Static Replication of Exotic Options

by

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Abstract

In the Black-Scholes model, stocks and bonds can be continuously traded to replicate the payoff of any derivative security. In practice, frequent trading is both costly and impractical. Static replication attempts to address this problem by creating replicating strategies that only trade rarely.

In this thesis, we will study the static replication of exotic options by plain vanilla options. In particular, we will examine barrier options, variants of barrier options, and lookback options. Under the Black-Scholes assumptions, we will prove the existence of static replication strategies for all of these options. In addition, we will examine static replication when the drift and/or volatility is time-dependent. Finally, we conclude with a computational study to test the practical plausibility of static replication.

Thesis Supervisor: Michael F. Sipser

Title: Professor of Mathematics

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I am indebted to many people in the completion of this thesis.[1]

First and foremost, I would like to thank Robert Merton for inspiring me to study analytical finance. His classes on continuous time finance and capital markets taught me that the application of advanced mathematics to financial problems could be interesting, challenging, and even practical.

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Chapter 1

Introduction

In 1994, the municipality of Orange County, CA, declared itself bankrupt after $1.7 billion in losses. As a result, many public services from hospitals to schools had to adopt austerity measures. The next year, Barings, a major British bank, became insolvent after losing over $1 billion. A 233 year old institution that had helped finance the Napoleonic wars was forced to seek an outside savior. Both of these catastrophes involved the mismanagement of financial instruments known as derivatives. Such diasters beg the following questions: what are derivatives, why would anyone use them, and how did they cause so much damage.

A derivative is a contract whose value is derived from the behavior of an underlying real asset such as a stock, currency, or bond. In their more primitives forms, derivatives have existed for hundreds of years. The 17th century Amersterdam stock exchange (as described by de la Vega[19]) was rich in derivatives. However, the most explosive growth in derivatives has occurred just recently. In the past twenty-five years, the uses, types, and volume of derivatives has increased tremendously. This extraordinary growth is due, in large part, to revolutionary pricing and hedging strategies that were developed in the 1970's.

As with most things in life, if properly used, derivatives can be beneficial, and if abused, derivatives can wreck havoc. Derivatives allow investors and institutions to tailor their exposures in sophisticated manners. They allows entities to reduce their risks and manage their cash flows. However, derivatives can be used to create speculative positions. In some cases, such speculation is warranted for well-informed investors or managers seeking high returns. If taken to extremes, excessive speculation can create devastating downside potentials, where moderate changes in the underlying securities can create enormous losses in the corresponding derivatives. In both Orange County and Barings, individuals took extremely speculative positions. If their guesses would have been correct, they would have made huge gains (or made up huge losses). As it turned out, fate was not so kind.

The current widespread use of derivatives owes much to mathematical models that have been developed over the past twenty-five years. In 1973,[1] papers by Black and Scholes[5], and Merton[36] introduced a new method for analyzing derivatives. This method was based upon a mathematical model that, coincidentally also yields the heat equation as found in physics. Since then, their model has found multiple interpretations using methods from such diverse areas as combinatorics and measure theory.

What these theories did was provide pricing formulas and hedging strategies for derivatives. Today, large financial banks such as J.P. Morgan, Goldman Sachs, and Morgan Stanley uses these theories to manage their derivative portfolios. These banks buy/sell derivatives from/to their corporate, government, and individual clients. In general, the clients are reducing their risk exposures, which means the financial banks are assuming risk. The banks, in turn, employ hedging strategies to virtually eliminate this risk. Essentially, these banks are providing a service (i.e. a market for derivatives) and are compensated via commissions and/or transaction costs. From these activities, the banks bear little or minimal risk (if properly managed).

Hence, the importance of the recent mathematical models was to provide pricing formulas and hedging strategies. The traditional methods of Black, Scholes, and Merton have a serious drawback. Their dynamic trading strategies, theoretically, require continuous trading. Practically, such a strategy is obviously impossible. Some kind of a discretization is necessary, which results in hedging errors and exposures. Furthermore, frequent trading is highly undesirable due to transaction and monitoring costs.

In this thesis, we study a relatively new approach called *static replication*. The purpose of static replication is to avoid continuous trading and instead, only trade infrequently. Such an approach has its pros and cons over the dynamic method. In the following chapters, we will review dynamic methods and describe static replication strategies for some types of derivatives. In addition, we will explore the computational plausability of static replication. It is the hope and purpose of this thesis to present static replication as a viable alternative

[1] Coincidentally, trading began on the Chicago Board of Options (a major market for derivatives) the same year.

to dynamic replication. In certain situations and markets, static replication can be the best way to hedge a derivative exposure.

1.1 Organization of Thesis

The rest of the introduction describes options, which are a particular type of derivative. We introduce the pricing and hedging of options and give a simple example of arbitrage. Next, we give a brief description of the two main types of replication schemes (dynamic and static) and discuss previous work. Those readers familiar with option theory may wish to skip directly to Chapter 3.

Chapter 2 presents background material. It describes the Black-Scholes model and presents several derivations of the Black-Scholes option pricing formula. In addition, we give background terminology and briefly list alternative models.

Chapter 3 is the beginning of our contributions. We introduce the concept of static replication and derive static replication schemes for single barrier options. We present several different derviations, which we hope will provide additional intuition.

Chapter 4 expands static replication to barriers more complex than the single barrier. In particular, we examine partial barriers, forward-starting barriers, double barriers, and roll-down barriers. In addition, we show a decomposition of lookback options into barrier options. Hence, we can apply static replication techniques to lookbacks.

Chapter 5 examines static replication with time-dependent drift. We first show that barrier options with non-flat barriers and/or time-dependent volatility can be converted into equivalent barrier options with flat barriers and time-dependent drift. Under time-dependent drift, we demonstrate the impossibility of some simple static replication schemes and show the existence of more complicated static replications.

Chapter 6 is a computational study of static replication. We examine out-of-the-money barrier options and test their static replication under some simple scenarios. We also study the volatility senstivity of static replication. Chapter 7 concludes.

1.2 Options

Derivatives come in many different types including forwards, futures, swaps, and options. In addition, many instruments have imbedded derivatives such as callable bonds, convertible securities, and mortgage loans. In this thesis, we will focus on options. For further information on other types of derivatives, we suggest the following references: Hull[31] and Nelken[39].

1.2.1 Plain Vanilla Options

An option is a contract that one party sells to another. The owner has the *option* to execute some transaction within some time frame. For example, a *European call option* gives the owner the right to buy a stock at a given price (the strike) at some time in the future (the maturity). It is strictly a right, and not an obligation. If the market price is below the strike, the owner will not execute the transaction. On the other hand, if the market price is above the strike, the owner can buy the stock at the strike and immediately sell it in the market. Thus, the payoff of a European call option is (see Figure 1-1):

$$\max(S - K, 0)$$

where S is the stock price at maturity and K is the strike price. A European put option gives the owner the right to sell a stock at a given price at some time in the future. By analogy, the payoff of a European put option is (see Figure 1-1):

$$\max(K - S, 0).$$

European calls and puts are the simplest type of options and are often referred to as *plain vanilla* (or simply vanilla) options. Their payoff depends only upon the stock price at maturity.

1.2.2 Uses of Options

The main purpose of options is hedging. They can also be used for speculative purposes. Small changes in the underlying stock price can cause large changes in the option's value. In that sense, options can be interpreted as a highly leverged positon. Furthermore, options provide an indirect market for volatility. Market makers often quote option prices in terms of Black-Scholes volatility. This facet will become more apparent in Chapter 2.

In the classical hedging example, put options are used for downside protection. Suppose

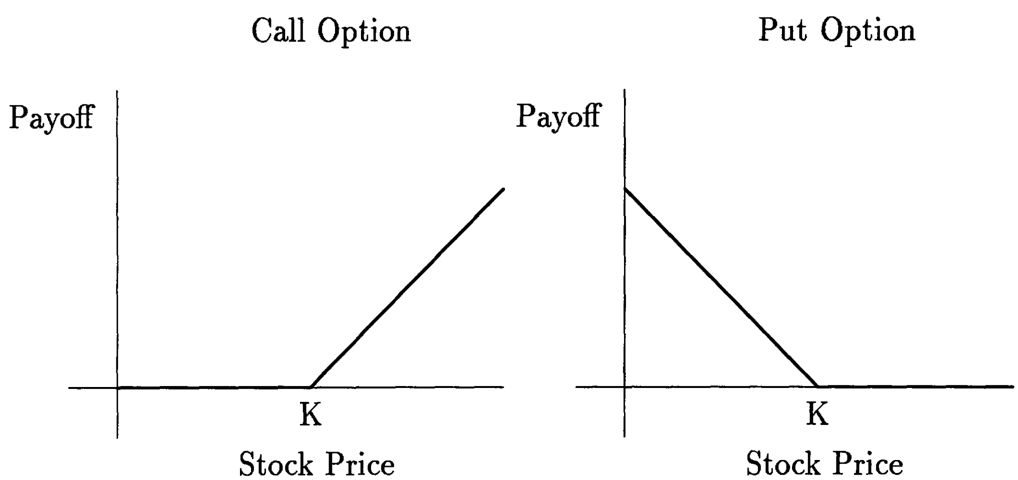

Figure 1-1: Payoffs for European Calls and Puts.

Carol is an investor in the stock market. Her money is in an index fund, and after the crash of 1987, she is concerned about the potential of another crash. She prefers the stock market over bonds, since she knows that the historical return is much greater. Of course, Carol realizes that the stock market is risky and is willing to bear some risk, but she would like to limit her losses to 10%.

One potential strategy is a stop-loss order. Suppose the price of Carol's fund is 100. If the price ever drops below 90, Carol will immediately sell. This strategy will limit Carol's losses to 10%. Carol can give this stop-loss order to her broker, and under normal market conditions, she will be protected. However, in a crash, Carol's order will probably not be executed at 90. The price will drop so fast, that Carol's broker will not be able to sell her portfolio at 90 and Carol could lose much more.

Carol really wants insurance against a crash. By buying a put option with a strike of 90, she will get her desired protection. In Figure 1-2, we illustrate the payoff of the index fund, the put option, and Carol's portfolio of the index fund and the put option. The index fund is shown in the upper left and consists of a straight line. The put option is non-linear payoff that has positive payoff when the stock price drops. The combined portfolio has limited downside, but unlimited upside. The cost of this insurance is the cost of the put option (pricing will be discussed later). This simple example[2] illustrates how options can

[2]Our discussion comparing stop-loss strategies and options is deceptively simplistic. Even with continuous

be used as insurance. Insurance is just one application of hedging with options. Many financial organizations have much more complicated exposures and will use options in far more sophisticated ways.

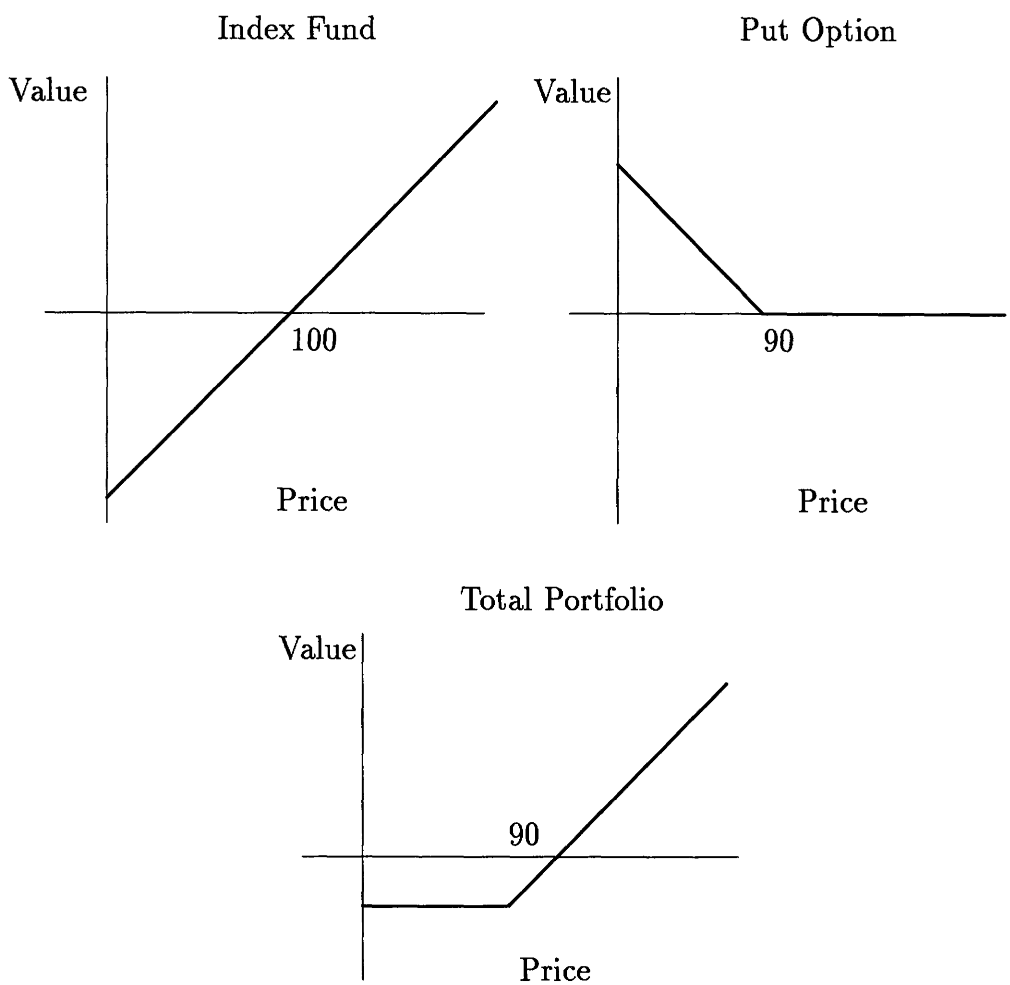

Figure 1-2: Value Profiles of Various Portfolios.

1.2.3 Exercise Types

The exercise of an option refers to the execution of the transaction specified by the option. Exercise types fall into three main catagories:

1. **European.** These options can only be executed on a fixed date.

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price movements and perfectly liquid markets, there are important differences between stop-loss (start-gain) strategies and options. For a more detailed discussion, see Carr and Jarrow[12].

2. **American.** These options can be executed at any time up to the expiration date, if any. Perpetual American options are those that never expire.

3. **Multi-European.** These options fall between American and European options. The owner may execute at a fixed set of exercise dates.

Of these different types, European options are the simpliest and the best understood. For both European and Multi-European calls and puts, closed form solutions exist. Currently, no closed form solution exists for American options. This question is still an active area of research as seen in Broadie and Detemple[6] and Carr[8]. In this thesis, we will exclusively focus on European options.

1.2.4 Exotic Options

Beyond calls and puts, a wide variety of other options exists. Collectively, these options are called *exotic options.* In this section, we briefly describe some of the various types.

- **Digitals.** These options are similar to European calls and puts. At maturity, they pay \$1 if the stock price is above a certain level and pay zero otherwise.

- **Binaries.** These options are similiar to digitals, except that they pay \$1 if the stock price ever goes above a certain level during the life of the option.

- **Barriers.** These options have an associated barrier. If the stock price ever reaches the barrier, the option is altered. If the barrier is never reached, the option retains its original character. A simple example is a knock out call option. Initially, the option is identical to an European call. However, if the barrier is ever reached, the option knocks out and becomes worthless.

- **Lookbacks.** The payoff of these options is a function of the maximum or minimum price realized during the life of the option. For example, a lookback put pays the difference between the maximum realized price and the price at maturity.

- **Asians.** The payoff of these options depend upon the average (arithmetic or geometric) stock price during the life of the option. For example, one type of Asian option pays the average price over the life of the option.

An important feature of many exotic options is path-depedency. Plain vanilla options are path-independent. Their payoff only depends upon the price at maturity. Except for digitals, all of the above options are path-dependent. American options are also, in general, path-dependent.

1.3 Pricing

In this section, we introduce the pricing and hedging of options. We will describe the basic theory, which was first presented in Black and Scholes[5] and Merton[36]. Building upon this idea, we will introduce the central theme of this thesis: *static replication.*

1.3.1 Arbitrage Pricing

The most fundamental question about options is: what should their price be? Prior to 1973, most models used the economic concept of equilibrium to determine price. The price was determined by supply and demand. The equilibrium price was the price that cleared the market by creating an equal number of buyers and sellers. To find the point of equilibrium, we must first determine investors' demand and supply for options. At what price would a rational investor want to buy/sell an option? From this viewpoint, two factors are critical. First, what does the investor expect the option to be worth? The investor has some probability distribution about the underlying stock price and uses that to compute a payoff distribution. Second, what are the investor's risk preferences? Most investors are risk averse and are willing to trade some expected value for protection against extreme movements.

In 1973, Black, Scholes, and Merton introduced the concept of *arbitrage pricing.* One of the amazing implications of this model was that the two previous fundamentals for determining price, investor expectations and risk aversion, are irrelevant! This result was so unusual, that most economists had difficulty accepting the Black, Scholes, and Merton approach. In fact, Black, Scholes, and Merton had to cast their results in an equilbrium model in order to get them published.

The driving force behind the Black-Scholes model is the preclusion of arbitrage. Arbitrage corresponds to a free lunch. It literally means a non-zero probability of gain with no chance of loss and no initial investment. A trivial example of arbitrage is as follows. Suppose one US dollar (USD) is worth 1.5 German Deutschmarks (DM) and one USD is worth 105 Japanese Yen. Then, it must be that one DM is worth $105/1.5 = 70$ Yen. Otherwise, by trading the various currencies, an investor could make unbounded, riskless profits.

In Black-Scholes option pricing, arbitrage takes the following form. Starting with an initial portfolio of the underlying stock and bonds, we will give a self-financing trading strategy such that the portfolio will exactly replicate the payoff of the option at maturity. A self-financing strategy is one that uses only internal funds without any capital inflows or outflows. Since we perfectly match the option payoff, the price of the option must, at all times, match the price of the replicating portfolio; otherwise, there would be an arbitrage opportunity. Since the portfolio consists of fundamental securities, we can always price the portfolio, and hence the option. In the following, we illustrate another simple example of arbitrage. The complete Black-Scholes argument is given in Chapter 2.

1.3.2 Forward Arbitrage

A forward contract is a simple type of derivative. It is an agreement to purchase an item at a future date at the forward price. There is no option: the parties must execute the transaction on the given date at the stated price. Furthermore, the forward price is set, so that forward contract is worth zero at initiation. For example, consider a forward contract on gold. Suppose the current price of gold G_0 is \$100 per ounce. For a one-year forward contract, what should the forward price be? Let K denote the forward price, then the payoff of the forward contract is:

$$G_1 - K$$

where G_1 denotes the price of gold one year from now.

Before we can determine the forward price by arbitrage, we first need some assumptions. We will assume there are no credit issues. Both sides of the forward contract have excellent credit rating and there is no probability of default. Thus, both parties can borrow and lend at the riskfree interest rate r, which we assume is 10% per year. Furthermore, we will assume that gold can be held for a year at zero cost. Security and/or storage costs are negligible.

We can construct a portfolio (consisting of gold and riskfree bonds) that will exactly match the payoff of the forward contract. Let our portfolio be:

- Buy one ounce of gold.
- Short one year bonds with a face value[3] of K.

In one year, the value of this portfolio will be $G_1 - K$, which exactly matches the forward contract. Thus, if we sold a forward contract and hedged with the above portfolio, our payoff in one year would be zero, regardless of the future price of gold. Since the forward contract cost zero, the portfolio must also be worth zero. To prevent arbitrage, a portfolio that has zero payoff in the future must be worth zero today. At initation, the price of the portfolio is:

$$G_0 - \frac{K}{1+r} = 100 - \frac{K}{1.1},$$

since the bonds must be discounted by the riskfree rate. Therefore, the forward price K is \$110.

We have shown the forward price must be \$110. The only information we used was the current price and the riskfree interest rate. Observe what information is conspicuously absent: investor expectations about the price of gold and risk preferences. Two parties may completely disagree about what will happen to price of gold, yet they must agree upon the forward price. Carol may think that gold is a great buy, and gold will be over \$200 a year from now. Ana, another investor, may think gold is a terrible buy and gold will be under \$70 in a year. Yet, both of them would agree that the forward price is \$110. Their expectations are irrelevant. Similarly, their risk preferences have no influence on the price.

Forward arbitrage is one of the simplest types of arbitrage. The Black-Scholes methodology applies the same idea to replicate option payoffs. However, the replicating strategy becomes more complicated. It requires continuous rebalancing of the portfolio. In forward arbitrage, the portfolio requires virtually no rebalancing. Only at initiation and maturity does the portfolio need to be rebalanced.

1.4 Types of Replication

The Black-Scholes replication of options uses a strategy of continuous rebalancing the underlying stock and riskless bonds. This technique can be used to price and hedge both

[3]The face value of a bond is how much it pays at maturity.

plain vanilla and exotic options. This type of replication is called *dynamic*, since it requires continuous rebalancing.

The central theme of this thesis is *static replication*. Static replication is replication with very few trades. In particular, we will focus on replicating exotic options with plain vanilla options. The advantage of this approach is that our portfolio does not need to be continuously rebalanced. Instead, our rebalancing is event-driven. Upon the occurence of certain events, our portfolio will be rebalanced. In later chapters, we will examine both types of replication in more detail. For now, we summarize

1. **Dynamic Replication.** Uses the underlying stock and bond as replicas and requires continuous rebalancing. Can be applied to all types of options.

2. **Static Replication.** Uses plain-vanilla options as replicas and requires event-driven rebalancing, which is rare in most cases. Is applicable to certain types of exotic options.

1.5 Previous Work

Option pricing theory can trace its origins back to Louis Bachelier's 1900 dissertation[1] on the theory of speculation. As those in the finance profession are proud to point out, Bachelier derived the basic mathematics of Brownian motion five years before Einstein's derivation in 1905. Unfortunately, this work was lost for over half a century.

In 1973, modern option theory was born. Independent of Bachelier's work, Fischer Black, Myron Scholes, and Robert Merton published their seminal works ([5], [36]). Today, these ideas are well-studied, and many excellent textbooks are available (such as Hull [31], Merton [38] and Wilmott *et al* [44]). Starting in the late 1970's, exotic options were studied intensively in several articles (e.g., [24], [23], [3] and [30]). For a more complete survey, we suggest the following to references: Nelken[39], Rubinstein[42], and Zhang[45].

Static replication was introduced by Bowie and Carr[7] and Derman, Ergner, and Kani([17], [18]). Bowie and Carr examined single barrier static replication under the condition that the interest rate equals the dividend rate. Derman *et al* created an algorithm for hedging single barriers in a binomial model. Carr, Ellis, and Gupta[11] extended these results to a symmetric volatility structure and several other instruments.

The contributions of this thesis are as follows. We study static replication in the more general case where the interest rate differs from the dividend rate. In doing so, we introduce several new techniques for determining static replication strategies. Furthermore, we examine some new structures beyond Carr, Ellis and Gupta and improve the static replication schemes for other instruments. Some of schemes in Carr *et al* require exotic options; all of our schemes exclusively use plain vanilla options. We also extend static replication to time-dependent drift (and/or volatility) and perform computational studies on the practical plausibility of static replication.

Chapter 2

Background

In this chapter, we present background material regarding the Black-Scholes model and Arrow Debreau securities. The presentation of the Black-Scholes model serves two purposes. It provides a summary of the mathematical approaches used in option's pricing, and it gives many of the necessary tools for understanding future chapters. Although static replication differs from the dynamic approach, they still have many connections. Arrow Debreau securities are a basic method of replication. They represent a fundamental decomposition of European options.

2.1 The Black-Scholes Model

The most celebrated formula in mathematical finance is the Black-Scholes formula for pricing options. It has tremendous theoretical and practical implications. A entire new line of research was created, and literally every financial institution that deals with options uses some variant of the Black-Scholes method. In this section, we present the Black-Scholes model and look at several of the various interpretations.

2.1.1 Assumptions

The Black-Scholes model is based upon the following set of assumptions. For simplicity, we will assume the underlying instrument is a stock.

1. The market for both stocks and bonds is always open. There are no transaction costs and continuous (in time) trading is possible. In addition, there is full divisibility of stock and bond units.

2. There are no credit issues. Short sales are permitted along with full use of proceeds. Investors can borrow or lend via the bond market.

3. The stock price S follows a geometric diffusion process:[1]

$$dS/S = A(S,t)dt + \sigma dZ \tag{2.1}$$

where $A(S,t)$ is an arbitrary bounded function, σ is a constant and dZ is a Wiener process.[2] If $A(S,t) = \alpha$, then S follows geometric Brownian motion.

4. The interest rate for bonds is a constant r, which is continuously compounded. In addition, the stock pays a continuous dividend rate ρ.

Within this framework, we have the necessary tools to price options. We will use the assumption of no arbitrage to derive the Black-Scholes pricing formula. There are multiple derivations of this formula, and we will present the three most important: the differential equation method, the binomial model, and the risk neutral probability measure. Our presentation will follow the historial development. The original derivation in 1973 used differential equations. In 1979, Cox, Ross, and Rubinstein[16] proposed the binomial model, which introduced the risk neutral probabilities. Harrison, Kreps, and Pliska (see [26], [27], and [28]) subsequently formalized this notion using measure. In this thesis, we will present a simplified sketch of the various interpretations.

2.1.2 Differential Equation Method

The differential equation method was the original method used to derive the Black-Scholes formula. Our presentation is based upon those given in Hull[31] and Merton[38].

We will begin by presenting a slightly informal derivation. In doing so, we will make additional assumptions, which will make the derivation more intuitive. Later, we will show how the derivation can be done directly without these additional assumptions.

[1] For those unfamiliar with the notation, it is really quite simple. We are writing the percentage change (dS/S) as the sum of deterministic drift component $A(S,t)dt$ and a random component σdZ.

[2] A Wiener process dZ is the limiting process (as $dt \to 0$) of $\epsilon\sqrt{dt}$ where ϵ is normally distributed (with mean zero and standard deviation one) and $\sqrt{dt}$ is a scaling factor. Note that a Wiener process is Markov. In addition, Wiener processes have many other interesting properties. For an introduction to Wiener processes, we suggest Chapter 9 of Hull[31].

Suppose we have a European option C with maturity T and payoff $V(S)$. Since the underlying process is Markov in S and t and the payoff only depends upon S, we can specify the price of the option as $C(S,t)$. We know at maturity that:

$$C(S,T) = V(S). \tag{2.2}$$

Assuming C is twice-differentiable, we can apply Ito's Lemma,[3] to obtain:

$$dC = \frac{1}{2}C_{SS}(dS)^2 + C_S dS + C_t dt \tag{2.3}$$

where $C_{SS} = \frac{\partial^2 C}{\partial S^2}$, $C_S = \frac{\partial C}{\partial S}$, and $C_t = \frac{\partial C}{\partial t}$. From (2.1),

$$dC = \frac{1}{2}C_{SS}S^2\sigma^2 dt + C_S S A(S,t)dt + C_S S\sigma dZ + C_t dt \tag{2.4}$$

Now, suppose we have a portfolio P consisting of

- 1 option
- w shares of the stock (w will be specified later)

The value of our portfolio is:

$$P = C + wS \tag{2.5}$$

The dynamics of P are:

$$dP = dC + wdS + wS\rho dt \qquad (wS\rho dt \text{ is from dividends}) \tag{2.6}$$

$$= [\frac{1}{2}C_{SS}S^2\sigma^2 + C_S S A(S,t) + C_t + wSA(S,t) + wS\rho]dt + [C_S S\sigma + wS\sigma]dZ \tag{2.7}$$

Recall that we get to choose w. Let $w = -C_S$. Then, (2.7) reduces to:

$$dP = [\frac{1}{2}C_{SS}S^2\sigma^2 + C_t - C_S S\rho]dt \tag{2.8}$$

Thus, the change in P is completely deterministic. We have chosen w to eliminate the

[3] Ito's Lemma is the fundamental rule for differentiating stochastic processes. Equation (2.3) is essentially the statement of Ito's Lemma. In addition, the following multiplication rules apply: $(dZ)^2 = dt$, $(dt)^2 = 0$, and $dZdt = 0$. For further discussions, we suggest Chapter 3 of Merton[38].

stochastic component dZ. Therefore, this portfolio must change at the riskless rate. In other words,

$$dP = rPdt \tag{2.9}$$

$$\implies \tfrac{1}{2}C_{SS}S^2\sigma^2 + C_t - C_S S\rho = rC - rC_S S \tag{2.10}$$

Rearranging and noting boundary conditions, we have

$$\tfrac{1}{2}C_{SS}S^2\sigma^2 + (r-\rho)C_S S + C_t = rC \tag{2.11}$$

$$C(S,T) = V(S) \tag{2.12}$$

The preceding equation is the Black-Scholes differential equation. It is identical to the heat equation from physics. Fortunately, this equation has been extensively studied and the solutions for many initial and boundary conditions are known. In particular, the solution for a European call option where $V(S) = \max(S-K,0)$ is:

$$Se^{-\rho(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{2.13}$$

$$\text{where} \qquad d_1 = \frac{\ln(S/K)+(r-\rho+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$\text{and} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

$N(\cdot)$ is the cumulative normal distribution with mean zero and standard deviation one. For a European put option, the Black-Scholes price is given by:

$$Ke^{-r(T-t)}N(-d_2) - Se^{-\rho(T-t)}N(-d_1) \tag{2.14}$$

At this point, we would like to make a few comments about the preceding derivation. First, note that $A(S,t)$ never enters (2.11) or (2.12). Therefore, the true drift of the stock process can never be part of the pricing formula as seen in (2.13) and (2.14). This fact is consistant with our claim that investor expectations are irrelevent. However, we do require agreement upon σ. Second, we made the assumption that $C(S,t)$ existed and that it was twice differentiable. Finally, our arbitrage argument was that a portfolio with no risky component must grow at the riskfree rate. This reasoning differs slightly from our previous arbitrage arguments, but it is, in fact, equivalent. We now present a slightly modified derivation that does not require an additional assumption and uses a self-financing trading strategy in the arbitrage argument.

Given the differential equation (2.11) and boundary condition (2.12), we find a solution $C(S,t)$. At $t=0$, we form a portfolio P with initial wealth $C(S,0)$. Our trading strategy is as follows:

- Always hold $C_S(S,t)$ shares of stock.
- Invest all remaining wealth in riskless bonds.

Let's examine the dynamics of our portfolio. We hold C_S shares of stock and $P - C_S S$ dollars in bonds. Therefore,

$$
\begin{aligned}
dP &= C_S dS + C_S \rho S dt + (P - C_S S) r dt && (2.15)\\
&= [C_S A(S,t)S + C_S \rho S + (P - C_S S)r]dt + C_S \sigma S dZ && (2.16)
\end{aligned}
$$

Since C satisfies (2.11), it is twice differentiable and we can apply Ito's Lemma to describe its dynamics (which are given in (2.3)).

Let's define a new variable $Q = P - C$, which represents the deviation of P from C. The dynamics of Q are:

$$
\begin{aligned}
dQ &= dP - dC && (2.17)\\
&= [C_S A(S,t)S + C_S \rho S + (P - C_S S)r]dt + C_S \sigma S dZ \\
&\quad -[\frac{1}{2}C_{SS}S^2\sigma^2 + C_S S A(S,t) + C_t]dt - C_S S \sigma dZ && (2.18)\\
&= rPdt - [\frac{1}{2}C_{SS}S^2\sigma^2 + (r-\rho)C_S S + C_t]dt && (2.19)\\
&= r(P - C)dt && (2.20)\\
&= rQdt && (2.21)
\end{aligned}
$$

Thus, $dQ = rQdt$ is an ordinary differential equation with solution:

$$Q = Q(0)e^{rt} \tag{2.22}$$

By construction, $Q(0) = 0$, so we have $Q = 0$. Thus, P perfectly tracks the value of C. In particular, P will match C at maturity, so our portfolio will exactly match the option payoff by the initial conditions. Therefore, arbitrage restrictions imply that the option value is P.

This derivation is technically superior to the first. However, it requires us to "guess" the Black-Scholes differential equation. This concludes our discussion on the differential equation method. All options must satisfy (2.11). Different options are specified by their initial and/or boundary conditions.

2.1.3 Binomial Model

The binomial model was introduced in 1979 as a discretization of the general stochastic process described in (2.1). This method has important practical applications, since it can provide numerical solutions. For those unfamiliar with stochastic processes or differential equations, this method provides a nice combinatorial interpretation of the Black-Scholes model.

We begin with a two period model. Suppose the current (period 0) stock price S is 100. In period 1, the stock price can either be uS or dS where $u > 1$ and $d < 1$ (see Figure 2-1). For concreteness, we let $d = 1/u$ and set $u = 1.25$. Denote the state where the stock price ends at $uS = 125$ as the *up* state. The *down* state is when the price ends at $dS = 80$. We also assume the interest rate is $r = 5\%$ between periods and the stock pays no dividends.

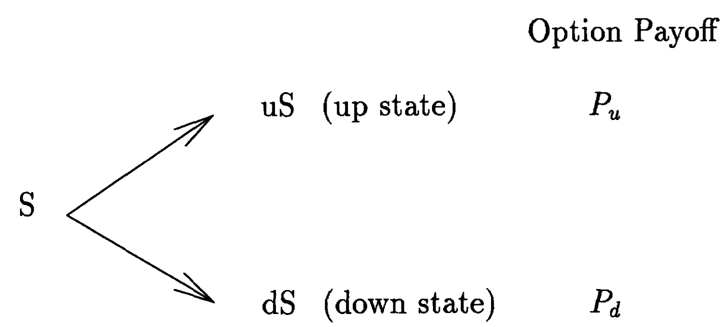


Figure 2-1: One Period Binomial Model.

Our goal is to price to a European call C with strike $K = 100$ which matures in period 1. The payoff of the European call is either 25 or 0. We will try to create a portfolio that matches this payoff. Let P consist of:

- x shares of stock
- Bonds with face value y

In both states, we want our portfolio to match the option payoff. In the up state,

$$P_U = uSx + y = 125x + y = 25 \tag{2.23}$$

Similiarly, for the down state,

$$P_D = dSx + y = 80x + y = 0 \tag{2.24}$$

We have linear system of equations, which we can solve with $x = 5/9$ and $y = -400/9$. Thus, we have a replicating portfolio. In period 0, the option C is worth

$$C = Sx + \frac{y}{1+r} = (5/9)(100) + \frac{-400/9}{1.05} = 13.23 \tag{2.25}$$

Observe that we never used the probability of entering the up or down state. This fact is similiar to the absence of $A(S,t)$ in (2.11) and (2.12). Investor expectations are irrelevant to option pricing. However, our choice of u and d is pertinent (which is analgous to the choice of σ in (2.1)).

In fact, if we solve (2.23) and (2.24) symbolically, we have:

$$x = \frac{P_u - P_d}{S(u-d)}, \; y = \frac{uP_d - dP_u}{u-d} \tag{2.26}$$

and

$$C = \frac{1}{1+r}\left[P_u\left(\frac{(1+r)-d}{u-d}\right) + P_d\left(\frac{u-(1+r)}{u-d}\right)\right] \tag{2.27}$$

We let $p = \frac{(1+r)-d}{u-d}$ and $q = \frac{u-(1+r)}{u-d}$, Note that $p + q = 1$. Arbitrage restrictions[4] require $u > 1 + r > d$. Therefore, p and q resemble probabilities and are called the ***risk-neutral probabilities.*** Using the risk-neutral probabilties, the expected stock price is:

$$p(uS) + q(dS) = \frac{(1+r)-d}{u-d}uS + \frac{u-(1+r)}{u-d}dS = (1+r)S \tag{2.28}$$

Thus, the expected stock return (under the risk-neutral probabilities) equals the riskfree

[4]The return of the stock can neither dominate nor be dominated by the riskless return. For example, if the stock strictly dominated the riskless return, one can create arbitrage by longing the stock and shorting bonds in equal dollar amounts. Such a portfolio costs zero today and is guaranteed to have positive future value. The symmetric argument applies if the stock return is dominated.

rate (which is why these probabilities are called risk-neutral). Rewriting (2.27), we have

$$C = \left(\frac{1}{1+r}\right)[pP_u + qP_d] \tag{2.29}$$

In other words, C is the discounted expected value of the future payoffs, where the expected value is computed using the risk-neutral probabilities. This computational trick gives a simple, intuitive method to price options. Note that the risk-neutral probabilities are derived by arbitrage arguments. They are completely artificial probabilities.

The next step is to extend the binomial model to multiple periods (see Figure 2-2). Since stock movements are Markov, the tree recombines, and the total number of nodes is polynomial in the number of time steps. At each level, we can repeat the previous argument and assign risk-neutral probabilities to every branch. If u and d are the same throughout the tree, the risk-neutral probabilites are consistent throughout the tree. Our risk-neutral distribution from the start of the tree to the leaves will be the binomial distribution. Using this distribution, European option prices are the discounted expected value of the payoff at maturity. For example, suppose we have an n period tree. The stock price at the leaves will be $S_F = u^k d^{n-k} S$ for $k = 0, \ldots, n$ and the payoff of the call is $\max(S_F - K, 0)$. Thus, the call option will have a price of:

$$C = \left(\frac{1}{1+r}\right)^n \sum_{i=0}^{n} \binom{n}{k} p^k q^{n-k} \max(u^k d^{n-k} S - K, 0) \tag{2.30}$$

where $\left(\frac{1}{1+r}\right)^n$ is the discount factor and $\binom{n}{k} p^k q^{n-k}$ is the risk-neutral probability.

To derive the Black-Scholes formula, we need to take the limit of the tree as it approaches the diffusion process in (2.1). For simplicity, let's assume $A(S,t) = \alpha$ is a constant. We introduce $\hat{p}, \hat{q}$ as the "real" probabilities of the corresponding up and down moves. The real probabilities are necessary to match the diffusion process.

Let dt corresponds to the time between steps in the tree. Then $n = T/dt$, where T is the time to maturity. We choose:

$$u = e^{\sigma\sqrt{dt}}, d = 1/u \tag{2.31}$$

$$\hat{p} = \tfrac{e^{\alpha dt} - d}{u - d}, \hat{q} = 1 - \hat{p} \tag{2.32}$$

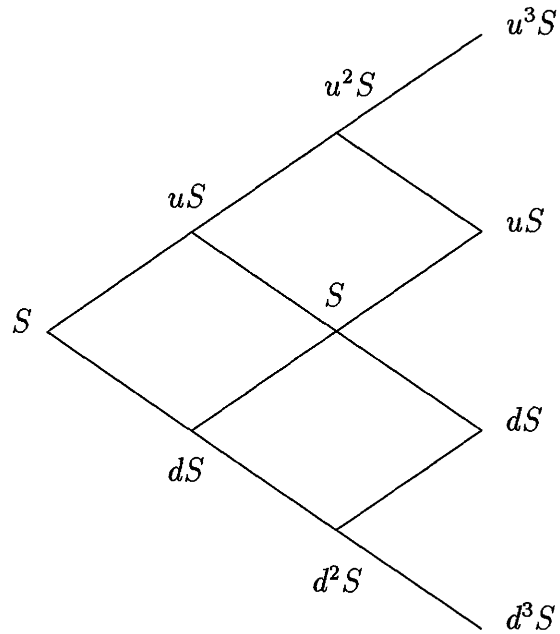


Figure 2-2: Multi-Period Binomial Model.

Then, the expected drift is

$$\hat{p}uS + \hat{q}dS = e^{\alpha dt}S \tag{2.33}$$

and the variance is

$$\hat{p}u^2S^2 + \hat{q}d^2S^2 - (e^{\alpha dt}S)^2 = (e^{\alpha dt}(e^{\sigma\sqrt{dt}} + e^{-\sigma\sqrt{dt}}) - e^{2\alpha dt} - 1)S^2 = \sigma^2S^2dt + o(dt) \tag{2.34}$$

where $o(\cdot)$ denotes higher order terms. In the limit as $dt \to 0$, we see that the instantaneous expectation and variance match those in the diffusion process.[5] Using the above limiting process, we find that (2.30) becomes (2.13).[6]

2.1.4 Risk Neutral Probability Measure

In the preceding model, we introduced the risk-neutral probabilities. This clever observation was formalized in a series of papers by Harrison and Kreps[26] and Harrison and Pliska[27]. A detailed discussion of these results are beyond the scope of this thesis. Instead, we will

[5] For a more detailed proof of convergence, see [21], [31], or [32].

[6] As a further technical detail, we would need to include dividends in the binomial model.

simply state the relevent result.

Recall that in the Black-Scholes model, the stock price follows the diffusion process given by (2.1). This diffusion corresponds to probability distribution (measure) of the future stock prices. The results of Harrison and Kreps state there exists an equivalent[7] measure, in which the price of all options is simply their discounted expected value in this new measure. In particular, this new measure can be described by the following diffusion:

$$dS/S = (r - \rho)dt + \sigma dZ^* \tag{2.35}$$

where r is the riskfree interest rate and ρ is the dividend rate.

This process is simply geometric Brownian motion with a drift of $r - \rho$. Since geometric Brownian motion follows a lognormal distribution, the distribution is:

$$p(S_T, S, T) = \frac{1}{S_T\sqrt{2\pi\sigma^2 T}} \exp\left[-\frac{(\ln(S_T/S) - (r - \rho - \frac{1}{2}\sigma^2)T)^2}{2\sigma^2 T}\right] \tag{2.36}$$

where $p(S_T, S, T)$ is the probability distribution of starting at S at time 0 and ending at S_T at time T.

This method gives us an incredible tool for pricing options. To price an option, we assume the stock price follows the process given in (2.35) and then calculate discounted expected value. The diffusion in (2.35) can be completely different from the true diffusion in (2.1), but we will, nevertheless, get the arbitrage-free option price.

For example, we can apply this method to price a call option. Let S be the current price, T be the time to maturity, and K be the strike. Then, the price of the call is:

$$\begin{aligned} C &= e^{-rT}\int_0^\infty p(S_T, S, T)\max(S_T - K, 0)dS_T && (2.37)\\ &= e^{-rT}\int_K^\infty p(S_T, S, T)(S_T - K)dS_T && (2.38)\\ &= Se^{-\rho T} N\left(\frac{\ln(S/K) + (r - \rho + \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\ &\quad -Ke^{-rT} N\left(\frac{\ln(S/K) + (r - \rho - \sigma^2/2)T}{\sigma\sqrt{T}}\right) && (2.39)\end{aligned}$$

[7] An equivalent measure is one which preserves null sets (i.e. those sets with measure zero).

2.2 Black-Scholes Terminology

Option theory has its unique language of terminology and jargon. In the following, we define some of the more common terms. In later chapters, we will use some of these terms.

- **The Greeks.** This term collectively refers to a portfolio's sensitivity to changes in various parameters. Each sensitivity is associated with a Greek letter.[8] Let Π denote the price function for a portfolio.

  - **Delta** – sensitivity of portfolio to changes in the underlying stock.

    $$\Delta = \frac{\partial \Pi}{\partial S}$$

    where S is the underlying stock price. Note that Δ corresponds to the number of shares held in the Black-Scholes replicating portfolio in section §2.1.2.

  - **Gamma** – sensitivity of delta to changes in the underlying stock.

    $$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 \Pi}{\partial S^2}$$

    Gamma is a measure of how fast delta changes. In practice, gamma is sometimes used to refer to second order and higher changes.[9]

  - **Vega** – sensitivity of portfolio to changes in volatility. Λ (an upside down V) is often used to represent Vega.

    $$\Lambda = \frac{\partial \Pi}{\partial \sigma}$$

  - **Theta** – sensitivity of portfolio to changes in time.

    $$\Theta = \frac{\partial \Pi}{\partial t}$$

    For options, Θ measures the decay in time value (defined below).

  - **Rho** – sensitivity of portfolio to changes in the interest rate.

    $$\rho = \frac{\partial \Pi}{\partial r}$$

[8] Technically, vega is not actually a Greek letter, but it seems like it should be one.

[9] In the continuous model, only first and second order changes are significant (see (2.3)). Under any discretization, higher order effects can matter, especially during violent price changes such as a crash.

When hedging, we would like to make all our Greeks as close to zero as possible. Any deviation from zero represents an exposure. For example, in a delta neutral portfolio ($\Delta = 0$), our portfolio is unaffected by changes (up to first order effects) in the underlying stock. Note that the Black-Scholes replicating portfolio is a delta neutral portfolio.

- **Delta Hedging.** Common phrase used to describe the actual process of performing the Black-Scholes replication. The term *delta* refers to the fact that the portfolio is delta neutral.

- **Intrinsic Value and Time Value.** These terms are associated with European options. The intrinsic value of a call option is $\max(S - K, 0)$, where S is the current stock price and K is the strike. For a put option, the intrinsic value is $\max(K - S, 0)$. The difference between the option price and its intrinsic value is called the time value.

- **Implied Volatility.** In practice, the instananeous volatility of a stock is the only unobservable parameter of the Black-Scholes formula. All other parameters (stock price, strike, maturity, interest rate, and dividend rate) are directly observable from the market or specified in the contract. Given the market price, we can reverse engineer the volatility necessary to match the Black-Scholes formula.

$$\text{Implied Volatility} = \sigma \text{ such that } BS(\sigma) = \text{Market price}$$

  where $BS(\cdot)$ refers to the Black-Scholes formula for the particular option.

  In reality, the Black-Scholes model is only an approximation. However, implied volatility is still used to quote prices. It provides a quick, simple (but imperfect) benchmark for comparing options.

2.3 Alternative Models

Beyond the Black-Scholes model, many variations or extensions have been studied. In the section, we list several of the variants and briefly describe them.

- **Stochastic Interest Rates.** In the Black-Scholes model, we assumed the interest rate was constant. We now allow the interest rate to be stochastic which may or may not be correlated to the underlying stock. Such a model was studied by Merton[36]. Essentially, the same dynamic replication argument still holds.

- **Jump Diffusion.** In the Black-Scholes model, we assumed the stock process was a pure diffusion process and hence continuous. In the jump diffusion model (due to Merton[37]), we allow the price process to have discontinuities (i.e. jumps). In general, it is impossible to hedge against jumps, so the perfect dynamic replication argument is no longer possible.

- **Nonconstant, Deterministic Drift and Volatility.** Here, we allow the drift (i.e. $r - \rho$) and instantaneous volatility of the risk-neutral process to be a function of the current spot and time. This type of model has been studied by Duprie[22], Derman and Kani[20], and Rubinstein[40]. The main motiviation for these models is to model the so-called "volatility smile".[10]

- **Stochastic Volatility.** Volatility is stochastic with possible correlation to the underlying stock. It is impossible to perfectly replicate an option using just the stock and bonds. However, by introducing another hedge instrument, namely other options, we can again perform perfect replication as in Kani[33].

2.4 Arrow Debreau Securities

In this section, we will define Arrow Debreau securities[11] and demonstrate their construction from call options. Arrow Debreau securities form a basis for European options and are a convenient means to represent such options.

In a discrete setting, Arrow Debreau securities pay \$1 in a particular state of the world. The continuous analog is a security $AD(K)$ that has a payoff function:

$$\delta(K - S) \tag{2.40}$$

where $\delta(x)$ is a Dirac delta function and S is the stock price at maturity. The above security

[10]The volatility smile is the empirical observation that the implied volatilities of options varies across strikes (fixing all other parameters). If the Black-Scholes model were accurate, the implied volatility would be constant across strikes.

[11]These securities are named after the economists Arrow and Debreau who introduced state-contingent claims. This material in this section is based upon p. 441-50 of Merton[38].

has nonzero payoff only if $S = K$.

We will use Arrow Debreau securities to replicate European options. For example, if we want to construct a portfolio that pays \$1 if $S \in (A, B)$, our portfolio will be:

dK shares of $AD(K)$ for $K \in [A, B]$

where dK is the infinitesimal differential of K. The payoff of this portfolio is the sum of the individual payoffs:

$$\int_A^B \delta(K - S)dK = \begin{cases} 1 & \text{if } S \in (A, B), \\ 0 & \text{otherwise} \end{cases} \tag{2.41}$$

Similiarly, we can replicate a call option strike $\hat{K}$ with the portfolio:

$(K - \hat{K})dK$ shares of $AD(K)$ for $K > \hat{K}$

We now discuss the pricing of Arrow Debreau securities. Consider the following option portfolio (which is often called a butterfly spread):

- Long $\frac{1}{\epsilon^2}$ calls with strike $\hat{K} - \epsilon$
- Short $\frac{2}{\epsilon^2}$ calls with strike $\hat{K}$
- Long $\frac{1}{\epsilon^2}$ calls with strike $\hat{K} + \epsilon$

The payoff of this portfolio is shown in Figure 2-3. By construction, the area under the triangle is always 1. Thus, as we take the limit as $\epsilon \to 0$, the payoff approaches a Dirac delta function.

Let $C(K)$ denote the price of a call with strike K. Then, the price of our portfolio is:

$$\frac{C(\hat{K} - \epsilon) - 2C(\hat{K}) + C(\hat{K} + \epsilon)}{\epsilon^2} \to \frac{\partial^2 C(\hat{K})}{\partial \hat{K}^2} \tag{2.42}$$

as $\epsilon \to 0$. Thus, the price of an Arrow Debreau security is the second derivative of the call pricing function with respect to strike. Since Arrow Debreau securities must have positive price, $C(K)$ must be convex. This derivation is model independent.

In the Black-Scholes model,

$$AD(K) = \frac{1}{K\sqrt{2\pi\sigma^2 T}} \exp\left[-\frac{(\ln(K/S) - (r - \rho - \frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}\right] \tag{2.43}$$

where S is the current stock price and T is the time left till maturity.

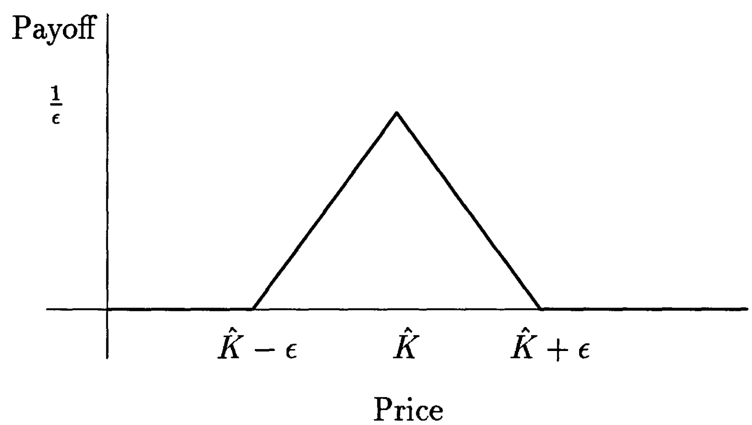


Figure 2-3: Creating an Arrow Debreau Security from a Butterfly Spread.

Chapter 3

Single Barrier Static Replication

We are now ready to present this thesis's contributions. In this chapter,[1] our study of static replication begins. For starters, we will define static replication and compare it to dynamic replication. Subsequently, we will derive the static replication of single barrier options.

3.1 Static Replication

The main insight of Black, Scholes, and Merton was that one could replicate option payoffs with a portfolio of the underlying stock and bonds. Unfortunately, this method requires continuous trading. Static replication attempts to address this problem. We loosely define static replication to encompass the replication of complex securities via simplier securites *without* continuous trading. Clearly, there is a wide spectrum of trading strategies that are not continuous. Some trading strategies may require a single trade, while others may use an arbitrary number of trades. In the next section, we will classify the types of static replication.

Thanks to the Black-Scholes model, plain vanilla options are well-understood, and fairly liquid vanilla option markets exist for many securities. In this thesis, we will focus on replicating exotic exposures using plain vanilla options. Specifically, we will statically replicate barrier options, variants of barrier options, and lookbacks. In these strategies, trading is *event-driven.* As certain events happen, some form of trading is required. This feature will become more transparent as we examine specific static replication schemes.

[1]This chapter is partially presented in Carr and Chou[9] and Chou, Moallemi, and Sundaram[14].

The purpose of this strategy is two-fold. First, we hope to gain insight into exotic options by using this non-traditional method of replication. In addition, static replication gives a new tool for creating valuation formulas for many exotic options. These alternative derivations will, hopefully, provide added intuition. Second, static replication, in certain situations, may be the best way to hedge an exotic exposure. There are both advantages and disadvantages to static replication over dynamic replication.

The most immediate advantage is frequency of trading. In reality, continuous trading is impossible. Even if it were, the associated transaction costs would make this strategy untenable.[2] The usual approach is to make a discrete approximation of the Black-Scholes replication. With any discretization, the replicating strategy becomes exposed to changes in delta (i.e. gamma). For options with high gamma (such as barrier options), this problem is serious. In static replication, there is no gamma exposure. Furthermore, dynamic replication is extremely suspectible to changes in volatility. When replicating with only the stock and bond, unexpected changes in volatility are completely unhedged, since the stock and bond are insensitive to volatility. However, by hedging with options, volatility exposure can be partially offset, since the hedge instrument is sensitive to volatility.

One disadvantage of static replication is higher transaction costs. In almost all situations, the market for the underlying stock is far more liquid than the vanilla option market. This effect is partially mitigated by fewer transactions and the fact that the notional amounts for dynamic replication are often much larger than for static replication. To obtain a meaningful comparison, we must look at total volume of trades. Another disadvantage is that many static replication schemes may require a large number of different options to obtain perfect replication. In dynamic replication, there are only two hedge instruments. With static replication, there are an arbitrary number of hedge instruments (i.e. vanilla options of different strikes and maturity). Finally, static replication schemes do not exist[3] for all types of exotic options (e.g., Asians). Most of the schemes in this thesis focus on barrier options and options that can be interpreted as barriers (e.g., lookbacks).

There is one problem that is common to both dynamic and static replication. Both schemes are vulnerable to sudden, drastic changes in the underlying stock. In dynamic

[2]In fairness, the Black-Scholes model assumes no transaction costs. The problem of finding optimal replicating strategies with transaction costs has been the topic of several papers (e.g., Leland[35] and Hodges and Neuberger[29]).

[3]In our static schemes, we do not permit continuous trading.

replication, the result is a large gamma exposure. In static replication, an event-driven trade may be missed.

3.2 Types of Static Replication

We will divide static replication strategies along two criteria. The first is the maximum number of trades. We use the following classification:

- **n-stage.** The maximum number of trades (excluding the initial trade) is at most n. For example, a strategy that uses at most one trade is a one-stage static replication.
- **Quasi-static.** The maximum number of trades is not finitely bounded. It is hard to call such a strategy static, so we denote it by quasi-static. Although we may trade infinitely often, we still prohibit continuous trading. For example, trading on an uncountable set of times with measure zero may be infinite, but it far less frequent (in a theoretical sense) than trading continuously.

Our second criteria is based upon the number of maturities. At any given time, our replicas consist of vanilla options, which may have different maturities. We classify static strategies by the maximum number of different maturities that can held at one time.

- **Single.** At any single time, replicas of only one maturity can be held.
- **n-tuple.** Replicas of up to n different maturities can be held at the same time.

We will find the preceding classification scheme very convenient for describing static strategies. Clearly, the most desirable strategy is a one-stage single-maturity strategy. For some complex exotic options, such a strategy is impossible. As we shall see, there are situations where we can tradeoff between number of trades and the number of maturities.

3.3 Barrier Options

In this thesis, we will spend a substantial amount of time on barrier options. The purpose of this section is familiarize the reader with the basic conventions associated with barrier options.

A single barrier option is like an European vanilla option with a twist. Associated with each option is barrier. If the price, at any time, reaches the barrier, the option fundamentally changes. There are two main kinds of barrier options:

1. **Knock outs** (or simply outs). Upon hitting the barrier, the option becomes worthless. At maturity, its payoff is identical to a plain vanilla option assuming the option never knocks out.

2. **Knock ins** (or simply ins). This option is the opposite of a knock out. If the barrier is never hit, the option's payoff is zero. Upon hitting the barrier, this option becomes identical to a plain vanilla option.

Additional termnology:

- **Up** – if the barrier is above the current stock price.

- **Down** – if the barrier is below the current stock price.

Combining these terms, we are able to name barrier options as up-and-out calls, down-and-in puts, etc.

One important observation is in-out parity. A portfolio that consists of knock in and a knock out is identical to a plain vanilla option:

$$\text{Knock in} + \text{Knock out} = \text{Plain Vanilla}$$

Hence, it suffices to study either knock ins or knock outs and then apply in-out parity.

Binary options have a similiar terminology. For European binaries, knock outs are called no-touch, and knock ins are called one-touch. Recall that European options pay $1 at maturity. In addition, there is an American variant of a one-touch. Upon hitting the barrier, the American binary pays $1 immediately.

3.4 Constructing the Static Replication

Currently, we have several derivations for the static replication of single barrier options. To a large extent, the various derivations correspond to the different interpretations of the Black-Scholes model. We present these derivations in sections §3.4.1, §3.4.2, and §3.4.3.

The first method uses the risk neutral probability measure, the second follows from the differential equation method, and the third method is based upon the binomial model.

In previous work, Bowie and Carr[7] solved the static replication in the special case where $r = \rho$, which is known as zero cost of carry.[4] Derman *et al*[18] presented an algorithmic approach to replicate barrier options using vanilla options of the same strike, but different maturity. Our methods uses vanilla options with the same maturity, but different strikes.

3.4.1 Symmetry in Probability Space

This derivation relies upon a symmetry found in the lognormal distribution given in (2.36). It can best be summarized by the following lemma:

Lemma 3.1 *In the Black-Scholes model, suppose X is an European option with maturity T and payoff:*

$$X(S_T) = \begin{cases} f(S_T) & \textit{if } S \in (A, B), \\ 0 & \textit{otherwise.} \end{cases}$$

For $H > 0$, let Y be European option with maturity T and payoff:

$$Y(S_T) = \begin{cases} (\frac{S_T}{H})^p f(H^2/S_T) & \textit{if } S_T \in (H^2/B, H^2A), \\ 0 & \textit{otherwise} \end{cases}$$

where the power $p \equiv 1 - \frac{2(r-\rho)}{\sigma^2}$ and r, ρ, and σ are the interest rate, dividend rate and instantaneous volatility.

Then, for $\tau < T$ with the stock price at H, options X and Y have the same price.

Proof. For $\tau < T$, let $t = T - \tau$. By risk-neutral pricing, the price of X at stock price H and time t is:

$$\begin{aligned} P_X &= e^{-rt} \int_A^B f(S_T) p(S_T, H, t) dS_T \\ &= e^{-rt} \int_A^B f(S_T) \frac{1}{S_T\sqrt{2\pi\sigma^2 t}} \exp\left[-\frac{(\ln(S_T/H) - (r - \rho - \frac{1}{2}\sigma^2)t)^2}{2\sigma^2 t}\right] dS_T \end{aligned}$$

Let $S = \frac{H^2}{S_T}$. Then, $dS_T = -\frac{H^2}{S^2} dS$ and

$$P_X = e^{-rt} \int_{H^2/B}^{H^2/A} f(H^2/S) \frac{1}{S\sqrt{2\pi\sigma^2 t}} \exp\left[-\frac{(\ln(H/S) - (r - \rho - \frac{1}{2}\sigma^2)t)^2}{2\sigma^2 t}\right] dS$$

[4] In (2.35), the term $r - \rho$ is the drift of the risk-neutral process and is referred to as the cost of carry.

$$= \quad e^{-rt}\int_{H^2/B}^{H^2/A}(S/H)^p f(H^2/S)\frac{1}{S\sqrt{2\pi\sigma^2 t}}\exp\left[-\frac{(\ln(S/H)-(r-\rho-\frac{1}{2}\sigma^2)t)^2}{2\sigma^2 t}\right]dS$$

where $p = 1 - \frac{2(r-\rho)}{\sigma^2}$. By inspection, P_X exactly matches the risk-neutral price of Y. ∎

Esssentially, this lemma allows us to reflect payoffs along barrier H, while preserving the option's price when the stock price is at H. This reflection incorporates both the geometric nature of the diffusion and the drift. The choice of p seems somewhat magical. In the Appendix, we give an informal derivation of p. Note that if the payoff of X is entirely above H, then the payoff of Y is entirely below H.

In the following theorem, we derive the static replication for down-and-in claims.

Theorem 3.2 *In a Black-Scholes economy, let W be a down-and-in claim with barrier H, maturity T, and payoff at maturity $f(S_T)$. Then, there exists a one-stage single-maturity static replication strategy for W, where the replicas mature at time T and have payoff at maturity:*

$$\hat{f}(S_T) \equiv \begin{cases} 0 & \textit{if } S_T > H, \\ f(S_T) + (\frac{S_T}{H})^p f\left(\frac{H^2}{S_T}\right) & \textit{if } S_T < H, \end{cases} \tag{3.1}$$

Proof. Suppose we have a down-and-in claim. If the barrier is never reached, it will expire worthless at maturity. Upon reaching the barrier, it becomes identical to a European claim. To replicate this exotic, we want a portfolio of vanilla options to imitate this behavior. If the barrier is never reached, our portfolio should be worthless at maturity. At the barrier, it should be equivalent to the appropriate European claim.

A down-and-in claim can have payoffs both above and below the barrier. For payoffs below the barrier, the requirement that the in-barrier be touched is superfluous, and so we can replicate with European options. For payoffs above the barrier, we use Lemma 3.1 to reflect these payoffs below the barrier. The reflected payoffs are constructed to have a value matching that of the original payoffs whenever the stock price is at the barrier. Thus, we can also replicate the reflected payoffs with vanilla options to complete our static hedge.

By applying the above argument, the static replicating portfolio is:

$$\hat{f}(S_T) \equiv \begin{cases} 0 & \text{if } S_T > H, \\ f(S_T) + (\frac{S_T}{H})^p f\left(\frac{H^2}{S_T}\right) & \text{if } S_T < H, \end{cases} \tag{3.2}$$

Barrier Security	Adjusted Payoff	
No-touch binary put	1	for $S_T > H$
	$-(S_T/H)^p$	for $S_T < H$
One-touch binary put (European)	0	for $S_T > H$
	$1 + (S_T/H)^p$	for $S_T < H$
Down-and-out call	$\max(S_T - K_c, 0)$	for $S_T > H$
	$-(S_T/H)^p \max((H^2/S_T) - K_c, 0)$	for $S_T < H$
Down-and-out put	$\max(K_p - S_T, 0)$	for $S_T > H$
	$-(S_T/H)^p \max(K_p - (H^2/S_T), 0)$	for $S_T < H$

$p = 1 - \frac{2(r-\rho)}{\sigma^2}$

Table 3.1: Adjusted Payoffs for Down Securities.

where the power $p \equiv 1 - \frac{2(r-d)}{\sigma^2}$. Note that the $(\frac{S_T}{H})^p f\left(\frac{H^2}{S_T}\right)$ term corresponds to the reflected payoff. ∎

We call $\hat{f}(S_T)$ the *adjusted payoff* for the down-and-in security. As an immediate corollary, we can derive static replication for down-and-out claims.

Corollary 3.3 *In a Black-Scholes economy, let W be a down-and-out claim with barrier H, maturity T, and payoff at maturity $f(S_T)$ for $S_T > H$. Then, there exists a one-stage single-maturity static replication strategy for W, where the replicas mature at time T and have payoff at maturity:*

$$\hat{f}(S_T) \equiv \begin{cases} f(S_T) & \text{if } S_T > H, \\ -\left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right) & \text{if } S_T < H. \end{cases} \tag{3.3}$$

Proof. Apply in-out parity. The sum of the adjusted payoffs for an down-and-in claim and down-and-out claim must equal the payoff of the European claim. One can also observe that this portfolio has zero value at the barrier and pays $f(S_T)$ if the barrier is never reached. ∎

Using a symmetric argument, we can show that up-and-in claims have one-stage single-maturity strategies with adjusted payoff:

$$\hat{f}(S_T) \equiv \begin{cases} f(S_T) + \left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right) & \text{if } S_T > H, \\ 0 & \text{if } S_T < H, \end{cases} \tag{3.4}$$

For an up-and-out claim, the adjusted payoff is:

$$\hat{f}(S_T) \equiv \begin{cases} -\left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right) & \text{if } S_T > H, \\ f(S_T) & \text{if } S_T < H. \end{cases} \tag{3.5}$$

To summarize, our one-stage single-maturity static replication strategies for a single barrier option are:

1. Upon initiation, purchase a European portfolio that matches the adjusted payoff.

2. At the first passage time of reaching the barrier, liquidate the current portfolio.

   (a) For knock outs, the portfolio will be worth zero.

   (b) For knock ins, use the proceeds to buy the corresponding European option.

In Table 3.1 and Figure 3-1, we show the adjusted payoff for some common securities. Upon inspection, the adjusted payoffs are usually not piecewise linear. Thus, an exact replication using a finite number of European puts and calls is usually not possible. However, as Figure 3-1 makes clear, the payoffs are close to linear. Furthermore, a few special cases are worth mentioning. When $r = \rho$, then $p = 1$ and all payoffs are linear. The resulting payoffs are identical to the results given in Bowie and Carr[7]. Also, for $r - \rho = \frac{\sigma^2}{2}$, then $p = 0$ and the binary payoffs are linear. In particular, a one-touch binary can be exactly replicated by two digitals.

Given the adjusted payoff, the value of the replicating portfolio can be determined by risk-neutral valuation:

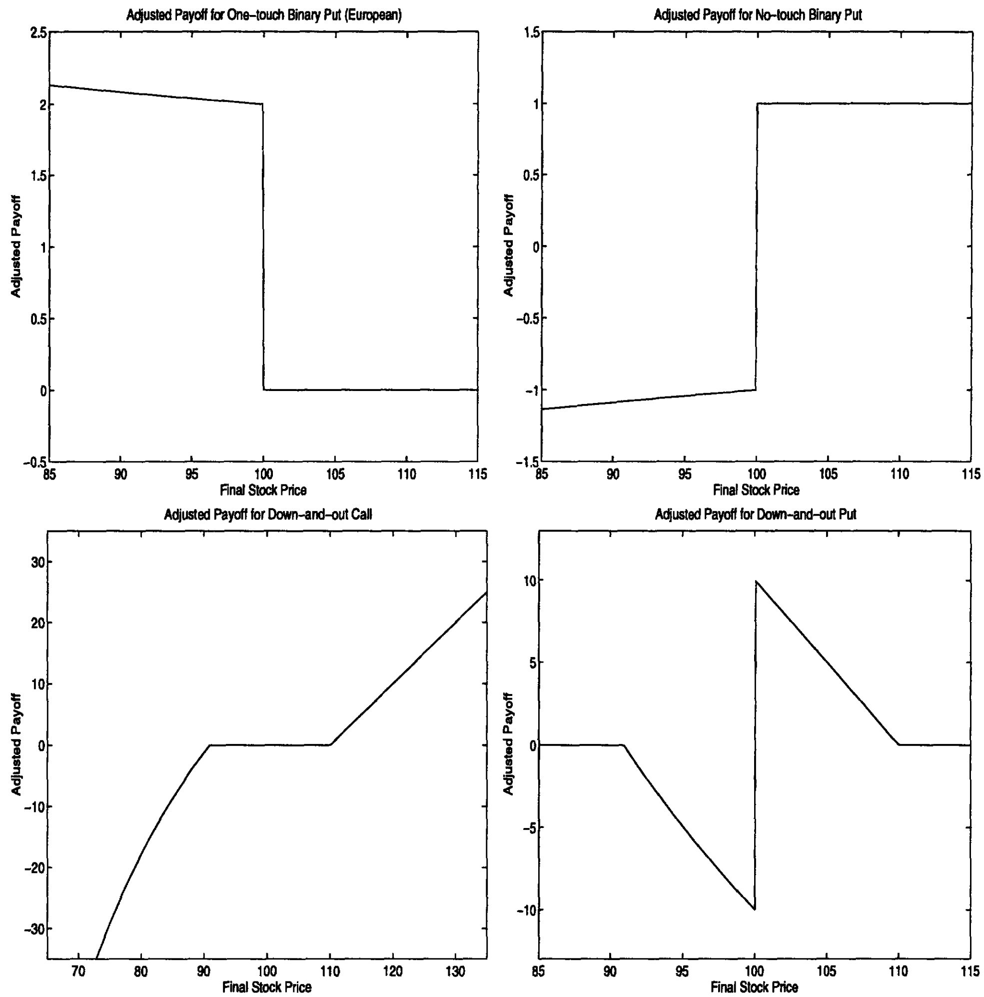
$$V(S,T) = e^{-rT} \int_0^\infty \hat{f}(S_T) p(S_T, S, T) dS_T. \tag{3.6}$$

For example, the price of a down-and-in call (with $K > H$) is:

$$\begin{aligned} & e^{-rT} \int_0^{H^2/K} (S_T/H)^p (H^2/S_T - K) p(S_T, S, T) dS_T && (3.7) \\ = & \left(\frac{H}{S}\right)^p \left[H e^{-\rho T} N(e_1) - \frac{SK}{H} e^{-rT} N(e_2)\right] && (3.8) \end{aligned}$$

where $e_1 = \frac{\ln(H^2/(SK)) - (r - \rho + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $e_2 = e_1 - \sigma\sqrt{T}$. For $K < H$, the price is:

$$Se^{-\rho T} N(d_1) - Ke^{-rT} N(d_2) - Se^{-\rho T} N(f_1) + Ke^{-rT} N(f_2) + \left(\frac{H}{S}\right)^p \left[Se^{-\rho T} N(g_1) - Ke^{-rT} N(g_2)\right] \tag{3.9}$$



$(r = 0.05, \rho = 0.03, \sigma = .15, K_c = K_p = 110, H = 100)$

Figure 3-1: Adjusted payoffs for down securities.

where $d_1 = \frac{\ln(S/K)+(r-\rho+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $f_1 = \frac{\ln(S/H)+(r-\rho+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $g_1 = \frac{\ln(H/S)+(r-\rho+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$, $g_2 = g_1 - \sigma\sqrt{T}$, and $f_2 = f_1 - \sigma\sqrt{T}$.

3.4.2 Derivation from Pricing Formula

In this section, we derive static replication in another manner. Suppose that a pricing formula for a barrier security is known, either because it exists in the literature (e.g., Rubinstein[41]), or because it has been derived using dynamic replication arguments. We then show how this formula can be used to generate a static hedge using vanilla options.

For simplicity, we again work with down securities only. We essentially work backwards from the results of last section. Thus, we assume we know the formula $D(S,T)$ for a down security as a function of the current stock price S and the time to maturity τ. The first step is to find the value of the replicating option portfolio for any initial stock price by simply removing the restriction that stock prices are above the barrier:

$$V(S,T) = D(S,T), \qquad S > 0. \tag{3.10}$$

The second step is to obtain the adjusted payoff which gave rise to this value. Since values converge to their payoff at maturity, simply take the limit of the value as the time to maturity approaches zero:

$$\hat{f}(S_T) = \lim_{T \downarrow 0} V(S,T), \qquad S > 0. \tag{3.11}$$

We illustrate this procedure with a down-and-in call struck at $K_c > H$. From Merton[36], the valuation formula is:

$$\begin{aligned} DIC(S,T;H) &= Se^{-\rho T}\left(\frac{S}{H}\right)^{p-2} N\left(\frac{\ln\left(\frac{H^2}{SK_c}\right) + (r-\rho+\sigma^2/2)T}{\sigma\sqrt{T}}\right) \\ &\quad -K_c e^{-rT}\left(\frac{S}{H}\right)^{p} N\left(\frac{\ln\left(\frac{H^2}{SK_c}\right) + (r-\rho-\sigma^2/2)T}{\sigma\sqrt{T}}\right), S > H. \end{aligned}$$

Removing the requirement that $S > H$, letting $T \downarrow 0$, and denoting the indicator function by $1(\cdot)$ gives:

$$\begin{aligned} \lim_{T\downarrow 0} DIC(S,T;H) &= S\left(\frac{S}{H}\right)^{p-2} 1\left(\frac{H^2}{S} > K_c\right) - K_c\left(\frac{S}{H}\right)^{p} 1\left(\frac{H^2}{S} > K_c\right) \\ &= \left(\frac{S}{H}\right)^{p}\left(\frac{H^2}{S} - K_c\right) 1\left(\frac{H^2}{S} > K_c\right) \\ &= \left(\frac{S}{H}\right)^{p} \max\left(0, \frac{H^2}{S} - K_c\right). \end{aligned}$$

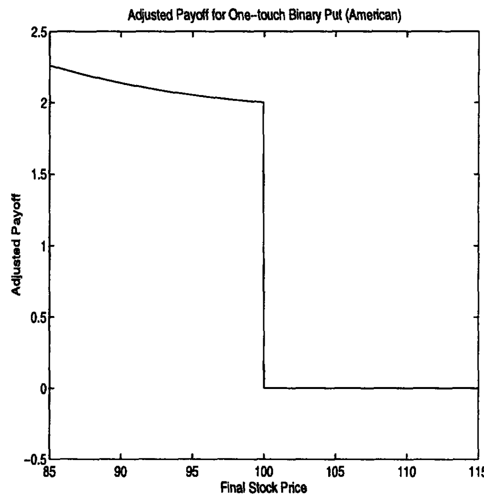
Thus, using in-out parity, the adjusted payoff for a down-and-out call agrees with Table 3.1 (recall $K_c > H$).

To show how this approach can be used to generate adjusted payoffs for other securities, consider the valuation of an American binary put, which pays \$1 dollar at the first passage time to H. From [41], the valuation formula is:

$$ABP(S,T;H) = \left(\frac{S}{H}\right)^{\gamma+\epsilon} N\left(\frac{\ln\left(\frac{H}{S}\right) - \epsilon\sigma^2 T}{\sigma\sqrt{T}}\right) + \left(\frac{S}{H}\right)^{\gamma-\epsilon} N\left(\frac{\ln\left(\frac{H}{S}\right) + \epsilon\sigma^2 T}{\sigma\sqrt{T}}\right), \tag{3.12}$$

for $S > H$, where $\gamma \equiv \frac{1}{2} - \frac{r-\rho}{\sigma^2}, \epsilon \equiv \sqrt{\gamma^2 + \frac{2r}{\sigma^2}}$. Removing the requirement that $S > H$ and letting $T \downarrow 0$ gives the adjusted payoff as (see Figure 3-2):

$$\lim_{T \downarrow 0} ABP(S,T;H) = \left[\left(\frac{S}{H}\right)^{\gamma+\epsilon} + \left(\frac{S}{H}\right)^{\gamma-\epsilon} \right] 1(S < H).$$



$(r = 0.05,\ \rho = 0.03,\ \sigma = .15,\ H = 100)$

Figure 3-2: Adjusted payoff for American binary put.

The derivation from the pricing formula follows naturally from the differential equation interpretation. All pricing formulas must satisfy the Black-Scholes differential equation (as given in (2.11)). Different options are created by imposing different initial value and boundary conditions. In this section, we are essentially transforming a Dirichlet problem (incomplete initial value problem with boundary conditions at the barrier) into a Cauchy problem (complete initial value problem). For single barrier options, both types of problems give rise to unique solutions. Given the final solution, it is straight forward (as illustrated above) to transform between the two types of problems.

3.4.3 Forward Chaining in the Binomial Model

In this section, we derive static replication from the binomial model. The basic technique is called forward chaining.[5] Consider a segment of the binomial tree as shown in Figure 3-3.

[5]Forward chaining was actually the first derivation of the static replication. It was originally presented in [14] and is based upon ideas in [13] and [15].

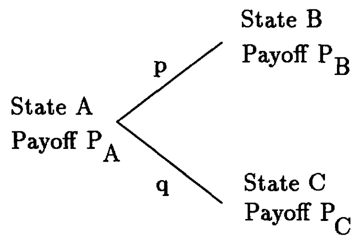


Figure 3-3: Forward Chaining. Determining P_B from P_A and P_C.

Typically, the payoffs of states B and C are known, and we use the risk-neutral probabilities to determine the payoff in state A as:

$$P_A = \frac{1}{R}[pP_B + qP_C].$$

where R is the one-period interest rate and p, q are the risk-neutral probabilities. However, we can also reverse the process. Suppose we know the payoffs in states A and C. By re-arranging the previous equation, we have:

$$P_B = \frac{1}{p}[RP_A - qP_C].$$

This relationship is derived from arbitrage. If this condition were violated, an arbitrage opportunity would exist.

In Figure 3-4, we illustrate the binomial model for an up-and-out claim with barrier H. Along the barrier, the payoffs are zero. Below the barrier, we have payoffs of x_1, x_2, ... at expiry. Our goal is to derive corresponding payoffs y_1, y_2, ... such that if we price any node along the barrier, the payoff of that node should be zero.

Now, consider node A_1 along the zero barrier. The payoff at A_1 is zero. Using risk-neutral pricing, we have

$$0 = \frac{1}{R^2}[p^2 y_1 + 2pq(0) + q^2 x_2] \Longrightarrow y_1 = -\left(\frac{q}{p}\right)^2 x_1$$

The payoff at A_2 is also zero, so

$$0 = \tfrac{1}{R^4}[p^4 y_2 + 4p^3 q y_1 + 6p^2 q^2 (0) + 4pq^3 x_1 + q^4 x_2]$$

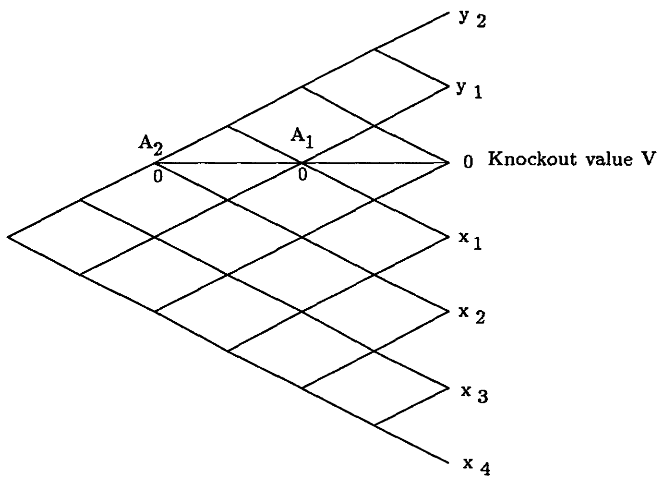


Figure 3-4: Zero Barrier Reflection.

$$\Longrightarrow \qquad y_2 = -\left(\tfrac{q}{p}\right)^4 x_2$$

By iterating this process, it follows that

$$y_i = -\left(\frac{q}{p}\right)^{2i} x_i.$$

Let $f(S_T)$ be the payoff at expiry if the spot price at maturity is S_T. Suppose $S_T > H$ and $S_T = Su^i d^{n-i}$ where S is the intial price. Also, suppose $H = Su^h d^{n-h}$. Then, the corresponding reflected price below the barrier is $\frac{H^2}{S_T}$ and

$$f(S_T) = -\left(\frac{q}{p}\right)^{2(i-h)} f(H^2/S_T).$$

In the continuous time limit (see Appendix), it follows that

$$\left(\frac{q}{p}\right)^{2(i-h)} \longrightarrow \left(\frac{S_T}{H}\right)^p$$

where $p = 1 - \frac{2(r-\rho)}{\sigma^2}$ and

$$f(S_T) \longrightarrow -\left(\frac{S_T}{H}\right)^p f(H^2/S_T).$$

For downward reflections, an identical result can be derived. Now, suppose $S_T < H$ and

$S_T = Su^i d^{n-i}$. Again, the corresponding reflected price is $\frac{H^2}{S_T}$ and

$$f(S_T) = -\left(\frac{p}{q}\right)^{2(h-i)} f(H^2/S_T). \tag{3.13}$$

Taking continuous limits, we find

$$f(S_T) \longrightarrow -\left(\frac{S_T}{H}\right)^p f(H^2/S_T).$$

Using these limits, we can derive Corollary 3.3. The other static replication results follow immediately.

3.5 Static Replication with Barrier Payoff

In some barrier options, the option pays a rebate upon reaching the barrier. In fact, we have already seen examples of such options. An American down-and-in binary put pays \$1 at the first passage time to barrier. The European down-and-in binary put pays \$1 at maturity if the barrier is ever reached. This payment is equivalent to paying $e^{-r(T-\tau)}$ at the first passage time to the barrier, where τ is the first passage time and T is the maturity date. In this section, we will find the static replication for a barrier option that has an arbitrary continuous payoff upon reaching the barrier. We begin with the following theorem, which calculates the static replication for a large class of exponential payoffs.

Theorem 3.4 *Let $\mu = r - \rho - \frac{1}{2}\sigma^2$ and $k \geq -r - \frac{\mu^2}{2\sigma^2}$. In a Black-Scholes economy, let W be a claim with maturity T that pays $e^{k(T-\tau)}$ at the first passage time τ to the barrier H and pays nothing if the barrier is never reached. Then, there exists a one-stage, single-maturity static replication using replicas that mature at time T and payoff:*

$$f(S_T) = \begin{cases} 0 & \text{if } S_T > H, \\ Q(S_T, H, k) & \text{if } S_T \leq H. \end{cases}$$

where

$$Q(S_T, H, k) = (S_T/H)^{\alpha_1} + (S_T/H)^{\alpha_2},$$
$$\alpha_1 = -\frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2(r+k)}{\sigma^2}},$$

$$\alpha_2 = -\frac{\mu}{\sigma^2} - \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2(r+k)}{\sigma^2}}.$$

Proof. As in previous static replication strategies, we will liquidate our portfolio at the first passage time to the barrier. Using risk-neutral pricing, the value of our portfolio at the barrier at time τ (let $t = T - \tau$) is:

$$e^{-rt}\int_0^H Q(S_T, H, k)p(S_T, H, t)dS_T =$$
$$e^{-rt}\left[\exp[(\mu + \frac{\alpha_1\sigma^2}{2})\alpha_1 t]N\left(\frac{-(\mu+\alpha_1\sigma^2)t}{\sigma\sqrt{t}}\right) + \exp[(\mu + \frac{\alpha_2\sigma^2}{2})\alpha_2 t]N\left(\frac{-(\mu+\alpha_2\sigma^2)t}{\sigma\sqrt{t}}\right)\right]$$

Observe that $\alpha_1 = -\frac{2\mu}{\sigma^2} - \alpha_2$, thus

$$\mu + \alpha_1\sigma^2 = -(\mu + \alpha_2\sigma^2).$$

and

$$(\mu + \frac{\alpha_1\sigma^2}{2})\alpha_1 t = (\mu + \frac{\alpha_2\sigma^2}{2})\alpha_2 t.$$

Therefore,

$$e^{-rt}\int_0^H Q(S_T, H, k)p(S_T, H, t)dS_T = \exp\left[\left(\frac{1}{2}\sigma^2\alpha_1^2 + \mu\alpha_1 - r\right)t\right] = e^{-k(T-\tau)}$$

At the barrier, our portfolio will match the desired payoff. Since our replicas are in-the-money only below the barrier, our portfolio will expire worthless if we never reach the barrier. ∎

From the replicas of the exponential payoffs, we can derive the replicas for an arbitrary polynomial.

Corollary 3.5 *Let τ be the first passage time to the barrier H. In a Black-Scholes economy, let W be a claim with maturity T which pays t^n where $t = T - \tau$. Then, there exists a one-stage, single-maturity static replication using replicas that mature at time T and payoff:*

$$f(S_T) = \begin{cases} 0 & \text{if } S_T > H, \\ \left.\frac{\partial^n Q(S_T,H,k)}{\partial k^n}\right|_{k=0} & \text{if } S_T \le H. \end{cases}$$

Proof. From Theorem 3.4, we have:

$$
\begin{aligned}
& e^{-rt}\int_0^H Q(S_T,H,k)p(S_T,H,t)dS_T = e^{kt} \\
\Longrightarrow \quad & \frac{d^n}{dk^n}\left(e^{-rt}\int_0^H Q(S_T,H,k)p(S_T,H,t)dS_T\right) = \frac{d^n e^{kt}}{dk^n} \\
\Longrightarrow \quad & e^{-rt}\int_0^H \left(\frac{d^n Q(S_T,H,k)}{dk^n}\right) p(S_T,H,t)dS_T = t^n e^{kt} \\
\Longrightarrow \quad & e^{-rt}\int_0^H \left(\left.\frac{d^n Q(S_T,H,k))}{dk^n}\right|_{k=0}\right) p(S_T,H,t)dS_T = t^n
\end{aligned}
$$

By inspection, we have created our desired replicas. ∎

As an example,

$$\left.\frac{dQ(S_T,H,k)}{dk}\right|_{k=0} = (S_T/H)^{\alpha_1}\frac{\ln(S_T/H)}{\delta_1\sigma^2} + (S_T/H)^{\alpha_2}\frac{\ln(S_T/H)}{\delta_2\sigma^2}$$

where α_1, α_2 are from Theorem 3.4 and

$$\delta_1 = \sqrt{\frac{\mu^2}{\sigma^4}+\frac{2r}{\sigma^2}}, \delta_2 = -\sqrt{\frac{\mu^2}{\sigma^4}+\frac{2r}{\sigma^2}},$$

Corollary 3.6 *In a Black-Scholes economy, let W be a claim with maturity T which pays $f(\tau)$ at the first passage time to barrier τ. Then, there exists a one-stage, single-maturity static replication using replicas that mature at time T.*

Proof. From Corollary 3.5, we can statically replicate all polynomial payoffs. By the Stone-Weierstrass theorem,[6] the set of polynomials is dense over the set of continuous functions on the compact set $[0,T]$. Therefore, we can statically replicate any arbitary continuous payoff. ∎

Combining Theorem 3.2 and Corollary 3.6, we have the following very general result:

Theorem 3.7 *In a Black-Scholes economy, suppose we have an exotic option W with down barrier H and maturity T. The exotic option W will pay at either the first passage time or at maturity (which ever comes first). At the first passage time τ, the option W pays*

[6] As given on p. 159 of Rudin[43].

$f(\tau)$ where f is continuous. At maturity, W pays $g(S_T)$ for $S_T > H$. Then, there exists a one-stage, single-maturity static replication strategy.

Proof. We can break W into two barrier options: X and Y. Option X is a down-and-out claim with payoff at maturity $g(S_T)$. Option Y is an option that pays at the first passage time $f(\tau)$ and pays nothing if the barrier is never reached. The portfolio of X and Y exactly replicates W. Using Corollary 3.3 and Corollary 3.6, we can replicate both X and Y, respectively. Note that their adjusted payoffs have the same maturity, and any rebalancing only occurs at the first passage time to the barrier. Thus, we have a one-stage, single-maturity static replication. ∎

Appendix

Informal Derivation of Reflection Coefficient in Lemma 3.1

We informally derive the statement of Lemma 3.1 from first principles. We consider Arrow-Debreau securities, since they form the building blocks for all European options.

Suppose we have dS_T Arrow-Debreau securities $AD(S_T)$ which have maturity T. Our goal is find a corresponding set of Arrow-Debreau securities which will match the price of $AD(S_T)dS_T$ at all times the stock price is H. Hence, we want to find $Q(S_T)$ and $F(S_T)$ such that

$$AD(S_T)dS_T = Q(S_T)AD(F(S_T))dF(S_T), \qquad \text{for } S = H \text{ at all times before } T$$

where the $Q(S_T)$ represents a scaling factor.

Let t be the time remaining till maturity ($t < T$). Then,

$$\begin{aligned}
\frac{1}{S_T\sqrt{2\pi\sigma^2 t}}\exp\left[-\frac{(\ln(S_T/H)-\mu t)^2}{2\sigma^2 t}\right]dS_T &= Q(S_T)\frac{1}{F(S_T)\sqrt{2\pi\sigma^2 t}}\exp\left[-\frac{(\ln(F(S_T)/H)-\mu t)^2}{2\sigma^2 t}\right]dF(S_T) \\
\frac{1}{S_T}\exp\left[-\frac{(\ln(S_T/H)-\mu t)^2}{2\sigma^2 t}\right]dS_T &= Q(S_T)\frac{1}{F(S_T)}\exp\left[-\frac{(\ln(F(S_T)/H)-\mu t)^2}{2\sigma^2 t}\right]F'(S_T)dS_T \\
-\ln(S_T) - \frac{(\ln(S_T/H)-\mu t)^2}{2\sigma^2 t} &= \ln\left(\frac{Q(S_T)F'(S_T)}{F(S_T)}\right) - \frac{(\ln(F(S_T)/H)-\mu t)^2}{2\sigma^2 t} \\
-2\sigma^2\ln(S_T)t - \ln^2(S_T/H) + 2\mu\ln(S_T/H)t &= 2\sigma^2\ln\left(\frac{Q(S_T)F'(S_T)}{F(S_T)}\right)t - \ln^2(F(S_T)/H) + 2\mu\ln(F(S_T)/H)
\end{aligned} \tag{3.14}$$

where $\mu = r - \rho - \frac{1}{2}\sigma^2$.

This relation must be invariant under t. Therefore, the derivative with respect to t must be zero. Hence,

$$\begin{aligned}
-2\sigma^2\ln(S_T) + 2\mu\ln(S_T/H) &= 2\sigma^2\ln\left(\frac{Q(S_T)F'(S_T)}{F(S_T)}\right) + 2\mu\ln(F(S_T)/H) \\
\Longrightarrow \qquad Q(S_T) &= (S/F(S_T))^{-1+\mu/\sigma^2}\frac{1}{F'(S_T)}
\end{aligned} \tag{3.15}$$

For any differentiable $F(S_T)$, we can find $Q(S_T)$ which makes the relationship invariant under t. By substituting $Q(S_T)$ back into (3.14), we get:

$$\ln^2(S_T/H) = \ln^2(F(S_T)/H) \tag{3.16}$$

Clearly, the only two solutions are: $F(S_T) = S_T$ and $F(S_T) = H^2/S_T$. The first solution is trivial. Using the second solution in (3.15), we obtain:

$$Q(S_T) = -(S_T/H)^{2\mu/\sigma^2}$$

which matches Lemma 3.1.

Continuous Time Limit of Forward Chaining in the Binomial Model

In this section, we compute the continuous time limit of forward chaining in the binomial model.

Theorem 3.8 *In the binomial model described in section §2.1.3, if we take the limit as the binomial model approachs a continuous diffusion process, then*

$$\left(\frac{q}{p}\right)^{2(h-i)} \longrightarrow \left(\frac{S_T}{H}\right)^{1-2(r-d)/\sigma^2}$$

where h, i are chosen such that

$$H = Su^h d^{n-h}, S_T = Su^i d^{n-i}$$

and S is the initial stock price.

Proof. We choose u, d, and R such that the binomial distribution converges to the lognormal distribution (as demonstrated in Duffie [1994], Ingersoll [1987], Merton [1992] or Wilmott, DeWynne and Howison [1993]). One possible choice is

$$u = e^{\sigma\sqrt{dt}}, d = 1/u = e^{-\sigma\sqrt{dt}}, R = e^{r\,dt}$$

as $dt \longrightarrow 0$. Then,

$$p = \frac{e^{(r-\rho)dt} - e^{-\sigma\sqrt{dt}}}{e^{\sigma\sqrt{dt}} - e^{-\sigma\sqrt{dt}}}, q = \frac{e^{\sigma\sqrt{dt}} - e^{(r-\rho)dt}}{e^{\sigma\sqrt{dt}} - e^{-\sigma\sqrt{dt}}}$$

and

$$\frac{q}{p} = \frac{e^{\sigma\sqrt{dt}} - e^{(r-\rho)dt}}{e^{(r-\rho)dt} - e^{-\sigma\sqrt{dt}}}$$

The Taylor's series for e^x is

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

where $o(\cdot)$ denotes lower order terms. So,

$$\begin{aligned}
\frac{q}{p} &= 1 + \frac{e^{\sigma\sqrt{dt}} + e^{-\sigma\sqrt{dt}} - 2e^{(r-\rho)t}}{e^{(r-\rho)t} - e^{-\sigma\sqrt{dt}}} \\
&= 1 + \frac{\sigma^2 dt - 2(r-\rho)dt + o(dt)}{\sigma\sqrt{dt} + o(\sqrt{dt})} \\
&= 1 + \sigma(1 - 2(r-\rho)/\sigma^2)\sqrt{dt} + o(\sqrt{dt})
\end{aligned}$$

Observe that

$$\frac{S_T}{H} = u^{h-i}d^{i-h} = (e^{\sigma\sqrt{dt}})^{2(h-i)} = (1 + \sigma\sqrt{dt} + o(\sqrt{dt}))^{2(h-i)}$$

For this term to converge to a constant, $2(h-i)$ must be bounded (above and below) by a constant multiple of $1/\sqrt{dt}$. Therefore, in the continuous limit,

$$\begin{aligned}
\left(\frac{q}{p}\right)^{2(h-i)} &= (1 + \sigma(1 - 2(r-\rho)/\sigma^2)\sqrt{dt} + o(\sqrt{dt}))^{2(h-i)} \\
&= (1 + \sigma\sqrt{dt} + o(\sqrt{dt}))^{2(h-i)(1-2(r-\rho)/\sigma^2)} \\
&= \left(\frac{S_T}{H}\right)^{1-2(r-\rho)/\sigma^2}
\end{aligned}$$

∎

Chapter 4

Complex Barrier Static Replication

In this chapter,[1] we continue our study of static replication. Our goal is to extend static hedging from single barrier options to more complex barrier options. In particular, we will examine the following types of barrier options:

1. **Partial Barrier Options:** For these options, the barrier is active only during an initial period. In other words, the barrier disappears at a prescribed time. The payoff at maturity may be a function of the contemporaneous stock price when the barrier disappears.

2. **Forward Starting Barrier Options:** For these options, the barrier is active only over the latter period of the option's life. The barrier level may be fixed initially, or alternatively, may be set at the forward start date to be a specified function of the contemporaneous stock price. The payoff may again be a function of the stock price at the time the barrier becomes active.

3. **Double Barrier Options:** Options that knock in or out at the first hitting time of either a lower or upper barrier (i.e. barriers below and above the current stock price).

4. **Roll-down (Roll-up) Options and Ladder Options:** These options are issued with a sequence of barriers, either all below (roll-down) or all above (roll-up) the initial stock price. Upon reaching each barrier, the option strike is ratcheted. For roll-downs and roll-ups, the option is knocked out at the last barrier.

[1] This chapter is largely presented in Carr and Chou[10].

5. **Lookback Options:** The payoff of these options depends upon the maximum or the minimum of the realized price over the lookback period. The lookback period may start before or after the valuation date but must end at or before the option's maturity.

We will show that the last two categories above may be decomposed into a sum of single barrier options. Consequently, they can be statically hedged using the results of the previous chapter. Furthermore, the decomposition is model-independent. Thus, as new static hedging results for single barrier options are developed, these results will automatically hold for these multiple barrier options.

The structure of this chapter is as follows. The first two sections examine the static replication of partial barrier options and forward starting barrier options respectively. The next section is concerned with static hedging of double barrier options. Finally, the hedging of rolldowns, ladders and lookbacks is examined in the final two sections.

4.1 Partial Barriers

A partial barrier option has a barrier that is active only during part of the option's life. Typically, the barrier is active initially, and then disappears at some point during the option's life. One could also imagine the opposite situation, where the barrier starts inactive and becomes active at some point. We denote these options as *forward-starting options* and discuss them in section §4.2.

We will present two different hedging strategies. In the first method, we will rebalance when the barrier disappears. This method is very general, in that the payoff of the option can depend upon the contemporaneous stock price when the barrier disappears. In the usual situation where the payoff depends only on the final stock price, we can apply a second hedging method, which is superior to the first method. The second method does not require rebalancing at the point where the barrier disappears. Instead, we will perform a static hedge with European options that mature with the barrier option and at the time the barrier disappears.

We will examine down-barriers. Nearly identical methods can be employed for up-barriers. In the following, we derive the replication of partial barrier options.

Theorem 4.1 *In a Black-Scholes economy, let W be a partial barrier option with maturity*

T_2, which knocks out at barrier H. Let T_1 denote the time where the barrier expires. The payoff of W at time T_2 may depend upon the stock price S_1 at time T_1. Then, there exists a two-stage single-maturity static replication.

Proof. At time T_1, either the option has knocked out or it becomes a European claim with some payoff at time T_2. Using risk-neutral pricing, we can always price this European claim as $V(S_1)$.

Define the adjusted payoff at time T_1 as:

$$\hat{f}(S_1) = \begin{cases} V(S_1) & \text{if } S_1 > H, \\ -\left(\frac{S_1}{H}\right)^p V(H^2/S_1) & \text{if } S_1 \leq H \end{cases}$$

Thus, our hedging strategy is as follows:

1. At initiation, purchase a portfolio of European options that gives the adjusted payoff $\hat{f}(S_1)$ at maturity date T_1.
2. If the barrier is reached before time T_1, liquidate our portfolio. From single barrier techniques, our portfolio is worth zero.
3. At the time T_1, if the barrier has not been reached, use the payoff to purchase the corresponding European claim maturing at time T_2.

∎

We describe the hedging strategy for an in-barriers in the following corollary.

Corollary 4.2 *In a Black-Scholes economy, let W be a partial barrier option with maturity T_2, which knocks in at barrier H. Let T_1 denote the time where the barrier expires. The payoff of W at time T_2 may depend upon the stock price S_1 at time T_1. Then, there exists a two-stage single-maturity static replication.*

Proof. We can apply in-out parity. The adjusted payoff at time T_1 is:

$$\hat{f}(S_1) = \begin{cases} 0 & \text{if } S_1 > H, \\ V(S_1) + \left(\frac{S_1}{H}\right)^p V(H^2/S_1) & \text{if } S_1 \leq H \end{cases}$$

Our hedging strategy is as follows:

1. At initiation, purchase a portfolio of European options that pays off $f(S_1)$ at time T_1.

2. If the barrier is reached before time T_1, then rebalance our portfolio to have payoff $V(S_1)$ at time T_1 *for all* S_1. By single barrier techniques, the value of the adjusted payoff term $(\frac{S_1}{H})^p \, V(H^2/S_1)$ exactly matches the value of the payoff $V(S_1)1_{S_1>H}$.

3. At time T_1, if the barrier has not been reached, our payoff is zero. Otherwise, we will receive payoff $V(S_1)$, which allows us to purchase the appropriate European claim maturing at T_2.

∎

The preceding hedging strategies used rebalancing points at the first passage time to the barrier and time T_1. We now present a second method that will only need to rebalance at the first passage time. However, we require the payoff at time T_2 to be independent of S_1. In addition, our replicating portfolio will use options that expire at both time T_1 and T_2.

Theorem 4.3 *In a Black-Scholes economy, let W be a partial barrier option with maturity T_2, which knocks out at barrier H. Let T_1 denote the time where the barrier expires. The payoff at maturity does not depend upon the stock price S_1 at time T_1. Then, there exists a one-stage double-maturity static replication.*

Proof. Let the payoff of W at time T_2 be $g(S_2)$ where S_2 is the stock price at time T_2. Suppose we have a portfolio of European options with payoff $g(S_2)$ at time T_2. We can value it at time T_1 (by using risk-neutral pricing) as $V(S_1)$. For times before T_1, the payoff $g(S_2)$ at time T_2 is always equivalent in value to the payoff $V(S_1)$ at time T_1. Thus, we will apply our barrier option techniques to the $V(S_1)$ payoff while really holding onto the $g(S_2)$ payoff.

Suppose our partial barrier is a knock out option. Then, we really want our payoff at time T_1 to be:

$$f(S_1) = \begin{cases} V(S_1) & \text{if } S_1 > H, \\ -\left(\frac{S_1}{H}\right)^p V(H^2/S_1) & \text{if } S_1 \leq H \end{cases}$$

Unfortunately, our current payoff is (equivalent) to $V(S_1)$ for all S_1. Thus, we'll simply add a portfolio of European options to make up this difference. Let our adjusted payoff at time T_1 be:

$$\hat{f}(S_1) = \begin{cases} 0 & \text{if } S_1 > H, \\ -V(S_1) - \left(\frac{S_1}{H}\right)^p V(H^2/S_1) & \text{if } S_1 \leq H \end{cases}$$

Our hedging strategy is as follows:

1. At initiation, purchase a portfolio of European options that:
   - Provide payoff $g(S_2)$ at maturity T_2.
   - Provide payoff $\hat{f}(S_1)$ at maturity T_1.
2. Upon reaching the barrier before time T_1, liquidate all options. Our portfolio will be worth zero.
3. If the barrier is not reached before time T_1, our payoff will be $g(S_2)$ at time T_2 as desired. Note that it is impossible for the options maturing at time T_1 to pay off without the barrier being reached.

■

Interestingly, the options maturing at T_1 never finish in-the-money. If the barrier is reached, they are liquidated. Otherwise, they expire out-of-the-money at time T_1. Thus, our only rebalancing point is the first passage time to the barrier.

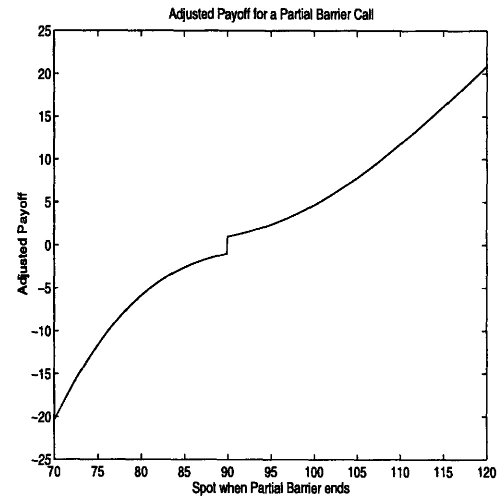
Corollary 4.4 *In a Black-Scholes economy, let W be a partial barrier option with maturity T_2, which knocks in at barrier H. Let T_1 denote the time where the barrier expires. The payoff at maturity does not depend upon the stock price S_1 at time T_1. Then, there exists a one-stage double-maturity static replication.*

Proof. Apply in-out parity. Our replicating portfolio is simply a portfolio of European options that provide payoff $-\hat{f}(S_1)$ at time T_1. If we ever hit the barrier before T_1, the value of our portfolio matches the value of a portfolio of European options that pays off $g(S_2)$ at time T_2. The options maturing at T_1 are sold and the proceeds are used to buy the options maturing at T_2. Otherwise, our replicas will expire worthless. We only need to rebalance at the first passage time to the barrier, if any. ■

As an example, consider a down-and-out partial barrier call with strike K, maturity T, partial barrier H, and barrier expiration T_1. Using the first hedging method, our initial replicating portfolio will have maturity T_1 and payoff (see Figure 4-1):

$$\hat{f}(S_1) = \begin{cases} C(S_1) & \text{if } S_1 > H, \\ -\left(\frac{S_1}{H}\right)^p C\left(\frac{H^2}{S_1}\right) & \text{if } S_1 < H \end{cases} \tag{4.1}$$

where $C(S_1)$ is the Black-Scholes call pricing formula for a call with stock price S_1, strike K, and time to maturity $T_2 - T_1$. As an alternative method to price this option, we can



$(r = 0.05,\ \rho = 0.03,\ \sigma = .15,\ H = 90,\ K = 100,\ T_2 - T_1 = .5)$

Figure 4-1: Adjusted payoff for a Partial Barrier Call Using First Hedging Method.

take the discounted expected value of $\hat{f}$ at time T_1. We find the price to be:

$$\begin{aligned}
&e^{-\rho T_2} S M(a_1, b_2, \gamma) - e^{-rT_2} K M(a_2, b_2, \gamma) \\
&- \left(\frac{S}{H}\right)^p \left[e^{-\rho T_2}(H^2/S) M(c_1, d_1, \gamma) - e^{-rT_2} K M(c_2, d_2, \gamma)\right]
\end{aligned}$$

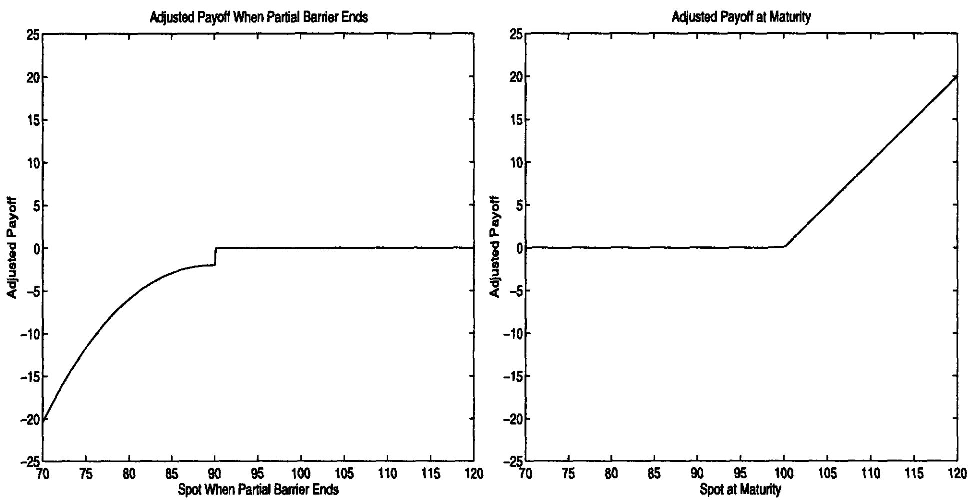
where $M(a, b, \gamma)$ denotes the standard cumulative bivariate normal with correlation $\gamma = \sqrt{T_1/T_2}$, and

$$\begin{aligned}
a_1 &= \frac{\ln(S/H) + (r - \rho + \sigma^2/2)T_1}{\sigma\sqrt{T_1}}, \qquad a_2 = a_1 - \sigma\sqrt{T_1}, \\
b_1 &= \frac{\ln(S/K) + (r - \rho + \sigma^2/2)T_2}{\sigma\sqrt{T_2}}, \qquad b_2 = b_1 - \sigma\sqrt{T_2}, \\
c_1 &= \frac{\ln(H/S) + (r - \rho + \sigma^2/2)T_1}{\sigma\sqrt{T_1}}, \qquad c_2 = c_1 - \sigma\sqrt{T_1}, \\
d_1 &= \frac{\ln(H^2/SK) - (r - \rho + \sigma^2/2)(T_2 - T_1) + (r - \rho + \sigma^2/2)T_1}{\sigma\sqrt{T_2}} \\
d_2 &= \frac{\ln(H^2/SK) - (r - \rho - \sigma^2/2)(T_2 - T_1) + (r - \rho - \sigma^2/2)T_1}{\sigma\sqrt{T_2}}.
\end{aligned}$$

The payoff of this option is independent of S_1, so we can also apply the second hedging method. The portfolio of options maturing at T_2 is just a call struck at K. The portfolio of options maturing at T_1 has the payoff (see Figure 4-2):

$$\hat{f}(S_1) = \begin{cases} 0 & \text{if } S_1 > H, \\ -C(S_1) - (\frac{S_1}{H})^p\, C(H^2/S_1) & \text{if } S_1 \leq H \end{cases} .$$

The value of the barrier option can be given by the sum of the values of the options maturing at T_1 and T_2.



$(r = 0.05,\, \rho = 0.03,\, \sigma = .15,\, H = 90,\, K = 100,\, T_2 - T_1 = .5)$

Figure 4-2: Adjusted payoffs for a Partial Barrier Call Using Second Hedging Method.

4.2 Forward Starting Options

For forward-starting options, the barrier is active only over the latter period of the option's life. The barrier level and payoff at maturity may be fixed initially, or alternatively, may be set at the forward start date to be a specified function of the contemporaneous stock price. As we shall see, forward start options are very similar to partial barrier options.

Again, we will present two different methods. The first method is more general and can be applied to cases where the barrier and/or payoff depend upon the contemporaneous stock price. This method possibly requires rebalancing when the barrier appears and at the first passage time to the barrier. The second method requires that the barrier and payoff be independent of the stock price when the barrier appears, but requires at most one rebalancing.

Theorem 4.5 *In a Black-Scholes economy, let W be a forward starting option with maturity T_2 and barrier H. Let T_1 denote the time when the barrier appears. The payoff at maturity may depend upon the stock price S_1 at time T_1. Then, there exists a two-stage single-maturity static replication.*

Proof. Consider a forward-starting option maturing at T_2, and let the barrier appear at time T_1. At time T_1, the exotic becomes identical to a single barrier option. Using existing single barrier techniques, we can price the exotic at time T_1 as $V(S_1)$.

Create a portfolio of European options that pays off $V(S_1)$ at time T_1. At time T_1, the payoff from these options will be used to buy a portfolio of options maturing at T_2 which replicates a single barrier option. Thus, our hedging strategy always requires rebalancing at time T_1. The subsequent single barrier replication may require an additional rebalancing. ∎

An important special case arises if $V(S_1)$ may be written as $S_1 \times n(\cdot)$, where $n(\cdot)$ is independent of S_1. This situation arises for barrier options where the strike and barrier are both proportional to S_1. In this case, the hedge is to buy $n(\cdot)e^{-\delta T_1}$ shares at time 0 and re-invest dividends until T_1. The shares are then sold and the proceeds are used to buy options providing the appropriate adjusted payoff at T_2.

We now discuss the second method, which is applicable when the barrier and payoff are independent of S_1. As before, we will examine down-barriers and leave it to the reader to apply the same techniques to up-barriers.

Theorem 4.6 *In a Black-Scholes economy, let W be a forward starting option with maturity T_2 and barrier H. Let T_1 denote the time where the barrier appears, which causes the option to knock out. The payoff at maturity does not depend upon the stock price S_1 at time T_1. Then, there exists a one-stage double-maturity static replication.*

Proof. Let $g(S_2)$ denote the payoff at time T_2 and let H be the barrier. At T_1, our situation is identical to a single barrier option, so we would like our adjusted payoff at time T_2 to be:

$$\hat{g}^{out}(S_2) = \begin{cases} g(S_2) & \text{if } S_2 > H, \\ -\left(\frac{S_2}{H}\right)^p g(H^2/S_2) & \text{if } S_2 \leq H. \end{cases}$$

We can value the adjusted payoff $\hat{g}^{out}(S_2)$ at time T_1 (using risk-neutral pricing) as

$V(S_1)$. Ideally, we would like our portfolio at time T_1 to be worth:

$$f(S_1) = \begin{cases} V(S_1) & \text{if } S_1 > H, \\ 0 & \text{if } S_1 \leq H. \end{cases}$$

The payoff of zero below the barrier arises because our forward-starting option is defined to be worthless if the stock price is below the barrier when the barrier is active. Thus, we will add options maturing at time T_1 with payoff:

$$\hat{f}^{out}(S_1) = \begin{cases} 0 & \text{if } S_2 > H, \\ -V(S_1) & \text{if } S_2 \leq H. \end{cases}$$

Our hedging strategy is:

1. At initiation, purchase a portfolio of European options that:
   - Provide payoff $\hat{g}^{out}(S_2)$ at maturity T_2.
   - Provide payoff $\hat{f}^{out}(S_1)$ at maturity T_1.
2. If the stock price at time T_1 is below H, our exotic has knocked out, so liquidate the portfolio.
3. Otherwise, we hold our portfolio. If we hit the barrier between time T_1 and T_2, we liquidate our portfolio. Otherwise, we receive payoff $g(S_2)$.

By construction, whenever we liquidate our portfolio, we will have zero value. The maximum number of rebalancings is at most one. ∎

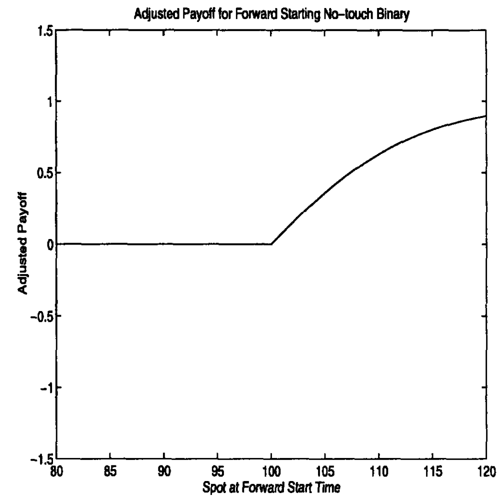
Corollary 4.7 *In a Black-Scholes economy, let W be a forward starting option with maturity T_2 and barrier H. Let T_1 denote the time where the barrier appears, which causes the option to knock in. The payoff at maturity does not depend upon the stock price S_1 at time T_1. Then, there exists a one-stage double-maturity static replication.*

Proof. We can apply in-out parity. Our replicating portfolio at time T is:

$$\hat{g}^{in}(S_2) = \begin{cases} 0 & \text{if } S_2 > H, \\ g(S_2) + (\frac{S_2}{H})^p \, g(H^2/S_2) & \text{if } S_2 \leq H. \end{cases}$$

and at time T_1

$$\hat{f}^{in}(S_1) = \begin{cases} 0 & \text{if } S_2 > H, \\ V(S_1) & \text{if } S_2 \leq H, \end{cases}$$



$(r = 0.05, \rho = 0.03, \sigma = .15, H = 100, T_2 - T_1 = .5)$.

Figure 4-3: Adjusted payoff for Forward Starting No-touch Binary Using First Hedging Method.

where V was defined previously as the time T_1 value of the payoff $\hat{g}^{out}$ at time T_2.

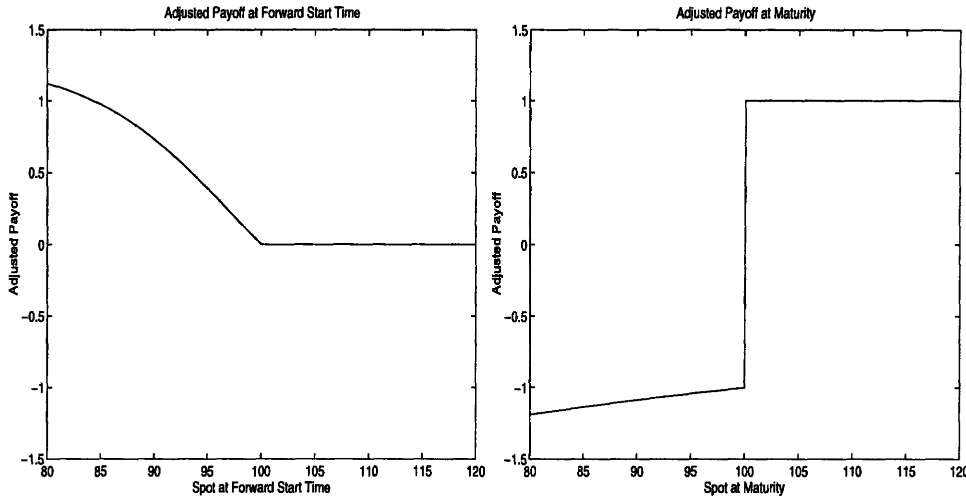
At time T_1, if $S_1 \leq H$, then the value of the down-and-in claim is that of a vanilla claim by definition. Our replicating portfolio consists of options maturing at both T_1 and T_2. By design, these options have a total value equal to the value of the vanilla claim. However, the short position in the options maturing at T_2 and struck below H must be changed to the appropriate long position, and the options maturing at T_1 provide exactly the necessary funds.

In the opposite case where $S_1 > H$ at T_1, then the $\hat{f}^{in}(S_1)$ replicas expire worthless. However, we now have the same replicas as in single barrier replication, and we can again apply single barrier replication techniques. Again, we only need to rebalance once. ∎

For example, consider a forward starting no-touch binary option with down barrier H, maturity T_2, and barrier start date T_1. Using the first method, the portfolio of options with maturity T_1 has payoff (as shown in Figure 4-3):

$$f(S_1) = \begin{cases} NTB(S_1) & \text{if } S_1 > H, \\ 0 & \text{if } S_1 < H \end{cases}$$

where $NTB(S_1)$ is the price of a Black-Scholes price of a no-touch binary with stock price S_1, time to maturity $T_2 - T_1$, and barrier H.



$(r = 0.05,\ \rho = 0.03,\ \sigma = .15,\ H = 100,\ T_2 - T_1 = .5)$.

Figure 4-4: Adjusted payoffs for Forward Starting No-touch Binary Using Second Hedging Method.

Since the barrier and payoff are independent of S_1, we can also apply the second method. The portfolio of options with maturity T_2 has payoff (see Figure 4-4):

$$\hat{g}^{out}(S_2) = \begin{cases} 1 & \text{if } S_2 > H, \\ -\left(\frac{S_2}{H}\right)^p & \text{if } S_2 \leq H. \end{cases}$$

and the replicating payoff with maturity T_1 is:

$$\hat{f}^{out}(S_1) = \begin{cases} 0 & \text{if } S_1 > H, \\ -NTB(S_1) & \text{if } S_1 \leq H. \end{cases}$$

where we extend the $NTB(\cdot)$ formula to values below H.

4.3 Double Barriers

A double barrier option has both an up and a down barrier. Double barrier calls and puts have been priced analytically in Kunitomo and Ikeda[34] and Beaglehole[2], and using Fourier series in Bhagavatula and Carr[4].

In analogy with the single barrier case, our goal is to find a portfolio of European options, so that at the earlier of the two first passage times and maturity, the value of the portfolio exactly replicates the payoffs of the double barrier option.

Theorem 4.8 *In a Black-Scholes economy, let W be a double knock out barrier option with down barrier D, up barrier U, and maturity date T. There exists a one-stage single-maturity static replication.*

Proof. Ideally, we would like to reflect the payoffs as in (3.3) and (3.5). However, we only know the adjusted payoff for the narrow region (D, U). To generate the adjusted payoff for the other regions, we will use multiple reflections.

We begin by dividing the interval $(0, \infty)$ into regions as in Figure 4-5. We can succinctly define the regions as:

$$\text{Region } k = \left(\left(\frac{U}{D}\right)^k D, \left(\frac{U}{D}\right)^k U \right)$$

To specify the adjusted payoff for a region i, we will use the notation:

$$\hat{f}_{(i)}(S_T).$$

We begin with $\hat{f}_{(0)}(S_T) = f(S_T)$

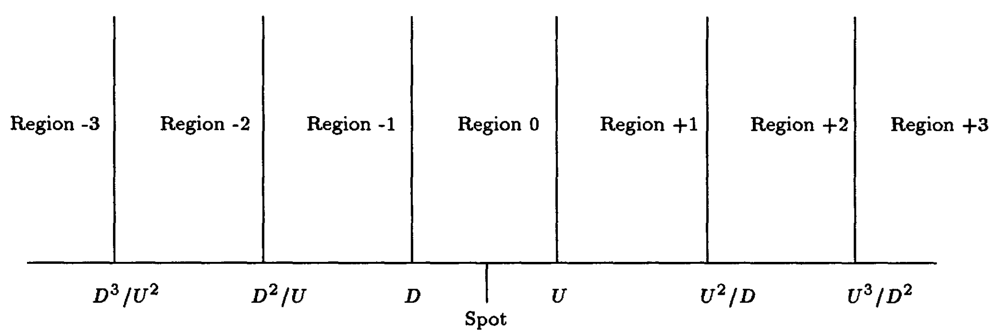


Figure 4-5: Dividing $(0, \infty)$ into regions.

From Lemma 3.1, we see that for a reflection along D, the region k (e.g. k=-2) would be the reflection of region $-k-1$ (e.g. -k-1=+1). Similarly, for reflection along U, region k would be the reflection of region $-k+1$.

Let's define the following two operators:

$$R_D(\hat{f}(S_T)) = -\left(\frac{S_T}{D}\right)^p \hat{f}(D^2/S_T) \text{ and } R_U(\hat{f}(S_T)) = -\left(\frac{S_T}{U}\right)^p \hat{f}(U^2/S_T)$$

It follows that

$$\hat{f}_{(k)}(S_T) = R_D(\hat{f}_{(-k-1)}(S_T)), \qquad \text{for } k < 0$$

and

$$\hat{f}_{(k)}(S_T) = R_U(\hat{f}_{(-k+1)}(S_T)), \qquad \text{for } k > 0.$$

Note that R_U and R_D bijectively map between the corresponding regions. Also, we are taking the negative of the reflection, so that the valuation of the payoffs will cancel. By induction, we can completely determine the entire adjusted payoff as:

$$\hat{f}_{(k)}(S_T) = \begin{cases} f(S_T) & \text{for } k = 0, \\ \underbrace{R_D \circ R_U \circ R_D \ldots}_{k \text{ operators}}(f(S_T)) & \text{for } k < 0, \\ \underbrace{R_U \circ R_D \circ R_U \ldots}_{k \text{ operators}}(f(S_T)) & \text{for } k > 0. \end{cases}$$

A portfolio of European options that delivers the above adjusted payoff replicates the payoff to a double barrier claim. If we never touch either barrier, then the adjusted payoff from region 0 matches the payoff of the original exotic. Upon reaching a barrier, the values of the payoff above the barrier are cancelled by the value of the payoff below the barrier. Therefore, our portfolio is worth zero at either barrier at which point we can liquidate our position. ∎

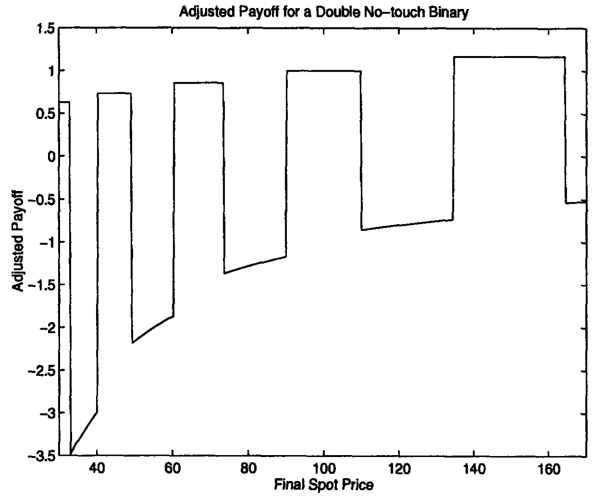
Corollary 4.9 *In a Black-Scholes economy, let W be a double knock in barrier option with down barrier D, up barrier U, and maturity date T. There exists a one-stage single-maturity static replication.*

Proof. To find the adjusted payoff for a knock in claim, we apply in-out parity. The adjusted payoff is given by:

$$\hat{f}_{(k)}(S_T) = \begin{cases} 0 & \text{for } k = 0, \\ f(S_T) - \underbrace{R_D \circ R_U \circ R_D \ldots}_{k \text{ operators}}(f(S_T)) & \text{for } k < 0, \\ f(S_T) - \underbrace{R_U \circ R_D \circ R_U \ldots}_{k \text{ operators}}(f(S_T)) & \text{for } k > 0. \end{cases}$$

∎

As an example, consider a no-touch binary option, which pays 1 at maturity if neither



$(r = 0.05,\ \rho = 0.03,\ \sigma = .15,\ D = 90,\ U = 110)$

Figure 4-6: Adjusted payoff for Double No-touch Binary.

barrier is hit beforehand. Then, $f(S_T) = 1$, and the adjusted payoff is (see Figure 4-6):

$$\hat{f}(S_T) = \begin{cases} -\left(\frac{S_T}{U}\right)^p \left(\frac{D}{U}\right)^{jp} & \text{in region } 2j+1, \\ \left(\frac{U}{D}\right)^{jp} & \text{in region } 2j \end{cases}$$

where j is an integer. Two special cases are of interest. For $r = d$, we have $p = 1$, and the adjusted payoff become piecewise linear. For $r - \rho = \frac{1}{2}\sigma^2$, we have $p = 0$, and the adjusted payoff is piecewise constant.

To compute the price of the double no-touch binary option, we simply compute the price of the adjusted payoff in each region and sum over all regions. The price can be found by taking discounted expected value in the risk-neutral measure. If the current stock price is S, the price of region k is:

$$V(S,k) = \begin{cases} -\left(\frac{S}{U}\right)^p \left(\frac{D}{U}\right)^{jp} e^{-rT} \left[N\left(\frac{\ln(x_1)-\mu T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln(x_2)-\mu T}{\sigma\sqrt{T}}\right)\right] & \text{in region } k = 2j+1, \\ \left(\frac{U}{D}\right)^{jp} e^{-rT} \left[N\left(\frac{\ln(x_1)+\mu T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln(x_2)+\mu T}{\sigma\sqrt{T}}\right)\right] & \text{in region } k = 2j, \end{cases}$$

where $x_1 = \frac{SD^{k-1}}{U^k}$, $x_2 = \frac{SD^k}{U^{k+1}}$, and $\mu = r - \rho - \frac{1}{2}\sigma^2$.

The price of the no-touch binary is the sum of the prices for each region.

$$NTB(S) = \sum_{k=-\infty}^{\infty} V(S,k).$$

Although this sum is infinite, we can get an accurate price with only a few terms. Intuitively, the regions far removed from the barriers will contribute little to the price. Therefore, we only need to calculate the sum for a few values of k near 0. In Table 4.1, we illustrate this fact.

Regions Used to Price	Price
$0 \leq k \leq 0$	0.80687
$-1 \leq k \leq 1$	0.62712
$-2 \leq k \leq 2$	0.62718
$-3 \leq k \leq 3$	0.62718
$-4 \leq k \leq 4$	0.62718
$-5 \leq k \leq 5$	0.62718

3 Month Option ($T = .25$)

Regions Used to Price	Price
$0 \leq k \leq 0$	0.47052
$-1 \leq k \leq 1$	0.03541
$-2 \leq k \leq 2$	0.07713
$-3 \leq k \leq 3$	0.07635
$-4 \leq k \leq 4$	0.07636
$-5 \leq k \leq 5$	0.07636

1 Year Option ($T = 1$)

($S = 100$, $r = 0.05$, $\rho = 0.03$, $\sigma = .15$, $U = 110$, $D = 90$)

Table 4.1: Price Convergence of No-Touch Binary Pricing Formula.

4.4 Roll-down Calls and Ladder Options

The static replication of rolldown calls and ladders was examined by Carr, Ellis, and Gupta[11]. In this section, we review their decomposition into single barrier options and then apply our techniques for barrier replication.

A rolldown call consists of a series of barriers: $H_1, H_2, \ldots, H_n$, which are all below the initial stock price. At conception, the roll-down call resembles a European call with strike K_0. If the first barrier H_1 is hit, the strike is rolled down to a new strike K_1. Upon hitting each subsequent barrier H_i, the strike is again rolled down to K_i. When the last barrier is hit, the option knocks out and becomes worthless.

Observe that a roll-down call can be written as:

$$RDC = DOC(K_0, H_1) - \sum_{i=1}^{n-1}[DOC(K_i, H_{i+1}) - DOC(K_i, H_i)]$$

This replication is model independent and works as follows. If the H_1 is never hit, then the first option provides the necessary payoff, while the terms in the sum cancel. If H_1 is reached, then $DOC(K_0, H_1)$ and $DOC(K_1, H_1)$ become worthless. We can re-write the portfolio as:

$$RDC = DOC(K_1, H_2) - \sum_{i=2}^{n-1} [DOC(K_i, H_{i+1}) - DOC(K_i, H_i)]$$

Thus, our replication repeats itself. If all the barriers are hit, then all the options knock out.

The hedging is straight forward. For each down-and-out call, use (3.3) to find the adjusted payoff. By summing the adjusted payoff, we can ascertain our total adjusted payoff. Every time a barrier is reached, we need to repeat the procedure to find our new adjusted payoff. Thus, the maximum number of rebalancings is the number of barriers.

As an example, consider a rolldown call with initial strike $K_0 = 100$. Suppose it has two rolldown barriers at 90 and 80 (i.e. $H_1 = 90$, $H_2 = 80$). Upon hitting the 90 barrier, the strike is rolled down to the barrier (i.e. $K_1 = 90$). If the stock price hits 80, the option knocks out. Then, our replicating portfolio is:

$$DOC(100, 90) - DOC(90, 80) + DOC(90, 90)$$

Each of these options can be statically replicated. The sum of the corresponding adjusted payoffs is (see Figure 4-7):

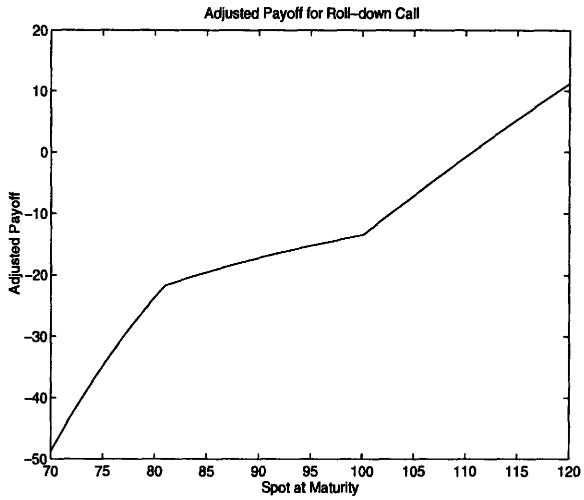
$$f(S_T) = (S_T - 100)^+ - \left(\frac{S_T}{90}\right)^p \left(\frac{90^2}{S_T} - 100\right)^+ + \left(\frac{S_T}{80}\right)^p \left(\frac{80^2}{S_T} - 90\right)^+ - \left(\frac{S_T}{90}\right)^p \left(\frac{90^2}{S_T} - 90\right)^+$$

We will need to rebalance this adjusted payoff upon hitting the barriers at 90 and 80.

Ladder options are similar to roll-down calls, except that instead of knocking out at the last barrier, the strike is rolled down for the last time. They can also be statically hedged.

4.5 Lookback

At first glance, a lookback option appears quite different from a barrier option. In this section, we will show how a lookback can be be decomposed into a portfolio of European binary options. For each binary option, we can create the appropriate adjusted payoffs. Thus, we can create the adjusted payoff of a lookback by combining the binary adjusted payoffs. This combined adjusted payoff will give us pricing and hedging strategies for the



$(r = 0.05,\ \rho = 0.03,\ \sigma = .15)$

Figure 4-7: Adjusted payoff for Roll-down Call.

lookback.

For simplicity, consider a lookback option that pays off $\min(S)$.

Theorem 4.10 *In a Black-Scholes economy, let W be a lookback option with maturity T. The payoff at maturity is the minimum realized price during the life of the option: $min(S)$. Then, there exists a quasi-static single-maturity replication.*

Proof. Let m be the current minimum price. At expiry, the lookback will payoff

$$m - \int_0^m bin(K)dK \tag{4.2}$$

where $bin(K)$ is the payoff of a one-touch down binary struck at K. Thus, our replicating portfolio is a zero coupon bond with face value m and dK one-touch binary options struck at K.

We can calculate the adjusted payoff of the lookback by adding the adjusted payoffs of the bond and binaries. The adjusted payoff of the bond is its face value, and the adjusted payoff of a one-touch binary with barrier K is (from (3.1)):

$$f_{bin}(S_T) = \begin{cases} 0 & \text{if } S_T > K, \\ 1 + (S_T/H)^p & \text{if } S_T < K \end{cases}.$$

Consequently, the adjusted payoff of a lookback option is:

$$f_{lb}(S_T) = m - \int_0^m f_{bin(K)}(S_T)dK \tag{4.3}$$

where $f_{lb}(\cdot)$ and $f_{bin(K)}(\cdot)$ are the adjusted payoffs for lookback and binary options respectively.

Note that the adjusted payoff of a binary struck at K is zero for values above K. Therefore:

$$\int_0^m f_{bin(K)}(S_T)dK = \begin{cases} \int_{S_T}^m \left[1 + \left(\frac{S_T}{K}\right)^p\right] dK & \text{for } S_T < m \\ 0 & \text{for } S_T > m. \end{cases} \tag{4.4}$$

The integral term depends upon the value of p. In particular:

$$\int_{S_T}^m \left[1 + \left(\frac{S_T}{K}\right)^p\right] dK = \begin{cases} m - S_T + S_T \ln(m/S_T) & \text{for } p = 1 \\ m - S_T + \frac{\sigma^2}{2c} S_T((m/S_T)^{2c/\sigma^2} - 1) & \text{for } p \neq 1 \end{cases} \tag{4.5}$$

where $c = r - \rho$. Combining (4.3), (4.4), and (4.5), we find the adjusted payoff of a lookback for $p \neq 1$ (see Figure 4-8) to be:

$$f_{lb}(S_T) = \begin{cases} S_T - \frac{\sigma^2}{2c} S_T((m/S_T)^{2c/\sigma^2} - 1) & \text{for } S_T < m \\ m & \text{for } S_T > m \end{cases} \tag{4.6}$$

and for $p = 1$, the adjusted payoff is:

$$f_{lb}(S_T) = \begin{cases} S_T - S_T \ln(m/S_T) & \text{for } S_T < m \\ m & \text{for } S_T > m \end{cases} \tag{4.7}$$

For $p = 0$ (i.e. $2c = \sigma^2$), the above payoff simplifies to:

$$f_{lb}(S_T) = \begin{cases} 2S_T - m & \text{for } S_T < m \\ m & \text{for } S_T > m. \end{cases} \tag{4.8}$$

In this case, the adjusted payoff is linear. Note that in all cases, the adjusted payoff is a function of m. ∎

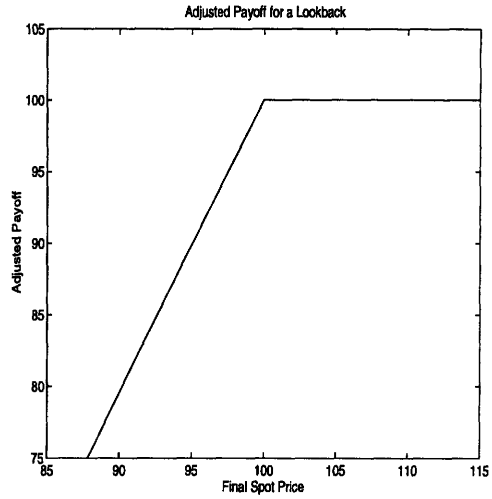


Figure 4-8: Adjusted payoffs for Lookback ($r = 0.05$, $\rho = 0.03$, $\sigma = .15$, $m = 100$).

4.5.1 Hedging

As shown in (4.2), a lookback is actually a continuum of binary options. Our hedging strategies require us to rebalance every time we hit a barrier, which happens every time the minimum changes. Therefore, to hedge a lookback, we will need to rebalance an infinite number of times.

This strategy is a quasi-static strategy. Rebalancing is certainly less frequent than in a continuous rebalancing strategy. In fact, the set of points where the minimum changes is almost certainly an uncountable set of measure zero[2]. In any practical implementation, the problem will be discretized, and rebalancing will occur at strikes of high liquidity.

4.5.2 Lookback Variants

Lookbacks comes in many variants, and our techniques are applicable to many of them. In the following list, we give several variants and show how they may be hedged. Let $m_T = \min(S)$ denote the minimum realized stock at expiry, and let S_T denote the price at expiry.

- **Lookback call.** The final payoff is $S_T - m_T$. The replication involves buying the underlying and shorting the lookback.

[2]In Harrison[25], it is shown that the set of times where the running minimum of a Brownian motion changes value is (almost surely) an uncountable set of measure zero.

- **Put on the Minimum.** The final payoff is $\max(K - m_T, 0)$. Let m denote the current achieved minimum. The replicating portfolio is:

$$\max(K - m, 0) + \int_0^{\min(m,K)} bin(S)dS$$

The adjusted payoff is:

$$f_{put-on-min} = \begin{cases} f_{lb}(\text{with } m = K) & \text{if } m > K, \\ K - m + f_{lb} & \text{if } m < K \end{cases}$$

where f_{lb} is the adjusted payoff of a lookback from (4.6). In the first case, we substitute $m = K$ in the formula for the adjusted payoff. Note that the adjusted payoff is fixed for $m > K$. Our hedge is static until the minimum goes below K, after which we need to rebalance at each new minimum.

- **Forward Starting Lookbacks.** These lookbacks pay m_{12}, the minimum realized price in the window from time T_1 to the maturity date T_2. In this situation, we can combine the methods from forward-starting options and lookbacks. At time T_1, we can value the lookback option with maturity T_2 as $LB(S_1)$.

  At initiation, we purchase a portfolio of European options with payoff $LB(S_1)$ at time T_1. At time T_1, we use the proceeds of the payoff to hedge the lookback as previously described. If $LB(S_1) = S_1 \times n(\cdot)$ where $n(\cdot)$ is independent of S_1, then the initial hedge reduces to the purchase of $n(\cdot)e^{-dT_1}$ shares. Once again, dividends are re-invested to time T_1 at which point the shares are sold and the lookback is hedged as before.

A similar analysis can be applied to the lookbacks that involve the maximum. We leave it to the reader to solve the analagous problem.

Chapter 5

Replication with Time-Dependent Drift

In this chapter, we will study static replication where the drift is time-dependent.[1] By drift, we mean the expected change in the risk-neutral process $(r - \rho)$ as given in (2.35). We should emphasize that we are looking at strictly time-dependent drift. The drift is *not* stochastic.

Static replication with time-dependent drift is equivalent to several other interesting problems. We will demonstrate that a non-flat boundary condition can be converted into an equivalent situation with a flat boundary and time-dependent drift. In addition, the issue of time-dependent volatility is reducible to time-dependent drift. Thus, we will focus our efforts on time-dependent drift with the knowledge that our results will generalize to non-flat boundaries and time-dependent volatility.

This chapter is organized as follows. We begin by showing the equivalence of flat boundaries and time-dependent volatility to time-dependent drift. Subsequently, we will study static replication with time-dependent drift.

5.1 Non-Flat Boundaries

In this section, we examine non-flat boundaries. By a change of variable, we can convert a non-flat boundary into a flat boundary with a different drift component. We will assume the

[1] Technically, the Black-Scholes model assumes constant drift. However, the model can easily be extended to time-dependent drift (see Merton[36]).

boundary curve is both positive and differentiable over a compact set. These assumptions are necessary to prevent infinite drifts.

Again, we will examine securities with down-barriers. The same techniques can be applied to up-barriers. We will consider very robust down-barrier claims. In particular, we allow the down-barrier claim to have an arbitrary continuous payoff at the first passage time to the barrier and, if the barrier is never reached, the down-barrier claim has an arbitrary payoff at maturity. Note that down-barrier claim only has one payoff: either at the first passage time or at maturity.

Theorem 5.1 *In a Black-Scholes economy with an underlying stock S, let W be a down-barrier claim on S with an arbitrary non-flat boundary, which is positive and differentiable.*

Then, there exists a derivative security Q and a down-barrier claim X on Q such that W and X are equivalent (i.e. the payoff of W and X are the same in all states of the world).

Proof. We specify the payoff of W by:

$$P_W = \begin{cases} f(\tau) & \text{if } S_\tau = B(\tau) \text{ and } \tau < T, \\ g(S_T) & \text{if } S_T > B(T). \end{cases}$$

where f is the payoff at the boundary, g is the payoff at maturity, and $B(\cdot)$ is the boundary, which is positive and differentiable over $[0, T]$.

Let $Q_t = D(t)S_t$ where $D(t) = \frac{B(T)}{B(t)}$. Since $B(t)$ is positive, $D(t)$ exists. Note that Q itself is a derivative security. By the Black-Scholes methodology, Q can be replicated from the underlying stock and bonds.

Then, let X be a down-barrier claim on Q with payoff:

$$P_X = \begin{cases} f(\tau) & \text{if } Q_\tau = H \text{ and } \tau < T, \\ g(Q_T) & \text{if } Q_T > H. \end{cases}$$

where $H = B(T)$. Thus, X has a flat barrier. By inspection, W and X have the same payoff in all states of the world. ∎

As stated, the preceding theorem does not seem remarkable. The interesting case arises when Q and S are conveniently related. In particular, exponential barriers are one such special case.

Corollary 5.2 *In a Black-Scholes economy, let W be a down-barrier claim with maturity T and exponential barrier $He^{k(T-t)}$. Then, there exists a one-stage single-maturity static replication for W.*

Proof. Let S be the underlying stock for W. The risk-neutral diffusion of S is given by:

$$dS_t/S_t = cdt + \sigma dZ$$

where $c = r - \rho$.

Apply the construction from Theorem 5.1. Thus, $D(t) = e^{-k(T-t)}$. By Ito's Lemma, the dynamics of Q are:

$$\begin{aligned} dQ_t/Q_t &= \frac{D(t)dS_t + S_tD'(t)dt}{S_tD(t)} \\ &= (c + \frac{D'(t)}{D(t)})dt + \sigma dZ \\ &= (c+k)dt + \sigma dZ. \end{aligned}$$

Hence, Q follows a lognormal diffusion. From Theorem 3.7, there is a one-stage single-maturity static replication of X. Observe that $Q_T = S_T$, and all replicas have maturity T. The replicas for X (with underlying Q) can be used as replicas for W (with underlying S). Therefore, the static replication from Theorem 3.7 can be applied to replicate W. ∎

As an example, consider a down-and-in call on the underlying stock S with boundary $B(t) = He^{-(r-\rho)(T-t)}$ and strike $K > H$. Then, $Q_t = e^{(r-\rho)(T-t)}S_t$ (i.e. Q is the forward price of S). The barrier for Q is H, the risk-neutral drift is $c + k = 0$ and adjusted payoff (from Theorem 3.2) is:

$$\hat{f}(Q_T) = \begin{cases} 0 & \text{if } Q_T > H, \\ \frac{K}{H}\max(H^2/K - Q_T, 0) & \text{if } Q_T < H. \end{cases}$$

The forward price equals the stock price at maturity, so we have the identical replica for S.

To summarize, the down-and-in call on S with boundary $He^{-(r-\rho)(T-t)}$ is equivalent to a down-and-in call on the forward price with a flat boundary H. Furthermore, this down-and-in call can be statically replicated with a single European put.

The observant reader will notice that the key property of exponential boundaries is: $\frac{D'(t)}{D(t)}$ is a constant. Thus, the drift of Q remains constant, which preserves the symmetry in the lognormal propagator. This feature is unique to exponential boundaries. In general, the conversion of a non-flat boundary to a flat boundary will introduce time-dependent drift.

5.2 Time-Dependent Volatility

We can convert between time-dependent volatility and time-dependent drift by using time-scaling. In essence, we stretch or shrink time, so that the volatility of the risk-neutral diffusion becomes constant. However, to maintain the same distribution, we will need to modify the drift. Usually, this entails making the drift time-dependent.

Suppose we have the following risk-neutral diffusion process:

$$dS_t/S_t = c(t)dt + \sigma(t)dZ_t$$

We assume that $\sigma(t) > 0$ for all t. Our goal is to rescale time, so that the volatility is a constant $\bar{\sigma} > 0$. Define the monotone increasing function $F(t)$:

$$F(0) = 0; \quad dF(t) = \frac{\sigma^2(t)}{\bar{\sigma}^2}\,dt$$

F represents our rescaling of time. Clearly, F is strictly monotone increasing and continuous, and thus, a bijection. Let

$$\bar{c}(F(t)) = \frac{\bar{\sigma}^2}{\sigma^2(t)}c(t)$$

Then,

$$dS_{F(t)}/S_{F(t)} = \bar{c}(F(t))dF(t) + \bar{\sigma}dZ_{F(t)}$$

For notation convenience, we write $T = F(t)$. Then, our new diffusion process is:

$$dS_T/S_T = \bar{c}(T)dT + \bar{\sigma}dZ_T \tag{5.1}$$

One special case occurs when $\frac{c(t)}{\sigma^2(t)}$ is a constant. In that case, $\bar{c}(\cdot)$ is a constant. The diffusion in (5.1) has both constant drift and volatility. Thus, we can apply existing static replication techniques.

Remark: The drift component $c(t)$ is the difference betwen the interest rate $r(t)$ and the dividend rate $\rho(t)$. It is natural to rescale $r(\cdot)$ and $\rho(\cdot)$ such that:

$$\bar{r}(F(t)) = \frac{\bar{\sigma}^2}{\sigma^2(t)}r(t); \quad \bar{\rho}(F(t)) = \frac{\bar{\sigma}^2}{\sigma^2(t)}\rho(t)$$

Under this rescaling, $\bar{r}(\cdot)$ is the appropriate discount factor under time measure T.

5.3 Time-Dependent Drift

In this section, we will examine time-dependent drift. In the preceding sections, we showed that options with non-flat barriers and time-dependent volatility could be converted to equivalent options with flat barriers and constant volatility. Thus, it suffices to examine time-dependent drift with a flat barrier and constant volatility.

We will demonstrate that one-stage single-maturity replication is impossible for time-dependent drift. Nevertheless, we will show many-stage single-maturity and one-stage multiple-maturity static replications are possible for time-dependent drift. These schemes are very similiar to the static replication of partial barriers and forward starting barriers.

5.3.1 Impossibility of One-Stage Single-Maturity Static Replication

We prove the impossibility of one-stage single-maturity static replication for piecewise constant drift. Since any non-constant drift is the limiting process of a piecewise constant process, this impossibility results holds for any arbitrary non-constant drifts. We begin with the following theorem regarding piecewise constant drift.

Theorem 5.3 *In a Black-Scholes economy with piecewise constant time-dependent drift, let W be a barrier option with a barrier H. If the drift differs in at least two points, then there does not exist a one-stage single-maturity replication.*

Proof. For our purposes, it suffices to consider only the last two regions of piecewise constant drift. Let T_0 and T_1 denote the last two times when the drift changes, and let T_2 denote the maturity of W (see Figure 5-1). We denote the drift between time T_0 and T_1 by c_1 and the drift between time T_0 and T_1 by c_2. We will prove that no one-stage single-maturity replication can exist starting at time T_0. Since it is possible to reach T_0 without reaching the barrier, there cannot exist a one-stage single-maturity replication for W.

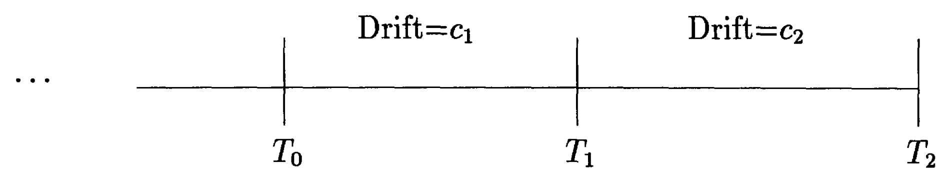


Figure 5-1: Time Seperation for Piecewise Constant Drift.

We proceed with proof by contradiction. Suppose a one-stage single-maturity replication exists for W. Clearly, the replicas must have maturity at time T_2. Let R denote the replicas that occur at time T_2.

Now, suppose we reach time T_1 without reaching the barrier. For the remaining time, the drift is constant c_2. We know the unique static hedge portfolio for replicating the barrier option (as given in Chapter 3). Therefore, R must match this portfolio. Since we never traded, this R must be our portfolio at time T_0. Note that R is independent of c_1. Clearly, this is impossible. It is easy to verify that our replicas will fail at times between T_0 and T_1. Therefore, no one-stage single-maturity replication can exist. ∎

This result has suprising implications regarding tree methods (i.e. the binomial model). We can incorporate non-constant drift into the binomial model in several ways. One method (due to Dupire[22]) is to use trinomial trees. Another method is to vary the vertical spacing of the tree (a variant of Derman and Kani[20]). In these modified trees, we can apply the forward chaining methodology from section §3.4.3. For any given tree, we will *always* find adjusted payoffs that provide exact replications.[2] However, the adjusted payoffs fail to converge. As we take finer refinements of the tree, we will observe the adjusted payoffs to exhibit non-convergent behavior (such as wide oscillations or unbounded growth). The important lesson to be learned is that ideas or methods which work in a tree may not necessary work in the continuous limit. It is always necessary to verify convergence.

For some added intuition, the Appendix demonstrates how the construction of the static replication given in the Appendix of Chapter 3 fails when the drift is time-dependent.

[2] Finding the adjusted payoffs involves solving a linear system of equations which has full rank. Therefore, an unique solution exists.

5.3.2 Existence of Static Replication Schemes

In this section, we will show the existence of static replication schemes for barrier options under time-dependent volatility. In particular, we will focus on piecewise constant time-dependent drift. As before, we will consider down options and leave it to reader to generalize to up options.

Theorem 5.4 *In a Black-Scholes economy with n-period piecewise constant time-dependent drift, let W be a down knock out option with barrier H. Then, there exists a n-stage single-maturity static replication.*

Proof. We will use induction. Our inductive hypothesis is the statement of the theorem with one additional strengthening. If the option does not knock out, then the number of trades is at most $n - 1$.

For $n = 1$, we can simply use Theorem 3.2. If the option does not knock out, then there are zero rebalancing.

For $n > 1$, let T_2 denote the maturity of the W and let T_1 denote the time when the drift last changed. Hence, the drift is a constant c from time T_1 to T_2. Consider a barrier option with maturity T_1 and the following payoff:

$$f(S_1) = \begin{cases} V(S_1) & \text{if } S_1 > H, \\ 0 & \text{if } S_1 \leq H \end{cases} \tag{5.2}$$

where $V(S_1)$ is the price of W of at time T_1 with spot S_1. Since the remaining drift is constant, we can price W using standard methods.

Let W' be a down knock out barrier option with maturity T_1, barrier H, and payoff f. Since W' has $n - 1$ periods of piecewise constant volatility, we can apply the inductive hypothesis. Thus, we can apply the static replication of W' until time T_1. If W' has knocked out, then W has also knocked out. Otherwise, if W' reaches time T_1 without reaching the barrier, we can uses the payoff of W' to construct the static replication of W in the remaining period. W' used at most $n - 2$ rebalances, so we can trade once at T_1 and again at the first passage time to the barrier, if necessary. The total number of rebalances is at most n. ∎

Using in-out parity, we get the following corollary.

Corollary 5.5 *In a Black-Scholes economy with n-period piecewise constant time-dependent drift, let W be a down knock in option with barrier H. Then, there exists a n-stage single-maturity static replication.*

We can also perform a static replication using one-stage multiple-maturities.

Theorem 5.6 *In a Black-Scholes economy with n-period piecewise constant time-dependent drift, let W be a down knock out option with barrier H. Then, there exists a one-stage n-maturity static replication.*

Proof. We will use induction. Our inductive hypothesis is the statement of theorem with the following additional strengthening. The maturity of our replicas only occur at times where the drift changes or at maturity. For $n = 1$, apply Theorem 3.2.

For $n > 1$, let T_0 denote the current time and T_1 be the time when the drift first changes. Thus, the drifts starts at a constant c_0 and changes at time T_1 to a constant c_1.

Suppose we alter the drift, so that the drift between T_0 and T_1 was actually c_1. Then, we would have an $n-1$ period piecewise constant drift. We can apply the inductive hypothesis to create a one-stage $(n-1)$-maturity static replication under this altered process. Let R denote the set of replicas. Note that the replicas in R mature after time T_1.

Now, let's return to the true drift process which has drift c_0 from time T_0 to T_1. If we ever reach time T_1, then we can uses the replicas in R to form a one-stage $n-1$ maturity static replication. However, at times before T_1, the replicas in R are not appropriate. By adding an additional replica which matures at time T_1, we will provide the necessary correction. Let

$$g(S_1) = V(S_1)$$

where V is the value of the replicas in R at time T_1 with spot S_1.

For the reflection in Lemma 3.1 to be valid, we would like our payoffs to be:

$$f(S_1) = \begin{cases} g(S_1) & \text{if } S_1 > H, \\ -(S_1/H)^p g(H^2/S_1) & \text{if } S_1 \le H \end{cases}$$

Therefore, let R' be a portfolio of European options with maturity T_1 and payoff:

$$\hat{f}(S_1) = \begin{cases} 0 & \text{if } S_1 > H, \\ -(S_1/H)^p g(H^2/S_1) - g(H) & \text{if } S_1 \le H \end{cases}$$

We have constructed $\hat{f}$ such that the portfolio of R and R' will have value $f(S_1)$ at time T_1. Therefore, if we ever reach the barrier before time T_1. Our combined portfolio is worth zero and we can liquidate.

If we reach time T_1 without reaching the barrier, R' will expire out-of-the-money, and we will be left with R, which are create a static replication after time T_1. Hence, our replication strategy uses n-maturities. ∎

Again, we can use in-out parity to obtain:

Corollary 5.7 *In a Black-Scholes economy with n-period piecewise constant time-dependent drift, let W be a down knock in option with barrier H. Then, there exists a one-stage n-maturity static replication.*

Hence, we have showed two different static replications for piecewise constant drift. Among the schemes, there is a clear tradeoff between more replicas at different maturities and additional rebalances. As the number of piecewise constant period increase, the complexity of our replication grows. In practice, we will need the drift to be fairly stable, or else static replication will not trade substantially less than dynamic schemes.

Appendix

Impossibility of Replicating Arrow-Debreau Securities under Time Dependent Drift

We follow the argument given in the Appendix of Chapter 3. Suppose that μ is time dependent. Differentiating (3.14) with respect to t and setting equal to zero, we get:

$$\begin{aligned}
& -2\sigma^2 \ln(S_T) + 2\mu(t)\ln(S_T/H) + 2\mu'(t)t\ln(S_T/H) \\
= \; & 2\sigma^2 \ln\left(\frac{Q(S_T)F'(S_T)}{F(S_T)}\right) + 2\mu(t)\ln(F(S_T)/H)2\mu'(t)t\ln(F(S_T)/H)
\end{aligned}$$

Solving for $Q(S_T)$, we get:

$$Q(S_T) = (S_T/F(S_T))^{-1+(\mu(t)+\mu'(t)t)/\sigma^2}\frac{1}{F'(S_T)} \tag{5.3}$$

By definition, $Q(S)$ must be time-invariant. Therefore,

$$\mu(t) + \mu'(t)t = k$$

for some constant k. It is easy to show the only solution is $\mu(t) = k$ (i.e. the drift must be constant). Thus, we cannot statically replicate Arrow-Debreaus with time-dependent drift as we did in the Appendix of Chapter 3.

Chapter 6

Approximate Replication

In this chapter, we will examine static replication as a practical technique. Up till this point, we have been interested in perfect replication (i.e. a strategy with zero hedging error). Unfortunately, this perfection has a cost, in that the exact replicating portfolio is often impractical or impossible to achieve in reality. We will attempt to trade off some of this perfection for a more pragmatic strategy.

6.1 Problem Statement

For simplicity, we will examine the static replication of a down-and-in call, whose strike is above the barrier (i.e. an out-of-the-money barrier). This option is, perhaps, the simpliest option we would care to examine. Thus, we would hope that its static replication is practical.

In Chapter 3, we derived the perfect static replication for the down-and-in call. For $r \neq \rho$, the payoff is non-linear, and thus, we would need an infinite number of European options to exactly replicate this payoff. Given the transaction costs for options, we prefer not to use many vanilla options to replicate a single barrier option. Our first task is to see how well can replicate using only one vanilla option. Our next goal is to test to the stability of our replication under a change in volatility. All replication schemes are sensitive to volatility, and we would like to address our exposure. During changes in volatility, static replications may have advantages over traditional dynamic methods.

6.2 Replication Error

In this section, we consider the important question: how do we measure replication error? To help motivate our choice, let's review our hedging strategy. The only time we trade is at the barrier. In addition, our replicating portfolio has non-zero payoff only below the barrier.[1] Thus, the only points of interest are our rebalancing points (i.e. when the stock price reaches the barrier). If we never reach the barrier, both our hedge portfolio and the down-and-in call expire worthless.

To measure replicating error, we use the following:

$$\epsilon = \max_{0 \leq t \leq T} e^{-rt} |DIC(H, T-t) - Hedge(H, T-t)| \tag{6.1}$$

where H is the barrier, T is the time to maturity, and $DIC(S,t)$ and $Hedge(S,t)$ are the values of the down-and-in call and hedge portfolio with stock price S and time t.

This measure represents the discounted maximum hedging error. The profit/loss (P/L) of our hedging strategy is strictly bounded by this number. This choice was further motivated by two other factors. First, we wanted to take a worst case approach. By using the maximum deviation, we can put strict bounds on our P/L. Second, this approach requires the fewest additional assumptions. Other measures (such as average or expected P/L) would require assumptions regarding the "true" probability distribution.

Note that when the stock price equals the barrier, the down-and-in call becomes a European call. If our hedge portfolio is a single put, our measure becomes:

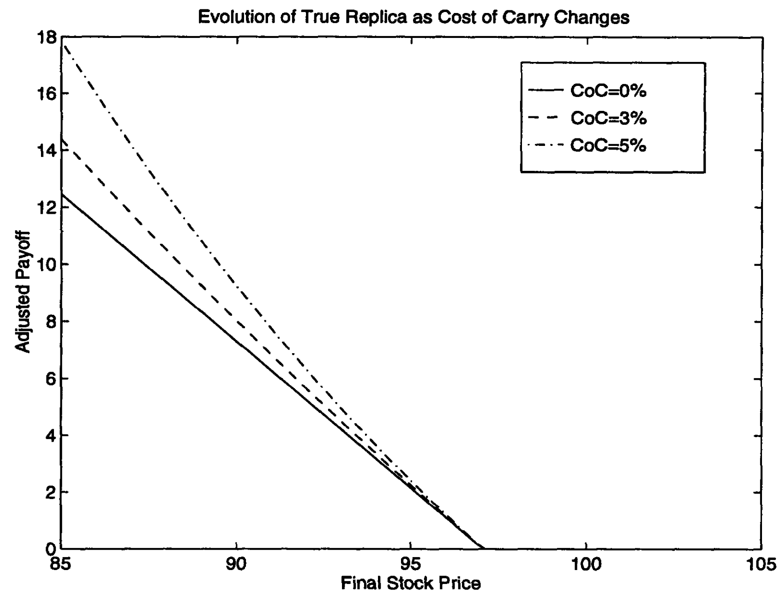
$$\epsilon = \max_{0 \leq t \leq T} e^{-rt} |Call(H, K_C, T-t) - N \cdot Put(H, K_P, T-t)| \tag{6.2}$$

where N denotes the put notional and K_C, K_P are the strikes of the call and put.

6.3 Finding the Optimal Replica

We restrict our replicating portfolio to a single put option, which is specified by the notional and strike. The optimal replica is the put option that minimizes our replication error, which is currently specified by (6.1). In the next section, we will use another measure of replication

[1] Recall that we are looking at out-of-the-money barrier options.



$(\sigma = .15,\ K_c = 103,\ H = 100)$

Figure 6-1: Adjusted Payoffs.

error.

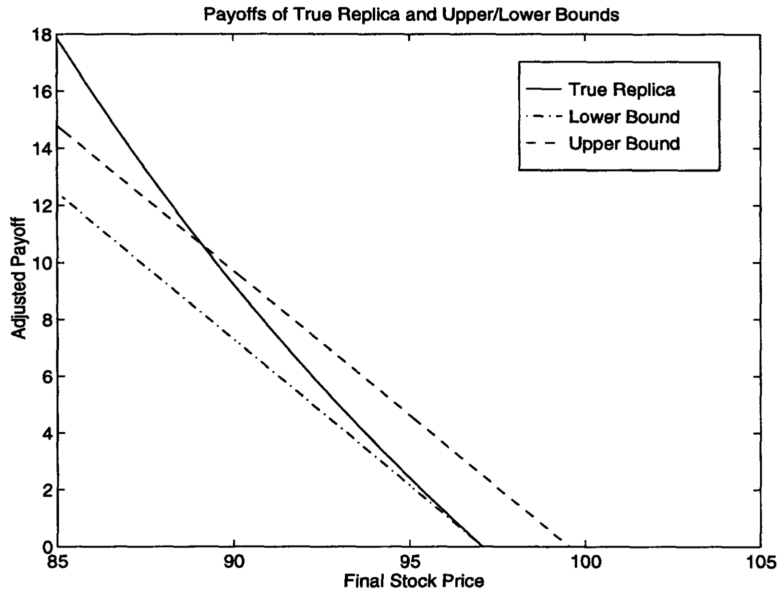
By in-out parity and Table 3.1, the adjusted payoff of a down-and-in call (with an out-of-the-money barrier) is:

$$\hat{f}(S_T) \equiv \begin{cases} (\frac{S_T}{H})^p \left((H^2/S_T) - K_C\right) & \text{if } S_T < H^2/K_C, \\ 0 & \text{if } S_T > H^2/K_C. \end{cases} \tag{6.3}$$

where $p = 1 - \frac{2(r-\rho)}{\sigma^2}$.

We denote the difference between the interest rate and the dividend rate $(r - \rho)$ as the cost of carry (CoC). In Figure 6-1, we plot the evolution of the adjusted payoff for different costs of carry. For zero cost of carry, the adjusted payoff is linear, and perfect replication is possible using a single vanilla put. We will be primarily interested in non-zero cost of carry, where perfect replication requires an infinite number of options. In particular, our replicas will be single European put, which is specified by a notional and strike. Note that Bowie and Carr[7] showed the existence of tight upper and lower bounds using single puts (see Figure 6-2). Clearly, the lower bound is valid, since it is dominated by the actual replica. In our strategy, we are only interested when the stock price is above the barrier. For those prices, the upper bound holds.

In Figure 6-3, we plot the replication error (as defined in (6.1)) as a function of the



$(r = .05,\ \rho = 0,\ \sigma = .15,\ K_c = 103,\ H = 100,\ T = .25)$

Figure 6-2: Adjusted Payoffs of Upper/Lower Bounds and True Replica.

replica, which we restrict to be a single put. Our optimal replica corresponds to the global minimum. Two dimensional minimization is a numerically difficult problem, and naive attempts to use MATLAB's optimization package were only partially successful. We were always able to find a point near the global minimum, but the answer was very sensitive to the initial guess. The standard MATLAB procedures had a difficult time locating the exact global minimum. This fact is all the more curious, since the graph does not appear to have local minimums, but rather only one global minimum.

To avoid these difficulties, we use a common technique from high dimensional optimization. We reduce the problem to a one dimensional problem and then show the optimal solution is near the one dimensional solution. Our search space is sufficiently reduced, so that we can use a brute force search to find the global minimum. In many cases, the one dimensional solution is, for all practical purposes, the global minimum.

Our reduction is as follows. We restrict our replicating put to have the same Black Scholes price as the down-and-in call. For a given strike, we set the notional such that:

$$N = \frac{DIC(S, K_C, T)}{Put(S, K_P, T)}$$

This new minimization has only one parameter: the strike. Upon finding the strike that

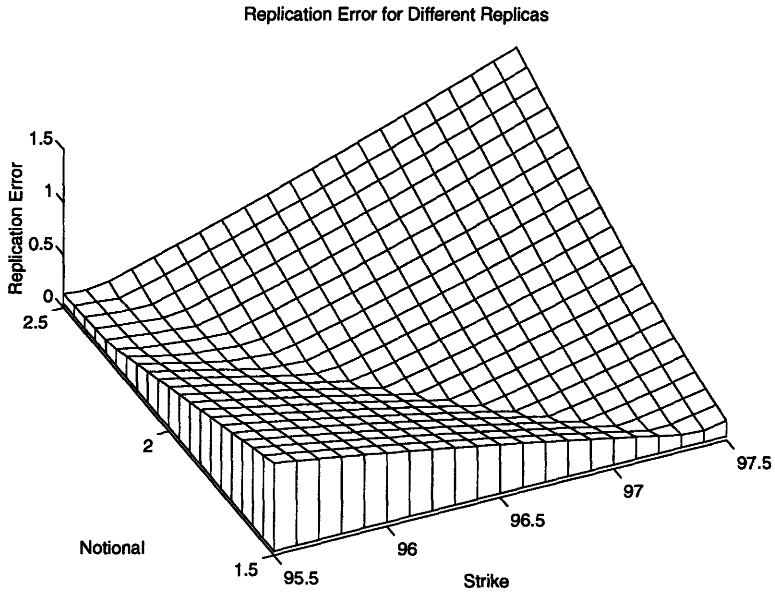


Figure 6-3: Replication Error for Various Linear Replicas.

minimizes the difference, we note the following arbitrage relationship:

$$N \cdot Put(S, K_H, T) - \epsilon \leq DIC(S, K, T) \leq N \cdot Put(S, K_H, T) + \epsilon$$

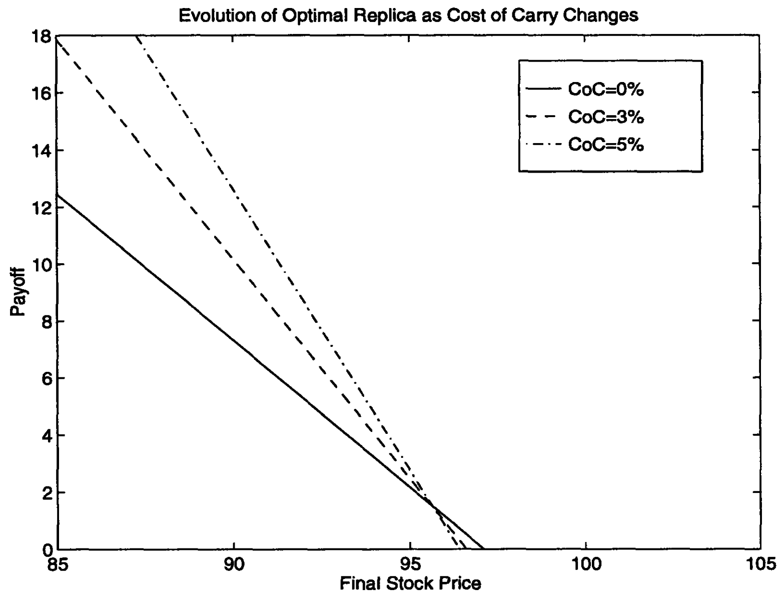
where ϵ is the replication error of the one-dimensional optimal solution. To avoid arbitrage, the price of the down-and-in call must be within the replication error of the price of the replica.

Given a replica with replication error ϵ, the global best replica must have a smaller replication error. Therefore, to find the global minimum, we only need to search for replicas whose price is in the range:

$$(DIC(S, K, T) - \epsilon, DIC(S, K, T) + \epsilon)$$

This area is sufficiently small that a brute force search can be applied.

In Figures 6-4 and 6-5 and Table 6.1, we present the results of this optimization. In Figure 6-4, we show how the optimal replicas evolve as the cost of carry adjusts. In Figure 6-1, the true replicas have the same "strike" (i.e. zero crossing) and they curve upward as the cost of carry increases. For the optimal replicas, the notational (i.e. slope) increases, but the strike decreases as the cost of carry increases. In Table 6.1, we explicitly give the



$(r = 0.05,\ \sigma = .15,\ K_c = 103,\ H = 100)$

Figure 6-4: Payoffs of Optimal Linear Replicas.

replication error for various parameter settings. Note that the replication error is expressed as a percentage of the Black Scholes price of the down-and-in call and is usually a small percentage of the option's price.

In Figure 6-5, we compare the true replica against the optimal replica for $CoC = 5\%$. Clearly, the optimal replica is **not** the best linear approximation[2] to the true replica. In fact, by simply looking at the graph, one would not expect these two payoffs to be closely related. The key point to remember is that we are only concerned when the stock price is at the barrier.

Finding the optimal replica is indeed a fruitful exercise. For example, the optimal replica demonstratively outperforms the upper and lower bounds. In Figure 6-6, we plot the difference in price (between the call and the replica) as a function of first passage time to the barrier for the optimal replica and upper/lower bounds. Clearly, the optimal replica is vastly superior.

The preceding results seem to indicate the static hedge strategy is indeed feasible. Given a liquid vanilla option market, it is possible to statically hedge our down-and-in call with a very small replication error. Since our replica is a single put option, our transactions costs

[2]Using typical linear approximation methods such as unweighted least squares.

T	σ .15	.2	.25
.25	2.4%	1.8%	1.4%
.5	2.8%	2.2%	1.7%
1	3.8%	2.9%	2.5%

(CoC = 3%, $K_C = 103$)

T	σ .15	.2	.25
.25	3.5%	2.7%	2.1%
.5	4.3%	3.3%	2.8%
1	5.8%	4.5%	3.9%

(CoC = 5%, $K_C = 103$)

T	σ .15	.2	.25
.25	1.4%	1.0%	0.8%
.5	1.8%	1.6%	1.2%
1	2.6%	2.2%	1.9%

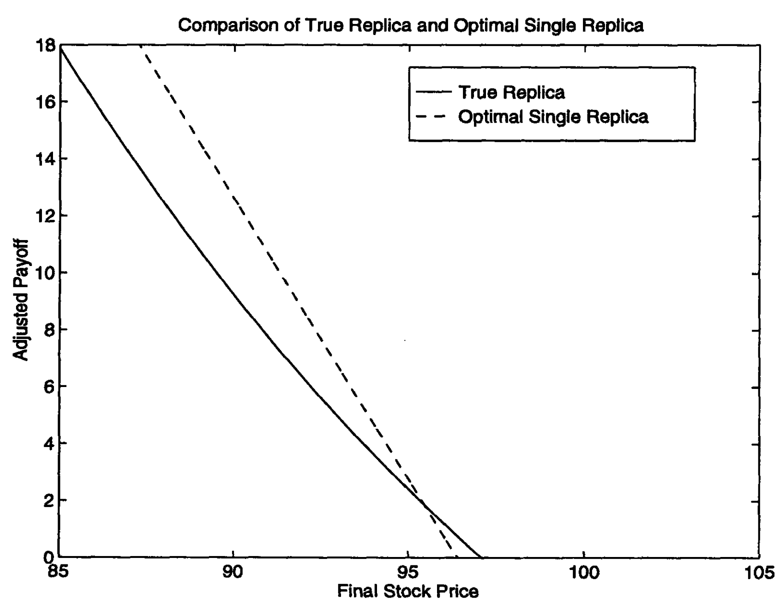
(CoC = 3%, $K_C = 106$)

T	σ .15	.2	.25
.25	2.1%	1.7%	1.5%
.5	2.8%	2.4%	2.0%
1	4.2%	3.4%	3.0%

(CoC = 5%, $K_C = 106$)

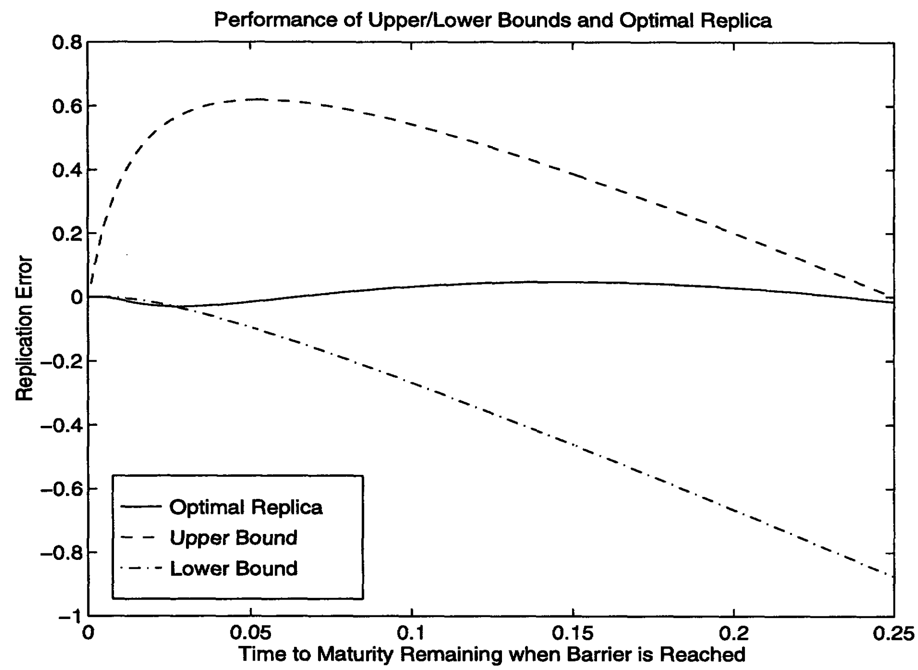
(Initial Stock Price = 103)

Table 6.1: Replication Error (as a percentage of the down-and-in call's price).



$(r = 0.05,\ \rho = 0,\ \sigma = .15,\ K_c = 103,\ H = 100)$

Figure 6-5: Payoffs of True and Optimal Linear Replicas.



$(r = 0.05,\ \rho = 0.0,\ T = .25,\ \sigma = .15,\ K_c = 103,$ Black Scholes Option Price $= 1.10)$

Figure 6-6: Relative Performance of Replicas.

should be modest. Next, we examine volatility sensitivity.

6.4 Shifts in Volatility

In this section, we will examine the exposure of our static hedge to a shift in volatility. Fundamentally, the diffusion process of the underlying stock is being altered (i.e. the value of σ in (2.1) changes). Thus, we are leaving the Black Scholes model[3] in order to better model "reality." For simplicity, we will make a rather tenuous assumption. In particular, we will assume the implied volatility of options *always* matches the instantaneous volatility of the stock price.[4] Thus, as the volatility changes in the diffusion processs, the option price changes as well. One possible interpretation of this scenario is that volatility risk is not priced in the market, and changes in volatility are completely unpredicted. Furthermore, this assumption imposes that implied volatilites are constant across all options. In some markets, this statement is blantantly false.

[3] In the Black Scholes model, volatility is assumed to be constant.

[4] Historically, implied volatilities are often higher than realized volatilities. This fact is often attributed to additional costs the hedger must bear such as volatility risk, gamma risk, and transaction costs.

Given this interpretation, static replication has another important advantage over dynamic approaches. In dynamic schemes, the replicas are the stock and bonds, which have no volatility sensitivity (assuming a simple delta hedging scheme). Thus, the hedger bears the full volatility risk.[5] In static replication, the replicas are other options, which have volatility sensitivity. In some cases, the volatility exposures of the replica and original security will offset, which in effect, reduces the hedger's exposure.

Finally, we need to further characterize the possible changes in volatility. We will take a rough, but robust, approach and allow volatilities to changes arbitrarily within a given range. In other words, volatilities are allowed to freely jump within some bounds. By allowing volatility to change so radically, our approach is a worst case approach, and our errors should be reliable bounds.

6.4.1 Zero Cost of Carry

Zero cost of carry is a very special situation. When $r = \rho$, the adjusted payoff (as given in (6.3)) is independent of σ. Thus, the replica is immune to shifts in volatility. In other words, static hedging has no volatility exposure whatsoever. Indeed, zero cost of carry is an ideal condition. Not only does the adjusted payoff match that of a European put, but the replica is immune to volatility shifts.

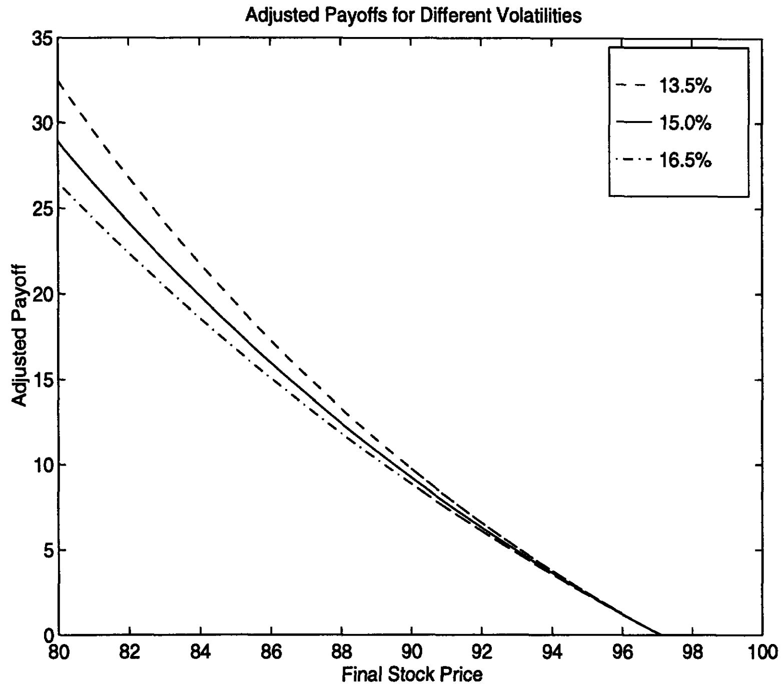
For most underlying securities, zero cost of carry is rare. One possible occurence would be in foreign exchange, when two countries have similiar interest rates. Another possibility are options on forwards. By construction, forwards have zero of cost of carry. In addition, if the cost of carry is close to very small (especially when compared to σ^2), the benevolent properties of zero cost of carry are closely preserved.

6.4.2 Non-Zero Cost of Carry

In Figure 6-7, we plot the adjusted payoffs for different values of implied volatility. We consider volatility shift of $\pm 10\%$ from the initial value of 15%. To create perfect replication, our payoff would have to match the adjusted payoff of the current volatility. Since volatility changes, this task is impossible.[6] In our model, even with an arbitrary number of European

[5]The hedger could try to reduce his risk by hedging his vega exposure with other options. In doing so, he uses options to hedge volatility, and stocks and bonds to hedge his remaining exposure. In static replication, we attempt to hedge all his exposures (vega, delta, gamma) with a single replica.

[6]Recall the uniqueness of the adjusted payoff.



$(r = 0.05, \rho = 0.0, T = .25, K_c = 103,)$

Figure 6-7: Adjusted Payoffs as a Function of Volatility.

replicas, it is impossible to create perfect replication.[7]

Again, we need a measure of our replication error. We will use

$$\epsilon = \max_{\sigma_{min} \leq \sigma \leq \sigma_{max}} \max_{0 \leq t \leq T} e^{-rt} |DIC(H, T-t, \sigma) - Hedge(H, T-t, \sigma)| \tag{6.4}$$

where $\sigma_{min}, \sigma_{max}$ are lower and upper bounds on volatility and $DIC(\cdot), Hedge(\cdot)$ are pricing formulas for the down-and-in call and the hedge portfolio.

In Table 6.2, we give two numbers for each parameter setting. The first number is the replication error (6.4) of our optimized replicas from §6.3 (i.e. the same replicas used to generate Table 6.1 are used in Table 6.2). We express this number as percentage of the price of the down-and-in call. The second number is a crude measure of volatility exposure to an unhedged option:

$$\frac{\max(|DIC(S,T,\sigma_{min}) - DIC(S,T,\sigma)|, |DIC(S,T,\sigma_{max}) - DIC(S,T,\sigma)|)}{DIC(S,T,\sigma)}. \tag{6.5}$$

[7]In fact, it is still an open question: what is the optimal replica (allowing for non-linear adjusted payoffs)?

T	σ .15	.2	.25
.25	9.6% 24%	6.0% 19%	4.0% 17%
.5	11% 18%	6.8% 15%	5.5% 14%
1	14% 14%	9.4% 13%	6.9% 12%

(CoC = 3%, K_C = 103)

T	σ .15	.2	.25
.25	18% 23%	10% 19%	6.9% 17%
.5	19% 17%	12% 15%	9.0% 14%
1	24% 14%	16% 12%	12% 11%

(CoC = 5%, K_C = 103)

T	σ .15	.2	.25
.25	15% 34%	6.7% 26%	4.6% 22%
.5	13% 24%	7.3% 19%	5.4% 17%
1	15% 17%	9.5% 15%	7.2% 14%

(CoC = 3%, K_C = 106)

T	σ .15	.2	.25
.25	22% 34%	12% 25%	7.9% 21%
.5	22% 23%	13% 18%	9.4% 16%
1	25% 16%	16% 14%	12% 13%

(CoC = 5%, K_C = 106)

(Initial Stock Price = 103)

Table 6.2: Replication Error (with volatility changes) and Maximum Volatility Exposure.

This number corresponds to percentage exposure (maximized over feasible volatility shifts) due to volatility changes at the given stock price.

As expected, our errors are much worse than without volatility shifts. For CoC=3% or $T = .25$, the original optimized replicas perform reasonably well. There is a roughly 50% reduction in the volatility exposure over the pure volatility exposure. However, for CoC=5% and $T \geq .5$, the replicas did not provide much in terms of volatility exposure. Often, the replicating error was substantially worse that the given exposure. This fact indicates that for high cost of carry, a linear replica may be insufficient.

One way to improve our replicating error is to optimize our replicating put while assuming the possibility of volatility shifts. In Table 6.3, we present the replicating error for replicas that are optimized under measure (6.4). Overall, the new replicating errors are marginally better (a few percentage points).

This concludes our computational study of replicating a (out-of-the-money barrier) down-and-in call with a single put. These preliminary results do indicate that static replication may be a feasible strategy, especially in low cost-of-carry markets. Certainly, there is substantial room for additional tests and experiments, which we leave for future research.

T	σ .15	.2	.25
.25	9.4%	5.5%	3.9%
.5	10.4%	6.8%	4.9%
1	12.5%	8.4%	6.2%

(CoC = 3%, $K_C = 103$)

T	σ .15	.2	.25
.25	16.2%	9.8%	6.7%
.5	17.9%	11.2%	8.1%
1	21.7%	14.2%	10.6%

(CoC = 5%, $K_C = 103$)

T	σ .15	.2	.25
.25	11.1%	6.5%	4.3%
.5	11.5%	7.2%	5.2%
1	13.5%	8.7%	6.5%

(CoC = 3%, $K_C = 106$)

T	σ .15	.2	.25
.25	19.2%	10.7%	7.3%
.5	19.8%	12.1%	8.7%
1	23.1%	14.9%	11.0%

(CoC = 5%, $K_C = 106$)

(Initial Stock Price = 103)

Table 6.3: Replication Error Using Volatility Optimized Replicas.

Chapter 7

Conclusions

In this thesis, we studied the static replication of barrier-type options using plain vanilla options. The advantages of static replication over the traditional dynamic methods are fewer transactions and a possible reduction in volatility exposure.

We classified static replications schemes based upon the number of rebalances and number of maturities in the replicating portfolio. Using this classification, we showed static replications for single barrier, partial barrier, forward-starting barrier, and double barrier options. In addition, we showed how rolldown and lookback options could be decomposed into barrier options, which allowed us to apply static replication methods to these options. For some options, we showed several static replications schemes, which traded off the number of rebalances against the number of maturities in the replicating portfolio.

In addition, we showed how to convert options with non-flat barriers into equivalent options with a flat barrier by modifying the drift. For exponential barriers, this transformation simply added a constant to the drift. Furthermore, we showed how time-scaling could be used to convert from time-dependent volatility to time-dependent drift. Under time-dependent drift, we showed the impossibility of one-stage single-maturity static replication, but also showed the existence of other types of static replication.

Finally, we presented a computational study of static replication. Under the right conditions, we found static replications schemes that were simple (used only one option) and had very small hedging errors. Further studies need to be completed to test if static replication is truly pragmatic.

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