Enumeration in Algebra and Geometry

by

ALEXANDER POSTNIKOV

Master of Science in Mathematics
Moscow State University, 1993

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1997

Author ....
Department of Mathematics
May 2, 1997

Certified by ....
Richard P. Stanley
Professor of Applied Mathematics
Thesis Supervisor

Accepted by ....
Hung Cheng
Chairman, Applied Mathematics Committee

Accepted by ....
Richard Melrose
Chair, Department Committee on Graduate Students

Enumeration in Algebra and Geometry

by

ALEXANDER POSTNIKOV

Submitted to the Department of Mathematics
on May 2, 1997, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Mathematics

Abstract

This thesis is devoted to solution of two classes of enumerative problems.

The first class is related to enumeration of regions of hyperplane arrangements. We investigate deformations of Coxeter arrangements. In particular, we prove a conjecture of Stanley on the numbers of regions of Linial arrangements. These numbers have several additional combinatorial interpretations in terms of trees, partially ordered sets, and tournaments. We study a more general class of truncated affine arrangements, counting their regions, giving formulas for their Poincaré polynomials, and proving a "Riemann hypothesis" on location of zeros of the latter. In addition, we find a couple of new interpretations for the Catalan numbers.

The second class of problems comes from enumerative algebraic geometry and Schubert calculus and is related to Gromov-Witten invariants of complex flag manifolds. We present a method for their calculation using a new construction for the quantum cohomology ring of the flag manifold. This construction provides quantum analogues of results of Bernstein, Gelfand, and Gelfand on this subject and of the theory of Schubert polynomials of Lascoux and Schützenberger. The quantum version of Monk's formula is established, and a general Pieri-type formula is derived.

While being remote from each other at first glance, both these subjects can be attacked with algebraic and combinatorial methods.

Thesis Supervisor: Richard P. Stanley
Title: Professor of Applied Mathematics

Acknowledgments

First of all, I thank my parents.

I am greatly indebted to my advisor Richard Stanley. His book *Enumerative Combinatorics,* vol. 1, deeply influenced me long before I had the privilege of meeting Professor Stanley personally.

My initial interest in mathematics was mainly the result of efforts of my high school teacher Sergey Grigorievich Roman, to whom my thanks are due.

I benefited greatly from a unique opportunity to attend the seminar of Israel Gelfand in Moscow. I wish to express my gratitude to him as well as to Askold Khovanskii, Vladimir Retakh, Andrei Zelevinsky, and many other people who contributed to the unforgettable mathematical environment.

I am grateful to Sergey Fomin and to Bertram Kostant. My warm regards are due to Gian-Carlo Rota. Communication with them was helpful and enriching.

I would also like to thank, possibly repeating names, my coauthors: Sergey Fomin, Israel Gelfand, Sergei Gelfand, Mark Graev, and Richard Stanley.

My thanks are due to Alexander Astashkevich, Arkady Berenstein, and Oleg Gleizer for helpful discussions related and not related to mathematics.

At last, I thank again the people who agreed to be the members of my thesis committee: Sergey Fomin, Richard Stanley, and Andrei Zelevinsky.

Contents

1 Hyperplane Arrangements **9**

1.1 Introduction . . . 9

1.2 Arrangements of Hyperplanes . . . 12

1.2.1 Regions and Poincaré polynomials . . . 12

1.2.2 Coxeter arrangements . . . 13

1.2.3 Whitney's formula . . . 14

1.2.4 Deformations of Coxeter arrangements . . . 16

1.2.5 The Orlik-Solomon algebra . . . 18

1.3 Catalan Miscellanea . . . 19

1.3.1 Semiorders . . . 19

1.3.2 Reciprocity for hyperplane arrangements . . . 21

1.3.3 Polyhedra and their triangulations . . . 23

1.4 Alternating Trees and the Linial Arrangement . . . 25

1.4.1 Counting alternating trees . . . 25

1.4.2 Local binary search trees . . . 27

1.4.3 On stability and fickleness . . . 28

1.4.4 The Linial arrangement . . . 29

1.4.5 Balanced graphs . . . 30

1.4.6 Sleek posets and semiacyclic tournaments . . . 31

1.5 Truncated Affine Arrangements . . . 34

1.5.1 Functional equations . . . 34

1.5.2 Formulae for the characteristic polynomial . . . 38

1.5.3 Roots of the characteristic polynomial . . . 40

1.6 Asymptotics and Random Trees . . . 41

1.6.1 Characteristic polynomials and trees . . . 41

1.6.2 Random trees . . . 42

1.6.3 Asymptotics of characteristic polynomials . . . 45

2 Quantum Cohomology of Flag Manifolds **49**

2.1 Introduction . . . 49

2.2 Background . . . 54

2.2.1 Flag manifold and Schubert cells . . . 54

2.2.2 NilHecke ring and Schubert polynomials . . . 55

2.2.3 Quantum cohomology and Gromov-Witten invariants . . . 57

2.3 Combinatorial Quantum Multiplication . . . 59

2.3.1 Commuting elements in the nilHecke ring 59
2.3.2 Combinatorial quantization 61
2.4 Standard Elementary Polynomials 63
2.4.1 Straightening 63
2.4.2 Deformation 65
2.4.3 Straightforward deformation 69
2.5 Quantum Schubert Polynomials 70
2.5.1 Simple properties 70
2.5.2 Orthogonality property 72
2.5.3 Axiomatic characterization 73
2.6 Monk's Formula and its Extensions 76
2.6.1 Quantum version of Monk's formula 76
2.6.2 Quadratic ring 77
2.6.3 General version of Pieri's formula 78
2.6.4 Action on the quantum cohomology 80
2.6.5 Proof of general Pieri's formula 81

What is this thesis about?

Enumerative problems come up in various areas of mathematical research. Some of them can be formulated in purely combinatorial terms, while for others even such a formulation can be the sole purpose of a highly nontrivial investigation.

In this thesis,[1] I concern with several problems that appear in two different fields of mathematics.

The first topic is related to the classical question: "On how many pieces a certain collection of hyperplanes subdivides a linear space?" It is usually not hard to answer this question for a generic collection of hyperplanes. But for some special hyperplane arrangements the answer can be much more interesting than for the generic case, and yet not so easy to gain.

The second task of this thesis belongs to the area of enumerative geometry and is similar to the (no less classical) question: "How many algebraic curves of a given degree pass through a given set of points, assuming that the conditions imply that this number is finite?" This, usually hard, question can sometimes be solved with the help of recently discovered algebraic structures.

Although our two aims seem to be far from each other, our means are close. These are the methods of algebraic combinatorics.

Two parts of the thesis are independent from each other and, consequently, the reader may peruse them in whatever order he or she prefers. A person more inclined to read about Schubert calculus and quantum cohomology may skip the first part and directly proceed to reading the second part. On the other hand, a person more at ease with hyperplane arrangements and combinatorics of trees and posets may choose to ignore the second part and concentrate entirely on the first part of the thesis.

When determined on which part to start with, the reader should first acquaint himself or herself with the corresponding introduction, afterwards keep on reading the remaining sections.

[1]The thesis contains the results obtained in the papers [17, 20, 42, 43, 44] written in collaboration with coauthors and without at various time during my graduate studies at M.I.T.

Chapter 1

Hyperplane Arrangements

This part of my thesis is based on a joint work with Richard Stanley [44]. It also contains the results of [42] as well as some results of [20] obtained in collaboration with Israel Gelfand and Mark Graev.

1.1 Introduction

The main objects in this chapter are arrangements of hyperplanes. The simplest invariant of a hyperplane arrangement $\mathcal{A}$ in a real vector space is its number of regions $r(\mathcal{A})$, i.e., the number of connected components, on which hyperplanes subdivide the space. Another invariant is the cohomology ring of the complement to the complexification of $\mathcal{A}$. It can be shown that the dimension of the cohomology ring is equal to $r(\mathcal{A})$.

The Coxeter arrangement of type A_{n-1} is the arrangement of hyperplanes

$$x_i - x_j = 0, \qquad 1 \le i < j \le n. \tag{1.1.1}$$

The regions of this arrangement, $n!$ in number, correspond different ways of ordering the sequence $x_1, \ldots, x_n$. The cohomology ring of the complement was calculated by Arnold [1]. In particular, he showed that its Poincaré polynomial, which is the generating function for the Betti numbers, is equal to $(1+q)\,(1+2q)\cdots(1+(n-1)q)$.

In this chapter we study a more general class of arrangements which can be viewed as deformations of the arrangement (1.1.1). One of them is the *Linial arrangement* $\mathcal{L}_n$ given by

$$x_i - x_j = 1, \qquad 1 \le i < j \le n. \tag{1.1.2}$$

A tree on the vertices labelled by integers is called *alternating* if the labels along every path alternate, i.e., form an up-down or down-up sequence. Our main result on Linial arrangements says that the number of regions of the arrangement $\mathcal{L}_n$ is equal to the number of alternating trees on the vertices $1, \ldots, n+1$.

The arrangement $\mathcal{L}_n$ was first considered by Linial and Ravid. They calculated the numbers of regions of $\mathcal{L}_n$ for several first values of n. The statement above was conjectured by Stanley on the base of their numerical results. Alternating trees earlier appeared in [20] in the context of a certain hypergeometric system and a related polyhedron, then they were studied in [42]. The formula for the number of alternating trees, proved in [42], thus provides the one for the number of regions of the Linial arrangements. Explicitly,

$$r(\mathcal{L}_n) = 2^{-n} \sum_{k=1}^{n} \binom{n}{k} (k+1)^{n-1}.$$

In addition, these numbers have several other combinatorial interpretations. For example, we show that $r(\mathcal{L}_n)$ is also equal to the number of binary trees on the vertices $1, \ldots, n$ such that left children are always less than their parents and right children are always bigger.

We study a more general class of arrangements called *truncated affine arrangements.* They are finite subarrangements of the affine type A_{n-1} hyperplane arrangement, and explicitly given by the following equations, where a and b are fixed integers,

$$x_i - x_j = k, \qquad 1 \le i < j \le n, -a < k < b. \tag{1.1.3}$$

For instance, the Linial arrangement $\mathcal{L}_n$ corresponds to the case of $a = 0$ and $b = 2$.

Remind that the characteristic polynomial of a hyperplane arrangement is related to the Poincaré polynomial by a simple transformation. For $0 \le a < b$, we prove that the characteristic polynomial $\chi_n^{ab}(q)$ of the truncated affine arrangement (1.1.3) equals

$$\chi_n^{ab}(q) = (b-a)^{-1}(S^a + S^{a+1} + \cdots + S^{b-1})^n \cdot q^{n-1},$$

where S is the shift operator $S : f(q) \mapsto f(q-1)$.

As a byproduct of this statement, a "Riemann hypothesis" on zeros of the characteristic polynomial is obtained. Namely, we demonstrate that if $a \ne b$ then all roots of the characteristic polynomial $\chi_n^{ab}(q)$ have the same real part equal to $(a+b-1)n/2$. In contrast, for $a = b$, the roots are real. If $a = b - 1$ then all roots are equal to na.

An asymptotics of characteristic polynomials of Linial arrangements is found. In particular, for "big" n, the distance between two adjacent roots of the characteristic polynomial is "close" to $\pi\alpha$, where $\alpha = 1.199678\ldots$ is the root of the equation

$$e^{2\alpha} = (\alpha+1)(\alpha-1)^{-1}, \qquad \alpha > 1.$$

We also investigate some arrangements related to the Catalan numbers, and prove a reciprocity result for certain deformations of Coxeter arrangements with and without central hyperplanes (1.1.1). In addition, we present several interpretations of the Catalan numbers.

In the rest of Introduction we outline how this chapter is organized. Section 1.2 is devoted to main definitions and general theorems from the theory of hyperplane arrangements. We discuss regions, Poincaré and characteristic polynomials, intersection poset, and Orlik-Solomon algebra. In Section 1.2.3 we review several general theorems on hyperplane arrangements, including a variant of the NBC Theorem, which is our main technical tool. Then in Section 1.2.4 we apply this theorem to deformations of Coxeter arrangements.

In Section 1.3 we study the hyperplane arrangements related, in a special case, to interval orders and the Catalan numbers. In Section 1.3.2 we prove a general reciprocity result for such arrangements. We also mention several new interpretations of the Catalan numbers.

A discussion of alternating trees, Linial arrangements, and other related objects is the purpose of Section 1.4. We give the main result on Linial arrangements and alternating trees (Theorem 1.4.5), and introduce several combinatorial objects whose numbers are equal to the number of regions of the Linial arrangement: local binary search trees, sleek posets, semiacyclic tournaments, FIS and SIF trees. In Section 1.4.6 we prove a theorem on characterization of sleek posets in terms of forbidden subposets.

In Section 1.5 we study truncated affine arrangements. We provide a proof to the result on numbers of regions of these arrangements and their characteristic polynomials (Theorem 1.5.7). To do that, we first establish in Section 1.5.1 a functional equation for the exponential generating function of the numbers of regions. Then we deduce a "Riemann hypothesis."

In Section 1.6 we study "random" trees and asymptotics of characteristic polynomials.

1.2 Arrangements of Hyperplanes

In this section we give main definitions and several general theorems from the theory of hyperplane arrangements. For more details, see [61, 38, 39]. We prove a generalized Whitney's theorem and its corollary—the NBC theorem. Then we apply them for calculation of the numbers of regions and the Poincaré polynomials of deformations of type A Coxeter arrangements. Finally, we recall the construction of the Orlik-Solomon algebra.

1.2.1 Regions and Poincaré polynomials

An *arrangement of hyperplanes* or *hyperplane arrangement* is a discrete collection of affine hyperplanes in a vector space. Let $\mathcal{A}$ be a finite arrangement of hyperplanes in a real vector space V. We will always assume[1] that the vectors dual to hyperplanes in $\mathcal{A}$ span the space V^* and call $\mathcal{A}$ a *nondegenerate* arrangement in this case. A *region* of $\mathcal{A}$ is a connected component of the complement to hyperplanes in the arrangement. Let $r(\mathcal{A})$ denote the number of regions of $\mathcal{A}$.

The Poincaré polynomial is a q-analogue for these numbers. Let $\mathcal{A}_{\mathbb{C}}$ denote the complexified arrangement $\mathcal{A}$, that is the collection of the hyperplanes $H \otimes \mathbb{C}$, $H \in \mathcal{A}$, in the complex vector space $V \otimes \mathbb{C}$. A let $C_{\mathcal{A}}$ be the complement to hyperplanes of $\mathcal{A}_{\mathbb{C}}$ in $V \otimes \mathbb{C}$. The *Poincaré polynomial* $\mathrm{Poin}_{\mathcal{A}}(q)$ of $\mathcal{A}$ is the generating function for the Betti numbers of $C_{\mathcal{A}}$:

$$\mathrm{Poin}_{\mathcal{A}}(q) = \sum_{k \geq 0} \dim \mathrm{H}^k(C_{\mathcal{A}}, \mathbb{C})\, q^k.$$

The *intersection poset*[2] $L_{\mathcal{A}}$ of the arrangement $\mathcal{A}$ is the collection of all nonempty intersections of hyperplanes in $\mathcal{A}$ ordered by reverse inclusion. Thus the poset $L_{\mathcal{A}}$ has a unique minimal element[3] $\hat{0} = V$. The *characteristic polynomial* of $\mathcal{A}$ is then defined by

$$\chi_{\mathcal{A}}(q) = \sum_{z \in L_{\mathcal{A}}} \mu(\hat{0}, z)\, q^{\dim z}, \tag{1.2.1}$$

where μ denotes the Möbius function of $L_{\mathcal{A}}$ (see [51, Section 3.7]). The general properties of geometric lattices [51, Proposition 3.10.1] imply, for example, that the sign of $\mu(\hat{0}, z)$ is equal to $(-1)^{\operatorname{codim} z}$.

The following fundamental result of Orlik and Solomon [38] establishes a relation between the Poincaré and characteristic polynomials and the number of regions $r(\mathcal{A})$ as well as the number of bounded regions of $\mathcal{A}$.

[1] without loss of generality

[2] Here and elsewhere the word "poset" stands for "partially ordered set."

[3] This poset has a unique maximal element if and only if the intersections of hyperplanes in $\mathcal{A}$ is nonempty. In this case $L_{\mathcal{A}}$ is a geometric lattice.

Theorem 1.2.1 *Assume that $\mathcal{A}$ is a nondegenerate arrangement in an l-dimensional vector space. Then*

$$\chi_{\mathcal{A}}(q) = q^l \operatorname{Poin}_{\mathcal{A}}(-q^{-1}). \tag{1.2.2}$$

The dimension of the cohomology ring $\dim \mathrm{H}^*(C_{\mathcal{A}}, \mathbb{C}) = \operatorname{Poin}_{\mathcal{A}}(1) = (-1)^l \chi_{\mathcal{A}}(-1)$ *is the number of regions $r(\mathcal{A})$ of $\mathcal{A}$. Likewise, the alternating sum of the Betti numbers* $\operatorname{Poin}_{\mathcal{A}}(-1) = \chi_{\mathcal{A}}(1)$ *is the number of bounded regions of $\mathcal{A}$.*

A combinatorial proof the last two statements of this theorem in terms of the characteristic polynomial was earlier given by T. Zaslavsky in [61].

1.2.2 Coxeter arrangements

Let V be a real l-dimensional vector space, and let Φ be a *root system* in V^* with a distinguished set of *positive* roots $\Phi_+ = \{\phi_1, \phi_2, \ldots, \phi_N\}$ (see [8, Ch. VI]). The *Coxeter arrangement* $\mathcal{A}(\Phi)$ is the arrangement of hyperplanes in V given by

$$\phi_i(x) = 0, \qquad 1 \leq i \leq N, \tag{1.2.3}$$

where $x \in V$.

The number of regions (Weyl chambers) of $\mathcal{A}(\Phi)$ is equal to the order of the corresponding Weyl group W.

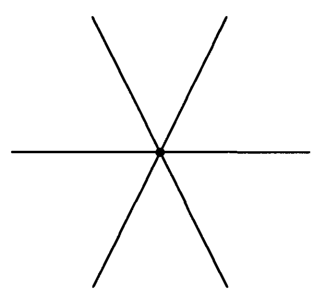

Figure 1-1: The Coxeter hyperplane arrangement $\mathcal{A}_3$.

In the case of a type A root system it is more convenient to use the augmented index $n = l + 1$. Let V_n denote the subspace (hyperplane) of all vectors $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that $x_1 + \cdots + x_n = 0$. The Coxeter arrangement[4] $\mathcal{A}_n = \mathcal{A}(A_{n-1})$ is the arrangement of hyperplanes in V_n explicitly given by

$$x_i - x_j = 0, \qquad 1 \leq i < j \leq n. \tag{1.2.4}$$

[4]or *Braid arrangement*

To compute the number of regions of this arrangement is not much harder than to compute the order of the symmetric group S_n—both these numbers are $n!$. Arnold [1] calculated the cohomology ring $\mathrm{H}^*(C_{\mathcal{A}_n}, \mathbb{C})$ (see Corollary 1.2.14). In particular, he demonstrated that the characteristic polynomial of $\mathcal{A}_n$ is equal to

$$\chi_{\mathcal{A}_n}(q) = (q-1)(q-2)\cdots(q-n+1). \tag{1.2.5}$$

Brieskorn [9] generalized Arnold's result to the case of any Coxeter arrangement. His formula for the characteristic polynomial of (1.2.3) involves the exponents $m_1, \ldots, m_l$ of the corresponding Weyl group W:

$$\chi_{\mathcal{A}(\Phi)}(q) = (q-m_1)(q-m_2)\cdots(q-m_l).$$

1.2.3 Whitney's formula

In this section we prove several essentially well-known results on hyperplane arrangements that will be useful in the sequel.

Consider the arrangement $\mathcal{A}$ of hyperplanes in $V \cong \mathbb{R}^l$ given by equations

$$h_i(x) = a_i, \qquad 1 \le i \le N, \tag{1.2.6}$$

where $x \in V$, the $h_i \in V^*$ are linear functionals on V, and the a_i are real numbers.

We call a subset I in $\{1, 2, \ldots, N\}$ *central* if the intersection of the hyperplanes $h_i(x) = a_i$, $i \in I$, is nonempty. For a subset $I = \{i_1, i_2, \ldots, i_m\}$, denote by $\mathrm{rk}(I)$ the dimension (rank) of the linear span of the vectors $h_{i_1}, \ldots, h_{i_m}$.

The following statement is a generalization of a classical Whitney's formula [57].

Theorem 1.2.2 [44, Theorem 4.1] *The Poincaré and characteristic polynomials of the arrangement $\mathcal{A}$ are equal to*

$$\mathrm{Poin}_{\mathcal{A}}(q) = \sum_I (-1)^{|I|-\mathrm{rk}(I)}\, q^{\mathrm{rk}(I)}, \tag{1.2.7}$$

$$\chi_{\mathcal{A}}(q) = \sum_I (-1)^{|I|}\, q^{l-\mathrm{rk}(I)}, \tag{1.2.8}$$

where I ranges over all central subsets in $\{1, 2, \ldots, N\}$. In particular, the number of regions of $\mathcal{A}$ is equal to

$$r(\mathcal{A}) = \sum_I (-1)^{|I|-\mathrm{rk}(I)},$$

and the number of bounded regions is equal to

$$\sum_I (-1)^{|I|}.$$

We need the well-known cross-cut theorem.

Lemma 1.2.3 [51, Corollary 3.9.4] *Let L be a finite lattice with the minimal element $\hat{0}$ and the maximal element $\hat{1}$, and let X be a subset of vertices in L such that* (a) $\hat{0} \notin X$, *and* (b) *if $y \in L$, $y \neq \hat{0}$ then $x \leq y$ for some $x \in X$ (such elements are called atoms). Then*

$$\mu_L(\hat{0}, \hat{1}) = \sum_k (-1)^k n_k, \tag{1.2.9}$$

where n_k is the number of k-element subsets in X with join equal to $\hat{1}$.

Now we can easily deduce Theorem 1.2.2.

Proof — Let z be any element in the intersection poset $L_{\mathcal{A}}$, and let $L(z)$ be the subposet of all elements $x \in L_{\mathcal{A}}$ such that $x \leq z$, i.e., the subspace x contains z. In fact, $L(z)$ is a geometric lattice. Let X be the set of all hyperplanes from $\mathcal{A}$ which contain z. If we apply Lemma 1.2.3 to $L = L(z)$ and sum (1.2.9) over all $z \in L_{\mathcal{A}}$, we get the formula (1.2.8). Then, by (1.2.2), we get (1.2.7). □

A *circuit* is a minimal subset I such that $\mathrm{rk}(I) = |I| - 1$. In other words, a subset $I = \{i_1, i_2, \ldots, i_m\}$ is a circuit if there exists a nonzero vector $(\lambda_1, \lambda_2, \ldots, \lambda_m)$, unique up to a nonzero factor, such that $\lambda_1 h_{i_1} + \lambda_2 h_{i_2} + \cdots + \lambda_m h_{i_m} = 0$. It is not difficult to see that a circuit I is central if, in addition, we have $\lambda_1 a_{i_1} + \lambda_2 a_{i_2} + \cdots + \lambda_l a_{i_l} = 0$. Thus, if $a_1 = \cdots = a_N = 0$ then all circuits are central, and if the a_i are generic then there are no central circuits.

A subset I is called *acyclic* if $|I| = \mathrm{rk}(I)$, i.e., I contains no circuits. It is clear that any acyclic subset is central.

Corollary 1.2.4 *In the case when the a_i are generic, the Poincaré polynomial equals*

$$\mathrm{Poin}_{\mathcal{A}}(q) = \sum_I q^{|I|},$$

where the sum is over all acyclic subsets I in $\{1, 2, \ldots, N\}$. In particular, the number of regions $r(\mathcal{A})$ is equal to the number of acyclic subsets.

Indeed, in this case a subset I is acyclic if and only if it is central.

Remark 1.2.5 The word "generic" in the corollary means no $l + 1$ distinct hyperplanes in (1.2.6) have a nonempty intersection. For example, it is sufficient to require that the a_i be linearly independent over rational numbers.

Let us fix a linear order ρ on the set $\{1, 2, \ldots, N\}$. We say that a subset I in $\{1, 2, \ldots, N\}$ is a *broken central circuit* if there exists $i \notin I$ such that $I \cup \{i\}$ is a central circuit and i is the minimal element in $I \cup \{i\}$ with respect to the order ρ.

The following, essentially well-known, theorem gives us the main tool for calculation of Poincaré (or characteristic) polynomials. We will later refer to it as the NBC Theorem.

Theorem 1.2.6 *We have*

$$\mathrm{Poin}_{\mathcal{A}}(q) = \sum_{I} q^{|I|},$$

where the sum is over all acyclic subsets I in $\{1, 2, \ldots, N\}$ without broken central circuits.

Proof — We will deduce this theorem from Theorem 1.2.2 using the *involution principle.* In order to do this we construct an involution $\iota : I \mapsto \iota(I)$ on the set of all central subsets I with a broken central circuit in such that for any I we have $\mathrm{rk}(\iota(I)) = \mathrm{rk}(I)$ and $|\iota \cdot I| = |I| \pm 1$.

This involution is defined as follows. Let I be a central subset with a broken central circuit, and let $s(I)$ be the set of all $i \in 1, \ldots, N$ such that i is the minimal element of a broken central circuit $J \subset I$. Note that $s(I)$ is nonempty. If the minimal element s_* of $s(I)$ lies in I, we define $\iota(I) = I \setminus \{s_*\}$. Otherwise, we define $\iota(I) = I \cup \{s_*\}$.

Note that $s(I) = s(\iota(I))$, thus ι is indeed an involution. It is clear now that all terms in (1.2.7) for I with a broken central circuit cancel each other and the remaining terms yield the formula in Theorem 1.2.6. □

Remark 1.2.7 Note that the number of subsets I without broken central circuits does not depend on the choice of the linear order ρ.

1.2.4 Deformations of Coxeter arrangements

In this section we apply the results of the previous section to hyperplane arrangements in V_n of the form

$$x_i - x_j = a_{ij}^{(1)}, \ldots, a_{ij}^{(k_{ij})}, \qquad 1 \le i < j \le n. \tag{1.2.10}$$

where k_{ij} are nonnegative integers and $a_{ij}^{(r)} \in \mathbb{R}$.

These arrangements can be viewed as deformations of the Coxeter arrangement of type A_l. We give an interpretations of these results in terms of (colored) graphs. It will be more convenient to use the index $n = l + 1$ instead of the index $l = \dim V$.

Let A denote the collection of the real numbers $a_{ij}^{(k)}$ that appear in (1.2.10). We say that G is an *A-colored graph* if G is a graph on the vertices $1, \ldots, n$ and each edge (i, j), $i < j$, of G is labelled by a number (color) $c \in \{a_{ij}^{(1)}, \ldots, a_{ij}^{(k_{ij})}\}$. We denote the edge (i, j) of color c by $(i, j)^c$. We will assume that $(i, j)^c = (j, i)^{-c}$. With a hyperplane $x_i - x_j = c$ in (1.2.10), we associate the edge $(i, j)^c$. Then a subset I of hyperplanes corresponds to an A-colored graph G. A graph G corresponds to an acyclic subset I if and only if G is a *forest.* We say that a circuit $(i_1, i_2)^{c_1}, (i_2, i_3)^{c_2}, \ldots, (i_m, i_1)^{c_m}$ in G is *central* if $c_1 + c_2 + \cdots + c_m = 0$ (cf. Section 1.2.3).

Fix a linear order on all edges $(i, j)^a$, $c \in \{a_{ij}^{(1)}, \ldots, a_{ij}^{(k_{ij})}\}$. We call an A-colored graph C a *broken A-circuit* if C is obtained from a central circuit by removing its minimal element. In the case when all $a_{ij}^{(r)}$ are zero, we get the classical notion of a broken circuit of a graph.

We summarize below several special cases of the NBC Theorem (Theorem 1.2.6). Here $|F|$ denotes the number of edges in a forest F.

Corollary 1.2.8 *The Poincaré polynomial of the arrangement (1.2.10) is equal to*

$$\mathrm{Poin}_{\mathcal{A}}(q) = \sum_F q^{|F|},$$

where the sum is over all A-colored forests F on the vertices $1, \ldots, n$ without broken A-circuits. The number of regions of arrangement (1.2.10) is equal to the number of such forests.

One special case is the arrangement (1.2.10) is the arrangement in V_l given by

$$x_i - x_j = a_{ij}, \qquad 1 \leq i < j \leq n, \tag{1.2.11}$$

where the a_{ij} are fixed real numbers.

In this case all $k_{ij} = 1$ and A-colored graphs are just usual graphs.

Corollary 1.2.9 *The Poincaré polynomial of the arrangement (1.2.11) is equal to*

$$\mathrm{Poin}_{\mathcal{A}}(q) = \sum_F q^{|F|},$$

where the sum is over all forests F on the vertices $1, \ldots, n$ without broken A-circuits. The number of regions of the arrangement (1.2.11) is equal to the number of such forests.

In the case when the $a_{ij}^{(r)}$ are generic these results become especially simple.

For a forest F on vertices $1, 2, \ldots, n$ we will write $k^F := \prod k_{ij}$, where the product is over all edges (i, j) in F.

Corollary 1.2.10 *Let $\mathcal{A}$ be an arrangement of type (1.2.10), where the $a_{ij}^{(r)}$ are generic real numbers. Then*

1. $\mathrm{Poin}_{\mathcal{A}}(q) = \sum k^F q^{|F|}$,

2. $r(\mathcal{A}) = \sum k^F$,

where the sums are over all forests F on the vertices $1, 2, \ldots, n$.

Corollary 1.2.11 *The number of regions of the arrangement (1.2.11) with generic a_{ij} is equal to the number of forests on n labelled vertices.*

This corollary is "dual" to the following well-known result (see, e.g., [51]).

Proposition 1.2.12 *Let* Perm_n *be the permutohedron, i.e., the polyhedron with vertices* $(w_1, \ldots, w_n) \in \mathbb{R}^n$*, where* $w_1, \ldots, w_n$ *ranges over all permutations of* $1, \ldots, n$*. Then the Erhart polynomial of* Perm_n *is equal to*

$$E_{\mathrm{Perm}_n}(q) = \sum_F q^{|F|},$$

where the sum is over all forests F *on* n *vertices. In particular, the number of integer points in* Perm_n *is equal to the number of forests on* n *vertices.*

The connected components of the $\binom{n}{2}$-dimensional space of all arrangements of type (1.2.11) correspond to (coherent) zonotopal tilings of the permutohedron, i.e., certain subdivisions of Perm_n into parallelepipeds. The regions of a generic arrangement (1.2.11) correspond to the vertices of the corresponding tiling, which are all integer points in Perm_n.

1.2.5 The Orlik-Solomon algebra

Orlik and Solomon [38] gave the following combinatorial description of the cohomology ring of an arbitrary hyperplane arrangement. Consider an arrangement $\mathcal{A}$ of affine hyperplanes $H_1, H_2, \ldots, H_N$ in a complex space $V \cong \mathbb{C}^l$ given by

$$H_i: \; h_i(x) = a_i, \qquad i = 1, \ldots, N,$$

where the $h_i(x)$ are linear forms on V and $a_i \in \mathbb{C}$.

Recall that subset of indices $I = \{i_1, \ldots, i_m\}$ is called central circuit if I is a minimal subset such that the codimension of the intersection $H_{i_1} \cap \cdots \cap H_{i_m}$ is equal to $m - 1$.

Let $e_1, \ldots, e_N$ be formal variables associated with the hyperplanes $H_1, \ldots, H_N$. The *Orlik-Solomon algebra* $\mathrm{OS}(\mathcal{A})$ of the arrangement $\mathcal{A}$ is generated over the complex numbers by $e_1, \ldots, e_N$ subject to the relations:

$$e_i e_j = -e_j e_i, \qquad 1 \le i < j \le N, \tag{1.2.12}$$

$$e_{j_1} \cdots e_{j_p} = 0, \qquad \text{if } H_{j_1} \cap \cdots \cap H_{j_p} = \emptyset, \tag{1.2.13}$$

$$\sum_{j=1}^{m} (-1)^j \, e_{i_1} \cdots \widehat{e_{i_j}} \cdots e_{i_m} = 0, \tag{1.2.14}$$

whenever $\{i_1, \ldots, i_m\}$ is a central circuit. (Here $\widehat{e_{i_j}}$ denotes that the term e_{i_j} is missing.)

Let $C_{\mathcal{A}} = V - \bigcup_i H_i$ be the complement to the hyperplanes H_i of $\mathcal{A}$.

Theorem 1.2.13 Orlik, Solomon [38] *Let* λ_i *be the cohomology class of the differential form* $dh_i/(h_i(x) - a_i)$ *in the (de Rham) cohomology* $\mathrm{H}^*(C_{\mathcal{A}}, \mathbb{C})$ *of* $C_{\mathcal{A}}$*. Then* *the map* $\phi : \mathrm{OS}(\mathcal{A}) \to \mathrm{H}^*(C_{\mathcal{A}}, \mathbb{C})$ *defined by*

$$\phi : \ e_i \longmapsto \lambda_i$$

is an isomorphism between the Orlik–Solomon algebra and the cohomology of $C_{\mathcal{A}}$.

As an example, consider the case of the Coxeter arrangement $\mathcal{A}_n$ of type A_{n-1} given by (1.2.4). The following well-know description of the corresponding cohomology was found by Arnold [1].

Corollary 1.2.14 [1] *The cohomology ring of the complement to the complexified Coxeter arrangement* $\mathcal{A}_n$ *is generated by anticommuting generators* e_{ij}, $1 \leq i < j \leq n$, *subject to the following "triangular" relations:*

$$e_{ij}e_{jk} - e_{ij}e_{ik} + e_{jk}e_{ik} = 0,$$

where $1 \leq i < j < k \leq n$.

1.3 Catalan Miscellanea

The sequence of Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{1.3.1}$$

is, probably, the most famous combinatorial sequence. Some interpretations of the numbers C_n are can be found in [52, Chapter 6, Exercises].

The best known combinatorial interpretation of the numbers C_n is given in terms of *Dyck words.* A sequence $w_1, w_2, \ldots, w_{2n}$ of 0's and 1's is said to be a Dyck word if, for any $k = 1, \ldots, 2n$, we have $w_1 + w_2 + \cdots + w_k \geq k$ and $w_1 + w_2 + \cdots + w_{2n} = n$. The number of Dyck words of length $2n$ is equal to C_n.

Recall that the generating function for the Catalan numbers is equal to

$$1 + \sum_{n \geq 1} C_n t^n = \frac{1 - \sqrt{1-4t}}{2t}. \tag{1.3.2}$$

In this section we give several new and old interpretations of these numbers in terms of hyperplane arrangements, posets, polyhedra, and trees.

1.3.1 Semiorders

A poset P on the vertices $1, 2, \ldots, n$ with the order relation $<_P$ is called a *semiorder* if there are real numbers $x_1, x_2, \ldots, x_n$ such that $i <_P j$ if and only if $x_i < x_j - 1$. The symmetric group S_n acts on semiorders on n vertices by permuting the vertices. Two semiorders are equivalent (isomorphic) if they are in the same S_n-orbit.

The following is a well-known result of Wine and Freund [58].

Theorem 1.3.1 [58] *The number of nonisomorphic semiorders on n vertices is equal to the Catalan number C_n.*

The set $\Phi_+ = \{\epsilon_{ij} \mid 1 \le i < j \le n\}$ of positive roots in the type A_{n-1} root system can be partially ordered as follows: $\epsilon_{ij} \le \epsilon_{kl}$ if and only if $\epsilon_{kl} - \epsilon_{ij}$ is a sum of positive roots, i.e., $k \le i < j \le l$.

In the equivalence class of a semiorder P there is a unique representative which is determined by a sequence $x_1, x_2, \dots, x_n$ satisfying $x_1 < x_2 < \cdots < x_n$. By I_P denote the subset in Φ_+ such that $\epsilon_{ij} \in I_P$ if and only if $x_i < x_j - 1$. The subset I_P is an order ideal in the poset Φ_+, i.e., $\epsilon_{ij} \in I_P$ implies $\epsilon_{kl} \in I_P$ for all $\epsilon_{kl} > \epsilon_{ij}$. It is easy to see that the map $P \mapsto I_P$ is a bijection between the equivalence classes of semiorders and order ideals in Φ_+. The latter are in an easy bijective correspondence with Dyck words, and thus their number is the Catalan number C_n.

Consider two arrangements of hyperplanes in $V_n \subset \mathbb{R}^n$: the first is given by the equations

$$x_i - x_j = \pm 1, \qquad 1 \le i < j \le n, \tag{1.3.3}$$

and the second is given by

$$x_i - x_j = 0, \ \pm 1, \qquad 1 \le i < j \le n. \tag{1.3.4}$$

The regions of (1.3.3) are in a bijective correspondence with semiorders on n vertices. This correspondence is described as follows: take a point $(x_1, x_2, \dots, x_n)$ in a region of (1.3.3), the sequence $x_1, x_2, \dots, x_n$ then determines a semiorder. The symmetric group S_n acts on the regions of the arrangement (1.3.4). Every S_n-orbit consists of $n!$ regions and has a unique representative in the *dominant chamber*, given by $x_1 < x_2 < \cdots < x_n$. The regions of (1.3.4) in the dominant chamber thus correspond to *unlabelled* (i.e., nonisomorphic) semiorders on n vertices. See [53] for more results and relations between hyperplane arrangements and *interval orders*, the latter generalize semiorders.

We can reformulate Theorem 1.3.1 as follows.

Proposition 1.3.2 *The number of regions of the arrangement (1.3.4) is $n!$ times the Catalan number C_n.*

The following expression for the generating function the number s_n of regions of the arrangement (1.3.3), i.e., the number of semiorders on n labelled vertices, can be derived from results of Chandon, Lemaire, and Pouget [11].

Theorem 1.3.3 *The generating function for the numbers s_n of semiorders on n labelled vertices is equal to*

$$1 + \sum_{n \ge 1} s_n t^n = \frac{1 - \sqrt{4e^{-t} - 3}}{2(1 - e^{-t})}.$$

Figure 1-2: Forbidden subposets for semiorders.

For example, $s_n = 1$, 3, 19, 183, 2371, 38703, 763099, for $n = 1, \ldots, 9$. This formula is a special case of a more general statement (Theorem 1.3.5) that we prove in the next section.

The following theorem, due to Scott and Suppes [47], presents a simple characterization of semiorders in terms of forbidden subposets (cf. Theorem 1.4.10).

Theorem 1.3.4 [47] *A poset P is a semiorder if and only if it contains no induced subposet of either of the two types shown on Figure 1-2.*

1.3.2 Reciprocity for hyperplane arrangements

Let us fix distinct real numbers $a_1, a_2, \ldots, a_m > 0$, and let $A = (a_1, \ldots, a_m)$. Let $\mathcal{C}_n = \mathcal{C}_n(A)$ be the arrangement of hyperplanes in $V_n = \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$ given by

$$x_i - x_j = a_1, a_2, \ldots, a_m, \quad i \neq j. \tag{1.3.5}$$

Also let $\mathcal{C}_n^0 = \mathcal{C}_n^0(A)$ be the arrangement obtained from $\mathcal{C}_n$ by adjoining the hyperplanes $x_i = x_j$. Explicitly $\mathcal{C}_n^0$ is given by

$$x_i - x_j = 0, a_1, a_2, \ldots, a_m, \quad i \neq j. \tag{1.3.6}$$

The exponential generating functions for the numbers of regions of these arrangements are given by

$$f_A(t) = \sum_{n \geq 0} r(\mathcal{C}_n) \frac{t^n}{n!},$$

$$g_A(t) = \sum_{n \geq 0} r(\mathcal{C}_n^0) \frac{t^n}{n!}.$$

The main result of this section is the following:

Theorem 1.3.5 [44, Theoreom 7.1] *We have $f_A(t) = g_A(1 - e^{-t})$ or, equivalently,*

$$r(\mathcal{C}_n^0) = \sum_{k \geq 0} c(n,k)\, r(\mathcal{C}_k),$$

where $c(n,k)$ is the signless Stirling number of the first kind, i.e., the number of permutations of $1,2,\dots,n$ with k cycles.

Note that Theorem 1.3.3 from the previous section is an immediate consequence of Theorem 1.3.5, for $A=(1)$, and formula (1.3.2).

We now proceed to the proof of Theorem 1.3.5. The symmetric group S_n acts on the regions of $\mathcal{C}_n$ and $\mathcal{C}_n^0$. Let R_n denotes the set of all regions of $\mathcal{C}_n$.

Lemma 1.3.6 *We have, $r(\mathcal{C}_n^0)$ is $n!$ times the number of S_n-orbits in R_n.*

Proof — Indeed, the number of regions of $\mathcal{C}_n^0$ is $n!$ times the number of those in the dominant chamber. They, in turn, correspond to S_n-orbits in R_n. $\square$

It was explained in [53] that the regions of $\mathcal{C}_n$ can be viewed as (labelled) generalized interval orders. On the other hand, the regions of $\mathcal{C}_n^0$ that lie in the dominant chamber, correspond to unlabelled generalized interval orders. Lemma 1.3.6 says that number of unlabelled objects is the number of S_n-orbits, which is, of course, a tautology.

We can apply the following well-known lemma of Burnside. Its proof is straightforward, and it is left to the reader.

Lemma 1.3.7 *Let G be a finite group which acts on a finite set M. Then the number of G-orbits in M is equal to*

$$\frac{1}{|G|}\sum_{g\in G}\mathrm{Fix}(g,M),$$

where $\mathrm{Fix}(g,M)$ is the number of elements in M fixed by $g\in G$.

By Lemmas 1.3.6 and 1.3.7 we have

$$r(\mathcal{C}_n^0)=\sum_{w\in S_n}\mathrm{Fix}(w,\mathcal{C}_n),$$

where $\mathrm{Fix}(w,\mathcal{C}_n)$ is the number of regions of $\mathcal{C}_n$ fixed by the permutation w.

Theorem 1.3.5 thus easily follows from the following statement.

Lemma 1.3.8 *Let $w\in S_n$ be a permutation with k cycles. Then the number of regions of $\mathcal{C}_n$ fixed by w is equal to the number of all regions of $\mathcal{C}_k$.*

Indeed, by Lemma 1.3.8, we have

$$r(\mathcal{C}_n^0)=\sum_{w\in S_n}\mathrm{Fix}(w,\mathcal{C}_n)=\sum_{k\geq 0}c(n,k)\,r(\mathcal{C}_k),$$

which is precisely the claim of Theorem 1.3.5.

Proof of Lemma 1.3.8 — We construct a bijection between the regions of $\mathcal{C}_n$ fixed by w and all regions of $\mathcal{C}_k$ as follows.

Suppose R is a region of $\mathcal{C}_n$ fixed by a permutation $w \in S_n$ and $(x_1, \ldots, x_n) \in R$. Then $x_i - x_j > a_s$ whenever $x_{w(i)} - x_{w(j)} > a_s$, for any i, j, s.

The permutation w is a product of several disjoint cycles: $w = c_1 \ldots c_k$. Let us denote by X_α the collection of the x_i for all elements i of the cycle c_α. We write $X_\alpha - X_\beta > a$ if $x_i - x_j > a$ for all $x_i \in X_\alpha$ and $x_j \in X_\beta$. The notation $X_\alpha - X_\beta < a$ has an analogous meaning. We show that for any two classes X_α and X_β and for any $s = 1, \ldots, m$ we have either $X_\alpha - X_\beta > a_s$ or $X_\alpha - X_\beta < a_s$.

Let x_i be the maximal element in X_α, and let x_j be the maximal element in X_β. Suppose that $x_i - x_j > a_s$. For any integer p, we have $x_{w^p(i)} - x_{w^p(j)} > a_s$, due to the w-invariance of the region R. Since $x_i \geq x_{w^p(i)}$ (x_i is maximal in X_α), we have $x_i - x_{w^p(j)} > a_s$. Thus for any q, $x_{w^q(i)} - x_{w^{p+q}(j)} > a_s$. This implies that $X_\alpha - X_\beta > a_s$.

Analogously, suppose that $x_i - x_j < a_s$. Then for any p, we have $x_{w^p(i)} - x_{w^p(j)} < a_s$. Since $x_j \geq x_{w^p(j)}$, we have $x_{w^p(i)} - x_j < a_s$. Finally, we obtain $x_{w^{p+q}(i)} - x_{w^q(j)} < a_s$, for any integer q. This implies that $X_\alpha - X_\beta < a_s$.

Let us choose elements $x_{1'} \in X_1, \ldots, x_{k'} \in X_k$. Then the point $x' = (x_{1'}, \ldots, x_{k'})$ belongs to some region R' of $\mathcal{C}_k$. Moreover, the region R' does not depend on the choice of the $x_{i'}$. Therefore we obtain a map $\phi : R \to R'$ from the set of regions of $\mathcal{C}_n$, invariant under w, to the regions of $\mathcal{C}_k$.

It is clear that ϕ is injective. To show that ϕ is surjective, take a point $x' = (x_{1'}, \ldots, x_{k'})$ in a region R' of $\mathcal{C}_k$. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be the point such that $x_i = x_{\alpha'}$ for i in cycle c_α. Then x belongs[5] to some region R of $\mathcal{C}_n$. According to the construction above, $\phi(R) = R'$. Thus the map ϕ is a bijection.

This completes the proof of Lemma 1.3.8 and Theorem 1.3.5. □

1.3.3 Polyhedra and their triangulations

Recall that $V_{n+1} = \{x \in \mathbb{R}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\}$. Let $\epsilon_{ij} = \epsilon_i - \epsilon_j$, $1 \leq i < j \leq n+1$, where $\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$. The polyhedron P_n in V_{n+1} is defined as the convex hull of the origin 0 and the vectors ϵ_{ij}, $i < j$.

The space V_{n+1} contains the n-dimensional integer lattice which is obtained by intersecting $\mathbb{Z}^{n+1}$ canonically embedded into $\mathbb{R}^{n+1}$ with V_{n+1}. The volume of any polytope with integer vertices is a multiple of $1/n!$.

Theorem 1.3.9 [20, Theorem 2.3(2)] *The volume of P_n is the Catalan number C_n divided by $n!$*

$$\mathrm{Vol}(P_n) = \frac{C_n}{n!}.$$

Below in this section we sketch a proof of this statement.[6] Let T be a tree on the vertices $1, \ldots, n+1$, and let $\Delta(T)$ denote the convex hull of the origin and the vectors ϵ_{ij} that correspond to edges (i, j), $i < j$, of T.

[5]We use here the condition that the a_s are nonzero.

[6]The statements below are fairly straightforward and their rigorous proofs are left to the reader.

Lemma 1.3.10 *The polyhedron $\Delta(T)$ is an n-dimensional simplex of volume $1/n!$. Every n-dimensional simplex in P_n with integer vertices which contains the origin is of the type $\Delta(T)$.*

We will study subdivisions of P_n into simplices $\Delta(T)$. Let us say that two trees T_1 and T_2 are *compatible* if the intersection of two simplices $\Delta(T_1)$ and $\Delta(T_2)$ is their common face. Define a *local triangulation* $\mathcal{T}$ of P_n as a collection of trees $\{T_1, \ldots, T_k\}$ such that the following two conditions holds:

- The union of all simplices $\Delta(T_i)$ is P_n.
- Any two trees T_i and T_j in $\mathcal{T}$ are compatible.

We call such triangulation local, because every simplex $\Delta(T_i)$ contain the origin, and therefore such triangulations are determined by their small neighborhood of 0.

First, we describe pairs of compatible trees. Let $G(T_1, T_2)$ be the ordered graph such that (i, j) is an edge of $G(T_1, T_2)$ if and only if (a) $i < j$ and (i, j) is an edge of T_1; or (b) $i > j$ and (j, i) is an edge of T_2.

Lemma 1.3.11 *Two trees T_1 and T_2 are compatible if and only if the graph $G(T, T')$ is acyclic.*

Not every simplex $\Delta(T)$ may appear in a local triangulation. A tree T on a linearly ordered set of vertices is called *alternating* if the vertices along every path[7] in T alternate: $\cdots < a > b < c > d < \cdots$.

We will study alternating trees in more detail in Section 1.4.1.

Lemma 1.3.12 *A tree T participates in some local triangulation if and only if T is an alternating tree.*

We say that an alternating tree T is *non-crossing* if there are no $i < j < k < l$ such that both (i, k) and (j, l) are edges of T. Analogously, we say that an alternating tree in *non-nesting* if there are no $i < j < k < l$ such that both (i, l) and (j, k) are edges of T.

Theorem 1.3.13 [20, Theorem 6.3] [20, Theorem 6.6]

1. *The set of all non-crossing alternating trees on the vertices $1, \ldots, n+1$ is a local triangulation of P_n.*
2. *The set of all non-nesting alternating trees on the vertices $1, \ldots, n+1$ is a local triangulation of P_n.*
3. *The number of non-crossing alternating trees on $n+1$ vertices is equal to the number of non-nesting alternating tree on $n+1$ vertices and is equal to the Catalan number C_n.*

It is an interesting problem to describe all (local) triangulations of P_n.

[7]It is sufficient to require this condition only for paths with three vertices.

1.4 Alternating Trees and the Linial Arrangement

In this section we study a sequence $f_0, f_1, f_2, \ldots$ of positive integer numbers which has several combinatorial interpretations. Here we summarize the main interpretations of this sequence. For definitions and proofs see corresponding subsections below. The number f_n is equal to:

- the number of regions of the Linial arrangement $\mathcal{L}_n$.
- the number of alternating trees with $n+1$ vertices,
- the number of local binary search trees with n vertices,
- the number of FIS trees with n vertices,
- the number of SIF trees with n vertices,
- the number of sleek posets with n vertices,
- the number of semiacyclic tournaments with n vertices,
- the alternating sum $\sum(-1)^{c(G)}$ over all balanced graphs G with n vertices, where $c(G)$ is the cyclomatic number of G.
- the sum
$$2^{-n}\sum_{k=0}^{n}\binom{n}{k}(k+1)^{n-1}.$$

1.4.1 Counting alternating trees

Recall that a tree T on a linearly ordered set of vertices is called *alternating* if the vertices in any path $i_1, \ldots, i_k$ in T alternate, i.e., we have $i_1 < i_2 > i_3 < \cdots i_k$ or $i_1 > i_2 < i_3 > \cdots i_k$. In other words, there are no $i < j < k$ such that both (i, j) and (j, k) are edges in T. Alternating trees first appear in [20] and were studied in [42], where they were called intransitive trees, see also [53] and [44].

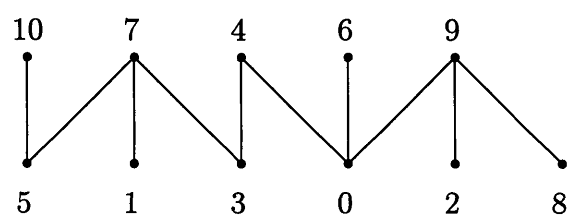

Figure 1-3: An alternating tree.

Let f_n be the number of alternating trees on the vertices $0, 1, 2, \ldots, n$, and let

$$f(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}$$

be the exponential generating function for the sequence f_n.

Theorem 1.4.1 [42, Theorems 1, 2] *For $n \geq 1$ we have*

$$f_n = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-1}. \qquad (1.4.1)$$

The series $f = f(x)$ satisfies the functional equation

$$f = e^{x(1+f)/2}. \qquad (1.4.2)$$

The first few numbers f_n are given below.

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
1	1	2	7	36	246	2104	21652	260720	3598120	56010096

We need some extra notation. We say that i is a *minimal vertex* in an alternating tree T if T contains an edge (i, j) for some $j > i$. A vertex is called *maximal* it is not a minimal vertex. If a vertex i is minimal (respectively, maximal) then for every edge (i, j) in T we have $j > i$ (respectively, $j < i$). For example, the tree on Figure 1-3 has minimal vertices $0, 1, 2, 3, 5, 8$ and maximal vertices $4, 6, 7, 9, 10$.

An alternating tree with a chosen vertex (root) is called a *rooted* alternating tree. If the root is a maximal vertex we call such a tree *top-rooted.*

Proof of Theorem 1.4.1 — First, we prove the formula (1.4.2).

Let $\widetilde{f}_n$ be the number of all top-rooted alternating trees on the vertices $1, \ldots, n$. It is clear that, for $n \geq 2$, the number $\widetilde{f}_n$ is half the number of all rooted alternating trees on $1, \ldots, n$. Thus $\widetilde{f}_n = n f_{n-1}/2$; also $\widetilde{f}_1 = (f_0 + 1)/2 = 1$. We obtain the following expression for the exponential generating function.

$$\widetilde{f}(t) := \sum_{n \geq 1} \widetilde{f}_n \frac{t^n}{n!} = t(f(t) + 1)/2.$$

To get an alternating tree on the vertices $0, 1, \ldots, n$, take a forest of top-rooted trees on the vertices $1, \ldots, n$ and connect 0 to each root. By the exponential formula (e.g., see [23, p. 166]), we have

$$f(t) = e^{\widetilde{f}(t)}.$$

This gives the formula (1.4.2).

Now we deduce (1.4.1). We have

$$\widetilde{f} = t\,(1 + e^{\widetilde{f}})/2.$$

By Lagrange's inversion formula (see [23, p. 17]), we get

$$[t^n]\widetilde{f} = \frac{1}{n\,2^n}[\lambda^{n-1}]\left(1 + e^{\lambda}\right)^n = \frac{1}{n\,2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{k^{n-1}}{(n-1)!},$$

where $[t^n]g$ denotes the coefficient of t^n in g. Thus, for $n \geq 2$,

$$f_{n-1} = \frac{1}{n\,2^{n-1}}\sum_{k=0}^{n}\binom{n}{k}k^{n-1}. \tag{1.4.3}$$

The formula (1.4.1) is equivalent to (1.4.3). □

Remark 1.4.2 It is also possible to calculate the inverse of the function $f(x)$. It follows from (1.4.2) that $x = 2\ln(f)/(1 + f)$. One can expand it as the following series:

$$x = -\sum_{n\geq 1}(1-f)^n\sum_{k=0}^{n}\binom{n}{k}^{-1}.$$

1.4.2 Local binary search trees

A plane binary tree on the vertices $1, 2, \ldots, n$ is called a *local binary search tree* (LBS tree, for short) if for any vertex i the left child of i is less than i and the right child of i is greater than i. These trees were first considered by Ira Gessel [21], and were studied in [42]. The name "local binary search tree" was suggested by Richard Stanley [53], see also [44].

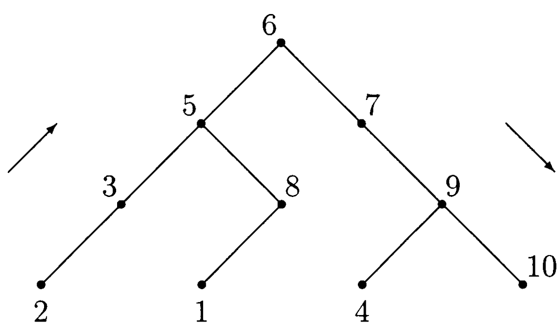

Figure 1-4: A local binary search tree.

Theorem 1.4.3 [42, Section 4.1] *For $n \geq 1$, the number of local binary search trees on the vertices $1, 2, \ldots, n$ is equal to the number f_n of alternating trees on the vertices $0, 1, 2, \ldots, n$.*

Proof — Let $\mathcal{R}_n$ be the set of rooted alternating trees on the vertices $0, 1, 2, \ldots, n$; and let $\mathcal{B}_n$ be the set of LBS trees on the vertices $0, 1, 2, \ldots, n$ such that the root has only one child (left or right).

Clearly, $|\mathcal{R}_n| = (n+1)f_n$. On the other hand, $|\mathcal{B}_n|$ is $n+1$ times the number of LBS trees on $1, 2, \ldots, n$. Indeed, for a LBS tree $B \in \mathcal{B}_n$, the root r of B can be any vertex $0 \leq r \leq n$. Deleting the root r, we get a LBS tree T' on $\{0, 1, \ldots, n\} \setminus \{r\}$. Conversely, we can always reconstruct T if we know T' and r. In the case when the roof r' of T' is less than r, we set r' to be the left child of r; otherwise, if $r' < r$, we set r' to be the right child of r.

In order to prove Theorem 1.4.3, we construct a bijection $\phi : \mathcal{R}_n \to \mathcal{B}_n$. Let T be a rooted alternating tree $T \in \mathcal{R}_n$. We construct the LBS tree $B = \phi(T)$ using the following simple procedure:

First, we orient the edges of T away from the root (e.g. the vertices adjacent to the root are children of the root).

If v is a minimal vertex in T and $i_1 < i_2 < \cdots < i_l$ are the children of v in T ($v < i_1$), then set i_1 to be the right child of v in B, and i_{k+1} to be the right child of i_k in B, for $k = 1, 2, \ldots, l-1$.

Analogously, if v is a maximal vertex in T and $j_1 > j_2 > \cdots > j_l$ are the children of v in T ($v > j_1$), then set j_1 to be the left child of v in B, and j_{k+1} to be the left child of j_k in B, for $k = 1, 2, \ldots, l-1$.

Clearly, the construction of the map ϕ is invertible. Thus ϕ is a bijection. □

1.4.3 On stability and fickleness

Let $\mathcal{O}_n$ denote the set of rooted trees on the vertices $1, 2, \ldots, n$ with edges oriented away from the root.

We say that a path $i_1, i_2, \ldots, i_k$ in a tree $T \in \mathcal{O}_n$ is *stable* if all edges (i_1, i_2), (i_2, i_3), (i_3, i_4), $\ldots$,(i_{k-1}, i_k) are oriented in the same direction, i.e., the path either approaches the root, or goes away from the root.

We also say that a path $j_1, j_2, \ldots, j_k$ is *fickle* if the vertices in it alternate, i.e, we have $j_1 < j_2 > j_3 < \ldots j_k$ or $j_1 > j_2 < j_3 > \ldots j_k$ (cf. Section 1.4.1).

A tree T in $\mathcal{O}_n$ is called a *"fickle is stable" tree* (FIS tree) if every fickle path in T is stable. Likewise, a tree T in $\mathcal{O}_n$ is called a *"stable is fickle" tree* (SIF tree) if every stable path in T is fickle.

Theorem 1.4.4 *The number of FIS trees in $\mathcal{O}_n$ is equal to the number of SIF trees in $\mathcal{O}_n$ and is equal to the number f_n of alternating trees on the vertices $0, 1, \ldots, n$.*

Proof — First, we notice that a tree T from $\mathcal{O}_n$ is a FIS tree if and only if every vertex i in T has none, one, or two children, and in the last case one of the children is less than i and the other is greater than i. We establish a simple bijection between FIS and local binary search trees. For a LBS tree, orient its edges away from the root, and then forget the structure of a binary tree, i.e., forget which child was left and which was right. We obtain a FIS tree. Thus, by merit of Theorem 1.4.3, the number of FIS trees in $\mathcal{O}_n$ is equal to the number f_n of alternating trees on the vertices $0, 1, 2, \ldots, n$.

SIF trees are more similar to alternating trees. In fact, they are almost alternating in the sense that the condition for alternating tree is satisfied in every vertex but the root (cf. Section 1.4.1).

A bijection between FIS trees and SIF trees can be constructed in a way similar to the proof of Theorem 1.4.3. $\square$

1.4.4 The Linial arrangement

Recall that V_n denotes the hyperplane $\{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = 0\}$ in $\mathbb{R}^n$. Consider the arrangement $\mathcal{L}_n$ of hyperplanes in V_n given by the equations

$$x_i - x_j = 1, \quad 1 \leq i < j \leq n. \tag{1.4.4}$$

This arrangement was first considered by Nati Linial and Shmulik Ravid. They calculated the numbers $r(\mathcal{L}_n)$ of regions of $\mathcal{L}_n$ and the Poincaré polynomials $\mathrm{Poin}_{\mathcal{L}_n}(q)$ for $n \leq 9$.

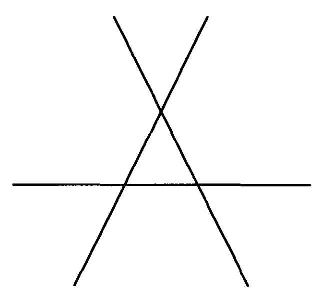

Figure 1-5: Seven regions of the Linial arrangement $\mathcal{L}_3$.

Our main result on the Linial arrangement is the following:

Theorem 1.4.5 [44, Theorem 8.2] *The number $r(\mathcal{L}_n)$ of regions of $\mathcal{L}_n$ is equal to the number f_n of alternating trees on the vertices $0, 1, 2 \ldots, n$ and, thus, to the number of local binary search trees on $1, 2, \ldots, n$.*

This theorem was conjectured by Richard Stanley, who used the numerical data provided by Linial and Ravid. In Section 1.5 we prove a more general result (see Theorems 1.5.1 and Corollary 1.5.9). A different proof of Theorem 1.4.5 was later given by C. Athanasiadis [3].

Corollary 1.4.6 *The Poincaré polynomial of the Linial arrangement is equal to*

$$\mathrm{Poin}_{\mathcal{L}_n}(q) = \sum_T q^{n - d_T(0)}, \tag{1.4.5}$$

where the sum is over all alternating trees T on the vertices $0, 1, \ldots, n$ and $d_T(0)$ denotes the degree of the vertex 0 in T.

1.4.5 Balanced graphs

Let $C = (c_1, c_2, \ldots, c_m)$ denote a cycle on a linearly ordered set of vertices which has the edges $(c_1, c_2), (c_2, c_3), \ldots, (c_{m-1}, c_m), (c_m, c_1)$. (Note that the sequence $(c_1, \ldots, c_m)$ is defined up to a cyclic permutation.) By convention, $c_0 = c_m$. We say that an index $1 \le i \le m$ is an *ascent* in C if $c_{i-1} < c_i$. Analogously, an index $1 \le j \le m$ is a *descent* if $c_{i-1} > c_i$.

We say that a cycle C is *balanced* if the number of ascents in C is equal to the number of descents in C. A graph G is called *balanced* if every cycle in G is balanced.

A graph G on the vertices $1, 2, \ldots, n$ corresponds to a subset of hyperplanes in (1.4.4): an edge (i, j) of G corresponds to the hyperplane $x_i - x_j = 1$, $i < j$. Geometrically, a graph is balanced if and only if the corresponding hyperplanes have a nonempty intersection.

Let $c(G)$ denotes the *cyclomatic number* of G, i.e., the number of edges minus the number of vertices plus the number of connected components. Theorem 1.2.2 implies the following statement.

Corollary 1.4.7 *For $n \ge 2$, the number of regions of the Linial arrangement is equal to the alternating sum*

$$r(\mathcal{L}_n) = \sum_G (-1)^{c(G)},$$

over all balanced graphs on the vertices $1, 2, \ldots, n$. The Poincaré polynomial of this arrangement is equal to

$$\mathrm{Poin}_{\mathcal{L}_n}(q) = \sum_G (-q^{-1})^{c(G)}\, q^{|G|},$$

where again the sum is over all balanced graphs on $1, \ldots, n$ and $|G|$ is the number of edges in G.

Orlik and Solomon's result (Theorem 1.2.13) allows us to describe the cohomology ring $\mathrm{H}^*(C_{\mathcal{L}_n}, \mathbb{C})$ of the complement $C_{\mathcal{L}_n}$ to the complexified Linial arrangement in terms of generators and relations.

Proposition 1.4.8 *The cohomology ring $\mathrm{H}^*(C_{\mathcal{L}_n}, \mathbb{C})$ is canonically isomorphic to the algebra (the Orlik-Solomon algebra) generated by $e_{ij} = e_{ji}$, $1 \le i, j \le n$, $e_{ii} = 0$, subject to the following relations:*

$$\begin{aligned}
&e_{ij}e_{kl} = -e_{kl}e_{ij},\\
&e_{r_1r_2}e_{r_2r_3}\cdots e_{r_{m-1}r_m}e_{c_mc_1} = 0,\\
&e_{ab}e_{bc}e_{ac} - e_{ab}e_{bc}e_{cd} + e_{ab}e_{ac}e_{cd} - e_{bc}e_{ac}e_{cd} = 0,\\
&e_{ac}e_{bc}e_{bd} - e_{ac}e_{bc}e_{ad} + e_{ac}e_{bd}e_{ad} - e_{bc}e_{bd}e_{ad} = 0.
\end{aligned} \tag{1.4.6}$$

where $i, j, k, l, r_1, \ldots, r_m \in \{1, \ldots, n\}$ *and* $1 \leq a < b < c < d \leq n$ *(cf. Figure 1-7).*

Proof — By Theorem 1.2.13, the Orlik-Solomon algebra of the Linial arrangement is generated by the e_{ij} which are subject to the first two relation in (1.4.6) and also the relation:

$$\sum_{j=1}^{p}(-1)^j e_{c_1c_2}e_{c_2e_2}\cdots\widehat{e_{c_jc_{j+1}}}\cdots e_{c_{p-1}c_p}e_{c_pc_1}, \tag{1.4.7}$$

where $C = (c_1, c_2, \ldots, c_p)$ is a balanced cycle (cf. 1.2.14). We will show by induction on p that the third and the fourth equations in (1.4.6) imply (1.4.7). If $p = 4$ then C is a cycle of one of the four types C_1, C_2, C_3, or C_4 shown on Figure 1-7. Thus C produces one of the relations (1.4.6). If $p > 4$, then we can find $r \neq s$ such that both $C' = (c_r, c_{r+1}, \ldots, c_s)$ and $C'' = (c_s, c_{s+1}, \ldots, c_r)$ are balanced. The equation (1.4.7) for C is the sum of the corresponding equations for C' and C''. □

This proposition is an analogue to Arnold's description of the cohomology of the complement to complexified Coxeter arrangement (see Corollary 1.2.14).

1.4.6 Sleek posets and semiacyclic tournaments

Let R be a region of the arrangement $\mathcal{L}_n$, and let $(x_1, \ldots, x_n)$ be any point in R. Define $P = P(R)$ to be the poset on the vertices $1, 2, \ldots, n$ such that $i <_P j$ if and only if $x_i - x_j > 1$ and $i < j$ in the usual order on $\mathbb{Z}$.

We call a poset P on the vertices $1, 2, \ldots, n$ *sleek* if P is the intersection of a semiorder (see Section 1.3.2) with the chain $1 < 2 < \cdots < n$.

The following proposition immediately follows from the definitions.

Proposition 1.4.9 *The map* $R \mapsto P(R)$ *is a bijection between regions of* $\mathcal{L}_n$ *and sleek posets on* $1, 2, \ldots, n$*. Hence the number* $r(\mathcal{L}_n)$ *is equal to the number of sleek posets on* $1, 2, \ldots, n$*.*

There is a simple characterization of sleek posets in terms of forbidden induced subposets (cf. Theorem 1.3.4).

Theorem 1.4.10 [44, Theorem 8.4] *A poset P on the vertices* $1, 2, \ldots, n$ *is sleek if and only if it contains no induced subposet of the four types shown on Figure 1-6, where* $a < b < c < d$*.*

In the remaining part of this section we prove Theorem 1.4.10.

First, we give another description of regions in $\mathcal{L}_n$ (or, equivalently, sleek posets). A *tournament* on the vertices $1, 2, \ldots, n$ is a directed graph T without loops such that for every $i \neq j$ either $(i, j) \in T$ or $(j, i) \in T$. For a region R of $\mathcal{L}_n$ construct a tournament $T = T(R)$ on the vertices $1, 2, \ldots, n$ such that for $(x_1, \ldots, x_n) \in R$ if we have $x_i - x_j > 1$, $i < j$, then $(i, j) \in T$, and if $x_i - x_j < 1$, $i < j$, then $(j, i) \in T$.

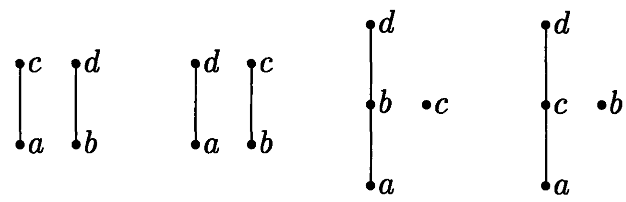

Figure 1-6: Obstructions to sleekness.

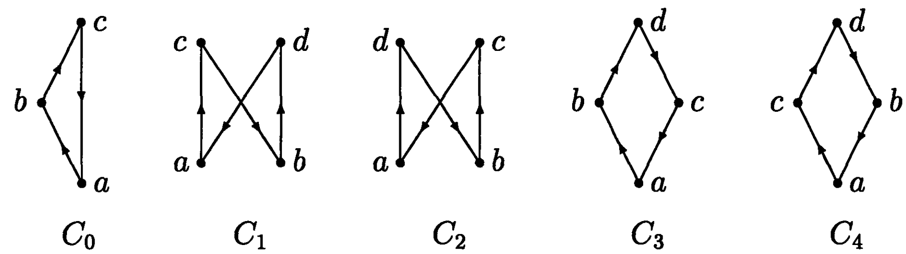

Figure 1-7: Ascending cycles.

Let $C = (c_1, \ldots, c_n)$ be a cycle; and let $\mathrm{asc}(C)$ denote the number of ascents and $\mathrm{des}(C)$ denote the number of descents in C. We say that a cycle C is *ascending* if $\mathrm{asc}(C) \geq \mathrm{des}(C)$. For example, the following cycles, shown on Figure 1-7, are ascending: $C_0 = (a, b, c)$, $C_1 = (a, c, b, d)$, $C_2 = (a, d, b, c)$, $C_3 = (a, b, d, c)$, $C_4 = (a, c, d, b)$, where $a < b < c < d$.

We call a tournament T on $1, 2, \ldots, n$ *semiacyclic* if it contains no ascending cycles. In other words, T is semiacyclic if for any directed cycle C in T we have $\mathrm{asc}(C) < \mathrm{des}(C)$.

Proposition 1.4.11 *A tournament T on $1, 2, \ldots, n$ corresponds to a region R in $\mathcal{L}_n$, i.e., $T = T(R)$, if and only if T is semiacyclic. Hence $r(\mathcal{L}_n)$ is the number of semiacyclic tournaments on $1, 2, \ldots, n$.*

This fact was independently found by Shmulik Ravid.

For any tournament T on $1, 2, \ldots, n$ without cycles of type C_0 we can construct a poset $P = P(T)$ such that $i <_P j$ if and only if $i < j$ and $(i, j) \in T$. The four ascending cycles C_1, C_2, C_3, C_4 in Figure 1-7 correspond to the four posets on Figure 1-6. Therefore, Theorem 1.4.10 is equivalent to the following result.

Theorem 1.4.12 [44, Theorem 8.6] *A tournament T on the vertices $1, 2, \ldots, n$ is semiacyclic if and only if it contains no ascending cycles of the types C_0, C_1, C_2, C_3, and C_4 shown in Figure 1-7, where $a < b < c < d$.*

Remark 1.4.13 This theorem is an analogue of a well-known fact that a tournament T is acyclic if and only if it contains no cycles of length 3. For semiacyclicity we have obstructions of lengths 3 and 4.

Proof — Let T be a tournament on $1, 2, \ldots, n$. Suppose that T is not semiacyclic. We will show that T contains a cycle of type C_0, C_1, C_2, C_3, or C_4. Let $C = (c_1, c_2, \ldots, c_m)$ be an ascending cycle in T of minimal length. If $m = 3$, or 4 then C is of type C_0, C_1, C_2, C_3, or C_4. Suppose that $m > 4$.

Lemma 1.4.14 *We have* $\mathrm{asc}(C) = \mathrm{des}(C)$.

Proof — Since C is ascending, we have $\mathrm{asc}(C) \geq \mathrm{des}(C)$. Suppose $\mathrm{asc}(C) > \mathrm{des}(c)$. If C has two adjacent ascents i and $i+1$ then $(c_{i-1}, c_{i+1}) \in T$ (otherwise we have an ascending cycle (c_{i-1}, c_i, c_{i+1}) of type C_0 in T). Then $C' = (c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_m)$ is an ascending cycle in T of length $m-1$, which contradicts our assumption that C is an ascending cycle of minimal length. So for every ascent i in C the index $i+1$ is a descent. Hence $\mathrm{asc}(C) \leq \mathrm{des}(C)$, and we get a contradiction. □

We say that c_i and c_j are *on the same level* in C if the number of ascents between c_i and c_j is equal to the number of descents between c_i and c_j.

Lemma 1.4.15 *We can find* $i, j \in \{1, 2, \ldots, m\}$ *such that* (a) i *is an ascent and* j *is a descent in* C*,* (b) $i \not\equiv j \pm 1 \bmod m$*, and* (c) c_i *and* c_{j-1} *are on the same level.*

Proof — We may assume that for any $1 \leq s \leq m$ the number of ascents in $\{1, 2, \ldots, s\}$ is greater than or equal to the number of descents in $\{1, 2, \ldots, s\}$ (otherwise take some cyclic permutation of $(c_1, c_2, \ldots, c_m)$). Consider two cases.

1. There exists $1 \leq t \leq m-1$ such that c_t and c_m are on the same level. In this case, if the pair $(i, j) = (1, t)$ does not satisfy conditions (a)–(c) then $t = 2$. On the other hand, if the pair $(i, j) = (t+1, m)$ does not satisfy (a)–(c) then $t = m-2$. Hence, $m = 4$ and C is of type C_1 or C_2 shown in Figure 1-7.

2. There is no $1 \leq t \leq m-1$ such that c_t and c_m are on the same level. Then 2 is an ascent and $m-1$ is a descent. If the pair $(i, j) = (2, m-2)$ does not satisfy (a)–(c) then $m = 4$ and C is of type C_3 or C_4 shown on Figure 1-7. □

Now we can complete the proof of Theorem 1.4.12. Let i, j be two numbers satisfying the conditions of Lemma 1.4.15. Then c_{i-1}, c_i, c_{j-1}, c_j are four distinct vertices such that (a) $c_{i-1} < c_i$, (b) $c_{j-1} > c_j$, (c) c_i and c_{j-1} are on the same level, and (d) c_{i-1} and c_j are on the same level. We may assume that $i < j$.

If $(c_{j-1}, c_{i-1}) \in T$ then $(c_{i-1}, c_i, \ldots, c_{j-1})$ is an ascending cycle in T of length less than m, which contradicts the requirement that C is an ascending cycle on T of minimal length. So $(c_{i-1}, c_{j-1}) \in T$. If $c_{i-1} < c_{j-1}$ then $(c_{j-1}, c_j, \ldots, c_m, c_1, \ldots, c_{i-1})$ is an ascending cycle in T of length less than m. Hence, $c_{i-1} > c_{j-1}$.

Analogously, if $(c_i, c_j) \in T$ then $(c_j, c_{j+1}, \ldots, c_p, c_1, \ldots, c_i)$ is an ascending cycle in T of length less than m. So $(c_j, c_i) \in T$. If $c_i > c_j$ then $(c_i, c_{i+1}, \ldots, c_j)$ is an ascending cycle in T of length less than m. So $c_i < c_j$.

Now we have $c_{i-1} > c_{j-1} > c_j > c_i > c_{i-1}$, and we get an obvious contradiction.

We have shown that every minimal ascending cycle in T is of length 3 or 4 and thus proved Theorem 1.4.12. □

1.5 Truncated Affine Arrangements

In this section we study a general class of hyperplane arrangements which contains, in particular, the Linial and Shi arrangements.

Let a and b be two integers such that $a+b \geq 2$. Consider the hyperplane arrangement $\mathcal{A}_n^{ab}$ in $V_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$ given by

$$x_i - x_j = k, \qquad 1 \leq i < j \leq n, \ -a < k < b. \tag{1.5.1}$$

We call $\mathcal{A}_n^{ab}$ *truncated affine arrangement* because it is a finite subarrangement of the affine arrangement of type $\widetilde{A}_{n-1}$ given by $x_i - x_j = r$, $r \in \mathbb{Z}$.

1.5.1 Functional equations

Let $f_n = f_n^{ab}$ be the number of regions of the arrangement $\mathcal{A}_n^{ab}$, and let

$$f(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!} \tag{1.5.2}$$

be the exponential generating function for f_n.

Theorem 1.5.1 [44, Theorem 9.1] *Suppose $a, b \geq 0$.*

1. *The generating function $f = f(x)$ satisfies the following functional equation:*

$$f^{b-a} = e^{x \cdot \frac{f^a - f^b}{1-f}}. \tag{1.5.3}$$

2. *If $a = b \geq 1$, then $f = f(x)$ satisfies the equation:*

$$f = 1 + x\, f^a, \tag{1.5.4}$$

Note that the equation (1.5.4) can be obtained from (1.5.3) by l'Hospital's rule in the limit $b \to a$.

In cases $a = b$ and $a = b \pm 1$ the functional equations (1.5.3) and (1.5.4) allow to calculate the numbers f_n^{ab} explicitly. The following statement was proved by P. Headley [24].

Corollary 1.5.2 *The number f_n^{aa} is equal to $(a\,n)(a\,n-1)\cdots(a\,n-n+2)$.*

The equation (1.5.3) is especially simple in the case $a = b \pm 1$. We call the arrangement $\mathcal{A}_n^{a,a+1}$ the *extended Shi arrangement.* In this case we get:

Corollary 1.5.3 *The number f_n of regions of the extended Shi arrangement given by*

$$x_i - x_j = -a+1, -a+2, \ldots, a, \qquad 1 \leq i < j \leq n$$

is equal to $f_n = (a\,n+1)^{n-1}$, and the exponential generating function $f = f(x)$ satisfies the functional equation $f = e^{x \cdot f^a}$.

In order to prove Theorem 1.5.1 we need several new definitions. A *graded graph* is a graph is a triple $G = (V, E, h)$, where V is a linearly ordered set of vertices, E is a set of (nonoriented) edges, and h is a function $h : V \to \mathbb{Z}_+$ called *grading*. The vertices v in V such that $h(v) = r$, $r = 0, 1, 2, \ldots$, form the r-th *level* of G. Let $e = (u, v)$ be an edge in G, $u < v$. We say that the *slope* of an edge $e = (u, v)$, $u < v$, in E is the integer $s(e) = h(v) - h(u)$. A graded graph G is *of type* (a, b) if the slopes of all edges in G are in the interval $[-a+1, b-1] = \{-a+1, -a+2, \ldots, b-1\}$. Analogously, we define graded trees, forests, and circuits of type (a, b).

Choose a linear order on the set $\{(u, s, v) \mid u, v \in V,\ u < v,\ -a < s < b\}$. Let C be a graded circuit of type (a, b). Every edge (u, v) in C corresponds to a triple (u, s, v), where s is the slope of the edge (u, v). Choose the edge e in C with the minimal triple (u, e, v). We say that $C \setminus \{e\}$ is a *broken circuit of type* (a, b).

We say that a graded forest is *planted* if each connected component contains a vertex on the 0-th level.

Proposition 1.5.4 *The number f_n^{ab} of regions of the arrangement (1.5.1) is equal to the number of planted graded forests of type (a, b) on the vertices $1, 2, \ldots, n$ without broken circuits of type (a, b).*

Proof — By Corollary 1.2.8, the number f_n is equal to the number of A-colored forests F without broken A-circuits. For an A-colored forest F, there is a unique planted graded forest $\widetilde{F} = (V, E, h)$ with the same sets of vertices and edges and such that the slope $s(e)$ of any edge $e \in E$ is equal to the color of e in the forest F. Then $\widetilde{F}$ is of type (a, b) and without broken circuits of type (a, b) if and only if F has no broken A-circuits. $\square$

From now on we fix the lexicographical order on triples (u, s, v), i.e., $(u, s, v) < (u', s', v')$ if and only if $u < u'$, or ($u = u'$ and $s < s'$), or ($u = u'$ and $s = s'$ and $v < v'$). Note the order of u, s, and v! We call a graded tree T *solid* if T is of type (a, b) and T contains no broken circuits of type (a, b).

Let T be a solid tree on $1, 2, \ldots, n$ such that the vertex 1 is on the r-th level. When we delete the minimal vertex 1, the tree T decomposes into connected components $T_1, T_2, \ldots, T_m$. Suppose that each component T_i is connected with 1 by an edge $(1, v_i)$ where v_i is on the r_i-th level.

Lemma 1.5.5 *Let $T, T_1, \ldots, T_m, v_1, \ldots, v_m$, and $r_1, \ldots, r_m$ be as above. The tree T is solid if and only if* (a) *all $T_1, T_2, \ldots, T_m$ are solid,* (b) *for all i the r_i-th level is the minimal nonempty level in T_i such that $-a+1 \le r_i - r \le b-1$, and* (c) *the vertex v_i is the minimal vertex on its level in T_i.*

Proof — First, we prove that if T is solid then the conditions (a)–(c) hold. Condition (a) is trivial, because if T_i contains a broken circuits of type (a, b) then T also contains this circuit. Assume that for some i there is a vertex v_i' on the r_i'-th level in T_i such that $r_i' < r_i$ and $r_i' - r \ge -a+1$. Then the minimal chain in T that connects the vertex 1 with the vertex v_i' is a broken circuit of type (a, b). Thus the condition (b) holds. Now suppose that for some i the vertex v_i is not the minimal vertex v_i'' on its level. Then the minimal chain in T that connects the vertex 1 with v_i'' is a broken circuit of type (a, b). Therefore, the condition (c) holds too.

Now assume that the conditions (a)–(c) are true. We prove that T is solid. For suppose not. Then T contains a broken circuit $B = C \setminus \{e\}$ of type (a, b), where C is a graded circuit and e is its minimal edge. If B does not pass through the vertex 1 then B lies in T_i for some i, which contradicts to the condition (a). We can assume that B passes through the vertex 1. Since e is the minimal edge is C, $e = (1, v')$ for some vertex v' on the level r' in T. Suppose $v \in T_i$. If v' and v_i are on different levels in T_i then, by (b), $r_i < r$. Thus the minimal edge in C is $(1, v_i)$ and not $(1, v')$. If v' and v_i are on the same level in T_i then, by (c), $v_i < v'$. Again, the minimal edge in C is $(1, v_i)$ and not $(1, v')$. Therefore, the tree T contains no broken circuit of type (a, b), i.e., T is solid. $\square$

Let l_i be the minimal nonempty level in T_i, and let L_i be the maximal nonempty level in T_i. By Lemma 1.5.5, the vertex 1 lies on the rth level for some $l_i - b < r < L_i + a$; for each r from this interval there is a unique way to connect T_i with the vertex 1 in the rth level.

Let p_{nkr} denote the number of solid trees (not necessarily grounded) on the vertices $1, 2, \ldots, n$ which are located on levels $0, 1, \ldots, k$ such that the vertex 1 is on the rth level, $0 \le r \le k$.

Let

$$p_{kr}(x) = \sum_{n \ge 1} p_{nkr} \frac{x^n}{n!}, \qquad p_k(x) = \sum_{r=0}^{k} p_{kr}(x).$$

By the exponential formula (see [23, p. 166]) and Lemma 1.5.5, we have

$$p'_{kr}(x) = \exp(b_{kr}(x)), \tag{1.5.5}$$

where $b_{kr}(x) = \sum_{n \ge 1} b_{nkr} \frac{x^n}{n!}$ and b_{nkr} is the number of solid trees T on n vertices located on the levels $0, 1, \ldots, k$ such that at least one of the levels $r - a + 1, r - a + 2, \ldots, r + b - 1$ is nonempty, $0 \le r \le k$. The polynomial $b_{kr}(x)$ enumerates the solid trees on the levels $1, 2, \ldots, k$ minus trees on the levels $1, \ldots, r - a$ and trees on levels the $r + b, \ldots, k$. Thus we obtain

$$b_{kr}(x) = p_k(x) - p_{r-a}(x) - p_{k-r-b}(x).$$

By (1.5.5), we get

$$p'_{kr}(x) = \exp(p_k(x) - p_{r-a}(x) - p_{k-r-b}(x)),$$

where $p_{-1}(x) = p_{-2}(x) = \cdots = 0$, $p_0(x) = x$, $p_k(0) = 0$ for $k \in \mathbb{Z}$. Hence

$$p'_k(x) = \sum_{r=0}^{k} \exp(p_k(x) - p_{r-a}(x) - p_{k-r-b}(x)).$$

Or, equivalently,

$$p_k'(x)\exp(-p_k(x)) = \sum_{r=0}^{k} \exp(-p_{r-a}(x))\,\exp(-p_{k-r-b})(x).$$

Let $q_k(x) = \exp(-p_k(x))$. We have

$$q_k'(x) = -\sum_{r=0}^{k} q_{r-a}(x)\,q_{k-r-b}(x), \tag{1.5.6}$$

$q_{-1} = q_{-2} = \cdots = 1$, $q_0 = e^{-x}$, $q_k(0) = 1$ for $k \in \mathbb{Z}$.

The following lemma describes the relation between the polynomials $q_k(x)$ and the numbers of regions of the arrangement $\mathcal{A}_n^{ab}$.

Lemma 1.5.6 *The quotient $q_{k-1}(x)/q_k(x)$ tends to $\sum_{n\geq 0} f_n \frac{x^n}{n!}$ as $k \to \infty$.*

Proof — Clearly, $p_k(x) - p_{k-1}(x)$ is the exponential generating function for the numbers of grounded solid trees of height less than or equal to k. By the exponential formula (see [23, p. 166]) $q_{k-1}(x)/q_k(x) = \exp(p_k(x) - p_{k-1}(x))$ is the exponential generating function for the numbers of grounded solid forests of height less than or equal to k. The lemma obviously follows from Proposition 1.5.4. □

All previous formulae and constructions are valid for arbitrary a and b. Now we take advantage of the condition $a, b \geq 0$. Let

$$q(x,y) = \sum_{k\geq 0} q_k(x) y^k.$$

By (1.5.6), we obtain the following differential equation for $q(x,y)$:

$$\begin{aligned} \frac{\partial}{\partial x} q(x,y) &= -\left(a_y + y^a q(x,y)\right)\cdot\left(b_y + y^b q(x,y)\right), \\ q(0,y) &= (1-y)^{-1}, \end{aligned}$$

where $a_y := (1-y^a)/(1-y)$.

This differential equation has the following solution:

$$q(x,y) = \frac{b_y \exp(-x\cdot b_y) - a_y \exp(-x\cdot a_y)}{y^a \exp(-x\cdot a_y) - y^b \exp(-x\cdot b_y)}. \tag{1.5.7}$$

Let us fix some small x. Since $Q(y) := q(x,y)$ is an analytic function of y, then $\gamma = \gamma(x) = \lim_{k\to\infty} q_{k-1}/q_k$ is the pole of $Q(y)$ closest to 0 (γ is the radius of convergence of $Q(y)$ if x is a small positive number). By (1.5.7), $\gamma^a \exp(-x\cdot a_\gamma) - \gamma^b \exp(-x\cdot b_\gamma) = 0$. Thus, by Lemma 1.5.6, $f(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!} = \gamma(x)$ is the solution of the functional equation

$$f^a e^{-x\,\cdot\,\frac{1-f^a}{1-f}} = f^b e^{-x\,\cdot\,\frac{1-f^b}{1-f}},$$

which is equivalent to (1.5.3).

This completes the proof of Theorem 1.5.1. □

1.5.2 Formulae for the characteristic polynomial

Let $\mathcal{A} = \mathcal{A}_n^{ab}$ be the truncated affine arrangement given by (1.5.1). Consider the characteristic polynomial $\chi_n^{ab}(q)$ of the arrangement $\mathcal{A}_n^{ab}$. Recall that it is equal to $q^{n-1}\mathrm{Poin}_{\mathcal{A}_n^{ab}}(-q^{-1})$.

Let $\chi^{ab}(x,q)$ be the exponential generating function

$$\chi^{ab}(x,q) = 1 + \sum_{n>0} \chi_{n-1}^{ab}(q)\, x^n/n!\,.$$

According to [53, Theorem 1.2], we have

$$\chi^{ab}(x,q) = f(-x)^{-q}, \tag{1.5.8}$$

where $f(x) = \chi^{ab}(-x,-1)$ is the exponential generating function (1.5.2) for numbers of regions of $\mathcal{A}_n^{ab}$.

Let S be the *shift operator* $S : f(q) \mapsto f(q-1)$.

Theorem 1.5.7 [44, Theorem 9.7] *Assume that* $0 \le a < b$. *Then*

$$\chi_n^{ab}(q) = (b-a)^{-n}(S^a + S^{a+1} + \cdots + S^{b-1})^n \cdot q^{n-1}. \tag{1.5.9}$$

Proof — The theorem can be deduced from Theorem 1.5.1 and (1.5.8) (using, e.g., Lagrange's inversion formula). □

There is an explicit formula for $\chi^{aa}(q)$. The following statement, found in [24], is not hard to derive from Corollary 1.5.2.

Proposition 1.5.8 *We have* $\chi_n^{aa}(q) = (q+1-an)(q+2-an)\cdots(q+n-1-an)$.

We can analytically extend $\chi_n^{ab}(q)$ to complex values of a and b, since

$$\chi_n^{ab}(q) = \left(S^a\,(S^{b-a}-1)/((S-1)(b-a))\right)^n \cdot q^{n-1}$$

In the limit $b \to a$, application of the l'Hospital's rule results in the expression[8]

$$\chi_n^{aa}(q) = \left(S^a \frac{\ln S}{S-1}\right)^n \cdot q^{n-1}.$$

[8]Comparing these two expressions for $\chi^{aa}(q)$, we obtain the formula

$$\left(\frac{D}{e^D-1}\right)^n \cdot q^{n-1} = (q-1)(q-2)\cdots(q-n+1)\,,$$

where $D = -\ln(S) = d/dq$. This formula yields an identity that involves the Bernoulli numbers, which are coefficients of the Taylor expansion of $x/(e^x-1)$, and the Stirling numbers of the first kind.

There are several equivalent ways to reformulate Theorem 1.5.7, as follows:

Corollary 1.5.9 *Let $r = b - a$.*

1. *We have*
$$\chi_n^{ab}(q) = r^{-n} \sum (q - \phi(1) - \cdots - \phi(n))^{n-1},$$
where the sum is over all functions $\phi : \{1, \ldots, n\} \to \{a, \ldots, b-1\}$.

2. *We have*
$$\chi_n^{ab}(q) = r^{-n} \sum_{s,l \geq 0} (-1)^l (q - s - an)^{n-1} \binom{n}{l} \binom{s + n - rl - 1}{n-1}.$$

3. *We have*
$$\chi_n^{ab}(q) = r^{-n} \sum \binom{n}{n_1, \ldots, n_r} (q - an_1 - \cdots - (b-1)n_r)^{n-1},$$
where the sum is over all nonnegative integers $n_1, n_2, \ldots, n_r$ such that $n_1 + n_2 + \cdots + n_r = n$.

Examples:

1. The Shi arrangement is the arrangement $\mathcal{A}_n^{12}$ given by
$$x_i - x_j = 0, 1, \qquad 1 \leq i < j \leq l + 1. \tag{1.5.10}$$
By Corollary 1.5.9.1, we get the following known formula (see [48, 49])
$$\chi_n^{12}(q) = (q - n)^{n-1}.$$

2. More generally, for the extended Shi arrangement $\mathcal{A}_n^{a,a+1}$, given by
$$x_i - x_j = -a + 1, -a + 2, \ldots, a, \qquad 1 \leq i < j \leq l + 1, \tag{1.5.11}$$
we have (cf. Corollary 1.5.3)
$$\chi_n^{a,a+1}(q) = (q - an)^{n-1}.$$

3. For the Linial arrangement $\mathcal{L}_n = \mathcal{A}_n^{02}$ (see Section 1.4), Corollary 1.5.9.3 gives
$$\chi_n^{02}(q) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} (q - k)^{n-1}, \tag{1.5.12}$$
(cf. Theorem 1.4.5)

4. More generally, for the arrangement $\mathcal{A}_n^{a,\,a+2}$, we have

$$\chi_n^{a,\,a+2}(q) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} (q - an - k)^{n-1}.$$

Formula (1.5.12) for the characteristic polynomial $\chi_n^{0\,2}(q)$ was earlier obtained by C. Athanasiadis [3, Theorem 5.2]. He used a different approach based on an interpretation of the value of $\chi_n(q)$ for sufficiently large primes q.

1.5.3 Roots of the characteristic polynomial

Theorem 1.5.7 has one surprising application concerning the location of roots of the characteristic polynomial $\chi_n^{ab}(q)$

We start with the case $a = b$. One can reformulate Proposition 1.5.8 in the following way:

Corollary 1.5.10 *The roots of the polynomial $\chi_n^{aa}(q)$ are the numbers $an-1, an-2, \ldots, an-n+1$ (each with multiplicity 1). In particular, the roots are symmetric to each other with respect to the point $(2a-1)n/2$.*

Now assume that $a \neq b$. We will prove the following "Riemann hypothesis."

Theorem 1.5.11 [44] *All the roots of the characteristic polynomial $\chi_n^{ab}(q)$ of the truncated affine arrangement $\mathcal{A}_n^{ab}$, $a \neq b$, have real part equal to $(a+b-1)\,n/2$. They are symmetric to each other with respect to the point $(a+b-1)\,n/2$.*

Thus in both cases the roots of the polynomial $\chi_n^{ab}(n)$ are symmetric to each other with respect to the point $(a+b-1)\,n/2$, but in the case $a = b$ all roots are real, whereas in the case $a \neq b$ the roots are on the same vertical line[9] in the complex plane $\mathbb{C}$. Note that in the case $a = b-1$ the polynomial $\chi_n^{ab}(q)$ has only one root $an = (a+b-1)n/2$ with multiplicity $n-1$.

The following lemma is implicit in a paper of Auric [4] and also follows from a problem posed by Pólya [40] and solved by Obreschkoff [37] (repeated in [41, Problem V.196.1, pp. 70 and 251]).

Lemma 1.5.12 *Suppose that a polynomial $f(q) \in \mathbb{C}[q]$ is such that every zero has real part a, and let λ be a complex number satisfying $|\lambda| = 1$. Then every zero of the polynomial $g(q) = (S-\lambda)f(q) = f(q-1) - \lambda f(q)$ has real part $a + 1/2$.*

Proof of Theorem 1.5.11. — All the zeros of the polynomial q^{n-1} have real part 0. The operator $(S^a + S^{a+1} + \cdots + S^{b-1})$ can be written as

$$S^a(S-\lambda_1)\cdots(S-\lambda_{b-a-1}),$$

[9] Let $\widetilde{\chi}_n(q) = \chi_n^{ab}(q - (a+b-1)n/2)$, its roots are purely imaginary. The following interlacing property of roots seems also to be valid: Between any two roots of $\widetilde{\chi}_n(q)$ there is a root of $\widetilde{\chi}_{n-1}(q)$.

where each λ_j is a complex number of absolute value one. The proof now follows from Theorem 1.5.7 and Lemma 1.5.12. $\square$

1.6 Asymptotics and Random Trees

1.6.1 Characteristic polynomials and trees

According to Theorem 1.5.7, the characteristic polynomial of a truncated affine arrangement can be easily expressed using the shift operator: $S : f(q) \mapsto f(q-1)$. Let us also introduce the differentiation operator: $D : f(q) \mapsto df/dq$. Then Taylor's theorem can be stated as

$$\exp(-D) = S.$$

Consider the exponential power series

$$h(t) = h_0 + h_1 t + h_2 t^2/2! + \cdots + h_k t^k/k! + \ldots,$$

where the h_i are some numbers and h_0 is nonzero.

Generalizing the expression (1.5.9), we define the polynomials $f_n(q)$, $n > 0$, by the formula

$$f_n(q) = (h(D))^n \, q^{n-1}. \tag{1.6.13}$$

The polynomials $f_n(q)$ are correctly defined even if the series $h(t)$ does not converge, since the expression for $f_n(q)$ involves only a finite sum of nonzero terms.

Let $\mathcal{T}_n$ be the set of trees on the vertices $0, 1, 2, \ldots, n$. We will assume that the edges of trees are oriented away for the root at the vertex 0. By $d_i = d_i(T)$ we denote the outdegree of a vertex i in a tree $T \in \mathcal{T}_n$. Define the weight of T by

$$w_q(T) = q^{d_0 - 1} h_{d_1} h_{d_2} \ldots h_{d_n}.$$

Proposition 1.6.1 *1. The polynomial $f_n(q)$ is the weight enumerator for trees on $n+1$ vertices, i.e.*

$$f_n(q) = \sum_{T \in \mathcal{T}_n} w_q(T).$$

2. The coefficient of q^k in $f_n(q)$ is equal to

$$\sum h_{k_1} \ldots h_{k_n} \binom{n-1}{k, k_1, \ldots, k_n},$$

where the sum is over all $k_1, \ldots, k_n \geq 0$ such that $k + k_1 + \cdots + k_n = n - 1$.

Proof — Let $D^{(k)} = D^k/k!$. Then, for $k > 0$, we have

$$D^{(k)} \, q^m = \binom{m}{k} q^{m-k},$$

if $m \geq k$ and 0 otherwise. By (1.6.13), we have

$$\begin{aligned} f_n(q) &= h(D)^n\, q^{n-1} = h(D)^{n-1} \sum_{k_1 \geq 0} h_{k_1} D^{(k_1)}\, q^{n-1} \\ &= h(D)^{n-1} \sum_{k_1 \geq 0} h_{k_1} \binom{n-1}{k_1} q^{n-1-k_1} = \dots \\ &= \sum_{k, k_1, \dots, k_n \geq 0} h_{k_1} \dots h_{k_n} \binom{n-1}{k, k_1, k_2, \dots, k_n} q^k \end{aligned}$$

where $k = n - 1 - k_1 - \dots - k_n$. Using Prüfer's coding [45], we obtain the first statement of the theorem. □

Let

$$f(z, q) = 1 + q \sum_{n \geq 1} f_n(q) \frac{z^n}{n!}.$$

Consider also the weighting on trees $T \in \mathcal{T}_n$ given by

$$\widetilde{w}(T) = h_{d_0} h_{d_1} \dots h_{d_n}.$$

Let $g_n = \sum_{T \in \mathcal{T}_n} \widetilde{w}(T)$ be the weighted sum of the trees, and let

$$g(x) = \sum_{n \geq 0} g_n \frac{x^{n+1}}{n!}$$

be the exponential generating function for the g_n. Note that $n g_n = f_{n+1}(0)$.

Proposition 1.6.2 *We have $f(x, q) = \exp(q\, g(x))$. The series $g = g(z)$ satisfies the functional equation*

$$g = x\, h(g). \tag{1.6.14}$$

The statement of the theorem is proved by a standard argument with the exponential formula.

1.6.2 Random trees

In this section we calculate the distribution of degrees of vertices of a "random infinite tree." We will need these calculations in Section 1.6.3.

We use the notation of the previous section with the assumption that $h_0, h_1, h_2, \dots$ are nonnegative integer numbers, and $h_0 > 0$.

Let I be the set of indices n for which $g_n > 0$. If h_i is nonzero for some $i \geq 1$, then I is an infinite set. For $n \in I$, consider the distribution on the set of trees on $n + 1$ vertices given by $P(T) = \widetilde{w}(T)/g_n$. Let $P_n(k)$ denote the probability that a random vertex of a random tree has outdegree k, i.e.,

$$P_n(k) = \frac{1}{(n+1)\, g_n} \sum_{T \in \mathcal{T}_n} m_k(T)\, \widetilde{w}(T),$$

where $m_k(T)$ is the number of vertices in T with outdegree k. Also let $P(k) = \lim_{n\to\infty} P_n(k)$, where the limit is taken over $n \in I$.

Here is an example when we need to be careful about the index set. Assume that $h_{2m} = 1$ and $h_{2m+1} = 0$, for $m \geq 0$, then $g_m = 0$ for odd n. In this case we have to take the limit over even n.

We can interpret $P(k)$ as the probability that a random vertex of a random infinite tree has outdegree k.

Theorem 1.6.3 *Assume that the series $h(t)$ converges, $h'(t)$ is unbounded on the interval $(0, +\infty)$, and $t = \alpha$ is the minimal positive solution of the equation*

$$t = h(t)/h'(t)\,. \tag{1.6.15}$$

Then

$$P(k) = \frac{h_k\, \alpha^k}{h(\alpha)\, k!}\,.$$

Before we prove this statement, consider several examples. In the case when $h_k = 1$, $k \geq 0$, we obtain the uniform distribution on trees.

Corollary 1.6.4 *The outdegrees of a random infinite tree have Poisson distribution:*

$$P(k) = e^{-1}/k!\,.$$

Proof — We have $h(t) = e^t$. The equation $t = e^t/e^t$ has a unique solution $t = 1$. □

Assume that $h_{2m} = 1$ and $h_{2m+1} = 0$, $m \geq 0$. We have the uniform distribution on trees with even outdegrees. We will call such trees *even.*

Corollary 1.6.5 *The outdegrees of a random infinite even tree have the following distribution:*

$$P(2m) = P(0)\, \alpha^{2m}/(2m)!\,,$$

where $\alpha = 1.199678640257733\ldots$ is a unique positive solution of the equation

$$e^{2t} = \frac{t+1}{t-1}, \quad t > 1, \tag{1.6.16}$$

and $P(0) = 1/\cosh(\alpha) = 0.552434124530883\ldots.$

Proof — In this case $h(t) = \cosh(t)$. The equation $t = \cosh(t)/\sinh(t)$ is equivalent to (1.6.16). □

Proof of Theorem 1.6.3 — Let

$$\begin{aligned} p(x) &= \sum_{n\geq 0}(n+1)\, P_n(k)\, g_n x^n/n!\,, \\ d(x) &= \sum_{n\geq 0}(n+1)\, g_n x^n/n!\,. \end{aligned}$$

Then, by definition, $P(k)$ is the limit of ratios of coefficients of x^n in $p(x)$ and $d(x)$ as $n \to \infty$. First, we note that if $x_0 > 0$ is the minimal positive pole of both $p(x)$ and $d(x)$, then $P(k) = \lim_{x\to x_0} p(x)/d(x)$.

We have $d(x) = g'(x)$. By (1.6.14),

$$\begin{aligned} d(x) &= h(g) + xh'(g)d(x)\,, \\ d(x) &= \frac{h(g)}{1 - xh'(g)}\,. \end{aligned} \tag{1.6.17}$$

Let $g_{(k)}(x, y)$ be the following exponential generating function

$$g_{(k)}(x, y) = \sum_{n\geq 0} \sum_{T\in\mathcal{T}_n} \widetilde{w}(T) y^{m_k(T)} x^{n+1}/n!.$$

Clearly,

$$p(x) = x^{-1} \left.\frac{\partial g_{(k)}}{\partial y}\right|_{y=1} (x).$$

The function $g_{(k)}$ satisfies the equation:

$$g_{(k)} = x\,(h(g_{(k)}) + (y-1)h_k g_{(k)}^k/k!).$$

Then

$$\begin{aligned} p(x) &= x\,h'(g)\,p(x) + h_k g^k/k!\,, \\ p(x) &= \frac{h_k g^k}{k!\,(1 - xh'(g))}\,. \end{aligned} \tag{1.6.18}$$

Let x_0 be the minimal positive number such that

$$1 - x_0\, h'(g(x_0)) = 0. \tag{1.6.19}$$

Then x_0 is the minimal positive pole of $p(x)$ as well as of $d(x)$. Assume not, then there is a pole x_1 of $g(x)$ (or $h(g(x))$) such that $0 < x_1 \leq x_0$ (cf. (1.6.17) and (1.6.18)). Since $h'(t)$ is unbounded, there is a root of (1.6.19) between 0 and x_1. Contradiction.

Therefore, by earlier remark,

$$P(k) = \lim_{x\to x_0} \frac{p(x)}{d(x)} = \frac{h_k\, \alpha^k}{h(\alpha)\, k!}\,,$$

where $\alpha = g(x_0)$. The equation (1.6.15) for α follows from (1.6.14) and (1.6.19). $\square$

1.6.3 Asymptotics of characteristic polynomials

It is convenient to introduce the following shift the characteristic polynomial of the Linial arrangement:

$$b_n(q) = 2^{n-1}\chi_n^{02}((q+n)/2).$$

Then, by Theorem 1.5.7,

$$b_n(q) = \left(\frac{S+S^{-1}}{2}\right)^n q^{n-1} = \cosh(D)^n\, q^{n-1}.$$

The first ten polynomials $b_n(q)$ are given below:

$$\begin{aligned}
b_1(q) &= 1,\\
b_2(q) &= q,\\
b_3(q) &= q^2+3,\\
b_4(q) &= q^3+12q,\\
b_5(q) &= q^4+30q^2+65,\\
b_6(q) &= q^5+60q^3+480,\\
b_7(q) &= q^6+105q^4+1995q^2+3787,\\
b_8(q) &= q^7+168q^5+6160q^3+41216q,\\
b_9(q) &= q^8+252q^6+15750q^4+242172q^2+427905,\\
b_{10}(q) &= q^9+360q^7+35280q^5+1021440q^3+6174720q,
\end{aligned}$$

Recall that $\alpha = 1.199678640\ldots$ is a unique solution of the equation

$$e^{2t} = (t+1)/(t-1), \quad t > 1. \tag{1.6.20}$$

Theorem 1.6.6 *The functions $b_{2m}(q)/b'_{2m}(0)$ converges to a limit $b_{\text{even}}(q)$ as m goes to infinity and*

$$b_{\text{even}}(q) = \sinh(\alpha q)/\alpha.$$

Theorem 1.6.7 *The functions $b_{2m+1}(q)/b_{2m+1}(0)$ converges to a limit $b_{\text{odd}}(q)$ as m goes to infinity and*

$$b_{\text{odd}}(q) = \cosh(\alpha q).$$

Consider the numbers $b_{n,k}$ given by

$$b_n(q) = b_{n,n-1}q^{n-1} + b_{n,n-3}q^{n-3} + b_{n,n-5}q^{n-5} + b_{n,n-7}q^{n-7} + \ldots,$$

and $b_{n,k} = 0$ if $n-k$ is even. Note that $b_{n,n-1} = 1$. Also let $b_n = b_{n,n-1} + b_{n,n-3} + b_{n,n-5} + \cdots = b_n(1)$.

For example, $b_1, b_2, \ldots, b_{10} = 1, 1, 4, 13, 96, 541, 588, 47545, 686080, 7231801$.

We say that $T \in \mathcal{T}_n$ is an *even tree* if the outdegrees $d_1, d_2, \ldots, d_n$ of all vertices T (the root 0 excluded) are even. Equivalently, the degrees of all vertices in T but the root are odd. Such trees are also called *odd degree trees.*

By Propositions 1.6.1 and 1.6.2, we obtain the following statement.

Corollary 1.6.8 *1. The number $b_{n,k}$ is equal to the number of all even trees on the vertices $0, 1, \ldots, n$ such the degree of the root 0 is equal to $k+1$.*

2. The number $b_{n,k}$ is equal to the sum of polynomial coefficients $\binom{n-1}{k, k_1, \ldots, k_n}$ over all even $k_1, \ldots, k_n \geq 0$ such that $k + k_1 + \cdots + k_n = n-1$.

3. Let

$$b(x,q) = 1 + q \sum_{n \geq 1} b_n(q) \frac{x^n}{n!}.$$

$$g(x) = \sum_{m \geq 0} b_{2m} \frac{x^{2m+1}}{(2m)!}$$

Then $b(x,q) = \exp(q\, g(x))$. The function $g = g(x)$ satisfies the functional equation

$$g = x \cosh(g).$$

Proof of Theorems 1.6.6 and 1.6.7 — It is enough to show that $b_{2m,2r+1}/b_{2m,2r+3}$ tend to $\alpha^{-2}(2r+3)(2r+2)$, and that $b_{2m+1,2r}/b_{2m+1,2r+2}$ tends to $\alpha^{-2}(2r+2)(2r+1)$ as m goes to infinity, where α is given by (1.6.20).

By Corollary 1.6.8.2, we have $b_{n,k} = \binom{n-1}{k} c_{n,k}$, where

$$c_{n,k} = \sum_{k_1, \ldots, k_n} \binom{n-k-1}{k_1, k_2, \ldots, k_n}$$

the sum over all even nonnegative $k_1, k_2, \ldots, k_n$ such that $\sum k_i = n-k-1$.

Asymptotically,

$$\frac{b_{n,k}}{b_{n,k+2}} \equiv (k+2)(k+1) \frac{c_{n,k}}{n^2\, c_{n,k+2}}$$

as n goes to the infinity with preserving its parity.

Let $M^{(1)}_{n,k}$ be the set of maps $\phi : \{1, \ldots, n-k-1\} \to \{1, \ldots, n\}$ such that $|\phi^{-1}(i)|$ is even for $i = 1, \ldots, n$. By definition, $c_{n,k}$ is equal to $|M^{(1)}_{n,k}|$.

Analogously, denote by $M^{(2)}_{n,k}$ the set of maps $\psi : \{1, \ldots, n-k-1\} \to \{1, \ldots, n\}$ such that $|\psi^{-1}(i) \cap \{1, \ldots, n-k-3\}|$ is even for $i = 1, \ldots, n$. It is clear that $n^2\, c_{n,k+2} = |M^{(2)}_{n,k}|$.

Also let $M^{(3)}_{n,k} \subset M^{(2)}_{n,k}$ be the subset of maps $\psi : \{1, \ldots, n-k-1\} \to \{1, \ldots, n\}$ such that $\psi(k) \in \{\psi(n-k-2), \psi(n-k-1)\}$ for some $k \in \{1, \ldots, n-k-3\}$.

The simple identity

$$\sum_{l\geq 0}\binom{m}{2l}=\sum_{l\geq 0}\binom{m}{2l+1}, \qquad m>0$$

implies that $|M_{n,k}^{(1)}|=|M_{n,k}^{(3)}|$.

Recall that $P(0)=0.552434\ldots$ is the "probability that a random vertex of a random even tree has outdegree 0," which is calculated in Corollary 1.6.5. The number of elements in $M_{n,k}^{(2)}\setminus M_{n,k}^{(3)}$ is asymptotically equal to $\alpha\,|M_{n,k}^{(2)}|$, as n goes to infinity (k is fixed). The proof of Theorems 1.6.6 and 1.6.7 now easily follows. $\square$

Chapter 2

Quantum Cohomology of Flag Manifolds

This part of my thesis contains the results of [17] obtained in collaboration with Sergey Fomin and Sergei Gelfand as well as the results of [43]. Our notation and exposition of results are close (though not always identical[1]) to that of [17].

2.1 Introduction

The purpose to this chapter is to present an algebro-combinatorial method for calculation of 3-point Gromov-Witten invariants of complex flag manifolds and to investigate its various consequences. These invariants are structure constants of the quantum cohomology ring of the flag manifold. In a special case, they are the Littlewood-Richardson coefficients, which are the intersection numbers of Schubert varieties.

A central open problem of Schubert calculus is to find an explicit rule for calculation of the Littlewood-Richardson coefficients of flag manifolds that would imply that they are integer nonnegative numbers. For example, one would like to have a combinatorial construction of a set, whose number of elements is equal to the corresponding Littlewood-Richardson coefficient. We extend this problem to that of finding a rule for the Gromov-Witten invariants. This extension may lead to a way to solve the problem, since the Gromov-Witten invariants seem to possess more symmetries.

We give below a brief account of definitions and results related to the classical and quantum cohomology rings of complex flag manifolds as well as formulate our main results. Although many of the constructions can be carried out in a more general setup of an arbitrary semisimple Lie group, only the case of type A_{n-1} is considered.

Let Fl_n denote the manifold of complete flags of subspaces in the n-dimensional linear space $\mathbb{C}^n$. There are several ways to describe the cohomology ring $\mathrm{H}^*(Fl_n, \mathbb{Z})$ of the flag manifold.

The additive structure of $\mathrm{H}^*(Fl_n, \mathbb{Z})$ can be obtained from the decomposition of Fl_n into Schubert cells, which are indexed by the elements of the symmetric

[1] One apparent difference is our extensive use of nilHecke rings.

group S_n. According to classical Ehresmann's result [15], the Schubert classes σ_w, $w \in S_n$, corresponding to these cells, form an additive $\mathbb{Z}$-basis of the cohomology ring of Fl_n.

The multiplicative structure of $\mathrm{H}^*(Fl_n, \mathbb{Z})$ can be recovered from Borel's theorem [7], which says that the cohomology of Fl_n is isomorphic, as a graded ring, to the quotient of the polynomial ring:

$$\mathrm{H}^*(Fl_n, \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, \dots, x_n] \,/\, \langle e_1^n, e_2^n, \dots, e_n^n \rangle, \tag{2.1.1}$$

where e_i^n is the i-th elementary symmetric polynomial in $x_1, \dots, x_n$ and $\langle e_1^n, \dots, e_n^n \rangle$ denotes the ideal generated by the e_i^n. (The somewhat unusual notation with upper indices will be handy when we use elementary symmetric polynomials in different number of variables.) The isomorphism is given by mapping the generators $x_1, \dots, x_n$ into the first Chern classes of n standard line bundles on Fl_n, which are 2-dimensional cohomology classes.

A way to relate these two approaches to the cohomology ring was found by Bernstein, Gelfand, and Gelfand in [5] and Demazure [14], using divided difference operators. Later, Lascoux and Schützenberger [31] further clarified this theory by introducing Schubert polynomials $\mathfrak{S}_w \in \mathbb{Z}[x_1, \dots, x_n]$, $w \in S_n$, whose images in the quotient (2.1.1) represent the Schubert classes σ_w. An algebraic formalization and extension of Bernstein-Gelfand-Gelfand operators was given by Kostant and Kumar [30], who studied the nilHecke ring.

A quantum version[2] of the story surfaced when mathematicians, motivated by ideas of physicists [60, 56], introduced the quantum cohomology ring $\mathrm{QH}^*(X, \mathbb{Z})$, for a Kähler manifold X (see, e.g., [46, 28, 19] and references therein). This ring is a deformation of the classical cohomology ring, its structure constants are 3-point Gromov-Witten invariants, which count the numbers of certain rational curves and play a role in enumerative algebraic geometry.

As a vector space, the quantum cohomology ring $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ of the flag manifold is essentially the same as the usual cohomology, and can be described via Ehresmann's result. More precisely,

$$\mathrm{QH}^*(Fl_n, \mathbb{Z}) \cong \mathrm{H}^*(Fl_n, \mathbb{Z}) \otimes \mathbb{Z}[q_1, \dots, q_{n-1}].$$

However, the multiplicative structure in $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ is different comparing to that of the classical cohomology and specializes to the latter when $q_1 = \cdots = q_{n-1} = 0$.

A quantum analogue of Borel's theorem was suggested by Givental and Kim [22], and then justified by Kim [27] and Ciocan-Fontanine [12]. Let $E_1^n, E_2^n, \dots, E_n^n \in \mathbb{Z}[x_1, \dots, x_n; q_1, \dots, q_{n-1}]$ be the coefficients of the characteristic polynomial

$$\det(1 + \lambda C_n) = 1 + \sum_{i=1}^{n} E_i^n \lambda^i \tag{2.1.2}$$

[2]The reader should not be confused by our use of the word "quantum." The quantization that we discuss in this chapter does not seem to be related, at least at first glance, to quantum groups.

of the following 3-diagonal matrix

$$C_n = \begin{pmatrix} x_1 & q_1 & 0 & \cdots & 0 & 0 \\ -1 & x_2 & q_2 & \cdots & 0 & 0 \\ 0 & -1 & x_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{n-1} & q_{n-1} \\ 0 & 0 & 0 & \cdots & -1 & x_n \end{pmatrix}. \tag{2.1.3}$$

The E_k^n are certain q-deformations of the elementary symmetric polynomials e_k^n and are equal to them when $q_1 = \cdots = q_{n-1} = 0$.

Givental, Kim, and Ciocan-Fontanine showed that the quantum cohomology ring of the flag manifold is isomorphic, as an algebra over $\mathbb{Z}[q_1, \ldots, q_{n-1}]$, to the quotient

$$\mathrm{QH}^*(Fl_n, \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n; q_1, \ldots, q_{n-1}] \,/\, \langle E_1^n, \ldots, E_n^n \rangle\,. \tag{2.1.4}$$

A natural problem is to find the expansion of the quantum product $\sigma_u * \sigma_w$ of two Schubert classes in the basis of Schubert classes, where "$*$" denotes the multiplication in the quantum cohomology ring. Equivalently, one would like to calculate the Gromov-Witten invariants of the flag manifold. We solve this problem, or at least reduce it to combinatorics. Our construction provides the quantum analogue of the result of Bernstein, Gelfand, and Gelfand and corresponding deformation of Schubert polynomials of Lascoux and Schützenberger. The solution to the above problem is essentially combinatorial, and only relies on a few geometrical properties of the quantum cohomology, which are obvious[3] from its definition.

Let I be a sequence $(i_1, \ldots, i_{n-1})$ such that $0 \le i_k \le k$ for all k. Define standard elementary polynomial e_I and quantum standard elementary polynomial E_I by the formulas

$$e_I = e_{i_1 \ldots i_{n-1}} = e_{i_1}^1 \cdots e_{i_{n-1}}^{n-1}\,, \tag{2.1.5}$$

$$E_I = E_{i_1 \ldots i_{n-1}} = E_{i_1}^1 \cdots E_{i_{n-1}}^{n-1}\,, \tag{2.1.6}$$

where, by convention, $e_0^k = E_0^k = 1$. The cosets of (quantum) standard elementary polynomials form linear bases in the quotient rings (2.1.1) and (2.1.4).

For $w \in S_n$, the Schubert polynomial $\mathfrak{S}_w$ can be uniquely expressed as a linear combination $\sum \alpha_I \, e_I$ of standard elementary polynomials. Define the *quantum Schubert polynomial* $\mathfrak{S}_w^q \in \mathbb{Z}[x_1, \ldots, x_n; q_1, \ldots, q_{n-1}]$ by

$$\mathfrak{S}_w^q = \sum_I \alpha_I \, E_I\,. \tag{2.1.7}$$

Our result on the quantum cohomology of the flag manifolds can now be stated as follows (cf. Theorem 2.3.6).

[3]almost

Theorem 2.1.1 [17, Theorem 1.2] *The image of a quantum Schubert polynomial $\mathfrak{S}^q_w$ in the quotient ring* (2.1.4) *represents the Schubert class σ_w in* $\mathrm{QH}^*(Fl_n, \mathbb{Z})$.

We also prove the quantum version of Monk's formula, which generalizes the classical Monk's result [36]. Let s_{ij} be the element of S_n that transposes i and j, and let $s_i = s_{i\,i+1}$, $i = 1, \ldots, n-1$, be the Coxeter generators of S_n. Let us also denote $q_{ij} = q_i q_{i+1} \cdots q_{j-1}$, for $i < j$.

Theorem 2.1.2 (Quantum Monk's formula) [17, Theorem 1.3] *For $w \in S_n$ and $1 \le k < n$, the quantum product of Schubert classes σ_{s_k} and σ_w is given by*

$$\sigma_{s_k} * \sigma_w = \sum \sigma_{ws_{ab}} + \sum q_{cd}\, \sigma_{ws_{cd}}\,, \tag{2.1.8}$$

where the first sum is over all transpositions s_{ab} such that $a \le k < b$ and $\ell(ws_{ab}) = \ell(w) + 1$ (as in the classical Monk's formula), and the second sum is over all transpositions s_{cd} such that $c \le k < d$ and $\ell(ws_{cd}) = \ell(w) - \ell(s_{cd}) = \ell(w) - 2(d-c) + 1$.

The formula (2.1.8) unambiguously determines the multiplicative structure of the quantum cohomology ring $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ with respect to the basis of Schubert classes, since this ring is generated by the 2-dimensional classes σ_{s_k}.

Our proof of Theorems 2.1.1 and 2.1.2 is based on the study of certain pairwise commuting elements in the nilHecke ring. They allow us to (combinatorially) deform a commutative ring[4] equipped with an action of the nilHecke ring. For instance, the deformation of the cohomology ring of Fl_n is shown to be equal to the corresponding quantum cohomology ring.

Another approach to the (quantum) cohomology ring of the flag manifold was recently highlighted by Fomin and Kirillov in [18], where they studied the ring $\mathcal{E}_n$ generated by the elements $\bar{\tau}_{ij}$, $1 \le i < j \le n$, subject to the relations

$$\begin{aligned}
&\bar{\tau}_{ij}\,\bar{\tau}_{ij} = 0\,, \\
&\bar{\tau}_{ij}\bar{\tau}_{jk} = \bar{\tau}_{jk}\bar{\tau}_{ik} + \bar{\tau}_{ik}\bar{\tau}_{ij}\,, \\
&\bar{\tau}_{jk}\bar{\tau}_{ij} = \bar{\tau}_{ik}\bar{\tau}_{jk} + \bar{\tau}_{ij}\bar{\tau}_{ik}\,, \\
&\bar{\tau}_{ij}\,\bar{\tau}_{kl} = \bar{\tau}_{kl}\,\bar{\tau}_{ij}\,, \quad \text{for distinct } i, j, k, \text{ and } l\,.
\end{aligned} \tag{2.1.9}$$

This ring also contains a family of pairwise commuting "Dunkl" elements defined by

$$\bar{\theta}_i = -\sum_{j<i} \bar{\tau}_{ji} + \sum_{k>i} \bar{\tau}_{ik} \qquad i = 1, 2, \ldots, n\,. \tag{2.1.10}$$

Fomin and Kirillov demonstrated that the subring generated by the $\bar{\theta}_i$ is isomorphic to the cohomology ring of Fl_n. The isomorphism is given by explicitly specifying that the generator x_i of (2.1.1) maps to $\bar{\theta}_i$.

[4]It deformation is also a commutative ring.

A deformation $\mathcal{E}_n^q$ of the ring $\mathcal{E}_n$ was also given in [18], along with the conjecture that its subring generated by the Dunkl elements is isomorphic to the quantum cohomology of the flag manifold. We provide a proof to this statement.

The ring $\mathcal{E}_n^q$ is generated by the elements $\hat{\tau}_{ij}$, $1 \le i < j \le n$, subject to the same relations with the $\bar{\tau}_{ij}$ replaced by the $\hat{\tau}_{ij}$, but instead of the relation (2.1.9) we have

$$\hat{\tau}_{ij}\,\hat{\tau}_{ij} = \begin{cases} q_i & \text{for } j = i+1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.11}$$

Define the elements $\hat{\theta}_i$ in the ring $\mathcal{E}_n^q$ by the same formula (2.1.10) with the $\bar{\tau}_{ij}$ replaced by the $\hat{\tau}_{ij}$. We can now formulate our result as follows.

Theorem 2.1.3 [43, Corollary 3.5] [18, Conjecture 13.4] *The pairwise commuting elements $\hat{\theta}_1, \ldots, \hat{\theta}_n$ generate the subring in $\mathcal{E}_n^q$ isomorphic to the quantum cohomology ring of Fl_n. The isomorphism is given by specifying that the generator x_i of* (2.1.4) *maps to $\hat{\theta}_i$.*

We deduce this theorem from a certain general Pieri-type formula. The latter also implies Pieri's formula for the product in $\mathrm{H}^*(Fl_n, \mathbb{Z})$ of any Schubert class σ_w with the class $\sigma_{c(k,m)}$, where $c(k,m) = s_{m-k+1}s_{m-k+2}\cdots s_m$. This rule was first formulated by Lascoux and Schützenberger [31] and proved geometrically by Sottile [50].

Another corollary is an analogue of Pieri's formula for the quantum cohomology ring that was recently proved by Ciocan-Fontanine [13], using nontrivial algebro-geometric techniques. By contrast, our proof is combinatorial, and does not rely upon geometry at all—once quantum Monk's formula (2.1.8) is established. Our proof seems to be new even in the classical case.

In the rest of Introduction we present the general outline of this chapter. In Section 2.2, we review the necessary background from the theory of classical cohomology of the flag manifold, the nilHecke ring, and Schubert polynomials, together with quantum cohomology definitions. In Section 2.3, we give a combinatorial construction of quantization for a ring equipped with an action of the nilHecke ring. This construction is based on a certain family of maximal commutative subrings in the nilHecke ring. In Section 2.4, we study the standard elementary polynomials and their quantum analogues. In Section 2.5, we define the quantum Schubert polynomials and give a combinatorial proof of their orthogonality property (Theorem 2.5.5). We prove an axiomatic characterization of these polynomials (Theorem 2.5.7), which implies Theorem 2.1.1. We also conjecture there even stronger statement (Conjecture 2.5.8). The proof of quantum Monk's formula, which is given in Section 2.6, becomes now almost tautological. In that section we also prove a general Pieri-type formula (Theorem 2.6.3). As corollaries we obtain Theorem 2.1.3 as well as several other conjectures from [18].

2.2 Background

2.2.1 Flag manifold and Schubert cells

We start with a short review of the basic results [5, 7] on the classical cohomology of the flag manifold. Most of the statements below can be extended to any semisimple Lie group.

Let Fl_n be the *flag manifold* whose points are the complete flags of subspaces

$$U_\cdot = (U_1 \subset U_2 \subset \cdots \subset U_n = \mathbb{C}^n)\ , \quad \dim U_i = i\ , \tag{2.2.1}$$

in the n-dimensional linear space $\mathbb{C}^n$. This is a projective algebraic variety.

A description of the additive structure of the cohomology ring $\mathrm{H}^*(Fl_n\,, \mathbb{Z})$ is based on a decomposition of Fl_n into even-dimensional cells indexed by the elements of the symmetric group S_n and called Schubert cells. These cells are described in terms of a relative position of a flag $U_\cdot$ with respect to a fixed reference flag $V_\cdot \in Fl_n$, as follows.

Let $v_1, \ldots, v_n$ be a basis in $\mathbb{C}^n$, and let V_r denote the r-dimensional subspace spanned by $v_1, v_2, \ldots, v_r$. For $w \in S_n$, define the *Schubert cell* Ω_w^o as the set of all flags $U_\cdot \in Fl_n$ such that, for all k, $r \in \{1, \ldots, n\}$,

$$\dim(U_k \cap V_r) = \#\{1 \le i \le k\,,\, n+1-w(i) \le r\}\,.$$

The cell Ω_w^o is homeomorphic to $\mathbb{R}^{n(n-1)-2l}$, where $l = \ell(w)$ is the length of w (the number of inversions). The collection of all Ω_w^o form a cell decomposition of Fl_n. The *Schubert variety* Ω_w is the closure of Ω_w^o in Zariski topology. Let $[\Omega_w] \in \mathrm{H}_{n(n-1)-2l}(Fl_n\,, \mathbb{Z})$ be the fundamental cycle of Ω_w . Define the *Schubert class*

$$\sigma_w = [\Omega_w]^* \in \mathrm{H}^{2l}(Fl_n\,, \mathbb{Z})$$

as the cohomology class corresponding to the fundamental cycle $[\Omega_w]$ under the natural isomorphism $\mathrm{H}_{n(n-1)-2l}(Fl_n\,, \mathbb{Z}) \cong \mathrm{H}^{2l}(Fl_n\,, \mathbb{Z})$. The following result of C. Ehresmann [15] is classical.

Theorem 2.2.1 *The Schubert classes σ_w, $w \in S_n$, form an additive basis of the cohomology $\mathrm{H}^*(Fl_n, \mathbb{Z})$ of the flag manifold. Thus the rank of $\mathrm{H}^*(Fl_n, \mathbb{Z})$ is $n!$.*

The manifold Fl_n is equipped with the flag of tautological vector bundles $0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_{n-1} \subset \mathcal{T}_n$; the fiber of $\mathcal{T}_i$ at the point (2.2.1) is the subspace U_i . Consider the ring homomorphism

$$p\,:\,\mathbb{Z}[x_1, \ldots, x_n] \longrightarrow \mathrm{H}^*(Fl_n, \mathbb{Z}) \tag{2.2.2}$$

given by $p(x_i) = -c_1(\mathcal{T}_i/\mathcal{T}_{i-1})$, where $c_1(\mathcal{T}_i/\mathcal{T}_{i-1}) \in H^2(Fl_n\,, \mathbb{Z})$, $i = 1, \ldots, n$, is the first Chern class of the line bundle $\mathcal{T}_i/\mathcal{T}_{i-1}$. Let $\mathcal{J}_n = \langle e_1^n, e_2^n, \ldots, e_n^n\rangle$ be the ideal in $\mathbb{Z}[x_1, \ldots, x_n]$ generated by the elementary symmetric polynomials $e_i^n = e_i(x_1, \ldots, x_n)$. Equivalently, $\mathcal{J}_n$ is generated by all symmetric polynomials without constant term. The following classical result is due to A. Borel [7].

Theorem 2.2.2 *The map p is epimorphism. The kernel of p is the ideal $\mathcal{J}_n$. Thus the map p induces the isomorphism of graded rings*

$$\mathbb{Z}[x_1, \ldots, x_n]/\mathcal{J}_n \cong \mathrm{H}^*(Fl_n\,, \mathbb{Z}). \tag{2.2.3}$$

In particular, $\mathrm{H}^2(Fl_n, \mathbb{Z})$ is spanned by the classes $p(x_i) = \sigma_{s_i} - \sigma_{s_{i-1}}$, $i = 1, \ldots, n$, where, by convention, $\sigma_{s_0} = 0$. There is an explicit rule for multiplying any Schubert class by a 2-dimensional class σ_k.

Theorem 2.2.3 (Monk's formula [36]; cf. also Chevalley [10]) *We have, for any $w \in S_n$ and $1 \leq k < n$,*

$$\sigma_{s_k}\, \sigma_w = \sum \sigma_{w s_{ij}}\ ,$$

where the sum is over all transpositions s_{ij} such that $i \leq r < j$ and $\ell(w s_{ij}) = \ell(w) + 1$.

2.2.2 NilHecke ring and Schubert polynomials

Bernstein, Gelfand, and Gelfand [5] and Demazure [14] suggested a procedure, based on divided difference recurrences, that can be used to compute the elements of the quotient ring $\mathbb{Z}[x_1, \ldots, x_n]/\mathcal{J}_n$ which correspond to the Schubert classes. Even more explicit combinatorial representatives called the Schubert polynomials were then discovered by Lascoux and Schützenberger [31]. In this section, we review the main definitions and basic facts of this theory. For more details see, e.g., Macdonald [35].

In the symmetric group S_n, let s_i denote the adjacent transposition (a Coxeter generator) that interchanges i and $i+1$. For a permutation $w \in S_n$, an expression $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ of minimal possible length is called a *reduced decomposition* of w, and $l = \ell(w)$ is the *length* of w. For example, the transposition s_{ij}, $i < j$, that interchanges i and j has a reduced decomposition $s_{ij} = s_i s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \ldots s_i$.

The *nilHecke ring* $\mathcal{NH}_n$ (see [30] for a general definition) is the ring with 1 generated by pairwise commuting elements $\chi_1, \chi_2, \ldots, \chi_n$ and the elements $\partial_1, \partial_2, \ldots, \partial_{n-1}$ satisfying the following nilCoxeter relations:

$$\begin{aligned} &\partial_i\, \partial_j = \partial_j\, \partial_i \ \text{ for } |i-j| > 1\,, \\ &\partial_i\, \partial_{i+1}\, \partial_i = \partial_{i+1}\, \partial_i\, \partial_{i+1}\,, \\ &\partial_i^2 = 0\,, \end{aligned} \tag{2.2.4}$$

and also the relations involving both sets of elements:

$$\begin{aligned} &\partial_i\, \chi_j = \chi_j\, \partial_i \ \text{ for } j \neq i,\, i+1\,, \\ &\partial_i\, \chi_i = \chi_{i+1}\, \partial_i + 1\,, \\ &\partial_i\, \chi_{i+1} = \chi_i\, \partial_i - 1\,. \end{aligned} \tag{2.2.5}$$

For a permutation w, define the element $\partial_w \in \mathcal{NH}_n$ by $\partial_w = \partial_{i_1} \cdots \partial_{i_l}$, where $s_{i_1} \cdots s_{i_l}$ is a reduced decomposition for w. It follows from relations (2.2.4) that ∂_w

does not depend on the choice of such reduced decomposition. Moreover, for any two permutations v and w

$$\partial_v \partial_w = \begin{cases} \partial_{vw} & \text{if } \ell(vw) = \ell(v) + \ell(w)\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.6}$$

Clearly the polynomial ring $\mathbb{Z}[\chi_1, \ldots, \chi_n]$ is a subring of $\mathcal{NH}_n$. Every element h in $\mathcal{NH}_n$ can be uniquely written as the sum $h = \sum_w f_w(\chi)\, \partial_w$, where the f_w are some polynomials in the χ_i. Analogously, it can be uniquely expressed as $h = \sum_w \partial_w\, g_w(\chi)$, with $g_w \in \mathbb{Z}[\chi_1, \ldots, \chi_n]$.

The symmetric group acts on polynomials $f = f(x_1, \ldots, x_n)$ by permuting the variables x_i. Explicitly, $w f = f(x_{w^{-1}(1)}, \ldots, x_{w^{-1}(n)})$, for $w \in S_n$. In the same fashion S_n acts on $\mathbb{Z}[\chi_1, \ldots, \chi_n]$.

The nilHecke ring also acts on the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ as follows. The element χ_i acts as the operator of multiplication[5] by x_i. The action of the element ∂_i is given by the *divided difference operator:*

$$\partial_i \cdot f = (x_i - x_{i+1})^{-1}(1 - s_i)\, f\,. \tag{2.2.7}$$

One easily checks that $\mathbb{Z}[x_1, \ldots, x_n]$ is invariant under ∂_k and that these operators satisfy the nilCoxeter relations (2.2.4). The operators corresponding to the elements ∂_w are also called the divided difference operators. The reader should not confuse $\partial_w\, f$, which stands for the product of ∂_w and f in $\mathcal{NH}_n$, with $\partial_w \cdot f$, the latter always denotes the result of applying the operator ∂_w to f. Note that the action of $\mathcal{NH}_n$ on the polynomial ring is exact.

The following "Leibniz formulas" hold the nilHecke ring $\mathcal{NH}_n$ (cf. [35, (2.2), 2.13]).

Proposition 2.2.4 • *For any polynomial $f \in \mathbb{Z}[\chi_1, \ldots, \chi_n] \subset \mathcal{NH}_n$ and any i,*

$$\partial_i\, f = \partial_i \cdot f(x_1, \ldots, x_n) + (s_i\, f)\, \partial_i\,, \tag{2.2.8}$$

where $f(x_1, \ldots, x_n) = f \cdot 1$ is the result of substituting the x_i in place of the χ_i in f. In particular, ∂_i commutes with any polynomial which is symmetric in χ_i and χ_{i+1}.

• *For a linear form $f = \sum \lambda_i \chi_i \in \mathcal{NH}_n$, we have*

$$\partial_w\, f = (wf)\, \partial_w + \sum (\lambda_i - \lambda_j)\, \partial_{ws_{ij}}\,, \tag{2.2.9}$$

where the sum is over all $i < j$ such that $\ell(ws_{ij}) = \ell(w) - 1$.

Let $\delta = \delta_n = (n-1, n-2, \ldots, 1, 0)$ and $x^\delta = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$. For each permutation $w \in S_n$, the *Schubert polynomial* $\mathfrak{S}_w \in \mathbb{Z}[x_1, \ldots, x_n]$ of Lascoux and

[5] Notational remark. To avoid confusion, we decided to use two different sets of variables: the x_i — the generators of the polynomial ring—and the $\chi_i \in \mathcal{NH}_n$, which act on the polynomial ring by multiplication by the x_i (cf. also the elements $\mathcal{X}_i$ defined in Section 2.3).

Schützenberger is defined by applying the divided difference operator to x^δ:

$$\mathfrak{S}_w = \partial_{w^{-1}w_o} \cdot x^\delta ,$$

where w_o is the longest element in S_n, given by $w_o(i) = n+1-i$. Equivalently,

$$\mathfrak{S}_{w_o} = x^\delta \quad \text{and} \quad \mathfrak{S}_{ws_i} = \partial_i \cdot \mathfrak{S}_w \quad \text{whenever} \quad \ell(ws_i) = \ell(w) - 1 . \tag{2.2.10}$$

More generally, for $v, w \in S_n$,

$$\partial_v \cdot \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{wv^{-1}} & \text{if } \ell(wv^{-1}) = \ell(w) - \ell(v) , \\ 0 & \text{otherwise} . \end{cases} \tag{2.2.11}$$

The following fundamental result is an immediate corollary of [5] (cf. also [14]).

Theorem 2.2.5 *The Schubert polynomials represent Schubert classes under the isomorphism* (2.2.3), *i.e.,* $p(\mathfrak{S}_w) = \sigma_w$.

The Schubert polynomials have the following orthogonality property (see, e.g., [35, (5.4)]). For a polynomial f, define

$$\langle f \rangle = (\partial_{w_o} \cdot f)(0, \dots, 0) . \tag{2.2.12}$$

Theorem 2.2.6 *For* $u, v \in S_n$ *,*

$$\langle \mathfrak{S}_u \, \mathfrak{S}_v \rangle = \begin{cases} 1 & \text{if } v = w_o u , \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.13}$$

2.2.3 Quantum cohomology and Gromov-Witten invariants

As an abelian group, the (small) quantum cohomology of the flag manifold Fl_n is nothing more than the usual cohomology tensored with a polynomial ring:

$$\mathrm{QH}^*(Fl_n, \mathbb{Z}) = \mathrm{H}^*(Fl_n, \mathbb{Z}) \otimes \mathbb{Z}[q_1, \dots, q_{n-1}] . \tag{2.2.14}$$

Everywhere in this chapter the letter q will denote the collection of $q_1, q_2, \dots, q_{n-1}$; and $\mathbb{Z}[q]$ will always stand for the ring $\mathbb{Z}[q_1, \dots, q_{n-1}]$. Likewise, x will abbreviate the collection of the x_i.

The multiplication in $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ is a $\mathbb{Z}[q]$-linear operation that is defined by specifying its structure constants. These can be expressed via Gromov-Witten invariants, to whose geometrical definition we now proceed, see [2, 6, 12, 16, 19, 22, 25, 26, 28, 29, 46] for details.

The homology classes $[\Omega_{w_o s_i}]$, $i = 1, \dots, n-1$, of two-dimensional Schubert varieties form a linear basis in $\mathrm{H}_2(Fl_n, \mathbb{Z})$. We say that an algebraic map $f : \mathbb{P}^1 \to Fl_n$ (or a rational curve in Fl_n) has *multidegree* $d = (d_1, \dots, d_{n-1})$ if $f_*[\mathbb{P}^1] = \sum d_i [\Omega_{w_o s_i}]$.

The d_i should be nonnegative integer numbers. The *moduli space* $\mathcal{M}_d(\mathbb{P}^1, Fl_n)$ of such maps is a smooth algebraic variety of dimension

$$D = \binom{n}{2} + 2\sum_{i=1}^{n-1} d_i \,. \tag{2.2.15}$$

For a subvariety $Y \subset Fl_n$ and a point $t \in \mathbb{P}^1$, let us denote

$$Y(t) = \{f \in \mathcal{M}_d(\mathbb{P}^1, Fl_n) \mid f(t) \in Y\}. \tag{2.2.16}$$

The codimension of $Y(t)$ in $\mathcal{M}_d(\mathbb{P}^1, Fl_n)$ equals the codimension of Y in Fl_n.

Let $w_1, \ldots, w_N \in S_n$. The *Gromov-Witten invariant*[6] associated to the classes $\sigma_{w_1}, \ldots, \sigma_{w_N}$ is defined as follows. Let $g_1, \ldots, g_N$ be generic elements of GL_n, and let $t_1, \ldots, t_N$ be distinct points in $\mathbb{P}^1$. Define

$$\langle \sigma_{w_1}, \ldots, \sigma_{w_N} \rangle_d = \begin{cases} \#\text{ of points in } \bigcap (g_i \Omega_{w_i})(t_i) & \text{if } \sum \ell(w_i) = D, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.17}$$

The condition $\sum \ell(w_i) = D$ ensures that this cardinality is finite. These invariants independent of the choice of points $t_i \in \mathbb{P}^1$ and generic linear transformations g_i.

In other words, the invariant $\langle \sigma_{w_1}, \ldots, \sigma_{w_N} \rangle_d$ is the number of of rational curves in Fl_n which have multidegree $d = (d_1, \ldots, d_{n-1})$ and pass through some general translates of Schubert varieties $\Omega_{w_1}, \ldots, \Omega_{w_N}$.

Only 3-point Gromov-Witten invariants (for $N = 3$) are needed to define the quantum product. The *(geometrical) quantum multiplication* in the space (2.2.14) is the $\mathbb{Z}[q]$-linear operation $*$ given in the basis of Schubert classes by

$$\sigma_u * \sigma_v = \sum_{w \in S_n} \sum_d q^d \, \langle \sigma_u, \sigma_v, \sigma_w \rangle_d \, \sigma_{w_o w} \,, \tag{2.2.18}$$

for any permutations u and v, where $d = (d_1, d_2, \ldots, d_{n-1})$, and $q^d = q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$. This operation is commutative and, remarkably, associative (see [46, 33]).

By definition, the *quantum cohomology ring* $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ of the flag manifold is the linear space (2.2.14) equipped with the operation $*$ of quantum multiplication, as defined above.

The Gromov-Witten invariants $\langle \sigma_u, \sigma_v, \sigma_w \rangle_{(0,0,0)}$ are the usual intersection numbers of Schubert varieties, thus the quotient of $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ modulo the ideal generated by the q_i is the ordinary cohomology ring (or, equivalently, the Chow ring) of Fl_n.

It can also be shown that the quantum product of several classes is expressed through Gromov-Witten invariants as follows. For any $w_1, w_2, \ldots, w_m \in S_n$,

$$\sigma_{w_1} * \cdots * \sigma_{w_m} = \sum_{w \in S_n} \sum_d q^d \, \langle \sigma_{w_1}, \ldots, \sigma_{w_m}, \sigma_w \rangle_d \; \sigma_{w_o w} \;, \tag{2.2.19}$$

[6]This is a special case of a more general Gromov-Witten mixed invariant defined in [46].

where, as before, $q^d = q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$ for $d = (d_1, \ldots, d_{n-1})$. Therefore, any Gromov-Witten invariant can be expressed via 3-point invariants using the associativity condition.

The following description of the quantum cohomology ring of the flag manifold was suggested by Givental and Kim [22], and then justified by Kim [25, 26] and Ciocan-Fontanine [12]. Let $\mathcal{J}_n^q$ be the ideal in the ring $\mathbb{Z}[q][x_1, \ldots, x_n]$ that is generated over $\mathbb{Z}[q]$ by the coefficients $E_1^n, \ldots, E_n^n$ of the characteristic polynomial of the matrix C_n given by (2.1.3).

Theorem 2.2.7 [22, 25, 26, 12] *The quotient $\mathbb{Z}[q][x_1, \ldots, x_n]/\mathcal{J}_n^q$ is isomorphic, as an algebra over $\mathbb{Z}[q]$, to the quantum cohomology ring $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ of the flag manifold. The isomorphism is determined by specifying that the polynomial $x_1 + x_2 + \cdots + x_i$ maps into the Schubert class σ_{s_i}, for $i = 1, \ldots, n$.*

2.3 Combinatorial Quantum Multiplication

In this section we give a combinatorial construction of quantization. First, we describe a certain commutative subring in the nilHecke ring. Then, using this subring, we show how to quantize a ring equipped with an action of $\mathcal{NH}_n$.

2.3.1 Commuting elements in the nilHecke ring

Let us recall that $\mathbb{Z}[q] = \mathbb{Z}[q_1, \ldots, q_{n-1}]$. Let $\mathcal{NH}_n^q = \mathcal{NH}_n \otimes \mathbb{Z}[q]$ be the nilHecke ring with coefficients in $\mathbb{Z}[q]$. It will be convenient to denote $q_{ij} = q_i q_{i+1} \cdots q_{j-1}$ for $i < j$. Let $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ be the elements of the nilHecke ring $\mathcal{NH}_n^q$ given by[7]

$$\mathcal{X}_k = \chi_k - \sum_{1 \le i < k} q_{ik}\, \partial_{(ik)} + \sum_{k < j \le n} q_{kj}\, \partial_{(kj)}\,, \qquad (2.3.1)$$

where $\partial_{(ij)} = \partial_{s_{ij}} = \partial_i \partial_{i+1} \cdots \partial_{j-2} \partial_{j-1} \partial_{j-2} \cdots \partial_i$ is the element of $\mathcal{NH}_n^q$ that corresponds to the transposition s_{ij}.

Notice that the $\mathcal{X}_i$ are homogeneous degree 1 elements in $\mathcal{NH}_n^q$ assuming that $\deg(\chi_i) = 1$, $\deg(\partial_i) = -1$, and $\deg(q_j) = 2$.

The following statement is essentially our Theorem 5.1 from [17], it is also related to Lemma 2.6.2 given in this chapter below.

---

[7] These elements and the theory developed below in this section can be extended to any semisimple Lie algebra (and, probably, any Kac-Moody algebra). In the associated nilHecke ring there is a family of commutative subrings generated by the elements

$$f + \sum q^\alpha \, \langle f, \alpha^\vee \rangle \, \partial_{s_\alpha}\,,$$

where $f \in \mathfrak{h}^*$ is an element of dual Cartan subalgebra, the sum is over positive roots $\alpha \in \Phi_+$ such that $\ell(s_\alpha) = 2|\alpha| - 1$, and $q^\alpha = q_1^{c_1} \cdots q_l^{c_l}$ for $\alpha = c_1\alpha_1 + \cdots + c_l\alpha_l$. Note that ∂_{s_α} is the product of $2|\alpha| - 1$ generators corresponding to a reduced decomposition of the reflection s_α.

Theorem 2.3.1 *The elements $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ in the nilHecke ring $\mathcal{NH}_n^q$ commute pairwise. They are algebraically independent over $\mathbb{Z}[q]$.*

To prove this result, we need the following lemma, in which $[x, y] = xy - yx$ is the usual commutator.

Lemma 2.3.2 *The following commutation relations hold in $\mathcal{NH}_n$.*

1. $[\partial_{(ac)}, \chi_b] = 0$ *unless* $a \leq b \leq c$.
2. $[\partial_{(ab)}, \chi_a + \chi_{a+1} + \cdots + \chi_b] = 0$.
3. $[\partial_{(ab)}, \partial_{(cd)}] = 0$ *unless* $b = c$ *or* $a = d$.
4. *For* $a < b < c$, *we have* $[\partial_{(ac)}, \chi_b] + [\partial_{(ab)}, \partial_{(bc)}] = 0$.

Proof — 1. The element χ_i commutes with ∂_j unless $j = i$ or $j = i - 1$.
2. Follows from $\chi_a + \cdots + \chi_b$ being a symmetric polynomial of $\chi_a, \ldots, \chi_b$.
3. Clearly, $[\partial_{(ab)}, \partial_{(cd)}] = 0$ unless $a \leq c < b \leq d$ or $c \leq a < d \leq b$. In the latter case $\ell(s_{ab}s_{cd}) < \ell(s_{ab}) + \ell(s_{cd})$ and thus $\partial_{(ab)}\partial_{(cd)} = \partial_{(cd)}\partial_{(ab)} = 0$ by (2.2.6).
4. From the "Leibniz formula" (2.2.9) with $w = s_{ac}$, we obtain

$$\partial_{(ac)}\, \chi_b = \chi_b\, \partial_{(ac)} - \partial_{s_{ac}s_{ab}} + \partial_{s_{ac}s_{bc}}\,,$$

which is equivalent to the claim. $\square$

Proof of Theorem 2.3.1 — By (2.3.1) and Lemma 2.3.2, we have, for $a < b$:

$$\begin{aligned}
[\mathcal{X}_a, \mathcal{X}_b] &= [\chi_a, \mathcal{X}_b - \chi_b] + [\mathcal{X}_a - \chi_a, \chi_b] + [\mathcal{X}_a - \chi_a, \mathcal{X}_b - \chi_b] \\
&= \Big[\chi_a, -\sum_{i \leq a} q_{ib}\, \partial_{(ib)}\Big] + \Big[\sum_{j \geq b} q_{aj}\, \partial_{(aj)}, \chi_b\Big] \\
&\quad + \sum_{i<a} q_{ib}\, [\partial_{(ia)}, \partial_{(ab)}] + \sum_{j>b} q_{aj}\, [\partial_{(ab)}, \partial_{(bj)}] - \sum_{a<i<b} q_{ab}\, [\partial_{(ai)}, \partial_{(ib)}] \\
&= -q_{ab}\, [\chi_a, \partial_{(ab)}] + q_{ab}\, [\partial_{(ab)}, \chi_b] + q_{ab} \sum_{a<i<b} [\partial_{(ab)}, \chi_i] \\
&= q_{ab}\, [\partial_{(ab)}, \chi_a + \chi_{a+1} + \cdots + \chi_b] = 0\,,
\end{aligned}$$

as desired.

The nilHecke ring $\mathcal{NH}_n^q$ and thus the elements $\mathcal{X}_1, \ldots, \mathcal{X}_n$, act on the polynomial ring $\mathbb{Z}[q][x_1, \ldots, x_n]$ via divided difference operators. Since the element $\mathcal{X}_i$ is equivalent, modulo the ideal generated by the q_j, to the element χ_i in the nilHecke ring, we have $\mathbb{Z}[q][\mathcal{X}_1, \ldots, \mathcal{X}_n] \cdot 1 = \mathbb{Z}[q][x_1, \ldots, x_n]$. The dimension argument shows that the $\mathcal{X}_i$ are algebraically independent over $\mathbb{Z}[q]$. $\square$

Theorem 2.3.1 implies that the elements $\mathcal{X}_i$ generate a commutative subring[8] $\mathbb{Z}[q][\mathcal{X}] = \mathbb{Z}[q][\mathcal{X}_1, \ldots, \mathcal{X}_n]$ in $\mathcal{NH}_n^q$ isomorphic to the polynomial ring in n variables with coefficients in $\mathbb{Z}[q]$.

2.3.2 Combinatorial quantization

Let R be a module over the nilHecke ring $\mathcal{NH}_n$ with an element v such that $\partial_i \cdot v = 0$ for all i and $\mathcal{NH}_n \cdot v = R$. The polynomial ring $\mathbb{Z}[\chi] = \mathbb{Z}[\chi_1, \ldots, \chi_n] \subset \mathcal{NH}_n$ then acts on R, and $\mathbb{Z}[\chi] \cdot v = R$, due to (2.2.5). The module R is a quotient of $\mathbb{Z}[\chi]$ and, thus, is endowed with a ring structure. In this ring $v = 1$, the identity element.

Equivalently, one can define R as the quotient ring $\mathbb{Z}[x_1, \ldots, x_n]/\mathcal{I}$, where $\mathcal{I}$ is an ideal invariant under divided difference operators given by (2.2.7). It can be shown that every such ideal $\mathcal{I}$ is generated by a sequence of symmetric polynomials. The nilHecke ring then acts on R; the ∂_i act by divided differences and the element χ_j acts as the operator of multiplication by x_j. By a slight abuse of notation, the x_j denote both generators of the polynomial ring and their cosets in R. Two basic examples are the polynomial ring $R = \mathbb{Z}[x_1, \ldots, x_n]$ and $R = \mathbb{Z}[x_1, \ldots, x_n]/\mathcal{J}_n$, the quotient (2.2.3).

Given this data, we construct a quantum deformation R^q of the ring R as follows. As a linear space, R^q is the tensor product $R \otimes \mathbb{Z}[q]$. The subring $\mathbb{Z}[q][\mathcal{X}]$ in the nilHecke ring $\mathcal{NH}_n^q$ acts on R^q by $\mathbb{Z}[q]$-linear transformations and $\mathbb{Z}[q][\mathcal{X}] \cdot 1 = R^q$. The linear space R^q is then isomorphic to a quotient of $\mathbb{Z}[q][\mathcal{X}]$ and, thus, inherits its multiplicative structure. We will denote this new product on R^q by $\bar{*}$.

We have actually proved the following statement.

Proposition 2.3.3 *There is a unique $\mathbb{Z}[q]$-linear associative operation $\bar{*}$ on $R^q = R \otimes \mathbb{Z}[q]$ such that, for any generator x_i and any $g \in R^q$,*

$$x_i \,\bar{*}\, g = \mathcal{X}_i \cdot g\,.$$

Moreover, the operation $\bar{*}$ is commutative.

Definition 2.3.4 The operation $\bar{*}$ that satisfies the conditions of the proposition above is called the *combinatorial quantum multiplication* (as opposed to the operation $*$ defined geometrically in Section 2.2.3). This operation makes the space $R^q = R \otimes \mathbb{Z}[q]$ into a commutative and associative ring called *combinatorial quantum deformation* of the ring R. The tautological map

$$R \otimes \mathbb{Z}[q] \longrightarrow R^q \tag{2.3.2}$$

is called the *quantization map.* This map is an isomorphism of $\mathbb{Z}[q]$-modules (but by no means a homomorphism of rings).

It is clear that the quotient of the ring R^q modulo the ideal generated by the q_i coincides with R.

[8] In fact, it is a maximal commutative subring in the nilHecke ring.

The combinatorial quantum deformation of the polynomial ring $\mathbb{Z}[x_1,\ldots,x_n]$ is isomorphic to the polynomial ring over $\mathbb{Z}[q]$. The quantization in this case is the map μ that maps a polynomial $f = f(x_1,\ldots,x_n)$ to the polynomial $\mu(f) = F(x_1,\ldots,x_n)$ such that

$$F(\mathcal{X}_1,\ldots,\mathcal{X}_n)\cdot 1 = f(x_1,\ldots,x_n)\,.$$

The $\bar{*}$-product of several generators is given by

$$x_{i_1}\,\bar{*}\,x_{i_2}\,\bar{*}\cdots\bar{*}\,x_{i_N} = \mathcal{X}_{i_1}\,\mathcal{X}_{i_2}\cdots\mathcal{X}_{i_N}\cdot 1.$$

For example,

$$x_1\,\bar{*}\,x_1 = x_1^2 + q_1\,,$$

$$x_1\,\bar{*}\,x_2 = x_2\,\bar{*}\,x_1 = x_1\,x_2 - q_1\,,$$

$$x_1\,\bar{*}\,x_1\,\bar{*}\,x_1 = x_1^3 + 2q_1x_1 + q_1x_2\,.$$

This implies the following formulas for the quantization map:

$$\mu(x_1^2) = x_1^2 - q_1\,,$$

$$\mu(x_1x_2) = x_1x_2 + q_1\,,$$

$$\mu(x_1^3) = x_1^3 - 2q_1x_1 - q_1x_2\,.$$

Recall that R is the quotient ring modulo an ideal $\mathcal{I}\subset\mathbb{Z}[x_1,\ldots,x_n]$ generated by a sequence of symmetric polynomials. (By Hilbert's basis theorem, it is always possible to find a finite sequence of symmetric generators.)

Proposition 2.3.5 *Suppose that $R = \mathbb{Z}[x_1,\ldots,x_n]/\mathcal{I}$ is the quotient ring modulo the ideal $\mathcal{I} = \langle f_1, f_2,\ldots\rangle$ generated by a sequence of symmetric polynomials f_i. The combinatorial quantum deformation of R is the ring $R^q = \mathbb{Z}[q][x_1,\ldots,x_n]/\mathcal{I}^q$, where $\mathcal{I}^q = \langle F_1, F_2,\ldots\rangle$ is the ideal generated by quantizations $F_i = \mu(f_i)$ of the f_i.*

Proof — Clearly R^q is the polynomial ring modulo the ideal $\mathcal{I}^q$ such that $F\in\mathcal{I}^q$ if and only if $F(\mathcal{X}_1,\ldots,\mathcal{X}_n)\cdot 1\in\mathcal{I}$. All polynomials $F_i = \mu(f_i)$ are in $\mathcal{I}^q$. The dimension argument shows that $\mathcal{I}^q = \langle F_1, F_2,\ldots\rangle$. $\square$

This proposition shows that $\mu(\mathcal{I}) = \mathcal{I}^q$. The quantization (2.3.2) can be described as the map that maps the coset of a polynomial f modulo the ideal $\mathcal{I}$ to the coset of the polynomial $\mu(f)$ modulo the ideal $\mathcal{I}^q$.

The most important for our purposes example is the cohomology ring of the flag manifold: $R = \mathrm{H}^*(Fl_n,\mathbb{Z})$. By Borel's result (2.2.3), it is isomorphic to the quotient $\mathbb{Z}[x_1,\ldots,x_n]/\mathcal{J}_n$. The nilHecke ring acts on R via divided differences. The quantization R^q of R is the quantum cohomology ring $\mathrm{QH}^*(Fl_n,\mathbb{Z})$ of Fl_n, as defined in Section 2.2.3, due to the following statement.

Theorem 2.3.6 *The operation $\bar{*}$ of combinatorial quantum multiplication on the space $\mathbb{Z}[q]\otimes\mathrm{H}^*(Fl_n,\mathbb{Z})$ coincides with the operation $*$ of geometrical quantum multiplication, as defined in Section 2.2.3.*

As we will see, this theorem is is essentially a reformulation of Theorem 2.1.1 from Introduction. Until we prove these theorems in Section 2.5.3, we will distinguish the geometric and combinatorial quantum multiplications.

The construction of the combinatorial quantum multiplication and the quantization map can be easily carried out for the polynomial ring $R = \mathbb{Z}[x_1, x_2, \dots]$ in infinitely many variables. Let us denote by $\hat{\mathcal{X}}_k$ the analogue of the element $\mathcal{X}_k$ in the infinite case. Explicitly,

$$\hat{\mathcal{X}}_k = \chi_k - \sum_{i=1}^{k-1} q_{ik}\, \partial_{(ik)} + \sum_{j=k+1}^{\infty} q_{kj}\, \partial_{(kj)}\,. \tag{2.3.3}$$

The $\hat{\mathcal{X}}_k$ involve infinite sums, but only finitely many terms survive in $\hat{\mathcal{X}}_k \cdot f$. Thus the action of the $\hat{\mathcal{X}}_k$ on the polynomial ring $R^q = \mathbb{Z}[q_1, q_2, \dots][x_1, x_2, \dots]$ is well-defined. This action allows us to define the combinatorial quantum multiplication $\bar{*}$ on R^q. The quantization map $\mu : R \otimes \mathbb{Z}[q_1, q_2, \dots] \to R^q$ is now given by

$$\mu : f \longmapsto F, \quad F(\hat{\mathcal{X}}_1, \hat{\mathcal{X}}_2, \dots) \cdot 1 = f(x_1, x_2, \dots)\,. \tag{2.3.4}$$

2.4 Standard Elementary Polynomials

In this section we give another description of the quantization map. First, the case of the polynomial ring in infinitely many variables is considered. Then we specialize results to finitely generated rings.

2.4.1 Straightening

Let $e_i^k = e_i(x_1, \dots, x_k)$ be the i-th *elementary symmetric polynomial* of $x_1, \dots, x_k$:

$$e_i^k = \sum_{1 \le j_1 < j_2 < \cdots < j_i \le k} x_{j_1} x_{j_2} \cdots x_{j_i}.$$

By convention, $e_0^k = 1$ for $k \ge 0$, and $e_i^k = 0$ unless $0 \le i \le k$.

The polynomials e_i^k generate the polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$ in infinitely many variables because $x_k = e_1^k - e_1^{k-1}$. They satisfy the following obvious recurrence:

$$e_i^k = e_i^{k-1} + x_k e_{i-1}^{k-1}. \tag{2.4.1}$$

Lemma 2.4.1 *For $i, j, k \ge 1$, the following relations hold:*

$$(e_i^{k+1} - e_i^k)\, e_{j-1}^k = (e_j^{k+1} - e_j^k)\, e_{i-1}^k\,, \tag{2.4.2}$$

$$e_i^k e_j^k = e_i^{k+1} e_j^k + \sum_{l \ge 1} e_{i-l}^{k+1} e_{j+l}^k - \sum_{l \ge 1} e_{i-l}^k e_{j+l}^{k+1}\,. \tag{2.4.3}$$

Proof — By (2.4.1), we have $(e_i^{k+1} - e_i^k)\, e_{j-1}^k = x_{k+1} e_{i-1}^k e_{j-1}^k = (e_j^{k+1} - e_j^k)\, e_{i-1}^k$. Equation (2.4.3) follows from (2.4.2). □

Lemma 2.4.2 *For $i \geq 0$ and $k, l \geq 1$ we have $\partial_l \cdot e_i^k = \delta_{kl}\, e_{i-1}^{k-1}$, where δ_{kl} is the Kronecker delta. In particular, ∂_l commutes with the multiplication by e_i^k if $k \neq l$.*

Proof — If $k \neq l$, then $\partial_l \cdot e_i^k = 0$ because e_i^k is invariant under interchanging x_k and x_{k+1}. For $k = l$ is it easy to check that $\partial_k \cdot e_i^k = e_{i-1}^{k-1}$. The second statement then follows by (2.2.8). □

For $I = (i_1, \ldots, i_m)$ such that $0 \leq i_k \leq k$, let

$$e_I = e_{i_1 \ldots i_m} = e_{i_1}^1 \cdots e_{i_m}^m \,. \tag{2.4.4}$$

We will call e_I a *standard elementary polynomial.* (These are the polynomials P_I of [31].) In other words, a standard elementary polynomial is any product of the e_i^k without repetitions of upper indices k.

Proposition 2.4.3 (Straightening) [17, Proposition 3.3] *The set of all standard elementary polynomials forms a linear basis in $\mathbb{Z}[x_1, x_2, \ldots]$.*

Proof — We will first show that every polynomial $f \in \mathbb{Z}[x_1, x_2, \ldots]$ is a linear combination of standard elementary polynomials. As noted above, f is a linear combination of some products of the e_i^k. Choose such a linear combination and apply to it the following straightening algorithm.

Suppose that some term in this combination is not standard. Find a term which has some of its upper indices k repeated, with the smallest possible value of k. Say, this term contains $e_i^k e_j^k$. Then substitute $e_i^k e_j^k$ by the right-hand side of (2.4.3). Note that, because of our choice of k, we will not create any new repetition of upper indices with a smaller k. Repeatedly using this procedure, we can express f as a combination of standard elementary polynomials.

Now let us show that all standard elementary polynomials are linearly independent. For suppose not. Find a nontrivial linear relation L with terms of minimal possible degree. Let k be the minimal index such that some e_i^k, $i > 0$, appears in some term in L. By Lemma 2.4.2, applying ∂_k annihilates every term not containing e_i^k, $i > 0$, whereas $\partial_k \cdot e_i^k e_j^{k+1} \cdots = e_{i-1}^{k-1} e_j^{k+1} \cdots$. Therefore applying ∂_k to L results in a nontrivial linear relation with terms of smaller degree. Contradiction. □

Recall that $\mathcal{J}_n$ is the ideal in the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ that is generated by $e_1^n, \ldots, e_n^n$. Let $H_n \subset \mathbb{Z}[x_1, \ldots, x_n]$ denote the $n!$-dimensional $\mathbb{Z}$-linear space spanned by all monomials $x_1^{a_1} x_2^{a_2} \cdots x_{n-1}^{a_{n-1}}$ such that $0 \leq a_k \leq n-k$ for $k = 1, \ldots, n-1$. The following result appears in [31] and [32, (2.6)-(2.7)]; see also [35, (4.13)].

Proposition 2.4.4 *The subspace H_n is complementary to the ideal $\mathcal{J}_n$. Each of the following families of polynomials is a $\mathbb{Z}$-linear basis of the space H_n:*

- *the monomials $x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$ such that $0 \leq a_k \leq n-k$;*

- *the standard elementary polynomials* $e_{i_1 i_2 \dots i_{n-1}}$ *;*

- *the Schubert polynomials* $\mathfrak{S}_w$ *for* $w \in S_n$ *.*

Thus, the corresponding cosets modulo $\mathcal{J}_n$ *form* $\mathbb{Z}$*-linear bases of* $\mathbb{Z}[x_1, \dots, x_n]/\mathcal{J}_n$ *.*

Proof — First note that $e_{i_1 i_2 \dots i_{n-1}} \in H_n$. By Proposition 2.4.3, these standard polynomials are linearly independent. As the number of them is $n! = \dim H_n$, they form a linear basis of H_n . The same arguments work for the Schubert polynomials $\mathfrak{S}_w$, which belong to H_n since $\mathfrak{S}_{w_o} = x_1^{n-1} x_2^{n-2} \cdots \in H_n$, and H_n is invariant under the ∂_i (cf. (2.2.10)).

Then observe that the quotient $\mathbb{Z}[x_1, \dots, x_n]/\mathcal{J}_n$ is equal to

$$\mathbb{Z}[x_1, \dots, x_n, x_{n+1}, x_{n+2}, \dots]/\langle e_1^n, \dots, e_n^n, x_{n+1}, x_{n+2}, \dots \rangle . \tag{2.4.5}$$

The ideal in (2.4.5) is generated by the standard elementary polynomials which are not of the form $e_{i_1 i_2 \dots i_{n-1}}$. It follows from Proposition 2.4.3 that the cosets of the polynomials $e_{i_1 i_2 \dots i_{n-1}}$, exactly $n!$ in number, form a basis in $\mathbb{Z}[x_1, \dots, x_n]/\mathcal{J}_n$. In particular, the dimension of the latter is $n!$. The same holds for the cosets of Schubert polynomials $\mathfrak{S}_w$, which are related to the standard elementary polynomials by a non-degenerate linear transformation. □

2.4.2 Deformation

We show how to quantize the basis of standard elementary polynomials. First we find the quantum deformations of the elementary symmetric polynomials e_i^k .

Recall that $E_i^k = E_i(x_1, \dots, x_k; q_1, \dots, q_{k-1})$ is the coefficient of λ^i in the characteristic polynomial (2.1.2) of the 3-diagonal matrix (2.1.3), where n is replaced by k. By convention, $E_i^k = 0$ unless $0 \le i \le k$. Alternatively, one can define the E_i^k via the following recurrence relations:

$$\begin{aligned} E_i^k &= E_i^{k-1} + x_k E_{i-1}^{k-1} + q_{k-1} E_{i-2}^{k-2} , \\ E_0^k &= 1 . \end{aligned} \tag{2.4.6}$$

for any $k \ge i \ge 1$, where we assume $q_0 = 0$.

It is not hard to calculate the E_i^k explicitly using the following monomer-dimer interpretation. Let us associate with each variable x_j the "monomer" $\{j\}$ and with each q_r the "dimer" $\{r, r+1\}$. Then E_i^k is the sum of all products of the x_j and q_r which correspond to disjoint collections of monomers and dimers covering i distinct elements of the set $\{1, 2, \dots, k\}$. The number of monomials in E_k^k is thus equal to the k-th Fibonacci number.

For a polynomial $F(x_1, x_2, \dots)$, we denote $F(\mathcal{X}) = F(\hat{\mathcal{X}}_1, \hat{\mathcal{X}}_2, \dots)$ the result of substituting the elements $\hat{\mathcal{X}}_i$ given by (2.3.3) in place of the x_i .

Theorem 2.4.5 [17, Proposition 5.4] *Let f be a polynomial in $\mathbb{Z}[x_1, x_2, \dots]$ which is symmetric in the variables $x_1, \dots, x_{k+1}$. Then $E_i^k(\mathcal{X}) \cdot f = e_i^k f$. Equivalently, $e_i^k \bar{*} f = e_i^k f$.*

Proof — Induction on k. If $k = 0$, then $E_0^0(\mathcal{X}) \cdot f = e_0^0 f = f$. Suppose $k > 0$. Then, using the induction hypothesis, Lemma 2.4.2, (2.4.1), and (2.4.6), we obtain:

$$\begin{aligned} E_i^k(\mathcal{X}) \cdot f &= (E_i^{k-1}(\mathcal{X}) + \hat{\mathcal{X}}_k E_{i-1}^{k-1}(\mathcal{X}) + q_{k-1} E_{i-2}^{k-2}(\mathcal{X})) \cdot f \\ &= e_i^{k-1} f + \hat{\mathcal{X}}_k \cdot (e_{i-1}^{k-1} f) + q_{k-1} e_{i-2}^{k-2} f \\ &= e_i^{k-1} f + x_k e_{i-1}^{k-1} f - q_{k-1} \partial_{k-1} e_{i-1}^{k-1} f + q_{k-1} e_{i-2}^{k-2} f \\ &= e_i^{k-1} f + x_k e_{i-1}^{k-1} f = e_i^k f , \end{aligned}$$

as desired. □

Corollary 2.4.6 *The polynomial E_i^k is the quantization $\mu(e_i^k)$ of the elementary symmetric polynomial e_i^k.*

Proof — Set $f = 1$ in Theorem 2.4.5. □

In particular, the quantization map sends the generators e_i^n, $i = 1, \dots, n$, of the ideal $\mathcal{J}_n$ to the generators E_i^n of the Givental-Kim ideal $\mathcal{J}_n^q$.

For a sequence $(i_1, \dots, i_m)$ such that $0 \le i_k \le k$, define the *standard quantum elementary polynomial* E_I by

$$E_I = E_{i_1 \dots i_m} = E_{i_1}^1 \cdots E_{i_m}^m .$$

Theorem 2.4.7 [17, Theorem 5.5] *For $I = (i_1, \dots, i_m)$, the polynomial E_I is the quantization $\mu(e_I)$ of the standard elementary polynomial e_I defined by* (2.4.4).

Proof — Repeatedly using Theorem 2.4.5, we obtain:

$$\begin{aligned} E_{i_1}^1 \cdots E_{i_m}^m(\mathcal{X}) \cdot 1 &= E_{i_1}^1 \cdots E_{i_{m-1}}^{m-1}(\mathcal{X}) \cdot e_{i_m}^m \\ &= E_{i_1}^1 \cdots E_{i_{m-2}}^{m-2}(\mathcal{X}) \cdot (e_{i_{m-1}}^{m-1} e_{i_m}^m) = \cdots = e_{i_1}^1 \cdots e_{i_m}^m , \end{aligned}$$

as needed. □

This theorem gives the following description of the quantization map (2.3.4). It is a unique map μ, linear over $\mathbb{Z}[q_1, q_2, \dots]$, that maps the basis elements e_I to the corresponding E_I:

$$\mu \, : \, e_I \longmapsto E_I \quad \text{for all } I = (i_1, \dots, i_m) .$$

The monomer-dimer combinatorial construction can be used to describe the quantization of any square-free monomial $x_a = x_{a_1}x_{a_2}\cdots$. Namely, consider the graph whose vertices are the a_i, and whose edges connect a_i and a_j if $|a_i - a_j| = 1$. Assign weight x_{a_i} to the vertex a_i and weight q_{a_i} to the edge $(a_i, a_i + 1)$. Then every matching in this graph (i.e., a collection of vertex-disjoint edges, or dimers) acquires a weight equal to the product of weights of its dimers multiplied by the weights of left out vertices. The sum of these weights, for all matchings, is the quantization of the monomial x_a. A similar rule for computing the inverse image (dequantization) of a square-free monomial can be obtained using Möbius inversion.[9] The only difference from the quantization rule is in replacing each q_i by $-q_i$.

Proposition 2.3.5 and Corollary 2.4.6 imply that the combinatorial quantum deformation of the cohomology ring $R = \mathrm{H}^*(Fl_n, \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_n]/\mathcal{J}_n$ is the quotient ring $R^q = \mathbb{Z}[q][x_1, \dots, x_n]/J_n^q$, which is canonically isomorphic to the quantum cohomology ring $\mathrm{QH}^*(Fl_n, \mathbb{Z})$, due to Theorem 2.2.7. Thus the quantization map establishes an isomorphism of $\mathbb{Z}[q]$-linear spaces

$$\mathrm{H}^*(Fl_n, \mathbb{Z}) \otimes \mathbb{Z}[q] \longrightarrow \mathrm{QH}^*(Fl_n, \mathbb{Z}).$$

This however does not prove Theorem 2.3.6, which now amounts to claiming that the quantization map is the tautological identification of the spaces in the right-hand side and the left-hand side of (2.2.14).

Recall that H_n is the $\mathbb{Z}$-span of the monomials $x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$ such that $0 \le a_k \le n - k$ for all k. Let us also denote $H_n^q = H_n \otimes \mathbb{Z}[q]$.

Corollary 2.4.8 *The space H_n^q is invariant under the quantization map, and is complementary to the ideal $\mathcal{J}_n^q$. The polynomials $E_{i_1 \dots i_{n-1}}$ form a $\mathbb{Z}[q]$-linear basis of H_n^q. Thus their cosets form a $\mathbb{Z}[q]$-basis of the quotient $\mathbb{Z}[q][x_1, \dots, x_n]/\mathcal{J}_n^q$.*

Proof — By Proposition 2.4.4, the space H_n is spanned by the standard elementary polynomials e_I, $I = (i_1, \dots, i_{n-1})$. Consider their quantizations $\mu(e_I) = E_I$. Each factor $E_{i_k}^k$ in E_I is a square-free polynomial in $x_1, \dots, x_k$. Hence every monomial $x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$ in the expansion of E_I satisfies the condition $a_k \le n - k$. Using Proposition 2.4.4, we conclude that $E_I \in H_n^q$. Hence this space is invariant under quantization. Since the quantization map is a $\mathbb{Z}[q]$-linear isomorphism that fixes H_n^q and sends the complementary ideal $\mathcal{J}_n$ to $\mathcal{J}_n^q$ (see Propositions 2.3.5 and 2.4.4), it follows that H_n^q is complementary to $\mathcal{J}_n^q$, and the E_I form a basis in H_n^q. $\square$

Proposition 2.4.9 [17, Proposition 6.2] *For any $g \in H_n^q$, and any polynomial f symmetric in $x_1, \dots, x_n$, we have $g \ast f = gf$.*

Proof — By Corollary 2.4.8, it is enough to consider the case when $g = E_{i_1 \dots i_{n-1}}$. The statement then follows by repeatedly applying Theorem 2.4.5. $\square$

[9] In general, for monomials with squares, this simple method of dequantization does not work.

Theorem 2.4.10 *In the nilHecke ring $\mathcal{NH}_n^q$ the element $e_i^n(\chi_1, \dots, \chi_n)$ coincides with the element $E_i^n(\mathcal{X}_1, \dots, \mathcal{X}_n)$. Thus in the ring $\mathbb{Z}[q][x_1, \dots, x_n]$ the combinatorial quantum multiplication by e_i^n coincides with the usual multiplication by e_i^n :*

$$e_i^n \bar{*} g = e_i^n g\,, \qquad \textit{for any} \;\; g \in \mathbb{Z}[q][x_1, \dots, x_n]\,.$$

Proof — Since the action of the nilHecke ring on the polynomial ring is exact, it suffice to show that $E_i^n(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot g = e_i^n g$ for any polynomial $g \in \mathbb{Z}[q][x_1, \dots, x_n]$.

The polynomial g belongs to the space H_N^q for some $N \geq n$. Let us expand g in the standard elementary polynomials $e_{i_1 \dots i_{N-1}}$. By Proposition 2.4.9, we have $e_i^N \bar{*} g = E_i^N(\hat{\mathcal{X}}_1, \dots, \hat{\mathcal{X}}_N) \cdot g = e_i^N g$. The statement easily follows, because $e_i^n \equiv e_i^N$, $E_i^N \equiv E_i^n$, and $\mathcal{X}_j \equiv \hat{\mathcal{X}}_j$ modulo the ideal $\langle x_{n+1}, \dots, x_N, q_n, \dots, q_{N-1} \rangle$, $\square$

More generally, the following identity holds in the nilHecke ring $\mathcal{NH}_n^q$.

Theorem 2.4.11 *For every $i \leq k < n$, we have*

$$\partial_1 \partial_2 \cdots \partial_k \left(E_i^k(\mathcal{X}_1, \dots, \mathcal{X}_k) - e_i^k(\chi_1, \dots, \chi_k) \right) = 0\,. \tag{2.4.7}$$

Proof — For a fixed k, let $\widetilde{\mathcal{X}}_a$ denote the element of the nilHecke ring $\mathcal{NH}_n^q$ given by

$$\widetilde{\mathcal{X}}_a = \chi_a - \sum_{1 \leq i \leq a} q_{ia}\, \partial_{(ia)} + \sum_{a < j \leq k} q_{aj}\, \partial_{(aj)}\,.$$

In other words, $\widetilde{\mathcal{X}}_a$ is the image of the element $\mathcal{X}_a$ in $\mathcal{NH}_k^q$ under the standard embedding $\mathcal{NH}_k^q \subset \mathcal{NH}_n^q$. Then $\mathcal{X}_a - \widetilde{\mathcal{X}}_a = \sum_{j=k+1}^n q_{aj}\, \partial_{(aj)}$. Let us substitute $\widetilde{\mathcal{X}}_a + (\mathcal{X}_a - \widetilde{\mathcal{X}}_a)$ instead of the $\mathcal{X}_a$ in (2.4.7) and then expand.

Theorem 2.4.10, with n replaced by k, implies

$$E_i^k(\widetilde{\mathcal{X}}_1, \dots, \widetilde{\mathcal{X}}_k) = e_i^k(\chi_1, \dots, \chi_k)\,.$$

To prove (2.4.7), it is thus sufficient to show that

$$\partial_1 \partial_2 \cdots \partial_k\, \widetilde{\mathcal{X}}_{i_1} \widetilde{\mathcal{X}}_{i_2} \cdots \widetilde{\mathcal{X}}_{i_r}\, \partial_{(aj)} = 0\,, \tag{2.4.8}$$

for any $1 \leq i_1 < i_2 < \cdots < i_r < a \leq k < j$.

Lemma 2.4.12 *For $c \leq d$, we have $(\partial_c \partial_{c+1} \cdots \partial_d)(\partial_c \partial_{c+1} \cdots \partial_d) = 0$.*

(The proof is left to the reader.[10])

Notice now that $\partial_{(aj)} = (\partial_a \partial_{a+1} \cdots \partial_{j-1}) \cdots$. The only term in $\widetilde{\mathcal{X}}_{i_r}$ which does not either commute with $\partial_a \partial_{a+1} \cdots \partial_{j-1}$ nor vanish upon composition with $\partial_a \partial_{a+1} \cdots \partial_{j-1}$

[10]Hint: Use the fact that $\ell(w^2) < 2\ell(w)$ for $w = s_c s_{c+1} \cdots s_d$.

is $q_{i_r\,a}\,\partial_{(i_r\,a)}$. Moving all irrelevant factors to the right, we can write the expression in the left-hand side of (2.4.8) as

$$\partial_1 \cdots \partial_k \, \widetilde{\mathcal{X}}_{i_1} \cdots \widetilde{\mathcal{X}}_{i_{r-1}} \, (\partial_{i_r} \partial_{i_r+1} \cdots \partial_{j-1}) \cdots .$$

Repeating this trick r times, we deduce that this expression is equal to

$$\partial_1 \cdots \partial_k \, (\partial_{i_1} \partial_{i_1+1} \cdots \partial_{j-1}) \cdots = 0 ,$$

as desired. □

The following quantum analogue of (2.4.2) can be used for the quantum straightening algorithm.

Lemma 2.4.13 *For $k \ge j \ge 0$, $k \ge i \ge 0$,*

$$E_i^k E_{j+1}^{k+1} + E_{i+1}^k E_j^k + q_k E_{i-1}^{k-1} E_j^k = E_j^k E_{i+1}^{k+1} + E_{j+1}^k E_i^k + q_k E_{j-1}^{k-1} E_i^k .$$

Proof — By (2.4.6),

$$E_i^k \, (E_{j+1}^{k+1} - E_{j+1}^k) = E_i^k \, (x_{k+1} E_j^k + q_k E_{j-1}^{k-1}) \, ,$$
$$E_j^k \, (E_{i+1}^{k+1} - E_{i+1}^k) = E_j^k \, (x_{k+1} E_i^k + q_k E_{i-1}^{k-1}) \, .$$

Subtracting the second equation from the first, we obtain the claim. □

2.4.3 Straightforward deformation

Let $R = \mathbb{Z}[x_1, \ldots, x_n]/\mathcal{I}$ be the quotient of the polynomial ring modulo the ideal $\mathcal{I}$ generated by a sequence of symmetric polynomials f_i. Proposition 2.3.5 says that the combinatorial quantum deformation of the ring R is the quotient ring $R^q = \mathbb{Z}[q][x_1, \ldots, x_n]/\mathcal{I}^q$ modulo the ideal $\mathcal{I}^q$ generated by quantizations F_i of the f_i. It is thus important to find the quantization of any symmetric polynomial.

For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l > 0$, let

$$e_\lambda^n = e_{\lambda_1}^n \, e_{\lambda_2}^n \cdots e_{\lambda_l}^n .$$

Analogously, we define

$$E_\lambda^n = E_{\lambda_1}^n \, E_{\lambda_2}^n \cdots E_{\lambda_l}^n .$$

Recall that the e_λ^n form a $\mathbb{Z}$-basis in the ring of symmetric polynomials of $x_1, \ldots, x_n$ (see, e.g., [34]). Thus any symmetric polynomial is a linear combination of the e_λ^n.

Corollary 2.4.14 *The quantization $\mu(e_\lambda^n)$ of the symmetric polynomial e_λ^n in the ring $\mathbb{Z}[x_1, \ldots, x_n]$ is the polynomial E_λ^n. The same, of course, holds for quantizations in the ring $R = \mathbb{Z}[x_1, \ldots, x_n]/\mathcal{I}$ of cosets of the e_λ^n.*

Proof — The polynomial e^n_λ is not standard, but it is equivalent to the standard elementary polynomial $e^n_{\lambda_1} e^{n+1}_{\lambda_2} \cdots e^{n+l-1}_{\lambda_l}$ in $\mathbb{Z}[x_1, \ldots, x_{n+l-1}]$ modulo the ideal generated by $x_{n+1}, \ldots, x_{n+l-1}$. The statement now follows from Theorem 2.4.7, since E^n_λ is equivalent to $E^n_{\lambda_1} \cdots E^{n+l-1}_{\lambda_l}$ modulo the ideal $\langle x_{n+1}, \ldots, x_{n+l-1}, q_n, \ldots, q_{n+l-2}\rangle$. □

It is thus possible to quantize any Schur polynomial via its expression as the Jacobi-Trudy determinant. Let λ' be the partition conjugate to λ, see [34].

Corollary 2.4.15 *In the ring $\mathbb{Z}[x_1, \ldots, x_n]$, the quantization of the Schur polynomial*

$$s_{\lambda'} = \det \left(e^n_{\lambda_i - i + j}\right)_{i,j \in \{1,\ldots,l\}}$$

is the polynomial $S_{\lambda'}$ given by an analogous expression

$$S_{\lambda'} = \det \left(E^n_{\lambda_i - i + j}\right)_{i,j \in \{1,\ldots,l\}} .$$

2.5 Quantum Schubert Polynomials

In this section we study quantum deformations of Schubert polynomials of Lascoux and Schützenberger. We prove the orthogonality property and give their axiomatic characterization, which implies Theorems 2.1.1 and 2.3.6.

2.5.1 Simple properties

Definition 2.5.1 For $w \in S_n$, the *quantum Schubert polynomial* $\mathfrak{S}^q_w$ is the quantization of the ordinary Schubert polynomial $\mathfrak{S}_w$. In other words, it is a unique polynomial in $\mathbb{Z}[q][x_1, \ldots, x_n]$ such that

$$\mathfrak{S}^q_w(\mathcal{X}_1, \ldots, \mathcal{X}_n) \cdot 1 = \mathfrak{S}_w(x_1, \ldots, x_n) .$$

The quantum multiplication of ordinary Schubert polynomials translates into the ordinary multiplication of the corresponding quantum Schubert polynomials.

We can apply Theorem 2.4.7 for explicit computation of the $\mathfrak{S}^q_w$. First, we express the ordinary Schubert polynomial $\mathfrak{S}_w$ as a linear combination of standard elementary polynomials

$$\mathfrak{S}_w = \sum_{I=(i_1,\ldots,i_{n-1})} \alpha_I \, e_I .$$

Then we replace each term by its quantum deformation[11]:

$$\mathfrak{S}^q_w = \sum_{I=(i_1,\ldots,i_{n-1})} \alpha_I \, E_I . \tag{2.5.1}$$

The expansions of Schubert polynomials in terms of the standard elementary polynomials can be computed recursively in the weak order of S_n, starting from

[11] This is our original definition of the $\mathfrak{S}^q_w$ given in Introduction.

$\mathfrak{S}_{w_o} = e_{12\ldots n-1}$, using the basic recurrence (2.2.10) together with the following rule for computing a divided difference of standard elementary polynomials, which is an immediate consequence of Lemmas 2.4.1 and 2.4.2.

Proposition 2.5.2 *We have, for $1 \le k < n$ and $I = (i_1, \ldots, i_{n-1})$,*

$$\partial_k \cdot e_I = \sum_{r \ge 0} e_{I'_r} - \sum_{r \ge 1} e_{I''_r} ,$$

where

$$I'_r = (i_1, \ldots, i_{k-2}, i_{k-1} + r, i_k - r - 1, i_{k+1}, \ldots, i_{n-1}) ,$$

$$I''_r = (i_1, \ldots, i_{k-2}, i_k - r - 1, i_{k-1} + r, i_{k+1}, \ldots, i_{n-1}) .$$

In more comprehensible terms, the proposition says that the divided difference ∂_k acts on the standard elementary polynomials $e_{i_1 \ldots i_{n-1}}$ in the same way as the following "divided sum" operator[12]

$$f \longmapsto (x_k - x_{k-1})^{-1} (1 + s_{k-1}) f \tag{2.5.2}$$

acts on the monomials $x_1^{i_1+n-1} x_2^{i_2+n-2} \cdots x_{n-1}^{i_{n-1}+1}$,

For example, we have in S_4:

$$\mathfrak{S}_{4321} = \mathfrak{S}_{w_o} = e_{123} ,$$

$$\mathfrak{S}_{3421} = \partial_1 \mathfrak{S}_{4321} = \partial_1 e_{123} = e_{023} ,$$

$$\mathfrak{S}_{3412} = \partial_3 \mathfrak{S}_{3421} = \partial_3 e_{023} = e_{022} - e_{013} ,$$

and so on. The corresponding quantum Schubert polynomials $\mathfrak{S}_w^q$ are then obtained by replacing each $e_i(x_1, \ldots, x_k)$ by its quantum analogue. For instance,

$$\mathfrak{S}_{3412}^q = E_{022} - E_{013} = x_1^2 x_2^2 + 2q_1 x_1 x_2 - q_2 x_1^2 + q_1^2 + q_1 q_2 .$$

Lemma 2.5.3 *The quantum Schubert polynomials form a $\mathbb{Z}[q]$-linear basis of H_n^q.*

Proof — The quantum Schubert polynomials are related to the $E_{i_1 \ldots i_{n-1}}$ by an invertible linear map, and thus form a basis of H_n^q, by Corollary 2.4.8. □

Let deg be the grading defined by $\deg(x_i) = 1$ and $\deg(q_j) = 2$.

Proposition 2.5.4 *The polynomial $\mathfrak{S}_w^q$ is of degree $\ell(w)$, with respect to the grading* deg *. Specializing $q_1 = \cdots = q_{n-1} = 0$ yields $\mathfrak{S}_w^q = \mathfrak{S}_w$, the classical Schubert polynomials.*

[12]Such operators, of course, satisfy the nilCoxeter relations (2.2.4).

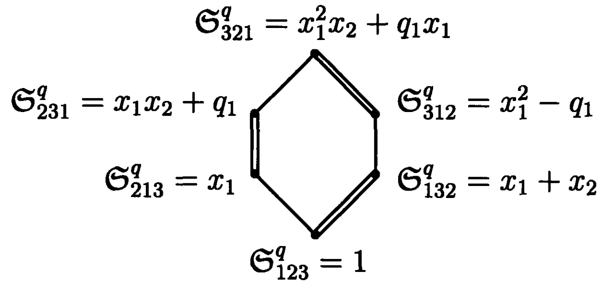

Figure 2-1: Quantum Schubert polynomials for S_3

It follows that the transition matrices between the bases $\{\mathfrak{S}^q_w\}$ and $\{\mathfrak{S}_w\}$ are unipotent triangular, with respect to any linear ordering that is consistent with the length function $\ell(w)$.

2.5.2 Orthogonality property

The orthogonality of Schubert classes is not hard to establish from the quantum cohomology definitions. At this point, however, we have not proved yet that quantum Schubert polynomials $\mathfrak{S}^q_w$ represent Schubert classes in the quantum cohomology ring. Moreover, the proof of this fact given in the following Section 2.5.3 relies on a combinatorial proof of the orthogonality of the $\mathfrak{S}^q_w$, provided below in this section.

For a polynomial $F \in \mathbb{Z}[q][x_1, \ldots, x_n]$, we define $\langle\!\langle F \rangle\!\rangle \in \mathbb{Z}[q]$ by

$$\langle\!\langle F \rangle\!\rangle = (\partial_{w_o} F(\mathcal{X}_1, \ldots, \mathcal{X}_n) \cdot 1)(0, \ldots, 0)\,. \tag{2.5.3}$$

If F is the quantization $\mu(f)$ of a polynomial f then $\langle\!\langle F \rangle\!\rangle = \langle f \rangle$, where $\langle f \rangle$ is given by (2.2.12). Note that $\langle\!\langle F \rangle\!\rangle$ depends only on the coset of F modulo the ideal $\mathcal{J}^q_n$.

From Corollary 2.4.8 and Lemma 2.5.3, we know that $\mathbb{Z}[q][x_1, \ldots, x_n]/\mathcal{J}^q_n$ has the following $\mathbb{Z}[q]$-linear bases given by cosets of:

- the monomials $x_1^{a_1} x_2^{a_2} \cdots x_{n-1}^{a_{n-1}}$ such that $0 \le a_k \le n-k$,
- the quantum standard elementary polynomials $E_{i_1 i_2 \ldots i_{n-1}}$,
- the quantum Schubert polynomials $\mathfrak{S}^q_w$.

Then $\langle\!\langle F \rangle\!\rangle$ is equal, respectively, to the coefficient of:

- the top monomial $x^\delta = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$
- the polynomial $E_{12\ldots n-1}$,
- the quantum Schubert polynomial $\mathfrak{S}^q_{wnot}$

in the expansion of the coset of F in each of these bases.

The following result is the quantum analogue of Theorem 2.2.6.

Theorem 2.5.5 (Orthogonality property) [17, Theorem 3.9] *For $u, v \in S_n$,*

$$\langle\!\langle \mathfrak{S}_u^q \mathfrak{S}_v^q \rangle\!\rangle = \begin{cases} 1 & \text{if } v = w_o u\,; \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.4}$$

By definition,

$$\langle\!\langle \mathfrak{S}_u^q \mathfrak{S}_v^q \rangle\!\rangle = (\partial_{w_o} (\mathfrak{S}_u^q \mathfrak{S}_v^q)(\mathcal{X}_1, \ldots, \mathcal{X}_n) \cdot 1)(0, \ldots, 0) =$$
$$= (\partial_{w_o} \mathfrak{S}_u^q(\mathcal{X}_1, \ldots, \mathcal{X}_n) \cdot \mathfrak{S}_v(x_1, \ldots, x_n))(0, \ldots, 0)\,.$$

The classical orthogonality property (2.2.13) together with the following identity, which holds in the nilHecke ring $\mathcal{NH}_n^q$, implies now Theorem 2.5.5.

Theorem 2.5.6 *We have, for any $u \in S_n$,*

$$\partial_{w_o} (\mathfrak{S}_u^q(\mathcal{X}_1, \ldots, \mathcal{X}_n) - \mathfrak{S}_u(\chi_1, \ldots, \chi_n)) = 0\,.$$

Proof — It is sufficient to show that, for any $I = (i_1, \ldots, i_{n-1})$,

$$\partial_{w_o} (E_I(\mathcal{X}_1, \ldots, \mathcal{X}_n) - e_I(\chi_1, \ldots, \chi_n)) = 0\,. \tag{2.5.5}$$

We prove this identity by induction on n. The case $n = 1$ is trivial. Assuming that $n > 1$ and omitting the variables $\mathcal{X}_j$ and χ_j for briefness, we can write the left-hand side of (2.5.5) as follows:

$$\partial_{w_o} e_{i_{n-1}}^{n-1} (E_{i_1 \ldots i_{n-2}} - e_{i_1 \ldots i_{n-2}}) + \partial_{w_o} (E_{i_{n-1}}^{n-1} - e_{i_{n-1}}^{n-1})\, E_{i_1 \ldots i_{n-2}}$$
$$= \cdots e_{i_{n-1}}^{n-1} \partial_{w_o'} (E_{i_1 \ldots i_{n-2}} - e_{i_1 \ldots i_{n-2}}) + \cdots \partial_1 \partial_2 \cdots \partial_{n-1} (E_{i_{n-1}}^{n-1} - e_{i_{n-1}}^{n-1}) \cdots\,,$$

where w_o' denotes the longest element in S_{n-1} acting on the first $n-1$ variables, thus $\partial_{w_o'}$ commutes with $e_{i_{n-1}}^{n-1}$—symmetric polynomial in $x_1, \ldots, x_{n-1}$. The first term in the last expression vanishes by the induction hypothesis and the second term vanishes, due to Theorem 2.4.11.

This completes the proof of Theorem 2.5.6 and of the orthogonality property. □

2.5.3 Axiomatic characterization

In this section we provide proof of Theorems 2.1.1 and 2.3.6.

Everywhere in this section QH denotes the quotient ring $\mathbb{Z}[q][x_1, \ldots, x_n]/\mathcal{J}_n^q$, which is canonically isomorphic to the quantum cohomology ring $\mathrm{QH}^*(Fl_n, \mathbb{Z})$. Recall that deg is the grading such that $\deg(x_i) = 1$ and $\deg(q_j) = 2$. Thus the ring QH acquires the structure of a graded ring since every generator E_i^n of the ideal $\mathcal{J}_n^q$ is a homogeneous degree i polynomial with respect to the grading deg. Let $\mathbb{Z}_+[q]$ denote the set of all polynomials in the q_j with nonnegative integer coefficients. Let us also denote by $\tilde{\mathfrak{S}}_w^q$ the image of the quantum Schubert polynomial $\mathfrak{S}_w^q$ in the quotient ring QH. Likewise, $\bar{E}_i^k \in$ QH is the image of E_i^k, etc. We have the following axiomatic characterization of the elements $\bar{\mathfrak{S}}_w^q$.

Theorem 2.5.7 [17, Theorem 9.1] *Suppose that the elements b_w, $w \in S_n$, form a $\mathbb{Z}[q]$-basis of the quotient* QH *and satisfy the following four axioms:*

1. (Degree condition) *The element b_w is homogeneous of degree $\ell(w)$ with respect to the grading* deg.
2. (Classical limit) *Modulo the ideal generated by the q_j, the element b_w coincides with the coset of the corresponding Schubert polynomial $\mathfrak{S}_w$.*
3. (Nonnegativity) *The product in the ring* QH *of any two elements b_u and b_v is a linear combination of the b_w with coefficients in $\mathbb{Z}_+[q]$.*
4. *Any element $\bar{E}_i^k$ is a linear combination of the b_w with coefficients in $\mathbb{Z}_+[q]$.*

Then elements b_w coincide with the corresponding $\bar{\mathfrak{S}}_w^q$.

Let $\bar{\sigma}_w \in$ QH be the element that corresponds to the Schubert class σ_w under the canonical isomorphism of QH and $\mathrm{QH}^*(Fl_n, \mathbb{Z})$. All four requirements of Theorem 2.5.7 hold for the elements $b_w = \bar{\sigma}_w$. Indeed, the first axiom (degree condition) is just the condition, clear from (2.2.17), that the Gromov-Witten invariant $\langle \sigma_u, \sigma_v, \sigma_w \rangle_d$ is zero unless $\ell(u) + \ell(v) + \ell(w) = \ell(w_o) + 2(d_1 + \cdots + d_{n-1})$. The second condition (classical limit) is equivalent to saying that the Gromov-Witten invariants $\langle \sigma_u, \sigma_v, \sigma_w \rangle_{(0,\ldots,0)}$ are the usual intersection numbers of Schubert varieties, which are the structure constants in the cohomology ring $\mathrm{H}^*(Fl_n, \mathbb{Z})$. The third condition (nonnegativity) is simply claiming that the Gromov-Witten invariants are nonnegative integer numbers, which is apparent from their geometrical definition as the number of certain curves. At last, the fourth condition is also satisfied, because a formula proved by Ciocan-Fontanine [12, formula (3)] implies that $\bar{E}_i^k = \bar{\sigma}_{c(i,k)}$, where $c(i,k) = s_{k-i+1} s_{k-i+2} \cdots s_k$.

Theorem 2.1.1 from Introduction, which claims that $\bar{\sigma}_w = \bar{\mathfrak{S}}_w^q$, is therefore a corollary of Theorem 2.5.7. Moreover, Theorem 2.1.1 implies[13] Theorem 2.3.6. Indeed, for any $u, v \in S_n$, the combinatorial quantum product $\sigma_u \bar{*} \sigma_v$ of two Schubert classes $\sigma_u, \sigma_v \in \mathrm{H}^*(Fl_n, \mathbb{Z}) \otimes \mathbb{Z}[q]$ coincides with the geometrical quantum product $\sigma_u * \sigma_v$, because both these products correspond to the usual product $\bar{\mathfrak{S}}_u^q \bar{\mathfrak{S}}_v^q$ of cosets of quantum Schubert polynomials—the former by definitions, the latter by Theorem 2.1.1.

Proof of Theorem 2.5.7 — Let us denote by QH_+ the $\mathbb{Z}_+[q]$-span of the elements b_w in QH. According to the nonnegativity condition, QH_+ is closed under multiplication. The fourth axiom implies that the $\bar{E}_i^k$, and thus all $\bar{E}_I = \bar{E}_{i_1}^1 \cdots \bar{E}_{n-1}^{i_{n-1}}$, are in QH_+.

Let us now fix a nonnegative integer $l \leq \ell(w_o)$. By Proposition 2.4.4, the polynomials $\mathfrak{S}_w$, $\ell(w) = l$, are related to the e_I with $|I| = i_1 + \cdots + i_{n-1} = l$, by a

[13] and is equivalent to

non-degenerate linear transformation. Moreover, each e_I is a nonnegative integer combination[14] of the $\mathfrak{S}_w$. Every $\mathfrak{S}_w$, $\ell(w) = l$, should enter the expansion of at least one e_I, $|I| = l$. Therefore

$$\sum_{I:\,|I|=l} e_I = \sum_{\ell(w)=l} \alpha_w \mathfrak{S}_w \ ,$$

with certain *positive* α_w. By (2.5.1) and the fact that $\bar{E}_I \in \mathrm{QH}_+$, we obtain:

$$\sum_{\ell(w)=l} \alpha_w \bar{\mathfrak{S}}^q_w \in \mathrm{QH}_+ \ . \tag{2.5.1}$$

The first two axioms imply that each $\bar{\mathfrak{S}}^q_w$ is equal to b_w plus a $\mathbb{Z}[q]$-linear combination of some b_v with $\ell(v) < \ell(w)$. It follows that

$$\sum_{\ell(w)=l} \alpha_w \bar{\mathfrak{S}}^q_w = \sum_{\ell(w)=l} \alpha_w \, b_w + \langle\, \text{linear combination of } b_v \text{ with } \ell(v) < \ell(w) \rangle \ ,$$

and (2.5.1) yields

$$\sum_{\ell(w)=l} \alpha_w (\bar{\mathfrak{S}}^q_w - b_w) \in \mathrm{QH}_+ \ . \tag{2.5.2}$$

Let $J = (j_1, \ldots, j_{n-1})$ be such that

$$j_1 + \cdots + j_{n-1} > \ell(w_\mathrm{o}) - l \ . \tag{2.5.3}$$

Since $\bar{E}_J \in \mathrm{QH}_+$, the nonnegativity condition implies that, for any w,

$$\langle\!\langle \bar{E}_J \, b_w \rangle\!\rangle \in \mathbb{Z}_+[q] \ . \tag{2.5.4}$$

Likewise, (2.5.2) gives $\sum_{\ell(w)=l} \alpha_w \langle\!\langle \bar{E}_J \, (\bar{\mathfrak{S}}^q_w - b_w) \rangle\!\rangle \in \mathbb{Z}_+[q]$. Using Theorem 2.5.5 (orthogonality property) and (2.5.3), we write the last statement as

$$- \sum_{\ell(w)=l} \alpha_w \langle\!\langle \bar{E}_J \, b_w \rangle\!\rangle \in \mathbb{Z}_+[q] \ . \tag{2.5.5}$$

Recall that the α_w are strictly positive. Comparing (2.5.4) with (2.5.5), we conclude that $\langle\!\langle \bar{E}_J \, b_w \rangle\!\rangle = 0$, for any l, any w of length l, and any J satisfying (2.5.3). Therefore $\langle\!\langle \bar{\mathfrak{S}}^q_{w_\mathrm{o} v} b_w \rangle\!\rangle = 0$, for any $v \in S_n$ satisfying $\ell(v) < \ell(w)$. Once again using orthogonality, we conclude that the expansion of b_w via the $\bar{\mathfrak{S}}^q_v$ contains no terms with $\ell(v) < \ell(w)$, meaning that $b_w = \bar{\mathfrak{S}}^q_w$, as desired.

This completes proof of Theorem 2.5.7 and thus of Theorems 2.1.1 and 2.3.6. □

It seems that a stronger statement than Theorem 2.5.7 is true, which does not include the last axiom—the only condition for $b_w = \sigma_w$ not immediately clear from

[14] Of course, this fact is well-know, but it also follows from axioms 2, 3, and 4.

definitions.

Conjecture 2.5.8 [17, Conjecture 9.3] *In terms of Theorem 2.5.7, the first, second, and third axioms imply that $b_w = \tilde{\mathfrak{S}}_w$, the coset of quantum Schubert polynomial.*

This conjecture has been verified for all S_n, $n \leq 4$.

2.6 Monk's Formula and its Extensions

In this section, we prove the quantum Monk's formula (Theorem 2.1.2), and then we investigate its consequences and extensions. We give a general Pieri-type formula following the approach developed by Fomin and Kirillov in [18] and obtain several conjectures posed in their paper. As corollaries, a new proof of classical Pieri's formula for cohomology of complex flag manifolds, and that of its analogue for quantum cohomology are provided.

2.6.1 Quantum version of Monk's formula

By the classical Monk's formula (Theorem 2.2.3), quantum Monk's formula (Theorem 2.1.2) can be formulated as follows.

Recall the notation $q_{ij} = q_i q_{i+1} \cdots q_{j-1}$, for $i < j$.

Theorem 2.6.1 *We have, for $w \in S_n$ and $1 \leq k < n$, the geometrical quantum product of σ_k and σ_w is equal to*

$$\sigma_{s_k} * \sigma_w = \sigma_{s_k} \sigma_w + \sum q_{ij} \, \sigma_{ws_{ij}}, \tag{2.6.1}$$

where the sum is over all transpositions s_{ij} such that $i \leq k < j$ and $\ell(ws_{ij}) = \ell(w) - \ell(s_{ij}) = \ell(w) - 2(j-i) + 1$.

We note that σ_{s_k} corresponds to $x_1 + \cdots + x_k$.

Proof — More generally, for any linear form $f = \sum \lambda_i x_i$, we have

$$f * \mathfrak{S}_w = f \,\bar{*}\, \mathfrak{S}_w = f \, \mathfrak{S}_w + \sum (\lambda_i - \lambda_j) \, q_{ij} \, \mathfrak{S}_{ws_{ij}},$$

summed over all $i < j$ such that $\ell(ws_{ij}) = \ell(w) - \ell(s_{ij})$. The fist equality holds by Theorem 2.1.1, which was at last proved in the previous section, and the second equality holds by the definition of combinatorial quantum product (Definition 2.3.4) and (2.2.11). □

2.6.2 Quadratic ring

Let $\mathcal{E}_n^p$ be the ring generated by the elements τ_{ij} and p_{ij}, $i, j \in \{1, 2, \ldots, n\}$, subject to the following relations:

$$\tau_{ij} = -\tau_{ji}\,, \quad \tau_{ii} = 0\,, \tag{2.6.2}$$

$$\tau_{ij}^2 = p_{ij}\,, \tag{2.6.3}$$

$$\tau_{ij}\tau_{jk} + \tau_{jk}\tau_{ki} + \tau_{ki}\tau_{ij} = 0\,, \tag{2.6.4}$$

$$[p_{ij}, p_{kl}] = [p_{ij}, \tau_{kl}] = 0\,, \quad \text{for any } i, j, k, \text{ and } l\,, \tag{2.6.5}$$

$$[\tau_{ij}, \tau_{kl}] = 0\,, \quad \text{for any distinct } i, j, k, \text{ and } l\,. \tag{2.6.6}$$

Here $[a, b] = ab - ba$ is the usual commutator. It follows from (2.6.2) and (2.6.3) that $p_{ij} = p_{ji}$ and $p_{ii} = 0$. This ring was defined[15] by Fomin and Kirillov [18, Section 15].

The commuting elements p_{ij} can be viewed as formal parameters. The quotient $\mathcal{E}_n$ of the ring $\mathcal{E}_n^p$ modulo the ideal generated by the p_{ij} was the main object of study in [18]. Also a ring $\mathcal{E}_n^q$ was introduced in that paper. It can be defined as the quotient of $\mathcal{E}_n^p$ by the ideal generated by the p_{ij} with $|i - j| \geq 2$. The image of $p_{i\,i+1}$ in $\mathcal{E}_n^q$ is denoted q_i.

Following [18, Section 5], define the *"Dunkl" elements* θ_i, $i = 1, \ldots, n$, in the ring $\mathcal{E}_n^p$ by

$$\theta_i = \sum_{j=1}^{n} \tau_{ij}. \tag{2.6.7}$$

The following important property of these elements is not hard to deduce from the relations (2.6.2)–(2.6.6).

Lemma 2.6.2 [18, Corollary 5.2 and Section 15] *The elements $\theta_1, \theta_2, \ldots, \theta_n$ commute pairwise.*

Let $x_1, x_2, \ldots, x_n$ be a set of commuting variables, and let p be a shorthand for the collection of p_{ij}'s. For a subset $I = \{i_1, \ldots, i_m\}$ in $\{1, 2, \ldots, n\}$, we denote by x_I the collection of variables $x_{i_1}, \ldots, x_{i_m}$. Define the *quantum elementary symmetric polynomial* $E_k(x_I; p) = E_k(x_{i_1}, x_{i_2}, \ldots, x_{i_m}; p)$ by the following recursive formulas:

$$E_0(x_{i_1}, x_{i_2}, \ldots, x_{i_m}; p) = 1\,, \tag{2.6.8}$$

$$\begin{aligned} E_k(x_{i_1}, x_{i_2}, \ldots, x_{i_m}; p) &= E_k(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}; p) \\ &+ \; E_{k-1}(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}; p)\, x_{i_m} \\ &+ \; \sum_{r=1}^{m-1} E_{k-2}(x_{i_1}, \ldots, \widehat{x_{i_r}}, \ldots, x_{i_{m-1}}; p)\, p_{i_r i_m}\,, \end{aligned} \tag{2.6.9}$$

[15]in slightly different notation

where the notation $\widehat{x_{i_r}}$ means that the corresponding term is omitted.

The polynomial $E_k(x_I;p)$ is symmetric in the sense that it is invariant under the simultaneous action of S_m on the variables x_{i_a} and the $p_{i_a\,i_b}$. One can directly verify from (2.6.8) and (2.6.9) that

$$E_1(x_{i_1}, x_{i_2}, \ldots, x_{i_m}; p) = x_{i_1} + x_{i_2} + \cdots + x_{i_m}\,,$$

$$E_2(x_{i_1}, x_{i_2}, \ldots, x_{i_m}; p) = \sum_{1\le a<b\le m} (x_{i_a} x_{i_b} + p_{i_a\,i_b})\,.$$

The polynomials $E_k(x_I;p)$ have the following elementary monomer-dimer interpretation (cf. Section 2.4.2). A *partial matching* on the vertex set I is a unordered collection of "dimers" $\{a_1,b_1\},\{a_2,b_2\},\ldots$ and "monomers" $\{c_1\},\{c_2\},\ldots$ such that all a_i,b_j,c_k are distinct elements in I. The *weight* of a matching is the product $p_{a_1\,b_1}p_{a_2\,b_2}\cdots x_{c_1}x_{c_2}\cdots$. Then $E_k(x_I;p)$ is the sum of weights of all matchings which cover exactly k vertices of I.

For example, we have

$$\begin{aligned} E_3(x_1,x_2,x_3,x_4;p) = {} & x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 \\ & + p_{12}\,(x_3+x_4) + p_{13}\,(x_2+x_4) + p_{14}\,(x_2+x_3) \\ & + p_{23}\,(x_1+x_4) + p_{24}\,(x_1+x_3) + p_{34}\,(x_1+x_2)\,. \end{aligned}$$

Specializing $p_{ij}=0$, one obtains $E_k(x_I;0) = e_k(x_I)$, the usual elementary symmetric polynomial. Assume that $p_{i\,i+1} = q_i$, $i=1,2,\ldots,n-1$, and $p_{ij}=0$, for $|i-j|\ge 2$. Then the polynomial $E_k(x_1,\ldots,x_n;q)$ is the quantum elementary polynomial E_k, which is a coefficient of the characteristic polynomial of the 3-diagonal matrix (2.1.3). Here and below the letter q stands for the collection of $q_1,q_2,\ldots,q_{n-1}$.

2.6.3 General version of Pieri's formula

For a subset $I=\{i_1,\ldots,i_m\}$ in $\{1,2,\ldots,n\}$, let θ_I denote the collection of the elements $\theta_{i_1},\ldots,\theta_{i_m}$, and let $E_k(\theta_I;p) = E_k(\theta_{i_1},\ldots,\theta_{i_m};p)$ denote the result of substituting the Dunkl elements (2.6.7) in place of the corresponding x_i in $E(x_I;p)$. This substitution is well defined, due to Lemma 2.6.2. We can state our result as follows.

Theorem 2.6.3 (General Pieri's formula) [43, Theorem 3.1] *Let I be a subset in $\{1,2,\ldots,n\}$, and let $J=\{1,2,\ldots,n\}\setminus I$. Then, for $k\ge 1$, we have in the ring $\mathcal{E}_n^p$:*

$$E_k(\theta_I;p) = \sum \tau_{a_1\,b_1}\tau_{a_2\,b_2}\cdots\tau_{a_k b_k}\,, \tag{2.6.10}$$

where the sum is over all sequences $a_1,\ldots,a_k,b_1,\ldots,b_k$ such that (i) *$a_j\in I$, $b_j\in J$, for $j=1,\ldots,k$;* (ii) *the $a_1,\ldots,a_k$ are distinct;* (iii) *$b_1\le\cdots\le b_k$.*

The proof of Theorem 2.6.3 will be given in Section 2.6.5. In the rest of this section we summarize several corollaries of Theorem 2.6.3.

First of all, let us note that specializing $p_{ij} = 0$ in Theorem 2.6.3 results in Conjecture 11.1 from [18].

Corollary 2.6.4 [43, Corollary 3.2] [18, Conjecture 15.1] *For $k = 1, 2, \dots, n$, the following relation in the ring $\mathcal{E}_n^p$ holds*

$$E_k(\theta_1, \theta_2, \dots, \theta_n; p) = 0 .$$

Proof — In this case, the sum in (2.6.10) is over the empty set. $\square$

Define a $\mathbb{Z}[p]$-linear homomorphism π by

$$\pi : \mathbb{Z}[x_1, x_2, \dots, x_n; p] \longrightarrow \mathcal{E}_n^p$$
$$\pi : x_i \longmapsto \theta_i .$$

Corollary 2.6.5 [43, Corollary 3.3] *The kernel of π is generated over $\mathbb{Z}[p]$ by*

$$E_k(x_1, x_2, \dots, x_n; p), \quad k = 1, 2, \dots, n . \tag{2.6.11}$$

Proof — All elements (2.6.11) map to zero, due to Corollary 2.6.4. The statement now follows from dimension argument (cf. [18, Section 7]). $\square$

In particular, we can define a homomorphism $\bar{\pi}$ by

$$\bar{\pi} : \mathbb{Z}[x_1, \dots, x_n] \longrightarrow \mathcal{E}_n ,$$
$$\bar{\pi} : x_i \longmapsto \bar{\theta}_i ,$$

where $\bar{\theta}_i$ is the image in $\mathcal{E}_n$ of the element θ_i.

Corollary 2.6.6 [18, Theorem 7.1] *The kernel of $\bar{\pi}$ is generated by the elementary symmetric polynomials*

$$e_k(x_1, x_2, \dots, x_n), \quad k = 1, 2, \dots, n .$$

Thus the subring in $\mathcal{E}_n$ generated by the $\bar{\theta}_i$ is isomorphic to the cohomology of Fl_n, which is isomorphic to the quotient (2.1.1).

Likewise, let $\hat{\theta}_i$ be the image in $\mathcal{E}_n^q$ of the element θ_i, and let $\hat{\pi}$ be the $\mathbb{Z}[q]$-linear homomorphism defined by

$$\hat{\pi} : \mathbb{Z}[x_1, \dots, x_n; q] \longrightarrow \mathcal{E}_n^q ,$$
$$\hat{\pi} : x_i \longmapsto \hat{\theta}_i .$$

Corollary 2.6.7 [43, Corollary 3.5] [18, Conjecture 13.4] *The kernel of the homomorphism $\hat{\pi}$ is generated over $\mathbb{Z}[q]$ by*

$$E_k(x_1, x_2, \dots, x_n; q), \quad k = 1, 2, \dots, n .$$

Thus the subring in $\mathcal{E}_n^q$ generated over $\mathbb{Z}[q]$ by the $\hat{\theta}_i$ is isomorphic to the quantum cohomology of Fl_n, the latter being isomorphic to the quotient (2.1.4).

2.6.4 Action on the quantum cohomology

Recall that s_{ij} is the transposition of i and j in S_n, $s_i = s_{i\,i+1}$ is a Coxeter generator, and $q_{ij} = q_i q_{i+1} \cdots q_{j-1}$, for $i < j$.

Let us define the $\mathbb{Z}[q]$-linear operators t_{ij}, $1 \leq i < j \leq n$, acting on the quantum cohomology ring $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ by

$$t_{ij}(\sigma_w) = \begin{cases} \sigma_{ws_{ij}} & \text{if } \ell(ws_{ij}) = \ell(w) + 1\,, \\ q_{ij}\,\sigma_{ws_{ij}} & \text{if } \ell(ws_{ij}) = \ell(w) - 2(j-i) + 1\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6.12}$$

By convention, $t_{ij} = -t_{ji}$, for $i > j$, and $t_{ii} = 0$.

Quantum Monk's formula (Theorem 2.1.2) can be stated as saying that the quantum product of σ_{s_m} and σ_w is equal to

$$\sigma_{s_m} * \sigma_w = \sum_{a \leq m < b} t_{ab}(\sigma_w)\,.$$

The relation between the ring $\mathcal{E}_n^q$ and quantum cohomology of Fl_n is justified by the following lemma, which is proved by a direct verification.

Lemma 2.6.8 [18, Proposition 12.3] *The operators t_{ij} given by* (2.6.12) *satisfy the relations* (2.6.2)–(2.6.6) *with τ_{ij} replaced by t_{ij}, $p_{i\,i+1} = q_i$, and $p_{ij} = 0$, for $|i-j| \geq 2$,*

Thus the ring $\mathcal{E}_n^q$ acts on $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ by $\mathbb{Z}[q]$-linear transformations

$$\tau_{ij} : \sigma_w \longmapsto t_{ij}(\sigma_w)\,.$$

Monk's formula is also equivalent to the claim that the Dunkl element $\hat{\theta}_i$ acts on the quantum cohomology of Fl_n as the operator of multiplication by x_i, the latter is defined via the isomorphism (2.1.4).

Let us denote $c(k,m) = s_{m-k+1}s_{m-k+2}\cdots s_m$ and $r(k,m) = s_{m+k-1}s_{m+k-1}\cdots s_m$. These are two cyclic permutations such that $c(k,m) = (m-k+1, m-k+2, \ldots, m+1)$ and $r(k,m) = (m+k, m+k-1, \ldots, m)$.

The following statement was geometrically proved in [12] (cf. also [17]). For the reader's convenience and for consistency we show how to deduce it directly from Monk's formula.[16]

Lemma 2.6.9 *The coset of the polynomial $E_k(x_1, \ldots, x_m; q)$ in the quotient* (2.1.4) *corresponds to the Schubert class $\sigma_{c(k,m)}$ under the isomorphism* (2.1.4). *Analogously, the coset of the polynomial $E_k(x_{m+1}, x_{m+2}, \ldots, x_n)$ corresponds to the class $\sigma_{r(k,m)}$.*

Proof — By (2.1.4) and (2.6.9), it is enough to check that

$$\sigma_{c(k,m+1)} = \sigma_{c(k,m)} + (\sigma_{s_{m+1}} - \sigma_{s_m}) * \sigma_{c(k-1,m)} + q_m \sigma_{c(k-2,m-1)}.$$

[16] A less obvious statement that, the opposite is true, i.e., that Monk's formula follows from the claim of Lemma 2.6.9 has been actually demonstrated in previous sections.

This identity immediately follows from Monk's formula:

$$(\sigma_{s_{m+1}} - \sigma_{s_m}) * \sigma_{c(k-1,m)} = \big(\sum_{b>m+1} t_{m+1\,b} - \sum_{a<m} t_{am} \big)(\sigma_{c(k-1,m)}) \, .$$

The claim about $\sigma_{r(k,m)}$ can be proved using a symmetric argument. □

It is clear now that Theorem 2.6.3 implies the following statement. This statement, though in a different form, was proved in [13].

Corollary 2.6.10 (Quantum Pieri's formulas) *For $w \in S_n$ and $0 \le k \le m < n$, the product in* $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ *of Schubert classes $\sigma_{c(k,m)}$ and σ_w is given by the formula*

$$\sigma_{c(k,m)} * \sigma_w = \sum t_{a_1\, b_1} t_{a_2\, b_2} \cdots t_{a_k\, b_m}(\sigma_w), \tag{2.6.13}$$

where the sum is over $a_1, \ldots, a_k, b_1, \ldots, b_k$ such that (i) $1 \le a_j \le m < b_j < n$ *for* $j = 1, \ldots, k$; (ii) *the $a_1, \ldots, a_k$ are distinct;* (iii) $b_1 \le \cdots \le b_k$.

Likewise, the quantum product of Schubert classes $\sigma_{r(k,m)}$ and σ_w is given by the formula

$$\sigma_{r(k,m)} * \sigma_w = \sum t_{c_1\, d_1} t_{c_2\, d_2} \cdots t_{c_k\, d_k}(\sigma_w), \tag{2.6.14}$$

where the sum is over $c_1, \ldots, c_k, b_1, \ldots, d_k$ such that (i) $1 \le c_j \le m < d_j < n$ *for* $j = 1, \ldots, k$; (ii) $c_1 \le \cdots \le c_k$; (iii) *the $d_1, \ldots, d_k$ are distinct.*

We would like to emphasize that Corollary 2.6.10 does not imply Theorem 2.6.3 (or even its weaker form for $\mathcal{E}_n^q$), since the representation $\tau_{ij} \mapsto t_{ij}$ of $\mathcal{E}_n^q$ in the quantum cohomology is not exact.

2.6.5 Proof of general Pieri's formula

For a subset I in $\{1, 2, \ldots, n\}$, let $\widetilde{E}_k(I)$ denote the expression in the right-hand side of (2.6.10). By convention, $\widetilde{E}_0(I) = 1$. For $k = 1$, Theorem 2.6.3 says that

$$\widetilde{E}_1(I) = \sum_{i \in I} \sum_{j \notin I} \tau_{ij} = \sum_{i \in I} \sum_{j=1}^{n} \tau_{ij} = E_1(\theta_I; p) \, ,$$

which is obvious by (2.6.2).

It suffices to verify that the $\widetilde{E}_k(I)$ satisfy the defining relation (2.6.9). Then the claim $E_k(\theta_I; p) = \widetilde{E}_k(I)$ will follow by induction on k. Specifically, we have to demonstrate that

$$\widetilde{E}_k(I \cup \{j\}) = \widetilde{E}_k(I) + \widetilde{E}_{k-1}(I)\, \theta_j + \sum_{i \in I} \widetilde{E}_{k-2}(I \setminus \{i\})\, p_{ij} \, , \tag{2.6.15}$$

where $I \subset \{1, 2, \ldots, n\}$ and $j \notin I$. To do this we need some extra notation. For a subset $L = \{l_1, l_2, \ldots, l_m\}$ and $r \notin L$, denote

$$\langle\!\langle L \mid r \rangle\!\rangle = \sum \tau_{u_1 r} \tau_{u_2 r} \cdots \tau_{u_m r},$$

where the sum is over all permutations $u_1, u_2, \ldots, u_m$ of $l_1, l_2, \ldots, l_m$.

For I and j as in (2.6.15), let $J = \{1, 2, \ldots, n\} \setminus I = \{j_1, j_2, \ldots, j_d\}$ with $j_1 = j$. Then the first term in the right-hand side of (2.6.15) can be written in the form

$$\widetilde{E}_k(I) = \sum_{I_1 \ldots I_d \subset_k I} \langle\!\langle I_1 \mid j_1 \rangle\!\rangle \, \langle\!\langle I_2 \mid j_2 \rangle\!\rangle \cdots \langle\!\langle I_d \mid j_d \rangle\!\rangle \,, \tag{2.6.16}$$

where the notation $I_1 \ldots I_d \subset_k I$ means that the sum is over all pairwise disjoint (possibly empty) subsets $I_1, I_2, \ldots, I_d$ of I such that $\sum_s |I_s| = k$. Let

$$\widetilde{E}_k(I) = A_1 + A_2 \,, \tag{2.6.17}$$

where A_1 is the sum of terms in (2.6.16) with $I_1 = \emptyset$ and A_2 is the sum of terms with $I_1 \neq \emptyset$. Likewise, we can split the left-hand side of (2.6.15) into two parts:

$$\begin{aligned} \widetilde{E}_k(I \cup \{j\}) &= \sum_{I'_2 \cdots I'_d \subset_k I \cup \{j\}} \langle\!\langle I'_2 \mid j_2 \rangle\!\rangle \, \langle\!\langle I'_3 \mid j_3 \rangle\!\rangle \cdots \langle\!\langle I'_d \mid j_d \rangle\!\rangle \\ &= B_1 + B_2 \,, \end{aligned} \tag{2.6.18}$$

where B_1 is the sum of the terms such that $j \notin I'_2 \cup \cdots \cup I'_d$, and B_2 is the sum of terms with $j \in I'_2 \cup \cdots \cup I'_d$. We also split the second term in the right-hand side of (2.6.15) into 3 summands:

$$\begin{aligned} \widetilde{E}_{k-1}(I)\, \theta_j &= \sum_{I''_1 \ldots I''_d \subset_{k-1} I} \langle\!\langle I''_1 \mid j_1 \rangle\!\rangle \cdots \langle\!\langle I''_d \mid j_d \rangle\!\rangle \sum_{s \neq j} \tau_{js} \\ &= C_1 + C_2 + C_3 \,, \end{aligned} \tag{2.6.19}$$

where C_1 is the sum of terms with $s \in I \setminus (I''_1 \cup I''_2 \cup \cdots \cup I''_d)$; C_2 is the sum of terms with $s \in I''_2 \cup I''_3 \cup \cdots \cup I''_d \cup J$; and C_3 is the sum of terms with $s \in I''_1$.

It is immediate from the definitions that $A_1 = B_1$. It is also not hard to verify that $A_2 + C_1 = 0$, since for $I_1 \neq \emptyset$

$$\langle\!\langle I_1 \mid j_1 \rangle\!\rangle = \sum_{i \in I_1} \langle\!\langle I_1 \setminus \{i\} \mid j_1 \rangle\!\rangle \, \tau_{i j_1} \,.$$

To prove the identity (2.6.15), it thus suffice to demonstrate that

$$B_2 = C_2 \,, \tag{2.6.20}$$

$$C_3 + \sum_{i \in I} \widetilde{E}_{k-2}(I \setminus \{i\})\, p_{ij} = 0 \,. \tag{2.6.21}$$

The following lemma implies the formula (2.6.20).

Lemma 2.6.11 *For any subset K in $\{1,2,\ldots,n\}$ and j, $l \notin K$, we have*

$$\langle\!\langle K \cup \{j\} \mid l \rangle\!\rangle = \sum_{L \subset K} \langle\!\langle L \mid l \rangle\!\rangle \, \langle\!\langle K \setminus L \mid j \rangle\!\rangle \sum_{s \in L \cup \{l\}} \tau_{js} \,. \tag{2.6.22}$$

Indeed, let $T = \langle\!\langle I_2' \mid j_2 \rangle\!\rangle \cdots \langle\!\langle I_d' \mid j_d \rangle\!\rangle$ be a term of B_2. Then $j \in J_r'$ for some r. By Lemma 2.6.11, T is equal the sum of all terms $\langle\!\langle I_1'' \mid j_1 \rangle\!\rangle \cdots \langle\!\langle I_d'' \mid j_d \rangle\!\rangle \, \tau_{js}$ in C_2 with fixed $I_u'' = I_u'$ for all $u \neq r$ such that $s \in I_r'' \cup \{j_r\}$ and the subsets $I_1'' \cup I_r'' = I_r' \setminus \{j\}$. Thus $B_2 = C_2$.

Proof of Lemma 2.6.11 — Induction on $|K|$. For $K = \emptyset$, the both sides of (2.6.22) are equal to τ_{jl}. For $|K| \geq 1$, the right-hand side of (2.6.22) is equal

$$\begin{aligned}
&\sum_{L \subset K} \langle\!\langle L \mid l \rangle\!\rangle \, \langle\!\langle K \setminus L \mid j \rangle\!\rangle \sum_{s \in L \cup \{l\}} \tau_{js} \\
&= \sum_{L \subsetneq K} \left(\sum_{i \in K \setminus L} \langle\!\langle L \mid l \rangle\!\rangle \, \tau_{ij} \, \langle\!\langle K \setminus L \setminus \{i\} \mid j \rangle\!\rangle \sum_{s \in L \cup \{l\}} \tau_{js} \right) + \langle\!\langle K \mid l \rangle\!\rangle \sum_{s \in K \cup \{l\}} \tau_{js} \\
&= \sum_{i \in K} \tau_{ij} \, \langle\!\langle (K \setminus \{i\}) \cup \{j\} \mid l \rangle\!\rangle + \langle\!\langle K \mid l \rangle\!\rangle \sum_{s \in K \cup \{l\}} \tau_{js} \\
&= \langle\!\langle K \cup \{j\} \mid l \rangle\!\rangle \,.
\end{aligned}$$

The second equality is valid by induction hypothesis; the remaining equalities follow from (2.6.4) and (2.6.6). $\square$

Using a similar argument to the one after Lemma 2.6.11, one can derive the formula (2.6.21) from the following lemma:

Lemma 2.6.12 *For any subset K in $\{1,2,\ldots,n\}$ and $j \notin K$, we have*

$$\sum_{s \in K} \left(\langle\!\langle K \mid j \rangle\!\rangle \, \tau_{js} + \sum_{L \subset K \setminus \{s\}} \langle\!\langle L \mid s \rangle\!\rangle \, \langle\!\langle K \setminus L \setminus \{s\} \mid j \rangle\!\rangle \, p_{js} \right) = 0 \,.$$

This statement, in turn, is obtained from the following "quantum analogue" of Lemma 7.2 from [18]. Its proof is a straightforward extension.

Lemma 2.6.13 *For $i, u_1, u_2, \ldots, u_m \in \{1, \ldots, n\}$, we have in the ring $\mathcal{E}_n^p$*

$$\begin{aligned}
&\sum_{r=1}^{m} \tau_{i\,u_r} \tau_{i\,u_{r+1}} \cdots \tau_{i\,u_m} \tau_{i\,u_1} \tau_{i\,u_2} \cdots \tau_{i\,u_r} \\
&= \sum_{r=1}^{m} p_{i\,u_r} \, \tau_{u_r\,u_{r+1}} \tau_{u_r\,u_{r+2}} \cdots \tau_{u_r\,u_m} \tau_{u_r\,u_1} \tau_{u_r\,u_2} \cdots \tau_{u_r\,u_{r-1}} \,,
\end{aligned} \tag{2.6.23}$$

where, by convention, the index u_{m+1} is identified with u_1.

Proof — Induction on m. The base of induction, for $m = 1$, is easily established by (2.6.3): $\tau_{i\,u_1}\tau_{i\,u_1} = p_{i\,u_1}$. Assume that $m > 1$. Applying (2.6.4) and (2.6.6) to the left-hand side of (2.6.23), we obtain:

$$\begin{aligned}
&\sum_{r=1}^{m} \tau_{i\,u_r}\tau_{i\,u_{r+1}}\cdots\tau_{i\,u_{m-1}}\left(\tau_{i\,u_m}\tau_{i\,u_1}\right)\tau_{i\,i_2}\cdots\tau_{i\,u_r} \\
&= \sum_{r=1}^{m} \tau_{i\,u_r}\tau_{i\,u_{r+1}}\cdots\tau_{i\,u_{m-1}}\left(\tau_{i\,u_1}\tau_{u_1\,u_m} + \tau_{u_m\,u_1}\tau_{i\,u_m}\right)\tau_{i\,u_2}\cdots\tau_{i\,u_r} \\
&= \left(\sum_{r=1}^{m-1} \tau_{i\,u_r}\tau_{i\,u_{r+1}}\cdots\tau_{i\,u_{m-1}}\tau_{i\,u_1}\tau_{i\,u_2}\cdots\tau_{i\,u_r}\right)\tau_{u_1\,u_m} \\
&\quad + \tau_{u_m\,u_1}\left(\sum_{r=2}^{m} \tau_{i\,u_r}\tau_{i\,u_{r+1}}\cdots\tau_{i\,u_m}\tau_{i\,u_2}\tau_{i\,i_3}\cdots\tau_{i\,u_r}\right).
\end{aligned}$$

By induction hypothesis, this expression is equal to

$$\begin{aligned}
&\left(\sum_{r=1}^{m-1} p_{i\,u_r}\,\tau_{u_r\,u_{r+1}}\tau_{u_r\,u_{r+2}}\cdots\tau_{u_r\,u_{m-1}}\tau_{u_r\,u_1}\tau_{u_r\,u_2}\cdots\tau_{u_r\,u_{r-1}}\right)\tau_{u_1\,u_m} \\
&\quad + \tau_{u_m\,u_1}\left(\sum_{r=2}^{m} p_{i\,u_r}\,\tau_{u_r\,u_{r+1}}\tau_{u_r\,u_{r+2}}\cdots\tau_{u_r\,u_m}\tau_{u_r\,u_2}\tau_{u_r\,u_3}\cdots\tau_{u_r\,u_{r-1}}\right) \\
&= p_{i\,u_1}\,\tau_{u_1\,u_2}\tau_{u_1\,u_3}\cdots\tau_{u_1\,u_m} + p_{i\,u_m}\,\tau_{u_m\,u_1}\tau_{u_m\,u_2}\cdots\tau_{u_m\,u_{m-1}} \\
&\quad + \sum_{r=2}^{m-1} p_{i\,u_r}\,\tau_{u_r\,u_{r+1}}\cdots\tau_{u_r\,u_{m-1}}\left(\tau_{u_r\,u_1}\tau_{u_1\,u_m} + \tau_{u_m\,u_1}\tau_{u_r\,u_m}\right)\tau_{u_r\,u_2}\cdots\tau_{u_r\,u_{r-1}}.
\end{aligned}$$

The latter expression coincides with the right-hand side of (2.6.23). □

This completes the proof of Theorem 2.6.3.

Bibliography

[1] V. I. Arnold, The cohomology ring of colored braid group, *Math. Notes* **5** (1969), 138–140.

[2] A. Astashkevich and V. Sadov, Quantum cohomology of partial flag manifolds $F_{n_1,\ldots,n_k}$, *Comm. Math. Phys.* **170** (1995), 503–528.

[3] C. A. Athanasiadis, *Algebraic combinatorics of graph spectra, subspace arrangements, and Tutte polynomials,* Ph. D. thesis, M.I.T., 1996.

[4] M. A. Auric, Généralisation d'un théorème de Laguerre, *C. R. Acad. Sci. Paris* **137** (1903), 967–969.

[5] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, Schubert cells and cohomology of the space G/P, *Russian Math. Surveys* **28** (1973), 1–26.

[6] A. Bertram, Quantum Schubert calculus, to appear in *Adv. Math.*

[7] A. Borel, Sur la cohomologie de espaces fibrés principaux et des espaces homogénes des groupes de Lie compacts, *Ann. of Math.* (2) **57** (1953), 115–207.

[8] N. Bourbaki, *Groupes et Algèbres de Lie,* 2ème partie, Ch. IV–VI, Paris, Hermann, 1968.

[9] E. Brieskorn, Sur les groupes de tress, in: *Séminaire Bourbaki* 1971/72, Lecture Notes in Math. **317**, Springer Verlag, 1973, pp. 21–44.

[10] C. Chevalley, Sur les décompositions cellulaires des espaces G/B, *Proc. Symp. Pure Math.* **56**, Part 1, 1–23, Amer. Math. Soc., Providence, RI, 1994.

[11] J. L. Chandon, J. Lemaire, and J. Pouget, Dénombrement des quasi-ordres sur un ensemble fini, *Math. Inform. Sci. Humaines* **62** (1978), 61–80, 83.

[12] I. Ciocan-Fontanine, Quantum cohomology of flag varieties, *Intern. Math. Research Notes* (1995), No. 6, 263–277.

[13] I. Ciocan-Fontanine, On quantum cohomology rings or partial flag varieties, preprint dated February 9, 1997.

[14] M. Demazure, Désingularization des variétés de Schubert généralisées, *Ann. Scient. Ecole Normale Sup.* (4) **7** (1974), 53–88.

[15] C. Ehresmann, Sur la topologie de certains espaces homogènes, *Ann. Math.* **35** (1934), 396–443.

[16] P. Di Francesco and C. Itzykson, Quantum intersection rings, in: *Progress in Mathematics* **129**, Birkhäuser, 1995, 81–148.

[17] S. Fomin, S. Gelfand, and A. Postnikov, Quantum Schubert polynomials, to appear in *J. Amer. Math. Soc.*

[18] S. Fomin and A. N. Kirillov, Quadratic algebras, Dunkl elements, and Schubert calculus, preprint AMSPPS #199703-05-001.

[19] W. Fulton and R. Pandharipande, Notes on stable maps and quantum cohomology, preprint alg-geom/9608011.

[20] I. M Gelfand, M. I. Graev, and A. Postnikov, Combinatorics of hypergeometric functions associated with positive roots, in: *Arnold-Gelfand Mathematical Seminars: Geometry and Singularity Theory,* Birkhäuser, 1997, 205–221.

[21] I. Gessel, private communication.

[22] A. Givental and B. Kim, Quantum cohomology of flag manifolds and Toda lattices, *Comm. Math. Phys.* **168** (1995), 609–641.

[23] I. P. Goulden, D. M. Jackson, *Combinatorial Enumeration,* John Wiley & Sons, 1983.

[24] P. Headley, *Reduced expressions in infinite Coxeter groups,* Ph. D. thesis, University of Michigan, 1994.

[25] B. Kim, Quantum cohomology of partial flag manifolds and a residue formula for their intersection pairing, *Intern. Math. Research Notes* (1995), No. 1, 1–16.

[26] B. Kim, On equivariant quantum cohomology, preprint q-alg/9509029.

[27] B. Kim, Quantum cohomology of flag manifolds G/B and quantum Toda lattices, preprint alg-geom/9607001.

[28] M. Kontsevich and Yu. Manin, Gromov-Witten classes, quantum cohomology, and enumerative geometry, *Comm. Math. Phys.* **164** (1994), 525–562.

[29] B. Kostant, Flag manifold quantum cohomology, the Toda lattice, and the representation with highest weight ρ, *Selecta Math. (N.S.)* **2** (1996), 43–91.

[30] B. Kostant, S. Kumar, The nil Hecke ring and cohomology of G/P for a Kac-Moody group G, *Proc. Nat. Acad. Sci. U.S.A.* **83** (1986), 1543–1545.

[31] A. Lascoux and M. P. Schützenberger, Polynômes de Schubert, *C. R. Ac. Sci.* **294** (1982), 447–450.

[32] A. Lascoux and M. P. Schützenberger, Fonctorialité de polynômes de Schubert, *Contemp. Math.* **88** (1989), 585–598.

[33] J. Li and G. Tian, The quantum cohomology of homogeneous varieties, to appear in *J. Algebraic Geom.*

[34] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press, Oxford, 1979.

[35] I. G. Macdonald, *Notes on Schubert polynomials*, Publications du LACIM, Montréal, 1991.

[36] D. Monk, The geometry of flag manifolds, *Proc. London Math. Soc.* **(3) 9** (1959), 253–286.

[37] N. Obreschkoff, *Jahresber. Deutsch. Math.-Verein.* **36** (1927), 43–45.

[38] P. Orlik, L. Solomon, Combinatorics and topology of complements of hyperplanes, *Invent. Math.* **56** (1980), 167–189.

[39] P. Orlik, H. Terao, *Arrangements of Hyperplanes,* Springer-Verlag, Berlin/-Heidelberg/New York, 1992.

[40] G. Pólya, Aufgabe 35, Section 2, *Jahresber. Deutsch. Math.-Verein.* **35** (1926), 48.

[41] G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, vol. II, Springer-Verlag, Berlin/Heidelberg/New York, 1976.

[42] A. Postnikov, Intransitive trees, to appear in *J. Combin. Theory Ser. A* **77** (1997).

[43] A. Postnikov, On a quantum version of Pieri's formula, preprint dated March 23, 1997.

[44] A. Postnikov and R. Stanley, Deformations of Coxeter hyperplane arrangements, preprint, version of February 1997.

[45] H. Prüfer, Neuer Beweis eines Satzes über Permutationen, *Arch. Math. Phys.* **27** (1918), 742–744.

[46] Y. Ruan and G. Tian, Mathematical theory of quantum cohomology, *J. Diff. Geom.* **42** (1995), no. 2, 259–367.

[47] D. Scott, P. Suppes, Foundational aspects of theories of measurement, *J. Symbolic Logic* **23** (1958), 113–128.

[48] J.-Y. Shi, *The Kazhdan-Lusztig cells in certain affine Weyl groups,* Lecture Notes in Mathematics, no. 1179, Springer-Verlag, Berlin/Heidelberg/New York, 1986.

[49] J.-Y. Shi, Sign types corresponding to an affine Weyl group, *J. London Math. Soc.* **35** (1987), 56–74.

[50] F. Sottile, Pieri's formula for flag manifolds and Schubert polynomials, *Annales de l'Institut Fourier* **46** (1996), 89–110.

[51] R. Stanley, *Enumerative Combinatorics,* vol. 1, Wadsworth & Brooks Cole, Belmont, CA, 1986.

[52] R. Stanley, *Enumerative Combinatorics,* vol. 2, to appear.

[53] R. Stanley, Hyperplane arrangements, interval orders, and trees, *Proc. Nat. Acad. Sci. U.S.A.* **93** (1996), 2620–2625.

[54] B. Sturmfels, *Algorithms in invariant theory,* Springer-Verlag, 1993.

[55] W. T. Trotter, *Combinatorics and Partially Ordered Sets,* The Johns Hopkins University Press, Baltimore and London, 1992.

[56] C. Vafa, Topological mirrors and quantum rings, in: *Essays on mirror manifolds* (S.-T. Yau, ed.), International Press, 1992.

[57] H. Whitney, A logical expansion in mathematics, *Bull. Amer. Math. Soc.* **38** (1932), 572–579

[58] R. L. Wine, J. E. Freund, On the enumeration of decision patterns involving n means, *Ann. Math. Statist.* **28** (1957), 256–259.

[59] R. Winkel, On the multiplication of Schubert polynomials, preprint dated January 1997.

[60] E. Witten, Two-dimensional gravity and intersection theory on moduli space, *Surveys in Diff. Geometry* **1** (1991), 243–310.

[61] T. Zaslavsky, Facing up to arrangements: face-count formulas for partitions of space by hyperplanes, *Mem. Amer. Math. Soc.,* vol. 1, no. 154, 1975.