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Nilpotent Orbits and the Affine Flag Manifold

by

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Abstract

Let G be a connected simple algebraic group over the complex numbers with Lie algebra $\mathfrak{g}$. Let N be a nilpotent element in $\mathfrak{g}$ and let $Z_G(N)$ be the centralizer in G of N. In general, $Z_G(N)$ is not connected and many applications require computing the group of components of $Z_G(N)$.

When G is of adjoint type, we give a unified description of the conjugacy classes in the group of components of $Z_G(N)$, generalizing the Bala-Carter classification of nilpotent orbits in $\mathfrak{g}$. Then we study how the group of components changes when we pass to the simply-connected cover of G. We conclude the first part of the thesis by showing that the irreducible representations of the group of components come from representations of a parabolic subgroup of G associated to N. This result should be useful for computing the G-module structure of the regular functions on any cover of the orbit in $\mathfrak{g}$ through N.

In the second part of the thesis, we study a family of representations U_t of the affine Weyl group W_a. The main result here identifies U_t as the representation of W_a on the total homology of the space of affine flags which contain a family of elements n_t in the affine Lie algebra. We also compute the Euler characteristic of the space of partial flags containing n_t and give a connection with the characteristic polynomials of hyperplane arrangements.

Thesis Supervisor: George Lusztig
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For Patty . . . Enjoy!

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Contents

Introduction

This thesis is divided into two independent parts.

Let G be a connected simple algebraic group of adjoint type defined over the complex numbers with Lie algebra $\mathfrak{g}$. For a nilpotent element N in $\mathfrak{g}$, one often needs to compute the finite group $A(N) := Z_G(N)/Z_G^0(N)$, where $Z_G(N)$ is the centralizer of N in G under the adjoint action and $Z_G^0(N)$ is its identity component. For example, these groups play an important role in the Springer correspondence [Sp1]. The computation of $A(N)$ appears in the literature [Al], [Mi] and in three texts on the subject [Ca2], [CM], [Hu2], but without a satisfactory unified approach.

In the first part of the thesis, we give a unified approach to the determination of the groups $A(N)$. The Bala-Carter classification of nilpotent orbits in $\mathfrak{g}$ states that nilpotent orbits are in bijection with conjugacy classes of pairs $(\mathfrak{l}, N)$, where N is a distinguished nilpotent element in the Levi subalgebra $\mathfrak{l}$ [BC1], [BC2]. In Chapter 2, we generalize the Bala-Carter classification by proving there is a bijection between conjugacy classes of pairs (N, C), where N is nilpotent and C is a conjugacy class in $A(N)$, and conjugacy classes of pairs $(\mathfrak{l}, N)$, where $\mathfrak{l}$ is the centralizer in $\mathfrak{g}$ of a single semisimple element in G and N is a distinguished nilpotent in $\mathfrak{l}$. We also study the conjugacy classes in $Z_G(N)/Z_G^0(N)$ when G is simply-connected. The structure of these groups is also already known [L1], [CM], but the methods presented here are more unified.

Let P be the parabolic subgroup of G determined by a Jacobson-Morozov triple for N. It is known that $Z_G(N)$ is contained in P. In Chapter 3, we show that all irreducible representations of $Z_G(N)/Z_G^0(N)$ arise from irreducible representations of P which are trivial on $Z_G^0(N)$. This fact should be helpful for determining the G-module structure of the regular functions on any cover of the nilpotent orbit in $\mathfrak{g}$ through N, in a manner similar to the work of McGovern [Mc] and Graham [Gr].

In the second half of the thesis, we study some aspects of the affine flag manifold. Let W_a be the affine Weyl group associated to the Weyl group W of G of rank n. For $w \in W_a$ let $s(w)$ be the least number of reflections needed to write w as a product of reflections. In Chapter 4, we construct a permutation representation U_t of W_a of dimension t^n for $t \in \mathbb{N}$. This representation has the property that for w of finite order, the trace of w on U_t equals $t^{n-s(w)}$ when t is not divisible by certain primes.

Let U_t also denote the restriction of U_t to W^J, the finite Weyl subgroup of W_a corresponding to a subset J of simple reflections for W_a. We show that U_t decomposes into a direct sum of representations induced from a parabolic subgroup of W^J. This fact leads to a new way to determine the characteristic polynomials of certain hyperplane arrangements which were computed by Orlik and Solomon in [OS2].

In Chapter 5, we recall the construction by Lusztig of certain regular, semisimple, nil-elliptic elements n_t in an affine Lie algebra. Fan has shown that the Euler characteristic of the space of affine flags containing n_t is t^n; this extends a result of Lusztig-Smelt in type A_n [F],[LS]. Generalizing to the space of partial affine flags associated to W^J containing n_t, our result is that the Euler characteristic is

$$\frac{(t+m_1)(t+m_2)...(t+m_j)t^{n-j}}{|W^J|}$$

where $m_1, \ldots, m_j$ are the exponents of W^J. D. S. Sage [Sa] independently proved this formula for the classical groups.

Finally, using work of Lusztig [L3] and Alvis-Lusztig [AL], we show that the virtual representation of W^J on the total homology of the space of affine flags containing n_t is U_t.

Chapter 1

Preliminaries

In this chapter we fix notation and review results from the theory of algebraic groups and Coxeter groups. Good references for such material are [Sp2], [OV], and [Hu1].

1.1 First Definitions

Let G be a complex reductive algebraic group and let $T \subset B \subset G$ be a maximal torus contained in a Borel subgroup in G. Let n be the dimension of T.

A (rational) character of G is a homomorphism of algebraic groups $\chi : G \to \mathbf{C}^*$. The set of all rational characters of G, denoted $X^*(G)$, becomes an abelian group under pointwise multiplication. In the case $G = T$, $X^*(T)$ is a free abelian group (also called a lattice) of rank n. We will often use $\hat{L}$ instead of $X^*(T)$ and refer to characters of T as weights.

A cocharacter of G is a homomorphism of algebraic groups $\chi : \mathbf{C}^* \to G$. When $G = T$, the set of cocharacters, denoted $X_*(T)$, becomes an abelian group under pointwise multiplication and is a lattice of rank n. Since every homomorphism from $\mathbf{C}^*$ to $\mathbf{C}^*$ is of the form $z \to z^k$, there is a pairing $\langle\ ,\ \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$. Explicitly, if $\chi \in X^*(T)$ and $\lambda \in X_*(T)$, then $\chi(\lambda(z)) = z^{\langle \chi, \lambda \rangle}$.

We will often use the next lemma (see [OV, Theorem 3.2.5])

Lemma 1. *There is a bijection φ between the closed subgroups of T and the subgroups of $X^*(T)$, which to a subgroup $Y \subset X^*(T)$ assigns the subgroup*

$$T^Y = \{x \in T |\ \chi(x) = 1 \text{ for all } \chi \in Y\} \subset T.$$

Moreover, let $r_1, \ldots, r_m (m \leq n)$ be the nonzero invariant factors of Y (as subgroup of the free abelian group $X^*(T)$). There is an isomorphism $\varphi : T \to (\mathbf{C}^*)^n$ such that

$$\varphi(T^Y) = \{(x_1, \ldots, x_n) \in (\mathbf{C}^*)^n |\ x_1^{r_1} = \cdots = x_m^{r_m} = 1\}.$$

The root system Φ of G is a subset of $X^*(T)$ and the coroot system $\Phi^\vee$ of G is a subset of $X_*(T)$. For $\alpha \in \Phi$, let $\alpha^\vee \in \Phi^\vee$ be the corresponding coroot. The choice of Borel subgroup determines a set of positive roots Φ^+ and a set of simple roots $\Pi = \{\alpha_i\}$ in Φ. Let $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$. When $i \neq j$, $a_{ij} \leq 0$ and $a_{ij}a_{ji}$ equals $0, 1, 2,$ or 3, and we can construct the Dynkin diagram from this information.

Any $\alpha \in \Phi^+$ can be expressed in $\hat{L}$ as $\sum_{i=1}^n a_i\alpha_i$ where the a_i are non-negative integers. The height of α, denoted $\mathrm{ht}(\alpha)$, equals $\sum a_i$. If G is simple, there exists a unique highest root $\theta \in \Phi$ for which $\mathrm{ht}(\theta) \geq \mathrm{ht}(\alpha)$ for all $\alpha \in \Phi$.

The key property of θ is that $\theta + \alpha_i$ is not a root for any simple root α_i.

Set $\alpha_0 = -\theta$ and let $\tilde{\Pi} = \Pi \cup \{\alpha_0\}$. The key property of θ implies $\langle \alpha_0, \alpha_i^\vee \rangle \leq 0$ for all $i \geq 1$; thus we can build the extended Dynkin diagram from the Dynkin diagram by adding an extra node which corresponds to α_0. Define the coefficients c_i of θ from the equation $\theta = \sum_{i=1}^n c_i\alpha_i$ and set $c_0 = 1$. We label the α_i-node in the extended Dynkin diagram with the number c_i as in the figures of Chapter 6.

For any subset J of $\tilde{\Pi}$ (always proper), let L_J be the lattice in $\hat{L} = X^*(T)$ generated by J. Instead of L_Π, we will write L for the lattice generated by Π (the root lattice). The intersection $\Phi \cap L_J$ is an abstract root system, denoted by Φ_J. The key property of θ implies that J is a set of simple roots for Φ_J.

For $J \subsetneq \tilde{\Pi}$, let d_J be the greatest common divisor of those c_i for which $\alpha_i \notin J$. Define $\tau_J \in L$ to be

$$\begin{aligned} \tau_J :=& \frac{1}{d_J} \sum_{\alpha_i \in J} c_i\alpha_i \\ =& -\frac{1}{d_J} \sum_{\alpha_i \in \tilde{\Pi} - J} c_i\alpha_i. \end{aligned} \tag{1.1}$$

Then the torsion subgroup of L/L_J is isomorphic to $\mathbb{Z}/d_J$ and is generated by the image of τ_J.

Let $W = N_G(T)/T$ be the Weyl group of G. For $w \in W$ denote by $\dot{w} \in N_G(T)$ any representative of w. For $\alpha \in \Phi$, let s_α be the unique Weyl group element which acts as the identity on the kernel of α. For $J \subsetneq \tilde{\Pi}$, let $S_J = \{s_\alpha |\ \alpha \in J\}$ and let W_J be the subgroup of W generated by the elements in S_J. So W_J is the Weyl group of the root system Φ_J.

The Weyl group acts on T and hence also on $X^*(T)$ and $X_*(T)$.

1.2 Pseudo-Levi subalgebras

Assume G is simple and connected. Let $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$ be the Lie algebras of $T \subset B \subset G$. Denote the adjoint action of G on itself and on $\mathfrak{g}$ by Ad and the adjoint action of $\mathfrak{g}$ on itself by ad.

By abuse of notation, we also view the characters of T as elements of $\mathfrak{t}^*$ (these are the differentials of the characters of T). In particular, Φ is also a subset of $\mathfrak{t}^*$. For $\alpha \in \Phi$, there exists a isomorphism x_α from $\mathbf{C}$ onto a unique closed subgroup U_α of G such that $tx_\alpha(u)t^{-1} = x_\alpha(\alpha(t)u)$ where $t \in T$ and $u \in \mathbf{C}$ [Sp2, 9.3.6]. Let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{t}\}.$$

The Lie algebra of U_α is $\mathfrak{g}_\alpha$.

If S is a subset of T, then the centralizer of S in G is denoted $Z_G(S)$ and the centralizer of S acting on $\mathfrak{g}$ via Ad is denoted $Z_\mathfrak{g}(S)$. The Lie algebra of $Z_G(S)$ is $Z_\mathfrak{g}(S)$ [Sp2, 4.4.7].

By [Ca2, Theorem 3.5.3] the identity component of $Z_G(S)$ is generated by T and those U_α with $\alpha(S) = 1$. Also $Z_{\mathfrak{g}}(S)$ is spanned by $\mathfrak{t}$ and those $\mathfrak{g}_\alpha$ with $\alpha(S) = 1$.

For $J \subsetneq \tilde{\Pi}$, we define G_J to be the subgroup of G generated by T and those U_α with $\alpha \in \Phi_J$. Then G_J is a connected reductive algebraic subgroup of G with root system Φ_J. The Lie algebra of G_J is

$$\mathfrak{g}_J := \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_J} \mathfrak{g}_\alpha.$$

Since G_J is reductive, the center Z of G_J is contained in T. Hence Z consists of the elements of T which lie in the kernel of all the roots in Φ_J. In other words, $Z = T^{L_J}$. Thus by Lemma 1, the identity component Z^0 of Z is isomorphic to a torus whose dimension is the rank of $\hat{L}/L_J$ and Z/Z^0 is isomorphic to the torsion subgroup of $\hat{L}/L_J$. In particular, if G is adjoint (meaning $\hat{L} = L$), then Z/Z^0 is cyclic of order d_J.

When $J \subset \Pi$, then G_J is a Levi subgroup of G and $\mathfrak{g}_J$ is a Levi subalgebra of $\mathfrak{g}$. For lack of better terminology when $J \subsetneq \tilde{\Pi}$, we call any G-conjugate of G_J a pseudo-Levi subgroup of G and any G-conjugate of $\mathfrak{g}_J$ a pseudo-Levi subalgebra of $\mathfrak{g}$. We call G_J (resp. $\mathfrak{g}_J$) a standard pseudo-Levi subgroup (resp. subalgebra). Note that a Levi subgroup/subalgebra of a pseudo-Levi subgroup/subalgebra is again a pseudo-Levi subgroup/subalgebra.

A Levi subgroup is characterized by the fact that it is the centralizer of a torus in G (and so it is automatically connected). By choosing an element as regular as possible in the torus, the Levi subgroup is the connected centralizer of that element. More generally,

Proposition 2. *Pseudo-Levi subgroups are the subgroups of G of the form $Z_G^0(x)$ where x is a semisimple element in G. Pseudo-Levi subalgebras are the subalgebras of $\mathfrak{g}$ which are of the form $Z_{\mathfrak{g}}(x)$ where x is a semisimple element in G.*

Proof. We prove the second statement, the statements being equivalent. We can assume that G is adjoint, so that $\hat{L} = L$. The proof follows Springer-Steinberg [SS]. Recall that every compact torus $\hat{S}$ contained in an algebraic torus S possesses an element y such that y^k ($k \in \mathbb{Z}, k \neq 0$) is not in the kernel of any non-trivial character of S. Such a y is called a topological generator for $\hat{S}$.

Let $\mathfrak{g}_J$ be a (standard) pseudo-Levi subalgebra. Since the torsion subgroup of L/L_J is cyclic of order d_J, we have by Lemma 1 that $T^{L_J} \simeq (\mathbf{C}^*)^r \times \mathbb{Z}/d_J$. Choose $x_1 \in (\mathbf{C}^*)^r$ to be a topological generator of the compact torus in $(\mathbf{C}^*)^r$ and x_2 to be a generator in $\mathbb{Z}/d_J$. By Lemma 1, L_J consists of the characters of T which are trivial on $x = x_1x_2$ and hence Φ_J consists of the roots of G which are trivial on x. It follows that $\mathfrak{g}_J = Z_{\mathfrak{g}}(x)$.

Conversely, let $x \in T$ and consider the subalgebra $Z_{\mathfrak{g}}(x)$. Let $Y = \{\chi \in X^*(T) | \chi(x) = 1\}$. Then as above, $T^Y \simeq (\mathbf{C}^*)^r \times A$ where A is some finite abelian group. Now $x \in T^Y$ so we can write $x = x_1x_2$ where $x_1 \in (\mathbf{C}^*)^r$ and $x_2 \in A$. Since A is finite, x_2 has finite order. It is now clear that A must be the subgroup generated by x_2, i.e. A is cyclic, for otherwise the characters of T which are trivial on x would be more than just Y.

Pick x_1' to be a topological generator in the compact torus $(S^1)^r \subset (\mathbf{C}^*)^r \subset T^Y$. Then $x' = x_1'x_2 \in (S^1)^n \subset T$ has the property that $Z_{\mathfrak{g}}(x') = Z_{\mathfrak{g}}(x)$.

Let $p : V \to (S^1)^n$ be the universal cover of the compact torus in T. Let $v \in V$ be a representative of x'. Conjugate v via the affine Weyl group W_a into the fundamental domain for the action of W_a on V (see Chapter 4 for definitions). Let J be the roots in $\tilde{\Pi}$ which are integral on v. Then Φ_J are the roots of Φ which are integral on v. Hence, Φ_J is W-conjugate to the roots which are trivial on x'. In other words, $Z_{\mathfrak{g}}(x')$ is conjugate under G to

$$\mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_J} \mathfrak{g}_\alpha = \mathfrak{g}_J,$$

which completes the proof. □

Remark 3. When G is adjoint, a corollary of the proof (repeated for the group version) is that the center Z of $Z_G^0(x)$ has the property that Z/Z^0 is cyclic and the image of x generates Z/Z^0.

The following proposition addresses the question of when $\mathfrak{g}_J$ is G-conjugate to $\mathfrak{g}_{J'}$ for $J, J' \subsetneq \tilde{\Pi}$.

Proposition 4. *For $J, J' \subsetneq \tilde{\Pi}$, the following are equivalent*

1. *J is W-conjugate to J'*
2. *Φ_J is W-conjugate to $\Phi_{J'}$*
3. *W_J is W-conjugate to $W_{J'}$*
4. *$\mathfrak{g}_J$ is G-conjugate to $\mathfrak{g}_{J'}$*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear.

For the implication (4) $\Rightarrow$ (1) suppose $\mathrm{Ad}(g)\mathfrak{g}_J = \mathfrak{g}_{J'}$ for some $g \in G$. Then $\mathrm{Ad}(g)\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}_{J'}$. Hence there exists $g' \in G_{J'}$ such that $\mathrm{Ad}(g'g)\mathfrak{t} = \mathfrak{t}$. Consequently, $g'g \in N_G(T)$ and we denote by w the image of $g'g$ in W. Then wJ is a set of simple roots for $\Phi_{J'}$. It follows that there exists $w' \in W_{J'}$ such that $w'wJ = J'$. □

1.3 Equivalence classes of subsets of $\tilde{\Pi}$

Assume G is simply-connected, simple, and connected. In order to apply Proposition 4, we need a way to determine the equivalence classes of subsets of $\tilde{\Pi}$, where J is equivalent to J' if and only if $J = wJ'$ for some $w \in W$. The next proposition is helpful for this.

Let $\Lambda = \hat{L}/L$ (the character lattice modulo the root lattice). Λ is a finite abelian group because the two lattices have the same rank.

For $J \subsetneq \tilde{\Pi}$, let $(w_0)_J$ be the longest element of the Weyl group W_J. We write w_0 instead of $(w_0)_\Pi$. Let $W_\Lambda = \{w \in W \,|\, w\tilde{\Pi} = \tilde{\Pi}\}$. The proof of the following proposition can be found in [IM].

Proposition 5 (Iwahori-Matsumoto). *The non-identity elements in W_Λ are of the form $w_0(w_0)_J$ where J is a maximal proper subset of Π and $c_i = 1$ for $\alpha_i \in \Pi - J$. Moreover W_Λ is isomorphic to Λ and acts simply-transitively on those $\alpha_i \in \tilde{\Pi}$ with $c_i = 1$.*

We indicate the action of W_Λ on $\tilde{\Pi}$ in the figures of Chapter 6.

We can now determine the equivalence classes of subsets of $\tilde{\Pi}$ under W. We repeatedly apply Proposition 5 both for $\tilde{\Pi}$ itself and for the extended Dynkin diagram associated to any proper subset of $\tilde{\Pi}$.

For A_n, C_n, G_2, F_4, E_6, and E_8, we find that the equivalence class of the subset J of $\tilde{\Pi}$ is determined by the isomorphism type of Φ_J and the length of the roots in J. We distinguish between different root lengths by placing a tilde over any summand of Φ_J containing only short roots.

For B_n the isomorphism type of Φ_J determines the equivalence class of J in $\tilde{\Pi}$ if we distinguish between the following pairs: A_1 and B_1; D_2 and $2A_1$; D_3 and A_3.

In E_7 there are three instances of subsets J', J'' (up to conjugacy) which are not conjugate under W but for which $\Phi_{J'} \simeq \Phi_{J''}$. This happens when $\Phi_{J'} \simeq \Phi_{J''}$ is of type A_5, $3A_1$, or $A_3 + A_1$. We can detect that J', J'' are not conjugate by computing the torsion subgroup of the abelian groups $\hat{L}/L_{J'}, \hat{L}/L_{J''}$. For one subset the torsion part is trivial, but for the other the torsion part is $\mathbb{Z}/2$ (see Section 2.4). When we want to distinguish between these two cases, we give the root system corresponding to the former situation one prime and the latter two primes.

In D_n the isomorphism type of Φ_J determines the equivalence class of J in $\tilde{\Pi}$ unless $\Phi_J \simeq A_{i_1} + A_{i_2} + \cdots + A_{i_k}$ where all i_j are odd and $\sum(i_j+1) = n$ (this is called the very even case). As in type B_n, we are distinguishing between the following pairs: D_2 and $2A_1$; D_3 and A_3. In the very even cases, however, there exists two subsets J', J'' (up to conjugacy) which are not conjugate but for which $\Phi_{J'} \simeq \Phi_{J''}$. We can detect that J', J'' are not conjugate because the images in $\hat{L}/L \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$ of the torsion subgroup of $\hat{L}/L_{J'}$ and $\hat{L}/L_{J''}$ are different.

Chapter 2

Component Group of the Centralizer of a Nilpotent

Let $N \in \mathfrak{g}$ be a nilpotent element. Let $Z_G(N)$ be the elements of G which centralize N and let $Z_G^0(N)$ be the identity component of $Z_G(N)$. We want to understand the finite group of components $Z_G(N)/Z_G^0(N)$, both when G is simply-connected and when G is adjoint. Let $\mathcal{O}_N \subset \mathfrak{g}$ be the G-orbit through N. When G is simply-connected, the group of components coincides with the fundamental group of $\mathcal{O}_N$, so we denote it by $\pi_1(\mathcal{O}_N)$. When G is of adjoint type, we denote the group of components by $A(N)$. So $\pi_1(\mathcal{O}_N)$ is a central extension of $A(N)$.

In this chapter we prove a generalization of the Bala-Carter theorem for nilpotent orbits in $\mathfrak{g}$. The generalization together with results in the next chapter will allow us to determine each $A(N)$ and write down its character table. We also study the conjugacy classes in each $\pi_1(\mathcal{O}_N)$ and write down its character table.

2.1 Review of results about nilpotent orbits

To each nilpotent orbit $\mathcal{O}$ there is a weighted Dynkin diagram which completely determines the orbit; we now recall how to construct the weighted Dynkin diagram.

For $N \in \mathcal{O}$, the Jacobson-Morozov theorem implies the existence of $M, H \in \mathfrak{g}$ such that $[H, N] = 2N$, $[H, M] = -2M$, and $[N, M] = H$. So $\{N, H, M\}$ generate a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbf{C})$. This implies in particular that H is semisimple in $\mathfrak{g}$ and conjugating $\{N, H, M\}$ via an element of G, we can assume that $H \in \mathfrak{t}$. By $\mathfrak{sl}_2(\mathbf{C})$-theory, it follows that $\alpha_i(H) \in \mathbb{Z}$ for $\alpha_i \in \Pi$. Conjugating $\{N, H, M\}$ via an element of W, we can assume that $\alpha_i(H) \geq 0$ for all $\alpha_i \in \Pi$. Assigning $\alpha_i(H)$ (which turns out to be $0, 1$, or 2) to the α_i-node $(1 \leq i \leq n)$ of the Dynkin diagram yields the weighted Dynkin diagram associated to $\mathcal{O}$.

We refer to H in the $\mathfrak{sl}_2$-triple $\{N, H, M\}$ as the neutral element.

Definition 6. A nilpotent element N in a reductive Lie algebra $\mathfrak{g}'$ is called **distinguished** if the conditions $X \in \mathfrak{g}'$ semisimple and $[X, N] = 0$ imply that X is in the center of $\mathfrak{g}'$. We also call a nilpotent orbit distinguished if any (hence all) of its elements are distinguished.

Note that N is distinguished in $\mathfrak{g}$ if and only if $Z_G^0(N)$ is unipotent.

In type A_n, only the regular nilpotent orbit is distinguished. For other simple $\mathfrak{g}$, the list of distinguished nilpotent orbits can be found in [Ca2].

Remark 7. We observe that distinguished nilpotent orbits in a simple Lie algebra $\mathfrak{g}$ are invariant under any automorphism of $\mathfrak{g}$ since their weighted Dynkin diagrams are unchanged under any automorphism of the Dynkin diagram.

This fact continues to hold for the reductive Lie algebras $\mathfrak{g}_J$, except for the cases in C_n and D_n where there are two isomorphic simple components of $\mathfrak{g}_J$ of type C_k ($k \geq 2$) and D_k ($k \geq 4$), respectively. In these cases, there exists an element $w \in W_\Lambda$ which induces an automorphism of $\mathfrak{g}_J$ and interchanges the two isomorphic simple components that are not of type A. We see that this automorphism will conjugate a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}_J$ to a different nilpotent orbit if and only if $\mathcal{O}$ intersects the two non-type A simple components in different nilpotent orbits.

2.2 Generalization of the Bala-Carter Theory

In this section G is assumed to be simple, connected, and of adjoint type, i.e. $\hat{L} = L$.

The Bala-Carter classification of nilpotent orbits in $\mathfrak{g}$ states that the nilpotent orbits in $\mathfrak{g}$ are in bijection with pairs $(\mathfrak{l}, N)$, where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and N is a distinguished nilpotent in $\mathfrak{l}$ (up to simultaneous conjugation of both factors by G) [BC1], [BC2].

We will extend the Bala-Carter classification in order to understand the conjugacy classes in $A(N)$. More precisely, we will establish a bijection between pairs (N, C), where N is a nilpotent element in $\mathfrak{g}$ and C is a conjugacy class in $A(N)$ (up to conjugation by G), and pairs $(\mathfrak{l}, N)$, where N is a distinguished nilpotent in the pseudo-Levi subalgebra $\mathfrak{l}$ (up to simultaneous conjugation of both factors by G).

We will need a series of propositions to establish the bijection. Our approach follows the exposition in [Ca2].

Proposition 8. *For $J \subsetneq \tilde{\Pi}$ there exists $w \in W$ such that $w(J) = J$ and the action of w on L/L_J generates the automorphism group of the torsion subgroup of L/L_J.*

Proof. We may assume $d_J \geq 3$, for otherwise there are no non-trivial automorphisms of the torsion subgroup of L/L_J. Note $\alpha_0 \in J$ since $c_0 = 1$ and otherwise we would have $d_J = 1$. The isomorphism types of Φ_J for the cases that arise are: A_2 in G_2; $A_2 + \tilde{A}_2$, $A_3 + \tilde{A}_1$ in F_4; $3A_2$ in E_6; $2A_3 + A_1$, $A_5 + A_2$, $3A_2$ in E_7; $2A_4$, $A_5 + A_2 + A_1$, $A_1 + A_7$, $D_5 + A_3$, $2A_3 + A_1$, A_8, $E_6 + A_2$, $A_2 + A_5$, $3A_2 + A_1$, $3A_2$ in E_8.

First, we note that in these cases the longest element w_0 takes J to $-J$. This is trivial in all exceptional groups except E_6 since w_0 is just multiplication by -1. In E_6, the action of w_0 on J is multiplication by -1 followed by interchanging α_1 with α_5 and α_2 with α_4. There is only one subset J in E_6 for which $d_J \geq 3$, namely $\tilde{\Pi} - \{\alpha_3\}$, and so indeed $w_0(J) = -J$. Since $(w_0)_J$ maps J to $-J$, it follows that $w = w_0(w_0)_J$ has the property that $w(J) = J$.

Let $\alpha_j \in J$ be such that $w(\alpha_j) = \alpha_0$ and let $J' = J - \{\alpha_0\}$.

Next, consider the action of w on τ_J, the generator of the torsion subgroup of L/L_J. We have from (1.1)

$$d_J \tau_J = \sum_{\alpha_i \in J, i \neq j} c_i \alpha_i \ + c_j \alpha_j. \tag{2.1}$$

Now $w(\alpha_i) \in L_{J'}$ for all $\alpha_i \in J$ with $i \neq j$. Therefore, applying w to both sides of (2.1) yields

$$w(d_J \tau_J) \equiv c_j \alpha_0 \ \text{ modulo } \ L_{J'}.$$

On the other hand, from (1.1) we see that $\alpha_0 \equiv d_J \tau_J$ modulo $L_{J'}$ and thus

$$w(d_J \tau_J) \equiv c_j d_J \tau_J \ \text{ modulo } \ L_{J'}$$

which means

$$d_J(w(\tau_J) - c_j \tau_J) \in L_{J'}.$$

But $L/L_{J'}$ is torsion free, which implies

$$w(\tau_J) - c_j \tau_J \in L_{J'}.$$

Hence $w(\tau_J) \equiv c_j \tau_J$ modulo $L_{J'}$ (and also modulo $L_J \supset L_{J'}$).

Now in each case we find (by inspection) that c_j is congruent to -1 modulo d_J. In other words, w is an automorphism of order 2 of the torsion subgroup of L/L_J. Thus the only case left unresolved is the one in E_8 where $J = \tilde{\Pi} - \{\alpha_4\}$. Here $d_J = 5$ and the automorphism group is isomorphic to $\mathbb{Z}/4$. To handle this, we consider the following permutation σ of the elements in J

$$\alpha_0 \to \alpha_1 \to \alpha_5 \to \alpha_8 \to \alpha_0$$
$$\alpha_2 \to \alpha_6 \to \alpha_3 \to \alpha_7 \to \alpha_2$$

and extend σ linearly to the real span of L. Then

$$\sigma(\alpha_4) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_8,$$

which is a root; hence σ preserves Φ. One checks that σ is actually an automorphism of Φ and consequently σ coincides with the action of an element $w \in W$ since all automorphisms of E_8 come from W. Finally, because $w(\alpha_8) = \alpha_0$ and $c_8 = 3$, we see that w generates the automorphism group of the torsion subgroup of L/L_J. □

For $J \subsetneq \tilde{\Pi}$ consider the standard pseudo-Levi subgroup G_J. As we noted in Section 1.2, the center Z of G_J coincides with T^{L_J} and Z/Z^0 is cyclic of order d_J. Let x be such an element of Z whose image generates Z/Z^0. Any such $x \in T$ is thus characterized by the fact that $\alpha(x) = 1$ for $\alpha \in J$ and $\tau_J(x)$ is a primitive d_J-th root of unity. Let N be a distinguished nilpotent element in $\mathfrak{g}_J$. Clearly, $x \in Z_G(N)$ which means that the image of x defines an element in $A(N)$. Although there are many choices for x, we have the following proposition.

Proposition 9. *The image of x in $A(N)$ is well-defined up to conjugacy in $A(N)$.*

Proof. Suppose $x_1, x_2 \in Z$ both generate Z/Z^0. Then $x_2 \equiv x_1^l$ modulo Z^0 for some l prime to d_J.

By Proposition 8 there exists $w^{-1} \in W$ such that $w^{-1}(J) = J$ and w^{-1} acts on the torsion subgroup of L/L_J by multiplying each element by l. We thus have $\mathrm{Ad}(\dot{w})x_1 \equiv x_1^l \equiv x_2$ modulo Z^0.

In addition $\mathrm{Ad}(\dot{w})$ is an automorphism of $\mathfrak{g}_J$. By Remark 7, the distinguished nilpotent orbits in $\mathfrak{g}_J$ for $\mathfrak{g}$ exceptional (which is the case we are considering) are invariant under any automorphism of $\mathfrak{g}_J$. Hence N and $\mathrm{Ad}(\dot{w})N$ are in the same nilpotent orbit in $\mathfrak{g}_J$ and so there exists $g \in G_J$ such that $\mathrm{Ad}(g\dot{w})N = N$, i.e. $g\dot{w} \in Z_G(N)$. Since $g \in G_J$, we have $\mathrm{Ad}(g\dot{w})x_1 \equiv x_2$ modulo Z^0. But $Z^0 \subset Z_G^0(N)$, thus x_1 and x_2 are conjugate in $A(N)$. □

Proposition 10. *Let K be a reductive algebraic subgroup of G. Let x, y be two semisimple elements in K whose images in K/K^0 are in the same conjugacy class. Let S be a maximal torus in the reductive group $Z_K^0(x)$.*

Then for some $g \in K$ and $s \in S$, we have $gyg^{-1} = xs$. In particular, $Z_{\mathfrak{g}}(x, S) \subset Ad(g)Z_{\mathfrak{g}}(y)$.

Proof. This is a result about the semisimple automorphisms of the connected reductive algebraic group K^0 and can be found in [OV, Chapter 4.4]. □

Let $N \in \mathfrak{g}$ be a nilpotent element and let $\mathfrak{m} = \{N, H, M\}$ be an $\mathfrak{sl}_2$-triple for N. The centralizer $Z_G(\mathfrak{m})$ of $\mathfrak{m}$ in G is reductive and there is a decomposition $Z_G(N) = Z_G(\mathfrak{m})U_N$ where U_N is the unipotent radical of $Z_G(N)$. Moreover, the natural map from $Z_G(\mathfrak{m})/Z_G^0(\mathfrak{m})$ to $A(N)$ is an isomorphism since U_N is connected (see [CM]).

Definition 11. Let L_1 be a pseudo-Levi subgroup with center Z and Lie algebra $\mathfrak{l}$. Given a conjugacy class C in $A(N)$, we say that $\mathfrak{l}$ has the **key property** for (N, C) if $N \in \mathfrak{l}$ and there exists $x \in Z$ such that

1. The image of x generates the cyclic group Z/Z^0.
2. The image of x in $A(N)$ belongs to C.

Given a pair (N, C) as in the definition above, we will now locate a pseudo-Levi subalgebra $\mathfrak{l}$ with the key property for (N, C). Let $x \in Z_G(\mathfrak{m})$ represent an element in the conjugacy class C in $A(N)$. Let $x = x_s x_u$ be the Jordan decomposition of x in $Z_G(\mathfrak{m})$. Since x_u is unipotent, we have $x_u \in Z_G^0(\mathfrak{m})$ which means that the image of x_s in $A(N)$ coincides with the image of x. In other words, we can assume x is semisimple.

Let $K = Z_G(\mathfrak{m})$. Certainly, $x \in Z_K(x)$ and there exists an integer k such that $x^k \in Z_K^0(x)$. Let S be a maximal torus in $Z_K^0(x)$ containing the semisimple element x^k and pick $s \in S$ a topological generator of the compact torus in S. Then

$$\mathfrak{l} := Z_{\mathfrak{g}}(x, S) = Z_{\mathfrak{g}}(xs) \tag{2.2}$$

is a pseudo-Levi subalgebra containing N with corresponding group $L_1 = Z_G^0(xs)$. Note that x generates the component group of the center of L_1. Hence, $\mathfrak{l}$ has the key property for (N, C).

Proposition 12. *The subalgebra $\mathfrak{l}$ in* (2.2) *is minimal among the pseudo-Levi subalgebras with the key property for (N, C). Moreover, any other minimal pseudo-Levi subalgebra with the key property for (N, C) is conjugate to $\mathfrak{l}$ by an element in $Z_G(N)$.*

Proof. Suppose $\mathfrak{l}'$ is another pseudo-Levi subalgebra with the key property for (N, C) and L_1' its corresponding group. Then there exists x' in the center of L_1' whose image generates the component group of the center of L_1' and whose image in $A(N)$ belongs to C. Multiplying x' by an appropriate element in the identity component of the center of L_1', we can just assume that $\mathfrak{l}' = Z_{\mathfrak{g}}(x')$.

Let $\mathfrak{m}' = \{N, H', M'\}$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{l}'$. Clearly, x' centralizes $\mathfrak{m}'$. By Kostant's theorem there exists $g \in Z_G^0(N)$ such that $\mathrm{Ad}(g)(\mathfrak{m}') = \mathfrak{m}$. Conjugating $\mathfrak{l}', L_1', x'$ by g, we can assume that x' and x are semisimple elements in $Z_G(\mathfrak{m})$ and they represent the same conjugacy class in the component group of $Z_G(\mathfrak{m})$. Applying Proposition 10, there exists $g \in Z_G(\mathfrak{m})$ such that $Z_{\mathfrak{g}}(x, S) = \mathfrak{l}$ is contained in $\mathrm{Ad}(g)Z_{\mathfrak{g}}(x') = \mathrm{Ad}(g)\mathfrak{l}'$.

Now if $\mathfrak{l}'$ is also a minimal pseudo-Levi subalgebra with the key property, then $\mathfrak{l} = \mathrm{Ad}(g)\mathfrak{l}'$. □

Proposition 13. *Let $\mathfrak{l}$ be a minimal pseudo-Levi subalgebra with the key property for (N, C). Then N is distinguished in $\mathfrak{l}$.*

Proof. Let $\mathfrak{c}$ be the center of $\mathfrak{l}$. Suppose N is not distinguished and let $X \in \mathfrak{l}$ be a semisimple element such that $[X, N] = 0$ but $X \notin \mathfrak{c}$.

Then the centralizer of $\mathfrak{c} \cup X$ is a proper Levi subalgebra of $\mathfrak{l}$ which contains N. Hence it is a pseudo-Levi subalgebra of $\mathfrak{g}$ and it has the key property for the conjugacy class C, contradicting the minimality of $\mathfrak{l}$. □

With these results we can extend the Bala-Carter bijection.

Theorem 14. *There is a bijection ϕ between G-conjugacy classes of pairs $(\mathfrak{l}, N)$, where $\mathfrak{l}$ is a pseudo-Levi subalgebra and N is a distinguished nilpotent in $\mathfrak{l}$, and G-conjugacy classes of pairs (N, C), where N is a nilpotent element in $\mathfrak{g}$ and C is a conjugacy class in $A(N)$.*

Proof. Given the pair $(\mathfrak{l}, N)$, let x be any element in the center Z of L which generates Z/Z^0. Then ϕ maps $(\mathfrak{l}, N)$ to (N, C) where C is the conjugacy class of the image of x in $A(N)$. This is well-defined by Proposition 9.

Proof of surjectivity: the construction preceding Proposition 12, together with Propositions 12 and 13 give surjectivity.

Proof of injectivity: suppose $\phi(\mathfrak{l}, N) = \phi(\mathfrak{l}', N')$. Then there exists $g \in G$ such that $\mathrm{Ad}(g)N' = N$. Now both $\mathrm{Ad}(g)\mathfrak{l}'$ and $\mathfrak{l}$ have the key property for the pair $\phi(\mathfrak{l}, N)$. Since N is distinguished in both of these subalgebras, they must both be minimal for the key property. Hence by Proposition 12 there exists $g_2 \in Z_G(N)$ such that $\mathrm{Ad}(g_2 g)\mathfrak{l}' = \mathfrak{l}$. Since $\mathrm{Ad}(g_2 g)N' = N$, that completes the proof of injectivity. □

Consider the trivial conjugacy class C in $A(N)$. Let $\mathfrak{l}$ be a minimal Levi subalgebra containing N with corresponding group L_1. Since the center of L_1 is connected, it follows that $\phi(\mathfrak{l}, N) = (N, C)$ under the bijection. We call the pair $(\mathfrak{l}, N)$ a Bala-Carter pair when $\mathfrak{l}$ is a Levi subalgebra. The fact that the trivial conjugacy class C is always represented by a Levi subalgebra leads to two easy corollaries of the theorem.

Corollary 15. *Any $\mathfrak{g}_J$ with $d_J = 1$ is a Levi subalgebra of $\mathfrak{g}$.*

Corollary 16. *Any pair $(\mathfrak{g}_J, N)$ with $d_J > 1$ gets mapped under ϕ to a non-trivial conjugacy class in $A(N)$.*

Now suppose $\phi(\mathfrak{g}_J, N)$ equals (N, C) for some non-trivial conjugacy class C in $A(N)$. What can we say about the order of an element in C? By construction, we represented C by an element x in the center Z of G_J such that the image of x generates the cyclic group $Z/Z^0 \simeq \mathbb{Z}/d_J$. Therefore the image of x in $A(N)$ has order different from 1 and dividing d_J. If d_J is prime or N is distinguished in $\mathfrak{g}$, then elements in C have order exactly d_J.

Remark 17. The G-conjugacy classes of pairs $(\mathfrak{l}, N)$, where N is a distinguished nilpotent in the pseudo-Levi subalgebra $\mathfrak{l}$, are in bijection with the equivalence classes under W of pairs (J, J_{dynkin}), where $J \subsetneq \tilde{\Pi}$ and J_{dynkin} is the weighted Dynkin diagram of a distinguished nilpotent orbit in $\mathfrak{g}_J$. This is a consequence of Proposition 2, Proposition 4, and Remark 7.

Remark 18. Proposition 9 implies that for every element $x \in A(N)$, x is conjugate to x^l for all l prime to the order of x. Consequently, the characters for the representations of $A(N)$ take their values in the integers ([DF, Exercise 20 in 15.4]). We will find the characters of the irreducible representations of $A(N)$ in the next chapter.

2.3 Computing the bijection

Given a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$, we would like to find all pairs $(\mathfrak{l}, N)$ up to G-conjugacy such that $N \in \mathcal{O}$ and N is a distinguished nilpotent in the pseudo-Levi subalgebra $\mathfrak{l}$. Then for any $N \in \mathcal{O}$, we would know the number of conjugacy classes of $A(N)$ and some information about the order of elements in each conjugacy class.

Let Ψ denote the map from pairs $(\mathfrak{l}, N)$ appearing in the bijection (up to conjugacy) to nilpotent elements in $\mathfrak{g}$ (up to conjugacy) given by $\Psi(\mathfrak{l}, N) = N$. To compute the fiber of Ψ above any element in the orbit $\mathcal{O}$, we only need to consider pairs $(\mathfrak{g}_J, N)$ such that N is distinguished in $\mathfrak{g}_J$ and $N \in \mathcal{O}$. Moreover by Remark 17, each conjugacy classes of these pairs is in bijection with pairs (J, J_{dynkin}) up to equivalence under W, where J_{dynkin} is the weighted Dynkin diagram of N in $\mathfrak{g}_J$.

We now present an algorithm for computing Ψ. For each pseudo-Levi subalgebra $\mathfrak{g}_J$, we make a list of the weighted Dynkin diagrams of the distinguished nilpotent orbits in $\mathfrak{g}_J$ by looking at tables in [Ca2]. Fix such a weighted Dynkin diagram for the distinguished nilpotent $N \in \mathfrak{g}_J$. This tells us the values of $\alpha_i(H)$ for $\alpha_i \in J$, where $H \in \mathfrak{t}$ is the neutral element of an $\mathfrak{sl}_2$-triple for some conjugate of N. Moreover, H belongs to the semisimple part of $\mathfrak{g}_J$ and this fact uniquely determines the values of $\alpha_i(H)$ for all $\alpha_i \in \Pi$. Then we locate $w \in W$ such that $\alpha_i(\text{Ad}(\dot{w})H) \geq 0$ for all $\alpha_i \in \Pi$. These positive integers yield the weighted Dynkin diagram of the nilpotent orbit in $\mathfrak{g}$ through N.

In fact, the location of w is not difficult. If $\alpha_i(H) < 0$ for some $\alpha_i \in \Pi$, then the number of positive roots α such that $\alpha(\text{Ad}(\dot{s}_i)H) < 0$ is one less than the number with $\alpha(H) < 0$. We also have $\alpha_j(\text{Ad}(\dot{s}_i)H) = s_i(\alpha_j)(H)$ for $\alpha_j \in \Pi$. The integers $\alpha_j(\text{Ad}(\dot{s}_i)H)$ for $\alpha_j \in \Pi$ yield a new labeled diagram and we continue applying simple reflections in this manner until all nodes of the diagram are non-negative, arriving at the desired weighted Dynkin diagram.

Example 19. Let G be of type G_2. Consider the example of the regular nilpotent in the pseudo-Levi subalgebra of type $A_1 + \tilde{A}_1$. For the neutral element H in the $\mathfrak{sl}_2$-triple corresponding to the regular nilpotent in $A_1 + \tilde{A}_1$, we have $\alpha_0(H) = \alpha_2(H) = 2$ (α_2 is the short simple root). Because $\alpha_1 = \frac{1}{2}(-\alpha_0 - 3\alpha_2)$, it follows that $\alpha_1(H) = -4$. We encode this information in a weighted diagram with respect to α_1, α_2; the diagram looks like $-4\ 2$. Following the algorithm above, we apply the simple reflection s_1 to H and the diagram becomes $4\ {-2}$ because $s_1(\alpha_2) = \alpha_2 + \alpha_1$. Then we apply s_2 and the diagram becomes $-2\ 2$ since $s_2(\alpha_1) = \alpha_1 + 3\alpha_2$. Finally applying s_1 yields $2\ 0$, the weighted Dynkin diagram of the subregular nilpotent orbit in G_2. For $\mathfrak{g}_J$ of type $A_1 + \tilde{A}_1$, we have $d_J = c_1 = 2$ and so the component group of a subregular nilpotent in G_2 contains a conjugacy class with elements of order 2.

The only other pseudo-Levi subalgebra in G_2 (up to conjugacy), which is not a Levi subalgebra, is of type A_2. Choosing the regular nilpotent in A_2 yields the weighted diagram $2\ {-2}$. Applying the element $w = s_1 s_2 s_1 s_2$ gives the weighted Dynkin diagram $2\ 0$. For $\mathfrak{g}_J$ of type A_2, we have $d_J = c_2 = 3$ and therefore the component group of a subregular nilpotent contains a conjugacy class with elements of order 3.

These calculations and Theorem 14 imply that when N is not a subregular nilpotent in G_2, we have $A(N) = 1$. When N is a subregular nilpotent, $A(N)$ contains two non-trivial conjugacy classes, one with elements of order 2 and the other with elements of order 3. Let $x \in A(N)$ have order 3. Since x^2 also has order 3, there exists $y \in A(N)$ such that $yxy^{-1} = x^2$. Conjugating both sides by y gives $y^2 x y^{-2} = x^4 = x$, which forces y to have even order. But then y must have order 2 and the subgroup generated by x and y is isomorphic to S_3, the symmetric group on 3 letters. Since this subgroup intersects all conjugacy classes in the finite group $A(N)$, we conclude that $A(N) = S_3$.

2.3.1 Exceptional groups

We carried out the algorithm for the exceptional groups. We use the notation S_m for the symmetric group on m letters and Bala and Carter's notation for a distinguished nilpotent in a semisimple Lie algebra. There are five cases that occur:

1. The fiber of Ψ consists of one element, namely a distinguished nilpotent in a Levi subalgebra (the Bala-Carter pair). Thus $A(N)$ is trivial.

2. The fiber of Ψ consists of two elements. In addition to the Bala-Carter pair, there is another pair which contributes a conjugacy class to $A(N)$. Thus $A(N) = S_2$ since $A(N)$ has only two conjugacy classes.

   We find that when $\mathfrak{g}_J$ is of type $2A_3 + A_1$ in E_7 or of type $2A_3 + A_1$, $A_1 + A_7$, or $D_5 + A_3$ in E_8, the regular nilpotent in $\mathfrak{g}_J$ gives rise to an element of order 2 in $A(N)$ even though $d_J = 4$.

3. The fiber has three elements. In addition to the Bala-Carter pair, there is one conjugacy class with elements of order 2 and another with elements of order 3. The same argument given in Example 19 shows that $A(N) = S_3$.

4. The distinguished nilpotent orbit $F_4(a_3)$ in F_4. Since $F_4(a_3)$ is distinguished, the conjugacy class in $A(N)$ corresponding to a pair $(\mathfrak{g}_J, N)$ contains elements of order exactly d_J. We find that there are four non-trivial conjugacy classes in $A(N)$ corresponding to $A_3+\tilde{A}_1$, $A_2+\tilde{A}_2$, $B_4(a_1)$, and $A_1+C_3(a_1)$ consisting of elements of order 4, 3, 2, and 2, respectively. I am not sure if it is possible to conclude immediately that $A(N)=S_4$, but we will see this in the next chapter.

5. The distinguished nilpotent orbit $E_8(a_7)$ in E_8. There are six non-trivial conjugacy classes in $A(N)$ corresponding to $A_5+A_2+A_1$, $2A_4$, $D_5(a_1)+A_3$, $E_6(a_3)+A_2$, $D_8(a_5)$, and $E_7(a_5)+A_1$ consisting of elements of order 6, 5, 4, 3, 2, and 2, respectively. We will see shortly that $A(N)=S_5$.

The results for the bijection in the exceptional groups are listed in the tables in Chapter 6.

2.3.2 Classical groups

In type A_n, d_J always equals one and hence all $A(N)$ are trivial.

For the other classical groups, $d_J=1$ or 2 and so every (non-identity) element in $A(N)$ has order two and therefore $A(N)$ must be an elementary abelian 2-group. Using the usual description of the classical groups and their nilpotents, we will now describe the fiber of the map Ψ above the nilpotent N.

Let $\varepsilon=0,1$. All congruences are modulo 2.

Consider a complex vector space V of dimension m (m is even if $\varepsilon=1$) with basis $e_1,e_2,\ldots,e_m$ and an inner product $(\,,\,)$ satisfying $(e_i,e_j)=0$ if $i+j\neq m+1$ and $(e_i,e_{m+1-i})=(-1)^{\varepsilon}(e_{m+1-i},e_i)=1$.

Let G_1 be the identity component of the subgroup of $GL(V)$ which preserves $(\,,\,)$ and let $G=G_{ad}$ be the quotient of G_1 by its center. Their Lie algebra $\mathfrak{g}$ consists of the elements $X\in\mathfrak{gl}(V)$ for which

$$(X.v_1,v_2)+(v_1,X.v_2)=0 \text{ for all } v_1,v_2\in V.$$

We choose $\mathfrak{t}$ to be the diagonal matrices in $\mathfrak{g}\subset\mathfrak{gl}(V)$ and $\mathfrak{b}$ to be the upper triangular matrices in $\mathfrak{g}\subset\mathfrak{gl}(V)$. The rank of $\mathfrak{g}$ is $n=\lfloor m/2\rfloor$. Let us describe the simple components of the pseudo-Levi subalgebras in $\mathfrak{g}$ containing $\mathfrak{t}$.

For $1\leq k<l\leq m/2$, let V_1 be the subspace of V spanned by $e_k,e_{k+1},\ldots,e_l$ and V_2 be the subspace of V spanned by $e_{m+1-k},e_{m-k},\ldots,e_{m+1-l}$. Then

$$\{X\in\mathfrak{g}|\ X.V_1\subset V_1, X.V_2\subset V_2, \text{ and } X.e_i=0 \text{ for all } e_i\notin V_1+V_2\}$$

is isomorphic to $\mathfrak{gl}_{l-k+1}$ and it is a summand of $\mathfrak{g}_J$ where $J=\{\alpha_k,\ldots,\alpha_{l-1}\}$.

On the other hand, for $k=1$ and $1\leq l\leq m/2$, we have

$$\{X\in\mathfrak{g}|\ X.(V_1+V_2)\subset V_1+V_2, \text{ and } X.e_i=0 \text{ for all } e_i\notin V_1+V_2\}$$

is isomorphic to $\mathfrak{so}_{2l}$ if $\varepsilon=0$ and is isomorphic to $\mathfrak{sp}_{2l}$ if $\varepsilon=1$. This is a summand of $\mathfrak{g}_J$ where $J=\{\alpha_0,\ldots,\alpha_{l-1}\}$ (we assume $l>1$ if $\varepsilon=0$).

Finally, if $l = \lceil m/2 \rceil$ and $1 \leq k \leq l$, then

$$\{X \in \mathfrak{g} |\ X.(V_1 + V_2) \subset V_1 + V_2, \text{ and } X.e_i = 0 \text{ for all } e_i \notin V_1 + V_2\}$$

is isomorphic to $\mathfrak{so}_{m-2k+2}$ if $\varepsilon = 0$ and is isomorphic to $\mathfrak{sp}_{m-2k+2}$ if $\varepsilon = 1$. This is a summand of $\mathfrak{g}_J$ where $J = \{\alpha_k, \ldots, \alpha_n\}$ (we assume $k < l$ if $\varepsilon = 0$ and m is even).

Each nilpotent $N \in \mathfrak{g}$ has a Jordan normal form in $\mathfrak{gl}(V)$ and so we can associate to N a partition $[p_1 \geq p_2 \cdots \geq p_k]$ of m, abbreviated $[p_j]$. Let $\mu(i)$ be the number of times i appears in the partition. The only partitions which actually arise are the ones where $i \equiv \varepsilon$ implies $\mu(i)$ is even. The partition completely determines the nilpotent orbit in $\mathfrak{g}$ except for the very even orbits in D_n, n even, where all the parts in the partition are even (see Section 1.3).

The distinguished nilpotents in $\mathfrak{g}$ correspond to partitions where $i \equiv \varepsilon$ implies $\mu(i) = 0$ and $i \not\equiv \varepsilon$ implies $\mu(i) = 1$.

Given a distinguished nilpotent element in a pseudo-Levi subalgebra $\mathfrak{g}_J$, corresponding to a partition in each simple component, which partition does it correspond to in $\mathfrak{g}$? The answer is clear: the parts in the partition for each $\mathfrak{sl}$ are doubled and taken together with the parts in the two simple components not isomorphic to $\mathfrak{sl}$. These yield a partition of m after we tack on the appropriate number of 1's. This partition automatically satisfies the condition on parts imposed by $\mathfrak{g}$.

Conversely, given the partition $[p_j]$ for a nilpotent $N \in \mathfrak{g}$, what is the fiber of the map Ψ above N? We now answer this question.

Define the following sets which depend on the partition $[p_j]$ and ε:

$$S_{odd} = \{i \in \mathbb{N} |\ i \not\equiv \varepsilon,\ \mu(i) \equiv 1\}$$
$$S_{even} = \{i \in \mathbb{N} |\ i \not\equiv \varepsilon,\ \mu(i) \equiv 0\}.$$

Let $s = \sum_{i \in S_{odd}} i$.

Choose $T_1 \subset S_{odd}$ and $T_2 \subset S_{even}$. Let $t_1 = \sum_{i \in T_1} i$ and $t_2 = \sum_{i \in T_2} i$. Define $a(i)$ as follows

$$a(i) = \begin{cases} \frac{\mu(i)}{2} & \text{if } i \equiv \varepsilon \text{ or } i \in S_{even} - T_2 \\ \frac{\mu(i)-1}{2} & \text{if } i \in S_{odd} \\ \frac{\mu(i)}{2} - 1 & \text{if } i \in T_2 \end{cases}$$

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Here s, t_1, t_2 are automatically even. To the triple (N, T_1, T_2) we can associate the standard pseudo-Levi subalgebra with simple components $\mathfrak{sp}_{t_1+t_2}$, $\mathfrak{sp}_{t_2+s-t_1}$, and $a(i)$ copies of type A_{i-1} for each $i \in \mathbb{N}$. Choose the regular nilpotent in each A_{i-1} component. Choose the distinguished nilpotent in $\mathfrak{sp}_{t_1+t_2}$ corresponding to the partition whose parts are the elements of the set $T_1 \cup T_2$ and choose the distinguished nilpotent in $\mathfrak{sp}_{t_2+s-t_1}$ corresponding to the partition whose parts are the elements of the set $T_2 \cup (S_{odd} - T_1)$.

In this manner, we obtain all the pairs $(\mathfrak{g}_J, N')$ in the fiber of Ψ above N. Note that interchanging T_1 and $S_{odd} - T_1$ yields conjugate pairs. This is the phenomenon discussed in Remark 7.

Example 20. Consider the nilpotent orbit in C_6 with partition $[4,4,2,2]$. So $\varepsilon = 1$ and $\mu(2) = \mu(4) = 2$. We have $S_{odd} = \emptyset$ and $S_{even} = \{2,4\}$. The possible choices for T_2 are $\emptyset, \{2\}, \{4\}$, or $\{2,4\}$.

If $T_2 = \emptyset$, then $a(2) = a(4) = 1$ and we get the regular nilpotent in $A_3 + A_1$ (this is the Bala-Carter pair).

If $T_2 = \{2\}$, then $a(2) = 0$ and $a(4) = 1$ and we get the regular nilpotent in $C_1 + A_3 + C_1$.

If $T_2 = \{4\}$, then $a(2) = 1$ and $a(4) = 0$ and we get the regular nilpotent in $C_2 + A_1 + C_2$.

Finally, if $T_2 = \{2,4\}$, then $a(2) = a(4) = 0$ and we get the pseudo-Levi subalgebra $C_3 + C_3$ with the subregular nilpotent $[4,2]$ in each C_3 factor.

We see that $A(N) = \mathbb{Z}/2 \times \mathbb{Z}/2$.

Type B_n

Here s is automatically odd. We require that $t_1 \equiv t_2$. To the triple (N, T_1, T_2) we can associate the standard pseudo-Levi subalgebra with simple components $\mathfrak{so}_{t_1+t_2}, \mathfrak{so}_{t_2+s-t_1}$ and $a(i)$ copies of type A_{i-1} for each $i \in \mathbb{N}$. Choose the regular nilpotent in each A_{i-1}. Choose the distinguished nilpotent in $\mathfrak{so}_{t_1+t_2}$ corresponding to the partition whose parts are the elements of the set $T_1 \cup T_2$ and choose the distinguished nilpotent in $\mathfrak{so}_{t_2+s-t_1}$ corresponding to the partition whose parts are the elements of the set $T_2 \cup (S_{odd} - T_1)$.

In this manner, we obtain all the pairs $(\mathfrak{g}_J, N')$ in the fiber of Ψ above N.

Example 21. Consider the nilpotent orbit in B_4 with partition $[5,3,1]$. So $\varepsilon = 0$ and $\mu(1) = \mu(3) = \mu(5) = 1$. We have $S_{odd} = \{1,3,5\}$ and $S_{even} = \emptyset$. The possible choices for T_1 are $\emptyset, \{1,3\}, \{1,5\}$, or $\{3,5\}$ since $t_2 = 0$ and so t_1 must be even. For all choices of T_1 and for all i, we have $a(i) = 0$.

If $T_1 = \emptyset$, then we get the nilpotent $[5,3,1]$ in B_4 (the Bala-Carter pair).

If $T_1 = \{1,3\}$, then we get the pseudo-Levi subalgebra $D_2 + B_2$ with the nilpotent $[3,1]$ in D_2 and the nilpotent $[5]$ in B_2.

If $T_1 = \{1,5\}$, then we get $D_3 + B_1$ with the nilpotent $[5,1]$ in D_3 and the nilpotent $[3]$ in B_1.

Finally, if $T_1 = \{3,5\}$, then we get the subregular nilpotent $[5,3]$ in D_4.

We see that $A(N) = \mathbb{Z}/2 \times \mathbb{Z}/2$.

Type D_n

Here s is automatically even. We require that $t_1 \equiv t_2$. To the triple (N, T_1, T_2) we can associate the standard pseudo-Levi with simple components $\mathfrak{so}_{t_1+t_2}, \mathfrak{so}_{t_2+s-t_1}$ and $a(i)$ copies of type A_{i-1} for each $i \in \mathbb{N}$. Choose the regular nilpotent in each A_{i-1}. Choose the distinguished nilpotent in $\mathfrak{so}_{t_1+t_2}$ corresponding to the partition whose parts are the elements of the set $T_1 \cup T_2$ and choose the distinguished nilpotent in $\mathfrak{so}_{t_2+s-t_1}$ corresponding to the partition whose parts are the elements of the set $T_2 \cup (S_{odd} - T_1)$.

We thus obtain all the pairs $(\mathfrak{g}_J, N')$ in the fiber of Ψ above N. As in type C_n, interchanging T_1 and $S_{odd} - T_1$ yields conjugate pairs. In the very even case, only a Bala-Carter pair maps to N under Ψ, so our carelessness above with partitions is harmless.

2.3.3 Relation to known results about $A(N)$ for classical groups

Assume $\mathfrak{g}$ is of type B_n, C_n, or D_n. We have already seen that $A(N)$ is an elementary abelian 2-group. We now recall the usual way to see this and to see which conjugacy classes correspond to the triples (N, T_1, T_2) described in the previous subsection.

Let $[p_j]$ be the partition of N and form an $\mathfrak{sl}_2$-triple $\mathfrak{m} = \{N, H, M\}$. The action of $\mathfrak{m}$ on $V = \mathbf{C}^m$ decomposes V into irreducible modules for $\mathfrak{m}$ of dimension equal to the parts of $[p_j]$. Denote by $M(r)$ the sum of the modules of dimension r.

For $i \in \mathbb{N}$, let K be the subgroup of $GL(V)$ which preserves $(,)$, commutes with $\mathfrak{m}$, and acts as the identity on $M(r)$ for $r \neq i$. Then K is determined by its action on the highest weight space of $M(i)$ and K is isomorphic to $O(\mu(i), \mathbf{C})$ if $i \not\equiv \varepsilon$ and to $Sp(\mu(i), \mathbf{C})$ if $i \equiv \varepsilon$ (see [CM, Chapter 5]).

Hence $Z_{G_1}(\mathfrak{m})$ is isomorphic to

$$\prod_{i \not\equiv \varepsilon} O(\mu(i), \mathbf{C}) \times \prod_{i \equiv \varepsilon} Sp(\mu(i), \mathbf{C}) \text{ if } \varepsilon \equiv 1 \tag{2.3}$$

$$\text{and } S(\prod_{i \not\equiv \varepsilon} O(\mu(i), \mathbf{C}) \times \prod_{i \equiv \varepsilon} Sp(\mu(i), \mathbf{C})) \text{ if } \varepsilon \equiv 0. \tag{2.4}$$

In type C_n and D_n, the center of G_1 has two elements which we need to ignore if we are interested in $A(N)$. Taking the center of G_1 into account, we see that the conjugacy classes in $A(N)$ are parameterized by subsets $T_1 \subset S_{odd}$ and $T_2 \subset S_{even}$ satisfying the conditions of the previous subsection. Namely, given such a pair of subsets, the corresponding conjugacy class in $A(N)$ is represented in $Z_{G_1}(\mathfrak{m})$ by an element which has determinant -1 on the highest weight space of $M(i)$ for $i \in T_1 \cup T_2$ and determinant 1 on the highest weight space of $M(i)$ for $i \notin T_1 \cup T_2$. In type B_n and D_n, the condition that $t_1 \equiv t_2$ is imposed by the determinant 1 condition above and in type C_n and D_n, the equivalence of interchanging T_1 and $S_{odd} - T_1$ comes from the presence of the center in G_1.

On the other hand, given the pair (T_1, T_2), define $g \in GL(V)$ to be multiplication on the basis vector e_i by the scalar

$$\begin{cases} -1 & \text{if } 1 \leq i \leq \frac{t_1+t_2}{2} \\ 1 & \text{if } \frac{t_1+t_2}{2} < i \leq m - \frac{t_1+t_2}{2} \\ -1 & \text{if } m - \frac{t_1+t_2}{2} < i \leq m \end{cases}$$

Then the image of g in $G = G_{ad}$ belongs to the center of the pseudo-Levi subgroup L_1 determined by the triple (N, T_1, T_2) in 2.3.2. Moreover, g does not lie in the identity component of the center of L_1 and so represents the conjugacy class in $A(N)$ corresponding to (N, T_1, T_2) under our generalized Bala-Carter bijection. We also see that g gives rise to same conjugacy class specified by the pair (T_1, T_2) of the previous paragraph. This is the relation between the classically known conjugacy classes in $A(N)$ and our bijection.

2.4 Conjugacy classes in $\pi_1(\mathcal{O})$

In this section, G is connected, simple, and simply-connected. We will write $\pi_1(\mathcal{O}_N)$ for $Z_G(N)/Z_G^0(N)$ and $A(N)$ for the same group when we have in mind G_{ad}, the quotient of G

by its center.

Although the groups $\pi_1(\mathcal{O}_N)$ have been computed in the literature [L1], [CM], we thought it would be fun to use the bijection and elementary facts about root systems to give a somewhat unified approach to the conjugacy classes in $\pi_1(\mathcal{O}_N)$. This at least has the advantage of making the thesis self-contained.

The center of G is isomorphic to the finite abelian group $\hat{L}/L$ (actually to the dual of $\hat{L}/L$, which is isomorphic to $\hat{L}/L$). If N is a distinguished nilpotent in $\mathfrak{g}$, then no element of the center (which consists of semisimple elements) lies in $Z_G^0(N)$ (which is unipotent). Hence $\pi_1(\mathcal{O}_N)$ is a central extension of $A(N)$ by a finite group isomorphic to $\hat{L}/L$.

Consider the following generalization

Proposition 22. *Let N be a distinguished nilpotent element in $\mathfrak{g}_J$, a Levi subalgebra. Then $\pi_1(\mathcal{O}_N)$ is a central extension of $A(N)$ by a finite group isomorphic to the torsion subgroup of $\hat{L}/L_J$.*

Proof. The center Z of G_J is isomorphic to $(\mathbf{C}^*)^r \times K$, where K is isomorphic to the torsion subgroup of $\hat{L}/L_J$, and we can view K as a quotient of the center of G.

Since G_J is a Levi subgroup and N is distinguished in $\mathfrak{g}_J$, the Bala-Carter classification implies that Z^0 is a maximal torus of $Z_G^0(N)$. If Z^0 were not maximal, N would be contained in the centralizer of the maximal torus in $Z_G^0(N)$, which would be a proper Levi subalgebra of $\mathfrak{g}_J$.

Therefore, the image of the center of G in $\pi_1(\mathcal{O}_N)$ generates a central subgroup precisely isomorphic to K. □

We can use a similar trick even when $\mathfrak{g}_J$ is not a Levi subalgebra. The previous proposition is a statement about how the trivial conjugacy class lifts from $A(N)$ to $\pi_1(\mathcal{O}_N)$. We can ask the same question for any other conjugacy class.

Suppose N is distinguished in $\mathfrak{g}_J$. By studying the torsion subgroup of $\hat{L}/L_J$, we can see how the conjugacy class C in $A(N)$ associated to the pair $(\mathfrak{g}_J, N)$ lifts to $\pi_1(\mathcal{O}_N)$. But we have to be careful, because when the identity component of the center of G_J is not a maximal torus in $Z_G^0(N)$, it can happen that the inclusion of L/L_J into $\hat{L}/L_J$ is not an isomorphism even when $\pi_1(\mathcal{O}_N) \simeq A(N)$.

We will use these ideas to study the conjugacy classes in $\pi_1(\mathcal{O}_N)$.

Type A_n

We have seen that $A(N) = 1$ and all pseudo-Levi subalgebras are actually Levi subalgebras. Thus $\pi_1(\mathcal{O}_N)$ is isomorphic to the torsion subgroup of $\hat{L}/L_J$ where N is a regular nilpotent in $\mathfrak{g}_J$ and $J \subset \Pi$.

Let $\Pi - J$ consist of the simple roots $\alpha_{i_1}, \ldots, \alpha_{i_k}$. Let $d = gcd(n+1, i_1, \ldots, i_k)$. If $d = 1$, then $\hat{L}/L_J$ is torsion free. If $d > 1$ then

$$\lambda = (\frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d}, 0, \frac{1}{d}, \ldots, 0, \frac{1}{d}, \ldots, \frac{d-1}{d})$$

(in the basis of simple roots) belongs to $\hat{L}$. Now $\lambda \notin L_J$ but $d\lambda \in L_J$ and in fact λ generates the torsion subgroup of $\hat{L}/L_J$. Hence $\pi_1(\mathcal{O}_N) \simeq \mathbb{Z}/d$.

In the notation of partitions, d also equals $gcd(p_1, \dots, p_k)$ where $[p_j]$ is the partition corresponding to N.

We have seen that $A(N)$ is an elementary abelian 2-group in types C_n, B_n, and D_n because all the coefficients of the highest root θ are 1 or 2.

Type C_n

The group $\hat{L}/L$ has order 2 and is generated by the weight $\frac{1}{2}\alpha_n \in \hat{L}$. Take $J \subsetneq \tilde{\Pi}$. There are three cases:

1. $\alpha_n, \alpha_0 \notin J$. Then $\hat{L}/L_J$ is torsion free.

2. $\alpha_n, \alpha_0 \in J$. Then the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

3. Otherwise, the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/2$.

Case (2) implies that the lifting of an element of order 2 from $A(N)$ to $\pi_1(\mathcal{O}_N)$ still has order two. Conclusion: $\pi_1(\mathcal{O}_N)$ is also an elementary abelian 2-group.

Let N be distinguished in the Levi subalgebra $\mathfrak{g}_J$ with $J \subset \Pi$. Then $\alpha_n \in J$ implies $\pi_1(\mathcal{O}_N) \simeq A(N) \times \mathbb{Z}/2$. If $\alpha_n \notin J$, then $\pi_1(\mathcal{O}_N) \simeq A(N)$. In the notation of partitions, the latter case happens only when all parts occur an even number of times in $[p_j]$.

Type B_n

The group $\hat{L}/L$ has order 2 and is generated by the weight

$$\tau_c = \frac{1}{2}\Big(\sum_{i \text{ odd}, i\geq 1} \alpha_i\Big) \in \hat{L}.$$

Let $\Pi^{odd} = \{\alpha_i \in \Pi |\ i \text{ is odd}\}$ and $\Pi^{even} = \Pi - \Pi^{odd}$. If $J \subset \Pi$, the cases that arise are:

1. $\Pi^{odd} \not\subset J$. Then $\hat{L}/L_J$ is torsion free.

2. $\Pi^{odd} \subset J$. Then the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/2$.

If J is not conjugate to a subset of Π (so $\alpha_0, \alpha_1 \in J$), the cases are:

3. $\Pi^{odd} \subset J$. Then the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

4. $\Pi^{even} \subset J$. Then the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/4$.

5. Neither (3) nor (4) holds. Then the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/2$.

Consider the non-identity element $w \in W_\Lambda$ defined in Proposition 5. Namely, $w = w_0(w_0)_{J'}$, where $J' = \Pi - \alpha_1$. Then w interchanges α_0 and α_1 and fixes all other $\alpha_i \in \tilde{\Pi}$, as noted in the figure.

Proposition 23. *In cases* (3) *and* (4)*, the element w acts by non-trivial automorphism on the torsion part of $\hat{L}/L_J$. Hence as in the proof of Proposition 9, the two liftings of the element corresponding to* $(\mathfrak{g}_J, N)$ *in* $A(N)$ *are in the same conjugacy class in* $\pi_1(\mathcal{O}_N)$.

Proof. In both cases $w(J) = J$ and $w(\tau_J) = \tau_J$. In case (4), we have

$$w(\tau_c) \equiv -\tau_c \text{ modulo } L_J,$$

proving the first statement since the image of τ_c generates the torsion subgroup in $\hat{L}/L_J \simeq \mathbb{Z}/4$. In case (3), we have

$$w(\tau_c) \equiv \tau_c + \tau_J \text{ modulo } L_J,$$

proving the first statement since the images of τ_c and τ_J generate the torsion subgroup in $\hat{L}/L_J \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

The second statement follows as in Proposition 9 because w induces an automorphism of $\mathfrak{g}_J$ preserving each simple component. Hence there exists $g \in Z_G(N)$ which interchanges the two lifts of an element in the conjugacy class C in $A(N)$. □

Case (2) tells us when $\pi_1(\mathcal{O}_N)$ is bigger than $A(N)$. The condition $\Pi^{odd} \subset J \subset \Pi$ implies $\mathfrak{g}_J$ contains no simple components of type A_k for k even. Thus in the notation of partitions, each odd part in $[p_j]$ appears only once.

Assume $\pi_1(\mathcal{O}_N)$ is bigger than $A(N)$ and let $x \in \pi_1(\mathcal{O}_N)$ be the image of the non-trivial central element in G. Let $y \in \pi_1(\mathcal{O}_N)$ be different from $1, x$. The previous proposition implies that y and yx are in the same conjugacy class in $\pi_1(\mathcal{O}_N)$. We conclude that $\pi_1(\mathcal{O}_N)$ has one more conjugacy class than $A(N)$. We will need these facts when we search for the irreducible representations of $\pi_1(\mathcal{O}_N)$ in Chapter 3.

Type D_n, n is odd

The group $\hat{L}/L \simeq \mathbb{Z}/4$ is generated by the weight

$$\tau_c = \frac{1}{2}\Big(\sum_{i \text{ odd}, i \geq 1} \alpha_i\Big) + \frac{1}{4}(\alpha_{n-1} - \alpha_n) \in \hat{L}.$$

Let $\Pi^{odd} = \{\alpha_i \in \Pi \mid i \text{ is odd}\} \cup \{\alpha_{n-1}\}$ and $\Pi^{even} = \Pi - \Pi^{odd}$. If $J \subset \Pi$, the appropriate cases are:

1. $\alpha_{n-1} \notin J$ or $\alpha_n \notin J$. Then $\hat{L}/L_J$ is torsion free.
2. $\Pi^{odd} \not\subset J$, but $\alpha_{n-1}, \alpha_n \in J$. Then the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/2$.
3. $\Pi^{odd} \subset J$. Then the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/4$.

Assume now that J is not conjugate to a subset of Π. We must have $\alpha_0, \alpha_1, \alpha_{n-1}, \alpha_n \in J$. The cases are:

4. $\Pi^{odd} \subset J$ or $\Pi^{even} \subset J$. Then the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/4 \oplus \mathbb{Z}/2$, with τ_c generating the $\mathbb{Z}/4$ and τ_J generating the $\mathbb{Z}/2$.
5. Otherwise, the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Assume J is as in case (4). Let $y \in \pi_1(\mathcal{O}_N)$ be a lift of an element in C, the conjugacy class in $A(N)$ determined by $(\mathfrak{g}_J, N)$. Let x be the element in the center of G determined by $\tau_c(x) = \xi$, where ξ is a primitive fourth root of unity.

As in type B_n, consider $w = w_0(w_0)_{J'}$ where $J' = \Pi - \alpha_1$. Then w interchanges α_0 and α_1 and interchanges α_{n-1} and α_n, but fixes all other $\alpha_i \in \tilde{\Pi}$. So in case (4), $w(J) = J$ and

$$w(\tau_c) \equiv \tau_c + \tau_J \text{ modulo } L_J. \tag{2.5}$$

As in Proposition 23, w gives rise to an element $g \in Z_G(N)$ which conjugates y to yx^2 and yx to yx^3 in $\pi_1(\mathcal{O}_N)$.

In addition, (3), (4) and (5) imply that in the group $G_1 = G/\langle x^2 \rangle$ (which is the special orthogonal group), all elements in $\tilde{A}(N) = Z_{G_1}(N)/Z^0_{G_1}(N)$ have order two. Conclusion: if $\pi_1(\mathcal{O}_N) \simeq \tilde{A}(N)$, then $\pi_1(\mathcal{O}_N)$ is an elementary abelian 2-group; if $\pi_1(\mathcal{O}_N)$ is bigger than $\tilde{A}(N)$, then $\pi_1(\mathcal{O}_N)$ has two more conjugacy classes than $\tilde{A}(N)$.

The latter situation occurs when (3) is satisfied. The condition $\Pi^{odd} \subset J \subset \Pi$ implies $\mathfrak{g}_J$ contains no simple components of type A_k for k even. In the notation of partitions, this means that each odd part in $[p_j]$ appears only once.

Type D_n, n is even

The group $\hat{L}/L \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is generated by the weights

$$\tau_1 = \frac{1}{2}\Big(\sum_{i \text{ odd}, i \geq 1} \alpha_i\Big)$$
$$\tau_2 = \frac{1}{2}(\alpha_{n-1} + \alpha_n).$$

Let $\Pi^{odd} = \{\alpha_i \in \Pi \mid i \text{ is odd}\} \cup \{\alpha_n\}$ and $\Pi^{even} = \Pi - \Pi^{odd}$. If $J \subset \Pi$, the appropriate cases are:

1. $\alpha_{n-1} \notin J$ or $\alpha_n \notin J$. Then $\hat{L}/L_J$ is torsion free.

2. $\Pi^{odd} \not\subset J$, but $\alpha_{n-1}, \alpha_n \in J$. Then the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/2$, generated by τ_2.

3. $\Pi^{odd} \subset J$. Then the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Assume that J is not conjugate to a subset of Π. We must have $\alpha_0, \alpha_1, \alpha_{n-1}, \alpha_n \in J$. The appropriate cases are:

4. $\Pi^{odd} \subset J$. Then the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$, with generators τ_1, τ_2, and τ_J.

5. $\Pi^{even} \subset J$. Then the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/4 \oplus \mathbb{Z}/2$, with generators τ_1 (of order 4) and τ_2 (of order 2).

6. Otherwise, the torsion subgroup of $\hat{L}/L_J$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Take J as in case (4) or (5). Let C be the conjugacy class in $A(N)$ determined by $(\mathfrak{g}_J, N)$. Let x be the element in the center of G defined by $\tau_1(x) = -1$ and $\tau_2(x) = 1$. Let $y \in \pi_1(\mathcal{O}_N)$ be a lift of an element in the conjugacy class C.

The Weyl group element w (defined as for n odd) has the same action on $\tilde{\Pi}$ as before. In case (4),

$$w(\tau_1) \equiv \tau_1 + \tau_J \text{ modulo } L_J \tag{2.6}$$

and in case (5),

$$w(\tau_1) \equiv -\tau_1 \text{ modulo } L_J. \tag{2.7}$$

The implication is that in cases (4) and (5), w gives rise to an element $g \in Z_G(N)$ as in Proposition 23 that conjugates y to yx. Furthermore, for $G_1 = G/\langle x \rangle$ (special orthogonal group), all elements in $Z_{G_1}(N)/Z^0_{G_1}(N)$ have order two. We have the same conclusions as for the case when n was odd.

Type E_6

Here $\hat{L}/L \simeq \mathbb{Z}/3$ and is generated by the weight

$$\tau_c = (\frac{1}{3}, \frac{2}{3}, 0, \frac{1}{3}, \frac{2}{3}, 0).$$

Let $J \subset \Pi$. We have $\hat{L}/L_J$ is torsion free unless $3\tau_c \in L_J$ in which case $\hat{L}/L_J \simeq \mathbb{Z}/3$. Now $3\tau_c \in L_J$ if and only if $\alpha_1, \alpha_2, \alpha_4, \alpha_5 \in J$, which means Φ_J is of type $2A_2, 2A_2{+}A_1, A_5,$ or E_6.

Only for the nilpotent $E_6(a_3)$ does it happen that $|A(N)| > 1$ and the center of G contributes to $\pi_1(\mathcal{O}_N)$. In this case our calculations show that $A(N)$ has only two conjugacy classes forcing $A(N) = S_2$. Then $\pi_1(\mathcal{O}_N)$ has 6 elements with a central subgroup isomorphic to $\mathbb{Z}/3$. So $\pi_1(\mathcal{O}_N) \simeq S_2 \times \mathbb{Z}/3$.

Remark 24. By looking at the weighted Dynkin diagram for the nilpotent orbits in E_6, we observe that if $\pi_1(\mathcal{O}_N)$ is bigger than $A(N)$, then the weighted Dynkin diagram for N has a non-zero value on the nodes corresponding to α_1 and α_5. Note that the fundamental weights ω_1 and ω_5 corresponding to α_1 and α_5 do not lie in the root lattice and their images in $\hat{L}/L \simeq \mathbb{Z}/3$ are the two generators.

In the next chapter we will see that the remark leads to another reason that $\pi_1(\mathcal{O}_N)$ is a split central extension of $A(N)$ when N is $E_6(a_3)$.

Type E_7

Here $\hat{L}/L \simeq \mathbb{Z}/2$ and is generated by the weight

$$\tau_c = (0, 0, 0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}).$$

Let $J \subset \Pi$. We have $\hat{L}/L_J$ is torsion free unless $2\tau_c \in L_J$ in which case $\hat{L}/L_J \simeq \mathbb{Z}/2$. The latter case occurs if and only if $\alpha_4, \alpha_6, \alpha_7 \in J$, which implies that Φ_J is of type $(3A_1)'', 4A_1, A_2 + 3A_1, (A_3 + A_1)'', A_3{+}2A_1, D_4{+}A_1, A_3{+}A_2{+}A_1, A_5'', A_5{+}A_1, D_5{+}A_1, D_6,$ and E_7.

Remark 25. By looking at the weighted Dynkin diagram for the nilpotents in E_7, we observe that $\pi_1(\mathcal{O}_N)$ is bigger than $A(N)$ if and only if the weighted Dynkin diagram for N has a non-zero value on one of the nodes corresponding to α_4, α_6, or α_7. Note that the fundamental weights ω_4, ω_6, and ω_7 are exactly the set of fundamental weights which do not lie in the root lattice.

In the next chapter we will use Remark 25 to show that $\pi_1(\mathcal{O}_N)$ is always a split central extension of $A(N)$ when G is of type E_7.

Remark 26. The entries for $\pi_1(\mathcal{O}_N)$ for the nilpotents $4A_1$ and A_5'' are incorrect in [CM].

Chapter 3

Representations of $\pi_1(\mathcal{O}_N)$

Now G is simply-connected, connected, and simple. We write $\pi_1(\mathcal{O}_N)$ for $Z_G(N)/Z_G^0(N)$ and $A(N)$ for the same group when we have in mind G_{ad}, the quotient of G by its center.

Let $N \in \mathfrak{g}$ be a nilpotent element and let $\mathfrak{m} = \{N, H, M\}$ be an $\mathfrak{sl}_2$-triple for N. We assume that $H \in \mathfrak{t}$ and $\alpha_i(H) \geq 0$ for $\alpha_i \in \Pi$. By $\mathfrak{sl}_2$-theory, $\mathrm{ad}(H)$ acts on $\mathfrak{g}$ with integral eigenvalues. Define

$$\mathfrak{g}_i = \{X \in \mathfrak{g} |\ [H, X] = iH\}.$$

We have $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$. The subalgebra $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}_i$ is a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{l} = \mathfrak{g}_0$ is the Levi subalgebra of $\mathfrak{p}$ containing $\mathfrak{t}$. Let $L_1 \subset P$ be the subgroups of G with Lie algebras $\mathfrak{l} \subset \mathfrak{p}$. Denote by U the unipotent radical of P. Recall that $Z_G^0(N) = Z_G^0(\mathfrak{m})U_N$ where U_N is the unipotent radical of $Z_G^0(N)$ and $U_N \subset U$. It is known that $Z_G(N) \subset P$.

For $\lambda \in X^*(T)$ (dominant with respect to the positive roots of G coming from L_1), we denote by V_λ the representation of L_1 of highest weight λ extended to P by letting U act trivially. In this chapter we prove

Theorem 27. *Let V be an irreducible representation of $\pi_1(\mathcal{O}_N)$. Then there exists $\lambda \in X^*(T)$ such that V_λ satisfies*

1. *V_λ is trivial on $Z_G^0(\mathfrak{m}) \subset L_1$. Since V_λ is trivial on U_N by construction, V_λ descends to a representation of $\pi_1(\mathcal{O}_N)$.*

2. *$V_\lambda \simeq V$ as representations of $\pi_1(\mathcal{O}_N)$.*

The proof will be by explicit construction in each case. In general there is more than one choice for λ, but for the moment we will be content to find one such λ in the classical groups and give enough information in the exceptional groups to find all such λ. Recall that ω_i are the fundamental weights for T corresponding to the coroots in $\Pi^\vee$. It will turn out that it is always possible to choose a λ that is W-conjugate to some ω_i.

We will first find all representations of $\pi_1(\mathcal{O}_N)$ which descend to $A(N)$, i.e. which are trivial on the image of the center of G. Therefore, for the time being, λ belongs to the root lattice.

Let $\lambda \in L$. We present an algorithm that checks whether V_λ descends to $A(N)$ and, in this case, computes the character of V_λ as a representation of $A(N)$.

Let $\lambda_1 = \lambda, \lambda_2, \dots, \lambda_k$ be the weights on T on V_λ. Checking that V_λ is trivial on $Z_G^0(\mathfrak{m})$ is equivalent to checking that $\lambda_1, \dots, \lambda_k$ are trivial on a maximal torus of $Z_G^0(\mathfrak{m})$. How do we locate a maximal torus in $Z_G^0(\mathfrak{m})$? Conjugate $\mathfrak{m}$ via an element $\dot{w} \in N_G(T)$ such that $\mathfrak{m}' = \mathrm{Ad}(\dot{w})\mathfrak{m} \subset \mathfrak{g}_J$ where $\mathfrak{g}_J$ is a Levi subalgebra and $N' = \mathrm{Ad}(\dot{w})N$ is distinguished in $\mathfrak{g}_J$. Then a maximal torus in $Z_G^0(\mathfrak{m}')$ is given by $S' = T^{L_J}$ and so a maximal torus in $Z_G^0(\mathfrak{m})$ is given by $S = \mathrm{Ad}(\dot{w}^{-1})S'$. Now $\lambda_1, \dots, \lambda_k$ are trivial on S precisely when $w\lambda_1, \dots, w\lambda_k \in L_J$.

Finding w is just as in Section 2.3. Given a Bala-Carter pair $(\mathfrak{g}_J, N')$ for N, then w^{-1} takes the weighted diagram for N' in $\mathfrak{g}_J$ to the weighted Dynkin diagram for N in $\mathfrak{g}$.

Example 28. Consider the example of the nilpotent $D_4(a_1)$ in E_6. Let H' be the neutral element of the $\mathfrak{sl}_2$-triple $\mathfrak{m}'$ corresponding to the nilpotent $D_4(a_1)$ in the Levi subalgebra D_4. The weighted Dynkin diagram for H' in D_4 is $\begin{smallmatrix} 2 & 0 & 2 \\ & 2 & \end{smallmatrix}$. Writing H' in terms of the simple coroots of E_6, we have $H' = 4\alpha_2^\vee + 6\alpha_3^\vee + 4\alpha_4^\vee + 4\alpha_6^\vee$. Therefore, the weighted diagram for H' in E_6 is $\begin{smallmatrix} -4 & 2 & 0 & 2 & -4 \\ & & 2 & & \end{smallmatrix}$. Applying $w^{-1} = s_3s_2s_4s_6s_3s_2s_4s_1s_5$ where $s_i = s_{\alpha_i}$ to H' yields the weighted Dynkin diagram $\begin{smallmatrix} 0 & 0 & 2 & 0 & 0 \\ & & 0 & & \end{smallmatrix}$ of $D_4(a_1)$ in E_6.

The fundamental weight ω_6 in a basis of simple roots is $\begin{smallmatrix} 1 & 2 & 3 & 2 & 1 \\ & & 2 & & \end{smallmatrix}$. Consider the representation V_{ω_6} of the Levi subgroup of type $A_2 + A_2 + A_1$ (extended trivially to P). Then V_{ω_6} is two dimensional with weights $\lambda_1 = \omega_6, \lambda_2 = \omega_6 - \alpha_6$. Applying w to λ_1 and λ_2 yields respectively $\begin{smallmatrix} 0 & 1 & 2 & 1 & 0 \\ & & 1 & & \end{smallmatrix}$ and $\begin{smallmatrix} 0 & 1 & 1 & 1 & 0 \\ & & 1 & & \end{smallmatrix}$, which both lie in L_J, the characters spanned by the roots in D_4. Hence V_{ω_6} is trivial on $Z_G^0(\mathfrak{m})$.

Our assumption remains that $\lambda \in L$. Assume that we have checked that V_λ is trivial on $Z_G^0(\mathfrak{m})$ and so descends to a representation of $A(N)$. We show how to compute the character of V_λ as a representation of $A(N)$.

To compute the character of the representation V_λ on the conjugacy class C in $A(N)$ represented by $(\mathfrak{g}_J, N')$, we proceed as follows. As above, let $\dot{w}$ conjugate $\mathfrak{m}$ to an $\mathfrak{sl}_2$-triple for N' in $\mathfrak{g}_J$. The fact that $\lambda_1, \dots, \lambda_k$ are trivial on a maximal torus of $Z_G^0(\mathfrak{m})$ implies that the images of $w\lambda_1, \dots, w\lambda_k$ in L/L_J are multiples of τ_J. This is because the identity component of T^{L_J} is a torus (not in general maximal) in $Z_G^0(\mathfrak{m}')$. Let $\lambda_i \equiv a_i\tau_J$ in L/L_J and let ξ be a primitive d_J-th root of unity. A representative of C in T is given by any element $x \in T^{L_J}$ such that $\tau_J(x) = \xi$. Therefore, the trace of x on V_λ equals $\sum \xi^{a_i}$. Repeating this process for each conjugacy class C in $A(N)$ we arrive at the character of V_λ.

Example 29. Let us compute the character of V_{ω_6} from the previous example on the conjugacy class represented by $3A_2$ (the notation means the regular nilpotent in the pseudo-Levi subalgebra of type $3A_2$). Here $\tau_J = -\alpha_3$ and $\xi^3 = 1$. The weighted diagram in E_6 looks like $\begin{smallmatrix} 2 & 2 & -6 & 2 & 2 \\ & & 2 & & \end{smallmatrix}$. We find that $w^{-1} = s_3s_2s_4s_6s_1s_3s_5s_2s_4s_6s_1s_3s_5s_2s_4s_6s_3$ maps $\mathfrak{m}'$ to $\mathfrak{m}$. Then $w\lambda_1 \equiv \tau_J$ and $w\lambda_2 \equiv 2\tau_J$ and therefore the character value is $\xi + \xi^2 = -1$.

By computing the character of V_{ω_6} on the conjugacy class represented by $A_3 + 2A_1$, we find that V_{ω_6} is the irreducible representation of $A(N) = S_3$ of dimension 2.

3.1 Exceptional Groups

We carried out this algorithm for the exceptional groups. In the tables we list all fundamental weights ω_i which lie in the root lattice such that V_{ω_i} descends to give a non-trivial irreducible representation of $A(N)$.

Certainly whenever the weighted Dynkin diagram of N has a non-zero value on the α_i-node, we automatically get a one-dimensional representation V_{ω_i} of P coming from the fundamental weight ω_i. There is no reason why V_{ω_i} should descend to $\pi_1(\mathcal{O}_N)$. Nevertheless, we checked that this always happens in G_2, F_4, E_7, E_8 (even when $\pi_1(\mathcal{O}_N) = 1$ and even when ω_i is not in the root lattice). This means that $Z_{\mathfrak{g}}(\mathfrak{m})$, which we know belongs to $\mathfrak{l}$, actually belongs to $[\mathfrak{l}, \mathfrak{l}]$. In many cases, this is true because $Z_{\mathfrak{g}}(\mathfrak{m})$ is semisimple, but in the remaining cases it seems surprising. In E_6, the same fact holds whenever $\pi_1(\mathcal{O}_N)$ is non-trivial (even when ω_i is not in the root lattice). Consequently, we omit the trivial representation from our tables since we get a trivial representation of $\pi_1(\mathcal{O}_N)$ for each node of the weighted Dynkin diagram with non-zero value which does not contribute a non-trivial representation of $\pi_1(\mathcal{O}_N)$.

We also checked in E_6 (resp. E_7) that if ω is a fundamental weight not in the root lattice and ω corresponds to a node with non-zero value in the weighted Dynkin diagram, then 3ω (resp. 2ω) descends to a trivial representation of $\pi_1(\mathcal{O}_N)$. This implies that the kernel of the representation V_ω is a normal subgroup of $\pi_1(\mathcal{O}_N)$ isomorphic $A(N)$ and hence $\pi_1(\mathcal{O}_N)$ is a split central extension of $A(N)$. But we observed in Remarks 24 and 25 that whenever $\pi_1(\mathcal{O}_N)$ is bigger than $A(N)$, there is always one such node in the weighted Dynkin diagram. Conclusion: $\pi_1(\mathcal{O}_N)$ is always a split central extension of $A(N)$ in E_6 and E_7.

It follows that by tensoring the one-dimensional representations coming from these fundamental weights (which are trivial on $A(N)$) with the representations that we found for $A(N)$ (which are trivial on the image of the center of G), we get all the irreducible representations for $\pi_1(\mathcal{O}_N)$ in E_6 and E_7.

It remains to show that the representations listed in the tables in Chapter 6 for $F_4(a_3)$ (respectively $E_8(a_7)$) are irreducible and that the component group is S_4 (resp. S_5).

Lemma 30. *Let χ be a character of a representation of a finite group K with the property that χ takes only the values 0,1,−1 at the non-identity elements of K and $\chi(1)^2 \leq |K|$. Then χ is an irreducible character.*

Proof. The inner product of χ with itself yields

$$\begin{aligned} \langle \chi, \chi \rangle &= \frac{1}{|K|} \sum_{g \in K} \chi(g)\chi(g^{-1}) \\ &< \frac{1}{|K|}(\chi(1)^2 + |K| - 1) < 2. \end{aligned}$$

Therefore, $\langle \chi, \chi \rangle = 1$ and χ is irreducible. □

$\mathbf{F_4(a_3)}$

Let C_1, C_2, C_2', C_3, C_4 be the conjugacy classes in $A(N)$ corresponding to $F_4(a_3)$, $A_1 + C_3(a_1)$, $B_4(a_1)$, $A_2 + \tilde{A}_2$, $A_3 + \tilde{A}_1$ (the subscript denotes the order of the elements in each conjugacy class). We abbreviate the representations V_{ω_i} listed in the table in Chapter 6 by V_i.

First, the order of $A(N)$ is divisible by 12 because $A(N)$ contains elements of order 3 and 4. The lemma then immediately implies that V_2, V_3, and V_4 are all irreducible since the order of $A(N)$ exceeds the square of the dimensions of these representations. Now V_1 is irreducible because the elements in C_3 are acting on V_1 with eigenvalues equal to primitive third roots of unity. Hence, if V_1 splits into 2 one-dimensional representations, the characters of these representations would have non-integral values, contradicting Remark 18. We have located as many irreducible representations of $A(N)$ as there are conjugacy classes in $A(N)$. By taking the sum of the squares of the dimensions of these representations, we find that $A(N)$ has order 24.

To determine the isomorphism type of $A(N)$, let $x \in C_3$. There exists $y \in A(N)$ such that $yxy^{-1} = x^2$. Then $y^2xy^{-2} = x^4 = x$. In other words, y^2 commutes with x and so y must have order 2 or 4. If y^2 is not the identity, then xy^2 would have order 6, which cannot happen. Hence $y^2 = 1$ and the subgroup K generated by x and y is isomorphic to S_3. Now the character table of $A(N)$ reveals that V_1 and V_2 are not faithful and therefore $A(N)$ has exactly two non-trivial normal subgroups: N_1 consisting of $C_1 \cup C_2' \cup C_3$ of order 12 and N_2 consisting of $C_1 \cup C_2'$. Since the inner product of the characters of V_1 and V_3 must be zero, we see that C_2' has 3 elements (and therefore N_2 has order 4). It follows that K does not contain any non-trivial normal subgroups of $A(N)$. Hence the permutation representation of $A(N)$ on the cosets of K gives a faithful embedding of $A(N)$ in S_4, which must be an isomorphism since $A(N)$ has order 24.

$\mathbf{E_8(a_7)}$

Let C_1, C_2, C_2', C_3, C_4, C_5, C_6 be the conjugacy classes corresponding to $E_8(a_7)$, $E_7(a_5) + A_1$, $D_8(a_5)$, $E_6(a_3)+A_2$, $D_5(a_1)+A_3$, $2A_4$, and $A_5+A_2+A_1$. (the subscript denotes the order of the elements in each conjugacy class). We abbreviate the representations V_{ω_i} listed in the table in Chapter 6 by V_i.

The order of $A(N)$ is divisible by 60 since $A(N)$ contains elements of order 5, 4, and 3. The lemma then implies that V_1, V_4, and V_8 are irreducible. For V_5 and V_7, the eigenvalues of elements in C_5 are primitive fifth roots of unity, forcing V_5 and V_7 to be irreducible by Remark 18. The representation V_6 is clearly not a sum of some combination of the six irreducible representations that we have already found. Hence if V_6 were reducible, the remaining irreducible representation of $A(N)$ would be a summand of V_6 and it would have dimension less than 6. But then the sum of the squares of the dimensions of the irreducible representations would be greater than 60, but less than 120, contradicting the fact that $A(N)$ is divisible by 60. We have thus located the seven irreducible representations of $A(N)$ and have shown that $A(N)$ has order 120.

The character table reveals that there is a single non-trivial normal subgroup K of $A(N)$ of order 60. We have $K = C_1 \cup C_2' \cup C_3 \cup C_5$. The orthogonality of the characters of V_6 and V_1 immediately implies that C_2' contains 15 elements. Moreover, C_2' remains a single conjugacy class in K since K has index 2 in $A(N)$ and therefore any conjugacy class which splits in K must split into 2 conjugacy classes of equal order.

Hence for any $x \in C_2'$, $H = Z_K(x)$ has order 4 and must be a 2-Sylow subgroup of K. Moreover, the non-identity elements in H belong to C_2'. It follows that the number of

2-Sylow subgroups of K is $15/(4-1) = 5$, which implies that the normalizer in K of H has order 12. On the other hand, any 2-Sylow subgroup of $A(N)$ has order 8 and it normalizes some conjugate of H (since H has index 2 in this group of order 8). Hence the normalizer in $A(N)$ of H has order divisible by 24. But $A(N)$ has no normal subgroup of order 4 and so the normalizer in $A(N)$ of H has order exactly 24. The permutation representation on the cosets of this normalizer gives a faithful embedding of $A(N)$ into S_5 since this normalizer does not contain the non-trivial normal subgroup of $A(N)$. Hence $A(N) = S_5$ by order considerations.

3.2 Classical Groups

In the classical groups we will need the description of $Z_G(\mathfrak{m})$ from Section 2.3.3.

Type A_n

Let $[p_j]$ be the partition in $\mathfrak{sl}_{n+1}(\mathbf{C})$ corresponding to N. The analysis in Section 2.4, translated into the language of partitions, says that $\pi_1(\mathcal{O}_N)$ is cyclic of order equal to the greatest common divisor d of the p_j's. Let x generate the center of $\mathfrak{sl}_{n+1}(\mathbf{C})$ and let $\xi = \omega_1(x)$ be a primitive $(n+1)$-st root of unity.

Let $\lambda_k = \omega_{\frac{k(n+1)}{d}}$ for $k = 0, 1, \dots, d-1$. Each V_{λ_k} is a one-dimensional representation of the parabolic subgroup P associated to N and is trivial on $Z_G^0(\mathfrak{m})$. Now the image of x also generates $\pi_1(\mathcal{O}_N)$. Finally, $\lambda_k(x) = \xi^{\frac{k(n+1)}{d}}$, which gives all d-th roots of unity as k runs through $0, 1, \dots, d-1$. Hence the V_{λ_k} yield all the irreducible representations of $\pi_1(\mathcal{O}_N)$.

Other classical groups

Let $[p_j]$ be the partition corresponding to N. As in Section 2.3.2 let $\mu(i)$ be the number of times i appears in the partition. Recall that $\mu(i)$ is even whenever $i \equiv \varepsilon$.

Let e be the largest even part of the partition and o the largest odd part. Let

$$E = \{e, e-2, \dots, 4, 2\} \text{ and}$$
$$O = \{o, o-2, \dots, 5, 3\}.$$

For $i \in E \cup O$ define the number $\sigma(i)$ to be

$$\sigma(i) = \sum_{j \geq i} \mu(j)(\lfloor \frac{j-i}{2} \rfloor + 1).$$

The weighted Dynkin diagram of N will have non-zero values on the nodes corresponding to $\alpha_{\sigma(i)}$ for $i \in E \cup O$. In fact, from the description of $Z_G^0(\mathfrak{m})$, all the one-dimensional representations $V_{\omega_{\sigma(i)}}$ descend to give representations of $\pi_1(\mathcal{O}_N)$.

Let $G = Sp_{2n}$. As i runs through the even parts of the partition $[p_j]$, the one-dimensional representations $V_{\omega_{\sigma(i)}}$, together with the tensor products among them, exhaust the irreducible representations of the elementary abelian 2-group $\pi_1(\mathcal{O}_N)$. We omit the details.

Let G_1 be a special orthogonal group. As i runs through the odd parts less than o of the partition $[p_j]$, the one-dimensional representations $V_{\omega_{\sigma(i)}}$, together with the tensor products among them, exhaust the irreducible representations of the elementary abelian 2-group $Z_{G_1}(N)/Z^0_{G_1}(N)$. We omit the details.

From the enumeration of the conjugacy classes of $\pi_1(\mathcal{O}_N)$ given in [L1] or in Section 2.4, when all odd parts in the partition appear only once, we are still missing one irreducible representation of $\pi_1(\mathcal{O}_N)$ in type B_n and two irreducible representations in type D_n.

Type B_n

Let $[p_j]$ be a partition for the nilpotent N with each odd part appearing once. Let k be the number of odd parts (k is necessarily odd) and let $l = \frac{k-1}{2}$. The Levi subgroup L_1 corresponding to N has a simple component of type B_l. Then V_{ω_n} is a representation of L_1 of dimension 2^l (an incarnation of the spin representation of B_l).

Since the odd parts in $[p_i]$ appear once, $Z^0_G(\mathfrak{m})$ has trivial intersection with the simple component of L_1 of type B_l. Therefore, V_{ω_n} is trivial on $Z^0_G(\mathfrak{m})$ and descends to a representation of $\pi_1(\mathcal{O}_N)$.

What is its character? Let x generate the center of G. Since ω_n does not lie in the root lattice, x acts as -1 on V_{ω_n} and therefore $\mathrm{tr}(x, V_{\omega_n}) = -2^l$. It follows that for any element $y \in \pi_1(\mathcal{O}_N)$, we have $\mathrm{tr}(y, V_{\omega_n}) = -\mathrm{tr}(xy, V_{\omega_n})$ since x is central. But for $y \neq 1, x$, we have seen that y and yx are in the same conjugacy class in $\pi_1(\mathcal{O}_N)$. Thus $\mathrm{tr}(y, V_{\omega_n}) = 0$ for all $y \neq 1, x$.

The order of $\pi_1(\mathcal{O}_N)$ is $2^k = 2^{2l+1}$ and so the inner product of the character of V_{ω_n} with itself is 1. Hence V_{ω_n} is the missing irreducible representation of $\pi_1(\mathcal{O}_N)$.

Type D_n, n odd

Keep the same notation as above for $[p_j]$, k, L_1. Now let $l = k/2$ (k is necessarily even). Then L_1 has a simple component of type D_l. Consider the representations $V_{n-1} = V_{\omega_{n-1}}$ and $V_n = V_{\omega_n}$ of L_1. These both have dimension 2^{l-1} and come from the half-spin representations of the component of L_1 of type D_l.

As above, V_{n-1} and V_n descend to give representations of $\pi_1(\mathcal{O}_N)$.

Let x be the generator of the center of G such that $\omega_{n-1}(x) = \xi$, a primitive fourth root of unity. Then we have

$$\mathrm{tr}(x, V_{n-1}) = \mathrm{tr}(x^3, V_n) = 2^{l-1}\xi$$
$$\mathrm{tr}(x, V_n) = \mathrm{tr}(x^3, V_{n-1}) = -2^{l-1}\xi.$$

Also we have seen that if $y \neq 1, x, x^2, x^3$, then y and x^2y are in the same conjugacy class in $\pi_1(\mathcal{O}_N)$. Thus $\mathrm{tr}(y, V_{n-1}) = \mathrm{tr}(y, V_n) = 0$ for all $y \neq 1, x, x^2, x^3$, since x^2 acts as -1 on V_{n-1} and V_n.

The order of $\pi_1(\mathcal{O}_N)$ is $2^k = 2^{2l}$ and so the inner product of the character of V_{n-1} (and of V_n) with itself is 1. Hence V_{n-1}, V_n are the missing irreducible representations of $\pi_1(\mathcal{O}_N)$.

Type D_n, n even

The situation here is the same as for n odd except that the center is not cyclic. Let x_2, x_1 be the non-identity elements of the center of G with

$$\omega_{n-1}(x_2) = \omega_n(x_2) = -1 \text{ and}$$
$$\omega_{n-1}(x_1) = -1 \text{ and } \omega_n(x_1) = 1$$

and let $x_3 = x_2 x_1$ be the other non-trivial element of the center of G.

Then we have

$$\begin{aligned}
\operatorname{tr}(x_2, V_{n-1}) &= \operatorname{tr}(x_2, V_n) = -2^{l-1} \\
\operatorname{tr}(x_1, V_{n-1}) &= \operatorname{tr}(x_3, V_n) = -2^{l-1} \\
\operatorname{tr}(x_3, V_{n-1}) &= \operatorname{tr}(x_1, V_n) = 2^{l-1}.
\end{aligned}$$

For $y \in \pi_1(\mathcal{O}_N)$ not in the center of $\pi_1(\mathcal{O}_N)$, y and $x_2 y$ are in the same conjugacy class. Thus $\operatorname{tr}(y, V_{n-1}) = \operatorname{tr}(y, V_n) = 0$ for all such y, since x_2 acts as -1 on V_1 and V_2.

As above, we find that V_{n-1} and V_n are the missing irreducible representations of $\pi_1(\mathcal{O}_N)$.

3.3 Applications to rings of functions

In later work we would like to use the results of Theorem 27 to study the (graded) G-module structure of the regular functions on any cover of $\mathcal{O}_N$. We will not say anything here except that in order to carry out this project, we must be more careful about which representation of P we choose in the theorem. On the one hand, we want λ to be a combination of simple roots with non-negative coefficients. On the other hand, Vogan has suggested that λ should be chosen to have minimal length in order that the higher cohomology of certain sheaves vanishes. This requirement seems natural and moreover, we checked that when we choose λ to have minimal length, λ turns out always to be W-conjugate to a fundamental weight.

Chapter 4

A family of representations of W_a

4.1 Notation

For the rest of the thesis, G is simply-connected, connected, and simple. We retain the notation from the previous chapters and make some more definitions. Let $V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ be the real span of the cocharacters of T and let $V^* = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ be the real span of the characters of T. We extend $\langle , \rangle$ to a pairing of V^* and V and extend the action of W to V, V^*. Then the pairing is W-invariant.

For a subset M of V or V^*, let $L(M)$ be the lattice generated by M. In Section 1.1, we defined $L, \hat{L} \subset V^*$. Similarly, let $L^\vee = L(\Phi^\vee)$ be the lattice in V generated by $\Phi^\vee$ and let

$$\hat{L}^\vee = \{v \in V \mid \langle \alpha, v\rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}.$$

Recall $\Lambda = \hat{L}/L$. We also have $\Lambda \simeq \hat{L}^\vee/L^\vee$. Let $f = |\Lambda|$. By Proposition 5, f equals the number of $\alpha_i \in \tilde{\Pi}$ with $c_i = 1$.

Let $L^\vee$ act on V by translation and form the affine Weyl group $W_a = W \ltimes L^\vee$. Let $H_{\alpha,k} = \{v \in V \mid \langle \alpha, v\rangle = k\}$ where $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Then it is known [Hu1] that W_a is generated by the reflections $s_{\alpha,k}$ in the hyperplanes $H_{\alpha,k}$. Let $s(w)$ be the least number of reflections needed to write $w \in W_a$ as a product of reflections. If w is of finite order, then w has a fixed point on V. In this case define $d(w)$ to be the dimension of the fixed point set of w. Then $d(w) = n - s(w)$, where recall that n is the rank of G [Ca1].

We study the natural action of W_a on the set $S_t = L^\vee/tL^\vee$ where $t \in \mathbb{N}$. Thus we get a representation U_t of W_a of dimension t^n on the space of complex functions on S_t. Similarly, we can consider the action of W_a on the set $\hat{S}_t = \hat{L}^\vee/tL^\vee$ and get a representation $\hat{U}_t$ of dimension ft^n.

4.2 Fixed points of W_a on S_t and $\hat{S}_t$

We want to know the number of fixed points of $w \in W_a$ of finite order on S_t and $\hat{S}_t$. When t is not divisible by certain primes associated to Φ, the answer is readily computed. But first we need some preliminaries.

Recall that a root subsystem of Φ is a subset of Φ which is itself a root system. Let $M \subset \Phi$. The root subsystem spanned by M is the collection of roots in Φ which are integral combinations of elements in M. The rational closure of a root subsystem Φ' denoted by $\bar{\Phi}'$ is the collection of roots in Φ which are rational combinations of elements in Φ'. Note that $\bar{\Phi}'$ is a root system of the same rank as Φ'. We will always assume our root subsystems are integrally closed.

Definition 31. A bad prime of Φ is a prime which divides the order of the torsion subgroup of $L/L(\Phi')$ for some root subsystem $\Phi' \subset \Phi$.

The following two results are found in [St].

Theorem 32. *The bad primes of Φ are precisely those primes which divide a coefficient of θ.*

Lemma 33. *Let Φ' be a root subsystem. Any set of simple roots for $\bar{\Phi}'$ can be extended to a set of simple roots for Φ.*

Corollary 34. *Let Φ' be a root subsystem. Then $L(\bar{\Phi}')/L(\Phi')$ is isomorphic to the torsion subgroup of $L/L(\Phi')$.*

Proof. Lemma 33 implies that $L/L(\bar{\Phi}')$ is torsion free. Now the corollary follows from the fact that $L(\bar{\Phi}')$ and $L(\Phi')$ have the same rank. □

Definition 35. We say t is good (for Φ) if it is prime to every bad prime. We say t is very good if it is also prime to f.

Remark 36. By inspection, when t is prime to the Coxeter number h of W, it is very good.

Lemma 37. *Let $\beta_1, \ldots, \beta_n$ be a set of linearly independent roots and let Φ' be the subsystem they span. Let $k_1, \ldots, k_n \in \mathbb{Z}$. If t is good, there exists $u \in \hat{L}^{\vee}$ with $\langle \beta_i, u \rangle \equiv k_i \pmod t$ for all i. If t is very good, there exists $u \in L^{\vee}$ with $\langle \beta_i, u \rangle \equiv k_i \pmod t$ for all i.*

Proof. $\langle , \rangle$ induces a pairing of $L(\Phi')$ and $\hat{L}^{\vee}$ which has determinant equal to $[L : L(\Phi')]$. The pairing of $L(\Phi')$ and $L^{\vee}$ has determinant equal to $f[L : L(\Phi')]$. The lemma follows since these determinants are invertible modulo t under the respective hypotheses. □

Definition 38. Let $\Phi' \subset \Phi$ be a root subsystem. Define

$$P(\Phi') = \{u \in S_t | \, \langle \alpha, u \rangle \equiv 0 \pmod t \text{ for all } \alpha \in \Phi'\}$$

and similarly

$$\hat{P}(\Phi') = \{u \in \hat{L}^{\vee}/t\hat{L}^{\vee} | \, \langle \alpha, u \rangle \equiv 0 \pmod t \text{ for all } \alpha \in \Phi'\}.$$

We can now prove the main result on the number of fixed points of $w \in W_a$ of finite order.

Proposition 39. *If t is good, then the number of fixed points of w on $\hat{S}_t$ is $ft^{d(w)}$. If t is very good, then the number of fixed points of w on S_t is $t^{d(w)}$.*

Proof. We give the proof for S_t, the case of $\hat{S}_t$ being similar.

Let $l = s(w)$ and let $w = s_{\beta_1,k_1}s_{\beta_2,k_2}...s_{\beta_l,k_l}$ be a minimal expression for w as a product of reflections. The roots β_i are necessarily linearly independent. Let Φ' be the root subsystem of rank l that they span. For any $u \in L$, continue to denote by u its image in S_t. By an easy induction on $s(w)$, we have

$$w(u) = u \text{ if and only if } \langle \beta_j, u\rangle \equiv k_j \pmod{t} \text{ for } j = 1, 2, ..., l.$$

Applying the previous lemma, we conclude that the number of fixed points of w is just $|P(\Phi')|$.

Because $f = [\hat{L}^\vee : L^\vee]$ and t is prime to f, the inclusion of $L^\vee$ into $\hat{L}^\vee$ induces an isomorphism of $L^\vee/tL^\vee$ and $\hat{L}^\vee/t\hat{L}^\vee$. This isomorphism maps $P(\Phi')$ onto $\hat{P}(\Phi')$.

Now t is not divisible by any bad prime, so Corollary 34 implies t is prime to $[L(\bar{\Phi}'):L(\Phi')]$. It is easy to see that in this case $\hat{P}(\Phi') = \hat{P}(\bar{\Phi}')$.

We are reduced to computing the cardinality of $\hat{P}(\bar{\Phi}')$. Extend a set of simple roots $\{\lambda_1, \ldots, \lambda_l\}$ of $\bar{\Phi}'$ to a set of simple roots $\{\lambda_1, \ldots, \lambda_l\} \cup \{\lambda_{l+1}, \ldots, \lambda_n\}$ of Φ as in Lemma 33. Let $\omega_1^\vee, \omega_2^\vee, \ldots, \omega_n^\vee \in \hat{L}^\vee$ be a corresponding set of fundamental coweights. That is, $\langle \lambda_i, \omega_j^\vee\rangle = \delta_{ij}$. Then we have

$$\hat{P}(\bar{\Phi}') = \{\sum x_i\omega_i^\vee \in \hat{L}^\vee/t\hat{L}^\vee | \, x_j \equiv 0 \pmod{t} \text{ for } j = 1, 2, \ldots, l\}.$$

Clearly, the cardinality of $\hat{P}(\bar{\Phi}')$ is just $t^{n-l} = t^{d(w)}$. □

Remark 40. The results in this section can be extended to root systems which are not irreducible. We will need this in the next section.

4.3 Stabilizers of elements in $\hat{S}_t$

Let $s_i \in W_a$ be the reflection in the hyperplane $H_{\alpha_i,0}$ for $i = 1, \ldots, n$ and let $s_0 \in W_a$ be the reflection in the hyperplane $H_{\theta,1}$. Let $I = \{s_0, s_1, \ldots, s_n\}$. For any proper subset J of I, the subgroup of W_a generated by the elements in J is a finite Weyl group W^J corresponding to a (not necessarily irreducible) root system Φ_J. Recall that a parabolic subgroup of W^J is a subgroup of W^J that is conjugate under W^J to $W^{J'}$ for some $J' \subset J$. Note that W^J is isomorphic to W_J from the previous chapters, but whereas $W_J \subset W$, we have $W^J \subset W_a$ (if we identify I with $\tilde{\Pi}$).

In this section we will prove

Proposition 41. *If t is good, the stabilizer in W^J of an element in $\hat{S}_t$ (or S_t) is a parabolic subgroup of W^J.*

First, we need some lemmas. Let $Q_1 = W^J, Q_2, \ldots, Q_k$ be representatives of the W^J-conjugacy classes of subgroups of W^J which are Weyl groups of root subsystems of Φ_J.

Lemma 42. *Let U be a representation of W^J. Suppose U has two expressions as a sum of induced representations*

$$U = \oplus_{i=1}^k f_i Ind_{Q_i}^{W^J}(1) = \oplus_{i=1}^k f_i' Ind_{Q_i}^{W^J}(1)$$

where f_i *and* f_i' *are non-negative integers and* $f_i = 0$ *whenever* Q_i *is not a parabolic subgroup of* W^J*. Then* $f_i = f_i'$ *for* $i = 1, 2, \ldots, k$*.*

Proof. Let w_j be a Coxeter element of Q_j. In general it is possible for w_j to be conjugate to $w_{j'}$ when $j \neq j'$. This occurs in B_4, for example. Nevertheless the elements w_j distinguish the subgroups Q_j of W^J enough to arrive at the conclusion of the lemma.

In fact, the following statements are true about w_j:

1. $\operatorname{tr}(w_j, Ind_{Q_i}^{W^J}(1)) \geq 0$ and $\operatorname{tr}(w_j, Ind_{Q_j}^{W^J}(1)) > 0$.

2. $\operatorname{tr}(w_j, Ind_{Q_i}^{W^J}(1)) = 0$ if rank $Q_i <$ rank Q_j.

3. If Q_j is actually a parabolic subgroup of W^J, then $\operatorname{tr}(w_j, Ind_{Q_i}^{W^J}(1)) = 0$ if rank $Q_i \leq$ rank Q_j and $j \neq i$.

Only the last statement is not obvious. It is equivalent to the statement that no conjugate of w_j belongs to any Q_i of the same rank as Q_j. This statement was checked by comparing the characteristic polynomial for w_j with the characteristic polynomials of elements in Q_i using the analysis of conjugacy classes in a Weyl group given in [Ca1].

The lemma follows from these statements by reverse induction on the rank of the subgroups Q_i by taking the trace of w_j on both expressions for U. □

Let $M \subsetneq \tilde{\Pi}$ and let Φ' be the root subsystem of Φ spanned by M. Recall from Section 1.1 that M is a set of simple roots for Φ'. It follows from Lemma 33 that M can be extended to a set of simple roots for Φ if and only if $\Phi' = \bar{\Phi}'$.

We begin by proving the proposition for the case $J = \{s_1, \ldots, s_n\}$, so that $W^J = W$. First we need some more definitions.

Denote by W_t the subgroup of W_a of the form $W \ltimes tL^\vee$. Note that W_t is isomorphic to $W_a = W_1$ for all t. Let

$$D_t = \{u \in V | \, \langle \alpha, u \rangle \geq 0 \text{ for } \alpha \in \Pi \text{ and } \langle \theta, u \rangle \leq t\}.$$

Recall that D_1 is a fundamental domain for the action of W_a on V (see [Hu1]). The same proof also shows that D_t is a fundamental domain for the action W_t on V for any t. Let $\hat{L}_t^\vee = D_t \cap \hat{L}^\vee$. Then each W-orbit on $\hat{S}_t$ contains a unique element of $\hat{L}_t^\vee$.

Proof of proposition when $W^J = W$**.** Choose $u \in \hat{S}_t$ and let W_u be the stabilizer of u in W. Without loss of generality, we can assume that $u \in \hat{L}_t^\vee$. Let

$$\Pi_u = \{\alpha \in \tilde{\Pi} | \, \langle \alpha, u \rangle \equiv 0 \pmod{t}\}$$

and let Φ' be the root subsystem spanned by Π_u. It is clear that W_u is just the Weyl group of Φ'. To show that W_u is a parabolic subgroup of W, we must show that a set of simple roots for Φ' extends to a set of simple roots for Φ, or equivalently that $\Phi' = \bar{\Phi}'$.

We may assume that u is non-zero and $\Pi_u \not\subset \Pi$, the result being clear otherwise. So $-\theta \in \Pi_u$. Since $u \in \hat{L}_t^\vee$ is non-zero and $-\theta \in \Pi_u$, we must have $\langle \theta, u \rangle = t$. Also $\langle \alpha, u \rangle = 0$ or t for each $\alpha \in \Pi_u \backslash \{-\theta\}$. But if $\langle \alpha', u \rangle = t$ for some $\alpha' \in \Pi_u \backslash \{-\theta\}$, then $\langle \theta, u \rangle = t$ forces $\langle \alpha, u \rangle = 0$ for all $\alpha \in \Pi \backslash \{\alpha'\}$. In other words, $\Pi_u = \tilde{\Pi}$ and $W_u = W$. Note that this situation can only occur when the coefficient of θ on α' is one (and this can only occur when $f > 1$). We are thus reduced to the case where $\langle \alpha, u \rangle = 0$ for each $\alpha \in \Pi_u \backslash \{-\theta\}$.

Now take $\beta \in \bar{\Phi}'$. Then

$$\beta = d(-\theta) + \sum_{\alpha \in \Pi_u \backslash \{-\theta\}} d_\alpha \alpha$$

where $d_\alpha, d \in \mathbb{Q}$. The fact that t is prime to the index of $L(\Phi')$ in $L(\bar{\Phi}')$ implies that $\langle \beta, u \rangle$ is a multiple of t. Taking the inner product with u on both sides of the expression for β reveals that d is actually an integer. Now it follows that all d_α are also integers. This means that β belongs to Φ' which is what we wanted.

Remark 43. The above proof extends to root systems which are not irreducible, a fact we will now use.

Proof of proposition for general W^J. Let $\hat{V}_t$ be the representation we called $\hat{U}_t$ in the case where Φ is replaced by Φ_J. Continue to denote by $\hat{V}_t, \hat{U}_t$ the restrictions of these representations to W^J. We note that if t is good for the root system Φ, then t is also good for the root system Φ_J [St]. Hence by the character formula of Proposition 39 (which only depended on the function $s(w)$ and the rank of W^J), we know that $\hat{U}_t$ is isomorphic to the direct sum of $t^{n-|J|}$ copies of $\hat{V}_t$.

Because both $\hat{U}_t$ and $\hat{V}_t$ were constructed as permutation representations (in different ways), we can express them as a direct sum of induced representations. Each W^J-orbit on the set used to define the permutation representation contributes a term $Ind_H^{W^J}(1)$ where H is the stabilizer of a point in the orbit.

Thus on the one hand, knowing the proposition for the case $W^J = W$ allows us to conclude that $\hat{V}_t = \oplus_{i=1}^k f_i \ Ind_{Q_i}^{W^J}(1)$ where $f_i = 0$ when Q_i is not a parabolic subgroup of W^J. On the other hand, it is clear that the stabilizer in W^J of an element $u \in \hat{S}_t$ is the Weyl group of a root subsystem of Φ_J. Hence, we can write $\hat{U}_t = \oplus_{i=1}^k f_i' \ Ind_{Q_i}^{W^J}(1)$. By Lemma 42, we can conclude that $f_i' = 0$ when Q_i is not a parabolic subgroup of W^J. In other words, the stabilizer in W^J of $u \in \hat{S}_t$ is actually a parabolic subgroup, which concludes the proof.

For the remainder of this section and the next section, we focus on the case $W^J = W$. Let $P_1 = W, P_2, \ldots, P_m$ be representatives of the W-conjugacy classes of parabolic subgroups of W, with $|P_i| \geq |P_j|$ for $i < j$. By the previous proposition we have

$$U_t = \oplus_{i=1}^m \chi_i(t) \ Ind_{P_i}^W(1) \text{ and} \tag{4.1}$$

$$\hat{U}_t = \oplus_{i=1}^m \hat{\chi}_i(t) \ Ind_{P_i}^W(1). \tag{4.2}$$

The functions $\chi_i(t)$ and $\hat{\chi}_i(t)$ are both well defined by Lemma 42. When t is very good, we have $\hat{\chi}_i(t) = f\chi_i(t)$ by comparison of the characters of U_t and $\hat{U}_t$. We will see in the next section that $\hat{\chi}_i(t)$ is a polynomial in t when t is good by relating it to the characteristic polynomial of a hyperplane arrangement. For now let us observe that $\hat{\chi}_i(t)$ has a nice combinatorial description when t is good.

Let M be a proper subset of $\tilde{\Pi}$. Define $p(M,t)$ to be the number of solutions $\mathbf{y}$ in strictly positive integers to the equation

$$\sum_{\alpha_i \in \tilde{\Pi} - M} c_i y_i = t.$$

Proposition 44. *Assume t is good. Let Π_j be a set of simple roots corresponding to P_j. Then $\hat{\chi}_j(t)$ is equal to*

$$\sum p(M,t)$$

where the sum is over the subsets M of $\tilde{\Pi}$ which are W-conjugate to Π_j.

Proof. Recall the definitions of W_u and Π_u in the first part of the proof of the previous proposition. We have

$$\begin{aligned} \hat{\chi}_j(t) &= \#\{u \in \hat{L}_t^\vee \mid W_u \text{ conjugate to } P_j\} \\ &= \#\{u \in \hat{L}_t^\vee \mid \Pi_u \text{ conjugate to } \Pi_j\} \\ &= \sum \#\{u \in \hat{L}_t^\vee \mid \Pi_u = M\} \end{aligned} \tag{4.3}$$

where the sum is over the subsets M of $\tilde{\Pi}$ which are W-conjugate to Π_j.

But $\#\{u \in \hat{L}_t^\vee \mid \Pi_u = M\}$ is easily determined. Let $\omega_1^\vee, \omega_2^\vee, \ldots, \omega_n^\vee$ be a set of fundamental coweights for $\hat{L}^\vee$ corresponding to Π. Express $u \in \hat{L}^\vee$ as $y_1\omega_1^\vee + y_2\omega_2^\vee + \cdots + y_n\omega_n^\vee$. Let $y_0 = t - \sum_{\alpha_i \in \Pi} c_i y_i$. Then $u \in \hat{L}_t^\vee$ if and only if $y_i \geq 0$ for $i = 0, 1, \ldots n$. In order for Π_u to equal M we must have $y_i = 0$ for $\alpha_i \in M$ and $y_i > 0$ for $\alpha_i \notin M$. Hence $\#\{u \in \hat{L}_t^\vee \mid \Pi_u = M\}$ is $p(M,t)$. □

4.4 Hyperplane arrangements

Let $\mathcal{A}$ be a set of hyperplanes in $V = \mathbf{R}^n$ such that $\cap_{H \in \mathcal{A}} H = 0$. Let $\mathcal{L} = \mathcal{L}(\mathcal{A})$ be the set of intersections of these hyperplanes. We consider $V \in \mathcal{L}$. Partially order $\mathcal{L}$ by reverse inclusion and define a Möbius function μ of $\mathcal{L}$ as follows: $\mu(X,X) = 1$ and $\sum_{X \leq Z \leq Y} \mu(Z,Y) = 0$ if $X < Y$ and $\mu(X,Y) = 0$ otherwise. The characteristic polynomial of $\mathcal{L}$ is

$$\chi(\mathcal{L},t) = \sum_{X \in \mathcal{L}} \mu(V,X) t^{\dim X}.$$

Let $\mathcal{M}$ be the complex manifold obtained by removing from $\mathbf{C}^n$ the complexification of the hyperplanes in $\mathcal{A}$. Orlik and Solomon have shown that the Poincaré polynomial

$$P(\mathcal{M},t) = \sum_{p \geq 0} \dim H^p(\mathcal{M},\mathbf{C})\, t^p$$

is equal to $(-t)^n \chi(\mathcal{L}, -t^{-1})$. For these results see [OS1].

In our case $\mathcal{A}$ is the set of hyperplanes $H_{\alpha,0}$ where $\alpha \in \Phi$. For any $X \in \mathcal{L}(\mathcal{A})$ let $\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \text{ and } X \not\subset H\}$. Let $\mathcal{L}^X = \mathcal{L}(\mathcal{A}^X)$ be the corresponding partially ordered set and $\chi(\mathcal{L}^X, t)$ its characteristic polynomial. In [OS2] it is proved that

$$t^{\dim X} = \sum_{Y \in \mathcal{L}, Y \geq X} \chi(\mathcal{L}^Y, t) \tag{4.4}$$

For each $X \in \mathcal{L}(\mathcal{A})$ let P_X be the elements of W which fix X pointwise. It is known that P_X is a parabolic subgroup of W. Clearly, if P_X and P_Y are conjugate, then $\mathcal{L}^X$ and $\mathcal{L}^Y$ are isomorphic and have the same characteristic polynomial.

The next proposition relates the functions $\hat{\chi}_j(t)$ to the characteristic polynomials of hyperplane arrangements. Let X_i be the fixed point set of P_i.

Proposition 45. *Assume t is good. Then*

$$\hat{\chi}_j(t) = \frac{f}{[N(P_j) : P_j]}\, \chi(\mathcal{L}^{X_j}, t)$$

where $N(P_j)$ is the normalizer of P_j in W.

Proof. Let w_j be a Coxeter element of P_j. Taking the trace of w_j on both sides of (4.2) yields

$$f t^{\dim X_j} = \sum_{i=1}^{m} \hat{\chi}_i(t) \operatorname{tr}(w_j, Ind_{P_i}^W(1)). \tag{4.5}$$

Moreover,

$$\begin{aligned} \operatorname{tr}(w_j, Ind_{P_i}^W(1)) &= \#\{gP_i \mid g^{-1}w_j g \in P_i\} \\ &= \#\{gP_i \mid g^{-1}P_j g \subset P_i\} \\ &= \#\{gP_i \mid P_j \subset gP_i g^{-1}\} \\ &= \#\{\text{Conjugates of } P_i \text{ containing } P_j\}[N(P_i) : P_i] \end{aligned} \tag{4.6}$$

On the other hand, we can write (4.4) in terms of parabolic subgroups which yields

$$\begin{aligned} t^{\dim X_j} &= \sum_{i=1}^{m} \chi(\mathcal{L}^{X_i}, t)\#\{Y \in \mathcal{L} \mid Y \geq X_j \text{ and } P_Y \text{ conjugate to } P_i\} \\ &= \sum_{i=1}^{m} \chi(\mathcal{L}^{X_i}, t)\#\{\text{Conjugates of } P_i \text{ containing } P_j\} \end{aligned} \tag{4.7}$$

Now putting (4.5) and (4.6) together, comparing with (4.7), and arguing by induction on j gives

$$\hat{\chi}_j(t) = \frac{f}{[N(P_j) : P_j]}\, \chi(\mathcal{L}^{X_j}, t).$$

□

In [OS2], Orlik and Solomon computed the roots of $\chi(\mathcal{L}^X, t)$, which turn out to be positive integers. When $X = V$, the roots are the exponents of W. Propositions 44 and 45 give a different, more elementary way to compute $\chi(\mathcal{L}^X, t)$. We illustrate this for the classical root systems.

Recall that Π_j is a set of simple roots corresponding to P_j and X_j is the fixed point set of P_j. A useful tool for finding the roots of $\chi(\mathcal{L}^{X_j}, t)$ is the following observation. Let m be good and assume m is less than $\sum_{\alpha_i \in \tilde{\Pi} - M} c_i$ for all subsets $M \subset \tilde{\Pi}$ conjugate to Π_j. It follows that $p(M, m) = 0$ for all $M \subset \tilde{\Pi}$ conjugate to Π_j. Hence $\hat{\chi}_j(m) = 0$ and so m is a root of the polynomial $\chi(\mathcal{L}^{X_j}, t)$. Incidentally, this can be taken as a generalization of the well-known fact that when m is prime to and less than h (which is just $\sum_{\alpha_i \in \tilde{\Pi}} c_i$), then m is an exponent for W.

Let n_j be the cardinality of Π_j.

Proposition 46. *The roots of* $\chi(\mathcal{L}^{X_j}, t)$ *are* $\{1, 2, \ldots, n - n_j\}$ *for* A_n *and* $\{1, 3, \ldots, 2(n - n_j) - 1\}$ *for* B_n.

In D_n *the roots are* $\{1, 3, \ldots, 2(n - n_j) - 1\}$ *if* Π_j *is not* W*-conjugate to a set of simple roots in* $A_{n-2} \subset D_n$ *and the roots are* $\{1, 3, \ldots, 2(n - n_j) - 3, n + r - n_j - 1\}$ *if* Π_j *is* W*-conjugate to a set of simple roots in* $A_{i_1} + A_{i_2} + \cdots + A_{i_r} \subset A_{n-2} \subset D_n$.

Proof. The results for A_n and B_n follow immediately from the observation. Similarly for D_n when Π_j is not conjugate to a set of simple roots in $A_{n-2} \subset D_n$. For the case in D_n when Π_j is conjugate to a set of simple roots in $A_{i_1} + A_{i_2} + \cdots + A_{i_r}$, the observation ensures that $\{1, 3, \ldots, 2(n - n_j) - 3\}$ are roots. The remaining root can be determined by noting that the sum of all the roots must equal the number of hyperplanes in X_j. In this particular case, the number of hyperplanes in X_j is seen to be $(n - n_j)(n - n_j - 1) + r$, whence the last root must equal $n + r - n_j - 1$. □

Chapter 5

Applications to the affine flag manifold

The motivation for defining U_t came from studying certain fixed point varieties on an affine flag manifold. We now describe this situation and compute the Euler characteristic of these varieties.

5.1 Euler Characteristic Computation

Let $\mathfrak{b}^{\text{opp}}$ be the opposite Borel subalgebra to $\mathfrak{b}$. For each $\phi \in \Phi$ choose a generator e_ϕ for the corresponding root space. Let $\Phi_k = \{\phi \in \Phi | \operatorname{ht}(\phi) = k\}$.

Let $F = \mathbf{C}((\epsilon))$ and $A = \mathbf{C}[[\epsilon]]$. Let $\hat{G} = G(F)$, $\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbf{C}} F$, and $\mathfrak{g}_A = \mathfrak{g} \otimes_{\mathbf{C}} A$. Let $p : \mathfrak{g}_A \to \mathfrak{g}$ be evaluation at $\epsilon = 0$ and define $\hat{\mathfrak{b}}_0 = p^{-1}(\mathfrak{b}^{\text{opp}})$. The $\mathbf{C}$-Lie subalgebras of $\hat{\mathfrak{g}}$ (other than $\hat{\mathfrak{g}}$ itself) which contain $\hat{\mathfrak{b}}_0$ are in bijection with proper subsets J of $I = \{s_0, s_1, \ldots, s_n\}$. Let $\hat{\mathfrak{b}}_0^J$ denote the subalgebra corresponding to the subset J.

Let $\hat{\mathcal{B}}^J$ denote the set of $\hat{G}$-conjugates of $\hat{\mathfrak{b}}_0^J$. This set can be given the topology of an increasing union of complex projective varieties. We refer to it as the partial affine flag manifold of type J. In the case J is the empty set, we write $\hat{\mathcal{B}}$ for $\hat{\mathcal{B}}^J$ and call it the affine flag manifold. There is a natural projection from $\hat{\mathcal{B}}$ to $\hat{\mathcal{B}}^J$ for any J with fiber equal to a finite dimensional partial flag manifold.

For any $n \in \hat{\mathfrak{g}}$ let $\hat{\mathcal{B}}_n^J$ be the subset of $\hat{\mathcal{B}}^J$ consisting of subalgebras which contain n.

In [F], Fan gives the construction of Lusztig of a family of regular, semisimple, nil-elliptic elements of Coxeter type (for definitions see [KL]) depending on a natural number t. We give this construction (up to conjugation by the longest element in the Weyl group). Write $t = ah + b$ where $0 \le b < h$. Define

$$n_t = \epsilon^a(\epsilon \sum_{\phi \in \Phi_{h-b}} e_\phi + \sum_{\phi \in \Phi_{-b}} e_\phi).$$

When t is relatively prime to the Coxeter number h, the fixed point space $\hat{\mathcal{B}}_{n_t}^J$ is a complex projective variety.

Let χ denote the Euler characteristic. In [LS], $\chi(\hat{\mathcal{B}}_{n_t}^J)$ is computed in type A_n for two partial affine flag manifolds and in [F], $\chi(\hat{\mathcal{B}}_{n_t})$ is computed in all types. We now give a proposition which computes $\chi(\hat{\mathcal{B}}^J_{n_t})$ in all cases. D. S. Sage proved this proposition for the classical groups, but the combinatorics in his proofs is different [Sa].

Proposition 47. *Let j be the cardinality of J. When t is prime to h,*

$$\chi(\hat{\mathcal{B}}^J_{n_t}) = \frac{(t+m_1)(t+m_2)...(t+m_j)t^{n-j}}{|W^J|}$$

where $m_1, \ldots, m_j$ are the exponents of W^J.

Before giving the proof, we want to be able to access the results of the previous sections. So we need to introduce some more ideas.

Our main tool is a $\mathbf{C}^*$ action on $\hat{\mathfrak{g}}$ which gives a $\mathbf{C}^*$ action on $\hat{\mathcal{B}}$. We recall the construction given in [F]. Let T be the maximal torus in G with Lie algebra $\mathfrak{t}$. Let $\hat{\rho} : \mathbf{C}^* \to T$ be the one parameter subgroup of T such that $\alpha(\hat{\rho}(\lambda)) = \lambda^{-2}$ for all $\alpha \in \Phi^+$. Denote by S the image of $\hat{\rho}$ in T. Let $\mathbf{C}^*$ act on $\mathfrak{g}$ through conjugation by S. Let $\lambda \in \mathbf{C}^*$ act on F by the rule $\lambda \circ f(\epsilon) = f(\lambda^{2h}\epsilon)$. Define the action of $\mathbf{C}^*$ on $\hat{\mathfrak{g}}$ by extending $\mathbf{C}$-linearly. Note that $\lambda \circ \epsilon^k e_\phi = \lambda^{2(hk-\mathrm{ht}(\phi))}\epsilon^k e_\phi$.

This $\mathbf{C}^*$ action has a number of key properties. First, it defines an (algebraic) action on any partial affine flag manifold $\hat{\mathcal{B}}^J$ and preserves the fixed point space $\hat{\mathcal{B}}^J_{n_t}$. Second, the fixed points of the $\mathbf{C}^*$ action on $\hat{\mathcal{B}}$ are of the form $w\hat{\mathfrak{b}}_0 w^{-1}$ where $w \in W_a$ (we do not distinguish here between elements in W_a and their representatives in $\hat{G}$ when there is no confusion).

Let $\hat{G}^J = \{g \in \hat{G} | g\hat{\mathfrak{b}}^J_0 g^{-1} = \hat{\mathfrak{b}}^J_0\}$. The quotient of $\hat{G}^J$ by its prounipotent radical is G^J which is a connected, reductive, algebraic group over $\mathbf{C}$. Let $\mathfrak{g}^J$ be the Lie algebra of G^J and let $p^J : \hat{\mathfrak{b}}^J_0 \to \mathfrak{g}^J$ be the canonical map. The last property of the $\mathbf{C}^*$ action that we will need in the last section is that $p^J(\lambda \circ n)$ is G^J-conjugate to $p^J(n)$ for $\lambda \in \mathbf{C}^*$ and $n \in \hat{\mathfrak{b}}^J_0$.

We now give another way to define the permutation representation U_t which is needed in the proof. Let $I_t = \{w \in W_a | \, wD_1 \subset D_t\}$. Since D_t is a fundamental domain for $W_t = W \ltimes tL^\vee$, we see that I_t is a set of right coset representatives for W_t in W_a. On the other hand, a set of coset representatives for $tL^\vee$ in $L^\vee$ also gives a set of right coset representatives for W_t in W_a. Hence there is a natural bijection between I_t and $S_t = L^\vee/tL^\vee$ which sends $w \in I_t$ to the element in S_t which represents the same right coset of W_t. Explicitly, the map sends w to $-w^{-1}(0)$. Furthermore, W_a acts on I_t by the inverse of right multiplication on the set of right cosets. This action, expressed in terms of S_t, is just induced from the action of W_a on $L^\vee$. As such it is the action we have been discussing.

Proof of the proposition. The $\mathbf{C}^*$ action preserves $\hat{\mathcal{B}}^J_{n_t}$. Let $\mathcal{F}$ be the points of $\hat{\mathcal{B}}^J_{n_t}$ fixed under the $\mathbf{C}^*$ action. A general principle implies that $\chi(\hat{\mathcal{B}}^J_{n_t}) = \chi(\mathcal{F})$. So our calculation reduces to determining the cardinality of the finite set $\mathcal{F}$.

In general $\mathcal{F} = \{w\hat{\mathfrak{b}}^J_0 w^{-1} | \, w \in W_a \text{ and } n_t \in w\hat{\mathfrak{b}}^J_0 w^{-1}\}$. But this set is in bijection with the set $\{w \in W_a/W^J | \, n_t \in w\hat{\mathfrak{b}}_0 w^{-1}\}$. This takes into account the fact that $w\hat{\mathfrak{b}}^J_0 w^{-1}$ stays the same if w is modified by an element of W^J on the right.

Define D^a to be

$$\{u \in V | \langle \alpha, u \rangle \geq -a \text{ for } \alpha \in \Phi_b \text{ and } \langle \alpha, u \rangle \geq -a - 1 \text{ for } \alpha \in \Phi_{b-h}\}.$$

A result in [F] implies the existence of $\tilde{w} \in W_a$ such that $\tilde{w}(D^a) = D_t$.

Now a calculation shows that $n_t \in w\hat{\mathfrak{b}}_0 w^{-1}$ if and only if $wD_1 \subset D^a$. And this is the case if and only if $\tilde{w}w \in I_t$. So the cardinality of the set $\{w \in W_a/W^J | n_t \in w\hat{\mathfrak{b}}_0 w^{-1}\}$ is just the number of orbits of W^J acting on I_t on the right. Under our bijection with S_t, this is the number of W^J-orbits on S_t.

In general, the number of orbits of a finite group H acting on a set is given by

$$\frac{1}{|H|}\sum_{h\in H}\sigma(h)$$

where $\sigma(h)$ is the number of fixed points of h on the set. By Proposition 39 we thus have

$$\begin{aligned}\chi(\hat{\mathcal{B}}^J_{n_t}) &= \frac{1}{|W^J|}\sum_{w\in W^J} t^{n-s(w)} \\ &= \frac{t^{n-j}}{|W^J|}\sum_{w\in W^J} t^{j-s(w)}.\end{aligned}$$

A theorem of Shepard and Todd [ShTo] states

$$\sum_{w\in W^J} q^{j-s(w)} = (q+m_1)(q+m_2)...(q+m_j),$$

whence the result.

Remark 48. We can also view the set S_t as the set of elements of order t in a fixed maximal torus of G. One maps $u \in \Phi^\vee$ to $u(\tau) \in T$ where τ is a primitive t-th root of unity. Then the W-orbits of S_t are in bijection with the conjugacy classes of elements of order t in G. This is the viewpoint of D. Peterson who earlier computed the cardinality of the conjugacy classes of elements of order t when t is prime to the Coxeter number.

5.2 Action of W^J on the homology of $\hat{\mathcal{B}}_{n_t}$

In [KL] and [L2], an action of the affine Weyl group is defined on the homology of $\hat{\mathcal{B}}_n$ for any $n \in \hat{\mathfrak{g}}$ such that $\hat{\mathcal{B}}_n \neq \emptyset$. As before, we choose a finite Weyl subgroup W^J of W_a. Here we compute explicitly the virtual representation of W^J on $H_*(\hat{\mathcal{B}}_{n_t}) = \sum_{i\geq 0}(-1)^i H_i(\hat{\mathcal{B}}_{n_t})$ when t is prime to h.

Let $\mathcal{B}^J$ be the flag variety of G^J which we consider to be the set of Borel subalgebras of $\mathfrak{g}^J$. For any $N \in \mathfrak{g}^J$, let $\mathcal{B}^J_N$ be the subvariety of $\mathcal{B}^J$ consisting of Borel subalgebras containing N. There is a Springer representation of W^J on $H_*(\mathcal{B}^J_N)$.

Theorem 49. *[AL] Let $N \in \mathfrak{g}^J$ be a nilpotent element, regular in a Levi subalgebra of a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}^J$. Then the representation of W^J on $H_*(\mathcal{B}^J_N)$ is isomorphic to* $\mathrm{Ind}_P^{W^J}(1)$ *where P is the parabolic subgroup of W^J corresponding to $\mathfrak{p}$.*

Let $N \in \mathfrak{g}^J$ be a nilpotent element. Define $Y_N \subset \hat{\mathcal{B}}^J_{n_t}$ to be the set of $g\hat{\mathfrak{b}}^J_0 g^{-1} \in \hat{\mathcal{B}}^J_{n_t}$ where $g \in \hat{G}$ such that $p^J(g^{-1}n_t g)$ is G^J-conjugate to N (this is well-defined). The Y_N are locally closed subvarieties of $\hat{\mathcal{B}}^J_{n_t}$ and we have $\hat{\mathcal{B}}^J_{n_t} = \cup Y_N$ where the union is over a set of representatives of the nilpotent orbits in $\mathfrak{g}^J$ (see [KL]).

Theorem 50. *[L3] Let t be prime to h. The representation of W^J on $H_*(\hat{\mathcal{B}}_{n_t})$ is isomorphic as a virtual W^J-module to*

$$\sum \chi(Y_N) H_*(\mathcal{B}^J_N)$$

where the sum is over a set of representatives of the nilpotent orbits in $\mathfrak{g}^J$.

Putting together these two theorems with our previous work we can conclude

Theorem 51. *Let t be prime to h. The virtual representation of W^J on $H_*(\hat{\mathcal{B}}_{n_t})$ is isomorphic to the restriction of U_t to W^J.*

Proof. The $\mathbf{C}^*$ action on $\hat{\mathcal{B}}^J_{n_t}$ preserves the subvarieties Y_N. Hence $\chi(Y_N) = \chi(Y_N^{\mathbf{C}^*})$. So to compute $\chi(Y_N)$ we only have to determine the cardinality of $Y_N^{\mathbf{C}^*}$.

Let $N \in \mathfrak{g}$ be a nilpotent element. Choose $w_1\hat{\mathfrak{b}}^J_0 w_1^{-1} \in Y_N^{\mathbf{C}^*}$. We may assume that $n_t \in w_1\hat{\mathfrak{b}}_0 w_1^{-1}$ by modifying w_1 by an element of W^J on the right.

Consider the map from $\pi : \hat{\mathcal{B}}_{n_t} \to \hat{\mathcal{B}}^J_{n_t}$. The fiber above the point $w_1\hat{\mathfrak{b}}^J_0 w_1^{-1}$ is isomorphic to $\mathcal{B}^J_N$. Since π is $\mathbf{C}^*$-equivariant, we get a $\mathbf{C}^*$ action on $\mathcal{B}^J_N$ with fixed points that can be identified with $\{w \in W^J | N \in w\mathfrak{b}_0 w^{-1}\}$. On the other hand, the analysis from the previous section shows that this set is just a set of right coset representatives for the stabilizer in W^J of $\tilde{w}w_1 \in I_t$. This stabilizer is a parabolic subgroup P of W^J by Proposition 41. Now an easy argument shows that if the set $\{w \in W^J | N \in w\mathfrak{b}_0 w^{-1}\}$ is a set of right coset representatives of a parabolic subgroup P of W^J, then N must be conjugate to a nilpotent which is regular in a Levi factor of a parabolic subalgebra in $\mathfrak{g}^J$ corresponding to P.

Thus when N is conjugate to a regular nilpotent in a Levi factor of a parabolic subalgebra in $\mathfrak{g}^J$ corresponding to P, $\chi(Y_N)$ equals the number of orbits in I_t (or S_t) with stabilizer conjugate to P. Moreover, if N is not such a nilpotent, then $\chi(Y_N) = 0$.

Putting this argument together with the previous two theorems and our analysis of the representation U_t yields the theorem. □

Remark 52. The virtual representation of W_a on $H_*(\hat{\mathcal{B}}_{n_t})$ is not isomorphic to U_t. This can be seen in A_1 with $t = 3$. Here the variety $\hat{\mathcal{B}}_{n_t}$ is two complex projective lines joined at a point. The representation of W_a on the homology of this variety is not completely reducible (using results from [Ka]), whereas the representation U_t is always completely reducible under W_a.

Remark 53. In type A_n, Lusztig and Smelt have shown that $\hat{\mathcal{B}}_{n_t}$ has no odd homology [LS].

Chapter 6

Tables and Figures

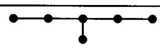
In the following tables, we show explicitly the bijection of Section 2.3.1 for the exceptional groups. We have listed only those N with non-trivial $A(N)$. The notation for a distinguished nilpotent in a semisimple Lie algebra follows Bala and Carter.

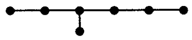
For those cases where $A(N) = S_2$, we have shown all fundamental weights of T which give rise to one-dimensional representations of P that restrict to the sign representation of $A(N)$. For $A(N) = S_3, S_4, S_5$, we gather the results of Theorem 27 in the next set of tables. In the tables, ω_i stands for V_{ω_i}, the representation of P of highest weight ω_i which is trivial on U.

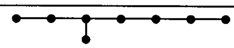
Let ω be any fundamental weight which gives rise to a one-dimensional representation V_ω of P not listed in the tables. Then V_ω always descends to a representation of $\pi_1(\mathcal{O}_N)$. If ω is not in the root lattice, then it has kernel isomorphic to $A(N)$; if ω is in the root lattice, then it has kernel isomorphic to $\pi_1(\mathcal{O}_N)$. This is the content of Section 3.1.

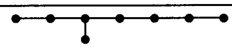
G_2			
	$A(N)$	pseudo-Levi	Class in $A(N)$
2 0	S_3	$G_2(a_1)$	1
		A_2	(123)
		$A_1 + \tilde{A}_1$	(12)

F_4				
	$A(N)$	pseudo-Levi	Class in $A(N)$	Sign rep
0 0 0 1	S_2	$\tilde{A}_1$ $A_1 + A_1$	1 -1	ω_4
2 0 0 0	S_2	A_2 $2A_1 + \tilde{A}_1$	1 -1	ω_1
2 0 0 1	S_2	B_2 A_3	1 -1	ω_4
1 0 1 0	S_2	$C_3(a_1)$ $A_1 + B_2$	1 -1	ω_3
0 2 0 0	S_4	$F_4(a_3)$ $A_3 + \tilde{A}_1$ $A_2 + \tilde{A}_2$ $B_4(a_1)$ $A_1 + C_3(a_1)$	1 (1234) (123) (12)(34) (12)	
0 2 0 2	S_2	$F_4(a_2)$ $A_1 + C_3$	1 -1	ω_2
2 2 0 2	S_2	$F_4(a_1)$ B_4	1 -1	ω_4

E_6				
	$A(N)$	pseudo-Levi	Class in $A(N)$	Sign rep
0 0 0 0 0 2	S_2	A_2 $4A_1$	1 -1	ω_6
0 0 2 0 0 0	S_3	$D_4(a_1)$ $3A_2$ $A_3 + 2A_1$	1 (123) (12)	
2 0 2 0 2 0	S_2	$E_6(a_3)$ $A_5 + A_1$	1 -1	ω_3

E_7				
	$A(N)$	pseudo-Levi	Class in $A(N)$	Sign rep
2 0 0 0 0 0 0	S_2	A_2 $4A_1$	1 -1	ω_1
1 0 0 0 1 0 0	S_2	$A_2 + A_1$ $5A_1$	1 -1	ω_1, ω_5
0 2 0 0 0 0 0	S_3	$D_4(a_1)$ $3A_2$ $A_3 + 2A_1$	1 (123) (12)	
0 1 0 0 0 1 1	S_2	$D_4(a_1) + A_1$ $A_3 + 3A_1$	1 -1	ω_2, ω_7-ω_6
0 0 1 0 1 0 0	S_2	$A_3 + A_2$ $D_4(a_1) + 2A_1$	1 -1	ω_3, ω_5
2 0 0 0 2 0 0	S_2	A_4 $2A_3$	1 -1	ω_1
1 0 1 0 1 0 0	S_2	$A_4 + A_1$ $A_1 + 2A_3$	1 -1	ω_1, ω_3
2 0 1 0 1 0 0	S_2	$D_5(a_1)$ $D_4 + 2A_1$	1 -1	ω_3, ω_5
0 2 0 0 2 0 0	S_2	$E_6(a_3)$ $A_5 + A_1$	1 -1	ω_2
0 0 2 0 0 2 0	S_3	$E_7(a_5)$ $A_5 + A_2$ $A_1 + D_6(a_2)$	1 (123) (12)	
2 0 2 0 0 2 0	S_2	$E_7(a_4)$ $A_1 + D_6(a_1)$	1 -1	ω_3
2 0 2 0 2 0 0	S_2	$E_6(a_1)$ A_7	1 -1	ω_1
2 0 2 0 2 2 0	S_2	$E_7(a_3)$ $A_1 + D_6$	1 -1	ω_3

E_8				
	$A(N)$	pseudo-Levi	Class in $A(N)$	Sign rep
0 0 0 0 0 0 2 0	S_2	A_2 $4A_1$	1 -1	ω_7
1 0 0 0 0 0 1 0	S_2	A_2+A_1 $5A_1$	1 -1	ω_1, ω_7
2 0 0 0 0 0 0 0	S_2	$2A_2$ A_2+4A_1	1 -1	ω_1
0 0 0 0 0 2 0 0	S_3	$D_4(a_1)$ $3A_2$ A_3+2A_1	1 (123) (12)	
0 0 0 0 0 1 0 1	S_3	$D_4(a_1)+A_1$ $3A_2+A_1$ A_3+3A_1	1 (123) (12)	
1 0 0 0 1 0 0 0	S_2	A_3+A_2 $D_4(a_1)+2A_1$	1 -1	ω_1, ω_5
2 0 0 0 0 0 2 0	S_2	A_4 $2A_3$	1 -1	ω_7
0 0 0 0 0 0 0 2	S_2	$D_4(a_1)+A_2$ $A_3+A_2+2A_1$	1 -1	ω_8
1 0 0 0 1 0 1 0	S_2	A_4+A_1 A_1+2A_3	1 -1	ω_5, ω_7
1 0 0 0 1 0 2 0	S_2	$D_5(a_1)$ D_4+2A_1	1 -1	ω_1, ω_5
0 0 1 0 0 0 1 0	S_2	A_4+2A_1 $D_4(a_1)+A_3$	1 -1	ω_3, ω_7
0 0 0 0 0 0 2 2	S_2	D_4+A_2 $D_5(a_1)+2A_1$	1 -1	ω_8
2 0 0 0 0 2 0 0	S_2	$E_6(a_3)$ A_5+A_1	1 -1	ω_6
0 1 0 0 0 1 0 1	S_2	$D_6(a_2)$ D_4+A_3	1 -1	ω_2, ω_6
1 0 0 1 0 1 0 0	S_2	$E_6(a_3)+A_1$ A_5+2A_1	1 -1	ω_4, ω_6
0 0 1 0 1 0 0 0	S_3	$E_7(a_5)$ A_5+A_2 $A_1+D_6(a_2)$	1 (123) (12)	

E_8				
	$A(N)$	pseudo-Levi	Class in $A(N)$	Sign rep
0 0 0 2 0 0 0 0	S_5	$E_8(a_7)$ $A_5+A_2+A_1$ $2A_4$ $D_5(a_1)+A_3$ $D_8(a_5)$ $E_7(a_5)+A_1$ $E_6(a_3)+A_2$	1 (123)(45) (12345) (1234) (12)(34) (12) (123)	
0 1 0 0 0 1 2 1	S_2	$D_6(a_1)$ D_5+2A_1	1 -1	ω_2, ω_8
0 0 1 0 1 0 2 0	S_2	$E_7(a_4)$ $A_1+D_6(a_1)$	1 -1	ω_3, ω_5
2 0 0 0 2 0 2 0	S_2	$E_6(a_1)$ A_7	1 -1	ω_7
0 0 0 2 0 0 2 0	S_2	D_5+A_2 $E_7(a_4)+A_1$	1 -1	ω_4
1 0 1 0 1 0 1 0	S_2	$D_7(a_2)$ D_5+A_3	1 -1	ω_1, ω_3
1 0 1 0 1 0 2 0	S_2	$E_6(a_1)+A_1$ A_7+A_1	1 -1	ω_7
2 0 1 0 1 0 2 0	S_2	$E_7(a_3)$ A_1+D_6	1 -1	ω_3, ω_5
0 0 2 0 0 0 2 0	S_3	$E_8(b_6)$ $E_6(a_1)+A_2$ $D_8(a_3)$	1 (123) (12)	
2 0 0 2 0 0 2 0	S_2	$D_7(a_1)$ $E_7(a_3)+A_1$	1 -1	ω_4
0 0 2 0 0 2 0 0	S_3	$E_8(a_6)$ A_8 $D_8(a_2)$	1 (123) (12)	
0 0 2 0 0 2 2 0	S_3	$E_8(b_5)$ E_6+A_2 $E_7(a_2)+A_1$	1 (123) (12)	
2 0 2 0 0 2 0 0	S_2	$E_8(a_5)$ $D_8(a_1)$	1 -1	ω_3, ω_6
2 0 2 0 0 2 2 0	S_2	$E_8(b_4)$ $E_7(a_1)+A_1$	1 -1	ω_3
2 0 2 0 2 0 2 0	S_2	$E_8(a_4)$ D_8	1 -1	ω_3, ω_7
2 0 2 0 2 2 2 0	S_2	$E_8(a_3)$ E_7+A_1	1 -1	ω_3

Representations of $A(N)$ for $A(N) = S_3, S_4, S_5$

$\mathbf{F_4}$

$F_4(a_3)$				
Conjugacy class	ω_1	ω_2	ω_3	ω_4
$F_4(a_3)$	2	1	3	3
$A_3 + \tilde{A}_1$	0	-1	1	-1
$A_2 + \tilde{A}_2$	-1	1	0	0
$B_4(a_1)$	2	1	-1	-1
$A_1 + C_3(a_1)$	0	-1	-1	1

$\mathbf{G_2}$

$G_2(a_1)$		
Conjugacy class	ω_1	ω_2
$G_2(a_1)$	1	2
A_2	1	-1
$A_1 + \tilde{A}_1$	-1	0

$\mathbf{E_6}$

$D_4(a_1)$		
Conjugacy class	ω_3	ω_6
$D_4(a_1)$	1	2
$3A_2$	1	-1
$A_3 + 2A_1$	-1	0

$\mathbf{E_7}$

$E_7(a_5)$		
Conjugacy class	ω_3	$\omega_7 - \omega_6$
$E_7(a_5)$	1	2
$A_5 + A_2$	1	-1
$A_1 + D_6(a_2)$	-1	0

$D_4(a_1)$		
Conjugacy class	ω_2	ω_1
$D_4(a_1)$	1	2
$3A_2$	1	-1
$A_3 + 2A_1$	-1	0

$\mathbf{E_8}$

$D_4(a_1)$		
Conjugacy class	ω_6	ω_7
$D_4(a_1)$	1	2
$3A_2$	1	-1
$A_3 + 2A_1$	-1	0

$D_4(a_1) + A_1$		
Conjugacy class	ω_6, ω_8	ω_7
$D_4(a_1) + A_1$	1	2
$3A_2 + A_1$	1	-1
$A_3 + 3A_1$	-1	0

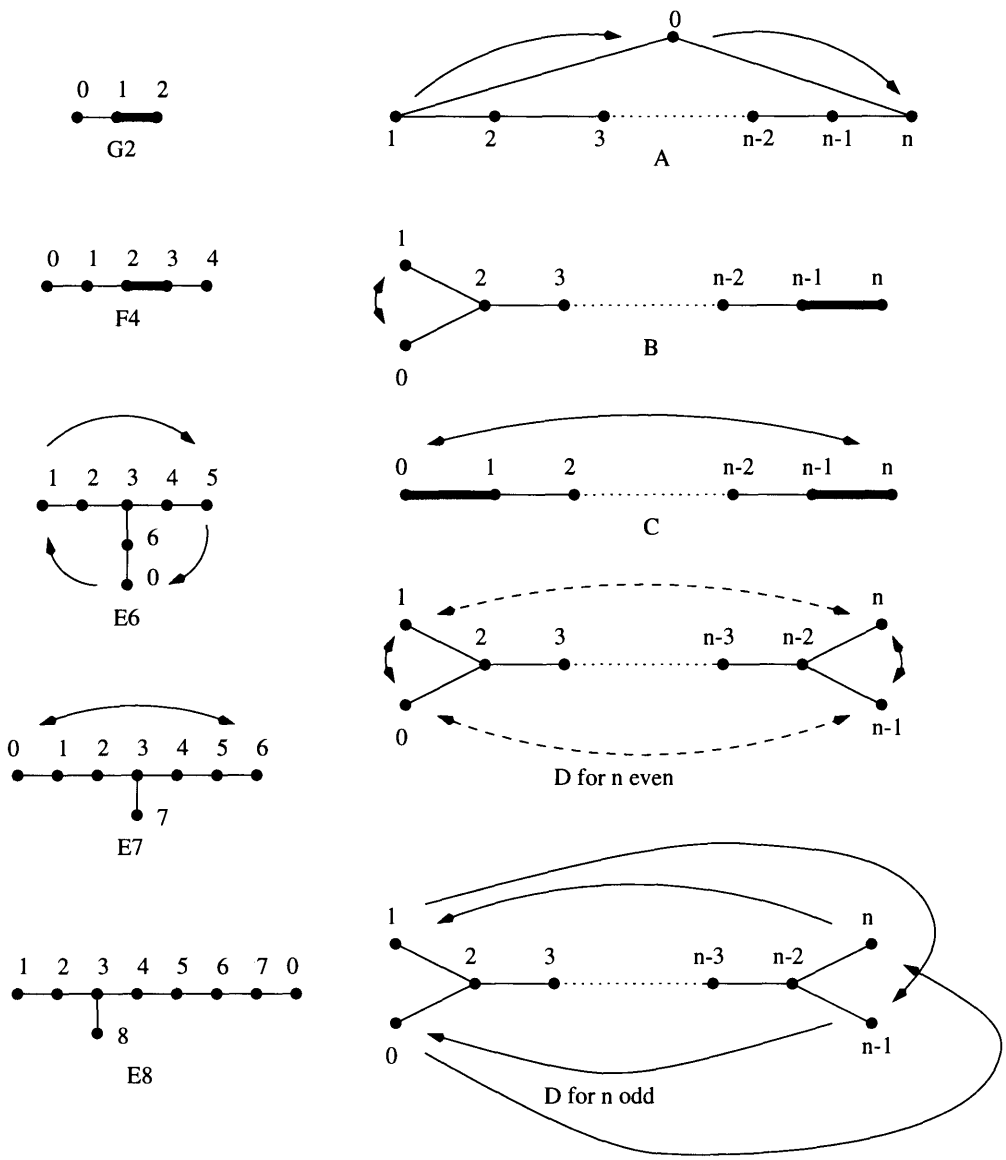
$E_7(a_5)$		
Conjugacy class	ω_3, ω_5	ω_8
$E_7(a_5)$	1	2
$A_5 + A_2$	1	-1
$D_6(a_2) + A_1$	-1	0

$E_8(b_6)$		
Conjugacy class	ω_3, ω_7	ω_8
$E_8(b_6)$	1	2
$E_6(a_1) + A_2$	1	-1
$D_8(a_3)$	-1	0

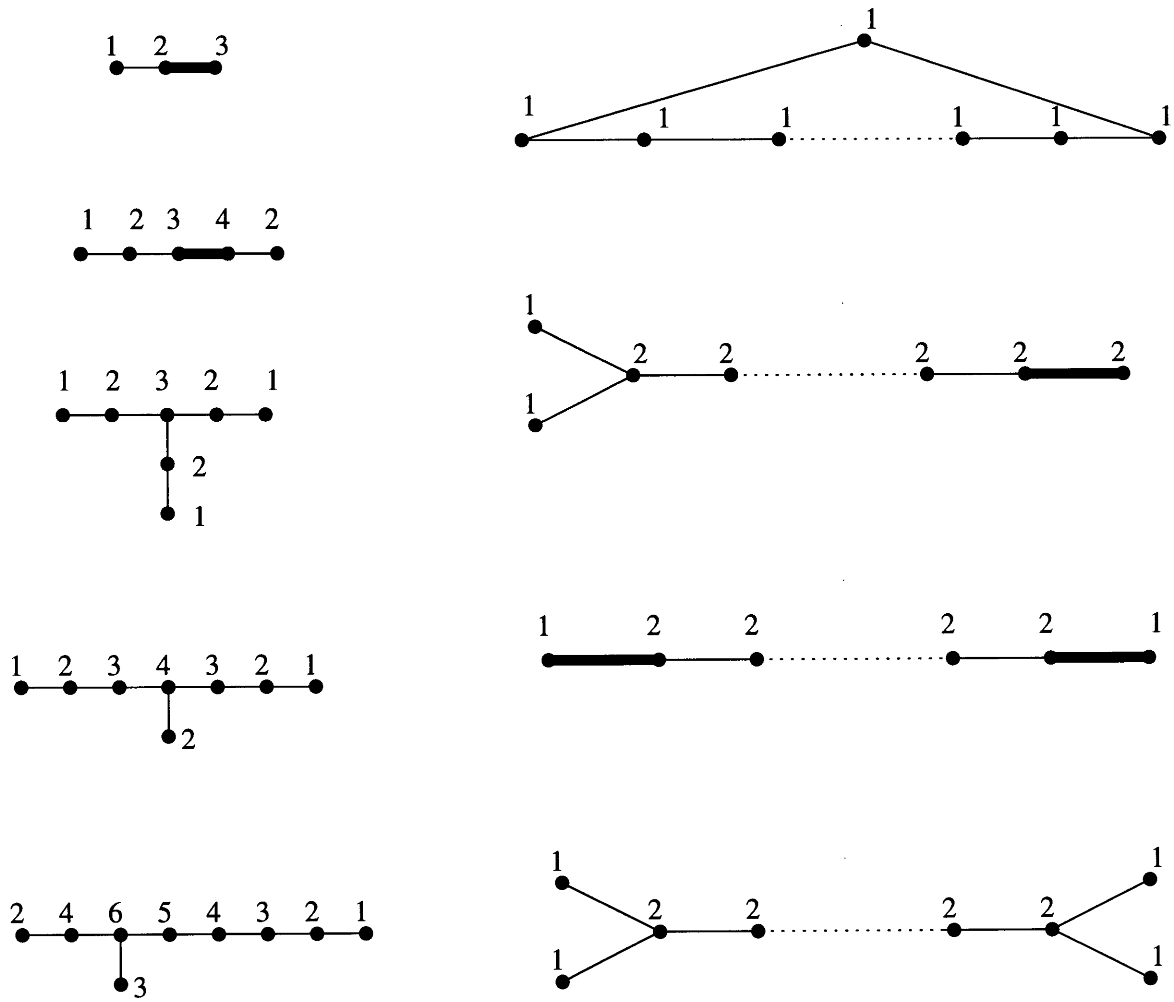
$E_8(a_6)$		
Conjugacy class	ω_3, ω_6	ω_7, ω_8
$E_8(a_6)$	1	2
A_8	1	-1
$D_8(a_2)$	-1	0

$E_8(b_5)$		
Conjugacy class	ω_3	ω_8
$E_8(b_5)$	1	2
$E_6 + A_2$	1	-1
$E_7(a_2) + A_1$	-1	0

$E_8(a_7)$						
Conjugacy class	ω_7	ω_5	ω_4	ω_6	ω_1	ω_8
$E_8(a_7)$	4	4	1	6	5	5
$A_5 + A_2 + A_1$	-1	1	-1	0	1	-1
$2A_4$	-1	-1	1	1	0	0
$D_5(a_1) + A_3$	0	0	-1	0	-1	1
$D_8(a_5)$	0	0	1	-2	1	1
$E_7(a_5) + A_1$	2	-2	-1	0	1	-1
$E_6(a_3) + A_2$	1	1	1	0	-1	-1



The extended Dynkin diagrams and their automorphisms in W



Coefficients of the highest root

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