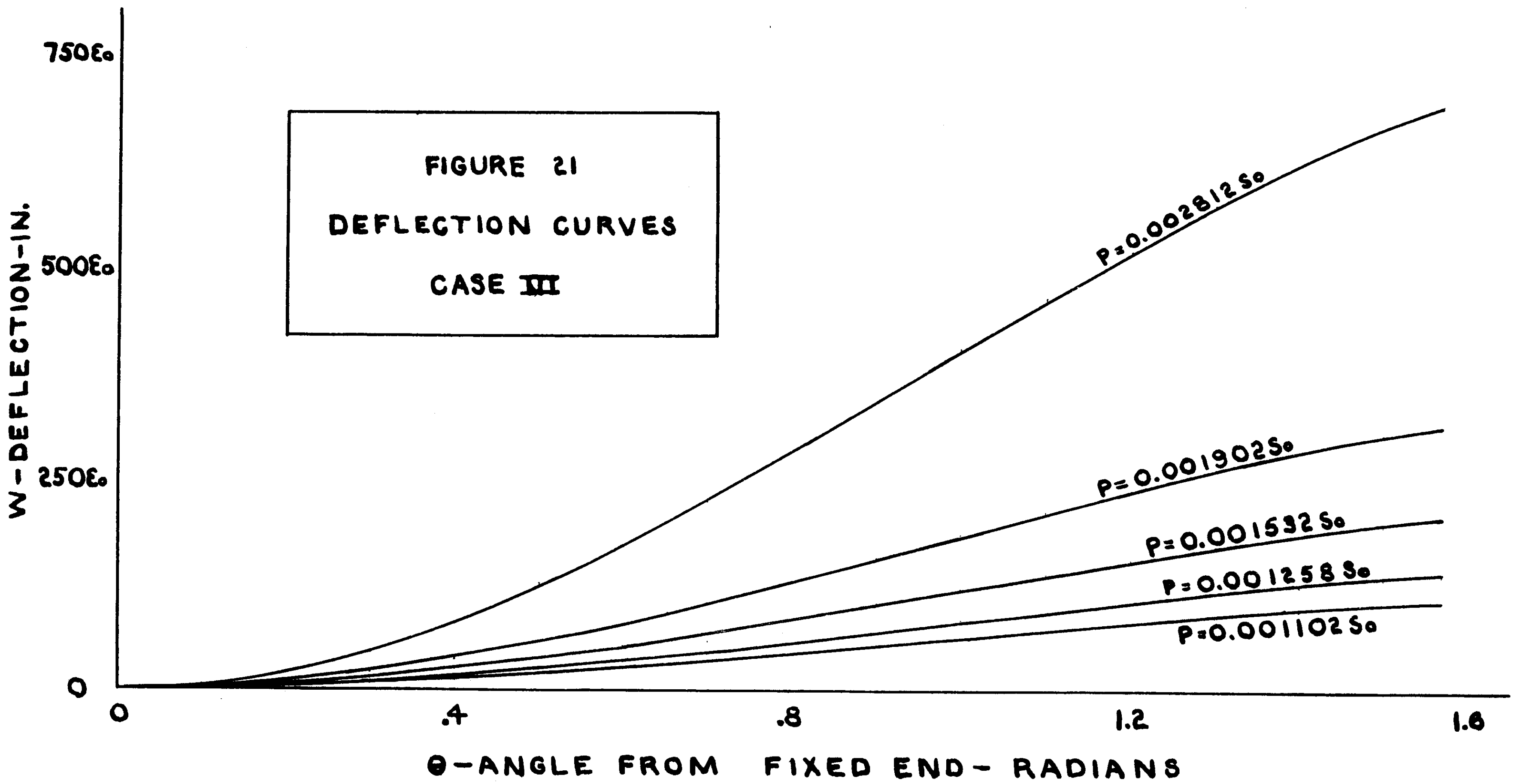


FIGURE 21
DEFLECTION CURVES
CASE III

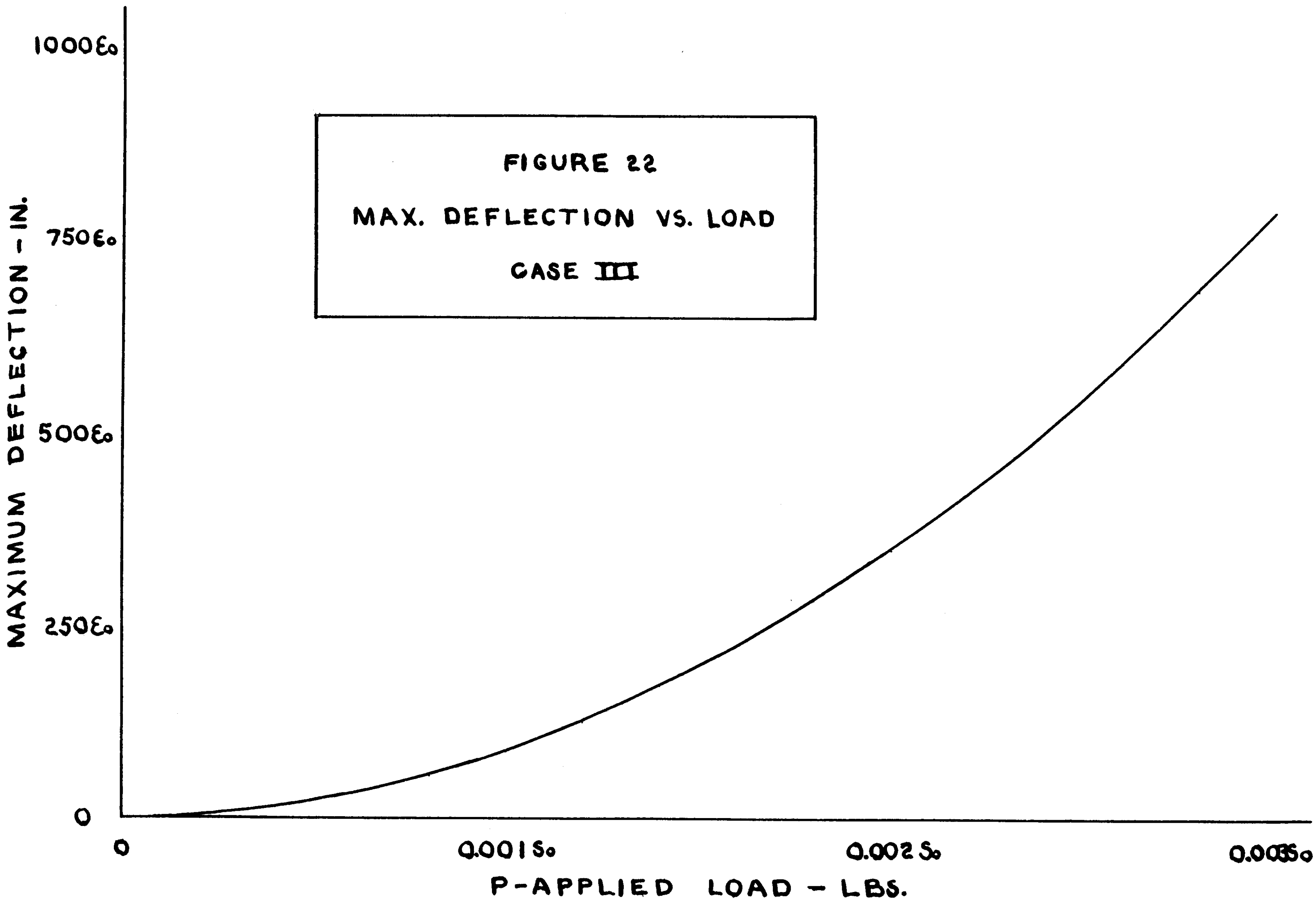


FIGURE 22

MAX. DEFLECTION VS. LOAD

CASE III

APPENDIX

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NOTES ON INTEGRATION

Integration of Equation 17a

$$\frac{d^2w}{d\theta^2} + w = -\rho_o \varepsilon_o \sqrt{1 + \frac{4e_o^2/3h^2}{1 - \frac{4Pe_o\cos\theta}{S_o bh^2}}}$$

This differential equation was integrated by the method of successive approximations using an interval in θ of 0.02 radians, which is an interval of very slightly over 1^o. The three point method was chosen. The basic reason for this choice was that since some approximations had been made at various points in arriving at the equation which was being integrated the accuracy was not sufficient to warrant the use of four or five point integration. Use was made of the integrating ahead coefficients to predict points in advance. The coefficients for three point integration are:

Δ_{-1}^{0}	Δ_{0}^{1}	Δ_{1}^{2}
0.41667	-0.08333	0.41666
0.66666	0.66666	-1.33333
-0.08333	0.41667	1.91667

At $\theta = \theta_1$, the equation

$$w' = C_1 \sin\theta + C_2 \cos\theta - \frac{Pe_o^3\,\theta \sin\theta}{2EI}$$

becomes effective. The constants, C_1 and C_2, are evaluated from the boundary conditions that when

$\theta = \theta_1$, $w = w'$, $dw/d\theta = dw'/d\theta$.

For the case where $P = 0.001236\ S_o$ lb, $\theta_1 = 37^\circ 17'$ which is 0.65 radians.

From numerical integration

$$w = -184.99\ \varepsilon_o$$

$$dw/d\theta = -518.03\ \varepsilon_o$$

Evaluation of C_1 and C_2

$$C_1 = -495.44\ \varepsilon_o$$

$$C_2 = 156.88\ \varepsilon_o$$

For the case where $P = 0.001396\ S_o$ lb, $\theta_1 = 45^\circ 13'$ which is 0.79 radians.

From numerical integration

$$w = -360.05\ \varepsilon_o$$

$$dw/d\theta = -752.02\ \varepsilon_o$$

Evaluation of C_1 and C_2

$$C_1 = -747.86\ \varepsilon_o$$

$$C_2 = 266.22\ \varepsilon_o$$

For the case where $P = 0.001493\ S_o$ lb, $\theta_1 = 48\ 47'$ which is 0.85 radians.

From numerical integration

$$w = -635.30\ \varepsilon_o$$

$$dw/d\theta = -1072.5\ \varepsilon_o$$

Evaluation of C_1 and C_2

$$C_1 = -1143.6\ \varepsilon_o$$

$$C_2 = 369.01\ \varepsilon_o$$

Integration of Certain Integrals in Cases I and II

Integrals containing terms of the form $\left(\frac{\pm y}{e_o + y}\right)^{\frac{1}{2}}$ can be readily integrated by substitution. For example, consider $\int y \left(\frac{-y}{e_o + y}\right)^{\frac{1}{2}} dy$

$$\text{LET } (-y)^{\frac{1}{2}} = x$$

$$-y = x^2$$

$$dy = -2x\,dx$$

$$e_o + y = e_o - x^2 \qquad \text{LET THIS} = X$$

$$X = a + bx + cx^2$$

$$a = e_o$$

$$b = 0$$

$$c = -1$$

$$\int y \left(\frac{-y}{e_o + y}\right)^{\frac{1}{2}} dy = 2 \int \frac{x^4\,dx}{\sqrt{X}}$$

This new integral is of a form which can be readily integrated by the use of any set of integral tables.