On bubble dynamics and gas dynamics in open tubes

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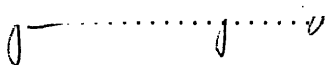in partial fulfillment of the requirements for the degree of

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Abstract

This thesis consists of two unrelated problems. In the first one, the thermal effects on the collapse and rebound (i.e. large amplitude free oscillations) of a spherical gas bubble are studied under the assumptions that the liquid is inviscid and incompressible, the mach number is zero (i.e. the pressure is uniform inside the bubble) and the gas is inviscid. Two nondimensional parameters characterize the problem. One is a measure of the strength of the collapse and the other is the coefficient of thermal diffusion. Three regions in time are identified through an asymtotic analysis (the inclusion of two nested boundary layers). In one of them the bubble behaves isothermally, in an other one adiabatically and in the third one neither. Solutions valid in each region are calculated and then matched to describe the dynamics of the radius and the fluid variables explicitly showing their dependence on the two nondimensional parameters mentioned above. Comparisons with the classical polytropic models are made.

In the second part of the thesis, by a weakly nonlinear analysis an equation for the first term in the asymptotic expansion of the acoustics waves of an isentropic gas in a tube with one end closed and the other one open is derived. This equation has a flux with an inflection point. It is shown that this asymptotic approximation is valid even for discontinuous solutions and explanations on how to interpret the different structures that this equation presents (contact discontinuities, shocks and rarefractions) are given.

Acknowledgments

Contents

Chapter 1

Introduction

In this thesis we study two unrelated problems. We then split the introduction into two sections. The first section is the introduction of Chapter two, the second section is the introduction of Chapter three.

1.1 Thermal effects on the collapse and rebound of a gas bubble

A bubble grows many times its equilibrium size if the pressure in the surrounding liquid remains negative for enough time. When the pressure becomes positive, the bubble, if it still exists, decreases in size violently, overshooting its equilibrium size and bouncing back (whenever it does not break). The ability of liquids to withstand tension makes the occurrence of this process (usually referred to as the collapse and rebound of the bubble) frequent and easy to produce. Cavitation damage and sonoluminescence are two good examples. In cavitation damage a jet is produced and a shock emitted into the liquid after the bubble collapses. They both produce strong pressures that are believed to cause erosion in nearby solids. In sonoluminescence the extremely high temperatures attained inside the bubble (presumably through the focusing of shocks more than just adiabatic compression) produce emission of light. The details of both phenomena are not completely understood but it is clear that any attempt to develop a quantitative theory concerning them (as well as others not mentioned here), needs to consider the details of the dynamics involved in the collapse and rebound of a bubble. This is an extremely complex task since many effects difficult to analyze have to be considered. For example the bubble loses spherical symmetry, the compressibility of the liquid and thermal effects become important, it is no longer true that the pressure inside the bubble is uniform, etc.. In chapter 2 we will study the thermal effects.

In the study of bubble dynamics, the polytropic approximation is widely used to determine the

pressure inside the bubble. Namely the pressure p is set to be proportional to $R^{-3\kappa}$ where R is the radius of the bubble and κ is a constant. The constant κ is taken to be one when the motion is isothermal and γ (the ratio of the specific heat of the gas) when the motion is adiabatic. This approximation has several deficiencies:

- It neglects the damping due to thermal effects (which is known to be the dominant form of damping in many situations)

- It is not clear how to choose the constant of proportionality in the adiabatic case (in fact this constant depends on the initial condition)

- If the motion is neither isothermal nor adiabatic, it is not clear how to choose the polytropic constant κ.

There has been a lot of effort to overcome these flaws. The main idea for many years was to consider small amplitude motion and linearize the complete set of equations (mass, momentum and energy both in gas and the liquid) to obtain the coefficient for thermal damping and an effective polytropic constant (see [41], [42], [45], [50], [10], [35], [19], [11], [2], [37], [12], [13], and [38]). These values can be then used in the nonlinear case hoping that it gives a good approximation, but this is not the case. More recently the nearly adiabatic (or isothermal) case , where simplified systems of equations can be derived without linearization or using the polytropic approximation, have been studied (see [29], [30], [36] and [22]).

If the pressure is assumed to be uniform inside the bubble and the gas to be inviscid, the three conservation equations (mass, momentum and energy) can be reduced to just one P.D.E.. The dynamics of the system can then be described by solving this P.D.E. coupled with the Rayleigh-Plesset equation. This has been recognized independently by two different groups (see [32], [33], [31], [34] and [39], [23], [36]). They have performed numerical calculations that show the inaccuracy of the polytropic approximation. In thesis we will go one step further by analyzing asymptotically the coupled system just mentioned in the case of large amplitude free oscillations (i.e. collapse and rebound).

Chapter 2 is organized as follows:

In Section 2.1 we explain the physical situation, introduce the complete set of equations of fluid motion, nondimensionalize those equations and reduce the system to a nonlinear P.D.E. coupled with the Rayleigh-Plesset equations after assuming that the liquid is inviscid and incompressible, that the speed of the bubble wall is much smaller than the speed of sound inside the bubble and that the gas is inviscid. To justify these assumptions we show in the appendix A some typical values of the different parameters involved.

In Section 2.2 we study the reduced model obtained in Section 2.1 when the initial conditions are of the form $R(0) = \delta^{-1}$ and $R_t(0) = 0$, where R is the nondimensional radius (with respect to the

radius in equilibrium conditions) and we assume δ to be small. The problem has two parameters: δ and μ, the nondimensional coefficient of thermal diffusion inside the bubble. We perform an asymptotic analysis and introduce two nested boundary layers in time identifying the region (in time) where the motion is isothermal, where the motion is adiabatic and the transition from one to the other. By doing this analysis we identify the order of magnitude of the different quantities (minimum radius, maximum temperature inside the bubble, etc.) as functions of the parameters δ and μ and the calculation has to be done only once (since through the asymptotics we eliminate the parameters in each region of time) and then the results can be used for any δ and μ. These two features are the main advantages over just performing numerical calculations from where, for example, it would be difficult to obtain the dependence of the minimum radius as a function of the maximum radius δ^{-1} and coefficient of thermal diffusion μ, since we would have to do a new computation each time δ and μ are changed and each computation would be expensive because small parameters are involved in the problem and boundary layers both in time and space occur when the motion is adiabatic around the minimum radius. At the end of section 2.2 we compute the thermal damping.

In Section 2.3 we make some final remarks and compare the results of this work with the classical polytropic models.

1.2 Weakly Nonlinear Gas Dynamics in Open Tubes

Forced acoustic waves can be found in a variety of physical situations. One example are the waves produce inside a tube filled with an inviscid gas, where a piston is driven periodically at one end and the other one is kept closed. This example has been studied extensively, both experimentally ([47], [46], [20], [26], [43] and [8]) and theoretically ([47], [3], [21], [1], [48], [24] and [7]). When the piston is driven at a natural frequency and with an amplitude much smaller than the length of the tube, linear theory predicts that the waves will grow indefinitely. Then, nonlinear effects can not be neglected and these are the effects that limit the amplitude of the waves. These two features:

1)Acoustic waves are exited

2)The growth of the amplitude of the waves are limited by nonlinear effects

are commonly encountered in many physical systems like combustion instabilities ([49]), thermoacoustics ([28], [5] and [4]), etc.

The forcing that produces the acoustics waves can be external, like the in the piston example or can be self sustained by the system like different kind of combustion instabilities. The dynamics depends on the kind of forcing but it will also depend on the geometry of the system, as can be seen by comparing the tube example already mention, with a tube that has an oscillating piston at one end but the other end is open ([46], [20], [26], [3], [1], [44] and [25]). For this reason, these kind of

instabilities are sometimes called chamber instabilities.

It is then of interest to understand the effects of the geometry of the system on the dynamics (i.e. the evolution of acoustic waves in different geometrical situations without any forcing). This is well understood in the case of a tube with both ends closed but not so when the tube has one end open and this is an important situation in combustion (like flames propagating in tubes, pulse combustors, etc. [49]). Some of the questions that one would want to answer are: Given initial waves that are small, how long does it take for the nonlinear effects to become important (if they become important at all)?. Do the waves decay?. If they do, in what time scale?. Do the waves produce shocks?

In chapter three we study the long term behavior of small but finite amplitude acoustic waves of an inviscid isentropic gas in a tube with one end closed and the other one open.

We have already mention some physical motivation to study this problem. Mathematically this is an example of a genuinely nonlinear hyperbolic system that has as a weakly nonlinear approximation an equation that is not genuinely nonlinear (it has a flux with an inflection point). This fact is due to the boundary condition at the open end.

It is well know that spatially periodic solutions of 2x2 genuinely nonlinear hyperbolic system decay to a constant state like t^{-1} ([14]). It is also known that this does not always hold for genuinely nonlinear hyperbolic system with constant boundary condition, since periodic solutions in time for some systems have been found ([16] and [17]). This paper is one example of a 2x2 genuinely nonlinear hyperbolic system with constant boundary condition that do not decay like t^{-1} but, it decays like $t^{-\frac{1}{2}}$ (as its asymptotic approximation does ([18], [6] and [9]).

In section 3.1 we state the governing equations.

By making the ratio of the radius of the tube with its length small enough in comparison with the amplitude of the acoustic waves, the effects of radiation at the open end become negligible. The boundary condition at the open end is derived under these conditions in section 3.2.

In section 3.3 we derive the following asymptotic approximation by the method of multiple time scales

$$\rho \sim 1 + \epsilon \left(\sigma(t-x) + \sigma(t+x) \right) \quad \text{and} \quad v \sim \epsilon \left(\sigma(t-x) - \sigma(t+x) \right) \tag{1.1}$$

where σ satisfies

$$\sigma(\phi - 1) + \sigma(\phi + 1) = 0 \tag{1.2}$$

$$\sigma_\tau + a \left(\sigma^3 \right)_\phi + b \left(\frac{1}{2} \int_{-1}^{1} \sigma^2(\phi) d\phi \right) \sigma_\phi = 0 \tag{1.3}$$

9

with a and b constants

$$a = \frac{3}{4} + \frac{5}{12}p^{(2)} + \frac{7}{48}\left(p^{(2)}\right)^2 - \frac{1}{12}p^{(3)} \qquad (1.4)$$

and

$$b = \frac{1}{4} - \frac{3}{4}p^{(2)} + \frac{7}{16}\left(p^{(2)}\right)^2 - \frac{1}{4}p^{(3)} \qquad (1.5)$$

where ρ is the non-dimensional density, v the non-dimensional velocity, $\tau = \epsilon^2 t$, σ is a function of the two variables (ϕ, τ), the quantities between brackets next to σ indicates where ϕ has to be evaluated and $p^{(i)}$ is the ith derivative of the non-dimensional pressure p with respect to ρ evaluated at $\rho = 1$.

In section 3.4 we explain the differences between open and closed tubes and the reflection of waves at the open and closed ends.

Motivated by the explanations in section 3.4, we introduce in section 3.5 a toy model that captures the essential qualitative behavior of the open tube system. We study extensively the toy model in this section.

In section 3.6 we go back to the original system and do a similar analysis than the one done for the toy model guided by the results obtain in section 3.5. In particular we show that the approximation (1.1)-(1.2)-(1.3) is valid even for discontinuous solutions. The discontinuities satisfy the usual jump conditions for conservation equations. We also interpret the meaning of the different structures that can be found in equations with a flux with an inflection point like equation (1.3) (like right contact discontinuities) that can not be found in genuinely nonlinear hyperbolic systems like the original open tube system.

In section 3.7 we make some final remarks.

Chapter 2

Thermal effects on the collapse and rebound of a gas bubble

2.1 Governing Equations

The physical situation that we study consists of a spherical gas bubble immersed in a liquid that extends to infinity in all directions. The bubble does not have translational motion and in the liquid the pressure tends to a constant value p_ℓ and the velocity to 0 as the distance to the bubble goes to infinity.

2.1.1 Complete set of equations

The full set of equations that govern the dynamics of fluids with spherical symmetry are:

- Mass

$$\bar{\rho}_{\bar{t}} + \frac{(\bar{r}^2 \bar{\rho}\bar{v})_{\bar{r}}}{\bar{r}^2} = 0 \tag{2.1}$$

- Momentum

$$\bar{\rho}(\bar{v}_{\bar{t}} + \bar{v}\bar{v}_{\bar{r}}) + \bar{p}_{\bar{r}} = \left(\bar{\xi}\frac{(\bar{r}^2\bar{v})_{\bar{r}}}{\bar{r}^2}\right)_{\bar{r}} + \frac{4}{3\bar{r}^3}\left(\bar{r}^4\bar{\eta}\left(\frac{\bar{v}}{\bar{r}}\right)_{\bar{r}}\right)_{\bar{r}} \tag{2.2}$$

- Energy

$$\bar{\rho}(\bar{e}_{\bar{t}} + \bar{v}\bar{e}_{\bar{r}}) + \frac{(\bar{r}^2\bar{v})_{\bar{r}}}{\bar{r}^2}\bar{p} = \frac{4}{3}\bar{\eta}\bar{r}^2\left(\left(\frac{\bar{v}}{\bar{r}}\right)_{\bar{r}}\right)^2 + \frac{\bar{\xi}}{\bar{r}^4}\left((\bar{r}^2\bar{v})_{\bar{r}}\right)^2 + \frac{(\bar{\lambda}\bar{r}^2\bar{T}_{\bar{r}})_{\bar{r}}}{\bar{r}^2} \tag{2.3}$$

At the bubble wall we have the interphase conditions

$$[\bar{v}]_\ell^g = 0 \quad [\bar{T}]_\ell^g = 0 \quad [\bar{\lambda}\bar{T}_{\bar{r}}]_\ell^g = 0 \tag{2.4}$$

$$\left[\bar{p} - \frac{4}{3}\bar{\eta}\bar{r}\left(\frac{\bar{v}}{\bar{r}}\right)_{\bar{r}} - \bar{\xi}\frac{(\bar{r}^2\bar{v})_{\bar{r}}}{\bar{r}^2}\right]_\ell^g = \frac{\bar{\sigma}}{R} \tag{2.5}$$

where $\bar{R}$ is the radius of the bubble and expressions of the form $[f]_\ell^g$ denote the value of f at the bubble wall on the side of the gas minus the value of f at the bubble wall on the side of the liquid. The different variables and coefficients in the equations are: the density $\bar{\rho}$, velocity $\bar{v}$ (which is in the radial direction), pressure $\bar{p}$, internal energy $\bar{e}$, temperature $\bar{T}$, bulk viscosity $\bar{\xi}$, shear viscosity $\bar{\eta}$, thermal conductivity $\bar{\lambda}$, surface tension times two $\bar{\sigma}$. We also have some thermodynamics relations, namely $\bar{\rho}$, $\bar{T}$ and $\bar{p}$ are related by an equation of state (i.e. only two of those variables are independent) and $\bar{e}$ depends on $\bar{T}$ and $\bar{p}$.

2.1.2 Nondimensionalization

We denote the speed of sound by a ($a^2 = d\bar{p}/d\bar{\rho}$), where the derivative is taken at constant entropy. We assume that the different diffusion coefficients (like viscosity, etc) and the surface tension are constants. We nondimensionalize as follows:

$$\bar{R} = R_e R, \quad \bar{r} = R_e r, \quad \bar{t} = \frac{R_e}{\epsilon a_\ell}t, \quad \bar{v} = \epsilon a_\ell v, \quad \bar{p} = p_0 + \rho_g a_g^2 p$$

where $p_0 = p_\ell$ in the liquid, $p_0 = p_g$ in the gas and $\epsilon = (a_g\sqrt{\rho_g})/(a_\ell\sqrt{\rho_\ell})$. R_e is the radius of equilibrium and we denote the values in equilibrium with the subscript g in the gas and with the subscript ℓ in the liquid. The rest of the variables are scale with respect at its equilibrium value. For example $\bar{\rho} = \rho_g\rho$ (in the gas), $\bar{\rho} = \rho_\ell\rho$ (in the liquid), etc.

The choice of the scale for $\bar{R}$ and $\bar{\rho}$ are natural. When R changes $O(1)$, ρ inside the bubble also changes $O(1)$. Then, the scale chosen for $\bar{p}$ also gives an $O(1)$ change in p (since $dp/d\rho = 1$ at constant entropy inside the bubble). The changes in pressure inside and outside the bubble are of the same order, then we use the same scale both in the liquid and in the gas. As the liquid is at first order incompressible and inviscid, we expect $\bar{\rho}\bar{v}\bar{v}_{\bar{r}}$ to be of the same order as $\bar{p}_{\bar{r}}$, which gives us the scale for the velocity. Having the characteristic length and velocity scales, the time scale is chosen in the natural way.

Inside the bubble

We assume that the gas inside the bubble is ideal with $\gamma = c_{pg}/c_{vg}$ (c_{pg} is the specific heat at constant pressure and c_{vg} is the specific heat at constant volume). The equation of state is $\bar{p} = R_0 \bar{\rho} \bar{T}$ (where $R_0 = c_{pg} - c_{vg}$) and the internal energy satisfies $\bar{e} = \bar{e_0} + c_{vg}\bar{T}$ ($\bar{e_0}$ is an arbitrary constant). After nondimensionalization the equations inside the bubble become:

$$\rho_t + \frac{(r^2 \rho v)_r}{r^2} = 0 \tag{2.6}$$

$$\epsilon^2 (a_\ell/a_g)^2 \rho(v_t + vv_r) + p_r = \frac{\nu_g}{r^3}\left(r^4\left(\frac{v}{r}\right)_r\right)_r + \kappa_g\left(\frac{(r^2 v)_r}{r^2}\right)_r \tag{2.7}$$

$$\rho(T_t + vT_r) + (\gamma - 1)\frac{(r^2 v)_r}{r^2}(1 + \gamma p) = \nu_g \gamma(\gamma - 1)r^2\left(\left(\frac{v}{r}\right)_r\right)^2 +$$
$$\kappa_g(\gamma - 1)\frac{\gamma}{r^4}\left((r^2 v)_r\right)^2 + \mu_g\frac{(r^2 T_r)_r}{r^2} \tag{2.8}$$

and the equation of state

$$1 + \gamma p = \rho T \tag{2.9}$$

where $\nu_g = (4\eta_g)/(3R_e a_g\sqrt{\rho_\ell \rho_g})$, $\kappa_g = \xi_g/(R_e a_g\sqrt{\rho_\ell \rho_g})$ and $\mu_g = (\lambda_g\sqrt{\rho_\ell})/(c_{vg} a_g R_e \rho_g\sqrt{\rho_g})$. We also have to keep in mind that the fact that there is no mass exchange between the liquid and the gas, and the way we have nondimensionalize the equations, imposes the restriction

$$\int_0^R 3\rho r^2 dr = 1 \tag{2.10}$$

In the liquid

After nondimensionalization and the use of thermodynamic identities, the equations in the liquid become:

$$\rho_t + \frac{(r^2 \rho v)_r}{r^2} = 0 \tag{2.11}$$

13

$$\rho(v_t + vv_r) + p_r = \frac{\nu_\ell}{r^3}\left(r^4\left(\frac{v}{r}\right)_r\right)_r + \kappa_\ell\left(\frac{(r^2v)_r}{r^2}\right)_r \tag{2.12}$$

$$\rho(T_t + vT_r) - \epsilon^2 c_1 T(p_t + vp_r) = \mu_\ell \frac{(r^2 T_r)_r}{r^2} +$$
$$\nu_\ell\epsilon^2 c_2 r^2\left(\left(\frac{v}{r}\right)_r\right)^2 + \kappa_\ell\epsilon^2 \frac{c_2}{r^4}\left((r^2v)_r\right)^2 \tag{2.13}$$

where $\nu_\ell = (4\eta_\ell)/(3R_e a_g\sqrt{\rho_\ell\rho_g})$, $\kappa_\ell = \xi_\ell/(R_e a_g\sqrt{\rho_\ell\rho_g})$, $c_1 = \beta_\ell a_\ell^2/c_{p\ell}$, where β_ℓ is the coefficient of thermal expansion ($\beta_\ell = -\bar{\rho}^{-1}\partial\bar{\rho}/\partial\bar{T}$ where the derivative is taken at constant pressure, as the other coefficients, we take β_ℓ to be constant), $\mu_\ell = \lambda_\ell/(c_{p\ell}a_g R_e\sqrt{\rho_\ell\rho_g})$ and $c_2 = a_\ell^2/(c_{p\ell}T_\ell)$.

At the bubble wall

After nondimensionalization the equations at the bubble wall become:

$$v(r = R^+) = v(r = R^-) = R_t \quad T(r = R^+) = T(r = R^-) \tag{2.14}$$

$$\lambda_\ell T_r(r = R^+) = \lambda_g T_r(r = R^-) \tag{2.15}$$

and

$$\left(p - \nu_g r\left(\frac{v}{r}\right)_r - \kappa_g\frac{(r^2v)_r}{r^2}\right)(r = R^-) =$$
$$\left(p - \nu_\ell r\left(\frac{v}{r}\right)_r - \kappa_\ell\frac{(r^2v)_r}{r^2}\right)(r = R^+) + s\left(\frac{1}{R} - 1\right) \tag{2.16}$$

where $s = \bar{\sigma}/(a_g^2\rho_g R_e)$.

2.1.3 Simplifying the system

Some typical values of the parameters are given in the appendix A. We are concerned with bubbles of the order of 10^{-3}cm. Having that in mind we will make a series of approximations

14

The energy interphase condition

The diffusion length scale for T in the liquid is $r = O(\sqrt{\mu_\ell})$. Then the variation of T in the liquid near the bubble are of the order

$$T - 1 = O(\sqrt{\mu_\ell}T_r(r = R^+)) = O(\sqrt{\mu_\ell}\frac{\lambda_g}{\lambda_\ell}T_r(r = R^-)) \tag{2.17}$$

We assume that the parameter $\sqrt{\mu_\ell}\lambda_g/\lambda_\ell$ is small enough so that we can replace the two interphase conditions involving the temperature and its gradient by

$$T(r = R^-) = 1 \tag{2.18}$$

Inviscid and incompressible liquid

We set the parameters ϵ, ν_ℓ and κ_ℓ to zero. This means that the liquid is considered to be incompressible (since at constant entropy $d\rho/dp = \epsilon^2$) and inviscid. The equations in the liquid then become:

$$\frac{(r^2 v)_r}{r^2} = 0 \quad \text{and} \quad v_t + vv_r + p_r = 0 \tag{2.19}$$

which, using the boundary conditions for the velocity at the bubble wall and for the pressure at infinity, imply

$$v = \frac{R^2 R_t}{r^2} \quad \text{and} \quad p = \frac{(R^2 R_t)_t}{r} - \frac{R^4 R_t^2}{2r^4} \tag{2.20}$$

Zero mach number and inviscid gas

We set the parameters $\epsilon^2(\frac{a_\ell}{a_g})^2$, ν_g and κ_g to zero. This means that we assume that the speed of sound in the gas is much greater than the speed of the bubble wall and that the gas is inviscid. The momentum equation inside the bubble reduces to $p_r = 0$. Then the energy equation becomes

$$\gamma p_t + \gamma\frac{(r^2 v)_r}{r^2}(1 + \gamma p) = \mu\frac{(r^2 T_r)_r}{r^2} \tag{2.21}$$

where we have dropped the subscript g in μ. This equation can be integrated to obtain v as a function of p and T

$$v = \frac{\mu}{\gamma}\frac{T_r}{(1 + \gamma p)} - \frac{r}{3}\frac{p_t}{(1 + \gamma p)} \tag{2.22}$$

15

Using the equation of state and the boundary condition $T(r = R) = 1$, we can write both p and T as functions of ρ and so (2.22) becomes

$$v = -\frac{\mu}{\gamma}\frac{\rho_r}{\rho^2} - \frac{r}{3\gamma}\frac{\rho_t(r = R)}{\rho(r = R)} \tag{2.23}$$

We now plug this expression for the velocity into the mass equation and we make the change of variables

$$u = R^3\rho \quad \text{and} \quad x = \frac{r}{R} \tag{2.24}$$

to get

2.1.4 The reduced system

$$u_t = \frac{(x^2 f)_x}{x^2} \tag{2.25}$$

where, defining $h = u(x = 1)$,

$$f = \left[\frac{1}{3\gamma}\frac{h_t}{h} + \frac{(\gamma - 1)}{\gamma}\frac{R_t}{R}\right]xu + \frac{\mu}{\gamma}R\frac{u_x}{u} \tag{2.26}$$

Equation (2.25) is subjected to the boundary conditions

$$f(x = 0) = f(x = 1) = 0 \tag{2.27}$$

The boundary condition at $x = 0$ comes from the fact that there is no flux of energy at the origin (i.e. $T_r = 0$ at $r = 0$) and the one at $x = 1$ is equivalent to $v(r = R) = R_t$.

Equation (2.25) is coupled to

$$RR_{tt} + \frac{3}{2}R_t^2 + s\left(\frac{1}{R} - 1\right) = \frac{h}{\gamma R^3} - \frac{1}{\gamma} \tag{2.28}$$

which is the interface condition at the bubble wall, (equation (2.5)) Equations (2.25–2.28) or variations of them that come from considering some effects neglected here (like compressibility or viscosity in the liquid or allowing vapor inside the bubble), have been considered (see [32, 33, 31, 34, 39, 23] and [36]). These studies include numerical calculations and also some analytical analysis when restricted to small disturbances around the equilibrium state (i.e. linear analysis) or when μ is either large or small. The case of small μ can also be found in [29, 30] and [22].

2.2 Collapse and Rebound of the Bubble

Assume that R attains a maximum at $t = 0$, $R(t = 0) = \delta^{-1}$, with $\delta \ll 1$. We will study in this section the evolution of R under these conditions. By introducing two nested boundary layers we will distinguish three regions in time. The first one is where the outer solution is valid. Here the bubble behaves isothermally and the dynamics of its radius is not affected by the pressure inside the bubble (i.e. the term $h/(\gamma R)$ in equation (2.28) is negligible). The second region is a boundary layer in time that needs to be included because the isothermal approximation fails to remain valid as the bubble gets too small. The radius still has enough inertia to decouple its dynamics with the gas. Eventually this decoupling fails as the bubble gets too compressed and we have a new boundary layer where the motion is adiabatic. We next procede with the details of the calculations

2.2.1 The outer solution. Isothermal behavior

We first rescale the variables (to choose the scale for t, we note that RR_{tt} should be order one at the maximum of R)

$$R = \delta^{-1}\bar{R} \quad \text{and} \quad t = \delta^{-1}\bar{t} \tag{2.29}$$

The equations (2.25–2.28) become

$$\delta u_{\bar{t}} = \frac{(x^2\bar{f})_x}{x^2} \quad \text{with} \quad \bar{f}(x = 0) = \bar{f}(x = 1) = 0 \tag{2.30}$$

$$\bar{R}\bar{R}_{\bar{t}\bar{t}} + \frac{3}{2}\bar{R}_{\bar{t}}^2 + s\left(\frac{\delta}{\bar{R}} - 1\right) = \delta^3\frac{h}{\gamma\bar{R}^3} - \frac{1}{\gamma} \tag{2.31}$$

where

$$\bar{f} = \delta\left[\frac{1}{3\gamma}\frac{h_{\bar{t}}}{h} + \frac{(\gamma - 1)}{\gamma}\frac{R_{\bar{t}}}{R}\right]xu + \delta^{-1}\frac{\mu}{\gamma}\bar{R}\frac{u_x}{u} \tag{2.32}$$

We assume that $\delta^2 \ll \mu$. Under this condition and having in mind that equation (2.10) is equivalent to $\int_0^1 3x^2 u \, dx = 1$, we have the following asymptotic approximation

$$\bar{R} \sim \bar{R}_0 \quad \text{and} \quad u \sim 1 \tag{2.33}$$

where $\bar{R}_0$ satisfies

$$\bar{R}_0\bar{R}_{0\bar{t}\bar{t}} + \frac{3}{2}\bar{R}_{0\bar{t}}^2 + \frac{1}{\gamma} - s = 0 \quad \text{with} \quad \bar{R}_0(0) = 1 \quad \text{and} \quad \bar{R}_{0\bar{t}}(0) = 0 \tag{2.34}$$

We multiply Equation (2.34) by $\bar{R}_0^2 \bar{R}_{0\bar{t}}$ and integrate once to obtain

$$\frac{\bar{R}_0^3 \bar{R}_{0\bar{t}}^2}{2} + \left(\frac{1}{\gamma} - s\right)\frac{\bar{R}_0^3}{3} = \frac{1}{3}\left(\frac{1}{\gamma} - s\right) \tag{2.35}$$

We assume $s < \gamma^{-1}$. Given the initial conditions for $\bar{R}_0$, Equation (2.34) implies that $\bar{R}_{0\bar{t}\bar{t}} < 0$ for all $\bar{t}$. As a consequence $\bar{R}_0$ goes to 0 in finite time (say as $\bar{t}$ goes to $\bar{t}_0$). We now use (2.35) to get

$$\bar{R}_0 \sim a(\bar{t}_0 - \bar{t})^{2/5} \quad \text{as} \quad \bar{t} \to \bar{t}_0 \quad \text{with} \quad a = \left[\frac{25}{6}\left(\frac{1}{\gamma} - s\right)\right]^{1/5} \tag{2.36}$$

To obtain the approximation for u in (2.33), we have assumed that $\delta u_t \ll \delta^{-1}\mu\bar{R}(u_x/u)_x$. Given (2.36) we have that $\delta u_t = O(\delta(\bar{t}_0 - \bar{t})^{-1})$ and $\delta^{-1}\mu\bar{R}(u_x/u)_x = O(\delta^{-1}\mu(\bar{t}_0 - \bar{t})^{2/5})$ as $\bar{t} \to \bar{t}_0$. Then this expansion fails for $\bar{t}_0 - \bar{t} = O(\delta^{10/7}\mu^{-5/7})$ and we need to introduce a boundary layer in time.

2.2.2 Boundary layer. Transition to adiabatic motion

Let

$$\bar{R} = a\alpha^{2/5}\hat{R} \quad \text{and} \quad \bar{t} = \bar{t}_0 + \alpha\hat{t} \quad \text{with} \quad \alpha = \left[\frac{\delta^2}{\mu a}\right]^{5/7} \tag{2.37}$$

Equations (2.30–2.32) now become

$$u_{\hat{t}} = \frac{(x^2\hat{f})}{x^2}x \quad \text{with} \quad \hat{f}(x=0) = \hat{f}(x=1) = 0 \tag{2.38}$$

$$\hat{R}\hat{R}_{\hat{t}\hat{t}} + \frac{3}{2}\hat{R}_{\hat{t}}^2 + \frac{s\delta\alpha^{4/5}}{a^3\hat{R}} - \frac{s\alpha^{6/5}}{a^2} = \frac{\delta^3 h}{\gamma a^5\hat{R}^3} - \frac{\alpha^{6/5}}{\gamma a^2} \tag{2.39}$$

where

$$\hat{f} = \left[\frac{1}{3\gamma}\frac{h_{\hat{t}}}{h} + \frac{(\gamma-1)}{\gamma}\frac{\hat{R}_{\hat{t}}}{\hat{R}}\right]xu + \frac{\hat{R}}{\gamma}\frac{u_x}{u} \tag{2.40}$$

Note that a is an order one constant and both δ and α are small parameters. Then, neglecting the small terms in Equation (2.39), solving for $\hat{R}$ and matching with the outer solution we have

$$\hat{R} \sim (-\hat{t})^{2/5} \quad \text{(for all } \hat{t}) \tag{2.41}$$

We now study the behavior of u as $\hat{t}$ goes to 0. We need

$$\left|\frac{1}{3\gamma}\frac{h_{\hat{t}}}{h} + \frac{(\gamma-1)}{\gamma}\frac{\hat{R}_{\hat{t}}}{\hat{R}}\right| \ll |\hat{t}|^{-1} \tag{2.42}$$

18

If (2.42) is not satisfied we would have that most of the mass inside the bubble gets accumulated near the bubble wall. Although a boundary layer does develop near the bubble wall as $\hat{t} \to 0$ where the density is much higher, most of the mass is still distributed throughout bubble. Equation (2.42) implies that

$$h \sim b(-\hat{t})^{-\frac{6(\gamma-1)}{5}} \quad \text{and} \quad u \sim u_0(x) \quad \text{(for } x \neq 1 \text{) as} \quad \hat{t} \to 0 \tag{2.43}$$

The constant b and the function u_0 are calculated numerically in the appendix B by solving (2.38) and (2.40), replacing $\hat{R}$ by its asymptotic approximation $(-\hat{t})^{2/5}$ and u subject to the asymptotic behavior $u \sim 1$ as $\hat{t} \to -\infty$. The assumption (2.42) is verified in the same appendix.

Equations (2.41) and (2.43) imply that $h\delta^3 \hat{R}^{-3}$ becomes of the same order of $\hat{R}\hat{R}_{\hat{t}\hat{t}}$ when $\hat{t} = 0 \left(\delta^{5/(2(\gamma-1))}\right)$ and the approximation then fails. We need to introduce a new boundary layer (inside the one we already have).

2.2.3   Thinner boundary layer. Adiabatic behavior

We define $\beta = \left(\delta^3 b\gamma^{-1}a^{-5}\right)^{\frac{5}{6(\gamma-1)}}$ and set

$$\hat{R} = \beta^{2/5}\tilde{R}, \quad \hat{t} = \beta\tilde{t} \quad \text{and} \quad h = b\beta^{-\frac{6}{5}(\gamma-1)}\tilde{h} \tag{2.44}$$

Equations (2.38–2.40) now become

$$u_{\tilde{t}} = \frac{(x^2\tilde{f})}{x^2}x \quad \text{with} \quad \tilde{f}(x=0) = \tilde{f}(x=1) = 0 \tag{2.45}$$

$$\tilde{R}\tilde{R}_{\tilde{t}\tilde{t}} + \frac{3}{2}\tilde{R}_{\tilde{t}}^2 + \frac{s\delta}{a^3}\frac{(\alpha\beta)^{4/5}}{\tilde{R}} - \frac{s}{a^2}(\alpha\beta)^{6/5} = \frac{\tilde{h}}{\tilde{R}^3} - \frac{(\alpha\beta)^{6/5}}{\gamma a^2} \tag{2.46}$$

where

$$\tilde{f} = \left[\frac{1}{3\gamma}\frac{\tilde{h}_{\tilde{t}}}{\tilde{h}} + \frac{(\gamma-1)}{\gamma}\frac{\tilde{R}_{\tilde{t}}}{\tilde{R}}\right]xu + \beta^{7/5}\frac{\tilde{R}}{\gamma}\frac{u_x}{u} \tag{2.47}$$

Then matching as $\tilde{t}$ goes to $-\infty$ and $\hat{t}$ goes to 0 and neglecting the small terms in (2.46) and (2.47) we obtain that

$$\tilde{R} \sim \tilde{R}_0, \quad u \sim u_0(x) \quad \text{and} \quad \tilde{h} \sim \tilde{R}_0^{-3(\gamma-1)} \tag{2.48}$$

19

where $\tilde{R}_0$ satisfies

$$\tilde{R}_0 \tilde{R}_{0\tilde{t}\tilde{t}} + \frac{3}{2} \tilde{R}_{0\tilde{t}}^2 = \tilde{R}_0^{-3\gamma} \quad \text{and} \quad \tilde{R}_0 \sim (-\tilde{t})^{2/5} \quad \text{as} \quad \tilde{t} \to -\infty \tag{2.49}$$

Note that the approximation for u is valid outside a boundary layer next to $x = 1$.

After multiplying Equation (2.49) by $\tilde{R}_0^2 \tilde{R}_{0\tilde{t}}$, we can integrate it and use the information that we have as $\tilde{t} \to -\infty$ to get

$$\frac{\tilde{R}_0^3 \tilde{R}_{0\tilde{t}}^2}{2} + \frac{\tilde{R}_0^{-3(\gamma-1)}}{3(\gamma-1)} = \frac{2}{25} \tag{2.50}$$

from where we infer that $\tilde{R}_0 \sim \tilde{t}^{2/5}$ as $\tilde{t} \to +\infty$.

2.2.4 The asymptotic approximation

With a similar analysis for $\bar{t} > \bar{t}_0$ we obtain

$$R(t) \sim \begin{cases} \delta^{-1} \bar{R}_0(\bar{t}) & \text{if} \quad 0 \le \bar{t} \le \bar{t}_0 - \alpha\beta\tilde{t}_0 \\ a(\alpha\beta)^{2/5} \delta^{-1} \tilde{R}_0(\tilde{t}) & \text{if} \quad |\tilde{t}| \le \tilde{t}_0 \\ \delta^{-1} \bar{R}_0(2\bar{t}_0 - \bar{t}) & \text{if} \quad \bar{t}_0 + \alpha\beta\tilde{t}_0 \le \bar{t} \le 2\bar{t}_0 \end{cases} \tag{2.51}$$

where $\tilde{t}_0$ is a constant the satisfies $1 \ll \tilde{t}_0 \ll (\alpha\beta)^{-1}$, $\bar{R}_0$ and $\tilde{R}_0$ are given by Equations (2.34) and (2.49) respectively and $\bar{t}$ and $\tilde{t}$ are related to t by (2.29), (2.37) and (2.44).

Note that we have derived

$$1 + \gamma p \sim \begin{cases} R^{-3} & \text{in the isothermal part} \\ \left(\delta^{-1} a\alpha^{2/5}\right)^{3(\gamma-1)} bR^{-3\gamma} & \text{in the adiabatic part} \end{cases} \tag{2.52}$$

if the transition from isothermal to adiabatic motion is given with

$$R \sim a\delta^{-3/5}(t_0 - t)^{2/5} \tag{2.53}$$

with t_0 a constant. Equation (2.53) is not restrictive at all. It will always be satisfied if the bubble collapses with enough inertia. This condition is equivalent to requiring that the gas becomes adiabatic before its contents can affect the dynamics of the radius. Note also that isothermal motion occurs while $\rho T_t = O(R_t R^{-4} T) \ll \mu (r^2 T_r)_r / r^2 = O(\mu R^{-2} T)$ (i.e. $1 \ll \mu R^2 / R_t$) and the motion is adiabatic when $\mu R^2 / R_t \ll 1$. Finally note that we have implicitly assume that μ is not exponentially large in δ^{-1} for the calculations in this section to work.

2.2.5 Dissipation time scale

The previous analysis has shown that as a first approximation, R is periodic with a period in the time scale of $\bar{t}$, but we still have dissipation that acts in a slower time scale that we calculate next. We define

$$\bar{E} = \frac{\bar{R}^3 \left(\bar{R}_{\bar{t}}\right)^2}{2} + \left(\frac{1}{\gamma} - s\right) \frac{\bar{R}^3}{3} + \delta s \frac{\bar{R}^2}{3} - \delta^3 \frac{\ln(\bar{R})}{\gamma} \tag{2.54}$$

We have two small parameters in equations (2.30–2.32), namely δ and δ^2/μ. We expand the variables in powers of δ^2/μ

$$u = u_0 + \frac{\delta^2}{\mu} u_1 + \cdots \quad h = h_0 + \frac{\delta^2}{\mu} h_1 + \cdots \quad \bar{R} = \bar{X}_0 + \frac{\delta^2}{\mu} \bar{X}_1 + \cdots \tag{2.55}$$

plug these anzats into the equations (2.30–2.32) and collect powers of δ^2/μ (without expanding in δ) to obtain

$$\bar{X}_0 \bar{X}_{0\bar{t}\bar{t}} + \frac{3}{2} \left(\bar{X}_{0\bar{t}}\right)^2 + s \left(\frac{\delta}{\bar{X}_0} - 1\right) = \frac{\delta^3}{\gamma \bar{X}_0^3} - \frac{1}{\gamma} \tag{2.56}$$

$$\bar{X}_0 \bar{X}_{1\bar{t}\bar{t}} + \bar{X}_1 \bar{X}_{0\bar{t}\bar{t}} + 3\bar{X}_{1\bar{t}} \bar{X}_{0\bar{t}} - s\delta \frac{\bar{X}_1}{\bar{X}_0^2} = -3 \frac{\delta^3}{\gamma} \frac{\bar{X}_1}{\bar{X}_0^4} - \delta^3 \frac{(\gamma - 1)}{5\gamma} \frac{\bar{X}_{0\bar{t}}}{\bar{X}_0^5} \tag{2.57}$$

Note that we have named $\bar{X}_i$ the terms in the expansion of $\bar{R}$ to avoid confusion with $\bar{R}_0$ previously defined.

The last two equations and the definition of $\bar{E}$ imply

$$\bar{E}_{\bar{t}} \sim -\frac{\delta^5}{\mu} \frac{(\gamma - 1)}{5\gamma} \frac{\left(\bar{X}_{0\bar{t}}\right)^2}{\bar{X}_0^3} \tag{2.58}$$

Let $\hat{t}_0$ be a positive constant such that $1 \ll \hat{t}_0 \ll \alpha^{-1}$. Then whenever $|\bar{t} - \bar{t}_0| \geq \alpha \hat{t}_0$, we are in the region where the approximation (2.33–2.34) is valid. In this region $\bar{X}_0 \sim \bar{R}_0$, and then, using (2.36) we obtain

$$\bar{E}(\bar{t}_0 - \alpha \hat{t}_0) - \bar{E}(0) \sim -\frac{4(\gamma - 1)}{175\gamma} \delta^3 \hat{t}_0^{-7/5} \tag{2.59}$$

and similarly

$$\bar{E}(2\bar{t}_0) - \bar{E}(\bar{t}_0 + \alpha \hat{t}_0) \sim -\frac{4(\gamma - 1)}{175\gamma} \delta^3 \hat{t}_0^{-7/5} \tag{2.60}$$

21

We now turn our attention to the region in time where the motion is not isothermal. We define

$$\hat{F} = \frac{\hat{R}^3 \left(\hat{R}_{\hat{t}}\right)^2}{2} + \frac{\alpha^{6/5}}{3a^2} \left(\frac{1}{\gamma} - s\right) \hat{R}^3 + \frac{s\delta\alpha^{4/5}}{2a^3}\hat{R}^2 + \frac{\delta^3}{3a^5\gamma(\gamma-1)}h \tag{2.61}$$

From the boundary condition at $x = 1$ (equation (2.38)) and equation (2.40) we get

$$\hat{F}_{\hat{t}} = -\frac{\delta^3}{a^5\gamma(\gamma-1)}\frac{\hat{R}}{h}u_x(x=1) \tag{2.62}$$

or in other words we have

$$\hat{F}(\hat{t}_0) - \hat{F}(-\hat{t}_0) = -\frac{\delta^3}{a^5\gamma(\gamma-1)}I \quad \text{with} \quad I = \int_{-\hat{t}_0}^{\hat{t}_0} \frac{\hat{R}}{h}u_x(x=1)d\hat{t} \tag{2.63}$$

As $\hat{t} \to 0^-$, a boundary layer develops. It is then not clear how big is the integrand of (2.63) is, but in the appendix B, we show that $\hat{R}u_x(x=1)/h = O(|\hat{t}|^{(3-6\gamma)/10})$ as $\hat{t} \to 0$, which means that although the integrand becomes large, the integral remains order 1 (since $13 - 6\gamma > 0$).

We now look at what happens as $\hat{t} \to \infty$. Again from equations (2.38) and (2.40) we have

$$\frac{\hat{R}}{h}u_x(x=1) \sim -\frac{2(\gamma-1)}{5\hat{t}} \quad \text{as} \quad \hat{t} \to \infty \tag{2.64}$$

Then, the integral I remains bounded for large $\hat{t}_0$ since the part of the integrand that is not integrable for $\hat{t}$ positive cancels with the one for $\hat{t}$ negative. On the other hand note that it is clear the integral is not small since $u_x(x=0) > 0$ always except when $\hat{t} \to +\infty$, then we conclude that $I > 0$ and $I = O(1)$.

To conclude with our initial goal, note that

$$\bar{E} = a^5\hat{F} - \frac{\delta^3}{\gamma}\ln(a\alpha^{2/5}\hat{R}) - \frac{\delta^3}{3\gamma(\gamma-1)}h \tag{2.65}$$

to get

$$\begin{aligned}
\bar{E}(2\hat{t}_0) - \bar{E}(0) &= \bar{E}(2\hat{t}_0) - \bar{E}(\hat{t}_0 + \alpha\hat{t}_0) + a^5\left(\hat{F}(\hat{t}_0) - \hat{F}(-\hat{t}_0)\right) - \\
&\quad \frac{\delta^3}{\gamma}\ln\left(a\alpha^{2/5}\frac{\hat{R}(\hat{t}_0)}{\hat{R}(-\hat{t}_0)}\right) - \\
&\quad \frac{\delta^3}{3\gamma(\gamma-1)}\left(h(\hat{t}_0) - h(-\hat{t}_0)\right) + \\
&\quad \bar{E}(\hat{t}_0 - \alpha\hat{t}_0) - \bar{E}(0) \sim -\frac{\delta^3}{\gamma(\gamma-1)}I \tag{2.66}
\end{aligned}$$

which is negative and $O(\delta^3)$. This means that $\bar{R}$ will decay in a time scale of order δ^{-3} (i.e.

22

$\bar{t} = O(\delta^{-3}))$.

2.3 Comparison with Classical Models and Remarks

In this section we compare the results of this work with the most classical models, namely the polytropic approximation. We then make some final remarks.

2.3.1 Isothermal Case

With our nondimensionalization, if we set $T = 1$ inside the bubble we get the following Rayleigh-Plesset equation

$$RR_{tt} + \frac{3}{2}R_t^2 + s\left(\frac{1}{R} - 1\right) = \frac{R^{-3}}{\gamma} - \frac{1}{\gamma} \tag{2.67}$$

Under the same initial conditions that we have considered, $R(0) = \delta^{-1}$ and $\dot{R}(0) = 0$, it can be deduced (by multiplying Equation (2.67) by $\dot{R}R^2$ and integrating once), that

$$R_{\min} = 0\left(\exp\left(-(1 - \gamma s)\,\delta^{-3}\right)\right)$$

and

$$v_{\max} = 0\left((SR_{\min})^{-3/2}\right) \tag{2.68}$$

where $R_{\min}$ is the minimum radius and $v_{\max}$ is the maximum velocity of the bubble wall attained. We clearly see that although a plot of R versus t of our approximation is very similar to the one obtained using the isothermal approximation, both models differ a lot for a very short (but also very important) time.

2.3.2 Polytropic approximation

The polytropic approximation consists of assuming that the pressure inside the bubble is proportional to $R^{-3\kappa}$. Under this assumption the Rayleigh-Plesset equation becomes

$$RR_{tt} + \frac{3}{2}R_t^2 + s\left(\frac{1}{R} - 1\right) = C\frac{R^{-3\kappa}}{\gamma} - \frac{1}{\gamma}$$

where C is a constant that depends on the initial conditions. Using the same initial conditions for the radius as in the rest of this work, we have

$$R_{\min} = 0\left(\delta^{\frac{1}{(\kappa-1)}}C^{\frac{1}{3(\kappa-1)}}\right), \qquad v_{\min} = 0\left(\delta^{-\frac{3}{2}\frac{\kappa}{(\kappa-1)}}C^{-\frac{1}{2(\kappa-1)}}\right),$$
$$T = 0\left(\delta^{-3}C^{-1}\right)$$

It is now clear that any calculation of the minimum radius, maximum velocity or thermodynamic quantities inside the bubble (like temperature), has to be done very carefully. Note that the terms neglected in Section (2.1) to reduce the full gas dynamics equations inside the bubble to one P.D.E. are guaranteed to be small for the initial conditions of order 1. Then the validity of the results in this paper are limited since eventually, for small enough δ, some of the neglected effects become important (see [15]). The same remarks apply to the liquid since for example, it is well known that the incompressible approximation fails for very violent collapses. Calculating the parameter region where the results in this paper are valid is very easy, one just has to compute the terms neglected and check when they are smaller than the terms kept. Let's finally mention that a similar analysis can be done when the equation of state in the gas is different.

Chapter 3

Weakly Nonlinear Gas Dynamics in Open Tubes

3.1 Governing Equations

After a proper nondimensionalization the equations of one dimensional isentropic gas dynamics can be written in the form

$$\rho_t + (\rho v)_x = 0 \tag{3.1}$$

$$(\rho v)_t + (\rho v^2 + p)_x = 0 \tag{3.2}$$

where ρ is the density of the gas, v the velocity and p the pressure. p depends only on ρ and this dependence is given by an equation of state.

The nondimensionalization is such that the closed end of the tube is at $x = 0$, the open one is at $x = 1$, the basic solution around which our expansion will be performed is $\rho = 1$, $v = 0$ and $p = p_0$, and $\frac{dp}{d\rho} = 1$ when $\rho = 1$. We will also assume that $\frac{d^2p}{d\rho^2} > 0$.

The boundary condition at the closed end is

$$v = 0 \quad \text{at} \quad x = 0 \tag{3.3}$$

and in the next section the following formula is derived for the boundary condition at the open end

$$\rho \frac{v^2}{2} + p = p_0 \quad \text{at} \quad x = 1 \tag{3.4}$$

3.2 Open end boundary condition

We will describe the solution inside and outside the tube. Then we will introduce a boundary layer at the open end to connect both states and by doing so we will derive the boundary condition (formula (3.4)).

Let R be the nondimensional radius of the tube (i.e. the ratio between the radius of the tube and its length). We assume that $R \ll 1$.

3.2.1 Outer solution inside the tube.

We consider the case when inside the tube ρ and v have asymptotic expansions of the form

$$\rho = 1 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \cdots \quad \text{and} \quad v = \epsilon v_1 + \epsilon^2 v_2 + \cdots \tag{3.5}$$

where $\epsilon \ll 1$ and ρ_i and v_i are functions of x and t.

3.2.2 Outer solution outside the tube.

We assume that the disturbances outside the tube are only due to the radiation at the open end. We denote by r the distance to the open end divided by the length of the tube and $\hat{r}$ the unit vector (in the scale of r) pointing from the open end. Note that r is of the same scale as x.

At distances of the scale of r outside the tube, the open end acts as a point source at first order because R is small. As a consequence ρ and v depend only on r (at first order).

Since the disturbances inside the tube are of order ϵ, we have that outside the tube they are of order ϵR^2. Then, linearizing the equations of isentropic gas dynamics with spherical symmetry and taking r small we get

$$\rho \sim 1 + \epsilon R^2 \frac{\dot{a}(t)}{r} \quad \text{and} \quad v \sim \epsilon R^2 \frac{a(t)}{r^2} \hat{r} \tag{3.6}$$

for $R \ll r \ll 1$, for some function $a(t)$ that depends on the field inside the tube.

3.2.3 Inner solution at the open end.

We now introduce a boundary layer at the open end.

Let $X = \frac{x-1}{R}$ and Y the distance to the X-axis (which coincides with the axis of the tube) divided by the radius of the tube. Since we have cylindrical symmetry the velocity has the form $u\hat{X} + w\hat{Y}$ and the equations become

$$\rho_t + R^{-1} \left((\rho u)_X + (\rho w)_Y + \frac{\rho w}{Y} \right) = 0 \tag{3.7}$$

$$(\rho u)_t + R^{-1}\left((\rho u^2)_X + (\rho u w)_Y + p_X + \frac{\rho u w}{Y}\right) = 0 \tag{3.8}$$

$$(\rho w)_t + R^{-1}\left((\rho u w)_X + (\rho w^2)_Y + p_Y + \frac{\rho w^2}{Y}\right) = 0 \tag{3.9}$$

The perturbations here are of the same order as those inside the tube. Then, using (3.7) we can write (3.8) as

$$\rho u u_X + \rho w u_Y + p_X = O(\epsilon R) \tag{3.10}$$

in particular, since $w(Y = 0) = 0$ we have

$$\rho\left(\frac{u^2}{2}\right)_X + p_X = O(\epsilon R) \tag{3.11}$$

at $Y = 0$.

We write

$$\rho \sim 1 + \epsilon\rho_1^{(in)} \quad \text{and} \quad u \sim \epsilon u_1^{(in)} \tag{3.12}$$

in the inner region to distinguish from (3.5)

3.2.4 Matching the outer solutions with the inner one.

Plugging the anzat (3.12) into (3.11) we obtain at order ϵ

$$\rho_{1X}^{(in)} = 0 \tag{3.13}$$

We now make the assumption that $R \ll \epsilon^2$. This assumption, (3.11) and (3.13) allow us to conclude that there is a constant k such that

$$\rho\frac{u^2}{2} + p = k + o(\epsilon^3) \tag{3.14}$$

at $Y = 0$ in the inner region, where $o(\epsilon^3)$ is a quantity that goes to 0 faster than ϵ^3.

Equations (3.6) imply that outside the tube, for $r \gg \frac{R^2}{\epsilon^2}$ and $r^4 \gg \frac{R^4}{\epsilon}$

$$\rho\frac{v^2}{2} + p = p_0 + o(\epsilon^3) \tag{3.15}$$

Then $k = p_0$ since both equations (3.14) and (3.15) are valid for $r \gg \frac{R^2}{\epsilon^2}$, $r^4 \gg \frac{R^4}{\epsilon}$, $r \ll 1$ and $Y = 0$. Finally, matching the inner solution with the outer solution inside the tube we get our

boundary condition which is the formula (3.15) evaluated at $x = 0$.

Remark. If we want to be precise we should say that the matching of the inner solution with the outer one inside the tube is performed in the region $R \ll -x \ll \epsilon^2$.

3.3 Asymptotic approximation

In this section we will derive an equation that describes the long term behavior of the first term in the asymptotic expansions of ρ and v (i.e. ρ_1 and v_1 in (3.5)).

We introduce a second time scale $T = \epsilon t$. We plug the anzats (3.5) into the equations (3.1) and (3.2) and the equations for the boundary conditions (3.3) and (3.4) and collect powers of ϵ.

At $O(\epsilon)$ we get:

$$\rho_{1t} + v_{1x} = 0 \tag{3.16}$$

$$v_{1t} + \rho_{1x} = 0 \tag{3.17}$$

with boundary conditions

$$v_1 = 0 \text{ at } x = 0 \quad \text{and} \quad \rho_1 = 0 \text{ at } x = 1 \tag{3.18}$$

The general solution of (3.16) and (3.17) is given by

$$\rho_1 = \sigma_1(\phi = t - x) - \sigma_2(\phi = t + x) \tag{3.19}$$

$$v_1 = \sigma_1(\phi = t - x) + \sigma_2(\phi = t + x) \tag{3.20}$$

where σ_1 and σ_2 are arbitrary functions of ϕ and T.

Applying the boundary conditions (3.18) we obtain that $\sigma_2 = -\sigma_1 = -\sigma$ and that σ satisfies the following linear equation:

$$\sigma(\phi - 1) + \sigma(\phi + 1) = 0 \tag{3.21}$$

At $O(\epsilon^2)$ we get:

$$\rho_{2t} + v_{2x} = -\rho_{1T} - (\rho_1 v_1)_x \tag{3.22}$$

$$v_{2t} + p_{2x} = -v_{2T} - (\rho_1 v_1)_t - (v_1^2)_x - \frac{p^{(2)}}{2}(\rho_1^2)_x \tag{3.23}$$

with boundary conditions

$$v_2 = 0 \text{ at } x = 0 \quad \text{and} \quad \rho_2 = -\frac{v_1^2}{2} - \frac{p^{(2)}}{2}\rho_1^2 \text{ at } x = 1 \tag{3.24}$$

where $p^{(i)}$ is the ith derivative of p with respect to ρ at $\rho = 1$.

The general solution of (3.22) and (3.23) is given by

$$\rho_2 = \alpha_1 - \alpha_2 \quad \text{and} \quad v_2 = \alpha_1 + \alpha_2 \tag{3.25}$$

where

$$
\begin{aligned}
\alpha_1 &= \alpha_1^0(t-x) - x\sigma_T(t-x) + \frac{1}{2}\left(1 + \frac{p^{(2)}}{2}\right)x\,(\sigma^2)_\phi(t-x) + \\
&\quad \frac{1}{2}\left(\frac{p^{(2)}}{2} - 1\right)\sigma_\phi(t-x)\,(\Sigma(t+x) - \Sigma(t-x)) + \\
&\quad \frac{1}{4}\left(1 - \frac{p^{(2)}}{2}\right)\left(\sigma^2(t+x) + 2\sigma(t+x)\sigma(t-x) - 3\sigma^2(t-x)\right)
\end{aligned} \tag{3.26}
$$

and

$$
\begin{aligned}
\alpha_2 &= \alpha_2^0(t+x) - x\sigma_T(t+x) + \frac{1}{2}\left(1 + \frac{p^{(2)}}{2}\right)x\,(\sigma^2)_\phi(t+x) + \\
&\quad \frac{1}{2}\left(\frac{p^{(2)}}{2} - 1\right)\sigma_\phi(t+x)\,(\Sigma(t+x) - \Sigma(t-x)) - \\
&\quad \frac{1}{4}\left(1 - \frac{p^{(2)}}{2}\right)\left(\sigma^2(t-x) + 2\sigma(t+x)\sigma(t-x) - 3\sigma^2(t+x)\right)
\end{aligned} \tag{3.27}
$$

where α_1^0 and α_2^0 are arbitrary functions of T and ϕ, Σ is a function of T and ϕ with the property that $\Sigma_\phi = \sigma$ and the argument between brackets next to any of the functions indicates where ϕ has to be evaluated. For example $\sigma(t-x)$ means $\sigma(\phi = t-x)$.

Applying the boundary conditions (3.24) we obtain that $\alpha_2^0 = -\alpha_1^0 = -\alpha$ and that α satisfies the following nonhomegeneos linear equation:

$$
\begin{aligned}
\alpha(\phi - 1) + \alpha(\phi + 1) &= \left(\frac{p^{(2)}}{2} - 1\right)\sigma_\phi(\phi + 1)\,(\Sigma(\phi + 1) - \Sigma(\phi - 1)) \\
&\quad - 2\sigma_T(\phi + 1) - p^{(2)}\sigma^2(\phi + 1)
\end{aligned} \tag{3.28}
$$

solving this equation we get

$$\alpha(\phi) = -\phi\sigma_T(\phi) + \frac{1}{2}\left(\frac{p^{(2)}}{2} - 1\right)\sigma_\phi(\phi)\,(\Sigma(\phi) - \Sigma(\phi - 2)) - \frac{p^{(2)}}{2}\sigma^2(\phi) \tag{3.29}$$

We now have ρ_2 and v_2 and in other to avoid that they grow linearly with time we need to require that

$$\sigma_T = 0 \tag{3.30}$$

which means that the variables do not depend on T.

Remark. We could add to α any solution of the homogeneous equation (3.21) but this would not change (3.30) and we are interested only in σ.

We introduce a slower time scale $\tau = \epsilon^2 t$.

At $O(\epsilon^3)$ we get:

$$\rho_{3t} + v_{3x} = -\rho_{1\tau} + R_1 \tag{3.31}$$

$$v_{3t} + \rho_{3x} = -v_{1\tau} + R_2 \tag{3.32}$$

with boundary conditions

$$v_3 = 0 \text{ at } x = 0 \quad \text{and} \quad \rho_3 = -v_1 v_2 \text{ at } x = 1 \tag{3.33}$$

where we have defined

$$R_1 = -(\rho_1 v_2)_x - (\rho_2 v_1)_x \tag{3.34}$$

$$R_2 = -(\rho_1 v_2)_t - (\rho_2 v_1)_t - (\rho_1 v_1^2)_x - 2(v_1 v_2)_x - p^{(2)}(\rho_1 \rho_2)_x - \frac{p^{(3)}}{6}(\rho_1^3)_x \tag{3.35}$$

and we have used the fact that $\rho_1 = 0$ at $x = 1$ in (3.33).

The general solution of (3.31) and (3.32) is given by

$$\rho_3 = \beta_1 - \beta_2 \quad \text{and} \quad v_3 = \beta_1 + \beta_2 \tag{3.36}$$

where

$$\begin{aligned}
\beta_1 &= \beta_1^0(t-x) - x\sigma_\tau(t-x) + \\
&\quad \frac{1}{2}\int_0^x \left(R_1(s, t-x+s) + R_2(s, t-x+s)\right) ds
\end{aligned} \tag{3.37}$$

30

and

$$\beta_2 = \beta_2^0(t+x) - x\sigma_\tau(t+x) +$$
$$\frac{1}{2}\int_0^x (R_1(s,t+x-s) - R_2(s,t+x-s))\,ds \qquad (3.38)$$

where β_1^0 and β_2^0 are arbitrary functions of τ and ϕ, the argument between brackets next to β_i^0 or σ indicates where ϕ has to be evaluated and the first argument in R_i corresponds to x and the second one corresponds to t. For example $R_1(s, t-x+s)$ means replace x by s and t by $t-x+s$ in R_1.

We now apply the boundary conditions (3.33) and obtain that $\beta_2^0 = -\beta_1^0 = -\beta$ and that β satisfies the following equation:

$$\beta(\phi - 1) + \beta(\phi + 1) = -2\sigma_\tau(\phi + 1) - v_1(1, \phi)v_2(1, \phi) +$$
$$\frac{1}{2}\int_0^1 (R_1(s, \phi + 1 - s) - R_1(s, \phi - 1 + s))\,ds - \qquad (3.39)$$
$$\frac{1}{2}\int_0^1 (R_2(s, \phi + 1 - s) + R_2(s, \phi - 1 + s))\,ds$$

where again the first variable corresponds to x and the second one to t for v_i and R_i and for β and σ the argument between brackets indicates where ϕ has to be evaluated.

The right hand side of (3.39), which we call R, satisfies the homogeneous equation. Then β grows linearly with ϕ unless R is 0. But that β grows linearly with ϕ is equivalent to say that ρ_3 and v_3 grow linearly with t and we want to avoid that. Then we set $R = 0$ and after some manipulation and dividing by -2 we get:

The asymptotic equation.

$$\rho \sim 1 + \epsilon\left(\sigma(t - x) + \sigma(t + x)\right) \quad \text{and} \quad v \sim \epsilon\left(\sigma(t - x) - \sigma(t + x)\right) \qquad (3.40)$$

where σ satisfies

$$\sigma(\phi - 1) + \sigma(\phi + 1) = 0 \qquad (3.41)$$

$$\sigma_\tau + a\left(\sigma^3\right)_\phi + b\left(\frac{1}{2}\int_{-1}^1 \sigma^2(\phi)d\phi\right)\sigma_\phi = 0 \qquad (3.42)$$

with a and b constants

$$a = \frac{3}{4} + \frac{5}{12}p^{(2)} + \frac{7}{48}\left(p^{(2)}\right)^2 - \frac{1}{12}p^{(3)} \qquad (3.43)$$

31

and

$$b = \frac{1}{4} - \frac{3}{4}p^{(2)} + \frac{7}{16}\left(p^{(2)}\right)^2 - \frac{1}{4}p^{(3)} \qquad (3.44)$$

3.4 Differences between closed and open ends

In this section we will understand the differences between open and closed tubes.

3.4.1 Asymptotics for closed tubes.

When the two ends of the tube are closed, this means replace the boundary condition (3.4) by $v = 0$ at $x = 1$, the asymptotic form of ρ and v remain the same ((3.40) is valid), but (3.41) and (3.42) have to be replaced. Instead σ satisfies:

$$\sigma(\phi - 1) - \sigma(\phi + 1) = 0 \qquad (3.45)$$

$$\sigma_T - \frac{1}{2}\left(1 + \frac{p^{(2)}}{2}\right)(\sigma^2)_\phi + \frac{1}{2}\left(1 - \frac{p^{(2)}}{2}\right)\left(\int_{-1}^{1}\sigma(\phi)d\phi\right)\sigma_\phi = 0 \qquad (3.46)$$

where T, as defined before, is $T = \epsilon t$.

3.4.2 Riemann invariant form of the system (3.1)-(3.2).

To understand these asymptotic equations let us take a closer look to the original system.

Defining the speed of sound c, the enthalpy h and the Riemann-invariants w and z as

$$c = \sqrt{\frac{dp}{d\rho}} \quad , \quad h = \int_{1}^{\rho}\frac{c(s)}{s}ds \qquad (3.47)$$

$$w = h - v \quad \text{and} \quad z = h + v \qquad (3.48)$$

we can write the system (3.1)-(3.2) in its Riemann-invariant form

$$w_t + \lambda_1 w_x = 0 \qquad (3.49)$$

$$z_t + \lambda_2 z_x = 0 \qquad (3.50)$$

where

$$\lambda_1 = v - c \quad \text{and} \quad \lambda_2 = v + c \qquad (3.51)$$

32

We introduce the two set of characteristics curves. The 1-characteristic are the level sets of $\eta(x, t)$ and the 2-characteristic are the level sets of $\xi(x, t)$, where $\eta(x, t)$ and $\xi(x, t)$ satisfy

$$\eta_t + \lambda_1 \eta_x = 0 \quad \text{and} \quad \xi_t + \lambda_2 \xi_x = 0. \tag{3.52}$$

With a proper set of initial and boundary conditions, η and ξ define a new set of independent coordinates. In these set of coordinates the equations (3.49) and (3.50) become

$$w_\xi = 0 \quad \text{and} \quad z_\eta = 0 \tag{3.53}$$

or in other words w is constant along the 1-characteristics and z is constant along the 2-characteristics.

Given (3.5) w and z also have expansions in powers of ϵ

$$w = \epsilon w_1 + \epsilon^2 w_2 + \cdots \quad \text{and} \quad z = \epsilon z_1 + \epsilon^2 z_2 + \cdots \tag{3.54}$$

and from (3.40) and (3.48) we see that

$$w_1 = 2\sigma(t + x) \quad \text{and} \quad z_1 = 2\sigma(t - x) \tag{3.55}$$

We will also need the expansion of the eigenvalues λ_i

$$\lambda_1 = -1 + \epsilon \left(\frac{(1 - c^{(1)})}{2} z_1 - \frac{(1 + c^{(1)})}{2} w_1 \right) + \cdots \quad \text{and}$$
$$\lambda_2 = 1 + \epsilon \left(\frac{(1 + c^{(1)})}{2} z_1 - \frac{(1 - c^{(1)})}{2} w_1 \right) + \cdots \tag{3.56}$$

where $c^{(1)} = \frac{dc}{d\rho}(\rho = 1)$.

3.4.3 Reflection of waves in closed tubes.

When we look at the system in its Riemann-invariant form, the boundary conditions for closed tubes become:

$$z = w \text{ at } x = 0 \quad \text{and} \quad w = z \text{ at } x = 1 \tag{3.57}$$

To gain insight, let us consider the particular situation where the initial conditions are :

$$z(t = 0) = \epsilon f(x) \quad \text{and} \quad w(t = 0) = 0 \tag{3.58}$$

where f is a function with the property that $f(0) = 0$, f increases in the interval $[0, x_1]$, decreases

33

in the interval $[x_1, x_2]$ and is 0 in $[x_2, 1]$ (see figure(3-1)).

Let us call $X\xi^{(x)}(t)$ the position of the 2-characteristic at time t that was at x when $t = 0$ (i.e $X\xi^{(x)}$ is a function of time that satisfies $X\xi^{(x)}(0) = x$ and $\dot{X}\xi^{(x)} = \frac{dX\xi^{(x)}}{dt} = \lambda_2$).

Then, from (3.52) and (3.56) $\dot{X}\xi^{(x)}(0) = \lambda_2 = 1 + \epsilon\frac{(1+c^{(1)})}{2}f(x) + \cdots$.

Since $c^{(1)} = \frac{1}{2}p^{(2)} > 0$, we see that $X\dot{\xi}^{(x)}(0)$, as a function of x, increases in the interval $[0, x_1]$, decreases in $[x_1, x_2]$ and is 1 in $[x_2, 1]$. The 2-characteristics starting in $[0, x_1]$ move apart from each other (expand) and the ones starting in $[x_1, x_2]$ get closer to each other (compress).

$\dot{X}\xi^{(x)}(t) = \dot{X}\xi^{(x)}(0)$ for $t < t_1$ where t_1 is when the 2-characteristic that starts at x_2 gets to the right end (i.e. $X\xi^{(x_2)}(t_1) = 1$). The shape of z at $t = t_1$ is given in figure(3-1) and w is still 0.

Let t_2 be the time at which $X\xi^{(0)}$ reaches the right end $(X\xi^{(0)}(t_2) = 1)$.

For t in $[t_1, t_2]$ the wave is being reflected at $x = 1$. The way this reflection occurs is given by the second equation in (3.57) and at $t = t_2$, when this process is completed, $z = 0$ and w will have a shape like the one plotted in figure(3-1)

We call $X\eta^{(x)}(t)$ the position of the 1-characteristic that satisfies $X\eta^{(x)} = 1$ at the same time as $X\xi^{(x)} = 1$.

At $t = t_2$, for x in $[x_1, x_2]$, $X\eta^{(x)}$ are closer to each other than $X\xi^{(x)}$ were at $t = t_1$. The reason for this is that $\frac{\partial\lambda_2}{\partial z} > 0$ and $\frac{\partial\lambda_1}{\partial w} < 0$ (which is the same reason for the compression of 2-characteristics for $t < t_1$).

We have not said anything about the dependence of λ_2 on w and λ_1 on z, but the overall effect of these terms (see equation (3.56)) in this case, is just to slow down the process of reflection (because $f(x) > 0$). In other words, the shape of w at $t = t_2$ is the same as the one obtained replacing λ_1 by $-1 - \epsilon\frac{(1+c^{(1)})}{2}w_1 + \cdots$ and λ_2 by $1 + \epsilon\frac{(1+c^{(1)})}{2}z_1 + \cdots$, only the time t_2 changes.

We now define t_3 and t_4 by $X\eta^{(x_2)}(t_3) = 0$ and $X\eta^{(0)}(t_4) = 0$. At t_4, the original wave have gone once back and forward in the tube, $w = 0$ and z would have a shape like the one display in figure(3-1).

We see that z has steepened in the region where it is decreasing and smoothed where it is increasing. It is developing a shock. This process is slow because the initial conditions are small, taking place in the scale of time of T, and is represented in the second term of equation (3.46). The third term in (3.46) corresponds, in our particular example, to the slow down of the reflection process at the ends of the tube due to the dependence of λ_1 on z and λ_2 on w (note that the this term is just σ_ϕ multiplied by a function that does not depend on ϕ).

Equation (3.45) reflects the fact that the wave takes (at first order) a time of 2 to go back and forward from one end to the other of the tube and that it has not change its value (as implied by the boundary conditions (3.57)).

We now understand equations (3.40), (3.45) and (3.46). This approximation is valid for times of order ϵ^{-1}. Shocks of the equation (3.45) correspond to shocks of the original system (3.1)-(3.2) and

the position of the shocks or the characteristics are also well approximated (with an error of order ϵ) for times of order ϵ^{-1}.

3.4.4 Reflection of waves at the open end.

We now turn back our attention to the tube with one end open.

The boundary conditions in Riemann-invariant form become:

$$z = w \text{ at } x = 0 \quad \text{and} \quad w = -z + O(w^2, z^2) \text{ at } x = 1 \tag{3.59}$$

We consider similar initial conditions as when we consider the closed tube (equation (3.58), plotted in figure(3-2)) and we also name position of the characteristics as before.

Until $t = t_1$ nothing changes (compare figure(3-1) and figure(3-2)). The reflection process will take place in the time interval $[t_1, t_2^*]$ (for some new t_2^* defined by $X\xi^{(0)}(t_2^*) = 1$). But now this reflection is governed by the second of the equations (3.59) , then, at $t = t_2^*$, w will be negative (see figure(3-2)). It's shape will be similar than in the closed end case but with it's sign changed.

Let t_3^* be the time at which $X\eta^{(x_2)}$ is 0. Then the regions that where compressing from $t = 0$ to $t = t_1$ will be expanding from $t = t_2$ to $t = t_3^*$ and vice versa. There is a compensation, compression waves get reflected as expansion waves and vice versa at the open end. Before going into a detail analysis we now introduce a simpler model that captures this behavior.

3.5 Toy model

In this section we will introduce and study a Toy model that has very similar features as the open end tube.

Suppose we look at w following the 1-characteristic from the open to the closed end. As explained in last section, there are two effects on w. One, due to itself, steepens its shape in the regions where w increases and smothens itself where w decreases. The other one, due to z, does not have an overall effect on its shape, it just changes slightly the time it will take the wave to go back and forward from one end of the tube to the other one twice (once if we are in a closed tube).

The first effect can be modeled by the single equation

$$w_t - \left(w + \frac{w^2}{2}\right)_x = 0 \tag{3.60}$$

and we forget about the second effect.

Similarly we can write for z

$$z_t + \left(z + \frac{z^2}{2}\right)_x = 0 \tag{3.61}$$

with boundary conditions

$$z = w \text{ at } x = 0 \quad \text{and} \quad w = -z \text{ at } x = 1 \tag{3.62}$$

where we have kept only the first order terms in (3.59). We now have obtained:

The toy equation.

$$w(x,t) = u(-x,t) \quad \text{and} \quad z(x,t) = u(x,t) \tag{3.63}$$

where u satisfies

$$u_t + \left(u + \frac{u^2}{2}\right)_x = 0 \tag{3.64}$$

this equation is solved in $[-1, 1]$ with boundary conditions

$$u(x = -1) = -u(x = 1) \tag{3.65}$$

3.5.1 Asymptotics for the toy equation

We now study the long term behavior of small amplitude solutions of the toy equation. We introduce a second time scale $T = \epsilon t$ and expand u in powers of ϵ

$$u = \epsilon u_1 + \epsilon^2 u_2 + \cdots \tag{3.66}$$

We plug this anzats in equations (3.64)-(3.65) and collect powers of ϵ.

At $O(\epsilon)$ we get:

$$u_{1t} + u_{1x} = 0 \quad \text{and} \quad u_1(x = -1) = -u_1(x = 1) \tag{3.67}$$

which has as general solution

$$u_1 = \alpha(\phi = x - t) \tag{3.68}$$

where α is an arbitrary function of ϕ and T that satisfies

$$\alpha(\phi - 1) + \alpha(\phi + 1) = 0 \qquad (3.69)$$

At $O(\epsilon^2)$ we get:

$$u_{2t} + u_{2x} = -u_{1T} - \left(\frac{u_1^2}{2}\right)_x \qquad (3.70)$$

with boundary condition

$$u_2(x = -1) = -u_2(x = 1) \qquad (3.71)$$

The general solution of (3.70) is

$$u_2 = \beta(x - t) - x\alpha_T(x - t) - x\left(\frac{\alpha^2}{2}\right)_\phi (x - t) \qquad (3.72)$$

where β is an arbitrary function of ϕ and T and the quantity inside the brackets indicate where ϕ has to be evaluated.

The boundary condition (3.71) imply that β satisfy

$$\beta(\phi - 1) + \beta(\phi + 1) = 2\alpha_T(\phi + 1) \qquad (3.73)$$

The dependence of α in T is given by the requirement that u_2 does not grow linearly with t or, which is equivalent, that β does not grow linearly with ϕ. Since α satisfies the homogeneous equation (3.69), from (3.73) we obtain:

$$\alpha_T = 0 \qquad (3.74)$$

and by taking $\beta = 0$, from (3.72) we have that

$$u_2 = -x\left(\frac{\alpha^2}{2}\right)_\phi (x - t) \qquad (3.75)$$

Equation (3.74) is the same as the one for the open tube system (see (3.30)).

We introduce a slower time scale $\tau = \epsilon^2 t$.

At $O(\epsilon^3)$ we get:

$$u_{3t} + u_{3x} = -u_{1\tau} - (u_1 u_2)_x \qquad (3.76)$$

37

with boundary condition

$$u_3(x = -1) = -u_3(x = 1) \tag{3.77}$$

The general solution of (3.76) is given by

$$u_3 = \gamma(x - t) - x\alpha_\tau(x - t) + x\left(\frac{\alpha^3}{3}\right)_\phi (x - t) + \frac{x^2}{2}\left(\frac{\alpha^3}{3}\right)_{\phi\phi}(x - t) \tag{3.78}$$

where γ is an arbitrary function of ϕ and τ.

The boundary con   ion (3.77) gives us an equation for γ

$$\gamma(\phi - 1) + \gamma(\phi + 1) = 2\alpha_\tau(\phi + 1) - 2\left(\frac{\alpha^3}{3}\right)_\phi (\phi + 1) \tag{3.79}$$

and requiring sublinearity in t of u_3 (or, what is the same, that γ does not grow linearly with ϕ) we obtain

The asymptotic approximation for the toy model

$$u \sim \epsilon\alpha(x - t) \tag{3.80}$$

where α satisfies

$$\alpha_\tau - \left(\frac{\alpha^3}{3}\right)_\phi = 0 \tag{3.81}$$

with the antiperiodicity condition

$$\alpha(\phi - 1) + \alpha(\phi + 1) = 0 \tag{3.82}$$

Remark. We see that equation (3.81) is very similar to the asymptotic approximation for the original system (equation (3.42)). Except for the values of the constants, the only difference is that now we do not have the last term in (3.42), but this is expected since that accounts for the effect we said at the beginning of this section we were going to ignore.

3.5.2   Solving the asymptotic equation

In this section we will first mention some of the properties of equation (3.81). For more details see ([22], [23] and [24])

In the regions where α is continuous, α is constant along the characteristics. These are curves $\phi(\tau)$ that satisfy $\phi_\tau = -\alpha^2$ (and as a consequence they are straight lines).

Discontinuities are admitted only along curves $\phi(\tau)$ that satisfy the Rankine Hugoniot condition

$$\frac{d\phi}{d\tau} = -\frac{1}{3}\frac{(\alpha^+)^3 - (\alpha^-)^3}{\alpha^+ - \alpha^-} \tag{3.83}$$

and the entropy condition

$$\alpha^+ > \alpha^- \geq -\frac{\alpha^+}{2} \quad \text{when} \quad \alpha^+ > 0 \tag{3.84}$$

and

$$\alpha^+ < \alpha^- \leq -\frac{\alpha^+}{2} \quad \text{when} \quad \alpha^+ < 0 \tag{3.85}$$

where $\alpha^+ = \alpha(\phi(\tau)^+, \tau)$ and $\alpha^- = \alpha(\phi(\tau)^-, \tau)$.

Under these rules for the discontinuities, there is existence and uniqueness of solutions once the initial condition $\alpha(\tau = 0)$ is prescribed.

There are two different kind of discontinuities: *genuine shocks* and *left contacts*.

A discontinuity is a genuine shock when $\alpha^- \neq -\frac{\alpha^+}{2}$ and it is a left contact when $\alpha^- = -\frac{\alpha^+}{2}$.

A genuine shock has the property that the characteristics run into it from both sides.

When the discontinuity is a left contact, only the characteristics coming from right run into it. At the left, it irradiates characteristics that are tangential to the discontinuity except when the left contact is a straight line, in which case it just does not intersect the characteristics at its left.

Figure(3-3) show some examples of solutions of (3.81) with different initial conditions. Note that a genuine shock can become a left contact and vice versa.

3.5.3 Understanding the validity of the approximation

We have used the method of multiple time scales to derive (3.80)-(3.81)-(3.82) from (3.64)-(3.65). In doing so we have assumed that α is continuous, but we know that even if we start with continuous initial conditions, α will develop discontinuities. The question that arises is:

Question 1: Does this approximation remain valid for discontinuous solutions and if this is the case, how do discontinuities move?.

The answer to this question is yes, this approximation is valid for discontinuous solutions and the discontinuities follow the entropy conditions (3.84) and (3.85) and, since u is conserved (and then α too), the discontinuities also follow the Rankine Hugoniot condition (3.83).

The toy equation (3.64) has a convex flux, and its approximation (3.81) has a flux with an inflection point. Then we expect that solutions of (3.64) will behave very different from solutions of (3.81). For example solutions of (3.64) has only genuine shocks as discontinuities. On the contrary, solutions of (3.81) have also left contacts, not only that, but no matter the initial conditions, it will

develop left contacts.

Then a second question is:

Question 2: In what sense is this approximation valid?. In particular, what do left contacts of α mean when $\epsilon\alpha(x-t)$ is interpreted as an approximation of $u(x,t)$?

The remaining of this section consists in answering questions 1 and 2.

Following a continuity point.

Let us suppose that at $(x,t) = (-1,t_0)$ u is continuous.

The characteristic $x(t)$ that passes through $(x,t) = (-1,t_0)$ is given by

$$x(t) = -1 + (1 + u(-1,t_0))\,(t - t_0) \tag{3.86}$$

Assuming that this characteristic does not run into any shock, let t_1 be the time at which it gets to $x = 1$. From (3.86)

$$t_1 = t_0 + \frac{2}{(1 + u(-1,t_0))} \tag{3.87}$$

Then, from (3.65) $u(-1,t_1) = -u(1,t_1) = -u(-1,t_0)$. If again the characteristic going through $(x,t) = (-1,t_1)$ do not run into any shock, it gets to $x = 1$ at t_2, where

$$t_2 = t_0 + \frac{2}{(1 + u(-1,t_0))} + \frac{2}{(1 - u(-1,t_0))} \tag{3.88}$$

and we have $u(-1,t_2) = u(-1,t_0)$.

It takes a time of $t_2 - t_0 = 4 + 4u(-1,t_0)^2 + O(u(-1,t_0)^3)$ to cover a distance of 4. Or in other words, the average speed of the characteristics is

$$average\ speed = 1 - u^2 + O(u^3) \tag{3.89}$$

(understanding that once we get to $x = 1$, we continue from $x = -1$).

The characteristics curves ϕ_τ of (3.81) satisfy $\phi_\tau = -\alpha^2$ from where, remembering that $\tau = \epsilon^2 t$ and (3.80), it is immediate that the *average speed* of the characteristics of the toy model is approximated, with an error of order ϵ^3, by the speed of the characteristics of its asymptotic approximation. This confirms that (3.80)-(3.81)-(3.82) is valid for times of order ϵ^{-2}, at least for the points that can be reached by tracing characteristics starting at $t = 0$.

Figure(3-4) shows a plot of $u(x,t)$ for different values of t with initial conditions given by $u(x,0) = \epsilon\sin(\frac{\pi}{2}x)$ where we take $\epsilon = 0.1$. The solid lines correspond to the real solution and the dotted lines to the approximation $\epsilon\alpha(x-t)$. The characteristic starting at $(x,t) = (-1,0)$ is plotted in figure(3-

5). Again the solid lines correspond to the real characteristics and the dotted lines to the asymptotic approximation. Note that the approximation is better for times multiples of four. This is because of averaging character of the approximation.

Following a shock.

Let us now suppose that u has a discontinuity that gets to $x = 1$ at $t = t_0$.

We define $\epsilon \bar{u}_1(x) = u(t = t_0)$.

At $t = t_0^-$, the value ahead of the shock is $u(1, t_0^-) = -u(-1, t_0^-) = -\epsilon \bar{u}_1(-1)$ and the value behind it is $u(1^-, t_0^-) = \epsilon \bar{u}_1(1)$. Then the entropy condition for (3.64) implies $\bar{u}_1(1) > -\bar{u}_1(-1)$.

On the other hand, $u(-1, t_0^-) = \epsilon \bar{u}_1(-1) > -\epsilon \bar{u}_1(1) = u(-1, t_0^+)$ and as a result a rare fraction is born at $(x, t) = (-1, t_0)$.

Each time a shock gets to $x = 1$, a rare fraction appears at $x = -1$. This is the same effect as the reflection of waves at the open end of the tube. When a compression waves gets to $x = 1$, because it changes sign, it becomes an expansion wave at $x = -1$ and vice versa.

As the rare fraction gets to $x = 1$, it will become a compression wave at $x = -1$ that may or may not produce a new shock. To understand the details of this process we need some calculations.

We want to solve (3.64)-(3.65) with initial conditions

$$u(t = 0) = \epsilon \bar{u}_1(x) \tag{3.90}$$

where $\bar{u}_1$ has the property that $\bar{u}_1(-1) + \bar{u}_1(1) > 0$ (i.e., the situation we just have described).

We perform our calculations in characteristics coordinates (ξ, t). In this set of coordinates, x and u satisfy

$$x_t = 1 + u \tag{3.91}$$

and

$$u_t = 0 \tag{3.92}$$

A set of characteristics will emanate from $(x, t) = (-1, 0)$ carrying values of u ranging in the interval $[-\epsilon \bar{u}_1(1), \epsilon \bar{u}_1(-1)]$.

We assign the value 0 to the characteristic along which $u = -\epsilon \bar{u}_1(1)$, the value $-\bar{u}_1(-1) - \bar{u}_1(1)$ to the one where $u = \epsilon \bar{u}_1(-1)$ and interpolate linearly in between. This translates to

$$x(\xi, t = 0) = -1 \quad \text{and} \quad u = \epsilon \left(-\bar{u}_1(1) - \xi \right) \tag{3.93}$$

41

for $-\bar{u}_1(-1) - \bar{u}_1(1) \leq \xi \leq 0$.

We continue to set the initial conditions in a natural way as follows

$$x(\xi, t = 0) = -1 - \bar{u}_1(-1) - \bar{u}_1(1) - \xi \tag{3.94}$$

and

$$u = \epsilon \bar{u}_1 \left(-1 - \bar{u}_1(-1) - \bar{u}_1(1) - \xi\right) \tag{3.95}$$

for $-2 - \bar{u}_1(-1) - \bar{u}_1(1) \leq \xi \leq -\bar{u}_1(-1) - \bar{u}_1(1)$.

We define $T(\xi)$ to be the time when the characteristic ξ is at $x = 1$. $T(\xi)$ is defined by the following implicit formula:

$$x(\xi, T(\xi)) = 1 \tag{3.96}$$

We want to think that once we get to $x = 1$, we continue from $x = -1$. Then we will assign the value $\xi + c$, where c is a constant, to the characteristic leaves $x = -1$ at $t = T(\xi)$. Because of the initial conditions (3.93) and (3.94), the natural choice of c is $c = 2 + \bar{u}_1(-1) + \bar{u}_1(1)$. In other words

$$x\left(\xi + 2 + \bar{u}_1(-1) + \bar{u}_1(1), T(\xi)\right) = -1 \tag{3.97}$$

and then (3.65) translates to

$$u\left(\xi + 2 + \bar{u}_1(-1) + \bar{u}_1(1)\right) = -u(\xi) \tag{3.98}$$

Equations (3.91) and (3.92) and the initial conditions (3.93), (3.94), (3.95), (3.97) and (3.98) define the problem in this set of coordinates.

We expand x, T and u in powers of ϵ.

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots, \quad T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots \quad \text{and}$$
$$u = \epsilon u_1(\xi) \tag{3.99}$$

where the last equation comes from (3.90) and (3.92).

We first find the general solution of (3.91) by plugging into it the anzats (3.99) and collecting powers of ϵ.

$$x_0(\xi, t) = t + \bar{x}_0(\xi), \quad x_1(\xi, t) = t u_1(\xi) + \bar{x}_1(\xi) \quad \text{and}$$
$$x_2(\xi, t) = \bar{x}_2(\xi) \tag{3.100}$$

where $\bar{x}_i(\xi)$ are arbitrary functions.

To go forward in time we will need $T(\xi)$ once we know $x(\xi, t)$. This is calculated by plugging (3.99) into (3.96).

$$T_0(\xi) = 1 - \bar{x}_0(\xi) , \quad T_1(\xi) = -\bar{x}_1(\xi) - u_1(\xi)T_0(\xi) \quad \text{and}$$

$$T_2(\xi) = -\bar{x}_2(\xi) - u_1(\xi)T_1(\xi) \tag{3.101}$$

Having $T(\xi)$ we get $\bar{x}_i(\xi + 2 + \bar{u}_1(-1) + \bar{u}_1(1))$ from (3.97) and (3.100).

$$\begin{aligned}
\bar{x}_0(\xi) &= -1 - T_0(\xi - 2 - \bar{u}_1(-1) - \bar{u}_1(1)) \tag{3.102} \\[1em]
\bar{x}_1(\xi) &= -T_1(\xi - 2 - \bar{u}_1(-1) - \bar{u}_1(1)) - \\
&\qquad T_0(\xi - 2 - \bar{u}_1(-1) - \bar{u}_1(1))\, \bar{u}_1(\xi) \\[1em]
\bar{x}_2(\xi) &= -T_2(\xi - 2 - \bar{u}_1(-1) - \bar{u}_1(1)) - \\
&\qquad T_1(\xi - 2 - \bar{u}_1(-1) - \bar{u}_1(1))\, \bar{u}_1(\xi)
\end{aligned}$$

where we have replaced ξ by $\xi - 2 - \bar{u}_1(-1) - \bar{u}_1(1)$ for future convenience.

$\xi = 0$ and $\xi = -\bar{u}_1(-1) - \bar{u}_1(1)$, the two extreme characteristics of the rare fraction, define three regions. If we now also consider the characteristics $\xi = 2$ and $\xi = 2 + \bar{u}_1(-1) + \bar{u}_1(1)$ we have five different regions (see figure(3-6)).

We name the different regions and describe them in characteristic coordinates.

$$\begin{aligned}
\text{Region 1} &= \{-2 - \bar{u}_1(-1) - \bar{u}_1(1) \le \xi \le -\bar{u}_1(-1) - \bar{u}_1(1)\} \\
\text{Region 2} &= \{-\bar{u}_1(-1) - \bar{u}_1(1) \le \xi \le 0\} \\
\text{Region 3} &= \{0 \le \xi \le 2\} \tag{3.103} \\
\text{Region 4} &= \{2 \le \xi \le 2 + \bar{u}_1(-1) + \bar{u}_1(1)\} \\
\text{Region 5} &= \{2 + \bar{u}_1(-1) + \bar{u}_1(1) \le \xi \le 4 + \bar{u}_1(-1) + \bar{u}_1(1)\}
\end{aligned}$$

Note that region 2 corresponds to the rare fraction and region 4 to its reflection.

We will assume that $\bar{u}_1$ has derivative of order one. As a consequence the only region where shock can occur is region 4.

We will solve the equations in each of the different regions.

Region 1.

We set $t = 0$ to get u_1 and x_i from (3.94), (3.95) and (3.100). then we use (3.101) to get:

$$u_1(\xi) = \bar{u}_1(-1 - \bar{u}_1(-1) - \bar{u}_1(1) - \xi) \tag{3.104}$$

$$x(\xi, t) = t - 1 - \bar{u}_1(-1) - \bar{u}_1(1) - \xi + \epsilon t u_1(\xi) \qquad (3.105)$$

and

$$
\begin{aligned}
T(\xi) \quad = \quad & 2 + \bar{u}_1(-1) + \bar{u}_1(1) + \xi - \epsilon(2 + \bar{u}_1(-1) + \bar{u}_1(1) + \xi)u_1(\xi) + \\
& \epsilon^2(2 + \bar{u}_1(-1) + \bar{u}_1(1) + \xi)u_1^2(\xi)
\end{aligned}
\qquad (3.106)
$$

Region 2.

Again setting $t = 0$ we get u_1 and x_i and then we calculate T.

$$u_1(\xi) = -\bar{u}_1(1) - \xi \qquad (3.107)$$

$$x(\xi, t) = t - 1 + \epsilon t u_1(\xi) \qquad (3.108)$$

$$T(\xi) = 2 - \epsilon 2 u_1(\xi) + \epsilon^2 2 u_1^2(\xi) \qquad (3.109)$$

Region 3.

Now we first calculate x_i using (3.102), u_1 using (3.98) and then we calculate T from (3.101)

$$u_1(\xi) = -\bar{u}_1(1 - \xi) \qquad (3.110)$$

$$x(\xi, t) = t - 1 - \xi + \epsilon(t - 2\xi)u_1(\xi) - \epsilon^2 2\xi u_1^2(\xi) \qquad (3.111)$$

$$T(\xi) = 2 + \xi + \epsilon(\xi - 2)u_1(\xi) + \epsilon^2(2 + \xi)u_1^2(\xi) \qquad (3.112)$$

Region 4.

Here and in region 5, we follow the same procedure as region 3.

$$u_1(\xi) = \xi - 2 - \bar{u}_1(-1) \qquad (3.113)$$

$$x(\xi, t) = t - 3 + \epsilon(t - 4)u_1(\xi) - \epsilon^2 4 u_1^2(\xi) \qquad (3.114)$$

$$T(\xi) = 4 + \epsilon^2 4 u_1^2(\xi) \qquad (3.115)$$

44

Region 5.

$$u_1(\xi) = \bar{u}_1(3 + \bar{u}_1(-1) + \bar{u}_1(1) - \xi) \tag{3.116}$$

$$x(\xi, t) = t - 1 - \xi + \bar{u}_1(-1) + \bar{u}_1(1) + \epsilon(t-4)u_1(\xi) - \epsilon^2 4u_1^2(\xi) \tag{3.117}$$

We now need to check whether there is any shock.

As said before, the only region where a shock can be developed, is region 4. If there is a shock, it starts at $t = t^*$ and $\xi = \xi^*$ given by

$$t^* = \min\{t : x_\xi(\xi, t) = 0 \text{ for some } \xi\} \tag{3.118}$$

and $x_\xi(\xi^*, t^*) = 0$. Understanding that ξ needs to be in region 4 (see (3.103)).

We then compute x_ξ in region 4

$$x_\xi = \epsilon(t-4) + \epsilon^2 8\left(\bar{u}_1(-1) - \xi + 2\right) \tag{3.119}$$

from where we immediately obtain that

$$t^* = 4 - \epsilon 8\bar{u}_1(-1) + O(\epsilon^2) \quad \text{and} \quad \xi^* = 2 \tag{3.120}$$

If we call x^* the location where the shock starts, we get:

$$x^* = x(\xi^*, t^*) = 1 - \epsilon 8\bar{u}_1(-1) + O(\epsilon^2) \tag{3.121}$$

Case 1: $\bar{u}_1(-1) < 0.$

When $\bar{u}_1(-1) < 0$, the last equation tell us that there is no shock formation since we would have $x^* > 1$. This means that the rare fraction wave in region 2 has expanded more than its reflection (region 4) has compressed.

Let us go back to the approximation (3.80)-(3.81)-(3.82). The initial conditions $u(t = 0) = \epsilon \bar{u}_1(x)$ corresponds to $\alpha_0(\phi) = \alpha(\tau = 0) = \bar{u}_1(x = \phi)$. We have discontinuity at $\phi = 1$ with the property that $0 < \alpha_0(1^+) < \alpha_0(1^-)$, since $\alpha_0(1^-) = \bar{u}_1(x = 1^-) = \bar{u}_1(1)$ and $\alpha_0(1^+) = -\bar{u}_1(x = -1^+) = -\bar{u}_1(-1)$.

This means that α has a rare fraction wave (in fact, due to its antiperiodicity, it has a rare fraction at $(\phi, \tau) = (2k+1, 0)$ for every integer k). This agrees with the fact that u does not produce a new shock. If we were to follow the rare fraction wave of u that started in region 2, we saw that

45

its reflection compresses, then after an other reflection it will expand again and so on, but overall, it expands more than it compresses and the region in space that this wave covers becomes wider with time. Region 2, corresponds to the rare fraction of α that is born at $(\phi, \tau) = (-1, 0)$, region 4 to the one born in $(-3, 0)$ and so on.

The same calculation than the one done for points where u is continuous, shows that not only qualitatively the behavior of the asymptotic approximation is correct, but also quantitatively (i.e. we can recover the location of the characteristic with an error of order ϵ for times of order ϵ^{-2}).

Figure(3-8) shows a plot of u for different t with $\epsilon = 0.1$ and $\bar{u}_1(x) = x + 0.5$. The solid lines correspond to the real solution and the dotted ones to the approximation. The extreme characteristics of the rare fraction are plotted in figure(3-8), where again, solid lines correspond to the real solution and the dotted ones to the approximation.

Case 2: $\bar{u}_1(-1) > 0$.

In this case a shock is born.

We now track the shock. First we note that ahead of the shock we are in region 3 and behind it in region 4. We use ξ from region 4 as independent variable and denote by $s(\xi)$ the position of the shock when it intersects the characteristic ξ, by $t(\xi)$ the time when this intersection occurs, and by $\xi_3(\xi)$ the characteristic that the shock is intersecting at that time ahead of it (i.e. in region 3). We also use the notation $u_1^{(3)}$ to denote u_1 in region 3 an similarly $u_1^{(4)}$ to denote u_1 in region 4.

The Rankine Hugoniot condition becomes

$$\frac{s_\xi}{t_\xi} = 1 + \epsilon \frac{u_1^{(3)} + u_1^{(4)}}{2} \tag{3.122}$$

and we have two more equations

$$s = x^{(3)}(\xi_3(\xi), t(\xi)) \quad \text{and} \quad s = x^{(4)}(\xi, t(\xi)) \tag{3.123}$$

where we have denoted $x^{(i)}$ the position x in the region i.

We expand s, t and ξ_3 in powers of ϵ.

$$s = 1 + \epsilon s_1 + \epsilon^2 s_2 \ , \quad t = 4 + \epsilon t_1 + \epsilon^2 t_2 \quad \text{and}$$
$$\xi_3 = 2 + \epsilon \xi_{31} + \epsilon^2 \xi_{32} \tag{3.124}$$

We multiply (3.122) by t_ξ, plug the anzats (3.124) into that equation and (3.123) and collect powers of ϵ to get:

At $O(\epsilon)$

$$s_{1\xi} = t_{1\xi} \ , \quad s_1 = t_1 - \xi_{31} \quad \text{and} \quad s_1 = t_1 \tag{3.125}$$

At $O(\epsilon^2)$

$$s_{2\xi} = t_{2\xi} + \frac{t_{1\xi}}{2}\left(\xi - 2 - 2\bar{u}_1(-1)\right) \tag{3.126}$$

$$s_2 = t_2 - \xi_{32} - (t_1 - 2\xi_{31})\bar{u}_1(-1) - 4\bar{u}_1^2(-1) \tag{3.127}$$

and

$$s_2 = t_2 + t_1\left(\xi - 2 - \bar{u}_1(-1)\right) - 4\left(\xi - 2 - \bar{u}_1(-1)\right)^2 \tag{3.128}$$

From equations (3.126) and (3.128) we get that

$$t_1 = \frac{16}{3}(\xi - 2) - 8\bar{u}_1(-1) \tag{3.129}$$

We define $\bar{\xi}$ as the solution of $s(\bar{\xi}) = 1$ (i.e. the characteristic in region 4 that intersects the shock when it gets to the right boundary). Equations (3.125) and (3.129) imply that

$$\bar{\xi} = 2 + \frac{3}{2}\bar{u}_1(-1) + O(\epsilon) \tag{3.130}$$

but this is going to be valid only if $\bar{\xi}$ belongs to region 4. Then we need to consider two different cases.

Case 2.1: $0 < \frac{1}{2}\bar{u}_1(-1) < \bar{u}_1(1)$

In this case $\bar{\xi}$ belongs to region 4. Equation (3.129) is valid all the way until the shock gets to the right boundary.

$t_1(\bar{\xi}) = s_1(\bar{\xi}) = 0$, then from (3.128), $t_2(\bar{\xi}) = \bar{u}_1^2(-1)$. Or, if we define $\bar{t} = t(\bar{\xi})$ (i.e. the time when the shock gets to the right boundary) we have

$$\bar{t} = 4 + \epsilon^2 \bar{u}_1^2(-1) + O(\epsilon^3) \tag{3.131}$$

Identifying this discontinuity with the one we started with, we can say that its average velocity is

$$average\ velocity = \frac{4}{\bar{t}} = 1 - \epsilon^2 \frac{\bar{u}_1^2(-1)}{4} + O(\epsilon^3) \tag{3.132}$$

From (3.113) we get that the value of u behind the shock when it gets to $x = 1$ is

$$u^-(\bar{t}) = \epsilon u_1^{(4)}(\bar{\xi}) = \epsilon \frac{\bar{u}_1(-1)}{2} + O(\epsilon^2) \tag{3.133}$$

47

We now compare u with α. Again the initial condition for α is $\alpha(\tau = 0) = \bar{u}_1(x = \phi)$. We have a discontinuity at $\phi = 1$ with the property $0 < -\frac{\alpha^+}{2} < \alpha^-$ (since $\alpha^+ = -\bar{u}_1(-1)$ and $\alpha^- = \bar{u}_1(1)$).

This means that α has a left contact followed by a rare fraction at $(\phi, \tau) = (1, 0)$ (and the same structure at $(2k + 1, 0)$ for all integers k). The velocity of the contact at $\tau = 0$ is

$$velocity \; of \; contact \; in \; (\phi, \tau) \; coordinates = -\frac{(\alpha^+)^2}{4} = -\frac{\bar{u}_1^2(-1)}{4} \tag{3.134}$$

and translated to (x, t) coordinates

$$velocity \; of \; contact \; in \; (x, t) \; coordinates = 1 - \epsilon^2 \frac{\bar{u}_1^2(-1)}{4} \tag{3.135}$$

As we can see this is the same value (except for an error of order ϵ^3) as the average velocity of the discontinuity of u (equation (3.132)).

Region 2 corresponds to the rare fraction-contact that are born in $(-1, 0)$, region 4 to the ones born in $(-3, 0)$, etc.

The value of α to the left of the contact (starting in $(-3, 0)$) at $\tau = \epsilon^2 4$ (which corresponds to $t = \bar{t}$ at first order) is $-\frac{\alpha(-3^+, 0)}{2} + O(\epsilon^2) = \frac{\bar{u}_1(-1)}{2} + O(\epsilon^2)$ and then we are properly approximating equation (3.133).

The boundary between regions 1 and 2 corresponds to the contact of α starting at $(-1, 0)$ and region 2 to the rare fraction next to that contact. The boundary between regions 3 and 4, that contains the shock, corresponds to the contact of α starting at $(-3, 0)$ and region 4 to the rare fraction next to that contact, and so on.

If we follow the part of region 4 that was not absorbed by the shock, it will expand with time as rare fractions expand for α.

Note that if we start the process again at $t = \bar{t}$, we do not know if we are in this case again. In other words, we do not know if $\frac{1}{2}u(-1^+, \bar{t}) < u(1^-, \bar{t})$. We really have an equality at first order, so it does not matter if we follow the rules just described or the one we will next explain for the case when inequality is reversed, up to the order of accuracy of our interest we will get the same result. The fact that at $t = \bar{t}$ the inequality became to first order an equality corresponds to the fact that for α we can only have rare fractions at $\tau = 0$, and the fact that after the wave of u has traveled twice from -1 to 1 we might switch from **Case 2.1** to **Case 2.2** and vice versa, corresponds to the fact that contact can become shocks and vice versa for α.

Figure(3-9) shows a plot of u for different t with $\epsilon = 0.2$ and $\bar{u}_1(x) = 1$. The solid lines correspond to the real solution and the dotted ones to the approximation. Figure(10) is a plot of the characteristics that correspond to the boundary of region 2, its reflection (region 4). The characteristics of the real solution are plotted in solid lines and its shock in dotted lines. The characteristics of the asymptotic approximation are plotted with dashed lines. Note that one of the

dashed lines corresponds to the contact discontinuity of α.

Case 2.2: $\frac{1}{2}\bar{u}_1(-1) > \bar{u}_1(1)$ **and** $\bar{u}_1(-1) > 0$

The shock hits the boundary between region 4 and 5 when $\xi = 2 + \bar{u}_1(-1) + \bar{u}_1(1)$. We define $\tilde{s} = s(2 + \bar{u}_1(-1) + \bar{u}_1(1))$, $\tilde{t} = t(2 + \bar{u}_1(-1) + \bar{u}_1(1))$ and $\tilde{\xi}_3 = \xi_3(2 + \bar{u}_1(-1) + \bar{u}_1(1))$. From (3.125), and (3.129) we have:

$$\tilde{s} = 1 + \epsilon \left(\frac{16}{3}\bar{u}_1(1) - \frac{8}{3}\bar{u}_1(-1) \right) + \epsilon^2 \tilde{s}_2 + \cdots \tag{3.136}$$

$$\tilde{t} = 4 + \epsilon \left(\frac{16}{3}\bar{u}_1(1) - \frac{8}{3}\bar{u}_1(-1) \right) + \epsilon^2 \tilde{t}_2 + \cdots \tag{3.137}$$

where from (3.128) $\tilde{s}_2$ and $\tilde{t}_2$ satisfy

$$\tilde{s}_2 = \tilde{t}_2 + \frac{4}{3}\bar{u}_1^2(1) - \frac{8}{3}\bar{u}_1(-1)\bar{u}_1(1) \tag{3.138}$$

which, together with (3.127), implies

$$\tilde{\xi}_3 = 2 - \epsilon^2 \frac{4}{3}\left(\bar{u}_1(-1) + \bar{u}_1(1)\right)^2 + \cdots \tag{3.139}$$

Let $\theta = \epsilon^{-1}(t - \tilde{t})$. We use θ as independent variable and consider s, t, ξ_3 and ξ_5 functions of θ (where ξ_5 is the value of the characteristic behind the shock, in region 5). Again we expand all the these variables in power of ϵ (the same as equation (3.124)). Using the same notation as before, in region 5, we denote the value of u_1 by $u_1^{(5)}$ and the position x by $x^{(5)}$.

The Rankine Hougoniot condition becomes

$$\frac{s_\theta}{\epsilon} = 1 + \epsilon \frac{u_1^{(3)} + u_1^{(5)}}{2} \tag{3.140}$$

and we also have the equations

$$s = x^{(3)}(\xi_3(\theta), t(\theta)) \quad \text{and} \quad s = x^{(5)}(\xi_5(\theta), t(\theta)) \tag{3.141}$$

Plugging the anzats for the different variables in powers of ϵ, multiplying equation (3.140) by ϵ and collecting powers of ϵ we get:

At $O(\epsilon)$

$$s_{1\theta} = 1 \;, \quad s_1 = t_1 - \xi_{31} \quad \text{and} \quad s_1 = t_1 - \xi_{51} \tag{3.142}$$

49

At $O(\epsilon^2)$

$$s_{2\theta} = \frac{\bar{u}_1(1) - \bar{u}_1(-1)}{2} \tag{3.143}$$

$$s_2 = t_2 - \xi_{32} - (t_1 - 2\xi_{31})\bar{u}_1(-1) - 4\bar{u}_1^2(-1) \tag{3.144}$$

$$s_2 = t_2 - \xi_{52} + t_1\bar{u}_1(1) - 4\bar{u}_1^2(1) \tag{3.145}$$

Solving these equations we get:

$$s = 1 + \epsilon\left(\frac{16}{3}\bar{u}_1(1) - \frac{8}{3}\bar{u}_1(-1) + \theta\right) + O(\epsilon^2) \tag{3.146}$$

$$t = 4 + \epsilon\left(\frac{16}{3}\bar{u}_1(1) - \frac{8}{3}\bar{u}_1(-1) + \theta\right) + O(\epsilon^2) \tag{3.147}$$

and

$$\xi_3 = 2 - \epsilon^2\left(\frac{4}{3}(\bar{u}_1(-1) + \bar{u}_1(1))^2 + \frac{\theta}{2}(\bar{u}_1(-1) + \bar{u}_1(1))\right) \tag{3.148}$$

We define $\bar{\theta}$ by $s(\bar{\theta}) = 1$ (i.e. the shock gets to the right boundary). From (3.146)

$$\bar{\theta} = \frac{8}{3}\bar{u}_1(-1) - \frac{16}{3}\bar{u}_1(1) + O(\epsilon) \tag{3.149}$$

and by denoting $\bar{\xi}_3$ the value of the characteristic ahead of the shock when it gets to $x = 1$, (3.148) and (3.149) imply

$$\bar{\xi}_3 = 2 + \epsilon^2\frac{4}{3}\left(\bar{u}_1^2(1) - \bar{u}_1(1)\bar{u}_1(-1) - 2\bar{u}_1^2(-1)\right) \tag{3.150}$$

Then from (3.112) we have that the shock reaches the right boundary at

$$\bar{t} = t(\bar{\theta}) = 4 + \epsilon^2\frac{4}{3}\left(\bar{u}_1^2(-1) - \bar{u}_1^2(-1)\bar{u}_1(1) + \bar{u}_1^2(1)\right) + O(\epsilon^3) \tag{3.151}$$

or the average velocity of the shock is

$$average\ velocity = 1 - \epsilon^2\frac{\bar{u}_1^2(-1) - \bar{u}_1^2(-1)\bar{u}_1(1) + \bar{u}_1^2(1)}{3} + O(\epsilon^3) \tag{3.152}$$

Let us compare this result with the one obtained using α. The initial condition for α is $\alpha(\tau =$

$0) = \bar{u}_1(x = \phi)$. The discontinuity at $\phi = 1$ has the property $\alpha^+ < \alpha^- < -\frac{\alpha^+}{2}$ with $\alpha^+ < 0$ since $\alpha^- = \bar{u}_1(1)$ and $\alpha^+ = -\bar{u}_1(-1)$.

Then, α has genuine shocks starting at $(\phi, \tau) = (2k + 1, 0)$ for all integers k, whose velocity at $\tau = 0$ is

$$\begin{aligned}
velocity\ of\ the\ shock \quad &= \quad -\frac{\left(\alpha^+\right)^2 + \alpha^+\alpha^- + \left(\alpha^-\right)^2}{3} \\
&= \quad -\frac{\bar{u}_1^2(-1) - \bar{u}_1(-1)\bar{u}_1(1) + \bar{u}_1^2(1)}{3}
\end{aligned} \tag{3.153}$$

which translated to (x, t) coordinates gives us the right approximation of the average velocity of the shock of u (see equation (3.152)).

$u(-1^+, \bar{t}) = \epsilon\bar{u}_1(-1) + O(\epsilon^3)$ and $u(1^-, \bar{t}) = \epsilon\bar{u}_1(1) + O(\epsilon^3)$. Then if $\frac{1}{2}\bar{u}_1(-1) - \bar{u}_1(1)$ is not too small, at $t = \bar{t}$ we are in the same case as we where at $t = 0$. We see that continuing this process, we will have a series of shocks that develop every 4 units of time approximately and at a distance of order ϵ of $x = 1$. All this series of shocks corresponds to the shocks of α. Region 2, corresponds to the shock of α that starts at $(\phi, \tau) = (-1, 0)$, the shock of u that gets to $x = 1$ at $t = 4 + O(\epsilon^2)$ corresponds to the shock of α that starts at $(\phi, \tau) = (-3, 0)$, and so on.

Figure(3-11) shows a plot of u for different t with $\epsilon = 0.05$ and $\bar{u}_1(x) = -x + 1$. The solid lines correspond to the real solution and the dotted ones to the approximation. Figure(3-12) is a plot of the characteristics that correspond to the boundary of region 2, its reflection (region 4) and the shock. The real solution is plotted with solid lines and the approximation with dotted lines.

3.5.4 Final remarks on the Toy model

We now understand how equations (3.80)-(3.81)-(3.82) approximate the toy system (3.64)-(3.65) and its validity for times of order ϵ^{-2}.

Then, as α, u decays like $t^{-\frac{1}{2}}$ (i.e. there exits a constant k such that $u < \frac{k}{t^{\frac{1}{2}}}$ for all x and t) and u will tend to a solution with only one of the rare fraction-shock structures described in section(3.5.3) corresponding to following a shock the **Case 2.1**. This is well illustrated in figures (3-13) and (3-14), where as an example we solve the toy equation for the case when the initial conditions are $u(t = 0) = \epsilon$ with $\epsilon = 0.2$. Figure(3-13) has plots of u for different times t_i, where t_i is the ith time that the the shock gets to $x = 1$. As we see the rare fraction expands each time more until finally its slowest boundary will be catched by the shock (since the shock has traveled through the interval $[-1, 1]$ one more time). By this time we have $u < \epsilon$ for all x. u will continue to decrease in time. In figure (3-14) the slowest characteristic of rare fraction is plotted in dashed lines, the characteristic that form the boundary of the region absorbed by the shock in solid lines and the shocks in dotted (or thicker) lines.

This situation corresponds to solving (3.80)-(3.81)-(3.82) with initial conditions $\alpha = 1$. Contacts

51

followed by rare fractions start at $(\phi, \tau) = (2k+1, 0)$. The rare fractions spread and eventually their left boundaries will be caught by the contacts. From this time on, characteristics will run into the contacts from ahead carrying some value of α say $\alpha = \alpha_0$ and at the same time, new characteristics leave the contacts tangentially from behind with $\alpha = -0.5\alpha_0$. As time goes on α will keep decreasing (see figures (3-15) and (3-16)).

Finally, let's remember that (3.64) is a conservation law. For example u might represent some density and the quantity we are really interested is the mass in any interval of space $[a, b]$ contained in $[-1, 1]$, $M(a, b) = \int_a^b u(x, t)dx$. Then if we define $\Gamma(a, b) = \int_a^b \alpha(\phi = x - t)dx$, we have

$$M(a, b) = \epsilon\Gamma(a, b) + O(\epsilon^2) \tag{3.154}$$

for any a and b in $[-1, 1]$ and t of order ϵ^{-2}. In this sense this approximation is as valid as for example the approximation for the closed tube (equations (3.45)-(3.46)), where from a genuinely nonlinear system, we derived an equation with a convex flux. Thinking in these terms, the validity of the approximation of the toy model should not be unexpected (it would be surprising if it were not valid or it would not satisfy the usual jump conditions for discontinuities).

3.6 Understanding the asymptotic approximation

In this section we will explore how the asymptotic system (3.40)-(3.41)-(3.42) approximates the open tube system (3.1)-(3.2)-(3.3)-(3.4). The analysis is very similar to the one done for the Toy model in last section. We will then be brief.

3.6.1 On continuous solutions

In this subsection we will consider a solution of the open tube system with no discontinuities.

We perform our calculations in characteristics coordinates (ξ, η) and we use as dependent variables the Riemann invariants w and z (see section 3.4).

The equations of motion become:

$$z_\eta = 0 \quad \text{and} \quad x_\xi - \lambda_1 t_\xi = 0$$
$$w_\xi = 0 \qquad x_\eta - \lambda_2 t_\eta = 0 \tag{3.155}$$

We expand all the dependent variables in powers of ϵ

$$z = \epsilon z_1 + \epsilon^2 z_2 + \cdots \quad w = \epsilon w_1 + \epsilon^2 w_2 + \cdots$$
$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \quad t = t_0 + \epsilon t_1 + \epsilon^2 t_2 + \cdots \tag{3.156}$$

52

The initial conditions in (x, t) coordinates are $z_i(t = 0) = z_{i0}(x)$ and $w_i(t = 0) = w_{i0}(x)$. In (ξ, η) coordinates, together with appropriate initial and boundary conditions we have:

at the left boundary ($\eta \geq 0$)

$$x(\eta = \xi) = 0 \quad \text{and} \quad z_i(\eta = \xi) = w_i(\eta = \xi) \tag{3.157}$$

at $t = 0$ ($0 \leq \eta \leq 1$)

$$t(\xi = -\eta) = 0 \quad \text{and} \quad z_i(\xi = -\eta) = z_{i0}(x = -\xi)$$
$$x(\xi = -\eta) = \eta \qquad w_i(\xi = -\eta) = w_{i0}(x = \eta) \tag{3.158}$$

and at the right boundary

$$x(\eta = \xi + 2) = 1 \;, \quad w_1(\eta = \xi + 2) = -z_1(\eta = \xi + 2) \quad \text{and}$$
$$w_2(\eta = \xi + 2) = -z_2(\eta = \xi + 2) - z_1^2(\eta = \xi + 2) \tag{3.159}$$

The fact that this solution is continuous, at least for some time, and the initial-boundary conditions imply the following restrictions on w_i and z_i.

$$w_1 = \beta(\eta) \quad \text{and} \quad z_1 = \beta(\xi)$$
$$w_2 = -\frac{\beta^2(\eta)}{2} \qquad z_2 = -\frac{\beta^2(\xi)}{2} \tag{3.160}$$

where β is a function of one variable that we call ϕ (as always the quantity between brackets indicates where ϕ has to be evaluated), β is defined by

$$\beta(\phi) = \begin{cases} w_{10}(x = \phi) & \text{if } 0 < \phi < 1 \\ z_{10}(x = -\phi) & \text{if } -1 < \phi < 0 \end{cases} \tag{3.161}$$

and the antiperiodicity condition $\beta(\phi - 1) + \beta(\phi + 1) = 0$.

Our goal is to calculate the time t^* at which the 1-characteristic that starts at $(x, t) = (1, 0)$ gets back to $x = 1$ after three reflections.

We consider the regions determined by $\eta = 1$, $\xi = 1$, $\eta = 3$, $\xi = 3$ and the boundaries of our domain $x = 0$, $x = 1$ and $t = 0$.

We first find the general solution of (3.155). This is done in appendix C. We calculate our particular solution applying the initial and boundary conditions region by region. As a result we

53

get:

$$
\begin{aligned}
t^* \ &= \ t(\eta = 5, \xi = 3) = 4 + \\
&\epsilon^2 \left(\frac{1}{8} - \frac{3}{4} c^{(1)} + \frac{5}{8} \left(c^{(1)} \right)^2 - \frac{c^{(2)}}{4} \right) \int_{-1}^{1} \beta^2(\phi) d\phi + \\
&\epsilon^2 \left(\frac{9}{4} + \frac{5}{4} \left(c^{(1)} \right)^2 + \frac{5}{2} c^{(1)} - \frac{c^{(2)}}{2} \right) \beta^2(1) + O(\epsilon^3)
\end{aligned}
\tag{3.162}
$$

where $c^{(i)}$ is the ith derivative of the speed of sound c with respect to the density ρ evaluated at $\rho = 1$.

We can then say that the average velocity of the characteristics is $\frac{4}{t^*}$. By noticing that $\beta(\phi) = 2\sigma(\phi)$ (see equation(3.55)) and remembering the relationship between the sound speed and the pressure, we confirm that the asymptotic approximation gives us the right speed of characteristics.

Figure(3-2) shows a typical example.

3.6.2 Following a shock

In this subsection we will analyze a solution with one discontinuity and for convenience we assume that the discontinuity is a 2-shock that gets to $x = 1$ at $t = 0$ (as a consequence a 1-rare fraction is born there).

The initial conditions are $z(t = 0) = \epsilon z_{10}(x)$ and $w(t = 0) = \epsilon w_{10}(x)$. The fact that the only discontinuity is a 2-shock at $x = 1$ implies $z_{10}(0) = w_{10}(0)$ and $z_{10}(1) > -w_{10}(1)$.

As in the previous calculations, we use characteristic coordinates. The equations to be solve are (3.155). We again expand x, t, z and w in powers of ϵ. For convenience we define β as in (3.161).

The initial-boundary conditions are:
at the left boundary or closed end $(\eta \geq 0)$

$$
x(\eta = \xi) = 0 \quad \text{and} \quad z_i(\eta = \xi) = w_i(\eta = \xi)
\tag{3.163}
$$

at $t = 0$ $(0 \leq \eta \leq 1)$

$$
\begin{aligned}
t(\xi = -\eta) = 0 \quad &\text{and} \quad x(\xi = -\eta) = \eta \\
z_1(\xi = -\eta) = \beta(\xi) \quad &\qquad w_1(\xi = -\eta) = \beta(\eta) \\
z_2(\xi = -\eta) = 0 \quad &\qquad w_2(\xi = -\eta) = 0
\end{aligned}
\tag{3.164}
$$

54

at the right end and $t = 0$ $(1 \leq \eta \leq 1 + a)$

$$t(\xi = -1) = 0 \quad \text{and} \quad x(\xi = -1) = 1$$
$$w_1 = \beta(1) + 1 - \eta \qquad w_2 = \frac{\beta^2(-1)}{a}(1 - \eta) \tag{3.165}$$

and at the right boundary or open end $(\eta \geq 1 + a)$

$$x(\eta = \xi + 2 + a) = 1 \ , \quad w_1(\eta = \xi + 2 + a) = -z_1(\eta = \xi + 2 + a) \quad \text{and}$$
$$w_2(\eta = \xi + 2 + a) = -z_2(\eta = \xi + 2 + a) - z_1^2(\eta = \xi + 2 + a) \tag{3.166}$$

where for notation convenience we have defined $a = \beta(-1) + \beta(1)$.

The characteristics that are the boundaries of the rare fraction born at $(x, t) = (1, 0)$, its reflections going back and forward in the tube twice and the boundaries $(x = 0$, $x = 1$ and $t = 0)$ determine twelve regions. These regions are plotted in figure(3-17). We have the general solution of (3.155) in appendix C. We now need to apply the initial and boundary conditions region by region.

The rare fraction expands in region 2. It gets reflected at the closed end and continues to expand (region 5). After a reflection at the open end, it starts to compress (region 8) and continues to do so in region 11 after an other reflection at the closed end (see figure(3-17)).

We first want to check whether a new shock if form or not. A simple calculation shows that if there is a shock, it is a 2-shock that starts in region 11 at $(\eta, \xi) = (\eta^*, \xi^*)$ defined as the pair (η, ξ) that minimizes t subject to the restriction $x_\xi(\eta, \xi) = t_\xi(\eta, \xi) = 0$.

We calculate η^* and ξ^* by expanding them in powers of ϵ and using the expression for x and t in region 11. Under the assumption that

$$\frac{9}{4} + \frac{5}{2}c^{(1)} + \frac{5}{4}\left(c^{(1)}\right)^2 - \frac{1}{2}c^{(2)} > 0 \tag{3.167}$$

we find that

$$\xi^* = 3 + a \quad \text{and} \quad \eta^* = 5 + 2a - \epsilon k \beta(1) + O(\epsilon^2) \tag{3.168}$$

where we have defined $k = 2\frac{\left(9 + 10c^{(1)} + 5\left(c^{(1)}\right)^2 - 2c^{(2)}\right)}{\left(1 + c^{(1)}\right)}$.

Case 1: $\beta(1) < 0$.

This case is analogous to the **Case 1** of the toy system.

When $\beta(1) < 0$ equation (3.168) tell us that there is no shock formation since $\eta^* > 5 + 2a$ (outside region 11).

Let us go back to the asymptotic approximation (3.40)-(3.41)-(3.42). The initial conditions for σ are $\sigma_0(\phi) = \sigma(\tau = 0) = \frac{\beta(\phi)}{2}$. Then we have that $\sigma_0(1^+) < \sigma_0(1^-) < 0$ which implies that we

55

have a rare fraction at $(\phi, \tau) = (2n + 1, 0)$ for all integers n. This agrees with the fact that there is no shock formation. The wave has expanded more in regions 2 and 5 than it has compressed in regions 8 and 11. From the formulas for x and t in region 11 we can calculate what we have called the average velocity of the characteristics that form the boundaries of the rare fraction (region 2) and its reflections. If we name avs the average velocity of the slowest set of characteristics avf the average velocity of the fastest set of characteristics we have:

$$avf \;=\; \frac{4}{t(\eta = 5 + 2a, \xi = 3 + a)} = 1 -$$
$$\epsilon^2 \frac{\left(1 - 6c^{(1)} + 5\left(c^{(1)}\right)^2 - 2c^{(2)}\right)}{32} \int_{-1}^{1} \beta^2(\phi)d\phi \;-$$
$$\epsilon^2 \frac{\left(9 + 10c^{(1)} + 5\left(c^{(1)}\right)^2 - 2c^{(2)}\right)}{16} \beta^2(1) + O(\epsilon^3) \tag{3.169}$$

and

$$avs \;=\; \frac{3 + x(\eta = 5 + 2a, \xi = 3 + 2a)}{t(\eta = 5 + 2a, \xi = 3 + 2a)} = 1 -$$
$$\epsilon^2 \frac{\left(1 - 6c^{(1)} + 5\left(c^{(1)}\right)^2 - 2c^2(2)\right)}{32} \int_{-1}^{1} \beta^2(\phi)d\phi \;-$$
$$\epsilon^2 \frac{\left(9 + 10c^{(1)} + 5\left(c^{(1)}\right)^2 - 2c^2(2)\right)}{16} \beta^2(-1) + O(\epsilon^3) \tag{3.170}$$

which are the same velocities than the one obtained by the asymptotic approximation (as can be easily checked remembering the relation between p and c and the relation between β and σ).

Case 2: $\beta(1) > 0$.

In this case a 2-shock is born in the boundary between region 11 and region 10 at a distance of order ϵ to the open end.

We now follow the shock. We use ξ from region 11 as independent variable. We denote by $s(\xi)$ the position of the shock when it intersects the characteristic ξ, $t(\xi)$ the time when this intersection occurs, by $\xi^{(10)}(\xi)$, $\eta^{(10)}(\xi)$, $w^{(10)}(\xi)$, $z^{(10)}(\xi)$, $x^{(10)}(\xi)$ and $t^{(10)}(\xi)$ the values of the 2-characteristic, the 1-characteristic (η), w, z, x and t ahead of the shock (in region 10). Similarly (replacing the superscript 10 by 11) we denote the values of the different variables behind the shock (in region 11).

The Rankine Hugonoit conditions become

$$\frac{s_\xi}{t_\xi} = \frac{[\rho v]}{[\rho]} = \frac{[\rho v^2 + p]}{[\rho v]} \tag{3.171}$$

where a quantity enclosed between brackets means the difference between its value ahead of the

shock and its value behind the shock. We also have the following equations

$$s = x^{(10)}(\eta^{(10)}(\xi), \xi^{(10)}(\xi)) = x^{(11)}(\eta^{(11)}(\xi), \xi)$$

$$t = t^{(10)}(\eta^{(10)}(\xi), \xi^{(10)}(\xi)) = t^{(11)}(\eta^{(11)}(\xi), \xi) \quad (3.172)$$

We expand the dependent variables s, t, $\xi^{(10)}(\xi)$, $\eta^{(10)}(\xi)$ and $\eta^{(11)}(\xi)$ in powers of ϵ

$$s = 1 + \epsilon s_1 + \epsilon^2 s_2 \qquad t = 4 + \epsilon t_1 + \epsilon^2 t_2$$

$$\xi^{(10)} = 3 + a + \epsilon \xi_1^{(10)} + \epsilon^2 \xi_2^{(10)} \quad \eta^{(10)} = 5 + 2a + \epsilon \eta_1^{(10)} + \epsilon^2 \eta_2^{(10)}$$

$$\eta^{(11)} = 5 + 2a + \epsilon \eta_1^{(11)} + \epsilon^2 \eta_2^{(11)} \quad (3.173)$$

plug this anzats in equations (3.171) and (3.172) and solve in a similar way than for the toy model to obtain

$$\eta_1^{(10)} = \eta_1^{(11)} = 2t_1 = 2s_1 \qquad \xi_1^{(10)} = 0 \quad (3.174)$$

$$\xi_2^{(10)} = -\frac{\left(9 + 10c^{(1)} + 5\left(c^{(1)}\right)^2 - 2c^{(2)}\right)}{12}(\xi - a - 3)^2 \quad (3.175)$$

$$\eta_2^{(10)} = \eta_2^{(11)} = -\frac{\left(9 + 10c^{(1)} + 5\left(c^{(1)}\right)^2 - 2c^{(2)}\right)}{12}(\xi - a - 3 - 2\beta(1))$$

$$(\xi - a - 3 - 6\beta(1)) + \frac{(1 - c^{(1)})}{2}$$

$$(\beta(1) - \beta(-1))s_1 + 2s_2 \quad (3.176)$$

$$t_2 = -\frac{\left(9 + 10c^{(1)} + 5\left(c^{(1)}\right)^2 - 2c^{(2)}\right)}{12}(\xi - a - 3 - \beta(1))$$

$$(\xi - a - 3 - 3\beta(1)) + \frac{\left(1 - 6c^{(1)} + 5\left(c^{(1)}\right)^2 - 2c^{(2)}\right)}{8}$$

$$\int_{-1}^{1} \beta^2(\phi)d\phi + \frac{(1 - c^{(1)})}{2}\beta(1)s_1 + s_2 \quad (3.177)$$

with

$$s_1 = \frac{\left(9 + 10c^{(1)} + 5\left(c^{(1)}\right)^2 - 2c^{(2)}\right)}{(1 + c^{(1)})}(\frac{2}{3}(\xi - 3 - a) - \beta(1)) \quad (3.178)$$

We define $\bar{\xi}$ as the solution of $s(\bar{\xi}) = 1$ (i.e. the characteristic in region 11 that intersects the

shock when it gets to the open end). Equation (3.178) implies

$$\bar{\xi} = 3 + a + \frac{3}{2}\beta(1) \tag{3.179}$$

This last formula is valid only when $\bar{\xi}$ belongs to region 11. Then we need to consider two different cases.

Case 2.1: $0 < \beta(1) < 2\beta(-1)$

This case is analogous to **Case 2.1** of the toy system.

In this case equation (3.179) is valid. Not all the region 11 is absorbed by the shock. Let $\bar{t} = t(\bar{\xi})$ (the time when the shock gets to the open end). By equations (3.174) and (3.177)

$$
\begin{aligned}
\bar{t} &= 4 + \epsilon^2 \frac{\left(1 - 6c^{(1)} + 5\left(c^{(1)}\right)^2 - 2c^{(2)}\right)}{8} \int_{-1}^{1} \beta^2(\phi)d\phi + \\
&\quad \epsilon^2 \frac{\left(9 + 10c^{(1)} + 5\left(c^{(1)}\right)^2 - 2c^{(2)}\right)}{16} \beta^2(1)
\end{aligned}
\tag{3.180}
$$

Identifying this shock with the one we started with, we can say that it has an average speed of $\frac{4}{\bar{t}}$. The value of z behind the shock (in region 11) at $t = \bar{t}$ is $z(x = 1^-, t = \bar{t}) = \epsilon\frac{\beta(1)}{2} + O(\epsilon^2)$.

If we now go back to the asymptotic approximation, the initial condition has a discontinuity at $\phi = 2n + 1$ (for all n integer) with the property that $0 < \sigma(1^-) < -2\sigma(1^+)$, which means that we have a right contact following a rare fraction. The velocity of the contact in (ϕ, τ) coordinates is

$$velocity\ of\ the\ contact = \frac{3}{4}a\sigma^2(1^-) + \frac{b}{2}\int_{-1}^{1}\sigma^2(\phi)d\phi \tag{3.181}$$

where a and b are the constants defined in (3.43) and (3.44). The value of σ to the right of the contact that starts at $(\phi, \tau) = (3, 0)$ is $\sigma(3^+, 0) = -\sigma(1^+, 0) = \frac{\sigma(1^-, 0)}{2}$. It can now be easily checked now that both the average speed of the discontinuity and the value of the 2-Riemann invariant z behind the shock (region 11) are well approximated by the asymptotic system. As in the toy model, the regions 2, 5, 8 and 11 are well approximated and correspond to the rare fractions of σ.

As in the case for the toy model, at $t = \bar{t}$ we do not know if we are again in this case or in case 2.2 to be described next. This is because σ can only have rare fractions at $\tau = 0$ and the fact that we can go from case 2.1 to case 2.2 corresponds to the fact that contacts can become genuine shock and vice-versa for σ.

Figure(3-17) correspond to this case.

Case 2.2: $-2\beta(1) < 2\beta(-1) < \beta(1)$.

This case is analogous to the **Case 2.2** for the toy model.

Now all the region 11 is absorbed by the shock. Before getting to the open end it will get to the boundary between regions 11 and 12. This happens when $\xi = 3 + 2a$. Equations (3.174)-(3.178)

allow as to calculate $\eta^{(10)}(3 + 2a)$, $\xi^{(10)}(3 + 2a)$ and $\eta^{(12)}(3 + 2a)$. We then solve equations (3.171)-(3.172) but replacing region 11 by region 12 and we use as independent variable $\frac{(t-\tilde{t})}{\epsilon}$, where $\tilde{t}$ is the time when the shock gets to region 12. These equations, as before, are solved by expanding in powers of ϵ. If we define $\bar{t}$ the time at which the shock gets to the right boundary we obtain:

$$\bar{t} = 4 + \epsilon^2 \frac{\left(1 - 6c^{(1)} + 5\left(c^{(1)}\right)^2 - 2c^{(2)}\right)}{8} \int_{-1}^{1} \beta^2(\phi) d\phi +$$

$$\epsilon^2 \frac{\left(9 + 10c^{(1)} + 5\left(c^{(1)}\right)^2 - 2c^{(2)}\right)}{12} (\beta^2(1) - \beta(1)\beta(-1) + \beta^2(-1)) \qquad (3.182)$$

Going back to the asymptotic approximation, we see that now the initial condition for σ has the property $-2\sigma(1^-) < -2\sigma(1^+) < \sigma(1^-)$ which means that we have genuine shocks. This is consistent with the fact that all region 11 disappears and again the speed of the shocks of σ is the correct one (i.e. the same with an error of order ϵ^3 as the average speed of the discontinuity, $\frac{4}{\bar{t}}$).

3.7 Concluding remarks

We now understand how the asymptotic equation approximates the open tube system and its validity for times of order ϵ^{-2}. In particular we can say that the waves will decay like $t^{-\frac{1}{2}}$ (different from the closed end tube where the waves decay like t^{-1}). And the solutions will tend in general to one of the shock rare fraction structures that corresponds to the **Case 2.1**. The situation is completely analogous as the toy model, all the remarks done in that section are still valid.

As explain for the toy model, it should not be a surprise that the asymptotic approximation is valid even for discontinuous solutions and that the discontinuities satisfy the usual jump conditions. In the sense of calculating the amount of mass and momentum in any section of the tube this approximation is as valid as the one for the closed tube.

To derive the open end boundary condition we have assumed that R, the ration between the radius of the tube and its length is much smaller than ϵ^2. It is well known that the decay of waves due to radiation at the open end takes place in a time scale of R^{-2} ([24] and [25]), in this paper we have shown that the decay of waves due to nonlinearities takes place in a time scale of ϵ^{-2}, then we expect that the behavior of the gas will still be like the one described in this paper for the less restricted case when $R \ll \epsilon$ although a simple formula for the open end boundary condition can not be derived.
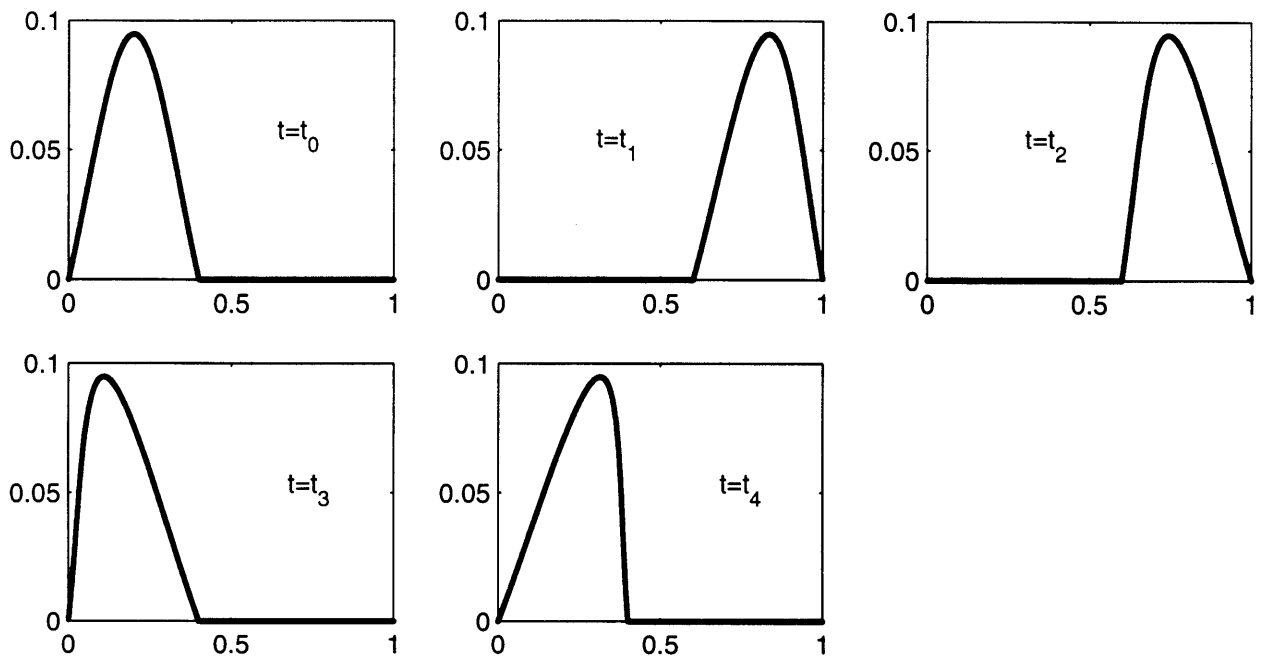
Figure 3-1: Evolution of a wave reflecting once at each end in a close tube. At $t = t_0$, $t = t_1$ and $t = t_4$ the plots correspond to the 2-Riemman invariant z. At $t = t_2$ and $t = t_3$ the plots correspond to the 1-Riemman invariant w.
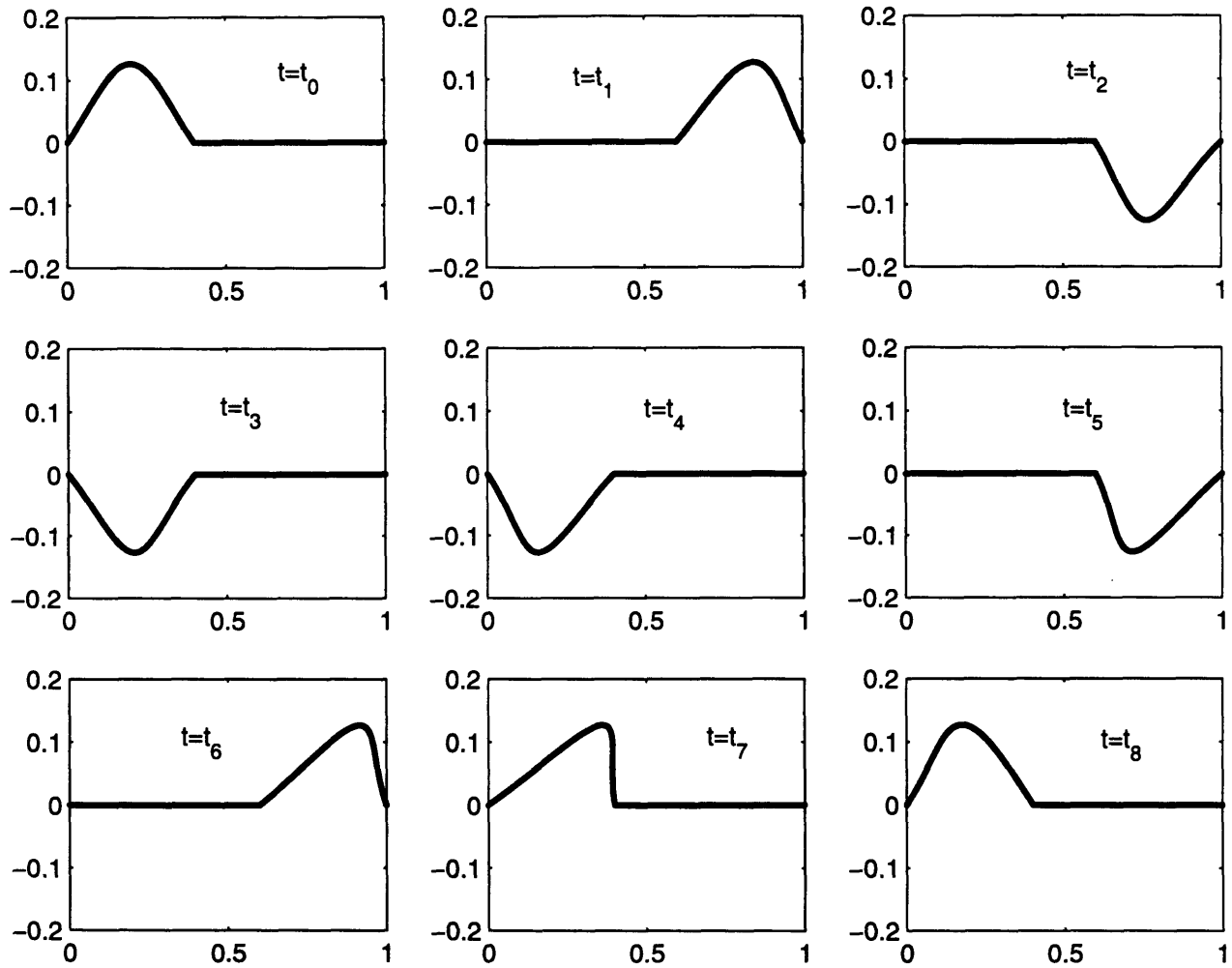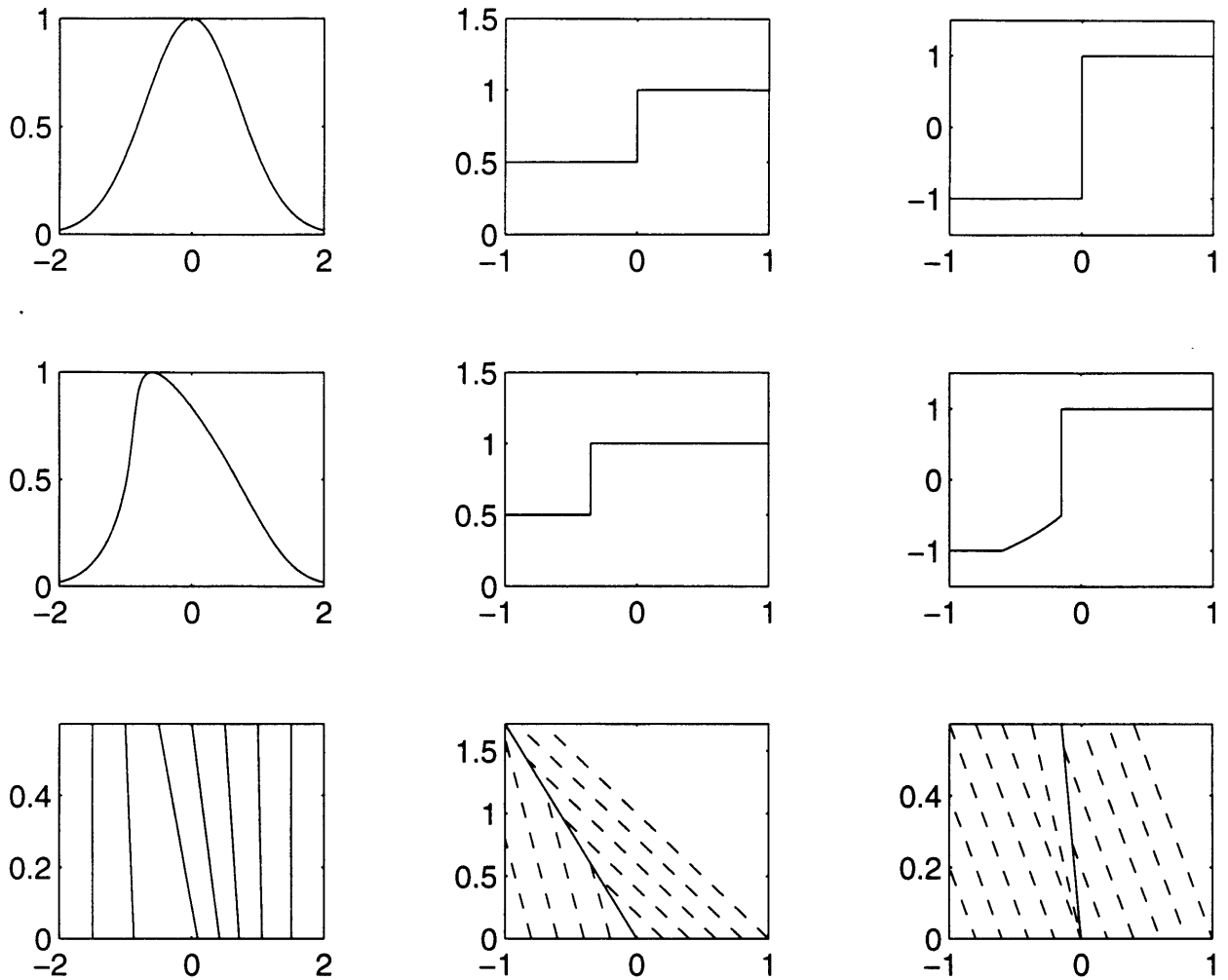
Figure 3-2: Evolution of a wave reflecting twice at each end in a tube with the right end open. At $t = t_0$, $t = t_1$, $t = t_4$, $t = t_5$ and $t = t_8$ the plots correspond to the 2-Riemman invariant z. At $t = t_2$, $t = t_3$, $t = t_6$ and $t = t_7$ the plots correspond to the 1-Riemman invariant w.

Figure 3-3: Three solutions of equation (3.81). the first row corresponds to the initial conditions. The second row to $\alpha(\tau = 0.6)$ and the third row to a plot of the characteristics and discontinuities. The first column is a continous solution, the second column is a shock and the third column a rarefraction followed by a contact.
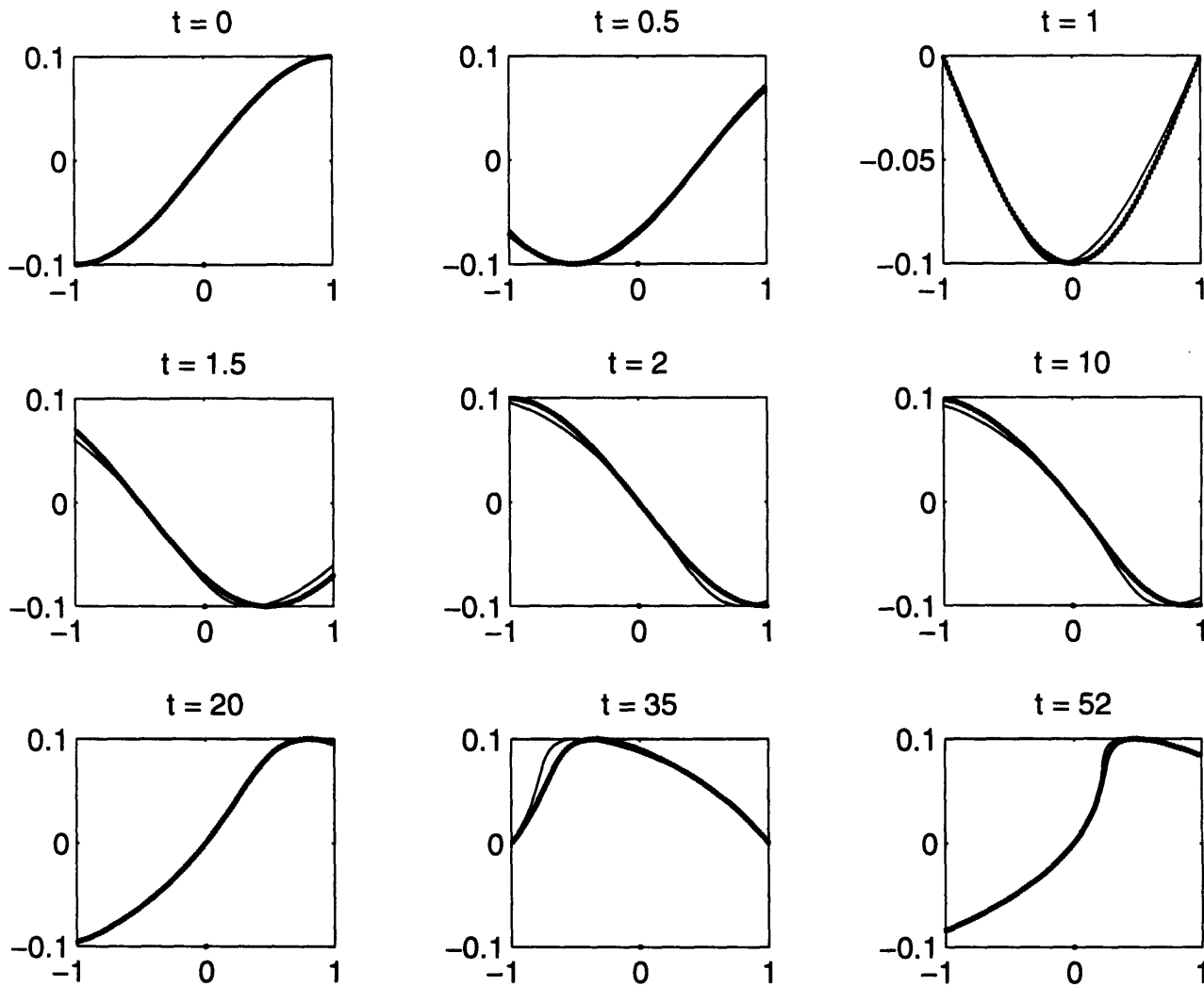
Figure 3-4: Plot of $u(x, t)$ for different t with initial conditions $u(x, t = 0) = 0.1\sin(\frac{\pi}{2}x)$. Solid lines correspond to the real solution and dotted lines (thicker lines) to the asymptotic approximation.
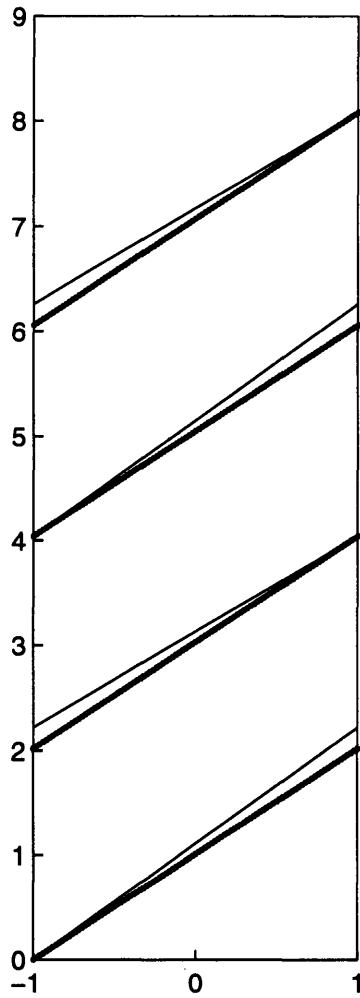
Figure 3-5: Plot of the characteristic that starts at $(x,t) = (-1,0)$ and its reflections when $u(x,t = 0) = 0.1\sin(\frac{\pi}{2}x)$. Solid lines correspond to the real solution and dotted lines (thicker lines) to the asymptotic approximation.
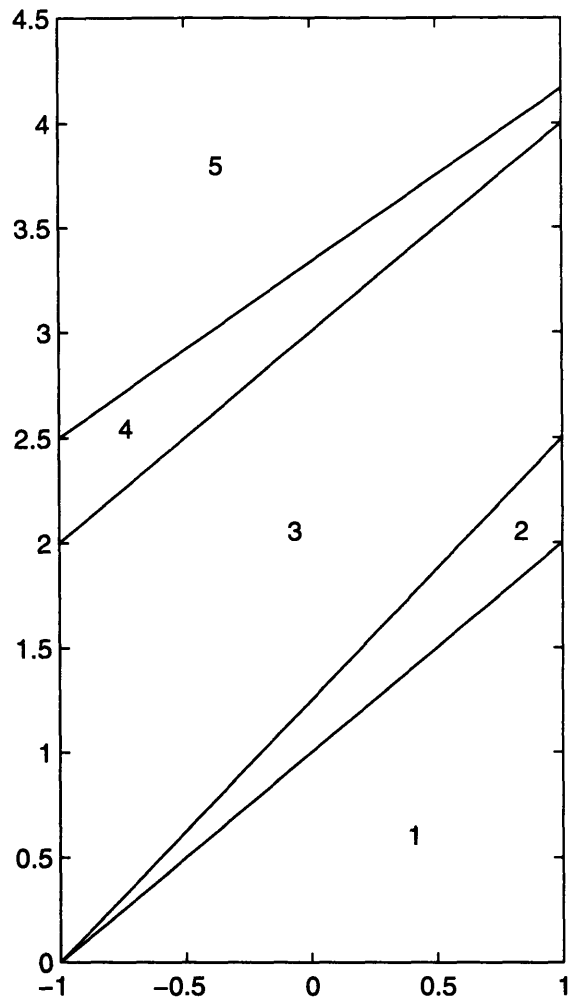
Figure 3-6: Plot of the different five regions described in section 6.3.2 (see equation (3.103)).
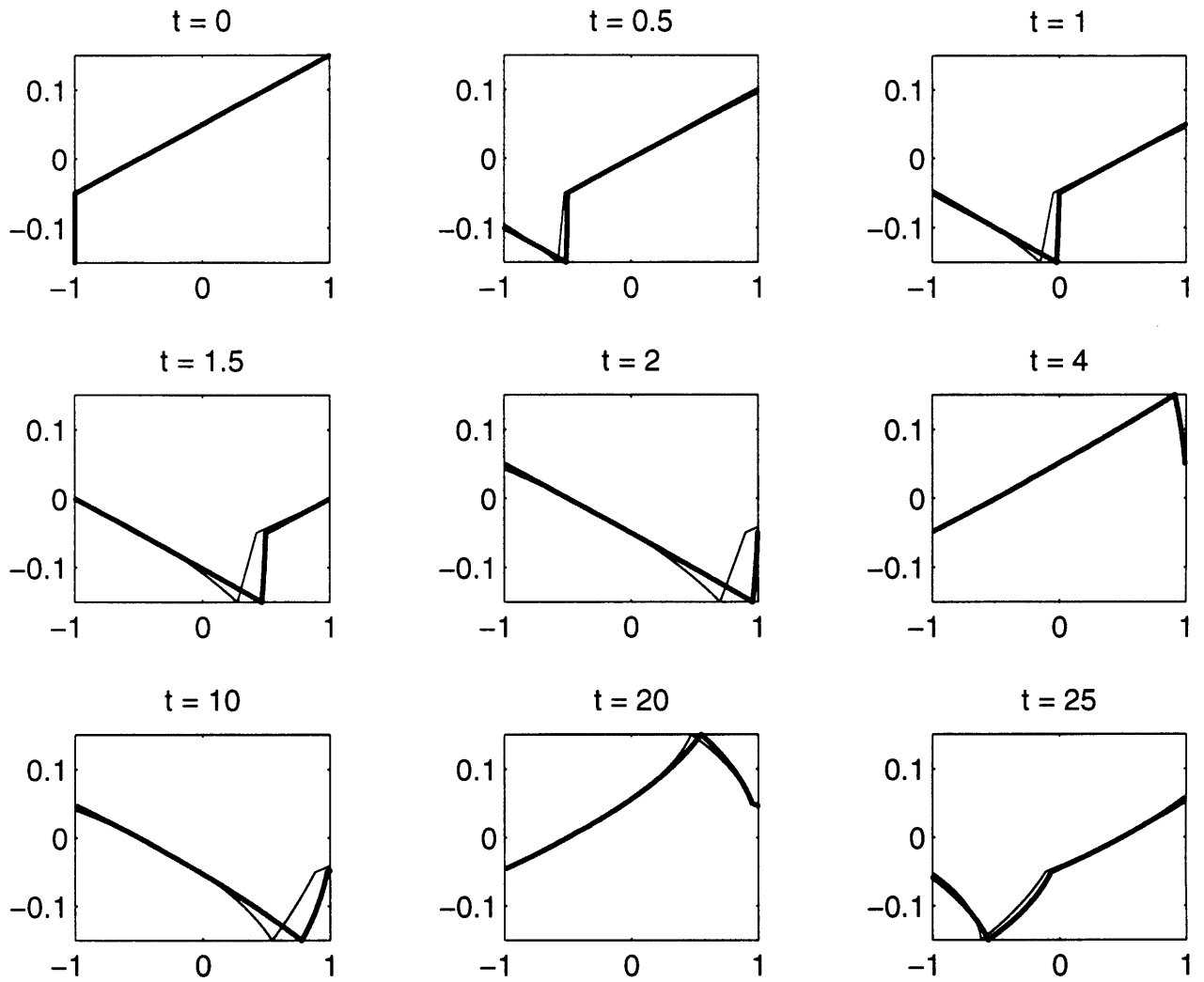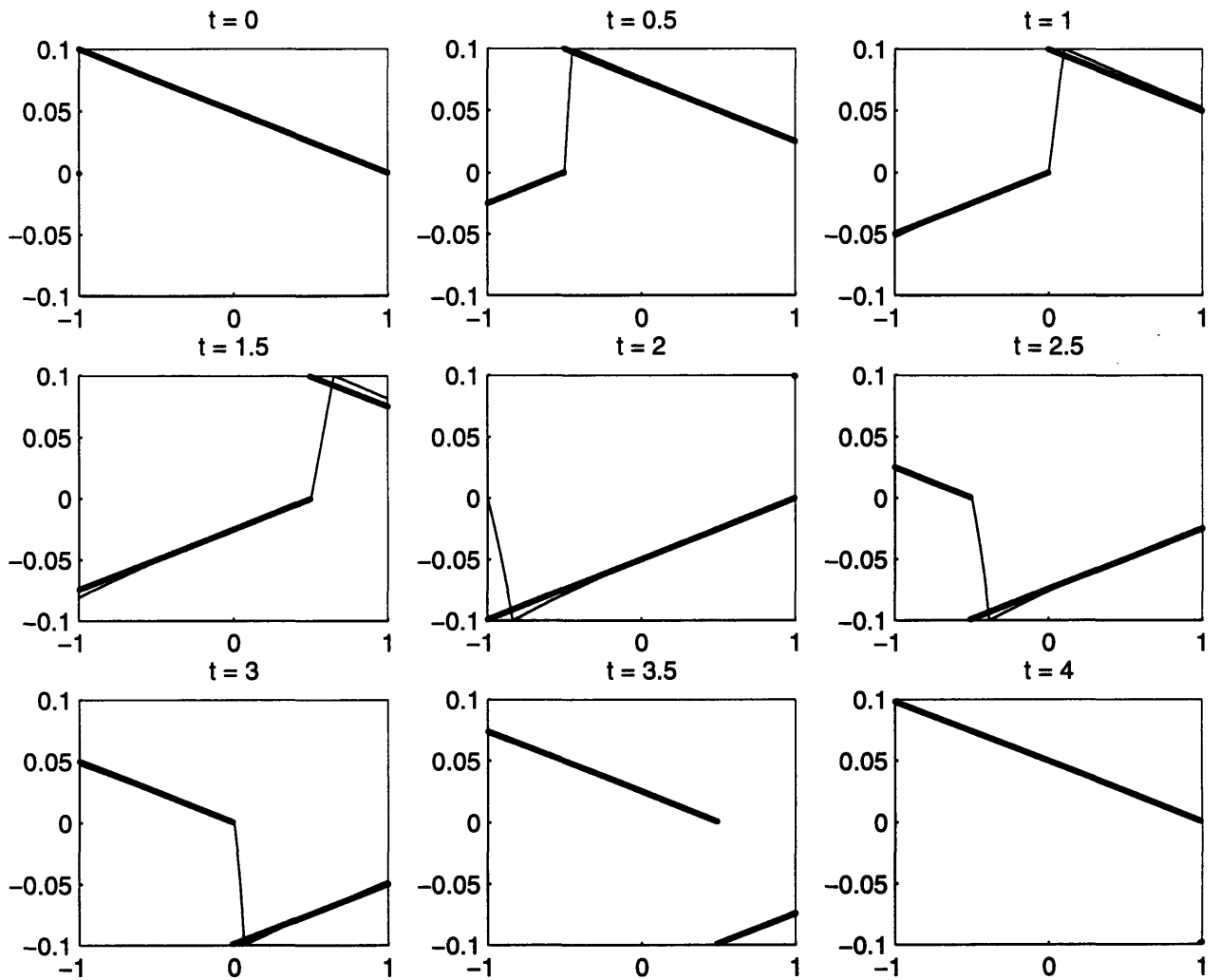
Figure 3-7: Plot of $u(x, t)$ for different t with initial conditions $u(x, t = 0) = 0.1(x + 0.5)$. Solid lines correspond to the real solution and dotted (thicker) lines to the asymptotic approximation.
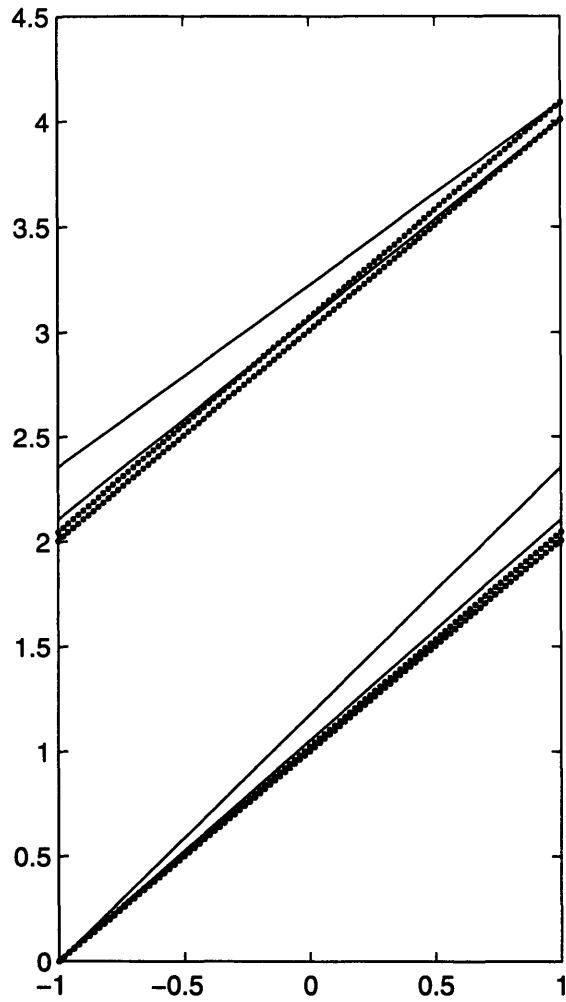
Figure 3-8: Plot of the characteristics that form the boundary of the rarefraction starting at $(x, t) = (-1, 0)$ and its reflection, where the initial conditions are $u(x, t = 0) = 0.1(x + 0.5)$. Solid lines correspond to the real solution and dotted lines to the asymptotic approximation.
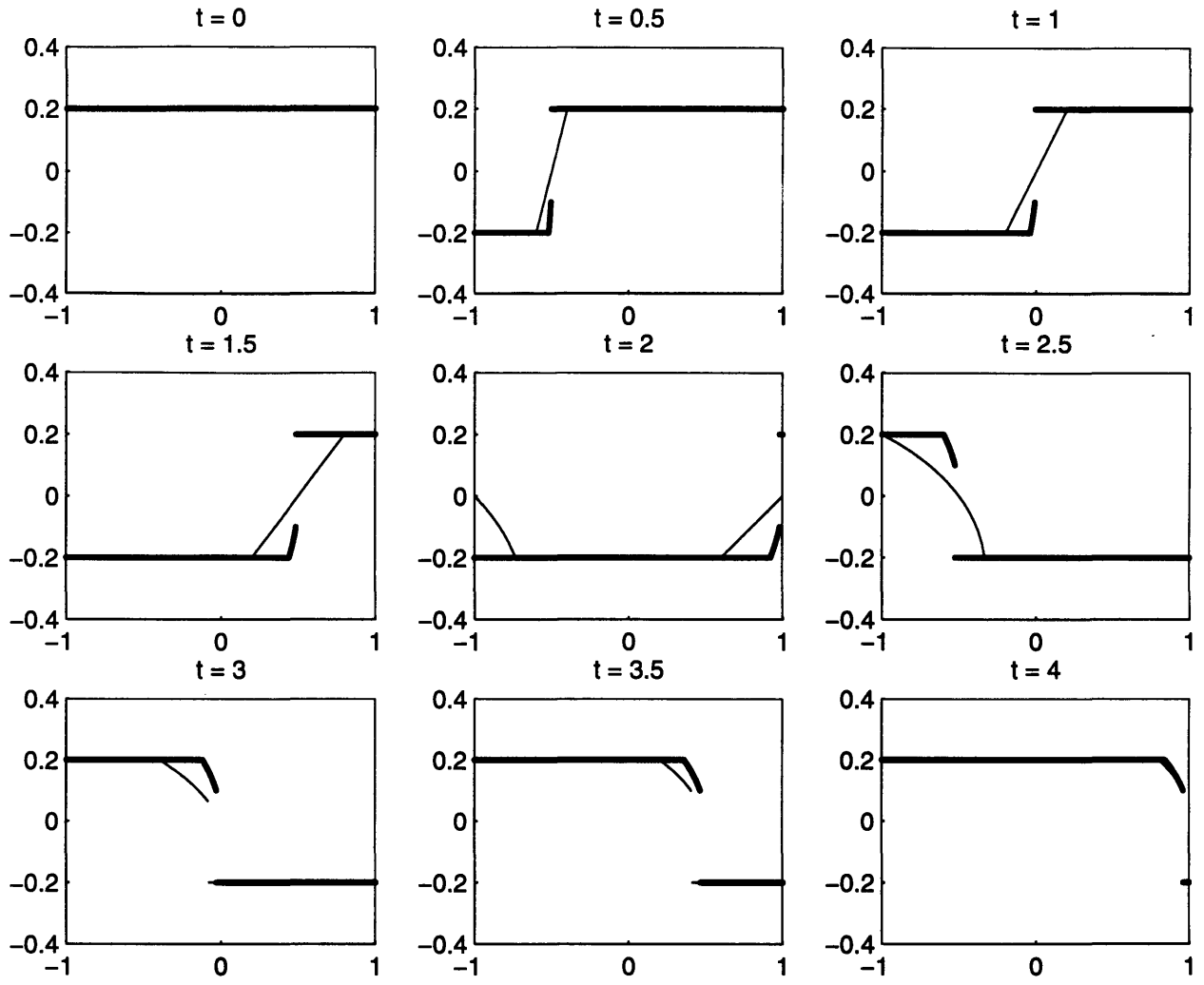
Figure 3-9: Plot of $u(x, t)$ for different t with initial conditions $u(x, t = 0) = 0.2$. Solid lines correspond to the real solution and dotted (thicker) lines to the asymptotic approximation.
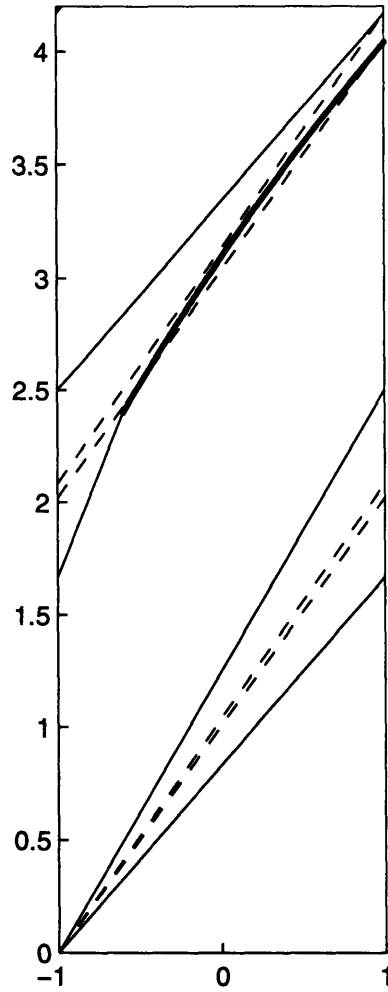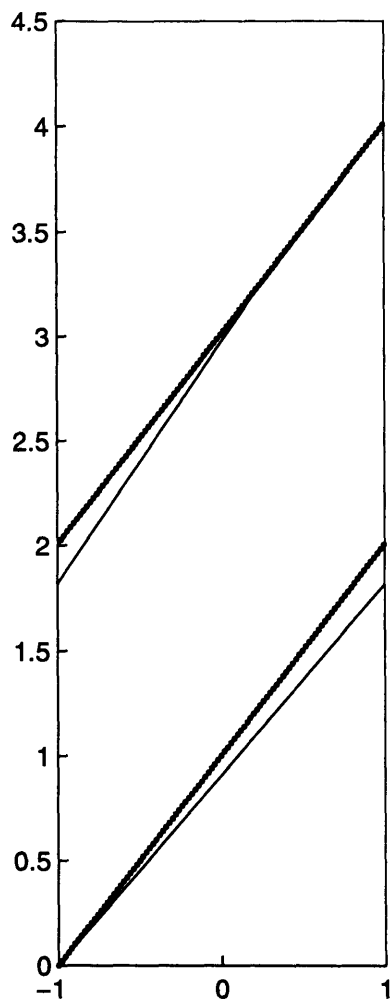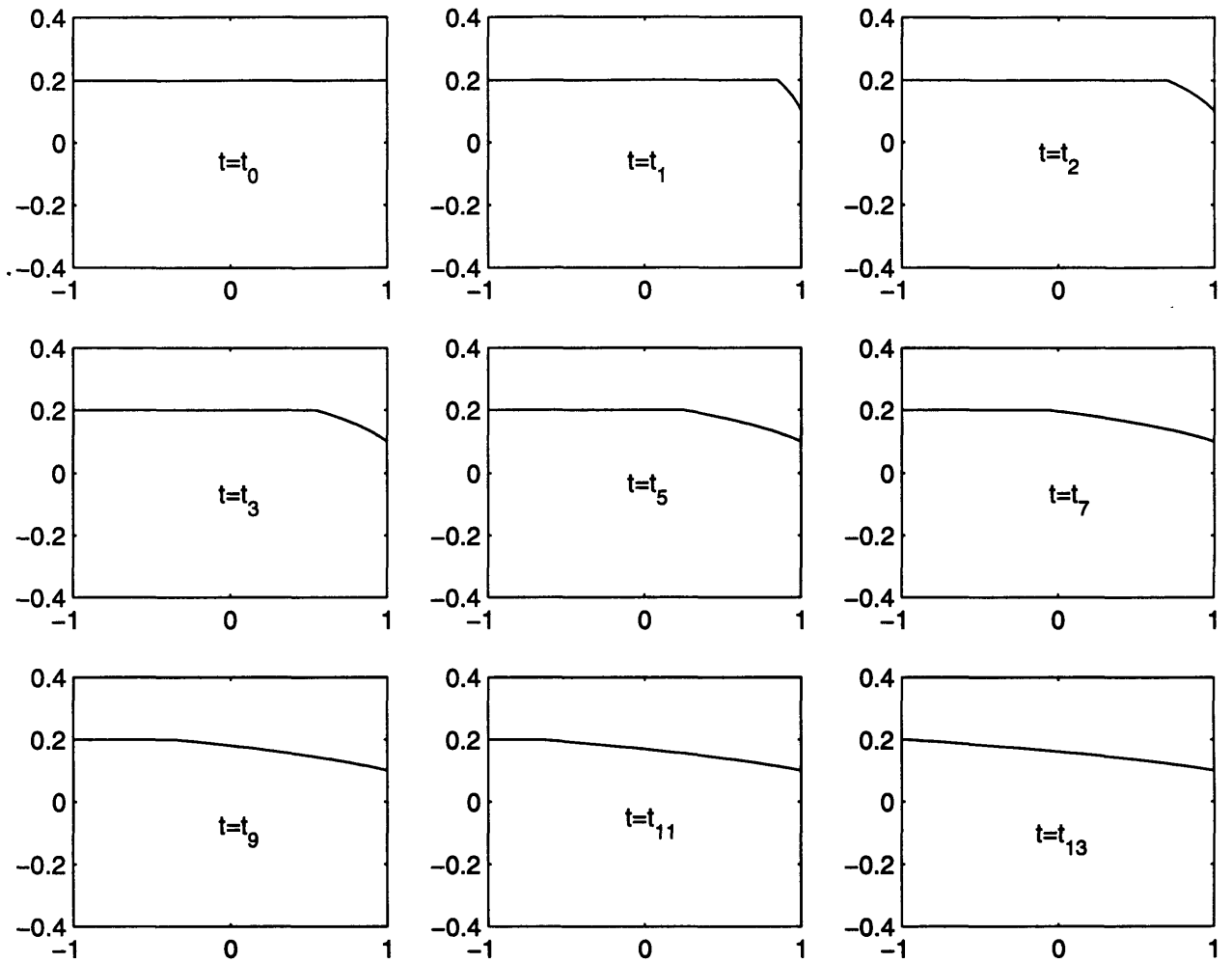
Figure 3-10: Plot of the characteristics that form the boundary of the rarefraction starting at $(x,t) = (-1,0)$ and its reflection, where the initial conditions are $u(x, t = 0) = 0.2$. Solid lines correspond to the real solution and dotted lines to the real shock and dashed lines to the asymptotic approximation.

Figure 3-11: Plot of $u(x,t)$ for different t with initial conditions $u(x,t=0) = 0.05(1-x)$. Solid lines correspond to the real solution and dotted (thicker) lines to the asymptotic approximation.

Figure 3-12: Plot of the characteristics that form the boundary of the rarefraction starting at $(x, t) = (-1, 0)$ and its reflection, where the initial conditions are $u(x, t = 0) = 0.05(x - 1)$. Solid lines correspond to the real solution and dotted lines to the asymptotic approximation.

Figure 3-13: Plots of u for different times when the initial conditions are $u(t = 0) = \epsilon$ with $\epsilon = 0.2$. The time t_i is the ith time that the the shock gets to $x = 1$.
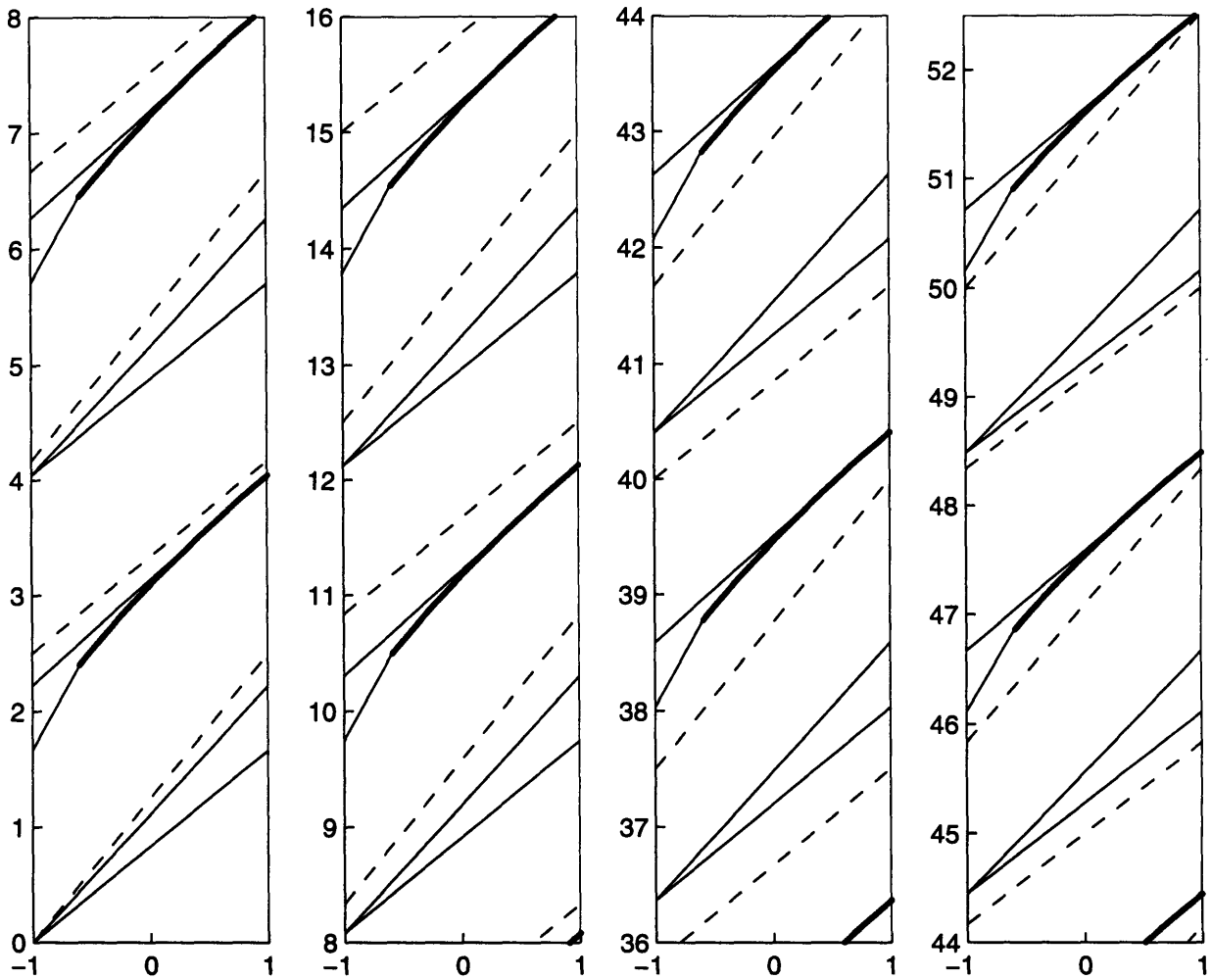
Figure 3-14: The characteristics that form the boundary of the regions absorved by the shocks are plotted with solid lines, the shocks with dotted (thicker) lines and the slowest boundary of the rarefraction with dashed lines. The inital conditions are $u(t = 0) = \epsilon$ with $\epsilon = 0.2$.
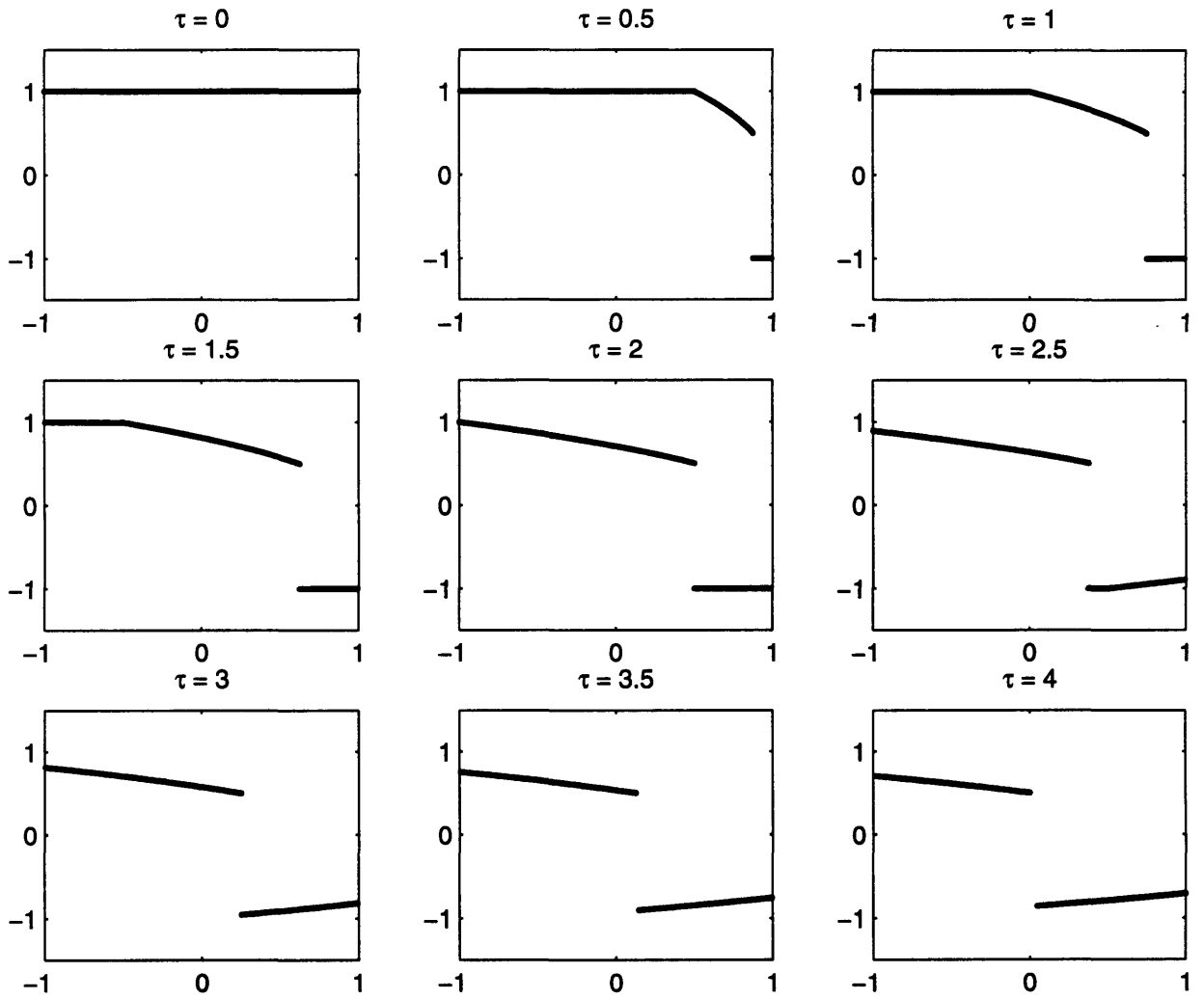
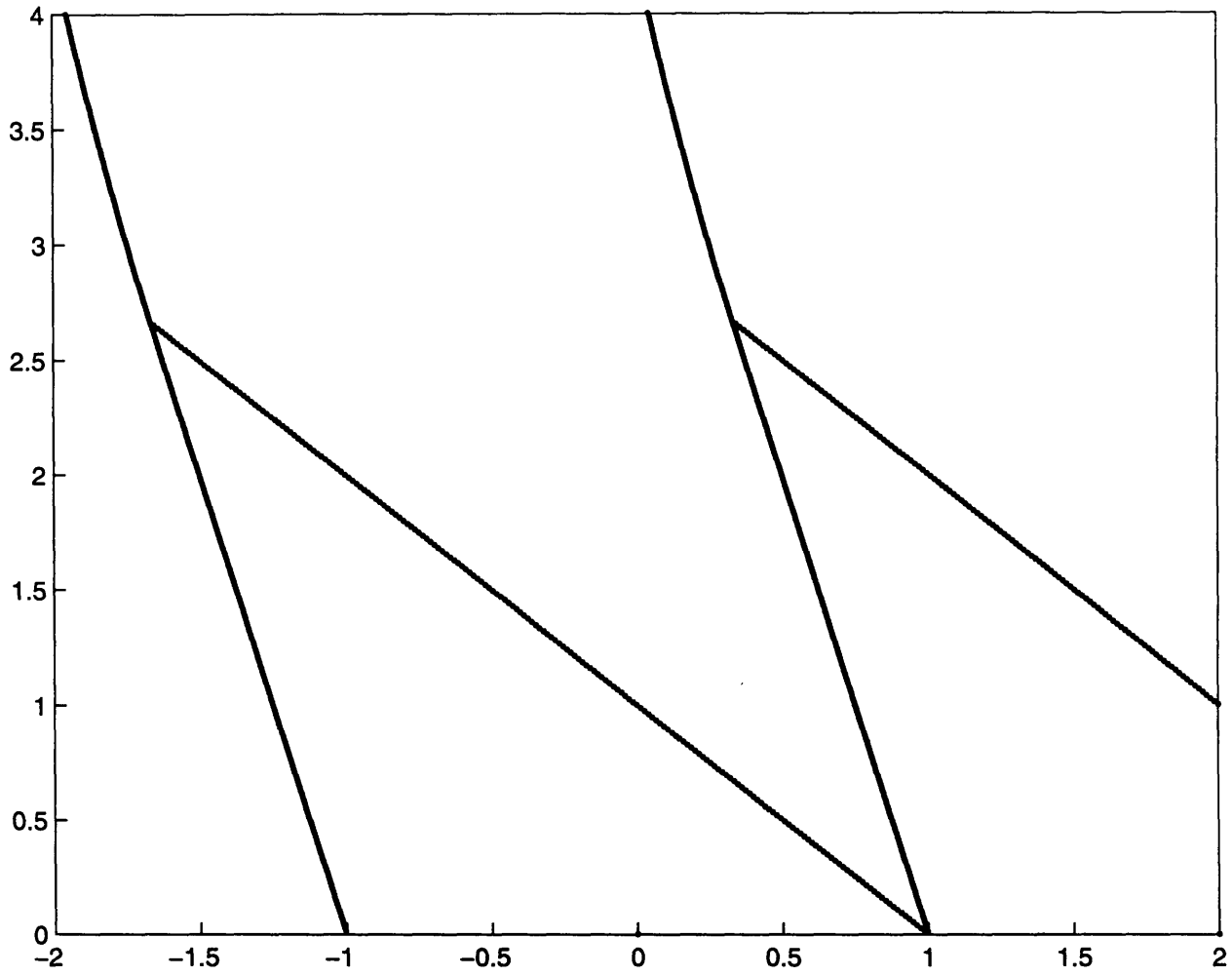Figure 3-15: Plots of α for differents τ. The initial conditions are $\alpha(\tau = 0) = 1$.

Figure 3-16: Plot of the boundary of the rarefractions and the contact discontinuity of α with $\alpha(\tau = 0) = 1$.
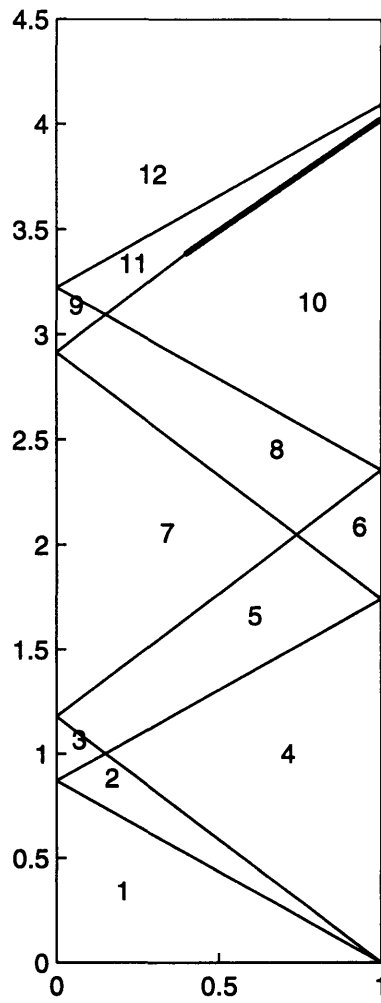
Figure 3-17: Plot of the different regions determined by a rerefraction and its reflection. A shock is born and it is plotted in dotted (thick) lines. The initial condition is $v = 0$ and $\rho = 1 + \epsilon$.

Appendix A

Numerical values of constants

These are some tipical numerical values of the constants in CGS taken at 1atm and 300K. The liquid is consider to be water and the gas air.

$$\rho_g = 10^{-3}, \quad \rho_l = 1, \quad a_g = 3.5 \ 10^4, \quad a_l = 1.5 \ 10^5, \quad \sigma = 70, \quad \eta_g = 2 \ 10^{-4},$$

$$\eta_l = 10^{-2}, \quad \lambda_g = 2 \ 10^3, \quad \lambda_l = 6 \ 10^4, \quad c_{vg} = 7 \ 10^6, \quad c_{pl} = c_{vl} = 4 \ 10^7,$$

$$\beta_l = 3 \ 10^{-4}, \quad \alpha_l = 5 \ 10^{-11}$$

where $\alpha_l = (1/\bar{\rho})(\partial \bar{\rho}/\partial \bar{p})$ (the partial deribative taken at constant $\bar{T}$) is the coefficient of isothermal compressibility. Then tipical values of the nondimensional parameters that appear in the equations are:

$$\epsilon = 7 \ 10^{-3}, \quad \nu_g = 2 \ 10^{-7}/R_e, \quad \mu_g = 2.5 \ 10^{-4}/R_e, \quad \nu_l = 10^{-5}/R_e,$$

$$\mu_l = 10^{-6}/R_e, \quad c_1 = 0.2, \quad c_2 = 2, \quad s = 5 \ 10^{-5}/R_e$$

Appendix B

Transition from Isothermal to Adiavatic Motion

In this appendix we calculate the prefactor b and then function u_0 from Equation (2.43). The equations to be solved (see Section 2.2.2, here we drop the hat on the variables) are

$$u_t = \frac{(x^2 f)_x}{x^2} \quad \text{with} \quad f(x=0) = f(x=1) = 0 \tag{B.1}$$

where

$$f = (-t)^{2/5} \frac{u_x}{\gamma u} + \left\{ \frac{1}{3\gamma} \frac{u_t(x=1)}{u(x=1)} + \frac{2(\gamma-1)}{5\gamma t} \right\} xu \tag{B.2}$$

under the asymptotic initial conditions

$$u \sim 1 \quad \text{as} \quad t \to -\infty \tag{B.3}$$

Then b and u_0 are defined as:

$$b = \lim_{t \to 0^-} u(x=1)(-t)^{\frac{6}{5}(\gamma-1)} \quad \text{and} \quad u_0(x) = \lim_{t \to 0^-} u(x,t) \tag{B.4}$$

$u_0(x)$ is defined for $x < 1$.

B.1 Initial Conditions

We first calculate more terms in the asymptotic expansion of u as $t \to -\infty$ in order to obtain a more accurate initial condition than Equation (B.3). This calculation, that has to be done under

the restriction that imposes the fixed amount of mass inside the bubble

$$
\begin{aligned}
u \quad \sim \quad & 1 + \frac{(\gamma - 1)}{5}(-t)^{-7/5}\left(x^2 - \frac{3}{5}\right) \\
& + \frac{(\gamma - 1)}{500}(27\gamma - 20)(-t)^{-14/5}\left(x^4 - \frac{3}{7}\right) \\
& - \frac{(\gamma - 1)}{750}(57\gamma - 22)(-t)^{-14/5}\left(x^2 - \frac{3}{5}\right)
\end{aligned}
\tag{B.5}
$$

B.2 Behavior as $t \to 0$

As $t \to 0$ we have $u(x = 1) \sim b(-t)^{-\frac{6}{5}(\gamma - 1)}$. Then there is a boundary layer at $x = 1$, where the diffusion term $(-t)^{2/5}\frac{u_{xx}}{\gamma u}$ is of the same order of u_t. We denote by β the thickness of the boundary. To estimate β we note that

$$
(-t)^{2/5}\frac{u_{xx}}{\gamma u} = 0\left((-t)^{2/5}\beta^{-2}\right) \quad \text{and} \quad u_t = 0\left((-t)^{\frac{(1-6\gamma)}{5}}\right)
$$

$$
\text{which implies} \quad \beta = 0\left((-t)^{\frac{(6\gamma + 1)}{10}}\right)
\tag{B.6}
$$

B.3 The Numerical Method

We will use a finite volume method to integrate Equation (B.1). Assume that at time $t = t_n$ we have the average of u in N disjoint subintervals that cover the interval $[0, 1]$ more precisely we have $v_i^n (i \leq i \leq N)$ and $x_i (0 \leq 1 \leq N)$ where

$$
v_i^n = \frac{\int_{x_{i-1}}^{x_i} x^2 u(x, t)\, dx}{\int_{x_{i-1}}^{x_i} x^2\, dx} \quad \text{and} \quad 0 = x_0 < x_1 < \cdots < x_N = 1
\tag{B.7}
$$

Assume also that we have $u_N^n = u(x = 1, t = t_n)$. We next explain how we compute Δt, v_i^{n+1} (the averages at $t = t_{n+1} = t_n + \Delta t$) and u_N^{n+1} (the value of $u(x = 1, t = t_{n+1})$).

B.3.1 Calculating the Averages

We first multiply Equation (B.1) by x^2, then we integrate the result over $[x_{i-1}, x_i] \times [t_n, t_{n+1}]$ and finally divide by the integral of x^2 over $[x_{i-1}, x_i]$ to get:

$$
v_i^{n+1} = v_i^n + \Delta t \frac{(x_i^2 g_i^n - x_{i-1}^2 g_{i-1}^n)}{\int_{x_{i-1}}^{x_i} x^2\, dx}
\tag{B.8}
$$

where

$$
g_i^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(x_i, t)\, dt
\tag{B.9}
$$

Formula (B.8) is used to compute v_i^{n+1}, but we need to approximate g_i^n first. This is done through the following steps:

1. We compute $u_{i-\frac{1}{2}}^n = u(x_{i-\frac{1}{2}}, t_n)$ from u_N^n and v_i^n (where $x_{i-\frac{1}{2}} = \frac{x_{i-1}+x_i}{2}$) by first approximating v_i^n as linear functions of $u_{i-\frac{1}{2}}^n$ and then inverting the system obtained.

   We approximate u linearly to get

   $$v_i^n = u_{i-\frac{1}{2}}^n - du_{i-\frac{1}{2}}^n x_{i-\frac{1}{2}} + du_{i-\frac{1}{2}}^n \frac{3}{4} \frac{x_i^4 - x_{i-1}^4}{x_i^3 - x_{i-1}^3} \tag{B.10}$$

   where $du_{i-\frac{1}{2}}^n = u_x(x_{i-\frac{1}{2}}, t_n)$ computed as $du_{i-\frac{1}{2}}^n = \frac{u_i^n - u_{i-1}^n}{x_i - x_{i-1}}$

   In the next step we explain how to compute u_i^n as functions of $u_{i-\frac{1}{2}}^n$ and u_N^n. Then the equations (B.10) form a tridiagonal linear system easy to invert.

2. We compute $u_i^n = u(x_i, t_m)$ for $1 \le i \le N-1$ from $u_{i-\frac{1}{2}}^n$ interpolating linearly

   $$u_i^n = \frac{\left(x_{i+\frac{1}{2}} - x_i\right)}{\left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}\right)} u_{i-\frac{1}{2}}^n + \frac{\left(x_i - x_{i-\frac{1}{2}}\right)}{\left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}\right)} u_{i+\frac{1}{2}}^n \tag{B.11}$$

3. We compute $du_i^n = u_x(x_i, t_n)$ for $1 \le i \le N$ from $u_{i-\frac{1}{2}}^n$ and u_N^n

   $$du_i^n = \frac{u_{i+\frac{1}{2}}^n - u_{i-\frac{1}{2}}^n}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \quad \text{and} \quad du_N^n = \frac{u_N^n - u_{N-\frac{1}{2}}^n}{1 - x_{N-\frac{1}{2}}} \tag{B.12}$$

4. We compute $u_{tN}^n = u_t(1, t_n)$ from the boundary condition at $x = 1$ (see Equation (B.1)).

   $$u_{tN}^n = -3(-t_n)^{2/5} \frac{du_N^n}{u_N^n} - \frac{6(\gamma-1)}{5t_n} u_N^n \tag{B.13}$$

5. We make the approximations

   $$f_{pi}^n \cong \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (-t)^{2/5} \frac{u_x(x_i, t)}{\gamma u(x_i, t)} dt \simeq (-t_n)^{2/5} \frac{du_i^n}{\gamma u_i^n} \tag{B.14}$$

   and the upwind approximation

   $$f_{hi}^n \simeq \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left\{ \frac{1}{3\gamma} \frac{u_t(x=1, t)}{u(x=1, t)} + \frac{2(\gamma-1)}{5\gamma t} \right\} x_i u(x_i, t) \, dt$$

   $$\simeq \text{vel} \left(u_i^n - \frac{\Delta t}{2} \text{vel} \, du_i^n \right) \tag{B.15}$$

   where $\text{vel} = \left\{ \frac{1}{3\gamma} \frac{u_{tN}^n}{u_N^n} + \frac{2(\gamma-1)}{5\gamma t_n} \right\} x_i$

Finally we have

$$g_i^n = f_{pi}^n + f_{hi}^n \tag{B.16}$$

B.3.2 Computing $u(1, t_{n+1})$

This is easily done. In the previous subsection we explain how to calculate $u_{tN}^n = u_t(1, t_n)$. Then we just set

$$u_N^{n+1} = u_N^n + \Delta t \ u_{tN}^n \tag{B.17}$$

B.3.3 Choosing the Nodes x_i

We said that there is a boundary layer of thickness of order $(-t)^{\frac{6\gamma+1}{10}}$ as $t \to 0^-$. To keep accuracy in our calculations we do a series of refinements of the grid.

Specifically we follow the following steps:

1. We choose L and we define the initial grid

$$x_i = \frac{i}{2^L} \qquad 0 \le i \le 2^L \tag{B.18}$$

2. We define the times at which we will refine the grid

$$t_i = -\left(\frac{1}{2}\right)^{\frac{10i}{6\gamma+1}} \qquad 1 \le i \le N_{ref} \tag{B.19}$$

so that at $t = t_i$ the boundary layer has thickness of order $\left(\frac{1}{2}\right)^i$

3. We integrate forward in time until $t = t_1$. Let $N+1$ be the number of grid points ($N = 2^L$, see Equation (B.18)). We introduce new grid points between x_{i-1} and x_i for $N - 2^{(L-1)} + 1 \le i \le N$ in the natural way

$$x_{i-\frac{1}{2}} = \frac{x_i + x_{i-1}}{2} \tag{B.20}$$

From v_i^n, the averages of u in $[x_{i-1}, x_i]$, we compute $u_{i-\frac{1}{2}}^n$ and $du_{i-\frac{1}{2}}^n$ and then we can compute the averages on $[x_i, x_{i+\frac{1}{2}}]$ (that we denote by $\tilde{v}_{i+\frac{1}{2}}^n$) and the averages on $[x_{i-\frac{1}{2}}, x_i](\tilde{v}_i^n)$. This is done

as before (see Equation (B.10))

$$\tilde{v}^n_{i+\frac{1}{2}} = u^n_{i+\frac{1}{2}} - du^n_{i+\frac{1}{2}} x_{i+\frac{1}{2}} + du^n_{i+\frac{1}{2}} \frac{3}{4} \frac{\left(x^4_{i+\frac{1}{2}} - x^4_i\right)}{\left(x^3_{i+\frac{1}{2}} - x^3_i\right)}$$

and

$$\tilde{v}^n_{i+\frac{1}{2}} = u^n_{i-\frac{1}{2}} - du^n_{i-\frac{1}{2}} x_{i-\frac{1}{2}} + du^n_{i-\frac{1}{2}} \frac{3}{4} \frac{\left(x^4_i - x^4_{i-\frac{1}{2}}\right)}{\left(x^3_i - x^3_{i-\frac{1}{2}}\right)}$$

We rearrange the indexes and the refinement is done. We continue forward in time until the next time we refine the grid and then do this same procedure again.

B.3.4   Choosing the Time Step Δt

The time step Δt is taken to be the minimum of Δt_1, Δt_2 and Δt_3, where $\Delta t_1 = t_i - t$, t_i being the next time we refine the grid and for stability reasons we choose

$$\Delta t_2 = 0.9 \frac{\gamma}{2(-t)^{2/5}} \min\left\{u^n_{i-\frac{1}{2}}(x_i - x_{i-1})^2\right\}$$

and

$$\Delta t_3 = 0.9 \frac{(x_N - x_{N-1})}{\text{vel}}$$

where vel was defined before (see Equation (B.15)).

B.3.5   The accuracy of the code

This code is second order in space and first order in time. Thanks to the time constarins, we can say that the errors in our computations are order $1/4^L$, where L was defined in (B.18), as long as the error in the initial conditions are smaller.

B.4   Numerical results

We have chosen the initial time to be $t = -10$, $L = 8$ and $N_{ref} = 10$ (see equations (B.18) and (B.19)). The evolution of u is plotted in figure (B-1). We computed $a_i = h(t_i)\, t_i^{\frac{6}{5}(\gamma-1)}$. We then used the value of a_8, a_9 and a_{10} to extrapolate and obtain b. To do this extrapolation we first assume

$$a_i = b + ct_i^\alpha \tag{B.21}$$

We have then three equations with three unknown so we can find b. The values of a_i are:

$$a_8 = 0.3059 \quad a_9 = 0.3022 \quad a_{10} = 0.3002 \tag{B.22}$$
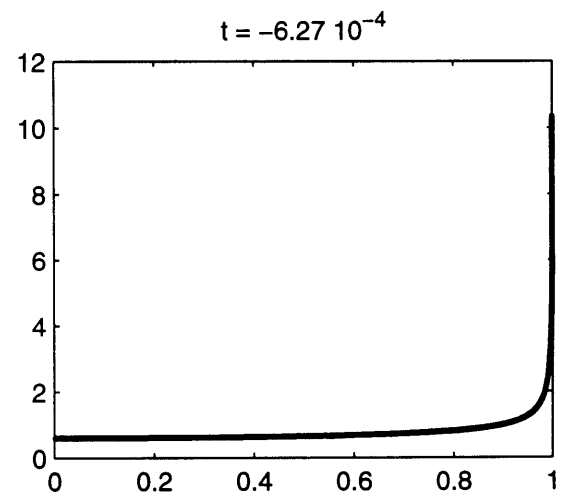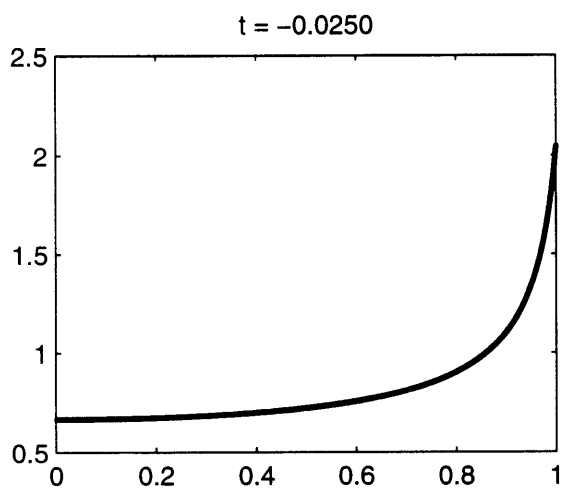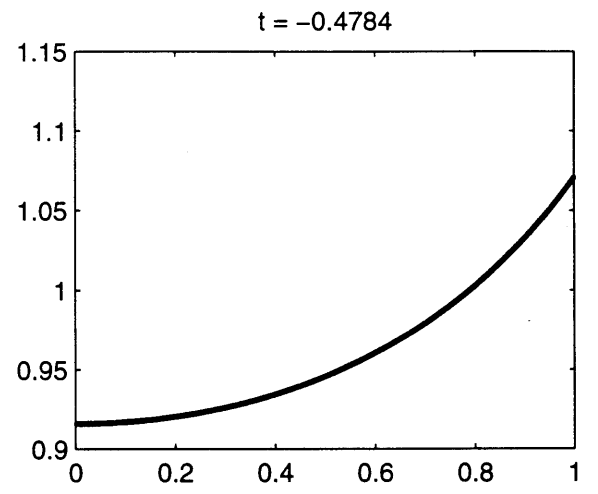
and the value of b obtained is
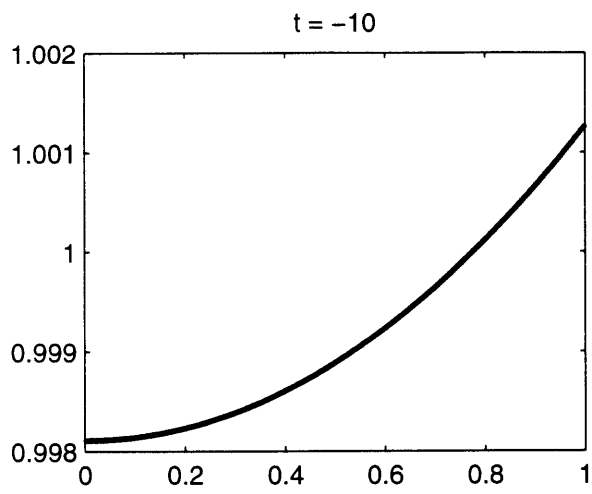
$$b = 0.2980 \qquad\qquad (B.23)$$

Figure B-1: Evolution of u

Appendix C

Formulas for x and t as functions of the characteristic variables

In this appendix we will solve the equations

$$x_\xi - \lambda_1 t_\xi = 0 \quad \text{and} \quad x_\eta - \lambda_2 t_\eta = 0$$

We expand all the dependent variables in powers of ϵ

$$z = \epsilon z_1 + \epsilon^2 z_2 + \cdots \quad w = \epsilon w_1 + \epsilon^2 w_2 + \cdots$$
$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \quad t = t_0 + \epsilon t_1 + \epsilon^2 t_2 + \cdots$$

where w_i depend only on η and z_i depend only on ξ.

Plugging the anzats into the equation and collecting powers of ϵ we obtain for x

$$x_0 = f_0(\eta) + g_0(\xi)$$

$$x_1 = \frac{\left(1 + c^{(1)}\right)}{4} \left(z_1(\xi)f_0(\eta) + w_1(\eta)g_0(\xi)\right) -$$
$$\frac{\left(1 - c^{(1)}\right)}{4} \left(\int w_1(\eta)f_0'(\eta)d\eta + \int z_1(\xi)g_0'(\xi)d\xi\right) + f_1(\eta) + g_1(\xi)$$

$$x_2 = \frac{\left(1 + c^{(1)} - 2\left(c^{(1)}\right)^2 + c^{(2)}\right)}{8} \left(w_1(\eta)\int z_1(\xi)g_0'(\xi)d\xi + z_1(\xi)\int w_1(\eta)f_0'(\eta)d\eta\right)$$
$$+ \frac{\left(2c^{(2)} - 3\left(c^{(1)}\right)^2 - 2c^{(1)} - 3\right)}{32} \left(w_1^2(\eta)g_0(\xi) + z_1^2(\xi)f_0(\eta)\right) +$$
$$\frac{\left(c^{(2)} + 3c^{(1)} - 2\left(c^{(1)}\right)^2 - 1\right)}{16} \left(\int z_1^2(\xi)g_0'(\xi)d\xi + \int w_1^2(\eta)f_0'(\eta)d\eta\right) +$$
$$\frac{\left(1 + c^{(1)}\right)^2}{16} z_1(\xi)w_1(\eta)\left(f_0(\eta) + g_0(\xi)\right) -$$
$$\frac{\left(1 - c^{(1)}\right)}{4} \left(\int w_1(\eta)f_1'(\eta)d\eta + \int z_1(\xi)g_1'(\xi)d\xi\right) +$$
$$\frac{\left(1 + c^{(1)}\right)}{4} \left(z_1(\xi)f_1(\eta) + w_1(\eta)g_1(\xi)\right) -$$
$$\frac{\left(1 - c^{(1)}\right)}{4} \left(\int w_2(\eta)f_0'(\eta)d\eta + \int z_2(\xi)g_0'(\xi)d\xi\right) +$$
$$\frac{\left(1 + c^{(1)}\right)}{4} \left(z_2(\xi)f_0(\eta) + w_2(\eta)g_0(\xi)\right) + f_2(\eta) + g_2(\xi)$$

and for t

$$t_0 = f_0(\eta) - g_0(\xi)$$

$$t_1 = \frac{\left(1 + c^{(1)}\right)}{4}\left(w_1(\eta)g_0(\xi) - z_1(\xi)f_0(\eta)\right) +$$
$$\frac{\left(1 - c^{(1)}\right)}{4}\left(\int w_1(\eta)f_0'(\eta)d\eta - \int z_1(\xi)g_0'(\xi)d\xi\right) + f_1(\eta) - g_1(\xi)$$

$$t_2 = \frac{\left(1 + c^{(1)} - 2\left(c^{(1)}\right)^2 + c^{(2)}\right)}{8}\left(w_1(\eta)\int z_1(\xi)g_0'(\xi)d\xi - z_1(\xi)\int w_1(\eta)f_0'(\eta)d\eta\right)$$
$$+\frac{\left(2c^{(2)} - 5\left(c^{(1)}\right)^2 - 2c^{(1)} - 1\right)}{32}\left(w_1^2(\eta)g_0(\xi) - z_1^2(\xi)f_0(\eta)\right) +$$
$$\frac{\left(c^{(2)} + 3c^{(1)} - 2\left(c^{(1)}\right)^2 - 1\right)}{16}\left(\int z_1^2(\xi)g_0'(\xi)d\xi - \int w_1^2(\eta)f_0'(\eta)d\eta\right) +$$
$$\frac{\left(1 + c^{(1)}\right)^2}{16}z_1(\xi)w_1(\eta)\left(f_0(\eta) - g_0(\xi)\right) +$$
$$\frac{\left(1 - c^{(1)}\right)}{4}\left(\int w_1(\eta)f_1'(\eta)d\eta - \int z_1(\xi)g_1'(\xi)d\xi\right) +$$
$$\frac{\left(1 + c^{(1)}\right)}{4}\left(w_1(\eta)g_1(\xi) - z_1(\xi)f_1(\eta)\right) +$$
$$\frac{\left(1 - c^{(1)}\right)}{4}\left(\int w_2(\eta)f_0'(\eta)d\eta - \int z_2(\xi)g_0'(\xi)d\xi\right) +$$
$$\frac{\left(1 + c^{(1)}\right)}{4}\left(w_2(\eta)g_0(\xi) - z_2(\xi)f_0(\eta)\right) + f_2(\eta) - g_2(\xi)$$

where f_i are arbitrary functions of η and g_i are arbitrary functions of ξ.

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