Exact and Asymptotic Enumeration of Permutations with subsequence conditions

by

Miklós Bóna

M.S., Eötvös Loránd University, Budapest (1992)

and

M.S., Paris 7 University, Paris (1992)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1997

Signature of Author ....

Department of Mathematics
May 2, 1997

Certified by ....

Richard Stanley
Professor of Mathematics
Thesis Supervisor

Accepted by ....

Hung Cheng
Chairman, Applied Mathematics Committe

Accepted by ....

Richard Melrose
Chairman, Departmental Committee for Graduate Students

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Abstract

We study permutations with a prescribed number of subsequences of a given type. First we consider the case when this number is zero and the subsequence is of length 4. We show that only very few permutations of length n, in fact simply an exponential number of them, have this property. An exact formula for the pattern 1342, and a link with labeled plane trees and planar maps is presented. We prove that some subsequences of length four are significantly easier to avoid than others. We also show recursive methods for treating longer patterns.

Then we consider the general case, that is, when this prescribed number is not necessarily zero. We prove that if the subsequence is 132, then the number of permutations of length n containing exactly r subsequences of the above type is a polynomially recursive function of n, for any natural number r. A surprisingly simple closed formula for the case of $r = 1$, and a recursive formula for $r = 2$ are included.

Finally, we consider the partially ordered set of all finite permutations and constructively prove that it contains an infinite antichain.

Thesis Supervisor: Richard Stanley
Title: Professor of Mathematics

To My Parents

To My Brother and My Sisters

Acknowledgements

In early February, 1995, when I was in my second year as a graduate student, I asked my advisor, Professor Richard Stanley for a new research project. He proposed the area of permutations avoiding certain patterns. This proved to be the most grateful topic of my mathematical life so far, providing a plethora of natural, easily understandable though deep and challenging problems.

Only days later Dan Spielman, who was a graduate student at MIT that time learned that I had started working in this field and told me about a closely related problem of his interest, that of the existence of an infinite antichain of permutations. Soon, we started thinking together about that question, which committed me to the area of pattern avoidance even more. Our joint work resulted in Chapter 4 of this thesis.

Then I had a fruitful summer in Hungary which gave rise to Chapter 2 of this thesis. I am grateful to my family who provided excellent working conditions during those months.

During the Rotafest Conference, in April 1996 Andrew Odlyzko told me about a flurry of interest and new developments concerning the enumeration of permutations with a given number of subsequences of a certain type. Two days later Herbert Wilf provided additional information on that subject. I instantly became interested and the results of the first three sections of Chapter 3 were found soon after.

In July of 1996 Gian-Carlo Rota encouraged me to work on the general Noonan-Zeilberger conjecture. I took his advice and, taking advantage of another enriching summer vacation at home, the main result of Chapter 3 was proved two weeks later.

Coming back to Cambridge in the Fall I received many valuable comments from Doron Zeilberger, who pointed out that my argument proved more than I had thought.

In November of 1996 I found a striking match between two unrelated-looking sequences while exploring N.J.A. Sloane's superseeker program. Thanks to Richard Stanley, Robert Cori and Gilles Schaeffer, I have soon learnt about a class of labeled trees relevant to my discovery, and two weeks later Section 2.1.3 was born.

Beyond this, fortunately, there is a long list of people who helped, encouraged and advised me in my efforts to count permutations. I am particularly grateful for this to Sergey Fomin, Ira Gessel, Rodica Simion, Christian Krattenthaler, Daniel Kleitman, Tom Roby, Julian West, Joseph Bonin and Gábor Hetyei. In my earlier years, László Babai, László Lovász and László Székely introduced me to the amazing, captivating world of combinatorics.

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Chapter 1

Introduction

Let $q = (q_1, q_2, \cdots, q_k) \in S_k$ be a permutation, and let $k \leq n$. We say that the permutation $p = (p_1, p_2, \cdots, p_n) \in S_n$ is *q-avoiding* if there is no $1 \leq i_{q_1} < i_{q_2} < \cdots < i_{q_k} \leq n$ such that $p(i_1) < p(i_2) < \cdots < p(i_k)$. Otherwise we say that p contains q as a pattern, or p contains a subsequence of type q. For example, a permutation is 12345-avoiding if it does not contain any increasing subsequence of length 5 in the above one-line notation. For another example, a permutation is 132-avoiding if it doesn't contain three elements among which the leftmost is the smallest and the middle one is the largest.

It is a natural and easy-looking question to ask how many permutations of length n (or, in the sequel, n-permutations) avoid a given pattern q. In the rest of this work this number will be denoted by $S_n(q)$. Throughout Chapter 2 we will study this function of n and q.

If the length of q is three, (and practically, only then) this question can indeed be satisfactorily answered fairly easily.

Theorem 1 *Let q be any permutation of length 3. Then $S_n(q) = c_n = \binom{2n}{n}/(n+1)$.*

Proof: We first show that $S_n(q)$ does not depend on q. It is clear that a permutation avoids 123, then its reverse avoids 321, thus $S_n(123) = S_n(321)$. Similarly, if a permutation avoids 132, then its reverse avoids 231, its complement (that is, the permutation obtained by subtracting the entries of the original permutation from $n+1$) avoids 312, and the reverse of its complement avoids 213. Therefore we also have $S_n(132) = S_n(231) = S_n(312) = S_n(213)$. All we need to show to prove our claim is that $S_n(123) = S_n(132)$. This is the content of the next lemma.

Lemma 1 $S_n(123) = S_n(132)$ *for all* n.

Proof: The are several ways to prove this first nontrivial result of the subject. We choose the one of Simion and Schmidt [25], as the machinery used by them will be useful for the purposes of the next chapter. Here and later in this work we will always use the one-line notation for permutations. An entry of a permutation which is smaller than all the entries that preceed it is called a *left-to-right minimum.* Note that the left-to-right minima form a decreasing subsequence.

We will construct a bijection f from the set of all 123-avoiding n-permutations onto the set of all 132-avoiding n-permutations which leaves all left-to-right minima fixed.

f is defined as follows. We take any 123-avoiding permutation p, and fix all its left-to-right minima. Then going from the left to the right, we put the elements which are not left-to-right minima into the empty slots between the left-to-right minima so that in each step we place the smallest element we haven't placed yet which is larger then the previous left-to-right minima.

For example, if $p = 4\,6\,5\,1\,3\,2$, then the left-to-right minima are the entries 4 and 1, thus we leave them in the first and fourth positions. The first empty slot is the second position and we put there the smallest entry which is larger than 4, that is to say, the entry 5. Similarly, we put 6 to the third position as it is the smallest of the entries not yet used which is larger than 4 (in fact, this is the only such entry). Then by the same reasoning we put 2 into the fifth position and 3 into the sixth position. This way we get the permutation $f(p) = 4\,5\,6\,1\,2\,3$

Clearly, $f(p)$ is 132-avoiding, because if there were a 132-pattern in it, then there would be one which starts with a left-to-right minimum, but that is impossible as elements smaller than any given left-to-right minimum are written in increasing order.

The inverse of f is even easier to describe: keep the left-to-right minima of p fixed and put all the other elements into the empty slots between them in decreasing order. Then we obtain a permutation which is the union of two decreasing subsequences and thus 123-avoiding. If we apply this operation to $f(p)$, then we must get p back, as the left-to-right minima haven't changed, and the other elements must have been in decreasing order in p, too, otherwise p wouldn't have been 123-avoiding. This completes the proof of the lemma. $\diamond$

To prove the theorem, it is therefore enough to show that $S_n(132) = \binom{2n}{n}/(n+1)$. Suppose we have an 132-avoiding n-permutation in which the entry n is in the ith position. Then it is clear that any entry to the left of n must be smaller than any entry to the right of n. Moreover, there are c_{i-1} possibilities for the substring of entries to the left of n and c_{n-i} possibilities for that to the right of n. Summing for all i we get the following recursion:

$$c_n = \sum_{i=0}^{n-1} c_{i-1} c_{n-i}. \tag{1.1}$$

Therefore, if $C(x) = \sum_{n=0}^{\infty} c_n x^n$ is the the ordinary generating function of the c_n, then (1.1) implies $C^2(x)x + 1 = C(x)$, which yields

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{1.2}$$

By standard methods this yields

$$C(x) = \sum_{k=1}^{\infty} \frac{1}{k}\binom{2k-2}{k-1} x^{k-1} = \sum_{n=0}^{\infty} \frac{1}{n-1}\binom{2n-4}{n-2} x^{n-2} \tag{1.3}$$

and the statement of the theorem is proved. $\diamond$

If q is longer than three, then the most exact result is due to Regev [21] and deals with monotonic patterns:

Theorem 2 *For all n, $S_n(1234\cdots k)$ asymptotically equals*

$$\lambda_k \frac{(k-1)^{2n}}{n^{(k^2-2k)/2}}.$$

Here

$$\lambda_k = \gamma_k^2 \int_{x_1\geq} \int_{x_2\geq} \cdots \int_{\geq x_k} [D(x_1, x_2, \cdots, x_k) \cdot e^{-(k/2)x^2}]^2 dx_1 dx_2 \cdots dx_k,$$

where $D(x_1, x_2, \cdots, x_k) = \Pi_{i<j}(x_i - x_j)$, and $\gamma_k = (1/\sqrt{2\pi})^{k-1} \cdot k^{k^2/2}$.

We will not need this strong version of the theorem, only the weaker statement saying that if $q = 1\,2\,3 \cdots k$, then $S_n(q) < (k-1)^{2n}$, thus we only prove this latter one here.

Let us say that an entry x of a permutation is of order i if it is the top of a rising subsequence of length i, but there is no rising subsequence of length $i+1$ it is the top of. Then for all i, elements of order i must form a descending subsequence. Therefore, a q-avoiding permutation can be decomposed into the union of $k-1$ descending subsequences. Clearly, there are k^{n-1} ways to partition the elements into $k-1$ classes and there are less than k^{n-1} ways to assign each position to one of the subsequences, completing the proof. $\square$

However, if q is not monotonic, then this problem turns out to be surprisingly hard. The following result from complexity theory [7] indicates that this is not by chance:

Lemma 2 *Let a be a given n-permutation. Then the problem of deciding whether an arbitrary permutation b is contained in a as a pattern is NP-complete. Moreover, the problem of counting the number of n-permutations avoiding b is #P-complete.*

That is why in the general case all we can expect is an upper bound or an asymptotic formula for $S_n(q)$, not an exact formula. The major problem of this area is to prove the conjecture of Wilf and Stanley [33] from 1990, stating that for each pattern q there is an absolute constant c so that $S_n(q) < c^n$ holds. In Section 2.1 we will prove this conjecture for all patterns of length 4.

For the case of $q = 1342$, however, we do better. We are going to present an *exact formula* for $S_n(q)$ by presenting interesting links between permutation pattern avoidance, labeled rooted trees, and planar maps. This formula will enable us to prove several conjectures for this pattern.

We have seen in Theorem 1 that $S_n(q)$ is the same for all patterns q of length 3. Numerical evidence computed by West in [34] shows, however, that this is not the case anymore if q is of length 4. Some of the values $S_n(q)$ for $n \leq 8$ from the above source are shown below.

- for $S_n(1342)$: 1, 2, 6, 23, 103, 512, 2740, 15485
- for $S_n(1234)$: 1, 2, 6, 23, 103, 513, 2761, 15767
- for $S_n(1324)$: 1, 2, 6, 23, 103, 513, 2762, 15793.

This observation rises the question whether it will always remain the case that $S_n(1423) < S_n(1234)$ if $n \geq 6$ and $S_n(1234) < S_n(1324)$ if $n \geq 7$. In Section 2.1 we answer this question in the affirmative. These are the first results we know of which prove that one pattern is more likely to occur in a random permutation than another one.

A general and probably very hard question is to decide whether $S_n(q_1) < S_n(q_2)$ for some n implies $S_n(q_1) < S_n(q_2)$ for *all* n. We point out that a positive answer to this question would imply a positive answer for the conjecture of Wilf and Stanley. Indeed, if q is any pattern of length k, then let q' be the monotonic pattern of length $k+1$. Then

$S_{k+1}(q) = (k+1)! - k^2 - 1 < S_{k+1}(q') = (k+1)! - 1$ would obviously hold, implying $S_n(q) < S_n(q') < k^{2n}$ (by Theorem 2) for all n.

The possible hardness of this problem is also underlined by the surprising fact that if the length of q is fixed, then the monotonic pattern does not provide the maximum or the minimum of the values $S_n(q)$. This remains the case when q is of length 5 or 6, too. Numerical evidence in [34] also suggests that it is much closer to the maximum than to the minimum. A complete understanding of this phenomenon would certainly boost the efforts to prove the $S_n(q) < c^n$ conjecture.

It has also been conjectured that while $S_n(q)$ is not the same for all patterns of length k if $k > 3$, these numbers will be at least asymptotically equal. We will disprove this conjecture by showing that $S_n(1234) < S_n(1324)$ even in the asymptotic sense. Even the weaker conjecture that $\lim_{n\to\infty} S_n(q)^{1/n} = (k-1)^2$ will be disproved in Section 2.1.3.

In section 2.2 we consider longer patterns. We will show a series of results, all of which show that if the conjecture of Wilf and Stanley is true for some pattern q, then it is true for some other patterns q' as well, where q' can be built out of q in a specified way.

In Chapter 3 we study permutations with a required number r (thus not necessarily 0) of occurences of a given pattern q. Here it would not be too interesting to ask the order of magnitude of the number of such permutations as it is clear that deleting r points from such a permutation we can always get a permutation with no subsequences of type q. Instead, we will adapt a more qualitative approach for this function $S_{q,r}(n)$.

Again, if q is of length 3, then exact enumeration is possible. Recently, Noonan has proved [19] that if $q = 1\,2\,3$, then $S_{q,1}(n) = \frac{3}{n}\binom{2n}{n-3}$. In section 3.2 we will prove a similar, yet even simpler formula for the case of $q = 1\ 3\ 2$ and $r = 1$, showing $S_{q,1}(n) = \binom{2n-3}{n-3}$. We point out that this formula is even simpler than that for $r = 0$, proved in Theorem 1. It is obtained by a generating function argument, and thus a direct combinatorial explanation for this formula is yet to be found. By obvious reasons of symmetry, this result completes the analysis of the case when q is of length 3 and $r = 1$. Then in section 3.3 we prove a recursive formula for the case $r = 2$.

However, the general conjecture of this area, made by Zeilberger and Noonan in [37] states much more. It says that for *any* fixed q and r the function $S_{q,r}(n)$ is P-recursive in n. (Recall that a function $f : N \to R$ is called P-recursive if there exists a natural number k and polynomials $p_0(n), p_1(n), p_2(n), \cdots p_k(n)$ so that for any positive integer n, there exist polynomials $P_0, P_1, \cdots, P_k \in Q[n]$, with $P_k \neq 0$ so that

$$P_k(n)f(n+k) + P_{k-1}(n)f(n+k-1) + \ldots + P_0(n)f(n) = 0$$

for all n. Some basic properties of P-recursive functions will be given in section 3.1).

To illustrate how far we are from the solution of this conjecture, we note that if q is longer than three, then we do not have a proof for any $r > 0$. If $r = 0$, then with the single exception of Theorem 9 of this work, which proves the conjecture for $q = 1342$, we have a proof only for the monotonic patterns of any length (see [38] or [13]) and of course, for those patterns for which it is known that they are avoided by as many n-permutations as for the monotonic patterns of the same length are.

In Section 3.4 we will prove this conjecture for the case of $q = 1\ 3\ 2$ and *any fixed* r. This is the first result we know of when the case of each r is solved for some given q. In fact, we show that this remains true even if we impose some restrictions on the permutations. We also show the stronger statement that the ordinary generating function $G_r(x)$ of $S_{132,r}(n)$ is algebraic, in fact, it is rational in the variables x and $\sqrt{1-4x}$. We use this information to show that the degree of the polynomial recursion satisfied by $S_{132,r}(n)$ is r.

In Chapter 4, the result of which has been obtained in joint work with Daniel A. Spielman, we take the set of all finite permutations and consider it as a partially ordered set in which $q_1 \leq q_2$ if and only if q_1 is a pattern contained in q_2. This will clearly be a graded poset. The conjecture we address in Chapter 2 and numerical evidence suggest that any element of this poset is covered by almost every element (that is, all but c^n) of any higher rank. Therefore, it could seem plausible that if we choose a large number of elements of this poset, then no other element will be incomparable to all of the chosen elements. However, this is not the case; in fact, we are going to construct an infinite antichain in this poset.

Chapter 2

The $S_n(q) < c^n$ conjecture

2.1 The case of length 4

2.1.1 Earlier results

Our goal in this chapter is to prove that for any pattern q of length 4 there exists a constant c so that $S_n(q) < c^n$ for all n. There are 24 patterns of length four; now we are going to give a survey of results previously obtained for them making it possible for us to restrict our attention to only two out of these 24 patterns. These patterns are 1 3 4 2 and 1 3 2 4.

It is clear that applying the symmetry arguments already seen in the proof of Theorem 1 (using the reverse and complement of permutations) we can restrict ourselves to those patterns of length four in which

- the first element is smaller than the last one and
- the first element is 1 or 2.

This still leaves us 11 patterns, namely 1 2 3 4, 1 2 4 3, 1 3 2 4, 1 3 4 2, 1 4 2 3, 1 4 3 2, 2 1 3 4, 2 1 4 3, 2 3 1 4, 2 3 4 1 and 2 4 1 3. Note that if p contains q, then the inverse of p

clearly contains the inverse of q (as the inverse of a permutation matrix is its transpose), so $S_n(q) = S_n(q^{-1})$. Therefore, we can drop 1 4 2 3, too, as its inverse 1 3 4 2 remains on the list. Similarly, we can drop 2 3 1 4 as its complement is 3 2 4 1 and the reverse of that is again 1 4 2 3. The next serious step in advance is the following theorem of West and Babson:

Theorem 3 *$S_n(123\cdots ra_{r+1}a_{r+2}\cdots a_{r+t}) = S_n(r\cdots 321a_{r+1}a_{r+2}\cdots a_{r+t})$, for any natural numbers r, t and n.*

Proof: See [34] for $r = 2$, see [1] for $r = 3$ and see [35] for $r > 3$. $\diamond$

In words, if the first r elements of a pattern are the smallest ones and they are in increasing order, their string can be reversed without changing the value of $S_n(q)$.

This theorem, together with its dual versions imply that we can drop 1 2 4 3, 1 4 3 2, 2 1 3 4, 2 1 4 3 and 2 3 4 1 as each of them is avoided by $S_n(1234)$ n-permutations. (And we know by Theorem 2 that $S_n(1234) < 9^n$). The only remaining pattern we have to deal with is 2 4 1 3. This is taken care of by the following lemma of Stankova.

Lemma 3 *[27] $S_n(1423) = S_n(2413)$ for all n.*

Therefore, the only patterns we still need to prove the conjecture are indeed 1 3 2 4 and 1 3 4 2. The next two sections deal with these two patterns.

2.1.2 The pattern 1324

Numerical evidence we presented in the introduction showed that if $n = 7$ or $n = 8$, then $S_n(1234) < S_n(1324)$. We are going to show that this remains the case as n grows. Recall that elements in a permutation which are smaller than any elements they are preceded by are called left-to-right minima. Similarly, we will say that an element is a *right-to-left* *maximum* if it is larger than any element it precedes. Note that the right-to-left maxima form a decreasing subsequence as the left-to-right minima do.

Theorem 4 *For all $n \geq 7$, $S_n(1234) < S_n(1324)$.*

Proof: We are going to classify all permutations of n according to the set and position of their left-to-right minima and right-to-left maxima. This definition is crucial in all this chapter, so we announce it on its own:

Definition 1 *Two permutations x and y are said to be in the same* class *if*

- *the left-to-right minima of x are the same as those of y*
- *they are in the same positions*
- *the same holds for the right-to-left maxima.*

For example, $x = 5\,1\,2\,3\,4$ and $y = 5\,1\,3\,2\,4$ are in the same class, but $z = 2\,4\,3\,1\,5$ and $v = 2\,4\,1\,3\,5$ are not, as the third entry of z is not a left-to-right minimum whereas that of v is.

The outline of our proof is going to be as follows: we show that each nonempty class contains *exactly* one 1234-avoiding permutation and *at least* one 1324-avoiding permutation. Then we exhibit some classes which contain more than one 1324-avoiding permutation and complete the proof.

Lemma 4 *Each nonempty class contains exactly one 1234-avoiding permutation.*

Proof: Suppose we have already picked a class, that is, we fixed the positions and values of all the left-to-right minima and right-to-left maxima. It is clear that if we put all the remaining elements into the remaining slots in decreasing order, then we get a 1234-avoiding permutation. (Indeed, the permutation obtained this way consists of 3 decreasing subsequences, that is, the left-to-right minima, the right-to-left maxima, and the remaining entries. Thus, if there were a 1234-pattern, then by the pigeon-hole principle two of its elements would be in the same decreasing subsequence, which would be a contradiction). On the other hand, if two of these elements, say a and b, were in increasing order, then together with the rightmost left-to-right minimum on the left of a and the leftmost right-to-left maximum on the right of b they would form a 1234-pattern. Finally, if the chosen class is nonempty, then we can indeed write the remaining numbers in decreasing order without conflicting with the existing constraints— otherwise the class would be empty. (In other words it is the decreasing order of the remaining elements that violates the least number of constraints). $\diamond$

Corollary 1 *The number of nonempty classes is asymptotically $c \cdot 9^n/n^4$, where c is as in Lemma 1.*

This is immediate by Lemma 4. $\square$

Lemma 5 *Each nonempty class contains at least one 1324-avoiding permutation.*

Proof: First note that if a permutation contains a 1324-pattern, then we can choose such a pattern so that its first element is a left-to-right minimum and and its last element is a right-to-left maximum. Indeed, we can just take any existing pattern and replace its first (last) element by its closest left (right) neighbor which is a left-to-right minimum (right-to-left maximum). Therefore, to show that a permutation avoids 1324, it is sufficient to show that it doesn't contain a 1324-pattern having a left-to-right minimum for its first element and a right-to-left maximum for its last element. (Such a pattern will be called a *good pattern*). Also note that a left-to-right minimum (right-to-left maximum) can only be the first (last) element of a 1324-pattern.

Now take any 1324-containing permutation. By the above argument, it has a good pattern. Interchange its second and third element. Observe that we can do this without violating the existing constraints, that is, no element goes on the left of a left-to-right minimum it is smaller than, and no element goes on the right of a right-to-left maximum it is bigger than. The resulting permutation is in the same class as the original because the left-to-right minima and right-to-left maxima have not been changed. Repeat this procedure as long as we can. Note that each step of the procedure decreases the number of inversions of our permutation by at least 1. Therefore, we will have to stop after at most $\binom{n}{2}$ steps. Then the resulting permutation will be in the same class as the original one, but it will have no good pattern and therefore no 1324-pattern, as we claimed. $\diamond$

Notation (by example): in the sequel we write $a_1 \; * \, a_2 \; * \; * \, b_1$ for the class of permutations of length 6 which have two left-to-right minima, a_1 and a_2, which are in the first and third position, and one right-to-left maximum, b_1, which is in the last position.

Finally, we must show that "at least one" in the above lemma doesn't always mean exactly one. If $n = 7$, then the class $3 * 1 * 7 * 5$ contains two 1324-avoiding permutations, $3\,6\,1\,2\,7\,4\,5$ and $3\,4\,1\,6\,7\,2\,5$. This proves $S_7(1234) < S_7(1324)$. For larger n we can extend this example in an easy way, such as taking the class $n\ (n-1) \cdots 8\ 3\ *\ 1\ *\ 7\ *\ 5$. This shows that there are more 1324-avoiding permutations than 1234-avoiding ones and completes the proof of the theorem. $\diamond$

Definition 2 *A class which contains more than one 1324-avoiding permutation is called a* large class*.*

Now we can attack the problem of asymptotics in a simple way. All we need to do is to evaluate the number of large classes. If there is a positive contant ϵ so that the number of large classes is at least ϵ times the number of all classes for all n, then we get that $S_n(1234)$ is asymptotically smaller than $S_n(1324)$. The following theorem shows that this is indeed the case.

Theorem 5 $S_n(1234)$ *is asymptotically smaller than* $S_n(1324)$.

Proof: We exhibit a set of large classes. They will be built up from our above example, that is, the class $3 * 1 * 7 * 5$ for $n = 7$. Now let $n > 7$ and let us choose any class C of permutations of length $n - 7$. (For example, let $n = 12$ and let C be $1 * * 5 2$). Now we define the *composition* of the class C and the class $3 * 1 * 7 * 5$ to a class C' of length n as follows. Simply add 7 to all left-to-right minima and right-to-left maxima of C and leave the empty slots between them as they are. Then write the class $3 * 1 * 7 * 5$ after this modified version of C. In this way our example results in the class $8 * * 12 9 3 * 1 * 7 * 5$. Clearly, this way we can define the compositon of *permutations* of these classes as well: if $p_1 = (a_1, a_2, \ldots, a_{n-7}) \in C$ and $p_2 = (3, b_1, 1, b_2, 7, b_3, 5) \in 3 * 1 * 7 * 5$, then let their composition be $p_1 = (a_1 + 7, a_2 + 7, \ldots, a_{n-7} + 7, 3, b_1, 1, b_2, 7, b_3, 5) \in C'$.

Now it is easy to see that if we have the permutations $p_1 \in C$ and $p_2 \in 3 * 1 * 7 * 5$, and both p_1 and p_2 are 1324-avoiding, then their composition is 1324-avoiding, too. (Indeed, it is not possible for a 1324-pattern to start somewhere among the first $n - 7$ entries and end somewhere among the last 7 entries). Therefore, every class obtained this way will be large because we have two different choices for p_2, and at least one choice for p_1.

(In our example, we get the permutations

- 8 12 10 11 9 3 6 1 2 7 4 5 and
- 8 12 10 11 9 3 2 1 6 7 2 5).

This shows that we can build up a large class of permutations of length n from every single class of permutations of length $n - 7$. The number of these classes equals $S_{n-7}(1234)$ by Lemma 4 and this is larger than $(1/9^7)S_n(1234)$ by Theorem 2. This immediately implies that $S_n(1324) \geq (1 + 1/9^7)S_n(1234)$, completing the proof of the theorem. $\diamond$

We have thus proved by the above theorem that $S_n(1324)$ is asymptotically larger than $S_n(1234)$, disproving the conjecture stating that all patterns of length k are equally likely to occur. We need more work to prove that $S_n(1324) < K^n$ for some constant K. This is the content of the next theorem. Before we start proving it, we introduce some new machinery that will be very useful in what follows.

Definition 3 *Two $n-$permutations x and y are said to be in the same* weak class *if the left-to-right minima of x are the same as those of y, and they are in the same positions.*

Thus here we don't require that the right-to-left maxima agree. For example, 3 4 1 2 5 and 3 5 1 2 4 are in the same weak class, though they are not in the same class.

The number of weak classes is easy to determine:

Lemma 6 *The number of nonempty weak classes is $c_n = \binom{2n}{n}/(n+1)$.*

Proof: Similar to the proof of Lemma 4. Each weak class contains exactly one 123-avoiding permutation which is obtained by writing all the entries which are not left-to-right minima in decreasing order. The number of 123-avoiding permutations is known to be $C_n = \binom{2n}{n}/(n+1)$ (see Theorem 1) and the proof is complete. ◇

Definition 4 *Let p be an n-permutation with m left-to-right minima, $a_m > a_{m-1}, \cdots > a_1 = 1$. Then an entry z of p which is not a left-to-right minima has rank i if $a_i < z < a_{i+1}$.*

Now we can state and proof our theorem on the upper bound for the number of 1324-avoiding permutations.

Theorem 6 *For all n, we have $S_n(1324) < 32^n$.*

Proof: As the number of weak classes is smaller than 4^n, it suffices to show that no weak class can contain more than 8^n 1324-avoiding permutations, and we will be done. Let W be any weak class of n-permutations with left-to-right minima $a_m > a_{m-1} > \cdots > a_1 = 1$. We are going to estimate the number of 1324-avoiding permutations in W. It is clear that entries of rank i must be on the right of a_i; otherwise some of them would be left-to-right minima. Therefore, in any 1324-avoiding n-permutation, the permutation of the entries of rank i must be 213-avoiding. We know that there are $b_i = a_{i+1} - a_i - 1$ such entries and thus $C_{b_i} < 4^{b_i}$ such permutations for each i. The problem is that there are several ways to merge these strings of elements of rank i into one single n-permutation. Thus we have to estimate the number of ways that merging can be done.

First suppose $m = 2$, thus W has only 2 left-to-right minima. Then the entries of any permutation in W can only have rank 1 or 2; we are going to call them *small* and *large* entries (respectively). Then it is easy to see that the following must hold in any 1324-avoiding permutation in W:

1. No 21-pattern of small entries can be followed by a large entry. Thus the set of small entries on the left of the rightmost large entry must form an increasing subsequence.

2. No small entry can be inserted between the two large entries of a 12-pattern both of which are on the right of $a_1 = 1$. Thus for any small entry s, the large entries put between 1 and s are larger than the large entries put on the right of s.

Indeed, if either condition is violated, $a_1 = 1$ and the three entries violating that condition would form a 1324-pattern.

If the first i elements of a permutation p form an increasing subsequence but the first $i+1$ don't, then we say that the *initial increasing subsequence* of p is of length i. If there are j ways to insert a bar among the entries of p so that everything before the bar is larger than everything after the bar, then we say that p has j cuts. We will call i and j the *parameters* of p.

The above observations suggest that in order to estimate the number of ways we can merge the strings of small and large entries together, we need to find an upper bound for the number of 213-permutations with long initial increasing subsequences as well as for those with many cuts. Now we are going to prove two lemmas which will provide these upper bounds.

Lemma 7 *Let x_i be the number of 213-avoiding n-permutations starting with an increasing subsequence of length at least i. Then $x_i < \frac{4^n}{2^i}$.*

Proof: It is easy to see that a 213-avoiding permutation is completely determined by the set and position of its right-to-left maxima. Indeed, once we know these, there is only one entry we can write into the rightmost empty slot: the one which is largest among those which are smaller than the closest right-to-left maximum to the right of that position. Thus it is enough to estimate the number of possible sets and positions of the right-to-left maxima to get an estimate for the number of 213-avoiding permutations with the given property.

If we wanted to estimate the number of *all* 213-permutations, then we could simply say that we have less than 2^n choices for the set of these right-to-left maxima and less than 2^n choices for their positions, so the number of these permutations is less than 4^n.

However, if we require that the first i entries form an increasing subsequence, then it is clear that no right-to-left maximum can be put into any of the first $i-1$ positions. Thus we have only 2^{n-i+1} choices for the positions of the right-to-left maxima. Note that we have less than 2^{n-1} choices for their set as the entry n must be part of that set and that set cannot have more than $n-i+1$ elements.

Therefore, the number of permutations with the given property is less than $2^{n-i+1} \cdot 2^{n-1} = 4^n/2^i$ as claimed. $\diamond$

Lemma 8 *Let y_i be the number of 213-avoiding n-permutations which have at least i cuts. Then $y_i < \frac{4^n}{2^i}$.*

Proof: Induction on n and i. The statement is obvious for $n = 1, 2, 3$ and $i = 0$. Suppose we know it for all pairs (m, j) which are smaller than (n, i) in the coordinate-wise ordering; that is, $j \leq i$ and $n_1 \leq n$ and at least one of these inequalities is strict.

Let p be an n-permutation in which the entry 1 is in the j-th position where $1 \leq j < n$. Then there cannot be any cuts after 1. Thus the induction hypothesis applies to the string of the first j entries, and we get that there are less than $4^j/2^i$ permutations which contain the entry 1 in the j-th place *and* have at least i cuts.

If $j = n$, then there is a cut before the last entry (which is 1), so there must be at least $i - 1$ other cuts in the permutation. We know by induction on i that there are less than $4^{n-1}/2^{i-1}$ permutations with that property.

This yields

$$y_i < \frac{4^{n-1}}{2^{i-1}} + \sum_{j=i}^{n-1} \frac{4^j}{2^i} = \frac{4^{n-1}}{2^{i-1}} + \frac{4^j}{2^i} \cdot \sum_{j=0}^{n-1-i} 4^j =$$

$$\frac{4^{n-1}}{2^{i-1}} + \frac{4^i}{2^i} \cdot \frac{4^{n-i}-1}{3} < \frac{4^{n-1}}{2^{i-1}} + \frac{4^n}{2^i \cdot 3} = \frac{2.5 \cdot 4^n}{3 \cdot 2^i} < \frac{4^n}{2^i}.$$

$\diamond$

Now we are in a position to complete the estimate of the number of ways to merge the strings of entries of rank r together. Recall that our weak class W has only two left-to-right minima, $a_1 = 1$ and a_2. Suppose we have a b_1-permutation on the entries of rank 1 whose initial increasing subsequence is of length i and a b_2-permutation on the entries of rank 2 which has j cuts. Then the constraints specified before the last two lemmas show that the only way to merge these two permutations is to insert the entries of the initial increasing subsequence of the first one into the cuts of the second one (in fact, only cuts on the right of a_1 are eligible), or on the left of all of them, or on the right of all of them, then to put all the other small entries at the end of the n-permutation. This can be done in $\binom{i+j}{i}$ ways. Therefore, by the two previous lemmas, there are less than

$$\binom{i+j}{i} \cdot \frac{4^{b_1}}{2^i} \cdot \frac{4^{b_2}}{2^j} < 4^{b_1+b_2} = 4^{n-2} \tag{2.1}$$

ways two pick two permutations with these parameters and merge them. Finally, we have to consider all possible choices for i and j. Clearly, $i+j<n$, so we have less than $\binom{n}{2}$ possibilities for the pair (i,j). Therefore, W has less than $\binom{n}{2} \cdot 4^{n-2} < 8^n$ 1324-avoiding permutations.

Now consider the general case, i.e., when W has $m>2$ left-to-right minima. Call entries of rank m *large* and the others *small.* We claim that it is sufficient to show that for any given choice of the vector of the parameters $(i_1, i_2, \cdots, i_m, j_1, j_2, \cdots, j_m)$, we have at most 4^n 1324-avoiding permutations with that vector of parameters. (Here i_k and j_k denote the length of the initial increasing subsequence and the number of cuts in the substring of elements of rank k).

Indeed, the initial increasing subsequence of the elements of rank r can contain at most one position after which a cut can be obtained. Thus $i_r + j_r \leq b_r + 1 = a_{r+1} - a_r$, and therefore $\sum_r (i_r + j_r) \leq n$. This implies that we have at most 2^{n-1} choices for the vector $(i_1, i_2, \cdots, i_r, j_1, j_2, \cdots, j_r)$, since the number of all compositions of the integer n is 2^{n-1}. (We can suppose that there is at least one element of each rank).

So let us estimate the number of 1324-avoiding permutations in W which have a given vector $(i_1, i_2, \cdots, i_r, j_1, j_2, \cdots j_r)$ of parameters. Suppose we already merged together the strings of all small entries and we want to merge the outcome with the string of large entries. As before, the large entries must form a 213-avoiding permutation, so the estimate of Lemma 8 holds. We must be a little bit more careful with the small entries. If a_{m-1} and a_{m-2} are not in consecutive positions, then the initial increasing subsequence on the small entries can only contain entries of rank $m-1$, thus we can simply apply our estimate to the permutation of entries of that rank. If $a_{m-1}, a_{m-2}, \cdots, a_t$ are in consecutive positions, then we can do the same for the 213-avoiding permutation of the entries of rank at least t and at most $m-1$. [as any 213-pattern on them would be entirely on the right of a_t and would thus form a 213-pattern]. This means that less than $1/2^i$th of all 213-avoiding permutations on the large entries have an initial increasing subsequence of length at least i, and less than $1/2^j$th of all 213-avoiding permutations on the small entries have at least j cuts. Thus we can always apply the method seen in the proof of formula 2.1 and get that we have less than 4^n 1324-permutations for any given vector of parameters. Thus W has less than 8^n 1324-avoiding permutations, as claimed. As we have only $c_n = \binom{2n}{n}/(n+1) < 4^n$ choices for W, the proof of the theorem is complete. $\diamond$

2.1.3 The pattern 1342

A correspondence between trees and permutations

In this section we are going to prove an exact formula for the number $S_n(1342)$ of 1342-avoiding permutations of length n showing that

$$S_n(1342) = \frac{(7n^2 - 3n - 2)}{2} \cdot (-1)^{n-1} + 3\sum_{i=2}^{n} 2^{i+1} \cdot \frac{(2i-4)!}{i!(i-2)!} \cdot \binom{n-i+2}{2} \cdot (-1)^{n-i},$$

by first proving that the ordinary generating function $H(x)$ for these numbers $S_n(1342)$ has the following simple form:

$$H(x) = \frac{32x}{-8x^2 + 12x + 1 - (1-8x)^{3/2}}.$$

This is the first result we know of which provides an exact formula for the number

$S_n(q)$ of permutations of length n avoiding a given pattern q if q is longer than three and is not 1234. Results concerning the case of length three can be traced back to two centuries; [8] already makes references to earlier work. The formula for $q = 1234$ is given in [13]. Until recently it has not even been known that $S_n(1342) < c^n$ for some constant c. In [2] this upper bound with $c = 9$ was proved. This formula pushes down this c to 8, and proves that it is optimal.

In our proof, we are going to present a new link between the enumeration of permutations avoiding the pattern 1342 and the that of $\beta(0,1)$-trees, a class of labeled trees recently introduced in [10]. We will show that the number $I_n(1342)$ of *indecomposable* 1342-avoiding permutations of length n is equal to the number of $\beta(0,1)$-trees on n nodes. The set $D_n^{\beta(0,1)}$ of $\beta(0,1)$-trees on n nodes is known to be [10] equinumerous to the set of rooted bicubic maps on $2(n+1)$ vertices, and an exact formula for the number t_n of these is provided in [32]. (These are planar maps with 2-colorable vertices which in addition all have degree three and a distinguished "root" edge and face). Therefore, $I_n(1342) = |D_n^{\beta(0,1)}| = t_n = 3 \cdot 2^{n-1} \cdot \frac{(2n)!}{(n+2)!n!}$. So combinatorially, the number of all n-permutations avoiding 1342 will be shown to be equal to that of plane forests on n vertices in which each component is a $\beta(0,1)$-tree. To our best knowledge, this is the first time when permutations avoiding a given pattern are shown to have such a close connection with some planar maps, though recently 2-stack-sortable permutations have been shown to be equinumerous to nonseparable planar maps [11] [12]. Examining the generating function $H(x)$ we will be able to prove and disprove several conjectures for the pattern 1342. $H(x)$ turns out to be algebraic, proving a conjecture of Zeilberger and Noonan [37] for the first time for a nonmonotonic pattern which is longer than three. We will see that $\sqrt[n]{S_n}(1342) \to 8$, which disproves a conjecture of Stanley and implies that $lim_{n\to\infty}(S_n(1342)/S_n(1234)) = 0$. We would like to point out the surprising nature of this result: while $S_n(q) = \binom{2n}{n}/(n+1)$ for *any* patterns q of length three, for the case of length four there are sequences $S_n(q)$ that are not only different from each other, but their quotient also converges to 0.

Definition 5 *[10] A rooted plane tree with nonnegative integer labels $l(v)$ on each of its vertices v is called a $\beta(0,1)$-tree if it satisfies the following conditions:*

- *if v is a leaf, then $l(v) = 0$,*
- *if v is the root and $v_1, v_2, \cdots, v_k$ are its children, then $l(v) = \sum_{i=1}^{k} l(v_k)$,*
- *if v is an internal node and $v_1, v_2, \cdots, v_k$ are its children, then $l(v) \leq 1 + \sum_{i=1}^{k} l(v_k)$.*

A *branch* of a rooted tree is a tree whose top is one of the root's children. Some rooted trees may have only one branch, which doesn't necessarily mean they consist of a single path.

We start by treating two special types of $\beta(0,1)$-trees on n vertices. These cases are fairly simple- they will correspond to 231-avoiding (resp. 132-avoiding) permutations, but they will be our tools in dealing with the general case.

First we set up a bijection f from the set of all 1342-avoiding n-permutations starting with the entry 1 and the set of $\beta(0,1)$-trees on n vertices consisting of one single path. In other words, the former is the set of 231-avoiding permutations of the set $\{2, 3, 4 \cdots, n\}$.

So let $p = (p_1 p_2 \cdots p_n)$ be an 1342-avoiding n-permutation so that $p_1 = 1$. Take an unlabeled tree on n nodes consisting of a single path and give the label $l(i)$ to its ith node $(1 \leq i \leq n-1)$ by the following rule:

$l(i) = \#\{j \leq i \text{ so that } p_j > p_s \text{ for at least one } s > i\}$.

Finally, let $l(n) = l(n-1)$. In words, $l(i)$ is the number of entries on the left of p_i (inclusive) which are larger than at least one entry on the right of p_i. We note that this way we could define f on the set of *all* n-permutations starting with the entry 1, but in that case, as we will see, f would not be a bijection. (For example, the images of 1342 and 1432 would be identical).

Example 1 If $p = 14325$, then the labels of the nodes of $f(p)$ are, from the leaf to the root, 0,1,2,0,0.

Lemma 9 *f is a bijection from the set of $D_n^{\beta(0,1)}$ of all $\beta(0,1)$-trees on n vertices consisting of one single path to the set of 1342-avoiding n-permutations starting with the entry 1.*

Proof: It is easy to see that f indeed maps into the set of $\beta(0,1)$-trees: $l(i+1) \leq l(i)+1$ for all i because there can be at most one entry counted by $l(i+1)$ and not counted by $l(i)$, namely the entry p_{i+1}. All labels are certainly nonnegative and $l(1) = 0$.

To prove that f is a bijection, it suffices to show that it has an inverse, that is, for any $\beta(0,1)$-tree T consisting of a single path, we can find the only permutation p so that $f(p) = T$. We claim that given T, we can recover the entry n of the preimage p. First note that p was 1342-avoiding and started by 1, so any entry on the left of n must have been smaller than any entry on the right of n. In particular, the node preceding n must have label 0. Moreover, n is the leftmost entry p_i of p so that $p_j > 0$ for all $j \geq i$ if there is such an entry at all, and $n = p_n$ if there is none. That is, n corresponds to the node which starts the uninterrupted sequence of strictly positive labels which ends in the last node, if there is such a sequence and corresponds to the last node otherwise. To see this, note that n is the largest of all entries, thus in particular it is always larger than at least one entry in any nonempty set of entries.

Once we located where n was in p, we can simply delete the node corresponding to it from T and decrement all labels after it by 1. (If this means deleting the last node, we just change $l(n-1)$ so it is equal to $l(n-2)$ to satisfy the root-condition). We can indeed do this because the node preceding n had label 0 and the node after n had a positive label, by our algorithm to locate n. Then we can proceed recursively, by finding the position of the entries $n-1, n-2, \cdots, 1$ in p. This clearly defines the inverse of f, so we have proved that f is a bijection. $\diamond$

Lemma 10 *The number $\beta(0,1)$-trees with all labels equal to zero is C_{n-1}.*

Proof: These $\beta(0,1)$-trees are in fact unlabeled plane trees. We prove that they are in one-to-one correspondence with the 132-avoiding permutations whose last entry is n. Suppose we already know this for all positive integers $k < n$. Let T be a $\beta(0,1)$-tree on n vertices with all labels equal to 0 and root r. Let r have t children, which are, from left-to-right, at the top of such unlabeled trees $T_1, T_2, \cdots, T_t$ on $n_1, n_2, \cdots, n_t$ nodes. Then by induction, each of the T_i corresponds to a 132-avoiding n_i-permutation ending with n_i. Now add $\sum_{j=i+1}^{t} n_j$ to all entries of the permutation p_i associated with T_i, then concatenate all these strings and add n to the end to get the permutation p associated with T. This is clearly a bijection as the blocks of the first $n-1$ elements determine the branches of T. $\diamond$

Example 2 The permutation 341256 corresponds to the $\beta(0,1)$-tree with all labels equal to 0 shown in Figure 2.

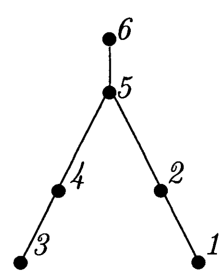

Figure 2

An easy way to read off the corresponding permutation once we have its entries written to the corresponding nodes is the well-known *preorder* reading: for every node, first write down the entries associated with its children from left to right, then the entry associated with the node itself, and do this recursively for all the children of the node.

Note that such a $\beta(0,1)$-tree has only one branch if and only if the next-to-last element of the indecomposable 132-avoiding n-permutation corresponding to it is $n-1$.

Let's introduce some more notions before we attack the general case. An entry of a permutation which is smaller than all the entries by which it is preceded is called a *left-to-right minimum.*

Recall definition 3 from the previous Section: Two n-permutations x and y are said to be in the same *weak class* if the left-to-right minima of x are the same as those of y, and they are in the same positions.

Proposition 1 *Each nonempty weak class W of n-permutations contains exactly one 132-avoiding permutation.*

Proof: Take all entries which are not left-to-right minima and fill all slots between the left-to-right minima with them as follows: in each step place the smallest element which has not been placed yet which is larger then the previous left-to-right minimum. The permutation obtained this way will be clearly 132-avoiding, and it will be the only one in this weak class because any time we deviate from this procedure, we create a 132-pattern. $\diamond$

Definition 6 *The normalization N(p) of an n-permutation p is the only 132-avoiding permutation in the weak class W containing p.*

Example 3 If $p = 32514$, then $N(p) = 32415$.

Definition 7 *The normalization $N(T)$ of a $\beta(0,1)$-tree T is the $\beta(0,1)$-tree which is isomorphic to T as a plane tree, with all labels equal to zero.*

Proposition 2 *A permutation p is indecomposable if and only $N(p)$ is indecomposable.*

Proof: Let W be the weak class containing p, given by the set and position of its left-to-right minima. It is clear that if $p \in W$ is decomposable, then the only way to cut it in two parts is to cut it immediately before a left-to-right minimum a. Now if there is a left-to-right minimum a so that it is in the $n+1-a$th position, then all entries which are larger than a must be placed on the left of a and so all such permutations in W are decomposable. If there is no such a, then for all left-to-right minima m, there will be an entry b so that $m < b$ and b is on the right of m, and so permutations in W will not be decomposable. $\diamond$

Corollary 2 *If p is an indecomposable n-permutation, then $N(p)$ always ends with the entry n.*

Proof: Note that the only way for a 132-avoiding n-permutation to be indecomposable is to end with n. Then the statement follows from Proposition 2. $\diamond$

Now we are in a position to prove our theorem about the number of indecomposable 1342-avoiding permutations.

Theorem 7 *The number of indecomposable n-permutations which avoid the pattern 1342 is*

$$I_n(1342) = t_n = 3 \cdot 2^{n-1} \cdot \frac{(2n)!}{(n+2)!n!}$$

Proof: We are going to set up a bijection F between these permutations and $D_n^{\beta(0,1)}$. This will be an extension of the bijection f of lemma 9.

Let p be an indecomposable 1342-avoiding n-permutation. Take $N(p)$. By corollary 2 its last element is n. Define $F(N(p))$ to be the $\beta(0,1)$-tree S associated to $N(p)$ by the bijection of lemma 10. Now write the entries of p to the nodes of S so that for all i, the p_i is written to the node where $N(p)_i$ was written in S. In particular, the left-to-right minima remain unchanged. Figure 3 shows how we associate the entries of the permutation 361542 to the nodes of $N(T)$, which is the image of $N(p) = 341256$.

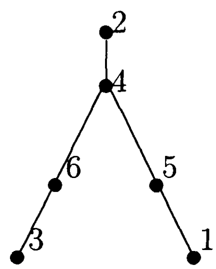

Figure 3

Now we are going to define the label of each node for this new $\beta(0,1)$-tree T and obtain $F(p)$ this way. (As an unlabeled tree, T will be isomorphic to S, but its labels will be different). Denote i the ith node of T in the preorder reading, thus p_i is the ith entry of p, (which is therefore associated to node i), while $l(i)$ is the label of this node. We say that p_i *beats* p_j if there is an element p_h so that p_h, p_i, p_j are written in this order and they form a 132-pattern. Moreover, we say that p_i *reaches* p_k if there is a subsequence $p_i, p_{i+a_1}, \cdots p_{i+a_t}, p_k$ of entries so that $i < i+a_1 < i+a_2 < \cdots < i+a_t < k$ and that any entry in this subsequence beats the next one. For example, in the permutation 361542, the entry 6 beats 5 and 4, 5 beats 4 and 2, and 4 beats 2, while 6 reaches 2 (of course, each entry reaches all those elements it beats, too). Then let

$l(i) = \#\{j$ descendants of i (including i itself) so that there is at least one $k > i$ for which p_j reaches $p_k\}$,

and let $F(p)$ be the $\beta(0,1)$-tree defined by these labels. (Recall that a descendant of i is an element of the tree whose top element is i). First, it is easy to see that F indeed maps into the set of $\beta(0,1)$-trees : if v is an internal node and $v_1, v_2, \cdots, v_k$ are its children, then $l(v) \leq 1 + \sum_{i=1}^{k} l(v_k)$ because there can be at most one entry counted by $l(v)$ and not counted by any of its children's label, namely v itself. All labels are certainly nonnegative and all leaves, that is, the left-to-right minima, have label 0.

If $p = 361542$, then $F(p)$ is the $\beta(0,1)$-tree shown in Figure 4b.

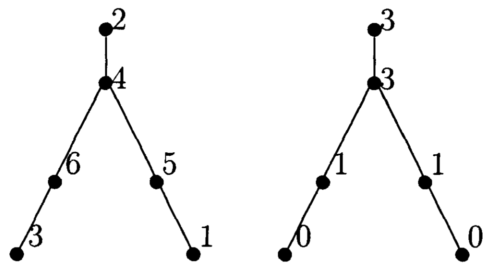

Figure 4a Figure 4b

To prove that F is a bijection, it suffices to show that it has an inverse, that is, for any $\beta(0,1)$-tree $T \in D_n^{\beta(0,1)}$, we can find the only permutation p so that $F(p) = T$. We again claim that given T, we can recover the node j which has the entry n of the preimage p associated to it, and so we can recover the position of n in the preimage.

Proposition 3 *Suppose $p_n \neq n$, that is, n is not associated to the root vertex. Then each ancestor of n, including n itself, has a positive label. If $p_n = n$, then $l(n) = 0$ and thus there is no vertex with the above property.*

Proof: If $p_n = n$, then nothing beats it, thus $p_n = 0$. Suppose p_n is not the root vertex.

To prove our claim it is enough to show that for any node i which is an an ancestor of j, there is an entry p_k so that p_k is an ancestor of p_i and $n = p_j$ reaches k. Indeed, this would imply that the entry $p_j = n$ is counted by the label $l(i)$ of i. Now let $a_m = p_1 > a_2 > \cdots a_1 = 1$ be the left-to-right minima of p, so that n is located between a_r and a_{r+1}. Then n certainly beats all elements located between a_r and a_{r+1} as a_r can play the role of 1 in the 132-pattern. Clearly, n must beat at least one entry y_1 on the right of a_{r+1} as well, otherwise p would be decomposable by cutting it right before a_{r+1}. If y_1 is on the right of i, then we are done. If not, then y_1 must beat at least one entry y_2 which is on the other side of a_{r_1+1}, where y is located between a_{r_1} and a_{r_1+1} for the same reason, and so on. This way we get a subsequence $y_1, y_2, \cdots$ so that n reaches each of the y_t, and this subsequence eventually gets to the right of i, as in each step we bypass at least one left-to-right minimum, and thus the proposition is proved. $\diamond$

Proposition 4 *Suppose $p_n \neq n$. Then n is the* leftmost *entry of p which has the property that each of its ancestors has a positive label.*

Proof: Suppose p_k and n both have this property and that p_k is on the left of n. (If there are several candidates for the role of p_k, choose the rightmost one). If p_k beats an element y on the right of n by participating in the 132-pattern $x\, p_k\, y$, then $x\, p_k\, n\, y$ is a 1342-pattern, which is a contradiction. So p_k does not beat such an element y. In other words, all elements after n are smaller than all elements before p_k. Still, p_k must reach elements on the right of n, so it beats some element v between p_k and n. This element v in turn beats some element w on the right of n by participating in some 132-pattern $t\, v\, w$. However, this would imply that $t\, v\, n\, w$ is a 1342-pattern, a contradiction, which proves our claim. $\diamond$

Therefore, we can recover the entry n of p from T. Then we can proceed as in the proof of Lemma 9: just delete n, subtract 1 from the labels of its ancestors and iterate this procedure to get p. If any time during this procedure we find that the current root is associated to the maximal entry that hasn't been associated to other vertices yet, and the tree has more than one branch at this moment, then deleting the root vertex will split the tree into smaller trees. Then we continue the same procedure on each of them. The set of the entries associated to each of these smaller trees is uniquely determined because T as an unlabeled tree determines the left-to-right minima of p. Therefore, we can always recover p in this way from T. This proves that F is a bijection.

Thus we have set up a bijection between the set of indecomposable 1342-avoiding n-permutations and $D_n^{\beta(0,1)}$. Therefore the Theorem is proven. $\diamond$

Note that in particular, F maps 132-avoiding permutations into $\beta(0,1)$-trees with all labels equal to 0 and permutations starting with the entry 1 into $\beta(0,1)$-trees consisting of a single path.

Corollary 3 $S_n(1342)$ *equals the number of plane forests on* n *vertices in which each component is a* $\beta(0,1)$*-tree.*

Consequences for Enumeration

Tutte [32] has obtained the numbers t_n by first computing a translate of their generating function

$$F(x) = \sum_{n=1}^{\infty} 3 \cdot 2^{n-1} \cdot \frac{(2n)!}{(n+2)!n!} x^n = \frac{8x^2 + 12x - 1 + (1-8x)^{3/2}}{32x}. \tag{2.2}$$

By theorem 7, the coefficients of this generating function are the numbers $I_n(1342)$. Therefore, the generating function of *all* 1342-avoiding permutations is given by the following theorem.

Theorem 8 *Let* $s_n = S_n(1342)$ *and let* $H(x) = \sum_{n=0}^{\infty} s_n x^n$. *Then*

$$H(x) = \sum_{i \geq 0} F^i(x) = \frac{1}{1 - F(x)} = \frac{32x}{-8x^2 + 12x + 1 - (1-8x)^{3/2}}. \tag{2.3}$$

Proof: Any 1342-avoiding permutation has a unique decomposition into indecomposable permutations. This can consist of $1, 2, \cdots$ blocks, implying that $s_n = \sum_{i=1}^{n} t_i s_{n-i}$, and the statement follows. $\diamond$

Theorem 9 *For all $n \geq 0$ we have*

$$S_n(1342) = S_n(1342) = \frac{(7n^2 - 3n - 2)}{2} \cdot (-1)^{n-1} + 3\sum_{i=2}^{n} 2^{i+1} \cdot \frac{(2i-4)!}{i!(i-2)!} \binom{n-i+2}{2} (-1)^{n-i}. \tag{2.4}$$

Proof: Multiply both the numerator and the denominator of $H(x)$ by $(-8x^2 + 20x + 1) + (1 - 8x)^{3/2}$. After simplifying we get

$$H(x) = \frac{(1-8x)^{3/2} - 8x^2 + 20x + 1}{2(x+1)^3}. \tag{2.5}$$

As $(1-8x)^{3/2} = 1 - 12x + \sum_{n \geq 2} 3 \cdot 2^{n+2} x^n \frac{(2n-4)!}{n!(n-2)!}$, formula (2.5) implies our claim. $\diamond$ So the

first few values of $S_n(1342)$ are 1,2,6,23,103,512,2740,15485,91245,555662. In particular, one sees easily that the expression on the right hand side of (2.5) is dominated by the last summand; in fact, the alternation in sign assures that this last summand is larger than the whole right hand side if $n \geq 8$. As $\frac{(2n-4)!}{n!(n-2)!} < \frac{8^{n-2}}{n^{2.5}}$ by Stirling's formula, we have proved the following exponential upper bound for $S_n(1342)$.

Corollary 4 *For all n, we have $S_n(1342) < 8^n$.*

It is straightforward to check that the numbers $I_n = t_n$ satisfy the following recurrence

$$t_n = (8n-4)t_{n-1}/(n+2). \tag{2.6}$$

In particular, $\sqrt[n]{t_n} \to 8$. Using this formula we can disprove a conjecture of Stanley claiming that for all permutation patterns q of length k, the sequence $\sqrt[n]{S_n(q)}$ converges to $(k-1)^2$.

Theorem 10 $\sqrt[n]{s_n} = \sqrt[n]{S_n(1342)} \to 8$ *when $n \to \infty$.*

Proof: This is true as clearly $t_n \leq s_n < 8^n$ by Corollary 4 and we know from (2.6) that $\sqrt[n]{t_n} \to 8$ if $n \to \infty$. $\diamond$

Corollary 5 $lim_{n\to\infty}(S_n(1342)/S_n(1234)) = 0.$

Proof: Follows from $lim_{n\to\infty}\sqrt[n]{S_n(1234)} = 9$ [13] [21]. $\diamond$

This Corollary certainly implies that $S_n(1342) < S_n(1234)$ if n is large enough. However, using the formulae of Theorem 9 and [13], we can easily show that this is true for *all* $n \geq 6$. (This has recently been shown by a long argument in [2]).

Corollary 6 *For all $n \geq 6$, we have $S_n(1342) < S_n(1234)$.*

Proof: It is known [13] that

$$S_n(1234) = 2 \cdot \sum_{i=0}^{n} \binom{2i}{i} \cdot \binom{n}{i}^2 \cdot \frac{3k^2 + 2k + 1 - n - 2kn}{(k+1)^2(k+2)(n-k+1)} \tag{2.7}$$

One sees easily that the dominant summand is the one with $i = 2n/3$, and that this summand is much larger than the last (and dominant) summand in (2.5) if $n \geq 9$. The proof then follows by checking the values of $S_n(1342)$ and $S_n(1234)$ for $n \leq 8$. $\diamond$

Formula (2.3) enables us to prove that the sequence $S_n(1342)$ is P-recursive in n, solving an instance of the conjecture of Zeilberger and Noonan mentioned in the Introduction. Indeed, $H(x)$ is certainly algebraic, thus in particular, it is d-finite and therefore $S_n(1342)$ is P-recursive as claimed. So we have proved the following theorem.

Theorem 11 *The sequence $S_n(1342)$ is P-recursive in n. Furthermore, its generating function $H(x)$ is algebraic and its only irrationality is $\sqrt{1-8x}$.*

2.2 Recursive results

2.2.1 Estimates for longer patterns

How can we attack this conjecture for longer patterns? A natural approach will be discussed in this section. It is based on the following idea: suppose we know that the conjecture on the exponential upper bound holds for a pattern q, then try to construct longer patterns for which it must hold as well. We are going to prove a series of lemmas to get more and more general methods to build such patterns.

To simplify notation, we define the *insertion* of an element into a pattern as follows:

Definition 8 *Let q be a pattern. Then* to insert *the entry y into the j-th position of q is to put y between the $j-1$-st and the $j-$th element of q and to increase the value of any entry v for which originally $v \geq y$ held by 1.*

We can define the *deletion* of an element similarly: just erase the element and decrease all elements larger than that by 1.

Example 4 Inserting 5 to the third position of the pattern 236154 results in the pattern 2357164. Then deleting 1 from this latter results in 124653.

The following definition simplifies our terminology even more:

Definition 9 *A pattern q is called* good *if the enumerative conjecture is true for it, that is, there exists a constant c so that $S_n(q) < c^n$ for all n.*

So all patterns of length 3 and 4 are good.

Notation: If $q = xz$ is a pattern and we insert the element y between them, then we get the pattern $q' = x\,y\,z$. So in particular, inserting 1 to the first position of q results in the pattern $1\,q$.

Lemma 11 *("adding an element to the front") Let q be a pattern with first entry 1 so that $S_n(q) < K^n$ for some absolute constant K. Then the there is some absolute constant K_1 so that $S_n(1q) < K_1^n$.*

Proof: Take any weak class W. One sees that if an n-permutation p avoids $1q$, then the string of its entries which are not left-to-right minima *must avoid* q. Indeed, if there were any copy of q on these entries, then one could complete it to a copy of $1q$ by joining the smallest left-to-right minimum s on the left of this copy of q and this copy of q. (By the definition of the left-to-right minima, s is smaller than the first entry of our copy of q, and therefore it is smaller than all entries of that copy because q starts with 1). So the number of $1q$-avoiding permutations in W is less than K^n as W has less than n entries which are not left-to-right minima. On the other hand, the number of weak classes is smaller than 4^n by Lemma 6, therefore we get $S_n(1q) < (4K)^n = K_1^n$ as claimed. $\diamond$

Remark: the estimate we have just used is not the best possible. For example, in the case when $S_n(q) < (k-1)^{2n}$, thus for all patterns q for which $S_n(q) \leq S_n(123\cdots k)$ we do better, in fact, we show that the estimate remains essentially the same. That is for the new, $(k+1)$-long pattern it will be k^{2n}. Indeed, if W has i left-to-right minima, then there are at most $\binom{n}{i}$ choices for the set of these minima and at most $\binom{n}{i}$ choices for their positions and less than $(k-1)^{2(n-i)}$ choices for the permutation induced on the other entries. Summing on all i we get that $S_n(1q) < \sum_{i=1}^{n} \binom{n}{i}^2 \cdot (k-1)^{2(n-i)} < (\sum_{i=0}^{n} \binom{n}{i} \cdot (k-1)^{(n-i)})^2 = k^{2n}$ as we have claimed.

Obviously, the dual versions of this lemma are also true, that is, we can take the reverse of all permutations occuring in the statement or the complement or the inverse of all permutations and the lemma remains true. Moreover, Theorem 3 will be applicable here if we iterate this lemma and prove this way that if q is as in the lemma, then for all r there is a constant K_r so that $S_n(123\cdots rq) < K_r^n$.

Corollary 7 *Let q be a pattern with first entry 1 so that $S_n(q) < K^n$ for some absolute constant K. Then for all r there is a constant K_r so that $S_n(123\cdots ra_{r+1}a_{r+2}\cdots a_{r+t}) = S_n(r\cdots 321a_{r+1}a_{r+2}\cdots a_{r+t}) < K_r^n$.*

Is there any way to modify a pattern q elsewhere, that is, not at its ends, and still get a good pattern? The following lemma is an example for that.

Lemma 12 *("inserting an element") Let q be a pattern and let y be an entry of q so that for any entry x preceeding y we have $x < y$ and for any entry z preceeded by y we have $y < z$. Suppose that $S_n(q) < K^n$ for some constant K and for all n. Let q' be the pattern obtained from q by inserting $y+1$ to the position right after y. Then $S_n(q') < (4K)^n$ for all n, thus q' is a good pattern.*

Example 5 If $q = 1\,3\,2\,4\,6\,5\,7$ and $y = 4$, then $q' = 1\,3\,2\,4\,5\,7\,6\,8$.

Proof: Take an n-permutation which avoids q'. Suppose it contains q. Then consider all copies of q in our permutation and consider their entries y. Clearly, these entries must form a *decreasing subsequence.* Indeed, if one of them, say y_1, would be smaller than an other one, say y_2, on its right, then y_1, y_2, the initial segment of the copy of q which contains y_1 and the ending segment of the copy of q which contains y_2 would form a copy of q'.

So the y_i form a decreasing subsequence of length at most $n-1$. Therefore, we have at most 4^{n-1} choices for the values and positions of the y_i. Deleting the y_i, we are left with a permutation which is shorter than n and avoids q. By our hypothesis, there are at most K^n such permutations. So we have had at most $4^{n-1}\cdot K^n$ possibilities for our permutation before the deletions. Finally, if our permutation doesn't even contain q, then we have only K^n choices for it. As $4^{n-1}\cdot K^n + K^n < (4K)^n$, the lemma is proved. $\diamond$

Note that Lemma 11 is a special case of this lemma, ie. the case when y is the first entry of q. This lemma can be applied repeatedly to show that for all r, we can insert the subsequence $y+1, y+2, y+3, \cdots y+r$ after y and still get a good pattern.

Can we insert anything else than increasing subsequences? The following lemma answers this question in the affirmative. To state the lemma, we need another definition:

Definition 10 *Let q be a pattern, and let y be an entry of q. Then* to replace *y by the pattern w is to add $y-1$ to all entries of w, then to delete y and to succesively insert the entries of w at its position.*

Example 6 Replacing the entry 1 in 1 4 2 3 by 1 3 2 4 results in the pattern 1 3 2 4 7 5 6.

Lemma 13 *("replacing an element by a pattern") Let q be a pattern and let y be an entry of q so that for any entry x preceeding y we have $x < y$ and for any entry z preceeded by y we have $y < z$. Suppose that $S_n(q) < K^n$ for some constant K and for all n.*

Let w be a pattern of length k starting with 1 and ending with k so that $S_n(w) < C^n$ holds for all n, for some constant C. Let q' be the pattern obtained by replacing the entry y by the pattern w in q. Then $S_n(q') < (2CK)^n$, thus q' is a good pattern.

Example 7 Thus the pattern of Example 6, 1 3 2 4 7 5 6, is a good pattern.

Proof: As before, take an n-permutation which avoids q'. Suppose it contains q. Then consider all copies of q in our permutation and consider their entries y. Clearly, these entries must form a permutation which does not contain w. For suppose they do, and denote y_1 and y_k the first and last elements of that purported copy of w. Then the initial segment of the copy of q which contains y_1 and the ending segment of the copy of q which contains y_k would form a copy of q'.

By similar argument to that of lemma 12, this shows that less than $(2C)^{n-1} \cdot K^n + K^n < (2CK)^n$ permutations of length n can avoid q'. $\diamond$

In some particular cases we do not need all of these restrictions made on w:

Lemma 14 *Let q be as in lemma 13 and let y be its* first *entry. Replace y by any good pattern w which ends with its largest entry. Then the pattern obtained this way is good.*

Similarly, if y is the last *entry of q, then it is enough to require that w be good and that w start with 1. With these conditions, the pattern obtained by replacing y by w is good.*

Proof: this can be proved exactly as lemma 13 was. The special values and positions of y obviate the omitted restrictions. $\diamond$

Example 8 If $p = 1\,3\,2\,4$, $w = 1\,3\,4\,2$, then by the second statement of this lemma we obtain that the conjecture is true for the pattern 1 3 2 4 6 7 5.

The following Proposition is obvious. However, our Lemmas 13 and 14 enable us to use it to prove the enumerative conjeture for patterns we may not be able to handle otherwise.

Proposition 5 *If q contains r as a pattern, then $S_n(q) \geq S_n(r)$. Thus if q is good, then so is r.* $\Box$

Example 9 Deleting 1 from Example 8 we get the pattern 2 1 3 5 6 4 is good.

Corollary 8 *Let q be any good pattern which starts with 1. Replace 1 by any pattern r of length 3 to get q'. Then q' is a good pattern as well.*

Proof: If r ends with 3, then this is immediate from lemma 14. If not, then add the entry 4 to the end of r. As all patterns of length 4 are good, lemma 14 applies. Then delete 4 and apply Proposition 5. $\diamond$

2.2.2 Relations between longer patterns

Theorem 4 can be easily generalized for longer patterns in the following sense:

Theorem 12 *Let q be a pattern of length k which starts with 1, ends with k and has only one inversion. Then $S_n(123\cdots k) < S_n(q)$ if n is sufficiently large.*

This can be proved as Theorem 4 is, in other words, we can define classes of permutations so that each of them contains exactly one $123\cdots k$-avoiding permutation and at least one q-avoiding one. If the inversion is due to the ith and $(i+1)$st elements, then such a classification is obtained as follows:

Definition 11 *For a permutation p, an element is said to be in the jth* basic subsequence *if it is the endpoint of an increasing subsequence of length j, but there is no increasing subsequence of length $j+1$ ending in that entry.*

It is clear that all basic subsequences are decreasing. Now let us define two permutations x and y to be in the same class if the set and position of their jth basic subsequences agree for all j except i. Then the proof is identical to that of Theorem 4. If q has only one inversion, but it doesn't start with 1 or doesn't end with k, then $S_n(q) = S_n(12\cdots k)$ as it is shown in Theorem 3.

A more general form of this theorem is as follows.

Theorem 13 *Let q_1 and q_2 be two patterns which both start with the entry 1 and $S_n(q_1) \leq S_n(q_2)$ for all n. Then $S_n(1q_1) \leq S_n(1q_2)$ for all n.*

Proof: Note that an n-permutation p will be $1q_i$ avoiding if and only if the substring formed by its entries which are not left-to-right minima is q_i avoiding, and the statement follows. $\diamond$

Example 10 For all $n \geq 0$ we have $S_n(12453) \leq S_n(12345)$.

Chapter 3

P-recursiveness

3.1 Background

3.1.1 On P-recursive functions

What else can we say about the functions $S_n(q)$ apart from that they are probably less than c^n? In this chapter we study questions of P-recursiveness, not only for the $S_n(q)$, but also for the number of permutations with a *prescribed* number r of the subsequence q, (thus not necessarily 0). As we mentioned in the Introduction, the general conjecture of this field [37] states that for *any* fixed q and r, this function $S_{q,r}(n)$ is P-recursive in n.

First, we need a brief survey of P-recursive functions. In this we are going to follow [29] closely. For simplicity, we are going to work over the field of complex numbers, denoted by $\mathbf{C}$, though a more general treatment is possible.

Definition 12 *A function $f : N \to R$ is called P-recursive if there exist polynomials $P_0(n), P_1(n), \cdots, P_k(n) \in Q[n]$, with $P_k(n) \neq 0$ so that*

$$P_k(n)f(n+k) + P_{k-1}(n)f(n+k-1) + \cdots + P_0(n)f(n) = 0 \tag{3.1}$$

for all natural numbers n.

Here P-recursive stands for "polynomially recursive". The continuous analogue of this notion is d-finiteness, which stands for "differentiably finite".

Definition 13 *Let $u(x) \in Q[[x]]$ be a power series. If there exist polynomials $p_0(n), p_1(n), \cdots p_d(n)$ so that $p_d \neq 0$ and*

$$p_d(x)u^{(d)}(x) + p_{d-1}(x)u^{(d-1)}(x) + \cdots + p_1(x)u'(x) + p_0(x)u(x) = 0, \qquad (3.2)$$

then we say that u is d-finite. (Here $u^{(j)} = \frac{d^j u}{dx^j}$).

Note that this is clearly equivalent to saying that the $Q(x)$-vectorspace spanned by u and all its derivatives is finite dimensional.

Theorem 14 *The function $f(n)$ is P-recursive if and only if its ordinary generating function $u(x) = F(x) = \sum_{n \geq 0} f(n)x^n$ is d-finite.*

Proof:

- First suppose u is d-finite, then (3.2) holds with $p_d \neq 0$. Differentiate both sides i times and multiply by x^j to get

$$x^j u^{(i)} = \sum_{n \geq 0} (n+i-j)f(n+i-j)x^n.$$

  (Here m_j denotes the falling factorial $m(m-1)\cdots(m-j+1)$). Equating the coefficients of x^{n+k} we get a polynomial recurrence for $f(n)$. This will not be 0=0 as $p_d \neq 0$.

- Now suppose $f(n)$ is P-recursive in n, so (3.1) holds. Note that for any fixed natural number i, the polynomials $(n+i)_j$, $j \leq 0$ form a Q-basis for the vectorspace $Q[n]$.

So $P_i(n)$ is a Q-linear combination of series of the form $\sum_{n\geq 0}(n+i)_j f(n+i)x^n$. Now note that the left hand side almost agrees with $x^{j-i}u^{(j)}$, they can only differ in finitely many terms with all negative coefficients. Let the sum of these terms be $R_i(x) \in x^{-1}K[x^{-1}]$, a Laurent-polynomial. Therefore, if we multiply (3.1) by x^n and sum for all nonnegative n, we get

$$0 = (\sum_i a_{ij}x^{j-i}u^{(j)}) + R(x). \tag{3.3}$$

Here the sum is finite by the definition of P-recursiveness and $R(x)$ is a Laurent-polynomial. If we multiply both sides by x^q for q sufficiently large, the terms with negative exponents will disappear and we get an equation of the form (3.2).

$\diamond$

Lemma 15 *Let $f(n)$ and $g(n)$ be P-recursive functions. Then the functions $f+g$ and the convolution $h(n) = \sum_{i=0}^{n} f(i)g(n-i)$ are P-recursive as well.*

Proof: Let $F(x), G(x)$ and $H(x)$ be the ordinary generating functions of f, g and h, respectively.

- For $f+g$ we don't even need them as it is clear that the sum of a polynomial recursion for f and a polynomial recursion for g is still a polynomial recursion, and $f+g$ satisfies it.

- For the convolution, we will show that $F \cdot G$ is d-finite. Let $K = V((x))$ be the quotient field of $\mathbf{C}[[x]]$ over $\mathbf{C}$, and for any power series $v \in \mathbf{C}[[x]]$, let V_v be the vector space over $\mathbf{C}(x)$ spanned by v and all its derivatives. Then let

$$\Phi : V_F \otimes_{K((x))} V_G \to V$$

be the unique linear transformation satisfying $\Phi(F^{(i)} \otimes G^{(j)}) = F^{(i)}G^{(j)}$ for all i and j. Now the image of Φ contains V_{FG} by the Leibniz rules. So $dim\, V_{FG} \leq dim(V_F \otimes V_G) = (dim V_F) \cdot (dim V_G) < \infty$, showing that $F \cdot G$ is d-finite. This implies our claim by Theorem 14. Indeed, one sees easily that $H(x) = F(x) \cdot G(x)$, so $H(x)$ is d-finite as so were $F(x)$ and $G(x)$ by Theorem 14 and therefore $h(n)$ is P-recursive by that same theorem.

$\diamond$

Corollary 9 *Let $f(n)$ be a P-recursive function and let $g(n)$ be a function which disagrees with f on a finite number of n's only. Then $g(n)$ is P-recursive as well.*

Proof: Let $p = g - f$. Then the ordinary generating function of p is a polynomial and thus d-finite, hence p is P-recursive and so is $p + f = g$. $\diamond$

3.1.2 Earlier results

If q is the trivial pattern 21, then we simply have to count the number of n-permutations with exactly r inversions. Let $i(p)$ be the number of inversions in the n-permutation p. Then it is well-known [28] that

$$\sum_p x^{i(p)} = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1}). \tag{3.4}$$

This shows that $S_{q,r}(n) = \sum_{j=0}^{r} S_{q,j}(n-1)$, which is P-recursive in n by an induction argument on r. Indeed, all but the last summands on the right-hand-side are P-recursive in n by induction on r. Let the ordinary generating function of their sum be $T(x)$, while that of $S_{q,r}(n)$ is $V(x)$. Then we have $T(x)/(1-x) = V(x)$ and the statement follows.

As a more serious application of this theory, we examine permutations having exactly zero $1\,2\,3 \cdots k$ patterns for some given k. As in Chapter 2, permutations avoiding these patterns are not too hard to deal with. This is the content of the next theorem, mentioned in [13] and [37].

Theorem 15 *$S_n(12 \cdots k)$ is P-recursive for all k.*

Proof: We know from the Robinson-Schensted correspondence [24] that permutations not having increasing subsequences of length k can be associated with pairs of standard Young-tableaux having at most $k-1$ columns. In other words, $S_n(12 \cdots k) = \sum_\lambda f_\lambda^2$, where λ runs through all Ferrer's shapes of size n which have at most $k-1$ columns, and f_λ denotes the number of Standard Young tableaux of shape λ. Let $\lambda = (m_1, m_2, \cdots, m_{k-1})$ be such a Ferrer's shape. Then $m_1 \geq m_2 \geq \cdots m_{k-1} \geq 0$, and $\sum_{i=1}^{k-1} = n$, m_i denoting the size of the ith column. It is well known by the famous Young-Frobenius ([38] or [24]) formula that

$$f_\lambda = [\Pi_{1 \leq i \leq j \leq k-1}(m_i - m_j + j - i)] \cdot \frac{(m_1 + m_2 + \cdots + m_{k-1})!}{(m_1 + k - 2)! \cdots (m_{k-1})!}. \tag{3.5}$$

If we repeatedly apply Lemma 15, we can easily see that the right hand side is P-recursive in each of its variables, and therefore so is its square. This implies that

$$\sum_{m_1, \cdots m_{k-1},\, m_1 + \cdots m_{k-1} = n} f_\lambda^2 = S_n(12 \cdots k) \tag{3.6}$$

is P-recursive in n. $\diamond$

Obviously, this result impies that $S_n(q)$ is P-recursive for all q for which $S_n(q) = S_n(12 \cdots k)$. (Theorem 3 provides a lot of patterns of this kind). What is more surprising is that these are almost the *only* patterns we can prove this conjecture! (The exceptions are, as we said, 1342, and patterns of length 3).

3.1.3 Disjoint subsequences

Unfortunately, if $r > 0$, then we cannot directly apply this method to show that the number of n-permutations containing exactly r copies of $1\,2\cdots k$ is P-recursive in n. However, if we also require that the r copies be *disjoint*, (and of maximum size) then we can proceed similarly. First we will need a lemma of Curtis Greene [14].

Lemma 16 *Let $a_1, a_2, \cdots a_k$ denote the length of the first, second, $\cdots$, k-th row of the P-tableau of a permutation p. Then for all i, $1 \leq i \leq k$, the maximum size of the union of i disjoint increasing subsequences is equal to $a_1 + a_2 + \cdots + a_i$.*

We point out that the fact that the maximum size of the union of i disjoint increasing subsequences is equal to $a_1+a_2+\cdots+a_i$ does not imply that the longest such subsequence must have length a_1, the second must have length a_2, and so on; we only have information of the sum of their length.

Theorem 16 *Let $D_{k,r}(n)$ denote the number of n-permutations p in which the longest increasing subsequences have size k and for which the r is the largest natural number so that there are r disjoint increasing subsequences of maximum size in p. Then $D_{k,r}(n)$ is a P-recursive function of n.*

Proof: Lemma 16 shows that necessarily $a_1 = k$ and $a_1 + a_2 + \cdots + a_r = r \cdot k$. Thus necessarily $a_1 = a_2 = \cdots = a_r = k$ and $a_{r+1} < k$ otherwise there would be $(r+1)$ increasing subsequences of length k which are disjoint. This means that the size of the last column is $m_k = r$. Applying (3.5) with k variables instead of $(k-1)$ and fixing $m_k = r$ we get the proof exactly as that of the last theorem. $\diamond$

One special case of Theorem 16 is of particular interest, namely, when $r = 1$. In this case the conditions mean that *any two* increasing subsequences of maximum size intersect. The following proposition shows that this implies a seemingly stronger property:

Proposition 6 *Let p be an n-permutation in which the longest increasing subsequences have size k and any two of them intersect. Then* all of them *have at least one entry in common.*

Proof: We construct a directed graph G_p associated to p. The vertices of G_p are the entries of p and there is an edge from the entry i to the entry j if and only if $i < j$ and i is on the left of j. So an increasing subsequence of length k in p corresponds to a path of length k in G_p. Now let us remove all edges not in any maximum-length-path from G_p, moreover add a "source" s and a "tail" t to get the graph G'_p. That is, s and t are vertices so that s has indegree zero, and there is an edge from s to all left-to-right minima of p, while t has outdegree zero and there is an edge to t from all right-to-left maxima of p. So each increasing subsequence of size k corresponds to an $s \rightarrow t$ path of maximum size in a natural way. Now suppose they do not have a vertex in common. Then we can delete any vertex v and still have an $s \rightarrow t$ path in G'_p. In other words, G'_p is 2-(s,t)-connected, which implies, by the famous theorem of Menger ([18]) that there are at least two vertex-disjoint $s \rightarrow t$ paths in G'_p. This is equivalent to saying that there are two increasing subsequences of size k in p which are disjoint, which is a contradiction and the proof is complete. $\diamond$

Corollary 10 *Let $D_k(n)$ be the number of n-permutations in which the longest increasing subsequences have size k and they all have at least one entry in common. Then $D_k(n)$ is a P-recursive function of n.*

3.2 An exact formula for $q = 1\,3\,2$ and $r = 1$

In the rest of this Section we will examine permutations with exactly r subsequences of type 1 3 2 with $r > 0$. In this Subsection we count n-permutations containing exactly one subsequence of type 132. (The same question for the pattern 1 2 3 has recently been solved by Noonan [19], who found the simple formula $\frac{3}{n}\binom{2n}{n-3}$). By a generating function argument we show that the number of these permutations is $\binom{2n-3}{n-3}$. This formula is even simpler than the one cited above and asks for a direct combinatorial proof. What is even more surprising is the fact that this formula is simpler than that for the number of 132-avoiding permutations, namely $S_n(132) = \binom{2n}{n}/(n+1)$. As mentioned in the Introduction, our result implies the same formula for the number of permutations containing exactly one subsequence of any nonmonotonic type, and so it completely arranges the problem for all subsequences of length 3. This formula was conjectured by Noonan and Zeilberger in [37]. We need one definition before starting the proof:

Definition 14 *Let p be an n-permutation. Its elements on the left of the entry n will be called* front elements, *whereas those on the right of n will be called* back elements.

Theorem 17 *Let b_n be the number of n-permutations having exactly one subsequence of type 132. Then $b_0 = b_1 = b_2 = 0$ and for all $n \geq 3$ we have*

$$b_n = \binom{2n-3}{n-3}. \tag{3.7}$$

Proof: Take any n-permutation p and suppose that the entry n is in the i-th position in p. Then there are three ways p can contain exactly one subsequence S of type 1 3 2.

1. When all elements of S are front entries. Then any front entry must be larger than any back entry for any pair violating this condition would form an additional 132-subsequence with n. Therefore, the i largest entries must be front entries n (these are the entries $n-1, n-2, \cdots, n-i+1$), while the $n-i$ smallest entries must be back entries (these are the entries $1, 2, \cdots n-i$). Moreover, there can be no subsequence of type 132 formed by back entries. So all we can do is to take a 132-avoiding permutation on the $n-i$ back entries in c_{n-i} ways and take a permutation

having having exactly one 132-subsequence on the $i-1$ front entries. This yields $b_{i-1}c_{n-i}$ permutations of the desired property.

2. When all elements of S are back entries. The argument of the previous case holds here, too, we must only swap the roles of the front and back entries. Then we get that in this case we have $c_{i-1}b_{n-i}$ permutations of the desired property.

3. Finally, it can happen that the leftmost element x of S is a front entry and rightmost element z of S is a back entry. This case is slightly more complicated. Note that here we must have $2 \leq i \leq n-1$, otherwise either the set of front entries or that of back entries would be empty.

   First note that there is exactly one pair (x, z) so that x is a front entry, z is a back entry and $x < z$. (For any such pair and n form a 132-subsequence). This implies that the front entries are $n-1, n-2, \cdots, n-i+2, n-i$ and the back entries are $1, 2, \cdots n-i-1, n-i+1$, the only pair with the given property is $(n-i, n-i+1) = (x, z)$, and any other front entry is larger than both x and z.

   Let us take these entries x and z. Clearly, all 132-subsequences of the given type must start with x and must end with z. We claim that the middle entry of S must be n. Indeed, if the middle element were some other w, then $x\,n\,z$ and $x\,w\,z$ would both be 132-subsequences. (Recall that $x < z$ and they both are smaller than any other front entry). Moreover, we claim that x must be the rightmost front entry, in other words, it must be in the position directly on the left of n. Indeed, if there were any entry y between x and n, then $x\,y\,z$ and $x\,n\,z$ would both be 132-subsequences for y is a front entry and thus larger than x and z.

   Therefore, all we can do is put the entry $n-i$ in the $i-1$-st position, then take any 132-avoiding permutation on the first $i-2$ elements in c_{i-2} ways and take any 132-avoiding permutation on the $n-i$ back entries in c_{n-i} ways. This gives us

$c_{i-2}c_{n-i}$ permutations of the desired property.

Summing for all permitted i in each of these three cases we get that

$$b_n = \sum_{i=1}^{n-1} b_{i-1}c_{n-i} + \sum_{i=1}^{n-1} c_{i-1}b_{n-i} + \sum_{i=2}^{n-1} c_{i-2}c_{n-i}. \tag{3.8}$$

Note that the first two sums are equal for they contain the same summands. Moreover, by (1.1) we can easily see that the last sum equals $c_{n-1} - c_{n-2}$. Thus the above recursive formula for b_n simplifies to

$$b_n = 2 \cdot (\sum_{i=1}^{n-1} b_{i-1}c_{n-i}) + c_{n-1} - c_{n-2}. \tag{3.9}$$

Now let $B(x) = \sum_{n=0}^{\infty} b_n x^n$, the ordinary generating function of the sequence $\{b_n\}$. Then one sees by (3.9) and by equating the coefficients of x^n that the following functional equation must hold:

$$B(x) = 2xB(x)C(x) + (x - x^2)C(x) - x.$$

This yields

$$B(x) = \frac{C(x)(x - x^2) - x}{1 - 2xC(x)} \tag{3.10}$$

Recall that (1.2) provides an explicit form for $C(x)$. Plugging it in (3.10) we get

$$B(x) = \frac{\frac{1-\sqrt{1-4x}}{2} \cdot (1 - x) - x}{\sqrt{1 - 4x}},$$

or, splitting the numerator into three parts,

$$B(x) = \frac{1}{2 \cdot (\sqrt{1 - 4x})} \cdot (1 - x) - \frac{x}{\sqrt{1 - 4x}} + \frac{1 - x}{2}. \tag{3.11}$$

We have already computed in the introduction, in the expansion of (1.2) that $1/\sqrt{1-4x} = \sum_{n\geq 0}\binom{2n}{n}x^n$. Therefore (3.11) is equivalent to

$$B(x) = \frac{1}{2}(1-x)\sum_{n\geq 0}\binom{2n}{n}x^n - x\sum_{n\geq 0}\binom{2n}{n}x^n + \frac{1-x}{2} =$$

$$= 1/2\sum_{n\geq 0}\binom{2n}{n}x^n - 1/2\sum_{n\geq 1}\binom{2n-2}{n-1}x^n - \sum_{n\geq 1}\binom{2n-2}{n-1}x^n + \frac{1-x}{2}$$

Equating coefficients of x^n, with $2 \leq n$ we get that

$$b_n = \frac{1}{2}\cdot\binom{2n}{n} - \binom{2n-2}{n-1} - \frac{1}{2}\cdot\binom{2n-2}{n-1} = \binom{2n-1}{n-1} - \binom{2n-2}{n-1} - \binom{2n-3}{n-2} =$$

$$= \binom{2n-2}{n-2} - \binom{2n-3}{n-2} = \binom{2n-3}{n-3},$$

and the theorem is proved. ◇

Corollary 11 *Let q be any pattern of length 3. Then $S_{q,1}(n)$ is P-recursive in n.*

The recursive method we have presented can also be used to give another proof for a classical polygon-partitioning problem first fully solved by Cayley [9].

Theorem 18 *The number of partitions of a convex $(n+1)$-gon into $(n-2)$ parts by noncrossing diagonals is $b_n = \binom{2n-3}{n-3}$.*

Proof: In other words, we are looking for the number of partitions of that polygon into one quadrilateral Q and $(n-3)$ triangles. We prove the statement by induction on n. Recall [28] that the number of partitions of a convex $(n+2)$-gons into n triangles by noncrossing diagonals is c_n.

If $n = 3$, then our statement is true. Suppose we know the statement for all integers larger than 2 and smaller than n. Let $A_1, A_2, \cdots, A_{n+1}$ denote the vertices of the polygon.

Let k be the smallest number so that there is a diagonal A_1A_k in our chosen partition. (If there is no such diagonal, then let $k = n + 1$. Thus $3 \leq k \leq n + 1$.) The diagonal A_1A_k cuts our polygon into two parts; the part containing the vertex A_2 is called the *front* whereas the other part is called the *back*. Now if the back contains Q (in b_{n-k+2} ways, by the induction hypothesis), then the front is partitioned into triangles in c_{k-3} ways as the diagonal A_2A_k must be contained in our triangulation. If the front contains Q, but A_1 is not a vertex of the quadrilateral, then again, A_2A_k is contained in the partition and we have c_{n-k+1} ways to triangulate the back and then b_{k-2} ways to partition the front. Finally, if A_1 is a vertex of Q, then $Q = A_1A_2A_{k-1}A_k$ and we have c_{k-3} ways to triangulate the rest of the front, in addition to the c_{n-k+1} ways to triangulate the back.

Replacing $k - 3$ by i and adding for all i we get the recursive formula of (3.8) and our claim is proved for n. $\diamond$

3.3 A recursive formula for $r = 2$.

In this section we provide a recursive formula for $d_n = S_{132,2}$, proving this way that $S_{132,2}$ is P-recursive in n. The method we use is similar to those of the last chapter, we have still found it worth including as this way we can see how the computation gets more and more complicated as r grows, and why we need a more general approach to treat the general case.

Theorem 19 *Let $d_n + S_{132,2}$ be the number of n-permutations containing exactly two subsequences of type 132. Then the sequence $\{d_n\}$ is P-recursive.*

Proof: We are going to distinguish three cases, according to the number of *bad pairs*, that is, pairs (x, y) where x is a front entry, y is a back entry and $x < y$. Cleary, each bad pair forms a 132-pattern with n, thus there can be at most two bad pairs. In each case, we are going to go through all possible permutations according to the position of n, but for shortness, we are not announcing it each time.

1. If there is no bad pair, then by an argument similar to that of the previous section, we see that there are

$$\sum_{i=0}^{n-1} d_{i-1}c_{n-i} + \sum_{i=0}^{n-1} c_{i-1}d_{n-i} + \sum_{i=0}^{n-1} b_{i-1}b_{n-i} = 2 \cdot \sum_{i=0}^{n-1} c_{i-1}d_{n-i} + \sum_{i=0}^{n-1} b_{i-1}b_{n-i} \quad (3.12)$$

   permutations containing exactly two 132-patterns.

2. If there is one bad pair (x, y), then there are two subcases. Note that x, n and y form a 132-pattern, so we must have exactly one more pattern of that kind. Furthermore, note that the position of n determines our only choice for the pair (x, y).

   - If x is directly on the left of n, then we can proceed as in the previous section and see that there are

$$\sum_{i=2}^{n-1} c_{i-2}b_{n-i} + \sum_{i=2}^{n-1} b_{i-2}c_{n-i} = 2 \cdot \sum_{i=2}^{n-1} c_{i-2}b_{n-i} = b_{n-1} - c_{n-2} + c_{n-3} \quad (3.13)$$

     permutations with the required property. (The last equality follows from (3.9)).

   - If there are some entries between x and n, then note that each such entry would form a 132-pattern with x and y, so there can be only one such entry, and therefore x must be exactly two positions to the left of n. There are no other restrictions and there can be no more 132-patterns, which yields

$$\sum_{i=3}^{n-1} c_{i-2}c_{n-i} = c_{n-1} - 2c_{n-2} \quad (3.14)$$

permutations with the required property. (Note that n must be preceded by at least two elements).

3. If there are two bad pairs, then there are two subcases again. Note that the two bad pairs with n provide the two 132-subsequences, thus there cannot be any additional subsequence of that type.

   - If the two bad pairs are (x, y) and (x, z), then x must be directly on the left of n to avoid additional 132-patterns. (Again, x, y and z are determined by the position of n). To ensure that no additional 132-patterns are formed, y and z must be in increasing order, otherwise x completes them to a 132-pattern. Observe that y and z are the largest two back elements, so we simply need to compute the number of 132-avoiding permutations on $n-i$ elements in which the largest two elements are in increasing order. It is easy to see that this number is $c_{n-i} - c_{n-i-1}$, since the only way those two elements can be in decreasing order is when the larger one is in the leftmost position of the permutation. Therefore, in this subcase we have

   $$\sum_{i=2}^{n-2} c_{i-2}(c_{n-i} - c_{n-i-1}) = c_{n-1} - 2c_{n-2} \tag{3.15}$$

   suitable permutations. (Note that n must precede at least two elements).

   - Finally, if the two bad pairs are (x, z) and (y, z), then x, y and z are again determined by the position of n. In order to avoid additional 132-patterns, x and y must be in the two positions directly on the left of n. They can be in either order as they are smaller than everything on their left and larger than everything on their right, except n and z. There are no other restrictions and no other 132-patterns, so this subcase gives us

$$2\sum_{i=3}^{n-1} c_{i-2}c_{n-i} = 2c_{n-1} - 4c_{n-2} \tag{3.16}$$

suitable permutations.

Summing for all cases we get that

$$d_n = 2 \cdot \sum_{i=1}^{n-1} c_{i-1}d_{n-i} + \sum_{i=1}^{n-1} b_{i-1}b_{n-i} + b_{n-1} + 4c_{n-1} - 9c_{n-2} + c_{n-3}. \tag{3.17}$$

We know from Lemma 15 that the sum and the convolution of two P-recursive sequences are P-recursive. So if h_n denotes the sum of all the terms on the right hand side of (3.17) except the first one, then h_n is P-recursive. Let $D(x)$ and $H(x)$ be the ordinary generating functions for $\{d_n\}$ and $\{h_n\}$, respectively. Then (3.17) gives rise to the functional equation

$$D(x) = 2 \cdot (C(x)D(x)x + 4x^4) + H(x),$$

that is,

$$D(x) = \frac{H(x) + 4x^4}{1 - 2xC(x)}, \tag{3.18}$$

thus $D(x)$ is d-finite as $1/(1-2xC(x)) = \sqrt{1-4x}$ is d-finite. Therefore, d_n is P-recursive, which was to be proved. $\diamond$

3.4 P-recursiveness for all r

3.4.1 The decomposition of the problem

In [37] Zeilberger and Noonan conjectured that for *any* given subsequence q and for *any* given r, the number of n-permutations containing exactly r subsequences of type q is a P-recursive function of n.

In this subsection we prove this conjecture for the subsequence 1 3 2. This is the first result we know of when the case of each r is solved for some given q.

The proof will be based on a carefully built induction on r. In fact, we prove something more general in Theorem 20: we prove that the statement remains true even if we restrict ourselves to n permutations which contain exactly r subsequences of type 132 *and* end in a subsequence of a given type, or whose largest elements form a subsequence of a given type.

We will proceed as follows. First we prove our statements for $r = 0$. Then we suppose that we know our statements, that is the statement without restrictions, and the ones with restrictions for all natural numbers smaller than r. In Part One of the proof of the induction step, we prove the statement without restrictions from the induction hypothesis on r. In Part Two we prove the statements with restrictions, from the induction hypothesis on r and the result of Part One, completing the proof. (In this last step we also use induction on k, the length of the restricting subsequence).

Recall that the sum, difference and convolution of two P-recursive functions are P-recursive as well. Therefore, the sum of *finitely many* such functions is P-recursive, too. We will use this fact throughout our proof. However, in order to be able to do so, we have to partition the set of all n-permutations with the desired property into a bounded number of classes. This decomposition is the subject of this subsection. Whenever we use the word "bounded", we mean an expression which is independent from n, that is, it depends only on the fixed number r and maybe also on k, the length of some fixed pattern q.

Definition 15 *Front elements of p which are smaller than the largest back element of p will be called* red elements, *whereas back elements of p larger than the smallest front element of p will be called* blue elements.

What is the advantage of this terminology? First, any red element is smaller than any front element which is not red, while any blue element is larger than any back element which is not blue. In other words, red elements are the smallest front elements, while blue elements are the largest back elements. Moreover, any red and any blue element is part of at least one 132-subsequence. Indeed, take any red element x, the entry n, and any back element larger than x. Dual argument applies for blue elements. Finally, if a 132-subsequence spans over the entry n, that is, it starts with a front element and ends with a back element, then it must start with a red one and end with a blue one.

Now we show several ways to partition the set of all n-permutations containing exactly r subsequences of type 132. Our point is that we can partition them into a *bounded* number of classes.

As we said above, any colored element is part of at least one subsequence of type 132. therefore if p has exactly r subsequences of type 132, and R (resp. B) denotes the number of red (resp. blue) elements, then $max(R, B) \leq r$. This implies that we have at most r^2 choices for the values of R and B.

Once the values of R and B are given and we know in which position the entry n is, then we only have a bounded number of choices for the set of red and blue elements. Indeed, if x is the smallest red element, then x is larger than all but B back elements. So if n is in the ith position, then $x \geq n - i - B$. On the other hand, x is the smallest front element, so $x \leq n - i + 1$. Similar argument applies for the largest blue element.

Finally, there is only a bounded number of positions where a red element can be. Indeed, if x is red and $y > x$ is a back element, then $x\ z\ y$ is a 132-pattern for all front elements on the right of x which are not red (and so, are larger than any back elements). We recall that $x\ n\ y$ is such a pattern as well. Thus, if t is the number of such (x, z) pairs, then we have $t + R \leq r$. In particular, the distance between any red element and the entry n cannot be larger than r.

The following definition makes use of the observations we have just made:

Definition 16 *We say that the n-permutations p_1 and p_2 are in the same* class *if they agree in all of the following:*

- *the position of the entry n*
- *the set of red elements*
- *the set of blue elements*
- *the pattern formed by the blue elements*
- *the position of the red elements*
- *the pattern starting with the leftmost red element and ending with the entry n.*

In other words, permutations of the same class agree in everything that can be part of a 132-pattern spanning through the entry n.

Our argument shows that there are only a bounded number of classes of n-permutations.

Definition 17 *Let p be an n-permutation. The subsequence of p consisting of*

- *all red and blue entries and*
- *all front entries which are preceded by at least one red entry and*
- *the entry n*

is called the fundamental subsequence *of p.*

This means that permutations of the same class have fundamental subsequences of identical type, and these subsequences are in the same positions in every permutation of a given class.

Definition 18 *The classes C and C' are called* similar *if their permutations have fundamental subsequences of the same type.*

Thus in this case the subsequences don't need to be in identical positions.

Example 11 The classes containing the permutations 3 4 1 5 2 and 4 2 5 3 1 are similar.

3.4.2 The Initial Step

First we prove two lemmas which will later be used as the initial step of our inductive proof. Recall the natural Definition 8 for inserting and deleting an element in or from a pattern.

Lemma 17 *Let q be any subsequence of length k. Then the number $C_q(n)$ of 132-avoiding n-permutations which end with a subsequence of type q is a P-recursive function of n.*

Proof: Induction on k. If $k = 0$, then $C_q(n) = C_n = \binom{2n}{n}/(n+1)$, the nth Catalan-number and we are done. Suppose we know the statement for all subsequences of length $k-1$ and prove it for the subsequence q, which has length k.

If q is not 132-avoiding, then clearly $C_q(n) = 0$. So we can suppose that q is 132-avoiding. Now we consider two separate cases.

1. If the last element y of q is *not* the largest one, then let x be the rightmost element of q which is larger than y. Then the entry 1 of our n-permutation p cannot be on the left of x, so in particular, the entry 1 of p is one of the last k entries, which

form a subsequence of type q. Then it must be the smallest of these last k entries, so we know exactly where the entry 1 of p is located. Let us erase the smallest entry of q to get the subsequence q'. Apply the induction hypothesis to q' to get that $C_{q'}(n-1)$ is P-recursive. Then insert 1 to its original place to see that $C_q(n)$ is P-recursive.

2. If y is the largest element of q, then it is easy to apply what we have just shown in the previous case. Let q'' be the subsequence obtained from q by deleting y. Moreover, let $q_1, q_2, \cdots q_{k-1}$ (respectively) be the k-subsequences whose first $k-1$ elements determine a subsequence of type q'' and whose last elements (respectively) are $1, 2, \cdots (k-1)$. Then it is obvious that

$$C_q(n) = C_{q''}(n) - \sum_{i=1}^{k-1} C_{q_i}(n) \tag{3.19}$$

   The first term of the right hand side is P-recursive by induction and the second one is P-recursive by the previous case, and the lemma is proven as the sum and difference of finitely many P-recursive functions are P-recursive.

$\diamond$

The method of case 2 will be called the *complementing method* for obvious reasons.

Lemma 18 *Let q be any subsequence of length k. Then the number $K_q(n)$ of 132-avoiding n-permutations in which the* largest *k elements form a subsequence of type q is a P-recursive function of n.*

Proof: As in the previous lemma, we proceed by induction on k. Again, if $k = 0$, then $K_q(n) = C_n$ and we are done. Suppose we know the statement for all subsequences of length $k-1$ and prove it for the subsequence q. We can suppose that q is 132-avoiding. We consider two separate cases again.

1. If the first element of q is *not* the smallest one, then the leftmost of the k largest elements of our n-permutation p must be at the very first position. (Otherwise some smaller element precedes it and we get a 132-pattern). In this way we know exactly what is the first element of p and can proceed as in case 1 of the proof of the previous lemma.

2. If the first element of q is the smallest one, then we are done by the complementing method of case 2 of the previous lemma.

$\diamond$

3.4.3 The Induction Step

Now we announce and then prove the main result of this section.

Theorem 20 *Let q be any subsequence of length k. Then the number $C_{q,r}(n)$ of n-permutations which contain exactly r subsequences of type 132 and end with a subsequence of type q is a P-recursive function of n. Similarly, the number $K_{q,r}(n)$ of n-permutations which contain exactly r subsequences of type 132 and in which the largest k elements form a subsequence of type q is a P-recursive function of n.*

Thus in particular, $S_r(n)$ is P-recursive in n.

Proof: Induction on r. If $r = 0$, then the two statements are equivalent to Lemmas (17) and (18). Now suppose we know both statements for $r - 1$.

PART ONE First we prove from this that $S_r(n)$ is P-recursive in n. Choose any class C of n-permutations. Suppose the fundamental subsequence type of C contains exactly s subsequences of type 132, where $s \leq r$.

- Suppose for now that the fundamental subsequence is nonempty, then $s \geq 1$ holds as well.

How can a permutation in C contain 132-patterns which are *not* contained in the fundamental subsequence? Clearly, they must be either entirely before the entry n or entirely after it. If i such subsequence is before and j is after, then $i+j+s=r$ must hold. Denote q_1 the pattern of all front entries in the fundamental subsequence and q_2 the pattern of all back entries there. Then with the previous notation we have

$$f(n_1, n_2, q_1, q_2, i, j, s) = C_{q_1,i}(n_1) \cdot K_{q_2,j}(n_2)$$

such permutations, where n_1 (resp. n_2) denotes the number of front (resp. back) entries which are not in the fundamental subsequence. Indeed, elements of the fundamental subsequence are either the rightmost front elements, or the largest back elements. We know by induction that $C_{q_1,i}(n_1)$ is P-recursive in n_1 and $K_{q_2,j}(n_2)$ is P-recursive in n_2. Therefore, their convolution

$$f(n, q_1, q_2, i, j, s) = \sum_{n_1+n_2=n} f(n_1, n_2, q_1, q_2, i, j, s) = \sum_{n_1+n_2=n} C_{q_1,i}(n_1) \cdot K_{q_2,j}(n_2) \tag{3.20}$$

is P-recursive in n. Clearly this convolution expresses the number of n-permutations with exactly r subsequences of type 132 in all classes similar to C. It is clear now that we have only a bounded number of choices for i, j and s so that $i+j+s=r$, so we can sum (3.20) for all these choices and still get that

$$f(n, q_1, q_2) = \sum_{i,j,s} f(n, q_1, q_2, i, j, s) \tag{3.21}$$

is P-recursive in n. (Recall that $s > 0$, thus we can always use the induction hypothesis). Summing (3.21) for all q_1 and q_2 we get that

$$f(n) = \sum_{q_1,q_2} f(n, q_1, q_2) \tag{3.22}$$

- Now suppose that the fundamental subsequence of the permutations in C is empty. Then any 132-subsequence must be either entirely on the left of the entry n or entirely on the right of n. Moreover, the position of n completely determines the set of the front and back elements. If n is in the ith position, and we have j 132-subsequences in the front and $r-j$ in the back, then this gives us

$$g(i,j) = S_j(i-1) \cdot S_{r-j}(n-i) \tag{3.23}$$

permutations of the desired kind. If $1 \leq j \leq r-1$, then the induction hypothesis applies for S_j and S_{r-j}, therefore, after summing (3.23) for all i

$$g(n) = \sum_i S_j(i-1) \cdot S_{r-j}(n-i) \tag{3.24}$$

is P-recursive in n. If $j=0$ of $j=r$, then we cannot apply the induction hypothesis. By similar argument as above we get nevertheless that in this case we have

$$2 \cdot \sum_i S_r(i-1) \cdot C_{n-i} \tag{3.25}$$

n-permutations with exactly r 132-subsequences. (Note that $S_0(n-i) = C_{n-i}$, the $(n-i)$th Catalan-number).

Summing (3.22), (3.24) and (3.25) we get

$$S_r(n) = f(n) + g(n) + 2 \cdot \sum_i S_r(i-1) \cdot C_{n-i}. \tag{3.26}$$

Let F, G, C and S denote the ordinary generating functions of $f(n)$, $g(n)$, C_n and $S_r(n)$. Then the previous equation yields

$$S(x) = F(x) + G(x) + 2x \cdot C(x)S(x),$$

that is,

$$S(x) = \frac{F(x) + G(x)}{1 - 2x \cdot C(x)}. \tag{3.27}$$

Therefore $S(x)$ is D-finite as the numerator is D-finite and $1/(1 - 2xC(x)) = 1/\sqrt{1-4x}$ is D-finite. So $S_r(n)$ is P-recursive and we are done with the first part of the proof.

PART TWO Now, using the induction hypothesis on r and the result of Part One, we prove the rest of the inductive step.

1. Let q be any subsequence of length k. We must prove that the number $C_{q,r}(n)$ of n-permutations which end with a subsequence of type q and contain exactly r subsequences of type 132 is a P-recursive function of n. Clearly, if q contains more than r 132-subsequences, then $C_{q,r}(n) = 0$ and we are done. Otherwise we will do induction on k, the case of $k = 1$ being equivalent to the result of Part One. We consider three different cases.

   - If q has more than r inversions, then it is obvious that no such permutation can have its entry 1 on the left of the last k elements. Therefore, this entry 1 must be a part of the q-subsequence formed by the last k elements, in fact, it is the minimal one among them. Now deleting this entry 1 we may or may not lose some 132-patterns as there may or may not be inversions on its right, but we can read off this information from q. (See the next example). Again, let q' be the pattern obtained from q by deleting its entry 1. If we don't lose any 132-patterns by this deletion, then we are left with an $(n-1)$-permutation ending with the pattern q' and having r subsequences of type 132. If we lose t such patterns, then we are left with an $(n-1)$-permutation ending with q' and having $r - t$ such subsequences. Both cases give rise to a P-recursive function of n by our induction hypothesis as q' is shorter than q.

Example 12 If $q = 3\,4\,1\,6\,5\,2$, then $q' = 2\,3\,5\,4\,1$ so we lose three subsequences of type 132 when deleting 1. Therefore, we can apply our inductive hypothesis for $r-3$, then reinsert the entry 1 to its place. If $q = 3\ 1\ 2\ 4$, then we don't lose any 132-patterns when deleting the entry 1 and getting $q' = 2\ 1\ 3$. So we still need to count permutations with r 132-patterns, but they must end with q', not with q. The pattern q' is shorter than q, thus the induction hypothesis on k can be applied.

- If q has at most r inversions, but q is not the monotonic pattern $1\ 2\ \cdots\ k$, then it can also happen that the entry 1 is not among the last k entries of our permutation. However, we claim that it cannot be too far away from them. Indeed, let y be an element from the last k elements of the permutation (so one of those elements which form the ending q) which is smaller than some other such element x on its left. Then clearly, if n is large enough, then y must be smaller than $r+k+1$, otherwise we would have too many 132-patterns of the form $w\,x\,y$. So y is bounded. If the entry 1 of the permutation were more than $2r+k+1$ to the left of y, then there would necessarily be more than r elements between 1 and y which are larger than y, a contradiction. So the distance between 1 and y is bounded. Therefore we can consider all possibilities for the position of the entry 1 of the permutation and for the subsequence on its right. In each case we can delete the entry 1 and reduce the enumeration to one with a smaller value of r, (as q has at least one inversion), then use the inductive hypothesis on r. So this case contributes a bounded number of P-recursive functions, too.
- Finally, if q is the monotonic pattern $1\ 2\ \cdots\ k$, then use the complementing method of lemma 17.

So we have proved that $C_{q,r}(n)$ is always P-recursive in n.

2. Finally, let again q be any subsequence of length k; we must then prove that the number $K_{q,r}(n)$ of n-permutations in which the k largest elements form a subsequence of type q and which contain exactly r subsequences of type 132 is P-recursive in n. As in the proof of the previous statement, we can suppose that q contains at most r subsequences of type 132. Then we proceed by induction on k, considering three different cases in a similar manner.

   - If q has more than r inversions, then it is clear that the leftmost of the k largest elements must be the leftmost element of the whole permutation. (Otherwise there would be too many 132-patterns containing the leftmost element of the permutation). In this case we can simply delete the first element of q to get the shorter pattern q'', apply the induction hypothesis on k, then reinsert the first element of q to the first position.

   - If q has at most r inversions, but q is not the monotonic pattern $1\,2 \cdots k$, then it can also happen that the leftmost of the k largest elements (say, x) is not the leftmost element of the permutation. However, we claim that x cannot be too far away from the front. In fact, if there were more than r elements preceeding x, then each of these elements would form a 132-pattern together with any inversion on the k largest elements, which would be a contradiction. So x is at the rth position at most. This means that the susbsequence Q consisting of the k largest elements and all the elements preceding x has at most $2r$ elements. Therefore, we can consider all the possibilities for its type as there are only a bounded number of them. In each of these possible cases we can delete the leftmost element of the permutation, which is also the leftmost element of Q and apply the induction hypothesis on r.

   - Finally, if q is the monotonic pattern $1\ 2\ \cdots\ k$, then we can use the complementing method again.

As we get a bounded number of P-recursive functions in all cases, their sum is P-recursive as well, and the statement is proved.

This completes the proof of Part Two.

We have shown that if the statement of Theorem 20 holds for all natural numbers smaller than r, then it holds for r as well. The initial step has been shown in Lemmas 17 and 18, thus the theorem is proved by induction. $\diamond$

3.5 Beyond P-recursiveness

This far we have studied the class of P-recursive power series. Another, smaller class of formal power series is that of *algebraic* series.

Definition 19 *We say that the series $v(x) \in \mathbf{C}[[x]]$ is* algebraic *if there exist polynomials $p_0(n), p_1(n), \cdots p_{d-1}(n)$ so that $p_{d-1} \neq 0$ and*

$$v^d(x) + p_{d-1}(x)v^{d-1}(x) + \cdots + p_1(x)v(x) + p_0(x) = 0. \tag{3.28}$$

The sum and product of two algebraic power series are algebraic, and again, if u and v differ in finitely many coefficients, and u is algebraic, then so is v. Any algebraic power series is necessarily D-finite.

Yet another, even smaller class of power series is that of *rational* functions, that is, elements of $\mathbf{C}(x)$, the fraction field of the polynomial ring $\mathbf{C}[x]$. In other words, the elements of this class are fractions in which both the numerator and denominator are polynomials of x. Clearly, rational functions are algebraic.

Now note that speaking in terms of ordinary generating functions, all operations we made throughout the induction step were either adding or multiplying a finite number of power series together. In particular, the ordinary generating function $C(x)$ of our inital c_n-sequence (that is, when $r = 0$ and $k = 0$) is $C(x) = \frac{1-\sqrt{1-4x}}{2x}$, that is an algebraic power series. Therefore, the ordinary generating function of $S_r(n)$, the power series $G_r(x)$ is algebraic, too.

Now note a bit more precisely that throughout or proof we have either added formal power series together, or, as in (3.23), multiplied two functions of type $S_j(i-1)$ together, or, as in (3.27), multiplied a power series by $1/(1 - 2x \cdot C(x)) = 1/\sqrt{1-4x}$. Therefore, the following proposition is immediate:

Proposition 7 $G_r(x) \in \mathbf{C}(x, \sqrt{1-4x})$. *Moreover, when written in smallest terms, the denominator of* $G_r(x)$ *is a power of* $(\sqrt{1-4x})$ *if* $r \geq 1$.

It is convenient to work in this setting as the square of $\sqrt{1-4x}$ is an element of $\mathbf{C}[x]$, which makes computations much easier.

We are going to determine the exponent $f(r)$ of $\sqrt{1-4x}$ appears in the denominator. Equations (3.23) and (3.27) show that

$$f(r) = max_{1 \leq i < r} f(i) + f(r-i) + 1. \tag{3.29}$$

We claim now that $f(r) = 2r-1$ if $r \geq 1$. It is easy to compute (see [4]) that $f(1) = 1$. Now suppose by induction that we know our claim for all positive integers smaller than r. Then (3.29) and the induction hypothesis yield that $f(r) \geq (2i-1) + (2r-2i-1) + 1 = 2r-1$, which was to be proved.

Recall now that $1/\sqrt{1-4x} = \sum_{n \geq 0} \binom{2n}{n} x^n$ and that the sequence $\binom{2n}{n}$ satisfies a linear recursion. Differentiate both sides of this equation several times. On the left-hand-side, each differentiation will add two to the exponent of $\sqrt{1-4x}$ in the denominator. On the right-hand-side, it will add one to the degree of the highest-degree polynomials appearing in the recursive formula for the coefficients. So differentiating $r-1$ times we get that the denominator of $G_r(x)$ gives rise to a polynomial recursion of degree r. The numerator of $G_r(x)$ cannot increase this degree.

We collect our observations in our last lemma:

Lemma 19 *Let $r \geq 1$ and write $G_r(x)$ in lowest terms. Then the denominator of $G_r(x)$ is equal to*

$$(\sqrt{1-4x})^{2r-1} = (1-4x)^{r-1} \cdot \sqrt{1-4x}.$$

Therefore, the sequence $S_n(r)$ satisfies a polynomial recursion with maximal degree r.

This result is useful for computational purposes: knowing the first few values of $S_r(n)$, one should be able to obtain these polynomial recursions of degree r.

Chapter 4

An Infinite Antichain of Permutations

When considering a partially ordered set with infinitely many elements, one should wonder whether it contains an infinite antichain (that is, a subset in which any two elements are incomparable). It is well known that all antichains of N^k (where $(x_1, x_2, \cdots, x_k) \leq (y_1, y_2, \cdots, y_k)$ if and only if $x_i \leq y_i$ for $1 \leq i \leq k$) are finite. (See [16]). Another basic result is that all antichains of the partially ordered set of the finite words of a finite alphabet are finite, where $x < y$ if one can delete some letters from y to get x. (This result is due to Higman and can be found in [17]). In this chapter we examine this question for the partially ordered set P of finite permutations with the following $<$ relation: if m is less than n, and p_1 is a permutation of the set $\{1, 2, \cdots, m\}$ and p_2 is a permutation of the set $\{1, 2, \cdots, n\}$, then $p_1 < p_2$ if and only if p_1 is contained in p_2 as a pattern.

We would like to point out that any answer to this question would be somewhat surprising. If there were no infinite antichains in this partially ordered set, that would be surprising because, unlike the two partially ordered sets we mentioned in the first paragraph, P is defined over an infinite alphabet and the "size" of its elements can be arbitrarily large. On the other hand, if there is an infinite antichain, and we will find one, then it shows that this poset is more complex in this sense than the poset of graphs ordered by the operations of edge contraction and vertex deletion. (That this poset of graphs does not contain an infinite antichain is a famous theorem of Robertson and Seymour [22, 23]). This is surprising too, as graphs are usually much more complex than permutations.

We are going to construct an infinite antichain, $\{a_i\}$. The elements of this antichain will be very much alike; they will in fact be identical at the beginning and at the end. Their middle parts will be very similar, too. These properties will help ensure that no element is contained in another one.

Let $a_1 = 13, 12, 10, 14, 8, 11, 6, 9, 4, 7, 3, 2, 1, 5$. We view a_1 as having three parts: a decreasing sequence of length three at its beginning, a long alternating permutation starting with the maximal element of the permutation and ending with the entry 7 at the fifth position from the right, (in this alternating part odd entries have only even neighbors and vice cersa; moreover, the odd entries and the even entries form two decreasing subsequences so that $2i$ is between $2i+5$ and $2i+3$), and a terminating subsequence 3 2 1 5.

To get a_{i+1} from a_i, simply insert two consecutive elements right after the maximum element m of a_i, and give them the values $(m-6)$ and $(m-3)$. Then make the necessary corrections to the rest of the elements, that is, increment all old entries larger than $(m-3)$ by 2, increment the old entries $(m-6), (m-5), (m-4)$ by 1, and leave the rest unchanged (see Figure 4-1).

Thus the structure of any a_i is very similar to that of a_1—only the middle part becomes two entries longer.

We claim that the a_i form an infinite antichain. Assume by way of contradiction that there are indices i, j so that $a_i < a_j$. How could that possibly happen? First, note that the rightmost element of a_j must map to the rightmost element of a_i, since this is the

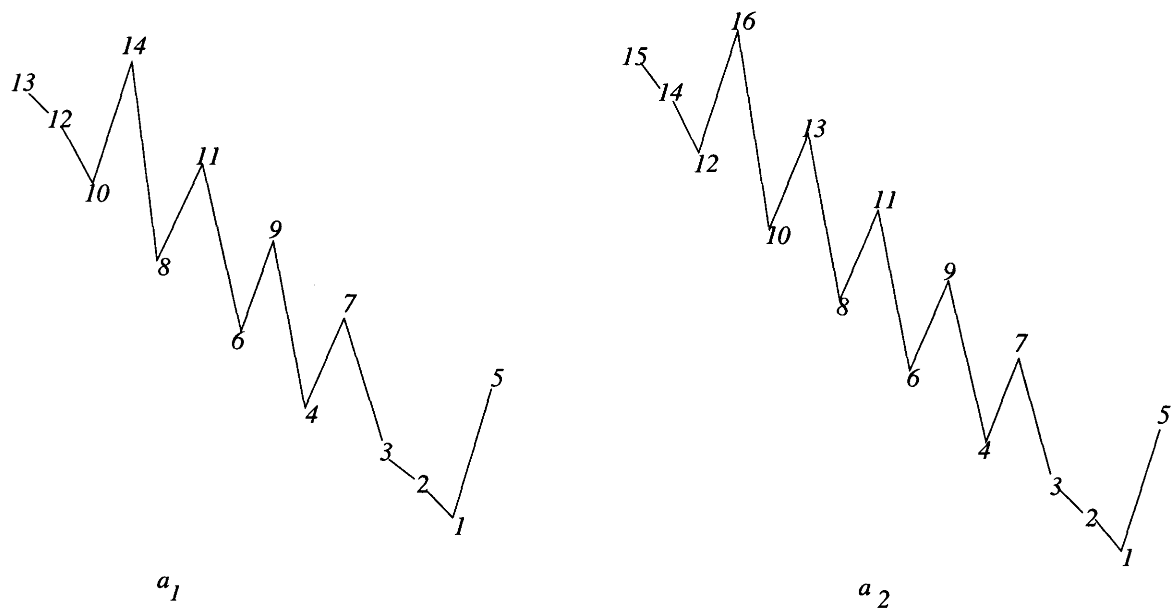

Figure 4-1: The pattern of a_i

only element in a_j preceded by four elements less than itself. Similarly, the maximal element of a_j must map to the maximal element of a_i, since, excluding the rightmost element, this is the only element preceded by three smaller elements. This implies that the first four and the last six elements of a_j must be mapped to the first four and last six elements of a_i, thus none of them can be deleted.

Therefore, when deleting elements of a_j in order to get a_i, we can only delete elements from the middle part, M_j. We have already seen that the maximum element cannot be deleted. Suppose we can delete a set D of entries from M_j so that the remaining pattern is a_i. First note that D cannot contain three consecutive elements, otherwise every element before those three elements would be larger than every element after them, and a_i cannot be divided in two parts with this property. Similarly, D cannot contain two consecutive elements in which the first is even. Thus D can only consist of separate single elements (elements whose neighbors are not in D) and consecutive pairs in which the first element is odd. Clearly, D cannot contain a separate single element as in that case the middle part of resulting permutation would contain a decreasing 3-subsequnce, but the middle part, M_i, of a_i does not. On the other hand, if D contained two consecutive elements x and y so that x is odd, then the odd element z on the right of y would not be in D as we cannot have three consecutive elements in D, therefore z would be in the remaining copy of a_i and z wouldn't be preceded by two entries smaller than itself. This is a contradiction as all odd entries of M_i have this property.

This shows that D is necessarily empty, thus we cannot delete any change elements from a_j to obtain some a_i where $i < j$.

We have shown that no two elements in $\{a_i\}$ are comparable, so $\{a_i\}$ is an infinite antichain. $\diamond$

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