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Ferrofluid Flow & Spin Profiles for Positive and Negative Effective Viscosities

by

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Abstract

This thesis continues research of ferrohydrodynamic pumping in spatially uniform sinusoidally time-varying magnetic fields. Earlier analysis has show that when the fluid spin velocity is small, the electromechanical coupling between magnetic field and flow can lead to an effective dynamic fluid viscosity, η_{eff}, that can be made zero or negative as a function of magnetic field strength and direction. When the effective viscosity changes sign from positive to negative, the earlier approximate theory of small spin velocity predicts flow reversal.

This thesis describes a method to numerically solve, without further approximation, the governing fluid and field equations in the viscous dominated limit. The one-dimensional equations of fluid flow in a planar duct that were solved include the case of an imposed uniform magnetic field along the duct axis and the case of an imposed uniform magnetic field transverse to the duct axis. Spin and flow velocity profiles were plotted for positive, zero, and negative values of effective viscosity as a function of frequency.

Unlike most past work which considered fluid pumping due to applied rotating or traveling magnetic fields, this thesis found that with alternating applied magnetic fields either perpendicular or parallel to the duct axis, time average flow and spin velocities result. The fluid spatial profiles have multi-valued regions where at one spatial position there can be more than one allowed flow and spin velocity.

Thesis Supervisor: Markus Zahn
Title: Professor

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I would also like to thank Professor Zahn, my advisor, for all his tremendous help, support, and understanding....even though I DID come from a "physics" background! (Thanks, too, to Mrs. Linda Zahn for letting me drop by after hours for some corrections!)

Anne Hunter deserves much thanks. She has helped me through MIT ever since freshman year when she was my freshman advisor. Without all her help, I would never have made it to this point. Part of this is hers!

I dedicate this thesis to my husband, Nick. He has been and will continue to be my inspiration in all I do. If only I could do things as well as he does!

Finally, and personally, I give thanks to God, for without Him, I can do nothing.

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Chapter 1

Introduction

Ferrohydrodynamics is a relatively new science dealing with the motion of fluids that are influenced by forces due to magnetic polarization. Before the 1960's, the only magnetizable liquids had magnetic susceptibilities less than 10^{-3}. In the mid-1960's, that situation changed with the production of colloidal magnetic fluids (ferrofluids) [1].

These ferrohydrodynamic materials are produced by taking small particles of iron, typically with 100 nm diameter, coating them with a surfactant, and suspending them in a continuous medium (like water). The coating prevents the magnetized particles from clumping together in magnetic fields, and thermal agitation keeps the particles suspended against gravity. Particles of colloidal size may either be grown (precipitation) or ground down from larger sizes (dispersion). Then the particles are transformed to a carrier liquid through peptization where the liquid and surfactant are added to the particles during the heating and agitation of the solution.

1.1 Background

Current primary applications of ferromagnetic fluids, such as rotating shaft seals and bearings, use DC magnetic fields. The equations of motion for ferrofluids in a DC field are straightforward, and the behavior of these fluids is predictable because the fluid magnetization is collinear with the magnetic field.

The motion of ferrofluids in a traveling wave magnetic field is sometimes contrary to intuition. The presence of a traveling magnetic field causes the fluid to flow, or pump, because fluid friction causes the magnetization to lag the traveling magnetic field. Because the magnetization is at an angle to the magnetic field, there is a body torque on the fluid. The direction of the flow should be determined by the direction of travel of the field. That is, the fluid is expected to pump in the same direction the magnetic field is traveling. However, experimenters have found that below some critical magnetic field, the fluid will flow opposite to the direction the magnetic field travels. This motion is referred to as backward pumping. The critical field strength depends on these physical parameters:

- the magnetic field's frequency, Ω, and amplitude
- the concentration of magnetic particles in the carrier fluid
- the viscosity, η, of the carrier fluid

1.2 Previous Analysis

Earlier analysis has shown that when the fluid spin velocity is small, the electromechanical coupling between magnetic field and flow can lead to an effective dynamic fluid viscosity, η_{eff}, that can be made zero or negative as a function of the magnetic field strength, direction, and frequency.

This previous work in the small spin-velocity limit has shown that both the flow and spin velocities of the ferrofluid depend on the effective viscosity. When the effective viscosity changes sign from a positive value to a negative value, the earlier approximate theory predicts flow reversal. However, mathematically (in the small spin-velocity approximation to be further discussed in Chapter 2), a singularity occurs in both the flow and spin velocities with zero spin viscosity so that these velocities become infinite, which violates the small spin-velocity approximation that was made.

A hypothesis explored in this thesis is whether the change in sign of the effective viscosity of the ferrofluid in a magnetic field is the cause of the observed backward pumping phenomena. Such a study will help to better understand ferrofluid pumping in AC and traveling wave magnetic fields. Solutions of the flow and spin velocity profiles for positive, zero, and negative effective viscosities provide a clearer understanding of how the effective viscosity of a ferrofluid can affect the dynamics of the fluid.

1.3 Scope of Thesis

This thesis continues the research of Professor Markus Zahn, which explored "Ferrohydrodynamic pumping in spatially uniform sinusoidally time-varying magnetic fields [3]." The purpose of this thesis is to numerically solve, without further approximation, the governing fluid and field equations in the viscous dominated limit. This allows fluid inertia to be neglected in order to better understand ferrofluid behavior under conditions of predicted infinite velocity in the small spin-velocity approximation. The equations of fluid flow in a planar duct that were solved include the case of an imposed uniform magnetic field component along the duct axis and the case of an imposed uniform magnetic field component transverse to the duct axis.

Chapter 2

Fundamentals of Ferrofluids

2.1 1-Dimensional Treatment

With the motion of ferrofluids being described by vector and tensor equations, it is useful to better understand complex phenomena under simple geometrical magnetic field conditions. To do this, the 1-dimensional case of a ferrofluid pumping in a planar duct with a spatially uniform applied magnetic field which sinusoidally varies with time is considered. Using a uniform magnetic field causes the motion of ferrofluids to depend on magnetic torque with zero magnetic force.

The setup is seen in Figure 2-1 in which the ferrofluid layer is being pumped between the two rigid walls of a planar duct.

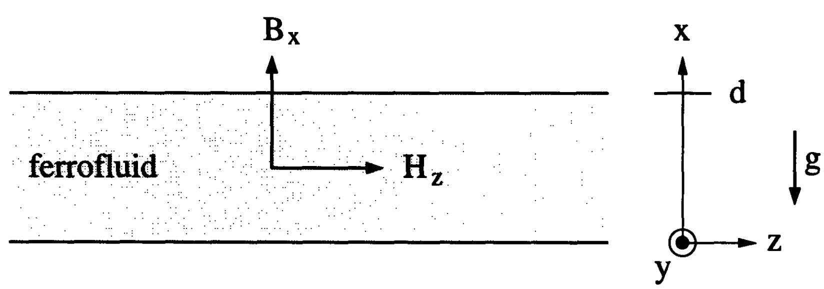


Figure 2-1: Experimental setup of ferrofluid pumping in a planar duct. The ferrofluid flows in the z-direction due to the applied uniform fields H_z and B_x which both vary sinusoidally in time.

The magnetic field H_z, along the duct axis, is uniformly z-directed, and the magnetic flux density transverse to the duct axis, B_x is uniformly x-directed. Each are externally imposed and independent of the ferrofluid magnetization. With this set up, all relevant variables become dependent only on the x coordinate.

2.2 Magnetic Fields and Forces

To describe the motion of ferrofluids, the principles of elementary hydrodynamics are used. By considering the ferrofluids a homogeneous liquid, equations of motion can be written using conservation of mass, momentum, and energy, including magnetization terms consistent with Maxwell's equations.

Ferrofluid particles have effectively constant magnetic moments, and the orientation of the particles in the absence of a magnetic field is random. The net effect is a zero magnetization for the entire system. The viscosity of the ferrofluid is largely determined by the viscosity of the carrier fluid. As an applied magnetic field attempts to align the particles in the direction of the field, this viscosity keeps the particles from moving freely. At the same time, the forces of hydrodynamics and thermal agitation cause disorientation. These factors effectively increase the viscosity of ferrofluids.

2.2.1 Magnetization and Magnetic Fields

The viscosity of the ferromagnetic solution causes the magnetic dipole moment, $\mathbf{m}$, of a ferrofluid particle to not always be aligned with the spatially uniform magnetic field $\mathbf{H}$ if it is time varying or rotating, as can be seen in Figure 2-2.

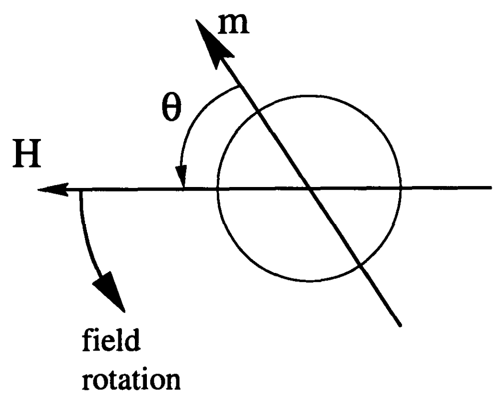


Figure 2-2: Fluid viscosity causes the magnetic dipole moment $\mathbf{m}$ to lag a rotating magnetic field $\mathbf{H}$ by an angle θ.

The magnetization of the ferrofluid is $\mathbf{M} = N\mathbf{m}$ where N is the number of magnetic dipoles per unit volume. As $\mathbf{H}$ changes direction, the retarding viscous force causes the orientation of the magnetic particles to lag behind the magnetic field. Because $\mathbf{M}$ is then not collinear with $\mathbf{H}$, a body torque density $\mathbf{T} = \mu_0(\mathbf{M} \times \mathbf{H})$ acts on the ferrofluid. In addition, the fluid cannot slip at boundaries, which introduces another source of drag, causing a phase lag between $\mathbf{M}$ and $\mathbf{H}$, as is illustrated in Figure 2-3.

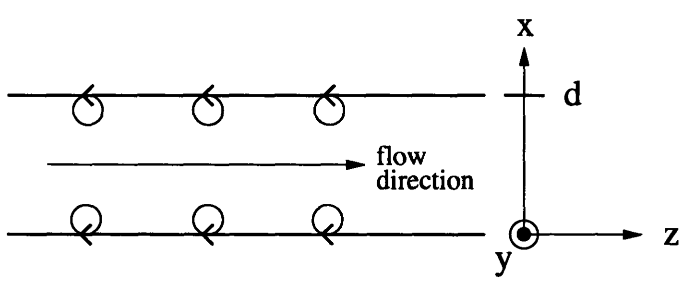


Figure 2-3: Fluid flow introduces hindered particle rotation near fixed boundaries.

Notice that the fluid flow velocity $\mathbf{v}$ in such a set up will be only z-directed, while the spin velocity $\boldsymbol{\omega}$ will only be y-directed. Each will vary only as a function of x. Thus,

$$\mathbf{v} = v_z(x)\mathbf{i}_z \qquad and \qquad \boldsymbol{\omega} = \omega_y(x)\mathbf{i}_y \tag{2.1}$$

A rotating or oscillating magnetic field and a fixed boundary cause particle rotation, which introduces a body coupling. The state of stress at any point in the fluid can be represented by a stress tensor, but that stress tensor is asymmetric due to this coupling.

Magnetization

As described above, the viscosity of the fluid causes the magnetization **M** to lag the field. The time it takes for **M** to reorient collinear to **H** is the magnetic relaxation time constant τ. The magnetization constitutive law for such fluids with velocity **v** and particle spin with angular velocity $\boldsymbol{\omega}$ is [2]

$$\frac{\partial \mathbf{M}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{M} - \boldsymbol{\omega} \times \mathbf{M} + \frac{1}{\tau}[\mathbf{M} - \chi_0 \mathbf{H}] = 0 \tag{2.2}$$

where χ_0 is the effective magnetic susceptibility. In general, χ_0 can be dependent on the magnetic field, but for simplicity, the work in this thesis assumes it is a constant, as varying the value would provide little insight to the fundamental motion of the ferrofluid. For all numerical case studies, $\chi_0 = 1$. In the 1-dimensional case, flow velocity **v** is in the z-direction only, and the spin velocity $\boldsymbol{\omega}$ can only be y-directed.

Magnetic Field and Flux Density

Because the imposed **B** and **H** fields are uniform with respect to the y and z coordinates, the fields can vary only with x. With no variations in y and z, Gauss's law requires the magnetic flux density B_x to be constant, and the current free Ampere's law requires the magnetic field H_z to be constant.

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \frac{dB_x}{dx} = 0 \quad \Rightarrow \quad B_x = constant, \tag{2.3}$$

$$\nabla \times \mathbf{H} = 0 \quad \Rightarrow \quad \frac{dH_z}{dx} = 0 \quad \Rightarrow \quad H_z = constant. \tag{2.4}$$

By applying Eq. (2.2) to the confined planar ferrofluid set up in Fig. 2-1 and considering the restrictions of Maxwell's equations in addition to the spatially uniform imposed magnetic field H_z and magnetic flux density B_x, fluid magnetization and motion give rise to magnetic field components B_z and H_x

$$\mathbf{B} = \Re\Big\{[\hat{B}_x\mathbf{i}_x + \hat{B}_z(x)\mathbf{i}_z]e^{i\Omega t}\Big\}, \tag{2.5}$$

$$\mathbf{H} = \Re\Big\{[\hat{H}_x(x)\mathbf{i}_x + \hat{H}_z\mathbf{i}_z]e^{i\Omega t}\Big\}, \tag{2.6}$$

where

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{2.7}$$

and Ω is the radian frequency of the sinusoidally varying magnetic fields. The "hat" character ($\hat{}$) denotes a complex amplitude which can depend on x, and $\mathbf{M}$ is the magnetization of the ferrofluid found in Eq. (2.2).

Because $\mathbf{B}$ has an x and z component, so does the magnetization $\mathbf{M}$. The magnetization components $\hat{M}_x$ and $\hat{M}_z$ can be solved in terms of the known imposed field amplitudes $\hat{H}_z$ and $\hat{B}_x$ by noting that the second term of Eq. 2.2 makes no contribution since $\mathbf{v}$ is only z-directed, while $\mathbf{M}$ varies only as a function of x. Thus,

$$\hat{M}_x = \frac{\chi_0\Big[\hat{H}_z(\omega_y\tau) + (i\Omega\tau + 1)\hat{B}_x/\mu_0\Big]}{\Big[(\omega_y\tau)^2 + (i\Omega\tau + 1)(i\Omega\tau + 1 + \chi_0)\Big]}, \tag{2.8}$$

$$\hat{M}_z = \frac{\chi_0\Big[(i\Omega\tau + 1 + \chi_0)\hat{H}_z - \hat{B}_x\omega_y\tau/\mu_0\Big]}{\Big[(\omega_y\tau)^2 + (i\Omega\tau + 1)(i\Omega\tau + 1 + \chi_0)\Big]}. \tag{2.9}$$

It can be seen from Eqs. (2.8) and (2.9) that the magnetization of the ferrofluid depends on the unknown spin velocity ω_y which can vary as a function of x. If the magnetization is not collinear with the field, a torque is produced, which causes fluid motion and a non-zero ω_y. The new ω_y produced by the torque then changes the magnetization of the fluid. It is this coupling that makes solving for the spin velocity complex, as both the magnetization equations and the mechanical equations must be simultaneously satisfied.

2.2.2 Magnetic Force and Torque Densities

Early studies of ferromagnetic fluid pumping in a planar duct were conducted using spatially non-uniform traveling wave magnetic fields. This introduces non-zero magnetic force densities and magnetic torque densities. By imposing a spatially uniform traveling wave magnetic field, the magnetic force density becomes zero. Thus, the effects of different magnetic field variations on the fluid flow can be studied more simply.

Magnetic Force Density

Between the walls of the planar duct ($0 < x < d$), the magnetic force density acting on the ferrofluid is

$$\mathbf{f} = \mu_0(\mathbf{M} \cdot \nabla)\mathbf{H}. \tag{2.10}$$

Because of the geometry, $\mathbf{f}$ has only x and z components which will either be constant or vary only with x. Using the x component of (2.7) in (2.3) with (2.10)

$$f_x = \mu_0 M_x \frac{dH_x}{dx} = -\mu_0 M_x \frac{dM_x}{dx} = -\frac{d}{dx}\left(\frac{1}{2}\mu_0 {M_x}^2\right), \tag{2.11}$$

$$f_z = \mu_0 M_x \frac{dH_z}{dx} = 0. \tag{2.12}$$

where the time average components are

$$\langle f_x \rangle = -\frac{d}{dx}\left(\frac{1}{4}\mu_0 |\hat{M}_x|^2\right), \tag{2.13}$$

$$\langle f_z \rangle = 0. \tag{2.14}$$

Magnetic Torque Density

The torque density, $\mathbf{T} = \mu_0(\mathbf{M} \times \mathbf{H})$, is only y-directed because $\mathbf{M}$ and $\mathbf{H}$ have only x and z components

$$\mathbf{T} = \mu_0(-M_x H_z + M_z H_x)\mathbf{i}_y. \tag{2.15}$$

Using $H_x = \frac{B_x}{\mu_0} - M_x$ and taking the time average of Eq. (2.15) yields

$$\langle \mathbf{T}_y \rangle = \frac{1}{2}\Re\left[\hat{M}_z \hat{B}_x^* - \mu_0 \hat{M}_x^* (\hat{H}_z + \hat{M}_z)\right]. \tag{2.16}$$

The superscript asterisks (*) denotes the complex conjugate of the complex field amplitude.

2.3 Ferrohydrodynamics

In addition to the magnetic field equations, the motion of ferrofluids is described by hydrodynamic linear and angular momentum equations driven by the magnetic force and torque densities on the fluid.

2.3.1 Fluid Mechanics

By definition of incompressible fluids,

$$\nabla \cdot \mathbf{v} = 0, \qquad \nabla \cdot \boldsymbol{\omega} = 0. \tag{2.17}$$

Also, there is a coupling between the linear and angular velocities in the momentum conservation equations. For a fluid in a gravity field, $-g\mathbf{i}_x$,

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right] = -\nabla p + \mathbf{f} + 2\zeta\nabla \times \boldsymbol{\omega} + (\zeta + \eta)\nabla^2\mathbf{v} - \rho g\mathbf{i}_x, \tag{2.18}$$

$$I\left[\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega}\right] = \mathbf{T} + 2\zeta(\nabla\times\mathbf{v} - 2\boldsymbol{\omega}) + \eta'\nabla^2\boldsymbol{\omega} \tag{2.19}$$

where

- ρ is the fluid mass density,
- p is the pressure,
- ζ is the vortex viscosity,
- η is the dynamic viscosity,
- I is the moment of inertia density,
- η' is the shear coefficient of spin-viscosity.

2.3.2 Combining the Magnetics with the Mechanics

To bring together the magnetic equations with the fluid equations, Eqs. (2.18) and (2.19) need to be applied to the planar duct setup. There are two assumptions made.

- The ferrofluid has viscous-dominated flow so the inertial terms are negligible.
- The ferrofluid is in the steady state so the fluid responds only to the time average force and torque densities.

A modified pressure can be defined from the magnetization force in Eq. (2.13) and the gravitational force density.

$$p' = p + \frac{1}{4}\mu_0|\hat{M}_x|^2 + \rho g x \tag{2.20}$$

With these assumptions and Eq. (2.20), the coupled linear and angular momentum conservation equations become:

$$(\zeta+\eta)\frac{d^2 v_z}{dx^2} + 2\zeta\frac{d\omega_y}{dx} - \frac{\partial p'}{\partial z} = 0, \tag{2.21}$$

$$\eta'\frac{d^2\omega_y}{dx^2} - 2\zeta\left(\frac{dv_z}{dx} + 2\omega_y\right) + \langle T_y\rangle = 0. \tag{2.22}$$

2.3.3 Normalization

Before continuing with the equations, it is useful to define dimensionless parameters. Space is normalized to the width of the planar duct, d. Time is normalized to the relaxation time constant, τ. The magnetic field and flux density are normalized to an arbitrary field strength, H_0. The tilde symbol ($\tilde{}$) is used to denote dimensionless quantities.

$$\tilde{x} = \frac{x}{d}, \quad \tilde{v}_z = v_z\frac{\tau}{d}, \quad \tilde{\Omega} = \Omega\tau \quad \tilde{\omega}_y = \omega_y\tau,$$

$$\tilde{H} = \frac{\hat{H}}{H_0}, \quad \tilde{B} = \frac{\hat{B}}{\mu_0 H_0}, \quad \tilde{M} = \frac{\hat{M}}{H_0}, \quad \tilde{T}_y = \frac{T_y}{\mu_0 H_0^2},$$

$$\tilde{\eta} = \frac{2\eta}{\mu_0 H_0^2\tau}, \quad \tilde{\eta}' = \frac{\eta' d^2}{\mu_0 H_0^2\tau}, \quad \tilde{\zeta} = \frac{2\zeta}{\mu_0 H_0^2\tau},$$

2.4 One Dimensional Governing Equations

There are four possible magnetic field variations that could be studied for imposed uniform magnetic fields:

- an axial component only
- a transverse component only
- in-phase axial and transverse magnetic fields (field at an angle to the duct axis)
- 90° phase difference between axial and transverse magnetic fields (rotating magnetic field)

Case studies will focus on the first two cases in this thesis, while the last two cases will be studied in the future.

2.4.1 Relevant Normalized Equations

By using normalized parameters, the flow and spin velocity equations (2.21) and (2.22) become dimensionless

$$\frac{1}{2}(\tilde{\zeta}+\tilde{\eta})\frac{d^2\tilde{v}_z}{d\tilde{x}^2}+\tilde{\zeta}\frac{d\tilde{\omega}_y}{d\tilde{x}}-\frac{\partial\tilde{p}'}{\partial\tilde{z}}=0, \tag{2.23}$$

$$\tilde{\eta}'\frac{d^2\tilde{\omega}_y}{d\tilde{x}^2}-\tilde{\zeta}\left(\frac{d\tilde{v}_z}{d\tilde{x}}+2\tilde{\omega}_y\right)+\langle\tilde{T}_y\rangle=0 \tag{2.24}$$

where

$$\langle\tilde{T}_y\rangle=\frac{1}{2}\Re\left[\tilde{M}_z\tilde{B}_x^*-\tilde{M}_x^*(\tilde{H}_z+\tilde{M}_z)\right] \tag{2.25}$$

and

$$\tilde{M}_x=\frac{\chi_0\left[\tilde{\omega}_y\tilde{H}_z+(i\tilde{\Omega}+1)\tilde{B}_x\right]}{\left[\tilde{\omega}_y^2+(i\tilde{\Omega}+1)(i\tilde{\Omega}+1+\chi_0)\right]}, \tag{2.26}$$

$$\tilde{M}_z=\frac{\chi_0\left[(i\tilde{\Omega}+1+\chi_0)\tilde{H}_z-\tilde{B}_x\tilde{\omega}_y\right]}{\left[\tilde{\omega}_y^2+(i\tilde{\Omega}+1)(i\tilde{\Omega}+1+\chi_0)\right]}. \tag{2.27}$$

These equations describe the motion of a ferrofluid confined between the rigid walls of a planar duct (seen in Figure 2-1) where the imposed magnetic fields are spatially uniform and sinusoidally time-varying. The simultaneous solution of the flow and spin velocity equations is complicated by the time average torque density $\langle\tilde{T}_y\rangle$, as it depends on the spin velocity $\tilde{\omega}_y$ which in turn depends on $\tilde{x}$ in a complex way.

To see the complexity of this dependence, substitute Eqs. (2.26) and (2.27) into Eq. (2.25), which gives

$$\langle\tilde{T}_y\rangle=\frac{\frac{\chi_0}{2}\left[-\tilde{\omega}_y\left[|\tilde{B}_x|^2\left(\tilde{\omega}_y^2-\tilde{\Omega}^2+1\right)+|\tilde{H}_z|^2\left[\tilde{\omega}_y^2-\tilde{\Omega}^2+(1+\chi_0)^2\right]\right]+2\Re\left[\left[\chi_0\left(\tilde{\omega}_y^2-\tilde{\Omega}^2\right)+i\tilde{\Omega}\left(\tilde{\omega}_y^2-\tilde{\Omega}^2-1-\chi_0\right)\right][H_zB_x^*]\right]\right]}{\left[\left[\tilde{\omega}_y^2-\tilde{\Omega}^2+1+\chi_0\right]^2+\left(2+\chi_0\right)^2\tilde{\Omega}^2\right]}. \tag{2.28}$$

It should be noted that for the two cases of field settings, $\tilde{H}_z=1$; $\tilde{B}_x=0$ and $\tilde{H}_z=0$; $\tilde{B}_x=1$, the $\Re$ (real part) term in the numerator is zero because of the multiplication of $[H_zB_x^*]$.

2.4.2 Zero Spin-Viscosity

This thesis will consider the simple limiting case of zero spin-viscosity, $\tilde{\eta}' = 0$, which simplifies Eq. (2.24) by reducing Eqs. (2.23) - (2.24) from a fourth order system to a second order system. In the fourth order system both the flow and spin velocities must be zero at the $x = 0$ and $x = d$ duct walls. The second order system allows the spin-velocity $\tilde{\omega}_y$ to be non-zero at the walls.

Resolving Eq. (2.24) in this limiting case ($\tilde{\eta}' = 0$) for $\frac{d\tilde{v}_z}{d\tilde{x}}$ gives

$$\frac{d\tilde{v}_z}{d\tilde{x}} = \frac{\langle \tilde{T}_y \rangle}{\tilde{\zeta}} - 2\tilde{\omega}_y. \tag{2.29}$$

Integrating Eq. (2.23) and substituting (2.29) into that equation, and differentiating with respect to $\tilde{x}$ gives

$$\frac{d\tilde{\omega}_y}{d\tilde{x}} = \frac{-\frac{1}{\tilde{\eta}}\frac{\partial \tilde{p}'}{\partial \tilde{z}}}{1 - \frac{\tilde{\zeta}+\tilde{\eta}}{2\tilde{\zeta}\tilde{\eta}}\frac{d\langle \tilde{T}_y \rangle}{d\tilde{\omega}_y}} \tag{2.30}$$

where we used the simplification that $\langle \tilde{T}_y \rangle$ explicitly depends only on $\tilde{\omega}_y$.

In addition, it is convenient to consider

$$\frac{d\tilde{v}_z}{d\tilde{\omega}_y} = \frac{d\tilde{v}_z}{d\tilde{x}}\frac{d\tilde{x}}{d\tilde{\omega}_y} \tag{2.31}$$

so that Eq. (2.29) can be rewritten as

$$\frac{d\tilde{v}_z}{d\tilde{\omega}_y} = \frac{\frac{\langle \tilde{T}_y \rangle}{\tilde{\zeta}} - 2\tilde{\omega}_y}{\frac{d\tilde{\omega}_y}{d\tilde{x}}} \tag{2.32}$$

These equation manipulations, with $\tilde{\omega}_y$ considered the independent variable, allow the flow and spin velocity profiles to be numerically integrated so that plots of $\tilde{v}_z(x)$ and $\tilde{\omega}_y(x)$ can be made to describe the motion of the ferrofluid in a planar duct varying in dimensionless $\tilde{x}$ from 0 to 1.

Spin Velocity Profiles

Because of the complex dependence $\langle\tilde{T}_y\rangle$ has on $\tilde{\omega}_y(x)$, Eq. (2.30) cannot be simply solved for $\tilde{\omega}_y$ in terms of $\tilde{x}$. However, it is possible to solve for $\tilde{x}$ as a function of $\tilde{\omega}_y$. By doing this, a parametric plot can be used to plot the spin velocity profile.

Resolving Eq. (2.30) for $d\tilde{x}$ gives

$$d\tilde{x} = \frac{1 - \frac{\tilde{\zeta}+\tilde{\eta}}{2\tilde{\zeta}\tilde{\eta}}\frac{d\langle\tilde{T}_y\rangle}{d\tilde{\omega}_y}}{-\frac{1}{\tilde{\eta}}\frac{\partial\tilde{p}'}{\partial\tilde{z}}}\, d\tilde{\omega}_y. \tag{2.33}$$

Both sides of this equation can be integrated, which introduces a constant of integration, C.

$$\tilde{x} = \frac{-\tilde{\eta}}{\frac{\partial\tilde{p}'}{\partial\tilde{z}}}\left[\int\left(1 - \frac{\tilde{\zeta}+\tilde{\eta}}{2\tilde{\zeta}\tilde{\eta}}\frac{d\langle\tilde{T}_y\rangle}{d\tilde{\omega}_y}\right)d\tilde{\omega}_y + C\right] \tag{2.34}$$

This value of C is determined by noting that the torque density is zero when $\tilde{\omega}_y = 0$ when either $\tilde{H}_z$ or $\tilde{B}_x$ are zero. Because $\tilde{\omega}_y$ is then an odd function around $\tilde{x} = 0.5$, $\tilde{\omega}_y(\tilde{x} = 0.5) = 0$.

When considering the two cases of field settings, $\tilde{H}_z = 1$; $\tilde{B}_x = 0$ and $\tilde{H}_z = 0$; $\tilde{B}_x = 1$, $\tilde{T}_y(\tilde{\omega}_y = 0) = 0$ and the integral with respect to $d\tilde{\omega}_y$ on the right side of Eq. (2.34) is zero when $\tilde{\omega}_y = 0$. Eq. (2.28) shows that these facts allow us to calculate

$$C = -.5\frac{\frac{\partial\tilde{p}'}{\partial\tilde{z}}}{\tilde{\eta}} \tag{2.35}$$

The relevant plot is $\tilde{\omega}_y$ as a function of $\tilde{x}$, although the equation is $\tilde{x}$ as a function of $\tilde{\omega}_y$. It is possible to use a parametric plot to force the plotting of the relevant relation. In general, instead of specifying the y coordinate of each point as a function of x, parametric plots allow specifying both the x and the y coordinates of each point as a function of a third parameter t. Specific to this work, parametric plots allow specifying both the $\tilde{x}$ and the $\tilde{\omega}_y$ coordinates of each point while varying the value of $\tilde{\omega}_y$. This can be seen in the Maple programs, found in Appendix A, which plotted the profiles.

Flow Velocity Profiles

The flow velocity $\tilde{v}_z$ is a function of $\tilde{x}$ and is coupled to the spin velocity $\tilde{\omega}_y$. Substituting Eq. (2.30) into Eq. (2.32) and integrating to solve for $\tilde{v}_z$ yields

$$\tilde{v}_z = -\frac{\tilde{\eta}}{\frac{\partial \tilde{p}'}{\partial \tilde{z}}} \int \left[\left(\frac{\tilde{T}_y}{\tilde{\zeta}} - 2\tilde{\omega}_y \right) \left(1 - \frac{\tilde{\zeta} + \tilde{\eta}}{2\tilde{\zeta}\tilde{\eta}} \frac{d\tilde{T}_y}{d\tilde{\omega}_y} \right) \right] d\tilde{\omega}_y \quad + D. \tag{2.36}$$

The value of the constant of integration D is determined by requiring the flow velocity $\tilde{v}_z$ to be zero at the boundaries. That is, $\tilde{v}_z = 0$ at $\tilde{x} = 0$ and $\tilde{x} = 1$. This calculation cannot be done in closed form, thus requiring numerical integration. A parametric plot is also necessary to plot $\tilde{v}_z$ as a function of $\tilde{x}$ by specifying both the $\tilde{x}$ and the $\tilde{v}_z$ coordinates of each point while varying the value of $\tilde{\omega}_y$. The constant of integration depends on the value of $\tilde{\omega}_y$ at the boundary of each case. Thus, for every changed parameter, a new calculation of $\tilde{\omega}_y(\tilde{x}=0) = -\tilde{\omega}_y(\tilde{x}=1)$ has to be made. Then, to find the true value of $\tilde{v}_z$, the first term on the right side of Eq. (2.36) must be evaluated at this particular value of $\tilde{\omega}_0$ and D is the negative of this value.

For clarity, Eq. (2.36) was redefined to be the function $\tilde{f}$ such that

$$\tilde{f} = -\frac{\tilde{\eta}}{\frac{\partial \tilde{p}'}{\partial \tilde{z}}} \int \left[\left(\frac{\tilde{T}_y}{\tilde{\zeta}} - 2\tilde{\omega}_y \right) \left(1 - \frac{\tilde{\zeta} + \tilde{\eta}}{2\tilde{\zeta}\tilde{\eta}} \frac{d\tilde{T}_y}{d\tilde{\omega}_y} \right) \right] d\tilde{\omega}_y \tag{2.37}$$

so that

$$\tilde{v}_z = \tilde{f} - \tilde{f}\Big|_{\tilde{\omega}_y = \tilde{\omega}_0}. \tag{2.38}$$

This, as well, can be seen in the Maple programs, found in Appendix A, which plotted the profiles. The Mathematica programs used to calculate $\tilde{\omega}_0$ for each case study can be found in Appendix B.

2.4.3 Effective Viscosity

It is useful to consider a small $\tilde{\omega}_y$ limit for Eq. (2.28). By taking the limit that $\tilde{\omega}_y$ is much less than 1, this equation can be written as a linear approximation. To first order in $\tilde{\omega}_y$,

$$\lim_{\tilde{\omega}_y \ll 1} \langle \tilde{T}_y \rangle \;\approx\; \tilde{T}_0 + \alpha \tilde{\omega}_y, \tag{2.39}$$

where

$$\tilde{T}_0 = -\frac{\chi_0 \Re\Big[\big[\chi_0 \tilde{\Omega}^2 + i\tilde{\Omega}(\tilde{\Omega}^2 + 1 + \chi_0)\big]\big[\tilde{H}_z \tilde{B}_x^*\big]\Big]}{\big[1 + \chi_0 + \tilde{\Omega}^2\big]^2 + \tilde{\chi}_0^2 \tilde{\Omega}^2}, \tag{2.40}$$

$$\alpha = \frac{\chi_0}{2} \frac{\Big[|\tilde{B}_x|^2(\tilde{\Omega}^2 - 1) + |\tilde{H}_z|^2\big[\tilde{\Omega}^2 - (1 + \chi_0)^2\big]\Big]}{\big[1 + \chi_0 + \tilde{\Omega}^2\big]^2 + \chi_0^2 \tilde{\Omega}^2}. \tag{2.41}$$

For the two cases, $|\tilde{B}_x| = 0;\ |\tilde{H}_z| = 1$ and $|\tilde{B}_x| = 1;\ |\tilde{H}_z| = 0$, the value of $\tilde{T}_0$ is zero.

By remanipulating Eqs. (2.23) and (2.24), the flow and spin velocities can be written as

$$\tilde{v}_z(\tilde{x}) = \frac{1}{\tilde{\eta}} \frac{\partial \tilde{p}'}{\partial \tilde{z}} \tilde{x}(\tilde{x} - 1) + \frac{1}{\tilde{\eta}} \left[\tilde{x} \int_0^1 \langle \tilde{T}_y \rangle d\tilde{x} - \int_0^{\tilde{x}} \langle \tilde{T}_y \rangle d\tilde{x} \right], \tag{2.42}$$

$$\tilde{\omega}_y = -\frac{1}{2\tilde{\eta}} \left[\frac{\partial \tilde{p}'}{\partial \tilde{z}} (2\tilde{x} - 1) - \frac{\tilde{\zeta} + \tilde{\eta}}{\tilde{\zeta}} \langle \tilde{T}_y \rangle + \int_0^1 \langle \tilde{T}_y \rangle d\tilde{x} \right]. \tag{2.43}$$

Substituting Eq. (2.38) into Eqs. (2.42) and (2.43), the approximate flow and spin velocity profiles in the small spin-velocity limit become

$$\tilde{v}_z(\tilde{x}) \approx \frac{\tilde{x}(\tilde{x} - 1)}{\tilde{\eta}_{eff}} \frac{\partial \tilde{p}'}{\partial \tilde{z}}, \tag{2.44}$$

$$\tilde{\omega}_y(\tilde{x}) \approx \frac{1}{(2\tilde{\zeta} - \alpha)} \left[\tilde{T}_0 - \frac{\tilde{\zeta}(2\tilde{x} - 1)}{\tilde{\eta}_{eff}} \frac{\partial \tilde{p}'}{\partial \tilde{z}} \right], \tag{2.45}$$

where the effective viscosity is defined to be

$$\tilde{\eta}_{eff} = \tilde{\eta} - \frac{\alpha \tilde{\zeta}}{2\tilde{\zeta} - \alpha} \tag{2.46}$$

With this definition, the effective viscosity can be positive, negative or zero. The effective viscosity is zero when

$$\alpha = \frac{2\tilde{\zeta}\tilde{\eta}}{\tilde{\zeta} + \tilde{\eta}} \tag{2.47}$$

However, this is inconsistent, as a zero effective viscosity makes the spin-velocity of Eq. (2.45) infinite, violating the small spin-velocity approximation made in Eq. (2.39).

Chapter 3

Transverse Magnetic Field

$|\tilde{B}_x| = 1;\ |\tilde{H}_z| = 0$

Both magnetic effects and hydrodynamic effects cause the effective viscosity of ferromagnetic fluids to be different than the viscosity of the carrier fluid. Therefore, not only does the effective viscosity depend on the vortex and dynamic viscosities, $\tilde{\zeta}$ and $\tilde{\eta}$ respectively, it also depends on the field frequency $\tilde{\Omega}$. For simplicity, this thesis assumes that $\tilde{\eta} = \tilde{\zeta}$. Thus, Eq. (2.46) becomes

$$\tilde{\eta}_{eff} = \tilde{\zeta} - \frac{\alpha\tilde{\zeta}}{2\tilde{\zeta} - \alpha} \tag{3.1}$$

Solving this equation for the variable α gives

$$\alpha = \frac{2\tilde{\zeta}(\tilde{\zeta} - \tilde{\eta}_{eff})}{2\tilde{\zeta} - \tilde{\eta}_{eff}} \tag{3.2}$$

Specifically for the case of transverse magnetic field, $|\tilde{B}_x| = 1;\ |\tilde{H}_z| = 0$, Eq. (2.41) reduces to

$$\alpha = \frac{\chi_0}{2}\frac{(\tilde{\Omega}^2 - 1)}{(1 + \chi_0 + \tilde{\Omega}^2)^2 + \chi_0^2\tilde{\Omega}^2} \tag{3.3}$$

By setting Eqs. (3.2) and (3.3) equal to each other, $\tilde{\Omega}$ can be solved for in terms of $\tilde{\zeta}$ and $\tilde{\eta}_{eff}$. This will allow plots of $\tilde{\Omega}(\tilde{\zeta})$ to be made for different values of effective viscosity. The result is a 4th order biquadratic equation in $\tilde{\Omega}$.

$$\tilde{\Omega}^4\Big[(2\tilde{\zeta})(\tilde{\zeta}-\tilde{\eta}_{eff})\Big] + \tilde{\Omega}^2\Big[(\chi_0^2+2\chi_0+2)(2\tilde{\zeta})(\tilde{\zeta}-\tilde{\eta}_{eff}) - \frac{\chi_0}{2}(2\tilde{\zeta}-\tilde{\eta}_{eff})\Big]$$

$$+\Big[(\chi_0^2+2\chi_0+1)(2\tilde{\zeta})(\tilde{\zeta}-\tilde{\eta}_{eff}) + \frac{\chi_0}{2}(2\tilde{\zeta}-\tilde{\eta}_{eff})\Big] = 0 \tag{3.4}$$

To plot $\tilde{\Omega}$ as a function of $\tilde{\zeta}$ for different values of $\tilde{\eta}_{eff}$, the quadratic formula was used to find $\tilde{\Omega}^2$, and then the square root was taken.

$$\tilde{\Omega}^2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \tag{3.5}$$

such that

$$\begin{aligned} a &= 2\tilde{\zeta}(\tilde{\zeta}-\tilde{\eta}_{eff}) & (3.6)\\ b &= (\chi_0^2+2\chi_0+2)(2\tilde{\zeta})(\tilde{\zeta}-\tilde{\eta}_{eff}) - \frac{\chi_0}{2}(2\tilde{\zeta}-\tilde{\eta}_{eff}) & (3.7)\\ c &= (\chi_0+1)^2(2\tilde{\zeta})(\tilde{\zeta}-\tilde{\eta}_{eff}) + \frac{\chi_0}{2}(2\tilde{\zeta}-\tilde{\eta}_{eff}) & (3.8) \end{aligned}$$

Once $\tilde{\Omega}^2$ was calculated, only those values that were both positive and real were plotted.

The values of $\tilde{\eta}_{eff}$ for numerical case studies were chosen to be

$$\tilde{\eta}_{eff} = \{-.1;\ -.05;\ -.025;\ -.01;\ 0;\ .01;\ .025;\ .05;\ .1\}. \tag{3.9}$$

These values were chosen so the plotted curves would be well spaced and show the trend of change as $\tilde{\eta}_{eff}$ goes from negative, through zero, and becomes positive. The resulting plot can be seen in Figure 3-1.

For simplification, the value of the effective magnetic susceptibility χ_0 was taken to be 1 for the calculations of this curve. $\chi_0 = 1$ will be assumed for the remainder of the discussion. This plot was made using a Matlab script file, which can be found in Appendix C.

The interesting feature of this plot is that each positive value of effective viscosity

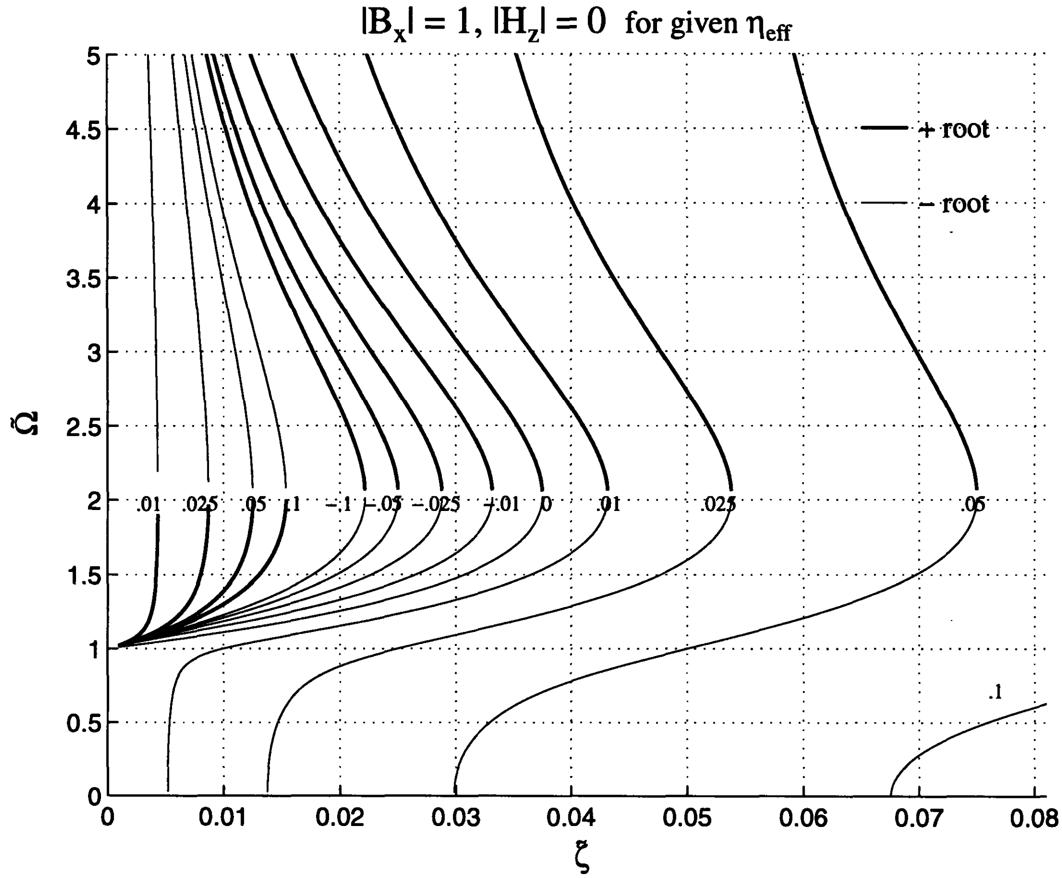


Figure 3-1: Frequency $\tilde{\Omega}$ as a function of viscosity $\tilde{\zeta}$ for nine values of $\tilde{\eta}_{eff}$. The bold lines represent the positive roots of the quadratic, and the plain lines represent the negative roots.

has four roots that are plotted, which look like two general curves. Each negative value of effective viscosity has only two roots plotted which look like one general curve. The other two roots resulted in $\tilde{\Omega}^2$ values that were negative or complex. These curves will be discussed separately as two distinct categories: the curves that fall to the right of the $\tilde{\eta}_{eff} = 0$ curve, and the curves that fall to the left of the $\tilde{\eta}_{eff} = 0$ curve.

3.1 Curves to the Right of $\tilde{\eta}_{eff} = 0$

All of the curves to the right of the $\tilde{\eta}_{eff} = 0$ demarcation curve are plots of positive values of effective viscosity. Although the $\tilde{\eta}_{eff} = 0$ curve follows a horizontal asymptote to $\tilde{\Omega} = 1$, the positive effective viscosity curves to its right range from a frequency of zero to infinity. Larger values of effective viscosity would follow the same curve characteristics, but would continue further and further to the right of the $\tilde{\eta}_{eff} = 0$ demarcation curve.

3.1.1 Spin Velocity $\tilde{\omega}_y(\tilde{x})$ Profiles

As was explained in Section 2.4.2, the spin velocity profile can be plotted using a parametric plot of $\tilde{x}$ and $\tilde{\omega}_y$ while varying $\tilde{\omega}_y$. Non-dimensional position $\tilde{x}$ is a function of six variables: $\tilde{\omega}_y$, $\tilde{\Omega}$, χ_0, $\tilde{\eta}$, $\tilde{\zeta}$, and $\frac{\partial \tilde{p}'}{\partial \tilde{z}}$. For simplicity, the following has been assumed:

- $\tilde{\eta} = \tilde{\zeta}$
- $\chi_0 = 1$
- $\frac{\partial \tilde{p}'}{\partial \tilde{z}} = 1$

Therefore, the important variables of $\tilde{x}$ in Eq. (2.34) become $\tilde{\omega}_y$, which is the ranging variable of the parametric plot, $\tilde{\Omega}$, and $\tilde{\zeta}$. To be able to plot $\tilde{\omega}_y(\tilde{x})$, values for $\tilde{\Omega}$ and $\tilde{\zeta}$ must be chosen.

Looking at the curves in Figure 3-1, six values of frequency were chosen. By choosing values of frequency, $\tilde{\Omega}$, the corresponding values of viscosity, $\tilde{\zeta}$ can be calculated, and both values can be used in the equations for plotting the spin velocity profiles. Instead of solving for $\tilde{\Omega}$ in Eq. (3.4), $\tilde{\zeta}$ is solved for.

$$2\tilde{\zeta}^2\left[\left(1+\chi_0+\tilde{\Omega}^2\right)^2+\chi_0^2\tilde{\Omega}^2\right]-\tilde{\zeta}\left[2\tilde{\eta}_{eff}\left[(1+\chi_0+\tilde{\Omega}^2)^2+\chi_0^2\tilde{\Omega}^2\right]+\chi_0\left[\tilde{\Omega}^2-1\right]\right]$$
$$+\frac{\chi_0}{2}\tilde{\eta}_{eff}\left(\tilde{\Omega}^2-1\right) = 0 \qquad (3.10)$$

To solve for $\tilde{\zeta}$, the quadratic equation is used, where

$$a = 2\Big[(1+\chi_0+\tilde{\Omega}^2)^2+\chi_0^2\tilde{\Omega}^2\Big] \quad (3.11)$$

$$b = -\Big[2\tilde{\eta}_{eff}\Big[(1+\chi_0+\tilde{\Omega}^2)^2+\chi_0^2\tilde{\Omega}^2\Big]+\chi_0(\tilde{\Omega}^2-1)\Big] \quad (3.12)$$

$$c = \frac{\chi_0}{2}\tilde{\eta}_{eff}\Big(\tilde{\Omega}^2-1\Big) \quad (3.13)$$

The values of $\tilde{\Omega}$ were chosen so that the $\tilde{\zeta}$ would be noticably different. They are

$$\tilde{\Omega} = \{1.05,\ 1.3,\ 2.0412,\ 3.25,\ 5.0,\ 10.0\} \quad (3.14)$$

The value of 1.05 was chosen to be just above the horizontal asymptote of the demarcation curve. The value of 2.0412 is the value of $\tilde{\Omega}$ where the positive and negative roots of the quadratic meet. To calculate the corresponding $\tilde{\zeta}$, a Matlab script was used, which can be found in Appendix D. The results are summarized by Table 3-1.

$\tilde{\Omega} = 1.05$			$\tilde{\Omega} = 1.3$			$\tilde{\Omega} = 2.0412$		
$\tilde{\eta}_{eff}$	$\tilde{\zeta}$		$\tilde{\eta}_{eff}$	$\tilde{\zeta}$		$\tilde{\eta}_{eff}$	$\tilde{\zeta}$	
0	.0048	0	0	.0225	0	0	.0375	0
.01	.0129	.0018	.01	.0286	.0039	.01	.0432	.0043
.025	.0276	.0022	.025	.0406	.0069	.025	.0538	.0087
.05	.0525	.0023	.05	.0637	.0088	.05	.0750	.0125
.1	.1024	.0023	.1	.1125	.0100	.1	.1222	.0154
$\tilde{\Omega} = 3.25$			$\tilde{\Omega} = 5.0$			$\tilde{\Omega} = 10.0$		
$\tilde{\eta}_{eff}$	$\tilde{\zeta}$		$\tilde{\eta}_{eff}$	$\tilde{\zeta}$		$\tilde{\eta}_{eff}$	$\tilde{\zeta}$	
0	.0284	0	0	.0159	0	0	.0047	0
.01	.0343	.0041	.01	.0224	.0036	.01	.0129	.0018
.025	.0456	.0078	.025	.0353	.0056	.025	.0276	.0021
.05	.0679	.0104	.05	.0592	.0067	.05	.0525	.0022
.1	.1162	.0122	.1	.1086	.0073	.1	.1024	.0023

Table 3.1: Matlab results of calculating $\tilde{\zeta}$ given some value of $\tilde{\Omega}$ and $\tilde{\eta}_{eff}$ for the positive effective viscosity curves to the right of the $\tilde{\eta}_{eff} = 0$ curve.

There are two values of $\tilde{\zeta}$ for each effective viscosity because there are two roots to the solution. The first column of $\tilde{\zeta}$ represents the curves that fall to the right of the $\tilde{\eta}_{eff} = 0$ demarcation curve. The second column of $\tilde{\zeta}$ represents those curves that fall to the left of the demarcation curve and will be discussed in section (3.2).

To find the range of $\tilde{\omega}_y$ in the parametric plot, the values must be known at the boundaries. Because $\tilde{\omega}_y$ is an odd function of $\tilde{x}$ around $\tilde{x} = 0.5$,

$$\tilde{\omega}_0 = \tilde{\omega}_y(\tilde{x} = 0) = -\tilde{\omega}_y(\tilde{x} = 1) \tag{3.15}$$

Thus, it was necessary to calculate $\tilde{\omega}_0$, which is the value of $\tilde{\omega}_y$ at the boundary of $\tilde{x} = 0$. For this calculation, Mathematica was used, and the results are summarized in Table 3-2. For the calculation, the general command is

$$Solve[\tilde{x}[\tilde{\omega}_0, \tilde{\Omega}, \chi_0, \tilde{\zeta}, \frac{\partial \tilde{p}'}{\partial \tilde{z}}] == 0, \tilde{\omega}_0] \tag{3.16}$$

which means "Solve for $\tilde{\omega}_0$ when $\tilde{x} = 0$." The appropriate values of $\tilde{\Omega}$ and $\tilde{\zeta}$ were plugged in, and the values $\chi_0 = 1$ and $\frac{\partial \tilde{p}'}{\partial \tilde{z}} = 1$ were used. The Mathematica file can be found in Appendix B.

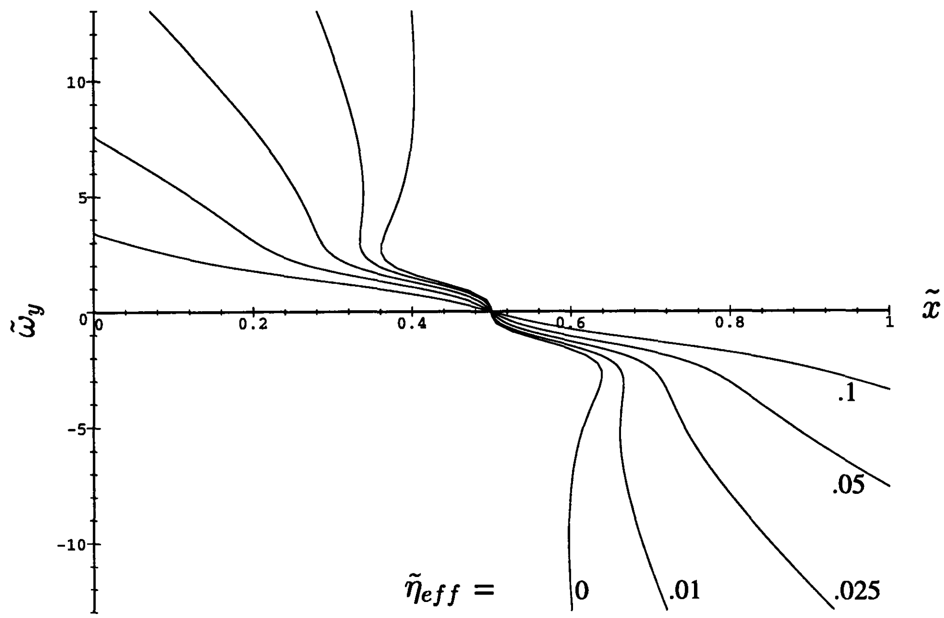
$\tilde{\Omega} = 1.05$		$\tilde{\Omega} = 1.3$		$\tilde{\Omega} = 2.0412$	
$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$
.0048	103.157	.0225	21.176	.0375	12.2371
.0129	37.7339	.0286	16.4234	.0432	10.4593
.0276	17.0611	.0406	11.2311	.0538	8.14266
.0525	8.42516	.0637	6.72355	.0750	5.45623
.1024	3.79722	.1125	3.39117	.1222	3.19134
$\tilde{\Omega} = 3.25$		$\tilde{\Omega} = 5.0$		$\tilde{\Omega} = 10.0$	
$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$
.0284	16.5104	.0159	30.3867	.0047	105.364
.0343	13.4498	.0224	21.2167	.0129	37.6553
.0456	9.75997	.0353	12.9145	.0276	16.4497
.0679	5.98232	.0592	6.77365	.0525	9.80991
.1162	3.7348	.1086	4.74664	.1024	5.18971

Table 3.2: Mathematica results of calculating spin velocity $\tilde{\omega}_0$ at the $\tilde{x} = 0$ planar duct wall for the positive effective viscosity curves to the right of the $\tilde{\eta}_{eff} = 0$ curve.

All of these values of $\tilde{\Omega}$, $\tilde{\zeta}$, and $\tilde{\omega}_0$ were used in Maple to plot the spin velocity profiles which are seen in Figure 3-2.

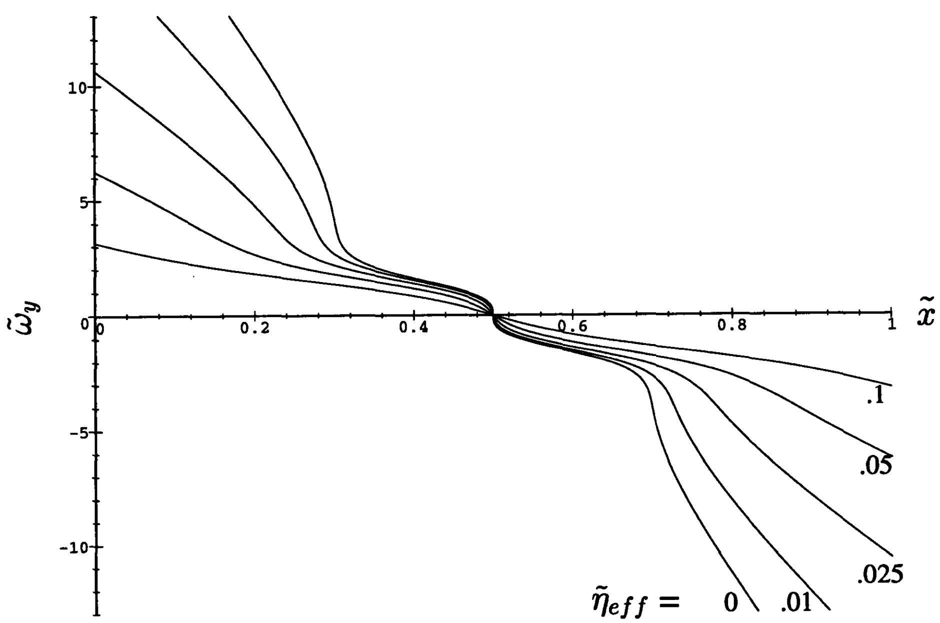
Figure 3-2: Spin velocity spatial distributions for various values of $\tilde{\Omega}$ and $\tilde{\eta}_{eff}$ to the right of the $\tilde{\eta}_{eff} = 0$ curve. a) $\tilde{\Omega} = 1.05$, b) $\tilde{\Omega} = 1.3$, c) $\tilde{\Omega} = 2.0412$, d) $\tilde{\Omega} = 3.25$, e) $\tilde{\Omega} = 5.0$, f) $\tilde{\Omega} = 10.0$.

$$|\tilde{B}_x| = 1, |\tilde{H}_z| = 0$$



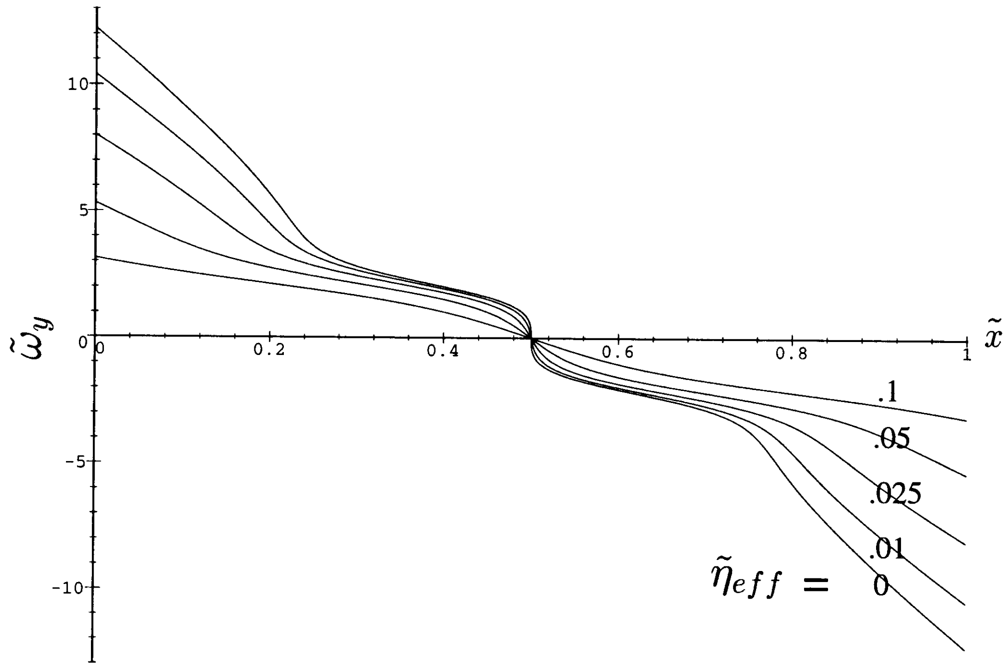
a: $\tilde{\Omega} = \mathbf{1.05}$

$$|\tilde{B}_x| = 1, |\tilde{H}_z| = 0$$



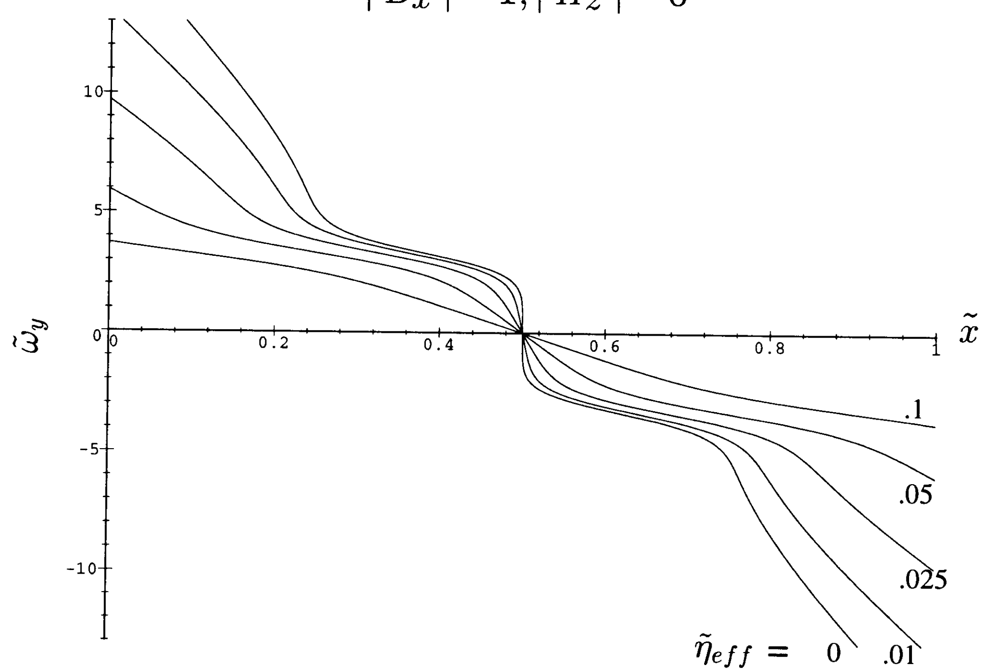
b: $\tilde{\Omega} = \mathbf{1.3}$

$\mid \tilde{B}_x \mid = 1, \mid \tilde{H}_z \mid = 0$



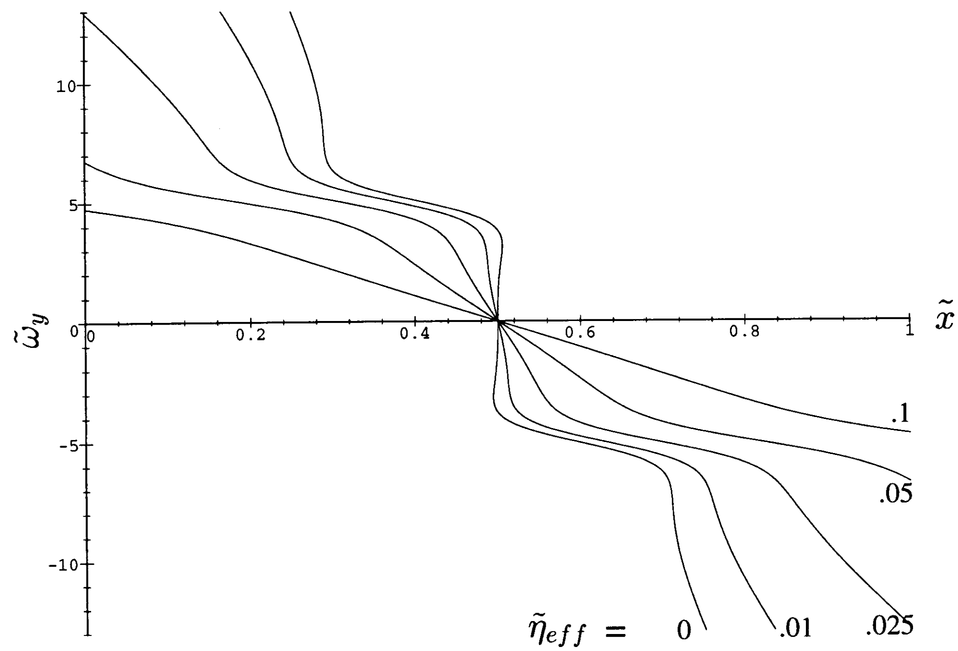
c: $\tilde{\mathbf{\Omega}} = \mathbf{2.0412}$

$\mid \tilde{B}_x \mid = 1, \mid \tilde{H}_z \mid = 0$



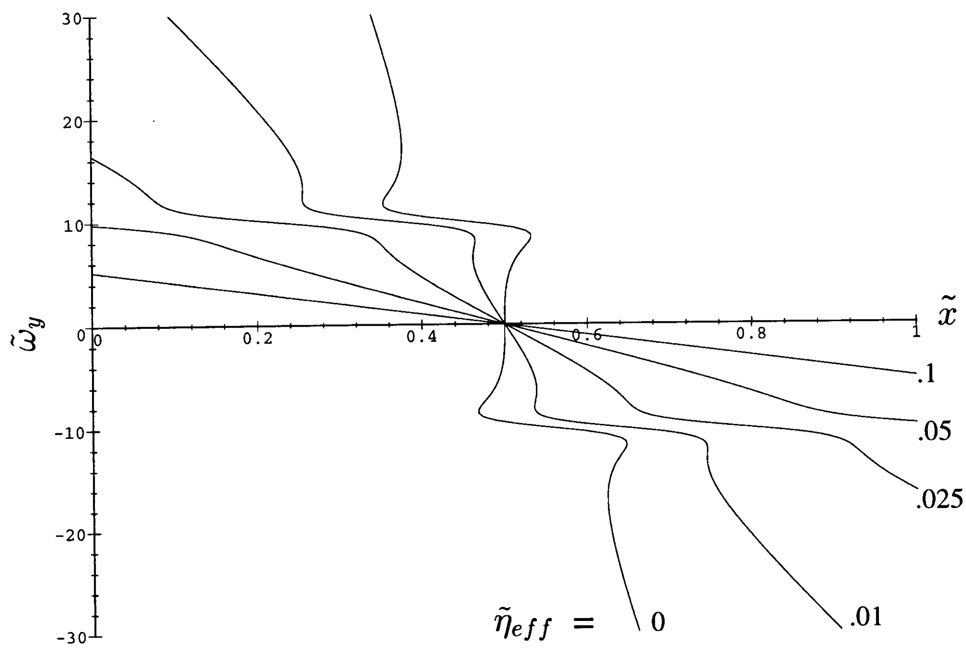
d: $\tilde{\mathbf{\Omega}} = \mathbf{3.25}$

$$|\tilde{B}_x| = 1, |\tilde{H}_z| = 0$$



e: $\tilde{\mathbf{\Omega}} = \mathbf{5.0}$

$$|\tilde{B}_x| = 1, |\tilde{H}_z| = 0$$



f: $\tilde{\mathbf{\Omega}} = \mathbf{10.0}$

The most interesting feature of these profiles is the fact that $\tilde{\omega}_y$ can be double-valued over a range of $\tilde{x}$ both at low frequencies and at high frequencies. There is a range around $\tilde{\Omega} = 2.0412$ where the function behaves as a single-valued function. This frequency is where the positive and negative roots of all curves seen in Figure 3-1 meet. At low frequencies, the function can be double-valued in two regions while at high frequencies, the function can be multi-valued in four regions with the middle regions being triple-valued.

At high enough effective viscosity, the spin velocity profiles are single-valued. For smaller values of effective viscosity, including zero, some profiles become double-valued.

The region around $\tilde{x} = 0.5$, half-way between the planar duct boundaries, is the most interesting. Here the spin velocity must be zero by symmetry where the small spin velocity approximation is valid. In this region, it is only the larger frequencies that are multi-valued.

3.1.2 Flow Velocity $\tilde{v}_z(\tilde{x})$ Profiles

Using Eqs. (2.36) - (2.38), the flow velocity equations can be plotted using a parametric plot of $\tilde{x}$ and $\tilde{f}$ while varying $\tilde{\omega}_y$. The flow velocity $\tilde{f}$ is effectively a function of seven variables: $\tilde{\omega}_y$ which is a function of $\tilde{x}$, $\tilde{\Omega}$, χ_0, $\tilde{\eta}$, $\tilde{\zeta}$, and $\frac{\partial \tilde{p}'}{\partial \tilde{z}}$. Again, for simplicity, we continue to use:

- $\tilde{\eta} = \tilde{\zeta}$
- $\chi_0 = 1$
- $\frac{\partial \tilde{p}'}{\partial \tilde{z}} = 1$

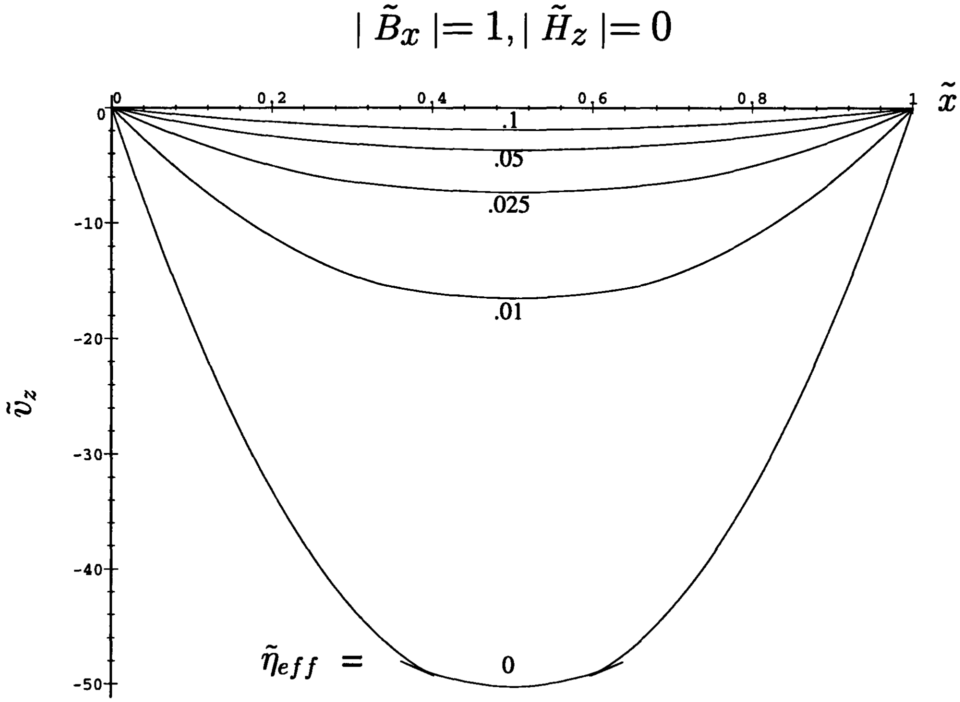
Thus, the important variables for calculating $\tilde{v}_z(\tilde{x})$ in Eq. (2.36) are $\tilde{\omega}_y$, which is the variable that is ranged in the parametric plot, $\tilde{\Omega}$, and $\tilde{\zeta}$. These values have already been calculated in Table 3-1. The boundary values of $\tilde{\omega}_0$ to use for the ranges have also been calculated in Table 3-2.

There is an extra step required before the flow velocity profiles can be plotted. The function $\tilde{f}$ represents the flow velocity as a function of $\tilde{x}$. However, the constant of integration D in Eq. (2.36) has not been calculated since it could not be done analytically. Each case must have its constant of integration calculated. The requirement is that the flow velocity, $\tilde{v}_z$ must be zero at the boundaries of $\tilde{x} = 0$ and $\tilde{x} = 1$. Therefore, the flow velocity of each case was calculated using the Maple equation

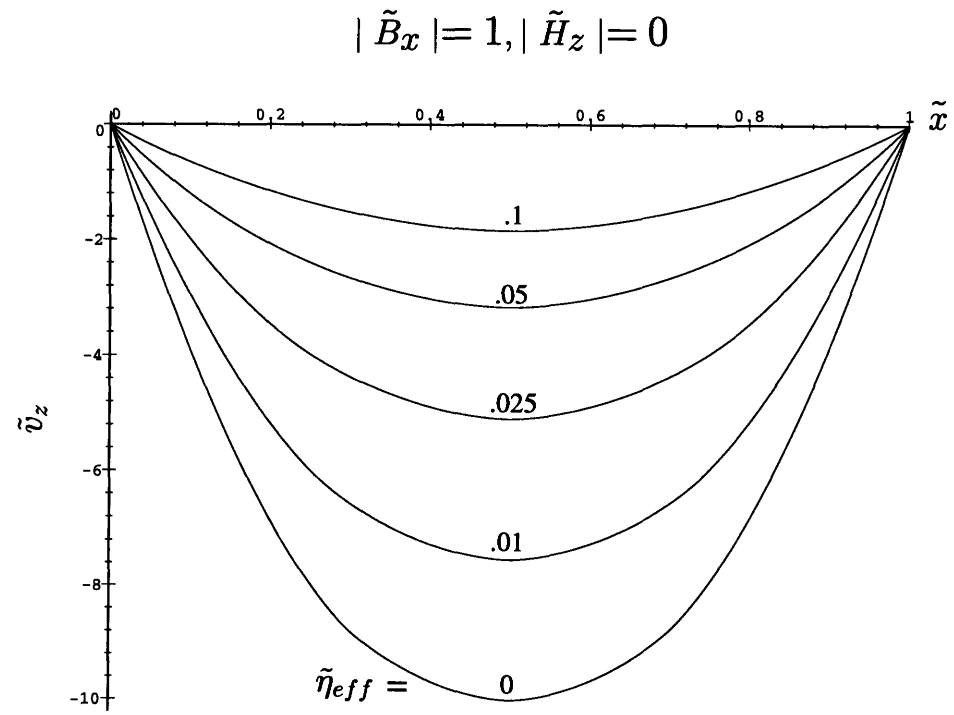
$$\tilde{v}_z(\tilde{\omega}_y) = \tilde{f}(\tilde{\omega}_y, \tilde{\Omega}, \tilde{\zeta}, \frac{\partial \tilde{p}'}{\partial \tilde{z}}) \; - \; eval(subs(\tilde{\omega}_0 = \text{value}, f(\tilde{\omega}_0, \tilde{\Omega}, \tilde{\zeta}, \frac{\partial \tilde{p}'}{\partial \tilde{z}})) \tag{3.17}$$

which means "$\tilde{v}_z$ equals the function $\tilde{f}$ minus that same function evaluated at the boundary by substituting $\tilde{\omega}_0$ for $\tilde{\omega}_y$." For each case, the appropriate values of $\tilde{\Omega}$ and $\tilde{\zeta}$ were used. Figure 3-3 shows the flow velocity profiles for the 6 values of frequency.

Figure 3-3: Linear flow velocity distributions for various values of $\tilde{\Omega}$ and $\tilde{\eta}_{eff}$ that fall to the right of the $\tilde{\eta}_{eff} = 0$ demarcation curve. a) $\tilde{\Omega} = 1.05$, b) $\tilde{\Omega} = 1.3$, c) $\tilde{\Omega} = 2.0412$, d) $\tilde{\Omega} = 3.25$, e) $\tilde{\Omega} = 5.0$, f) $\tilde{\Omega} = 10.0$.

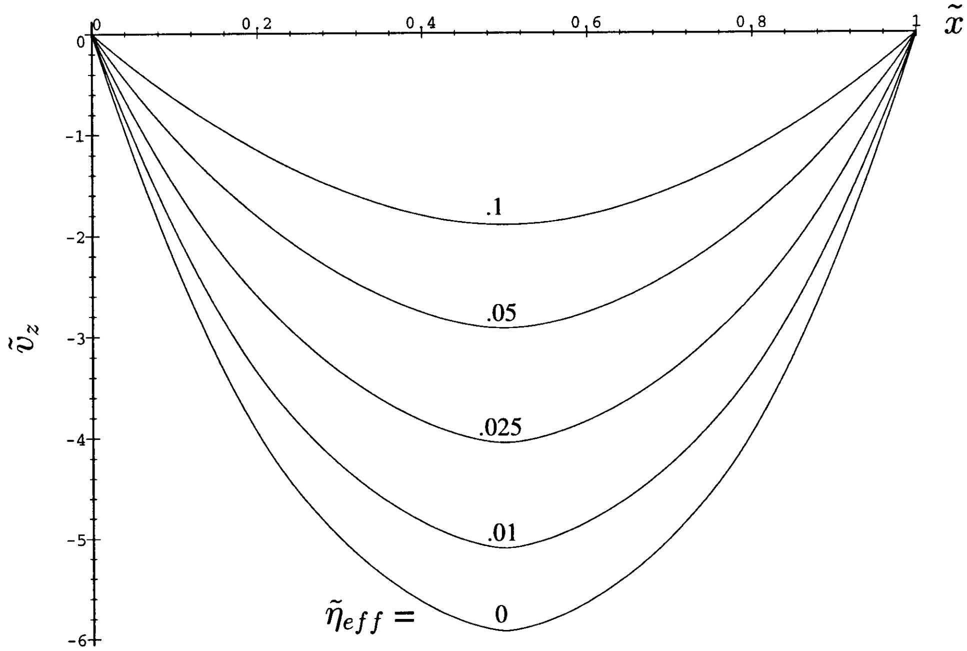


a: $\mathbf{\tilde{\Omega} = 1.05}$



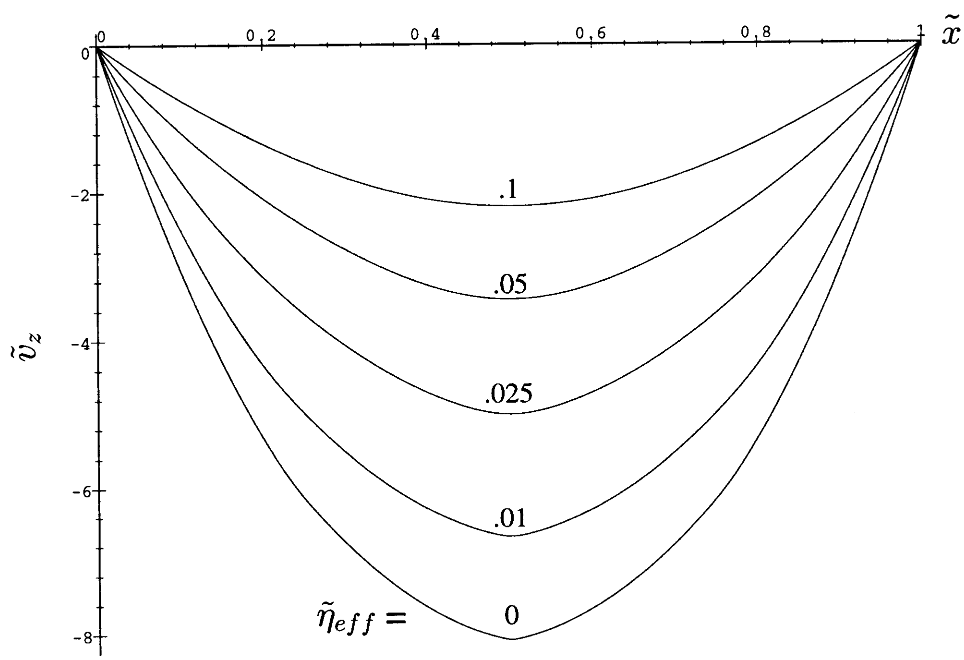
b: $\mathbf{\tilde{\Omega} = 1.3}$

$\mid \tilde{B}_x \mid = 1, \mid \tilde{H}_z \mid = 0$



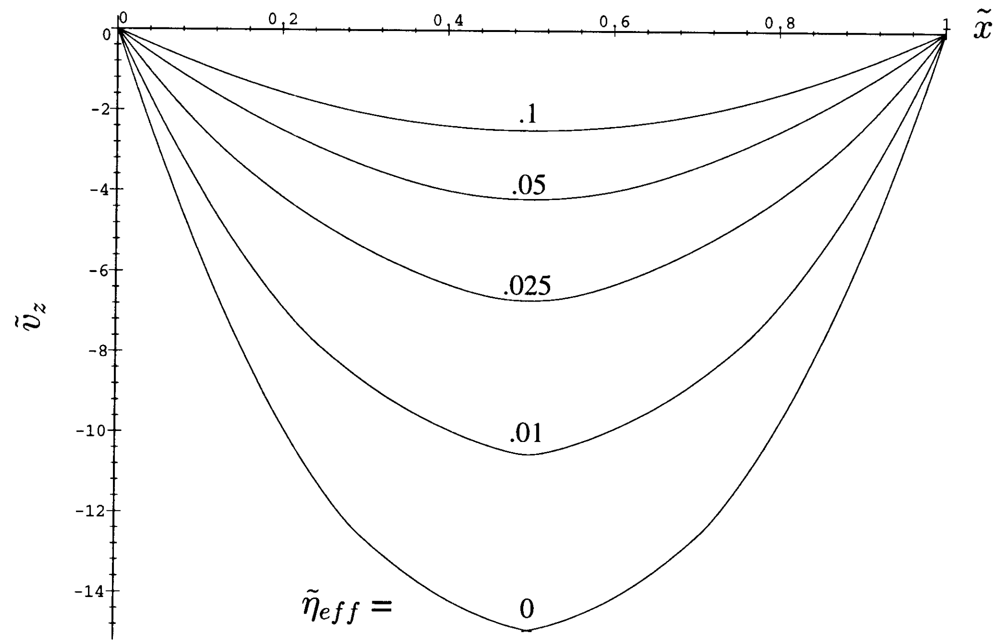
c: $\tilde{\mathbf{\Omega}} = \mathbf{2.0412}$

$\mid \tilde{B}_x \mid = 1, \mid \tilde{H}_z \mid = 0$



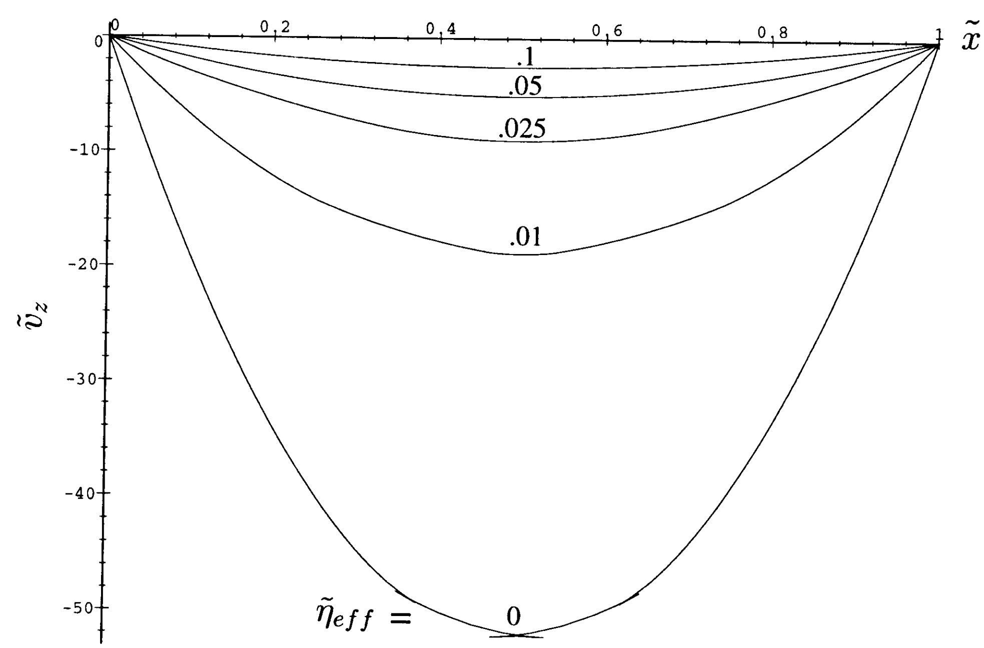
d: $\tilde{\mathbf{\Omega}} = \mathbf{3.25}$

$\mid \tilde{B}_x \mid = 1, \mid \tilde{H}_z \mid = 0$



e: $\tilde{\Omega} = \mathbf{5.0}$

$\mid \tilde{B}_x \mid = 1, \mid \tilde{H}_z \mid = 0$



f: $\tilde{\Omega} = \mathbf{10.0}$

With these profiles, we see that the flow velocity is large in the center of the duct for very small frequencies. Then the flow velocity decreases as the frequency increases to the value $\tilde{\Omega} = 2.0412$ which is the value where the positive and negative roots meet in Figure 3-1. Increasing the frequency further causes the flow velocity to become larger again.

These figures show that the larger the effective viscosity, the more constant the peak value of $\tilde{v}_z$ at $\tilde{x} = 0.5$ with changes in $\tilde{\Omega}$. For $\tilde{\eta}_{eff} = .1$, we see that the nondimensional peak flow velocity is always about 2. This agrees with intuition which requires that a larger viscosity would resist change in velocity. The smaller the effective viscosity, the more the flow velocity changes with a change in frequency.

We also see that these flow velocity profiles can be multi-valued as well. The scaling of the figures do not allow this fact to be easily seen. For the frequency $\tilde{\Omega} = 1.05$ we can see that for $\tilde{\eta}_{eff} = 0$, the flow velocity is triple-valued in two places. Upon increasing the scale of the figures, we find that the $\tilde{\eta}_{eff} = .01$ plot is triple-valued in two places as well. This can be seen in Figure 3-4.

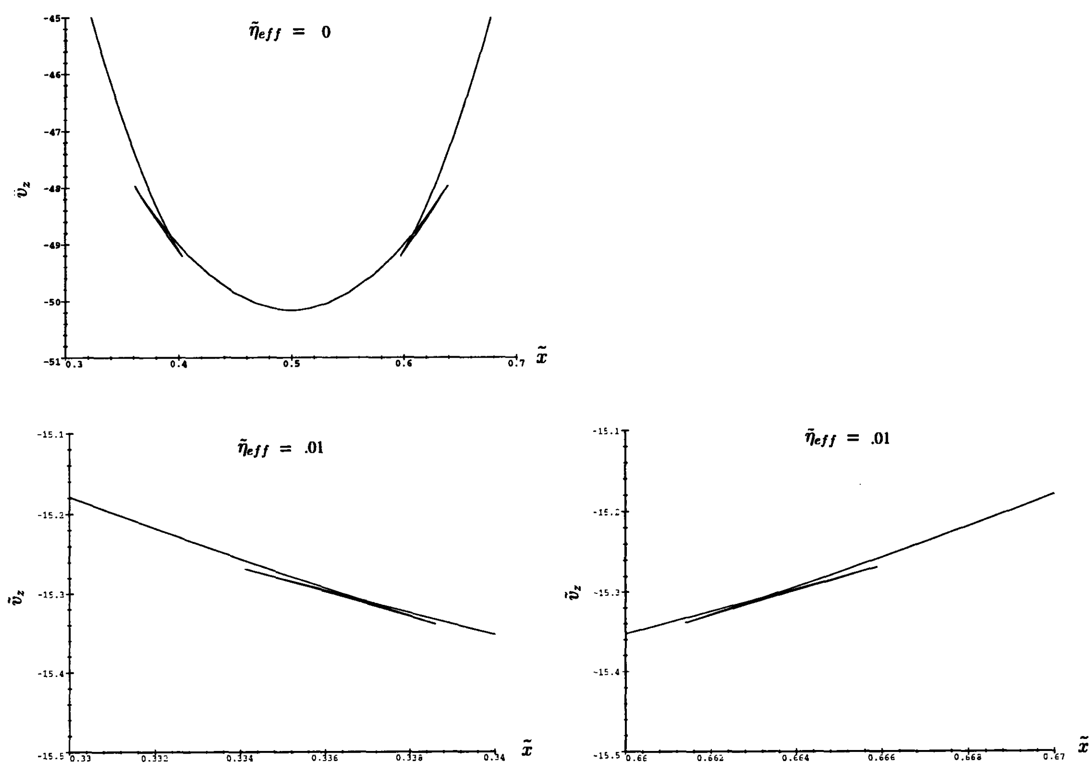


Figure 3-4: Increased scaling of flow velocity profiles for $\tilde{\Omega} = 1.05$ shows that $\tilde{\eta}_{eff} = .01$ is multi-valued as well as $\tilde{\eta}_{eff} = 0$.

Then, as the frequency increases, the profiles become single-valued for all values of effective viscosity. Further increasing the frequency above $\tilde{\Omega} = 2.0412$ causes the profiles of small effective viscosity to be multi-valued again. At $\tilde{\Omega} = 5$ we see that $\tilde{\eta}_{eff} = 0$ has become triple-valued again, but this time it is a small region in the center of the duct at $\tilde{x} = .5$. At $\tilde{\Omega} = 10$ we see that $\tilde{\eta}_{eff} = 0$ becomes triple-valued in three small regions. It is difficult to see whether the $\tilde{\eta}_{eff} = .01$ plot has become triple-valued. Thus we look back to the spin velocity profile for $\tilde{\Omega} = 10$ and $\tilde{\eta}_{eff} = .01$ in Figure 3-2 (f). From this figure, we see that the spin-velocity for $\tilde{\eta}_{eff} = .01$ is indeed multi-valued, possibly in 4 places. So we infer that the flow-velocity is multi-valued as well, but it is multi-valued over such a small region that it is not visible on the plot in Figure 3-3 (f).

Looking only in the region around $\tilde{x} = .5$, where the small spin-velocity approximation is valid, the flow velocity does not become multi-valued until larger frequencies (greater than $\tilde{\Omega} = 2.0412$). For the largest value of frequency considered, $\tilde{\Omega} = 10$, only the $\tilde{\eta}_{eff} = 0$ curve is multi-valued at $\tilde{x} = .5$. If the frequency is increased enough, will the other effective viscosity curves become multi-valued in this region? This question is difficult to conclude by extrapolating the trend of the flow velocity profiles. Looking back at the trends of spin velocity as the frequency is increased, Figures 3-2 (a) through (f), it is possible to deduce that sufficiently large frequencies will cause the spin velocities to be multi-valued for some effective viscosities that are greater than zero. The smaller effective viscosities will become multi-valued in this region before the larger effective viscosities.

3.2 Curves to the Left of $\tilde{\eta}_{eff} = 0$

The curves to the left of the $\tilde{\eta}_{eff} = 0$ demarcation curve in Figure 3-1 are plots of both positive and negative values of effective viscosity. All of these curves follow a horizontal asymptote at $\tilde{\Omega} = 1$. The left-most curves are positive effective viscosity, and as the value of effective viscosity is increased, the curves continue to the right with smaller and smaller spacing until a limit curve of $\tilde{\eta}_{eff} = +\infty$ is reached. In addition, as the value of the negative viscosity curves continues to decrease, the curves continue to the left with smaller and smaller spacing until they reach the same limit curve from the other side with $\tilde{\eta}_{eff} = -\infty$.

3.2.1 Spin Velocity $\tilde{\omega}_y(\tilde{x})$ Profiles

The same method of plotting the spin velocity profiles in section 3.1 was used for the profiles to the left of the $\tilde{\eta}_{eff} = 0$ demarcation curve. This time, the second column of viscosities in Table 3.1 represent the positive effective viscosity curves to the left of the demarcation curve and are repeated in Table 3.3. In addition, the viscosities of the negative effective viscosity curves were also calculated with the same program found in Appendix D and are also listed in Table 3.3. There is only one real value of $\tilde{\zeta}$ for each effective viscosity even though the equation is second-order. The second root is complex.

$\tilde{\Omega} = 1.05$		$\tilde{\Omega} = 1.3$		$\tilde{\Omega} = 2.0412$	
$\tilde{\eta}_{eff}$	$\tilde{\zeta}$	$\tilde{\eta}_{eff}$	$\tilde{\zeta}$	$\tilde{\eta}_{eff}$	$\tilde{\zeta}$
0	.0048	0	.0225	0	.0375
.01	.0018	.01	.0039	.01	.0043
.025	.0022	.025	.0069	.025	.0087
.05	.0023	.05	.0088	.05	.0125
.1	.0023	.1	.0100	.1	.0154
-.1	.0024	-.1	.0125	-.1	.0222
-.05	.0025	-.05	.0137	-.05	.0250
-.025	.0026	-.025	.0156	-.025	.0288
-.01	.0029	-.01	.0186	-.01	.0332
$\tilde{\Omega} = 3.25$		$\tilde{\Omega} = 5.0$		$\tilde{\Omega} = 10.0$	
$\tilde{\eta}_{eff}$	$\tilde{\zeta}$	$\tilde{\eta}_{eff}$	$\tilde{\zeta}$	$\tilde{\eta}_{eff}$	$\tilde{\zeta}$
0	.0284	0	.0159	0	.0047
.01	.0041	.01	.0036	.01	.0018
.025	.0078	.025	.0056	.025	.0021
.05	.0104	.05	.0067	.05	.0022
.1	.0122	.1	.0073	.1	.0023
-.1	.0162	-.1	.0086	-.1	.0024
-.05	.0179	-.05	.0092	-.05	.0025
-.025	.0206	-.025	.0103	-.025	.0026
-.01	.0243	-.01	.0124	-.01	.0029

Table 3.3: Matlab results of calculating $\tilde{\zeta}$ given some $\tilde{\Omega}$ and $\tilde{\eta}_{eff}$ for the curves of positive and negative effective viscosity to the left of the $\tilde{\eta}_{eff} = 0$ curve.

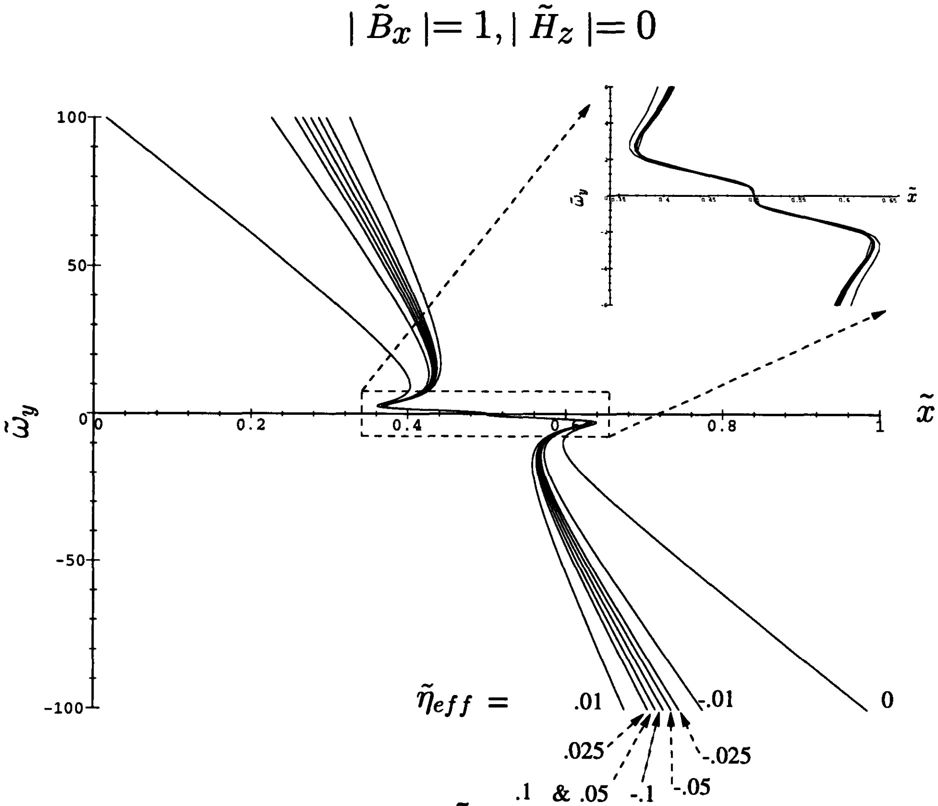
Because $\tilde{\omega}_y$ is the variable ranged in the parametric plot, the values of $\tilde{\omega}_y(\tilde{x} = 0) = \tilde{\omega}_0$ were calculated with the same Mathematica program that calculated the values in Table 3.2 using Eq. (3.16). The appropriate values of $\tilde{\Omega}$ and $\tilde{\zeta}$ from Table 3.3 were used. The file can be found in Appendix D.

$\tilde{\Omega} = 1.05$		$\tilde{\Omega} = 1.3$		$\tilde{\Omega} = 2.0412$	
$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$
.0048	103.157	.0225	21.176	.0375	12.2371
.0018	276.774	.0039	127.197	.0043	115.27
.0022	226.268	.0069	71.4498	.0087	56.4529
.0023	216.387	.0088	55.8004	.0125	38.9729
.0023	216.387	.0100	48.9797	.0154	31.4335
.0024	207.329	.0125	38.9746	.0222	21.4711
.0025	198.995	.0137	35.4685	.0250	18.941
.0026	191.302	.0156	31.0194	.0288	16.2913
.0029	171.408	.0186	25.8436	.0322	13.9774
$\tilde{\Omega} = 3.25$		$\tilde{\Omega} = 5.0$		$\tilde{\Omega} = 10.0$	
$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$
.0284	16.5104	.0159	30.3867	.0047	105.364
.0041	120.942	.0036	137.88	.0018	276.773
.0078	63.0845	.0056	88.2714	.0021	237.089
.0104	47.0517	.0067	73.6089	.0022	226.266
.0122	39.953	.0073	67.4731	.0023	216.385
.0162	29.8204	.0086	57.1147	.0024	207.326
.0179	26.8831	.0092	53.3206	.0025	198.992
.0206	23.2116	.0103	47.512	.0026	191.3
.0243	19.5003	.0124	39.2813	.0029	171.405

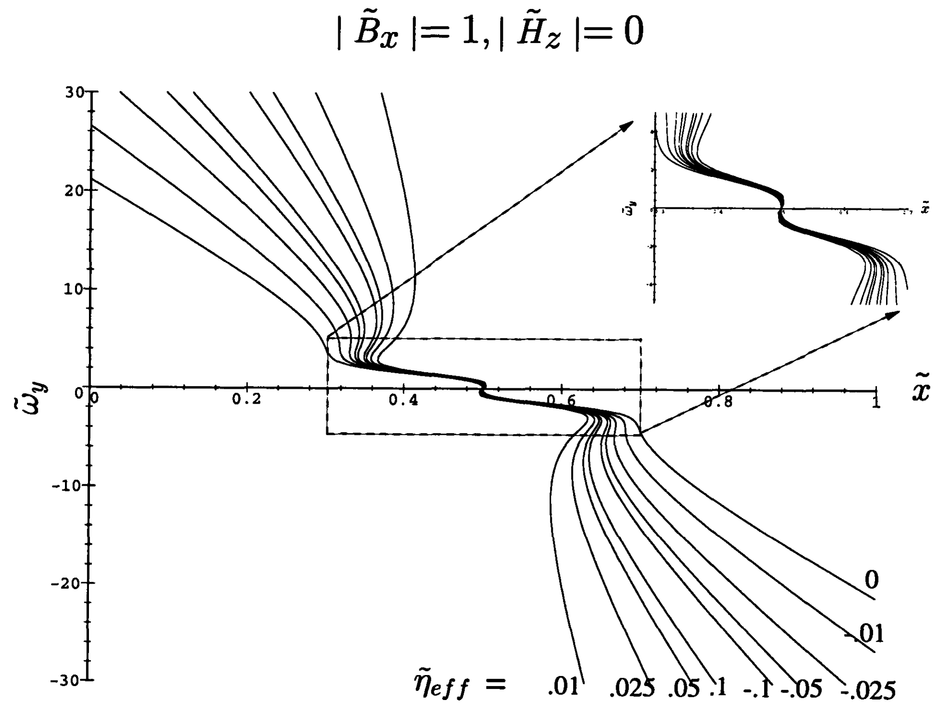
Table 3.4: Mathematica results of calculating spin velocity $\tilde{\omega}_0$ at the $\tilde{x} = 0$ planar duct wall for the positive and negative effective viscosity curves to the left of the $\tilde{\eta}_{eff} = 0$ curve.

These values of $\tilde{\Omega}, \tilde{\zeta}$, and $\tilde{\omega}_0$ in Tables 3.3 and 3.4 were used in Maple to plot the spin velocity profiles which are seen in Figure 3-5.

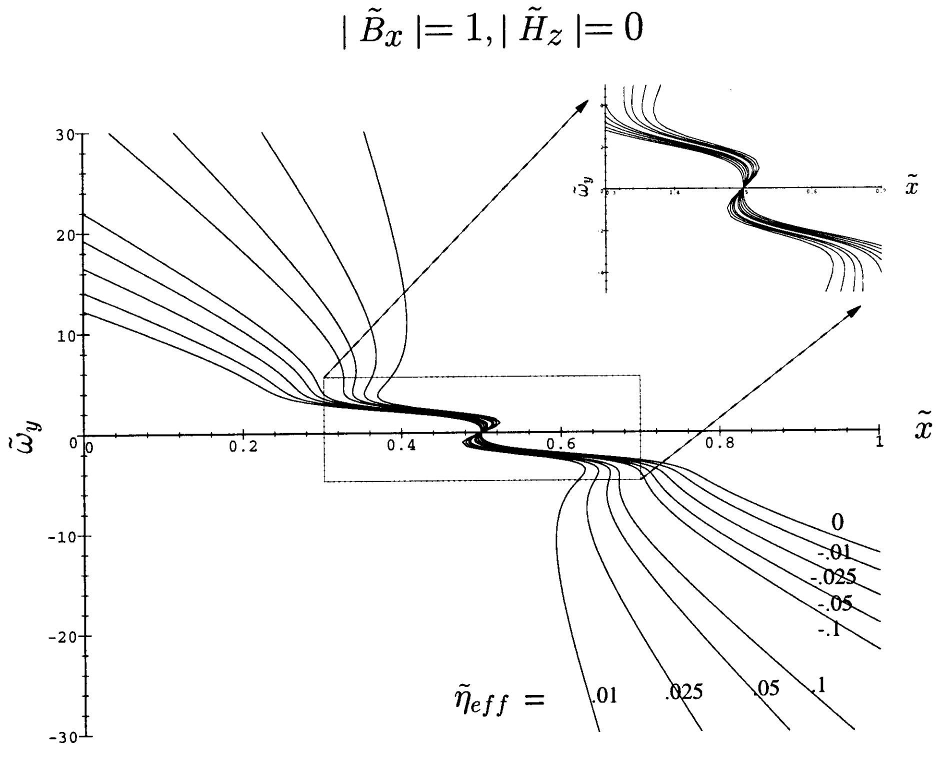
Figure 3-5: Spin velocity spatial distributions for various values of $\tilde{\Omega}$ and $\tilde{\eta}_{eff}$ to the left of the $\tilde{\eta}_{eff} = 0$ curve. a) $\tilde{\Omega} = 1.05$, b) $\tilde{\Omega} = 1.3$, c) $\tilde{\Omega} = 2.0412$, d) $\tilde{\Omega} = 3.25$, e) $\tilde{\Omega} = 5.0$, f) $\tilde{\Omega} = 10.0$.



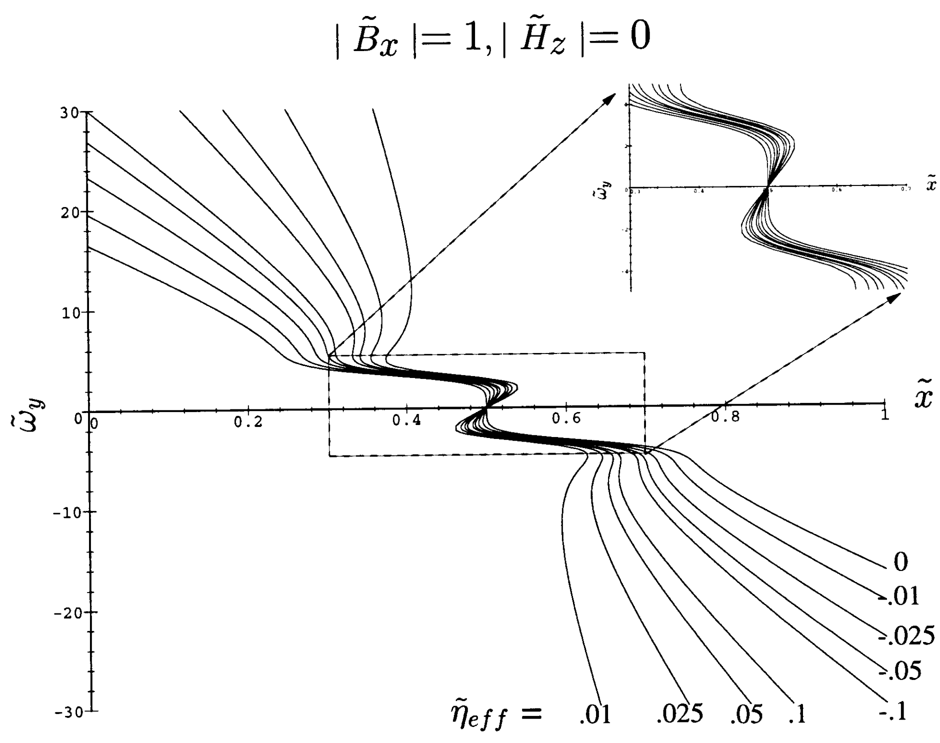
a: $\tilde{\Omega} = 1.05$



b: $\tilde{\Omega} = 1.3$

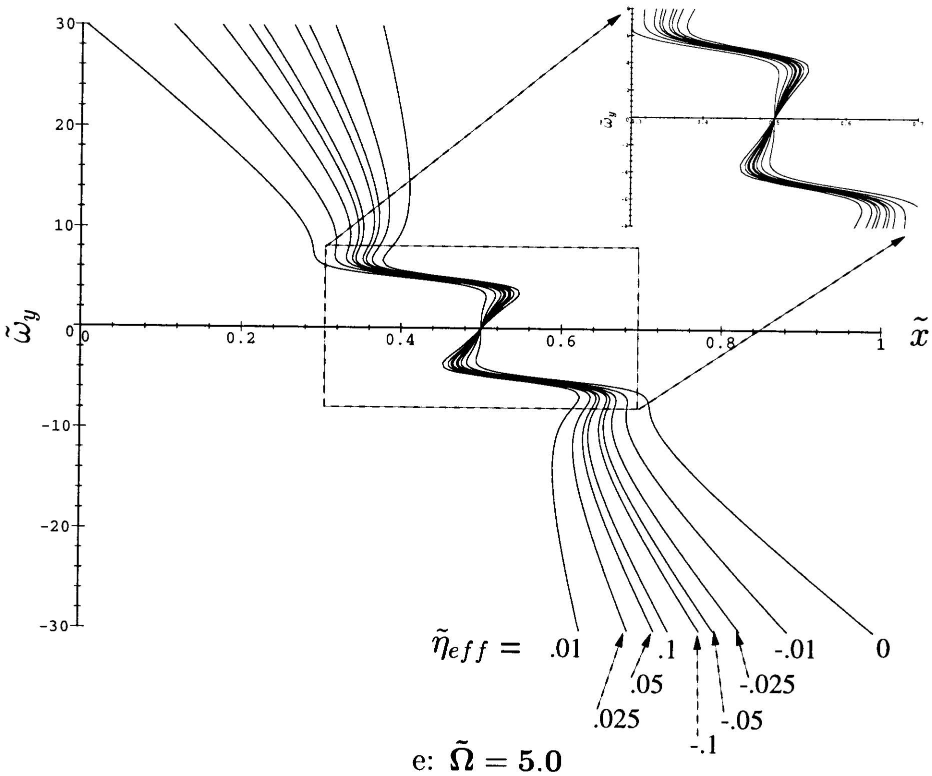


c: $\tilde{\boldsymbol{\Omega}} = \mathbf{2.0412}$



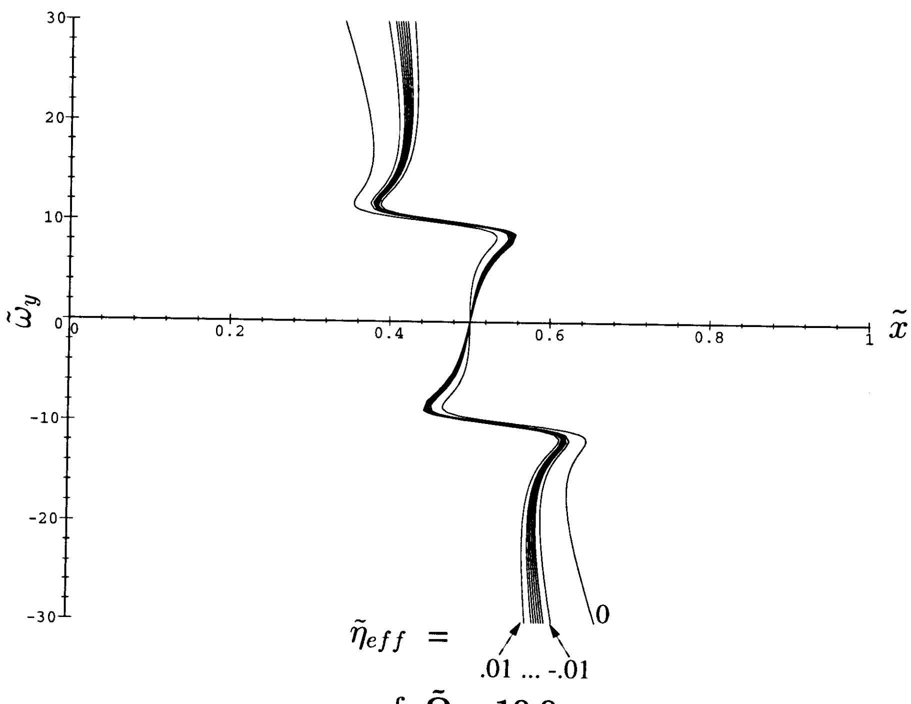
d: $\tilde{\boldsymbol{\Omega}} = \mathbf{3.25}$

$$| \tilde{B}_x | = 1, | \tilde{H}_z | = 0$$



e: $\tilde{\boldsymbol{\Omega}} = \mathbf{5.0}$

$$| \tilde{B}_x | = 1, | \tilde{H}_z | = 0$$



f: $\tilde{\boldsymbol{\Omega}} = \mathbf{10.0}$

All of these profiles are multi-valued in $\tilde{\omega}_y$. The first profile, $\tilde{\Omega} = 1.05$, is multi-valued in two regions. But a slight increase in $\tilde{\Omega}$ causes most of the effective viscosity curves to be multi-valued in four regions. Here it is $\tilde{\eta}_{eff} = 0$ that is last to become multi-valued in the middle regions. The positive effective viscosities are first to become triple-valued in the middle regions, whereas $\tilde{\eta}_{eff} = 0$ has ceased to be multi-valued in any region. Among the frequencies examined, the profiles for $\tilde{\Omega} = 1.3$ and greater have effective viscosity curves that are triple-valued in the middle regions near $\tilde{x} = 0.5$, and double-valued in the outer regions. It is not until $\tilde{\Omega} = 5.0$ that all effective viscosity curves are triple-valued in the middle regions.

The region around $\tilde{x} = 0.5$ is most interesting since the small spin velocity approximation is valid. The curves of positive effective viscosity become triple-valued at significantly lower frequencies than the positive effective viscosity curves that fall to the right of the $\tilde{\eta}_{eff} = 0$ demarcation curve of Figure 3-1. The negative effective viscosity curves follow the same trend as the positive effective viscosity curves that fall to the left of the $\tilde{\eta}_{eff} = 0$ curve. These curves are spaced the same way the curves in Figure 3-1 are spaced. That is, the positive effective viscosity curves approach a limit curve at $\tilde{\eta}_{eff} = +\infty$, which is the same limit curve of the negative effective viscosity curves at $\tilde{\eta}_{eff} = -\infty$.

The trend as the frequency increases is for the middle triple-valued regions to spread out in $\tilde{x}$. Most of the outer regions become double-valued at larger frequencies. With this trend, it is possible for frequencies greater than the examined $\tilde{\Omega} = 10$ that the curves become quadruple-valued as the middle region spreads to the outer region. Also, looking at the $\tilde{\eta}_{eff} = 0$ curve in Figure 3-5 (f) for $\tilde{\Omega} = 10$, we see that the zig-zag behavior in a very small region around $\tilde{x} = 0.5$ becomes more pronounced. It is also possible for larger frequencies that the value of $\tilde{\omega}_y$ have 5 values in this small region. If this trend does continue, for very large frequencies, the spin velocity profiles will become vertical curves, zig-zagging in the central region between $\tilde{x} \approx 0.4$ and $\tilde{x} \approx 0.6$ with multiple-values of $\tilde{\omega}_y$ for a given $\tilde{x}$, including large values of $\tilde{\omega}_y$.

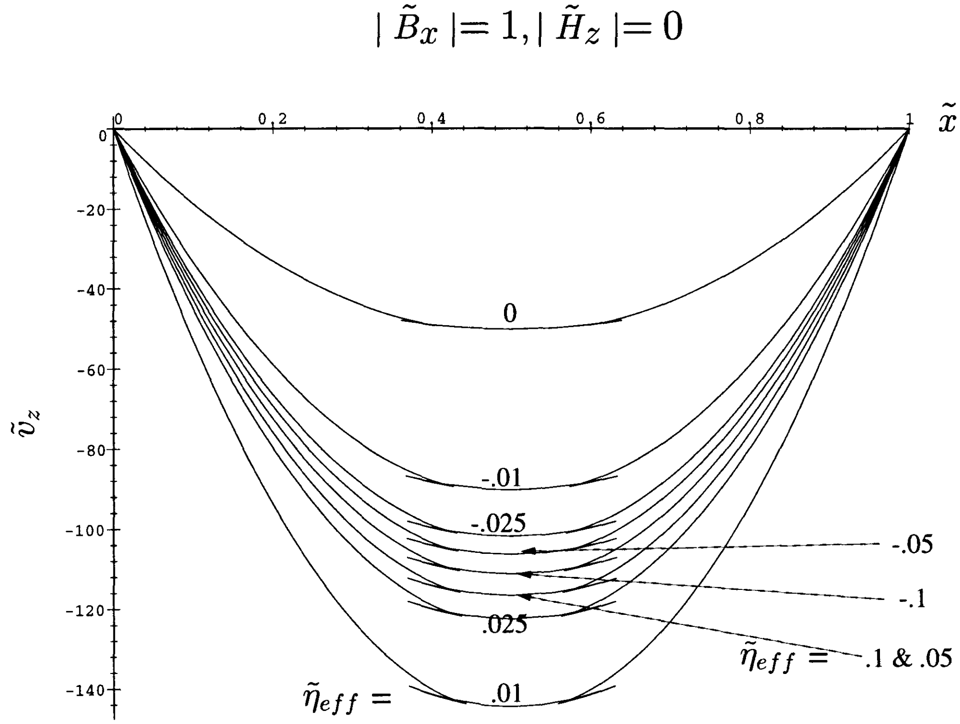
3.2.2 Flow Velocity $\tilde{v}_z(\tilde{x})$ Profiles

Similar to section 3.1.2, Eqs. (2.36) - (2.38) were used to plot the flow velocity profiles, seen in Figure 3-6, for the effective viscosity curves that fall to the left of the $\tilde{\eta}_{eff} = 0$ demarcation curve. The appropriate values of $\tilde{\Omega}$, $\tilde{\zeta}$, and $\tilde{\omega}_0$ from Tables 3.3 and 3.4 were used.

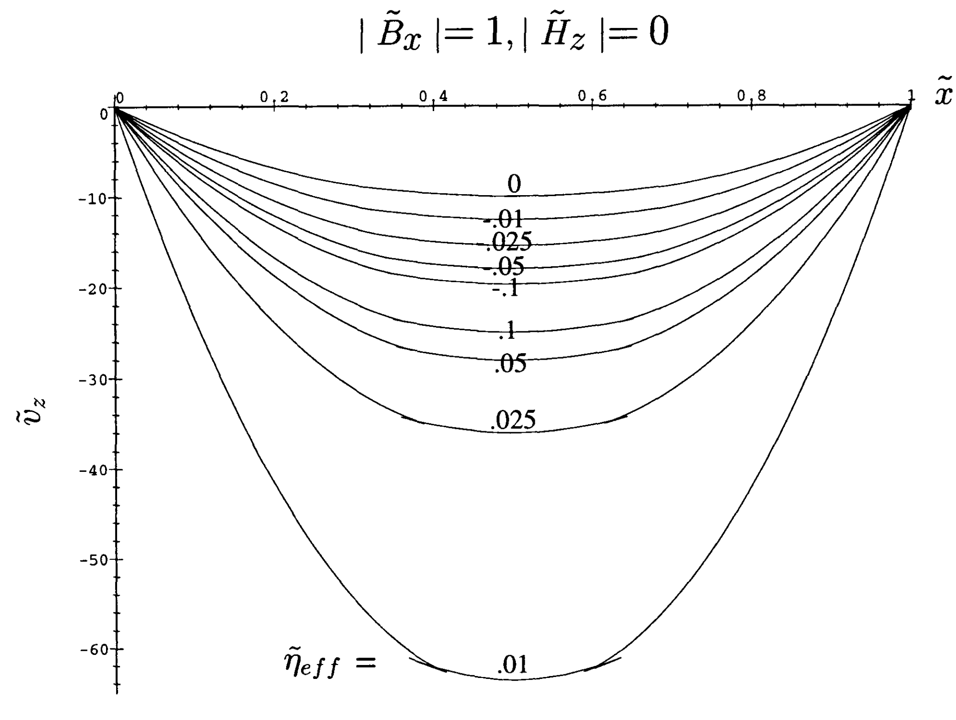
Similar to the flow velocity profiles for the effective viscosity curves to the right of the $\tilde{\eta}_{eff} = 0$ demarcation curve, these profiles show that the flow velocity is large in the center of the duct for very small frequencies. As the frequency increases to the value $\tilde{\Omega} = 2.0412$, which is the value where the positive and negative roots meet in Figure 3-1, the peak value of the flow velocity decreases. Then, increasing the frequency further causes the flow velocity to become larger in the center of the duct.

These figures show how a change in frequency affects the profiles of different effective viscosities relative to each other. For a small frequency of $\tilde{\Omega} = 1.05$, the curves of the positive and negative effective viscosities are grouped together without much distinction between the two signs. However, they are significantly different than the $\tilde{\eta}_{eff} = 0$ curve. Then, as the frequency is increased, the curves begin to group closer to the $\tilde{\eta}_{eff} = 0$ curve, while the $\tilde{\eta}_{eff} = 0.01$ curve becomes more distinct from the others. At a frequency of $\tilde{\Omega} = 2.0412$, the negative effective viscosity curves are close to the $\tilde{\eta}_{eff} = 0$ curve, and a distinction between the positive and negative effective viscosity curves can be seen. They follow the same trend as seen in Figure 3-1 where the $\tilde{\eta}_{eff} = +\infty$ limit curve is the same as the $\tilde{\eta}_{eff} = -\infty$ limit curve. As the frequency is increased further, the trend reverses again and the large frequency curve of $\tilde{\Omega} = 10$ looks similar in spacing to the $\tilde{\Omega} = 1.05$ curve in that the positive and negative effective viscosity curves are grouped together, separate from the $\tilde{\eta}_{eff} = 0$ curve.

Figure 3-6: Linear flow velocity distributions for various values of $\tilde{\Omega}$ and $\tilde{\eta}_{eff}$ that fall to the left of the $\tilde{\eta}_{eff} = 0$ demarcation curve. a) $\tilde{\Omega} = 1.05$, b) $\tilde{\Omega} = 1.3$, c) $\tilde{\Omega} = 2.0412$, d) $\tilde{\Omega} = 3.25$, e) $\tilde{\Omega} = 5.0$, f) $\tilde{\Omega} = 10.0$.

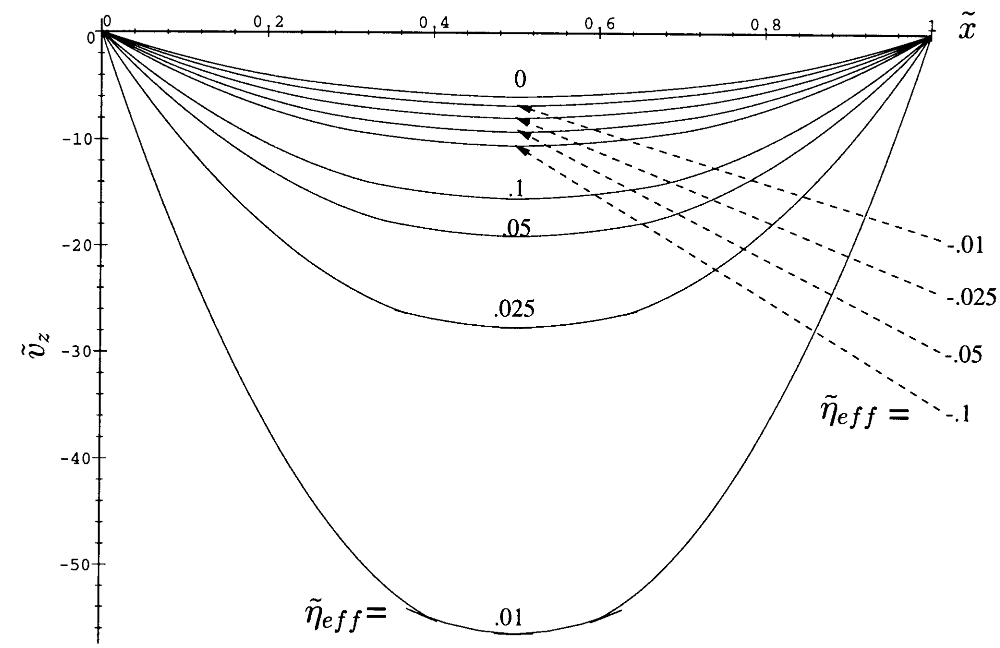


a: $\tilde{\Omega} = 1.05$



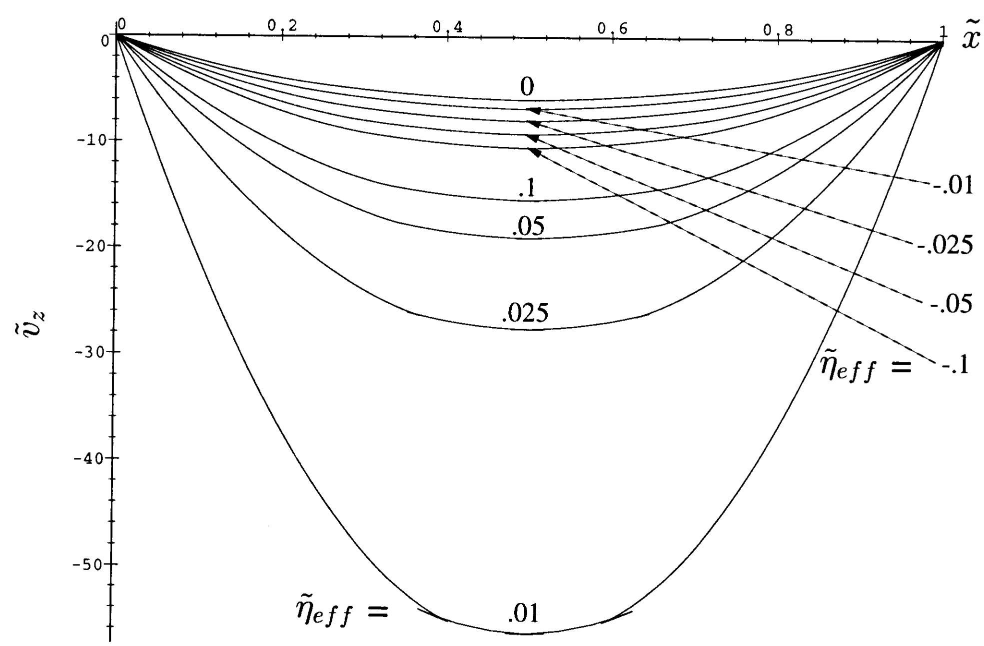
b: $\tilde{\Omega} = 1.3$

$| \tilde{B}_x |= 1, | \tilde{H}_z |= 0$



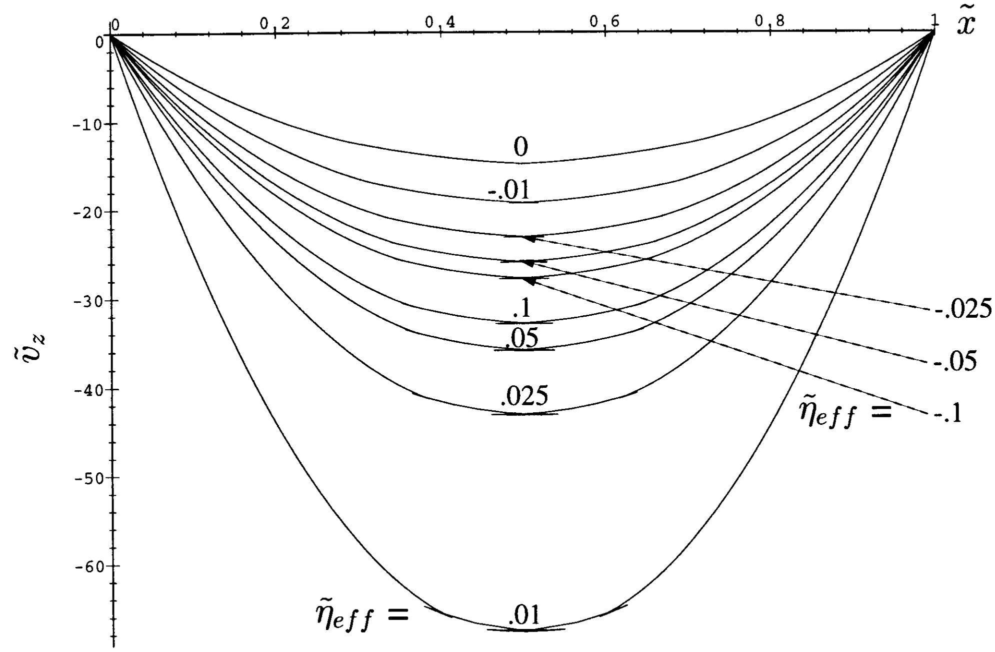
c: $\tilde{\mathbf{\Omega}} = \mathbf{2.0412}$

$| \tilde{B}_x |= 1, | \tilde{H}_z |= 0$



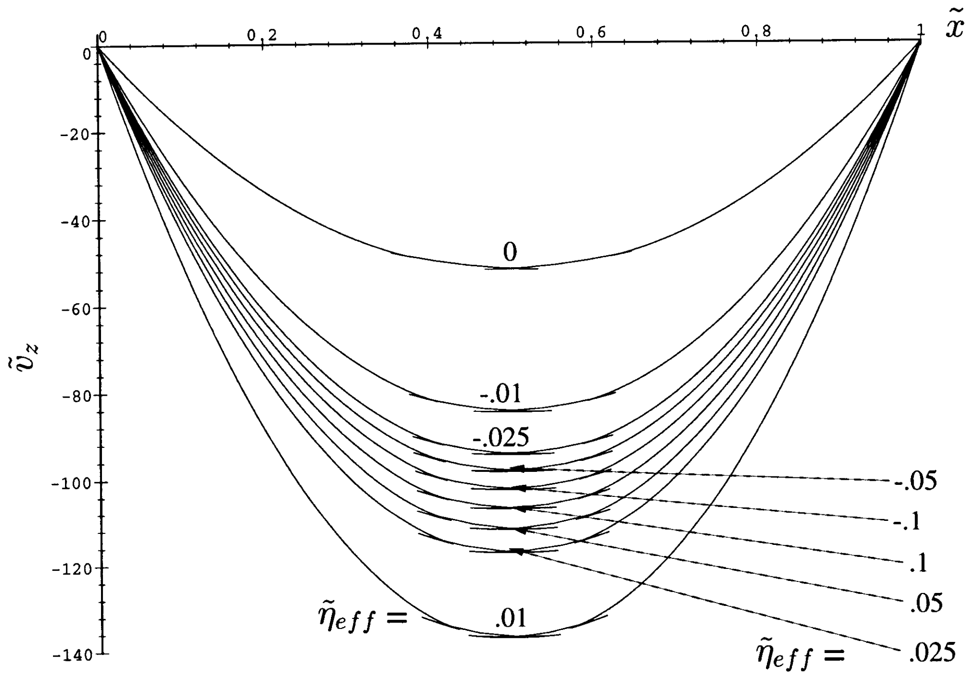
d: $\tilde{\mathbf{\Omega}} = \mathbf{3.25}$

$| \tilde{B}_x |= 1, | \tilde{H}_z |= 0$



e: $\tilde{\mathbf{\Omega}} = \mathbf{5.0}$

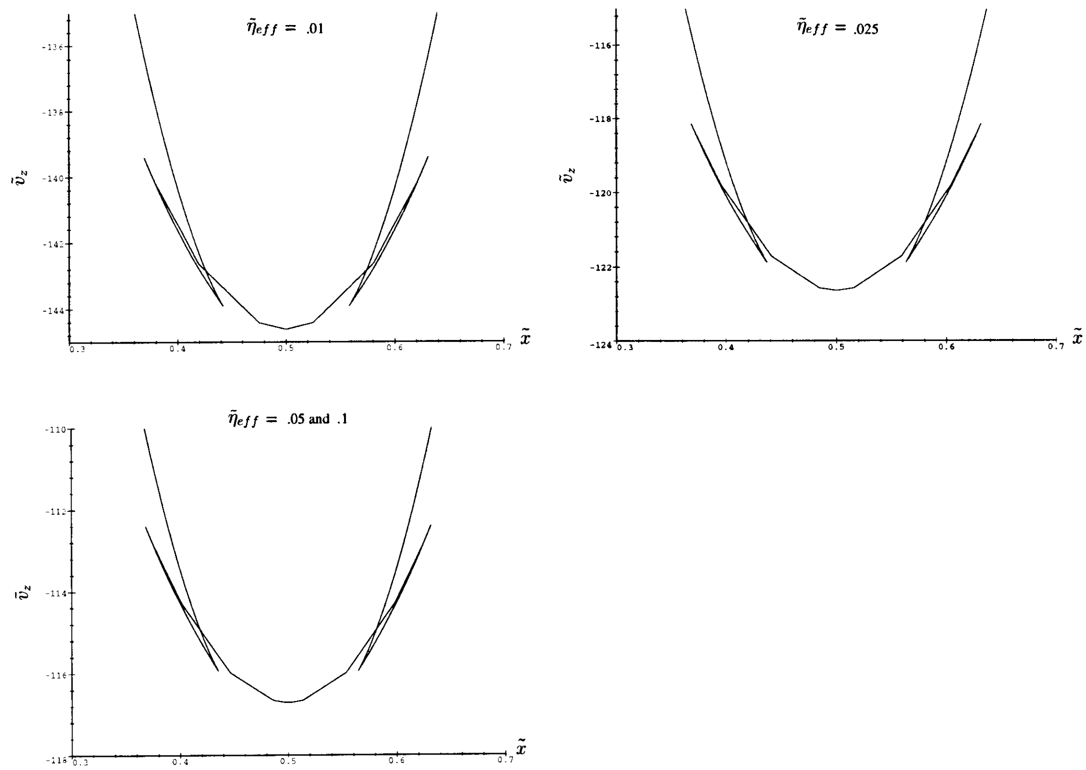
$| \tilde{B}_x |= 1, | \tilde{H}_z |= 0$



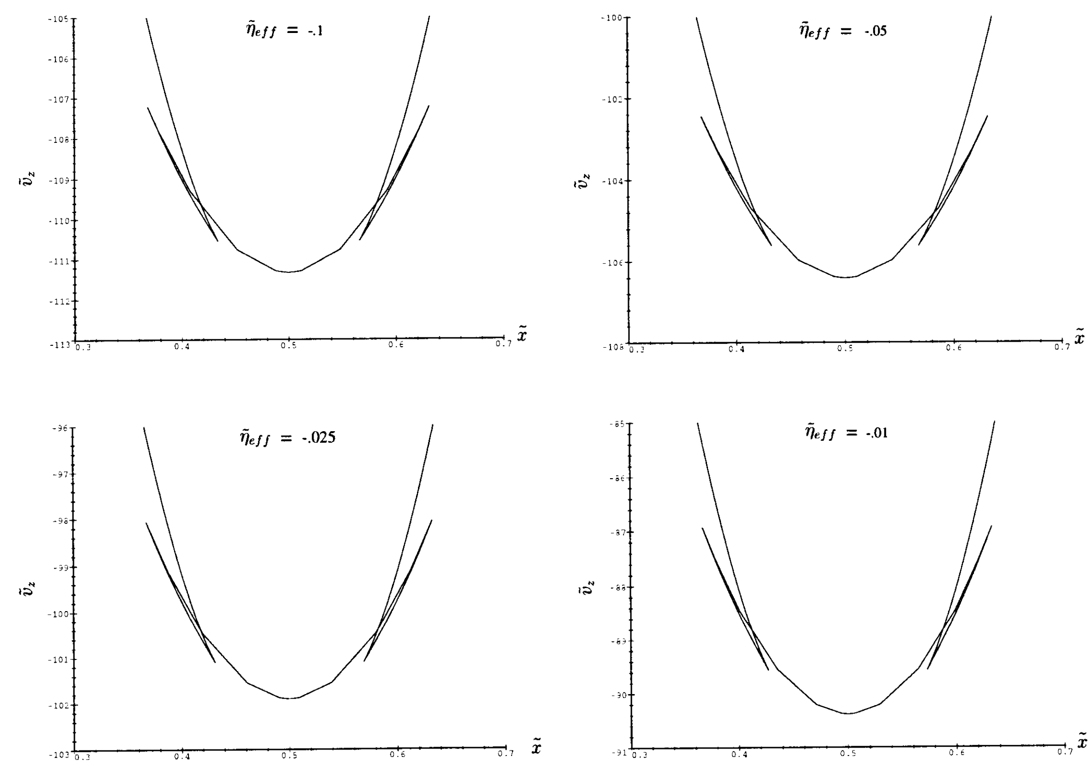
f: $\tilde{\mathbf{\Omega}} = \mathbf{10.0}$

Again, the region around $\tilde{x} = 0.5$ is most interesting since the small spin velocity limit is valid there. As the above described transition from small frequencies to large frequencies happens, there is a change in shape of this central region. Because of the scaling used in Figure 3-5, it is difficult to see the behavior of the curves in this region. It was necessary to examine them more closely, which is done in Figures 3-7 through 3-12 .

All of the effective viscosity curves for $\tilde{\Omega} = 1.05$ look essentially the same. They are triple-valued in two places in the outer regions. Slightly increasing the frequency to $\tilde{\Omega} = 1.3$ changes this behavior noticably. The multi-valued regions can be seen only in the positive effective viscosity curves. The larger the effective viscosity, the smaller the region of $\tilde{x}$ that is triple-valued. This is seen in Figure 3-8.



a: $\tilde{\boldsymbol{\Omega}} = \mathbf{1.05}$ positive effective viscosities.



b: $\tilde{\boldsymbol{\Omega}} = \mathbf{1.05}$ negative effective viscosities.

Figure 3-7: $\tilde{\Omega} = 1.05$ curves of Figure 3-6 (a) separated and increased to see the multi-valued behavior of the curves.

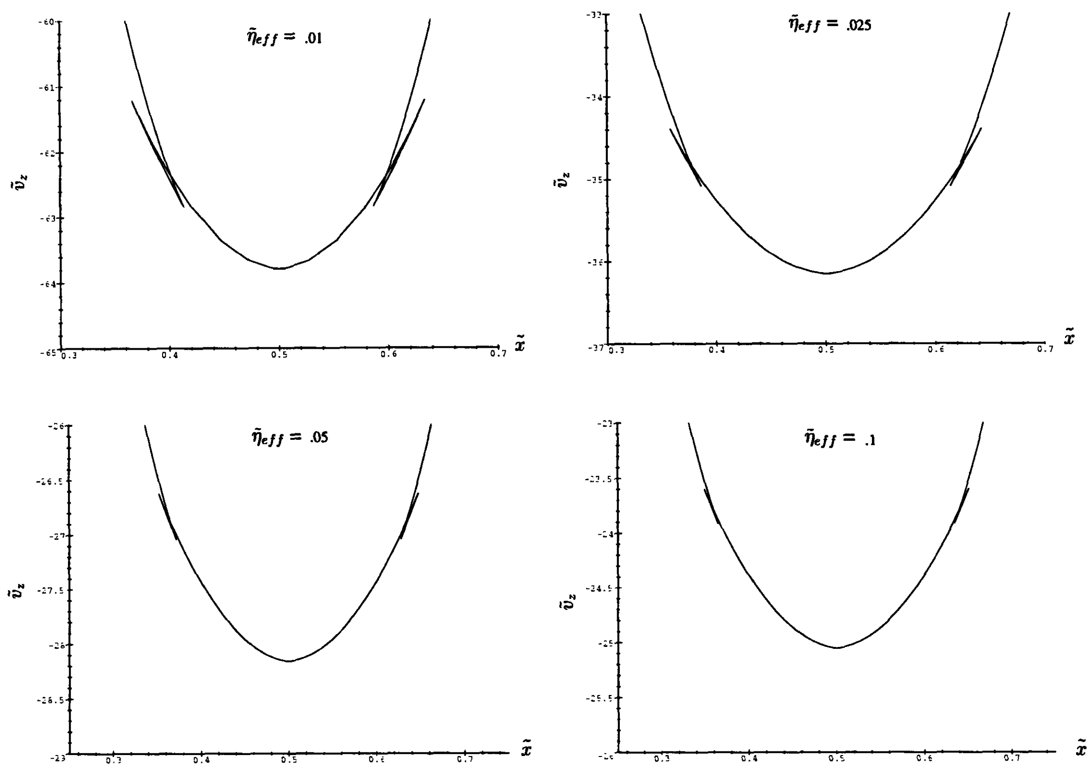
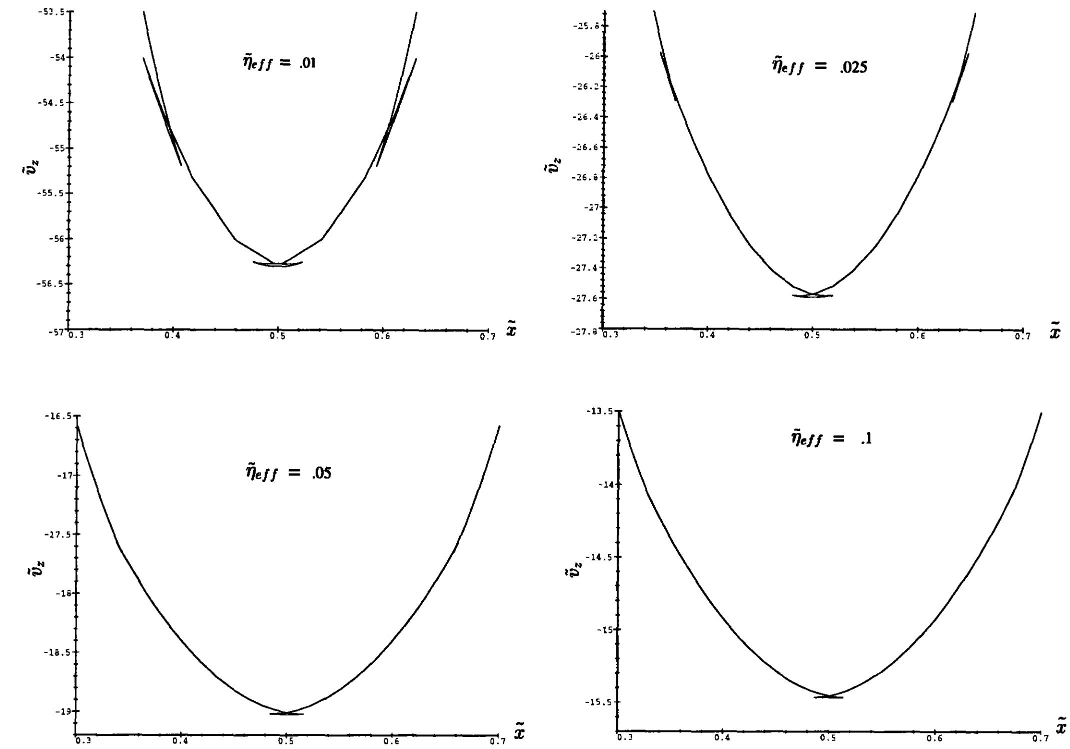


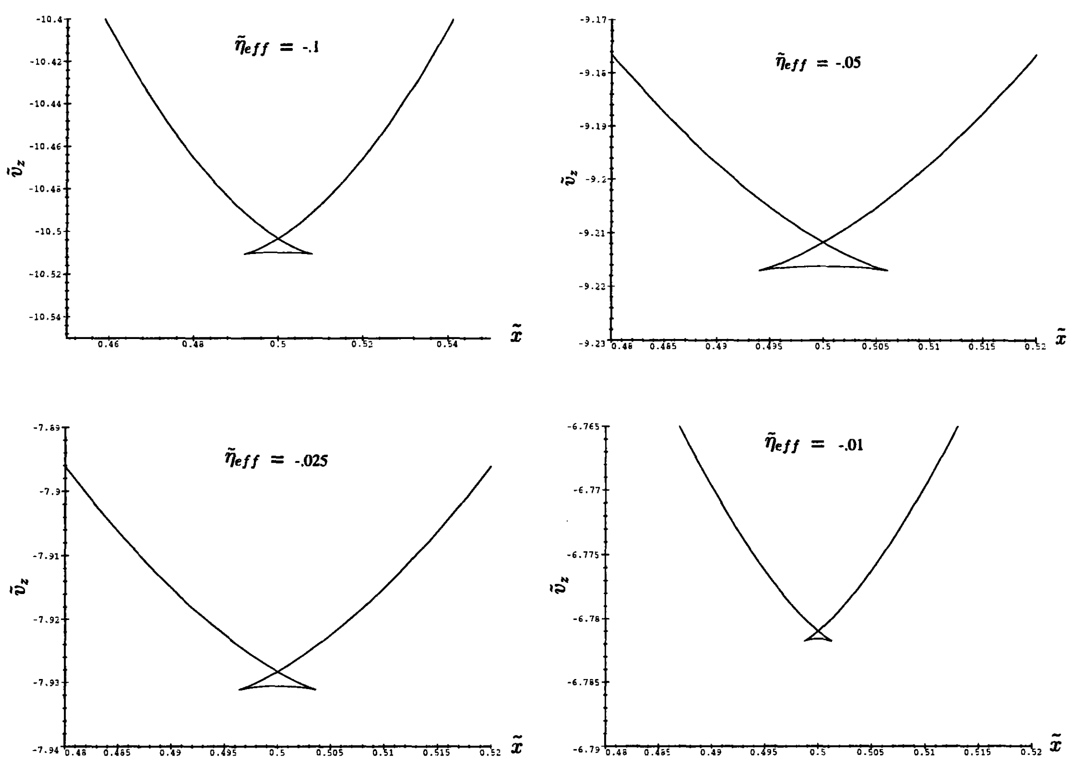
Figure 3-8: $\tilde{\Omega} = 1.3$ curves of Figure 3-6 (b) separated and increased to see the behavior of the curves. Only the positive effective viscosities show any multi-valued regions.

Further increasing the frequency to $\tilde{\Omega} = 2.0412$ causes both the positive and negative effective viscosities to become triple-valued in a very small region around $\tilde{x} = 0.5$. Also, the larger values of effective viscosity cease being triple-valued in the outer regions. The smaller values of effective viscosity, $\tilde{\eta}_{eff} = .01$ and .025 and possibly .05 are still triple-valued in the outer regions. This can be seen in Figure 3-9.

Figure 3-10 shows the individual effective viscosity curves for $\tilde{\Omega} = 3.25$. The behavior looks very much like the curves for $\tilde{\Omega} = 2.0412$. The negative effective viscosity curves appear to be multi-valued in the middle regions only, while the smaller values of positive effective viscosity are multi-valued both in the middle regions and in the outer regions, over a small range in $\tilde{x}$.

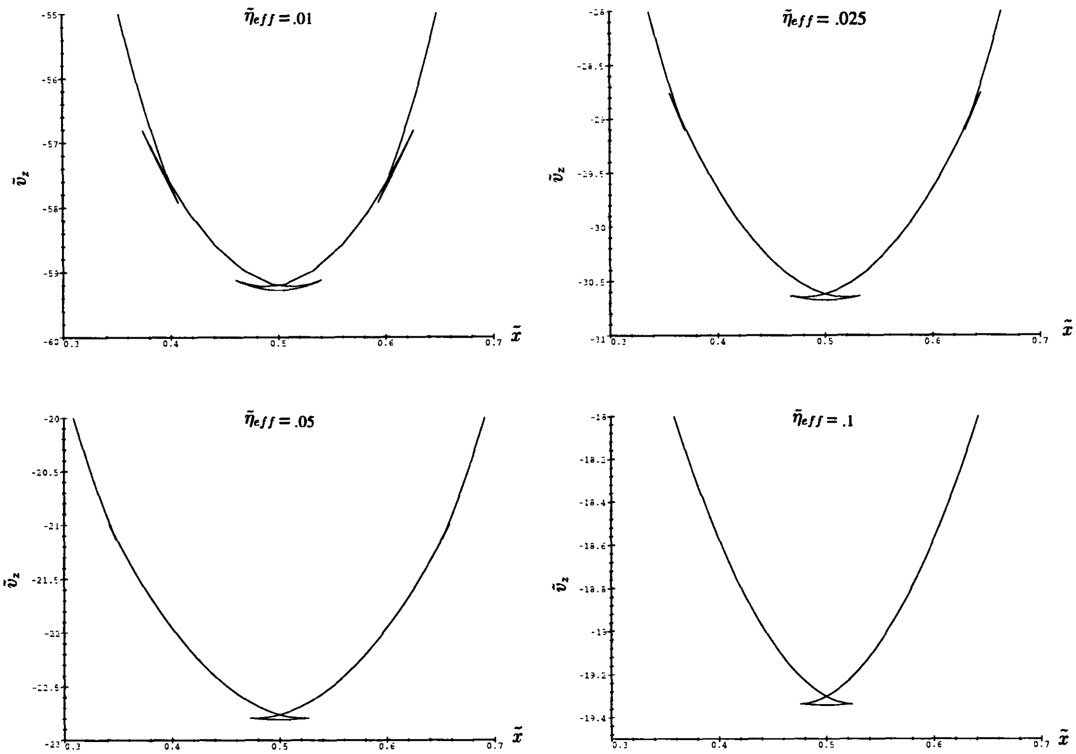


a: $\tilde{\Omega} = \mathbf{2.0412}$ positive effective viscosities.

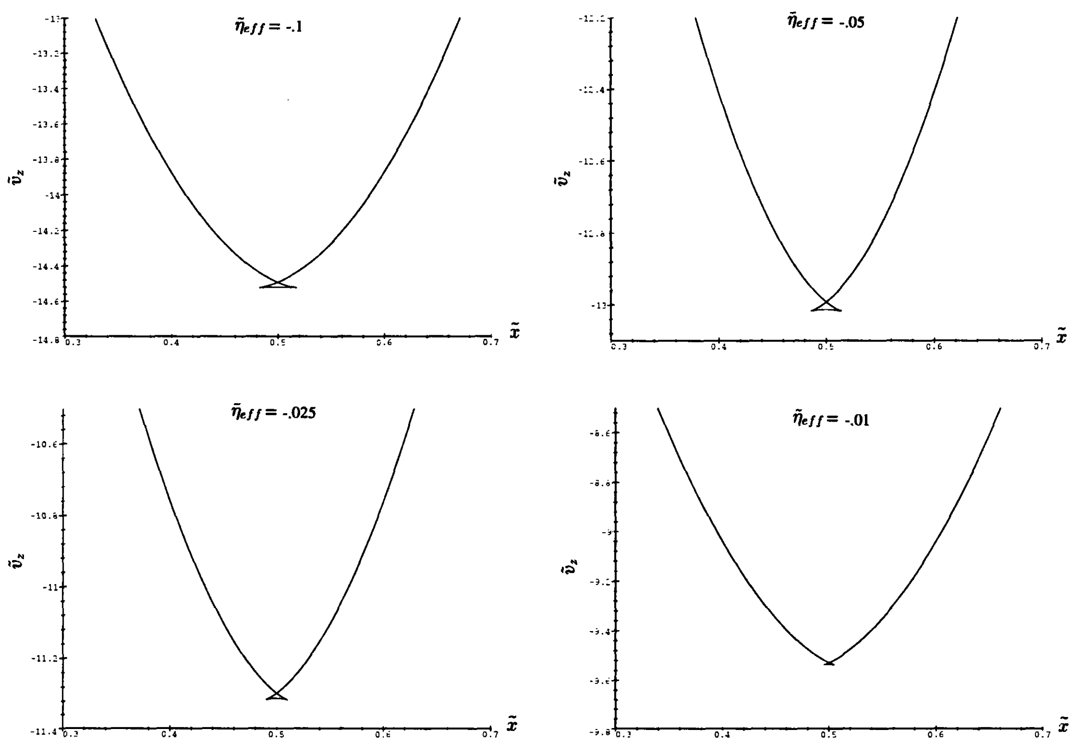


b: $\tilde{\Omega} = \mathbf{2.0412}$ negative effective viscosities.

Figure 3-9: $\tilde{\Omega}$ = 2.0412 curves of Figure 3-6 (c) separated and increased to see the multi-valued behavior of the curves.



a: $\tilde{\boldsymbol{\Omega}} = \mathbf{3.25}$ positive effective viscosities.

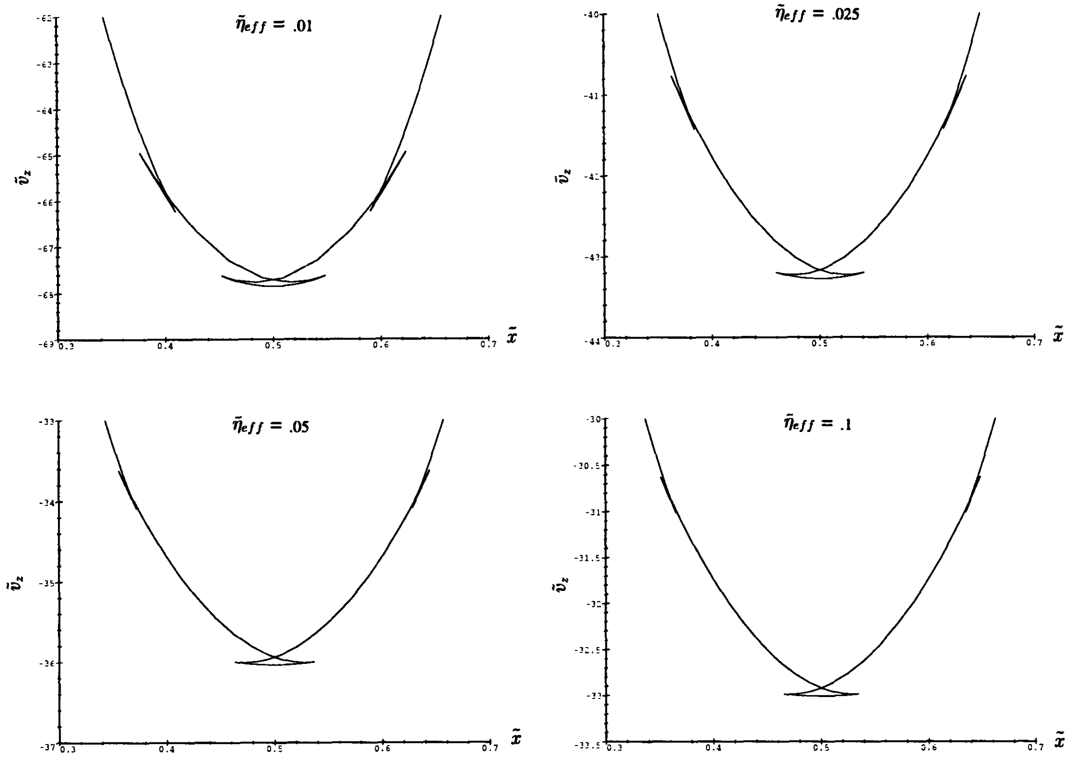


b: $\tilde{\boldsymbol{\Omega}} = \mathbf{3.25}$ negative effective viscosities.

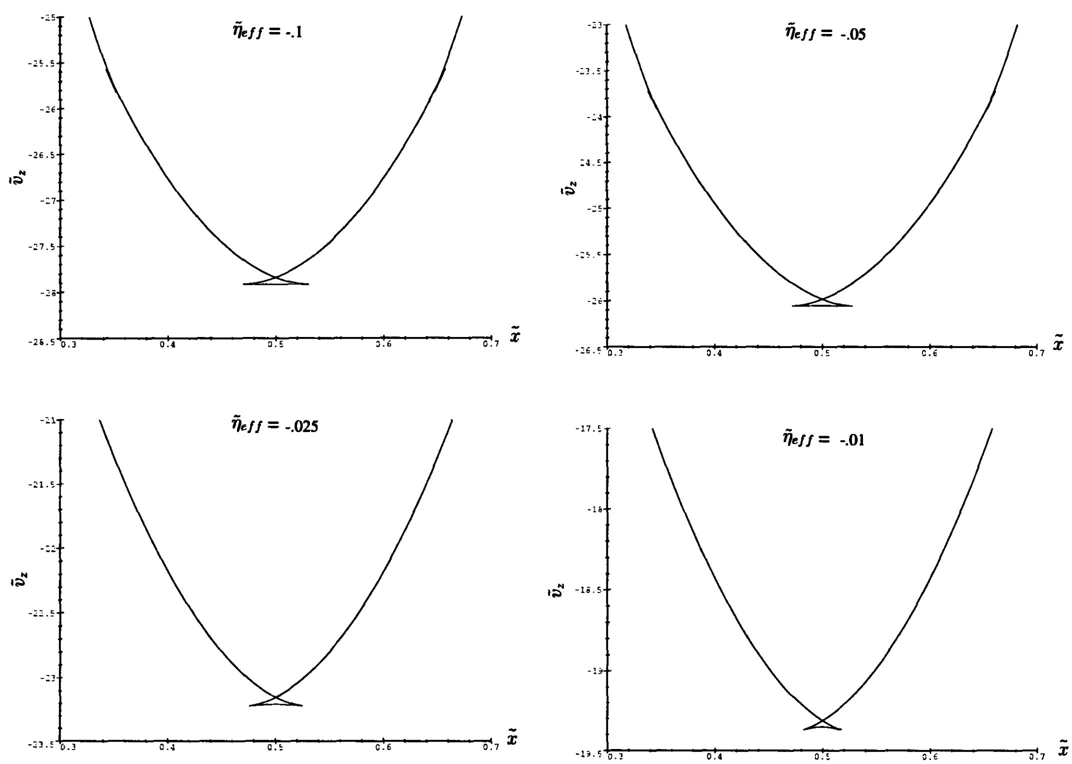
Figure 3-10: $\tilde{\Omega} = 3.25$ curves of Figure 3-6 (d) separated and increased to see the multi-valued behavior of the curves.

Further increasing the frequency to $\tilde{\Omega} = 5$, as seen in Figure 3-11, shows that the middle region around $\tilde{x} = 0.5$ is triple-valued over a large range of $\tilde{x}$. At the same time, the outer regions become triple-valued again for all of the positive effective viscosities and for some of the negative effective viscosities.

Finally, Figure 3-12 shows the enlarged curves for $\tilde{\Omega} = 10$. These curves are triple-valued in all 4 regions. The center region around $\tilde{x} = 0.5$ continues to grow larger with larger frequency. As well, it grows larger from $\tilde{\eta}_{eff} = 0$ to $\tilde{\eta}_{eff} = .01$ in the order that the effective viscosity curves range in Figure 3-1: $\tilde{\eta}_{eff} = \{0, -.01, -.025, -.05, -.1, .1, .05, .025, .01\}$. If the trend continues, it is possible that these inner and outer regions will overlap given a frequency larger than $\tilde{\Omega} = 10$ or a smaller effective viscosity than $\tilde{\eta}_{eff} = .01$ but larger than 0.

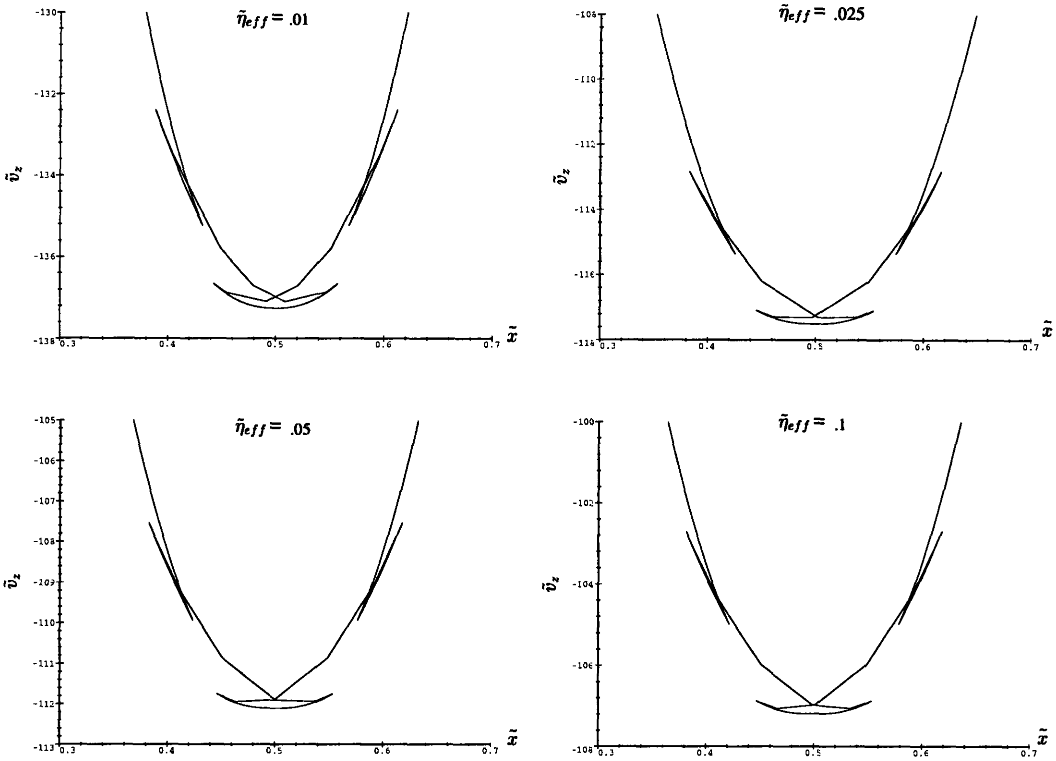


a: $\tilde{\boldsymbol{\Omega}} = \mathbf{5.0}$ positive effective viscosities.

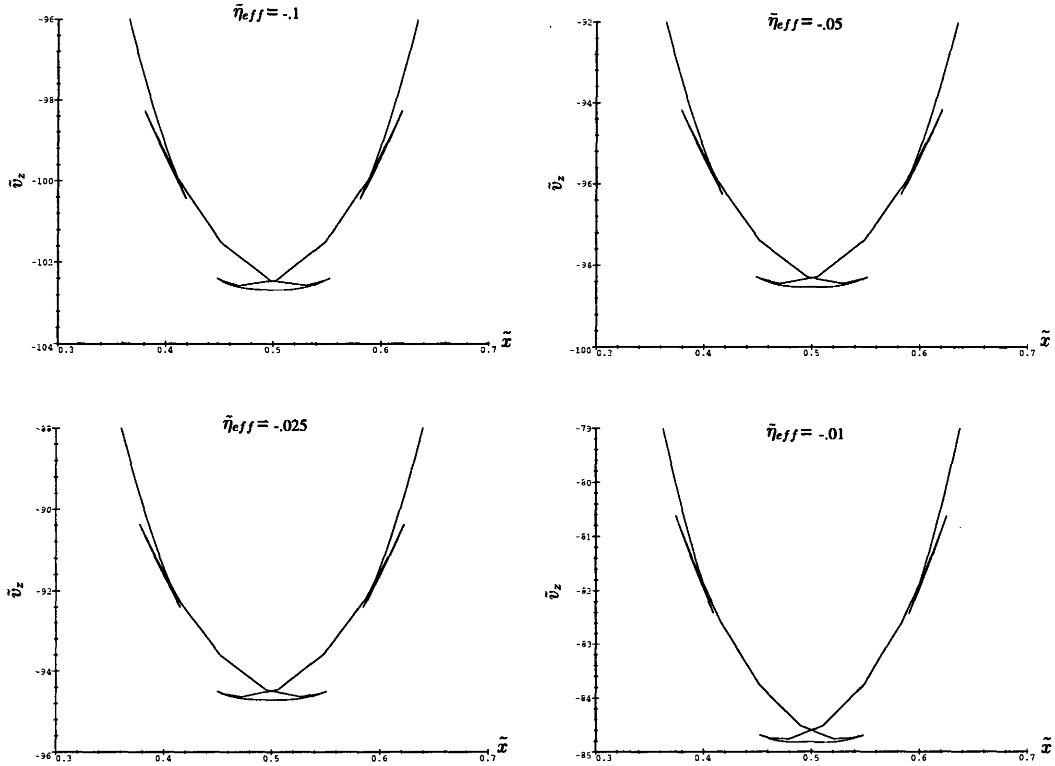


b: $\tilde{\boldsymbol{\Omega}} = \mathbf{5.0}$ negative effective viscosities.

Figure 3-11: $\tilde{\Omega} = 5.0$ curves of Figure 3-6 (e) separated and increased to see the multi-valued behavior of the curves.



a: $\tilde{\Omega} = 10.0$ positive effective viscosities.



b: $\tilde{\Omega} = 10$ negative effective viscosities.

Figure 3-12: $\tilde{\Omega} = 10$ curves of Figure 3-6 (f) separated and increased to see the behavior of the curves.

Chapter 4

Tangential Magnetic Field

$|\tilde{B}_x| = 0;\ |\tilde{H}_z| = 1$

Both magnetic effects and hydrodynamic effects cause the effective viscosity of ferromagnetic fluids to be different than the viscosity of the carrier fluid. Therefore, not only does the effective viscosity depend on the vortex viscosity $\tilde{\zeta}$ and dynamic viscosity $\tilde{\eta}$ but it also depends on the field frequency $\tilde{\Omega}$. We continue to assume $\tilde{\eta} = \tilde{\zeta}$, and repeat Eq. (3.2) here:

$$\alpha = \frac{2\tilde{\zeta}(\tilde{\zeta} - \tilde{\eta}_{eff})}{2\tilde{\zeta} - \tilde{\eta}_{eff}}. \tag{4.1}$$

Specifically for the tangential magnetic field, $|\tilde{B}_x| = 0$; $|\tilde{H}_z| = 1$, Eq. (2.41) reduces to

$$\alpha = \frac{\chi_0}{2}\frac{\left[\tilde{\Omega}^2 - (1+\chi_0)^2\right]}{(1+\chi_0+\tilde{\Omega}^2)^2 + \chi_0^2\tilde{\Omega}^2}. \tag{4.2}$$

By setting Eqs. (4.1) and (4.2) equal to each other, $\tilde{\Omega}$ can be solved for in terms of $\tilde{\zeta}$ and $\tilde{\eta}_{eff}$. This will allow plots of $\tilde{\Omega}(\tilde{\zeta})$ to be made for different values of effective viscosity. The result is a 4th order equation in $\tilde{\Omega}$.

$$\tilde{\Omega}^4 \ + \ \tilde{\Omega}^2\Big[2 + 2\chi_0 + \chi_0^2 - \frac{\chi_0}{2\alpha}\Big] + (1+\chi_0)^2\Big[1 + \frac{\chi_0}{2\alpha}\Big] \ = 0 \tag{4.3}$$

where α, defined in Eq. (3.2), depends on $\tilde{\zeta}$ and $\tilde{\eta}_{eff}$.

To plot $\tilde{\Omega}$ as a function of $\tilde{\zeta}$ for different values of $\tilde{\eta}_{eff}$, the quadratic formula was used to find $\tilde{\Omega}^2$, and then the square root was taken.

$$\tilde{\Omega}^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{4.4}$$

such that

$$\begin{aligned} a &= 1 && (4.5) \\ b &= 2 + 2\chi_0 + \chi_0^2 - \frac{\chi_0}{2\alpha} && (4.6) \\ c &= (1+\chi_0)^2\Big[1 + \frac{\chi_0}{2\alpha}\Big] && (4.7) \end{aligned}$$

Once $\tilde{\Omega}^2$ was calculated, only those values that were both positive and real were plotted.

The values of $\tilde{\eta}_{eff}$ were chosen to be

$$\tilde{\eta}_{eff} = \{-.1;\ -.05;\ -.025;\ -.01;\ 0;\ .01;\ .025;\ .05;\ .1\}. \tag{4.8}$$

These values were chosen so the plotted curves would be well spaced and show the trend of change as $\tilde{\eta}_{eff}$ goes from negative values, through zero, and becomes positive. The resulting plot can be seen in Figure 4-1. For simplification, the value of the effective magnetic susceptibility χ_0 was taken to be 1 for the calculations of this curve. This plot was made using a Matlab script file, which can be found in Appendix C.

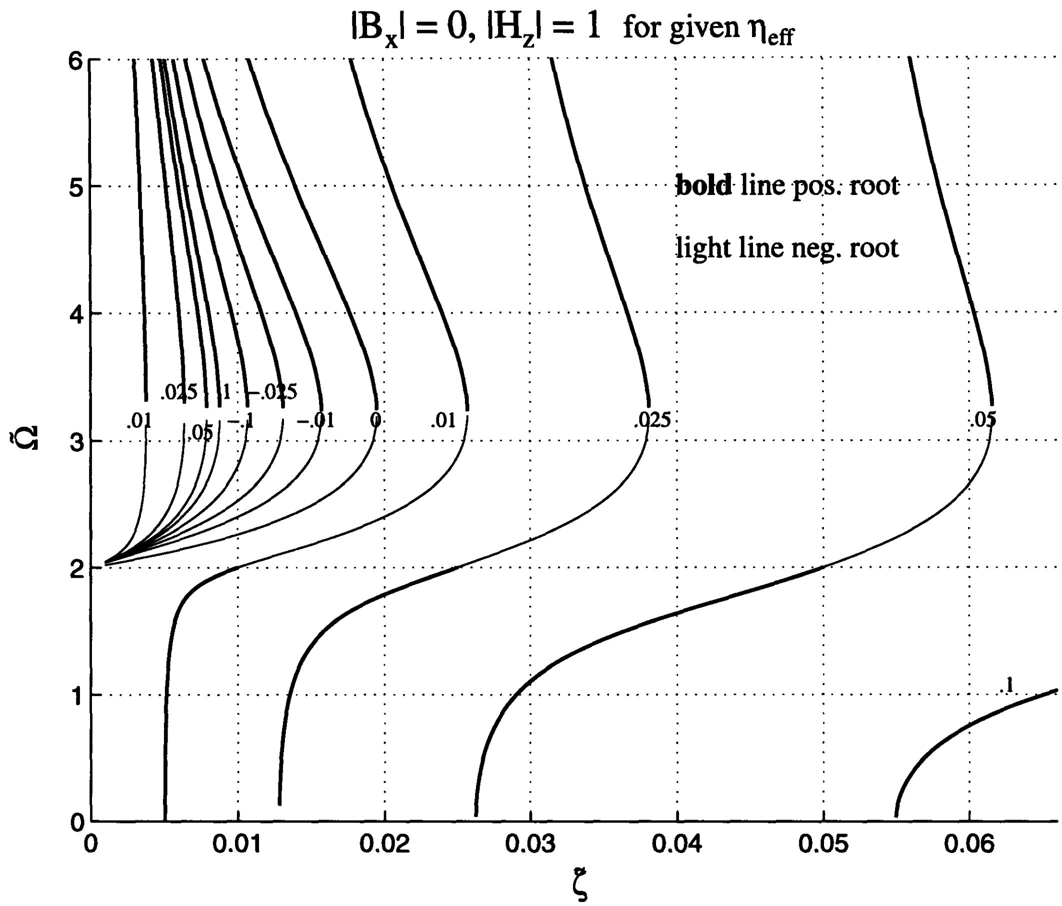


Figure 4-1: Frequency $\tilde{\Omega}$ as a function of viscosity $\tilde{\zeta}$ for nine values of $\tilde{\eta}_{eff}$. The bold lines represent the positive roots of the quadratic, and the plain lines represent the negative roots.

Similar to the case studied in Chapter 3, each positive value of effective viscosity has four roots that are plotted, which look like two general curves. Each negative value of effective viscosity has only two roots plotted which look like one general curve. These curves will be discussed separately as two distinct categories: the curves that fall to the right of the $\tilde{\eta}_{eff} = 0$ curve, and the curves that fall to the left of the $\tilde{\eta}_{eff} = 0$ curve.

4.1 Curves to the Right of the $\tilde{\eta}_{eff} = 0$ curve

All of the curves to the right of the $\tilde{\eta}_{eff} = 0$ demarcation curve are plots of positive values of effective viscosity. Although the $\eta_{eff} = 0$ curve follows a horizontal asymptote to $\tilde{\Omega} = 2$, the positive effective viscosity curves to its right range from a frequency of zero to infinity. Larger values of effective viscosity would follow the same curve characteristics, but would continue further and further to the right of the demarcation curve.

4.1.1 Spin Velocity $\tilde{\omega}_y(\tilde{x})$ Profiles

As was explained in Section 2.4.2, the spin velocity profile can be plotted using a parametric plot of $\tilde{x}$ and $\tilde{\omega}_y$ while varying $\tilde{\omega}_y$. Nondimensional position $\tilde{x}$ is a function of 6 variables, $\tilde{\omega}_y$, $\tilde{\Omega}$, χ_0, $\tilde{\eta}$, $\tilde{\zeta}$, and $\frac{\partial \tilde{p}'}{\partial \tilde{z}}$. We continue the assumptions from Chapter 3 which are:

- $\tilde{\eta} = \tilde{\zeta}$
- $\chi_0 = 1$
- $\frac{\partial \tilde{p}'}{\partial \tilde{z}} = 1$

Therefore, the important variables of $\tilde{x}$ become $\tilde{\omega}_y$, which is the ranging variable of the parametric plot; $\tilde{\Omega}$; and $\tilde{\zeta}$. To be able to plot $\tilde{\omega}_y(\tilde{x})$, the values for $\tilde{\Omega}$ and $\tilde{\zeta}$ must be chosen.

Looking at the curves in Figure 4-1, six values of frequency were chosen. By choosing values of frequency, $\tilde{\Omega}$, the corresponding values of viscosity, $\tilde{\zeta}$, can be calculated. Both values can be used in the equations for plotting the spin velocity profiles. Instead of solving for $\tilde{\Omega}$ in Eq. (4.3), $\tilde{\zeta}$ is solved for.

$$2\tilde{\zeta}^2\Big[(1+\chi_0+\tilde{\Omega}^2)^2+\chi_0^2\tilde{\Omega}^2\Big]-\tilde{\zeta}\Big[2\tilde{\eta}_{eff}[(1+\chi_0+\tilde{\Omega}^2)^2+\chi_0^2\tilde{\Omega}^2]+\chi_0(\tilde{\Omega}^2-(1+\chi_0)^2)\Big]$$
$$+\frac{\chi_0}{2}\tilde{\eta}_{eff}\Big(\tilde{\Omega}^2-(1+\chi_0)^2\Big) \quad (4.9)$$

To solve for $\tilde{\zeta}$, the quadratic equation is used, where

$$a = 2\Big[(1+\chi_0+\tilde{\Omega}^2)^2+\chi_0^2\tilde{\Omega}^2\Big] \tag{4.10}$$

$$b = -\Big[2\tilde{\eta}_{eff}\Big[(1+\chi_0+\tilde{\Omega}^2)^2+\chi_0^2\tilde{\Omega}^2\Big]+\chi_0(\tilde{\Omega}^2-(1+\chi_0)^2)\Big] \tag{4.11}$$

$$c = \frac{\chi_0}{2}\tilde{\eta}_{eff}\Big(\tilde{\Omega}^2-(1+\chi_0)^2\Big) \tag{4.12}$$

The values of $\tilde{\Omega}$ were chosen so the differences in $\tilde{\zeta}$ would be noticable. They are:

$$\tilde{\Omega} = \{2.05,\ 2.5,\ 3.2126,\ 4.5,\ 6,\ 10\} \tag{4.13}$$

The value of 2.05 was chosen to be just above the horizontal asymptote of the demarcation curve. The value of 3.2126 is the value of $\tilde{\Omega}$ where the positive and negative roots of the quadratic meet. To calculate the corresponding $\tilde{\zeta}$, a Matlab script was used, which can be found in Appendix D. The results are summarized by Table 4-1.

$\tilde{\Omega} = 2.05$			$\tilde{\Omega} = 2.5$			$\tilde{\Omega} = 3.2126$		
$\tilde{\eta}_{eff}$	$\tilde{\zeta}$		$\tilde{\eta}_{eff}$	$\tilde{\zeta}$		$\tilde{\eta}_{eff}$	$\tilde{\zeta}$	
0	.0024	0	0	.0151	0	0	.0195	0
.01	.0113	.0010	.01	.0216	.0035	.01	.0259	.0038
.025	.0262	.0011	.025	.0347	.0055	.025	.0381	.0064
.05	.0512	.0012	.05	.0587	.0064	.05	.0616	.0079
.1	.1012	.0012	.1	.1081	.0070	.1	.1107	.0088
$\tilde{\Omega} = 4.5$			$\tilde{\Omega} = 6$			$\tilde{\Omega} = 10$		
$\tilde{\eta}_{eff}$	$\tilde{\zeta}$		$\tilde{\eta}_{eff}$	$\tilde{\zeta}$		$\tilde{\eta}_{eff}$	$\tilde{\zeta}$	
0	.0158	0	0	.0108	0	0	.0046	0
.01	.0222	.0035	.01	.0178	.0030	.01	.0128	.0018
.025	.0352	.0056	.025	.0315	.0043	.025	.0275	.0021
.05	.0591	.0067	.05	.0560	.0048	.05	.0524	.0022
.1	.1085	.0073	.1	.1057	.0051	.1	.1023	.0022

Table 4.1: Matlab results of calculating $\tilde{\zeta}$ given some value of $\tilde{\Omega}$ and $\tilde{\eta}_{eff}$ for the positive effective viscosity curves to the right of the $\tilde{\eta}_{eff} = 0$ curve.

There are two values of $\tilde{\zeta}$ for each effective viscosity because there are two roots to the solution. The first column of $\tilde{\zeta}$ represents the curves that fall to the right of the $\tilde{\eta}_{eff} = 0$ demarcation curve. The second column of $\tilde{\zeta}$ represents those curves that fall to the left of the demarcation curve and will be discussed in section (4.2).

To range $\tilde{\omega}_y$ in the parametric plot, the values must be known at the boundaries. Because $\tilde{\omega}_y$ is an odd function of $\tilde{x}$ around $\tilde{x} = 0.5$,

$$\tilde{\omega}_0 = \tilde{\omega}_y(\tilde{x} = 0) = -\tilde{\omega}_y(\tilde{x} = 1). \tag{4.14}$$

Mathematica was used to calculate $\tilde{\omega}_0$, which is the value of $\tilde{\omega}_y$ at the boundary of $\tilde{x} = 0$. The same command in Eq. (3.16) was used. The results are listed in Table 4-2. The Mathematica file can be found in Appendix B.

$\tilde{\Omega} = 2.05$		$\tilde{\Omega} = 2.5$		$\tilde{\Omega} = 3.2126$	
$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$
.0024	207.328	.0151	32.0739	.0195	24.58
.0113	43.2218	.0216	22.0868	.0257	18.3633
.0262	18.0109	.0347	13.2873	.0381	11.9454
.0512	8.56627	.0587	7.20961	.0616	6.63099
.1012	3.42359	.1081	3.30883	.1107	3.58458
$\tilde{\Omega} = 4.5$		$\tilde{\Omega} = 6$		$\tilde{\Omega} = 10$	
$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$
.0158	30.5884	.0108	45.2551	.0046	107.677
.0222	21.4238	.0178	26.996	.0128	37.9572
.0352	12.9697	.0315	14.5751	.0275	16.4881
.0591	6.61044	.0560	6.99514	.0524	9.71204
.1085	4.33254	.1057	5.10211	.1023	5.18128

Table 4.2: Mathematica results of calculating spin velocity $\tilde{\omega}_0$ at the $\tilde{x} = 0$ planar duct wall for the positive effective viscosity curves to the right of the $\tilde{\eta}_{eff} = 0$ curve.

It is interesting to note that for $\tilde{\Omega} = 10$, the values of $\tilde{\zeta}$ and $\tilde{\omega}_0$ are virtually the same as those values in Table 3-2 for the same frequency. For large enough frequencies, the behavior of ferrofluids in a transverse magnetic field ($|\tilde{B}_x| = 1,\ |\tilde{H}_z| = 0$) is the same as in a tangential magnetic field ($|\tilde{B}_x| = 0,\ |\tilde{H}_z| = 1$).

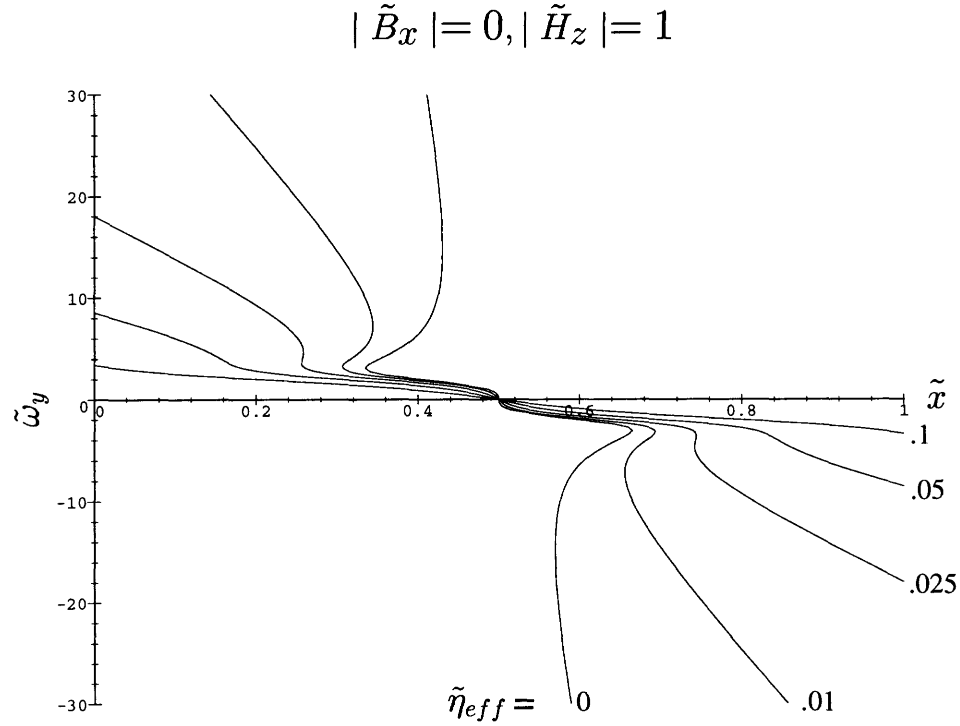
All of these values of $\tilde{\Omega}$, $\tilde{\zeta}$, and $\tilde{\omega}_0$ were used in Maple to plot the spin velocity profiles which are seen in Figure 3-2. The Maple program is found in Appendix A.

These profiles show that $\tilde{\omega}_y$ can be double-valued over a range of $\tilde{x}$ for any frequency. Unlike the transverse-only magnetic field profiles in Chapter 3 (section 3.1.1), there is no examined frequency where all of the functions behave as single-valued. At low frequencies, the function can be double-valued in two regions while at high frequencies, the function can be multi-valued in four regions with the middle regions being triple-valued.

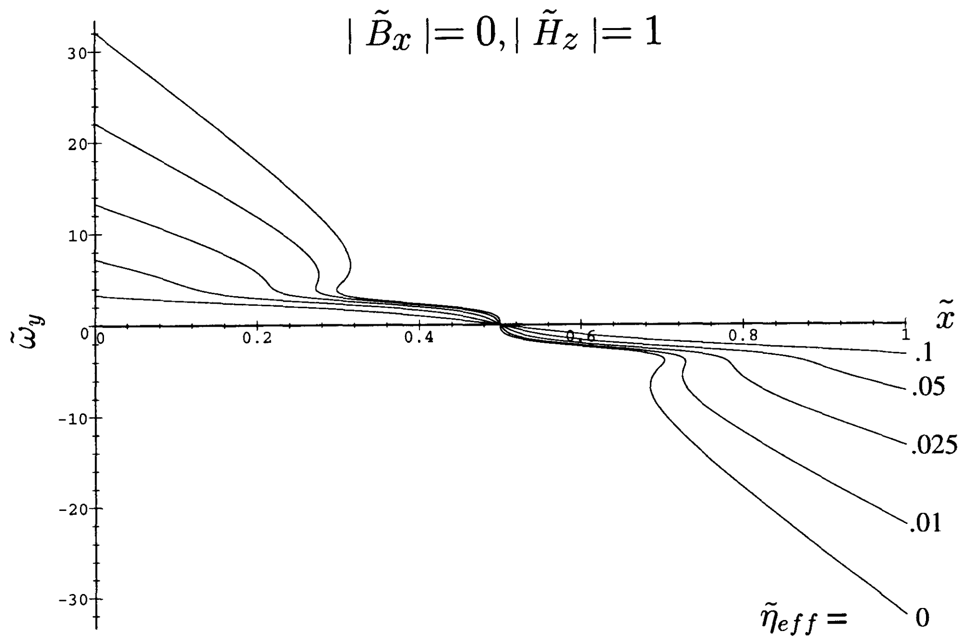
The spin velocity profiles for the larger values of effective viscosity behave as single-valued functions. The effect of increasing frequency does not change the shape of the function much. For smaller values of effective viscosity, including zero, some of the profiles are double-valued for any frequency.

The region around $\tilde{x} = 0.5$, which is half way between the planar duct boundaries, is the most interesting. It is here that the small spin velocity approximation is valid. At low frequencies, this region behaves as single-valued, but as frequency increases, this region becomes triple valued immediately to the left and right of $\tilde{x} = 0.5$.

Figure 4-2: Spin velocity spatial distributions for various values of $\tilde{\Omega}$ and $\tilde{\eta}_{eff}$ to the right of the $\tilde{\eta}_{eff} = 0$ curve. a) $\tilde{\Omega} = 2.05$, b) $\tilde{\Omega} = 2.5$, c) $\tilde{\Omega} = 3.2126$, d) $\tilde{\Omega} = 4.5$, e) $\tilde{\Omega} = 6.0$, f) $\tilde{\Omega} = 10.0$.

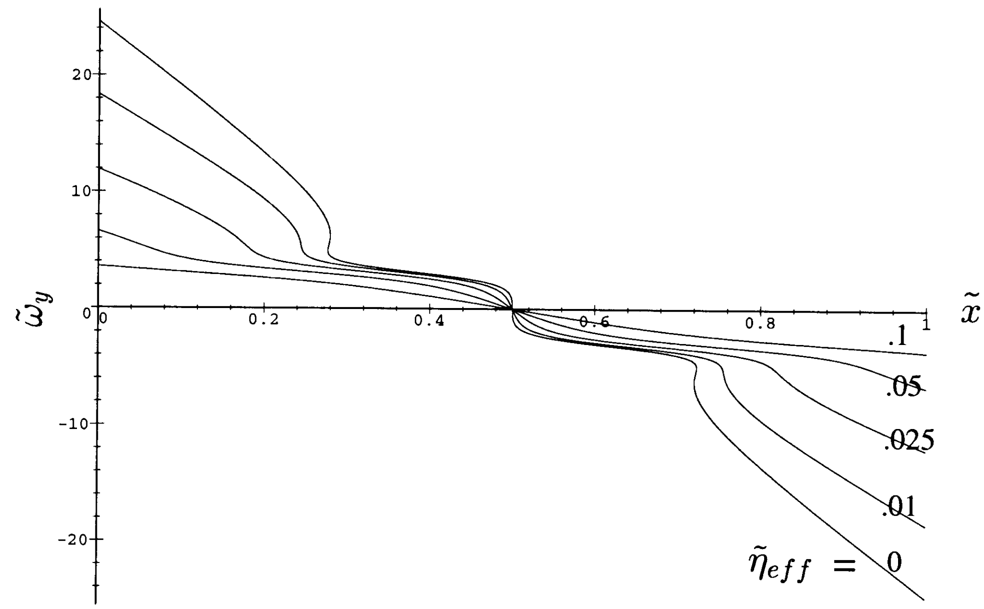


a: $\tilde{\mathbf{\Omega}} = \mathbf{2.05}$



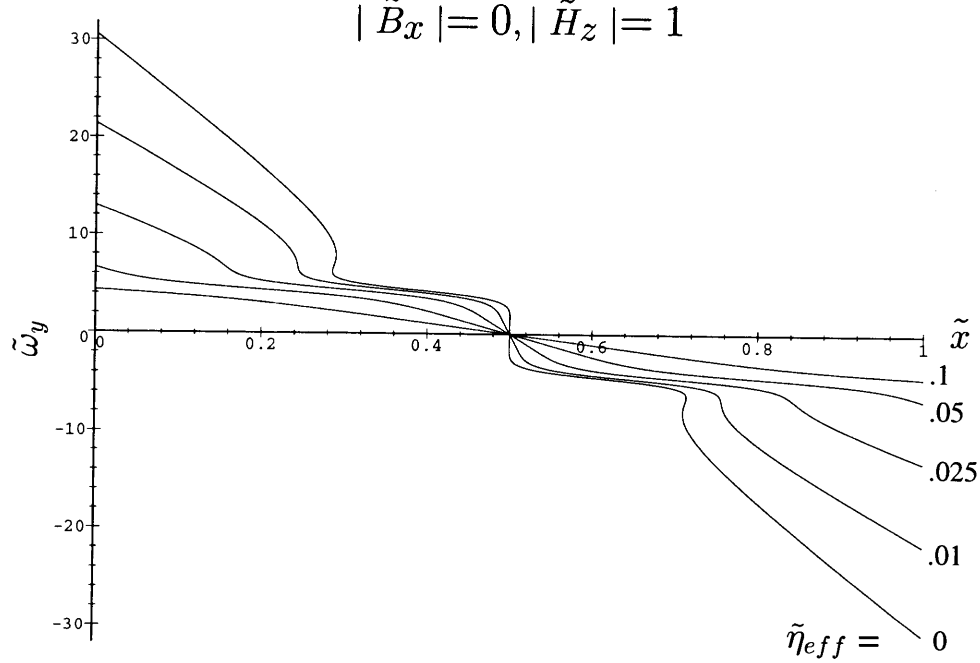
b: $\tilde{\mathbf{\Omega}} = \mathbf{2.5}$

$| \tilde{B}_x |= 0, | \tilde{H}_z |= 1$

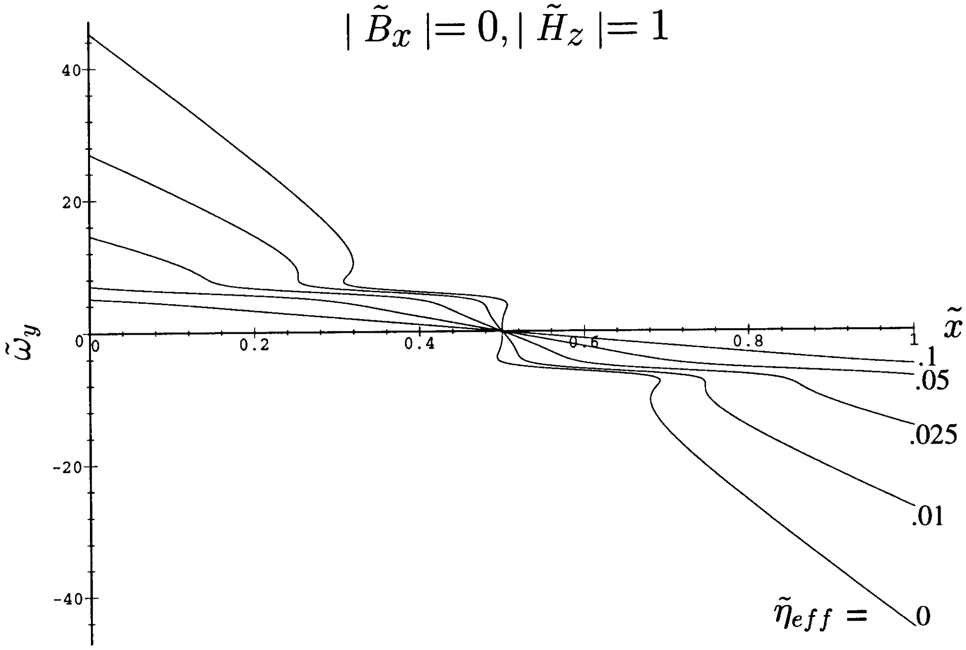


c: $\tilde{\mathbf{\Omega}} = \mathbf{3.2126}$

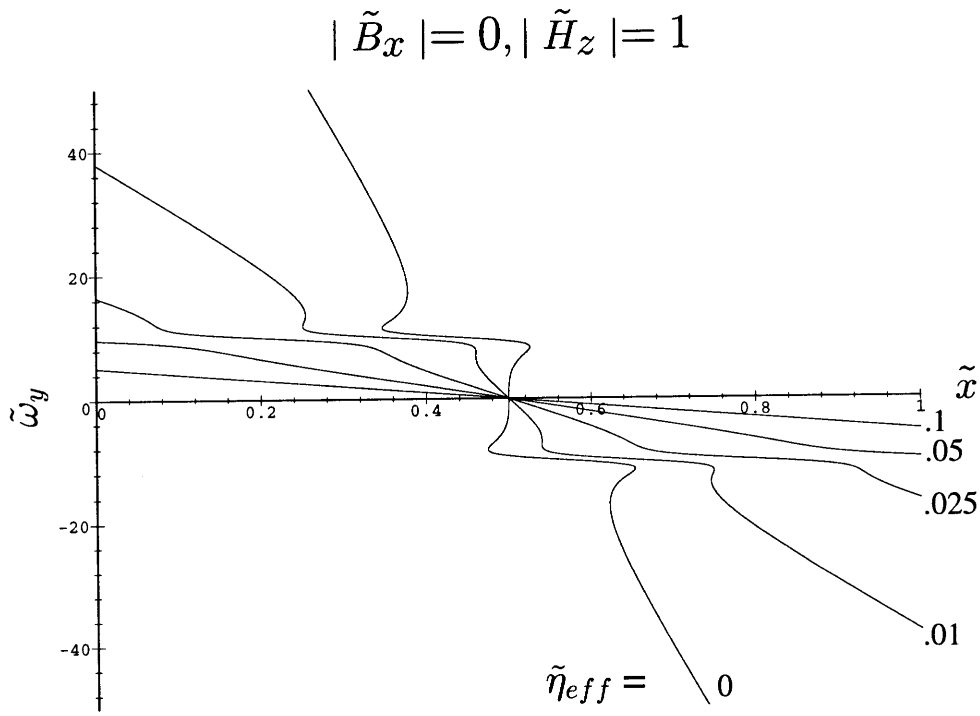
$| \tilde{B}_x |= 0, | \tilde{H}_z |= 1$



d: $\tilde{\mathbf{\Omega}} = \mathbf{4.5}$



e: $\tilde{\boldsymbol{\Omega}} = \mathbf{6.0}$



f: $\tilde{\boldsymbol{\Omega}} = \mathbf{10.0}$

4.1.2 Flow Velocity $\tilde{v}_z(\tilde{x})$ Profiles

Using Eqs. (2.36) - (2.38), the flow velocity equations can be plotted using a parametric plot of $\tilde{x}$ and $\tilde{f}$ while varying $\tilde{\omega}_y$. The flow velocity $\tilde{f}$ is effectively a function of seven variables: $\tilde{\omega}_y$ which is a function of $\tilde{x}$; $\tilde{\Omega}$; χ_0; $\tilde{\eta}$; $\tilde{\zeta}$; and $\frac{\partial \tilde{p}'}{\partial \tilde{z}}$. Again, for simplicity, we continue to use:

- $\tilde{\eta} = \tilde{\zeta}$
- $\chi_0 = 1$
- $\frac{\partial \tilde{p}'}{\partial \tilde{z}} = 1$

Thus, the important variables for calculating $\tilde{v}_z(\tilde{x})$ are $\tilde{\omega}_y$, which is the variable that is ranged in the parametric plot, $\tilde{\omega}$, and $\tilde{\zeta}$. These values have already been calculated in Table 4-1. The boundary values of $\tilde{\omega}_0$ to use for the ranges have also been calculated in Table 4-2.

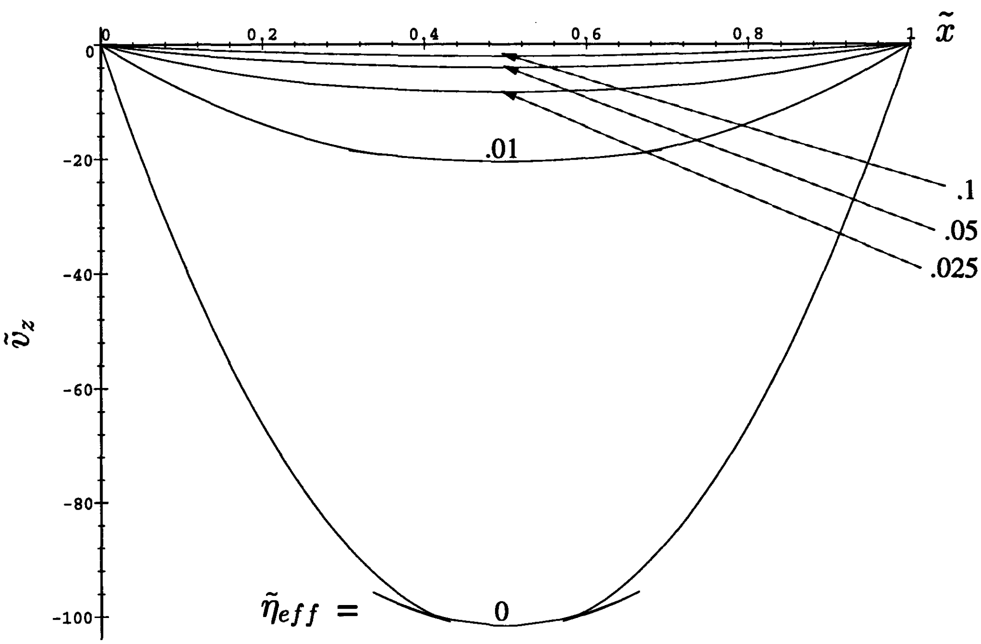
Because the constant of integration D in Eq. (2.36) cannot be solved analytically, it has been calculated for each case using the Maple equation of Eq. (3.17). Figure 3-3 shows the flow velocity profiles numerically integrated by Maple. The Maple program can be found in Appendix A.

With these profiles, we see that the flow velocity is large in the center of the duct for very small frequencies. Then the flow velocity decreases as the frequency increases to the value $\tilde{\Omega} = 3.2126$ which is the value where the positive and negative roots meet in Figure 4-1. Increasing the frequency further causes the flow velocity to become larger again.

These figures show that the larger the effective viscosity, the more constant the peak value of $\tilde{v}_z$ at $\tilde{x} = 0.5$ with changes in $\tilde{\Omega}$. For $\tilde{\eta}_{eff} = .1$, we see that the nondimensional peak value of the flow velocity is always about 2. The smaller the effective viscosity, the more the flow velocity changes with a change in frequency. This behavior is similar to the cases studied in Chapter 3.

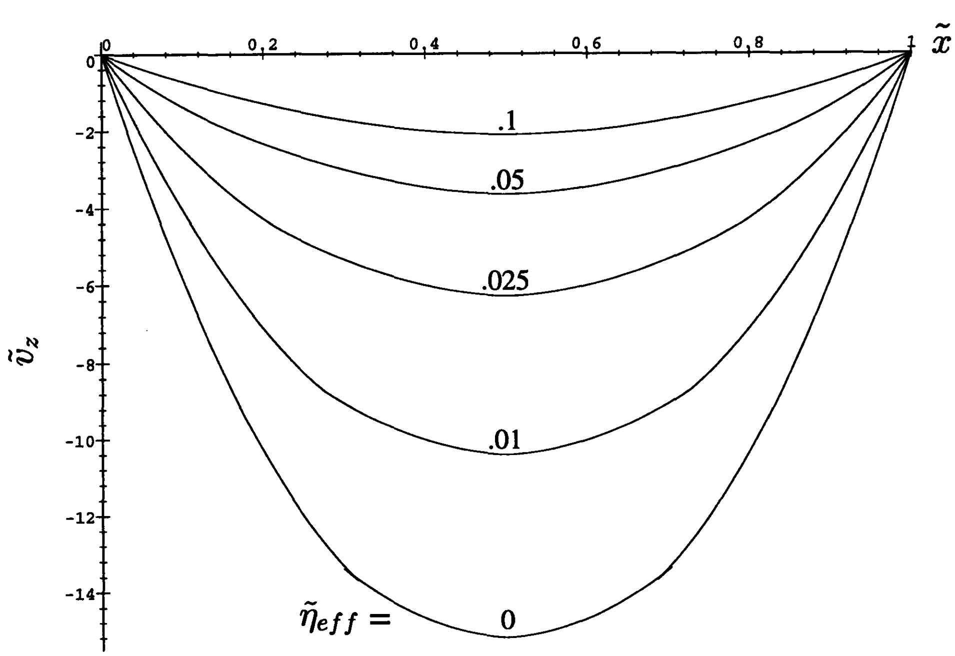
Figure 4-3: Linear flow velocity distributions for various values of $\tilde{\Omega}$ and $\tilde{\eta}_{eff}$ that fall to the right of the $\tilde{\eta}_{eff} = 0$ demarcation curve. a) $\tilde{\Omega} = 2.05$, b) $\tilde{\Omega} = 2.5$, c) $\tilde{\Omega} = 3.2126$, d) $\tilde{\Omega} = 4.5$, e) $\tilde{\Omega} = 6.0$, f) $\tilde{\Omega} = 10.0$.

$$| \tilde{B}_x | = 0, | \tilde{H}_z | = 1$$



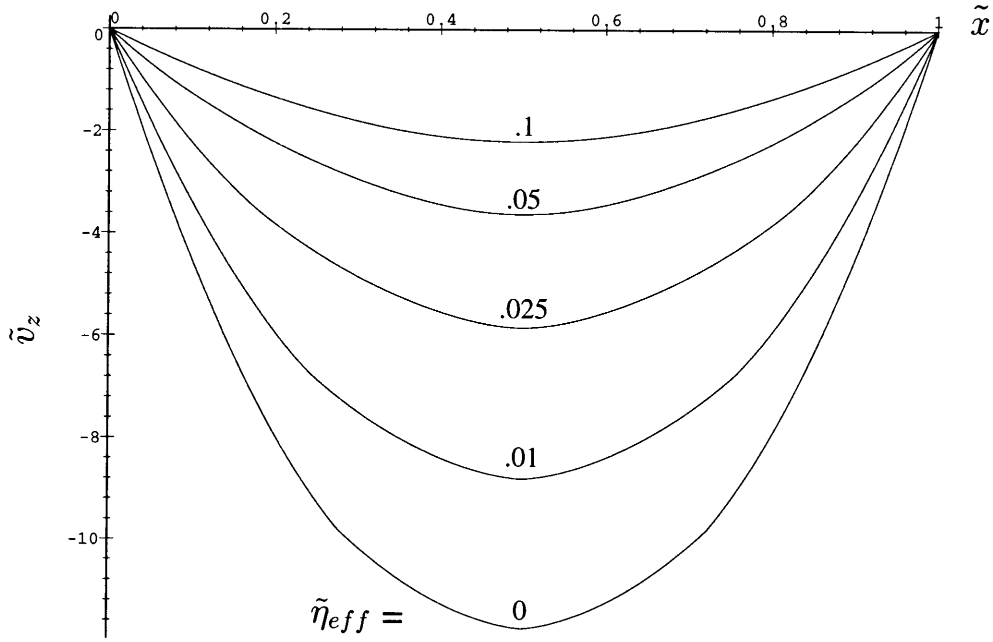
a: $\mathbf{\tilde{\Omega} = 2.05}$

$$| \tilde{B}_x | = 0, | \tilde{H}_z | = 1$$



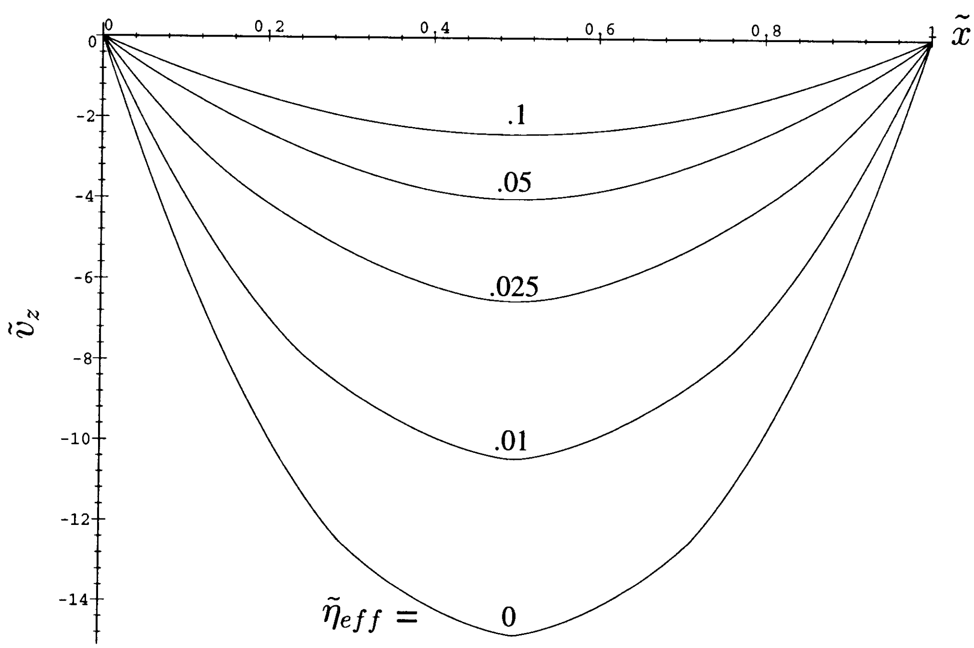
b: $\mathbf{\tilde{\Omega} = 2.5}$

$\mid \tilde{B}_x \mid = 0, \mid \tilde{H}_z \mid = 1$



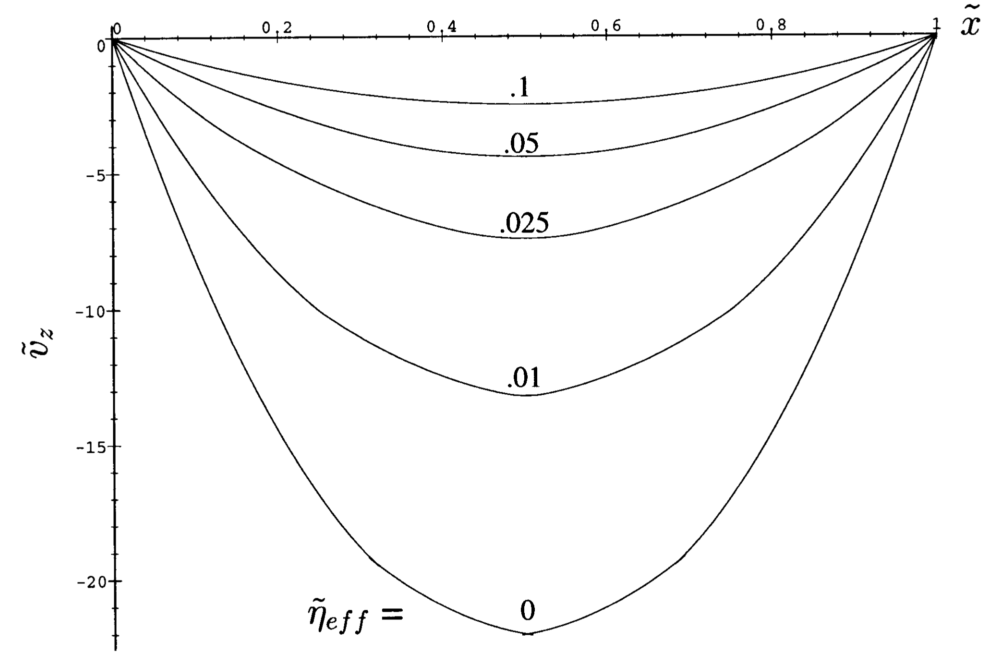
c: $\tilde{\mathbf{\Omega}} = \mathbf{3.2126}$

$\mid \tilde{B}_x \mid = 0, \mid \tilde{H}_z \mid = 1$



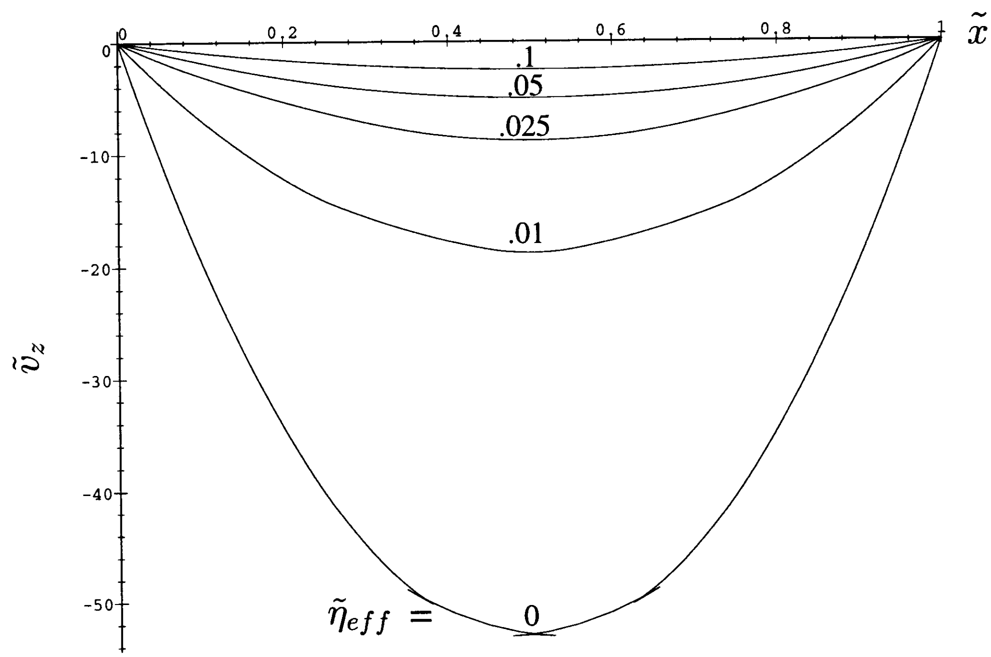
d: $\tilde{\mathbf{\Omega}} = \mathbf{4.5}$

$| \tilde{B}_x |= 0, | \tilde{H}_z |= 1$



e: $\tilde{\boldsymbol{\Omega}} = \mathbf{6.0}$

$| \tilde{B}_x |= 0, | \tilde{H}_z |= 1$



f: $\tilde{\boldsymbol{\Omega}} = \mathbf{10.0}$

We also see that these flow velocity profiles can be multi-valued as well. The scaling of the figures does not always allow this fact to be easily seen. For the frequency $\tilde{\Omega} = 2.05$ we can see that for $\tilde{\eta}_{eff} = 0$ and for $\tilde{\eta}_{eff} = .01$, the profile is triple-valued in two places. For the frequency $\tilde{\Omega} = 2.5$, the range of $\tilde{x}$ which is triple-valued for $\tilde{\eta}_{eff} = 0$ has decreased. Upon increasing the scale of the figures, we find that the $\tilde{\eta}_{eff} = .01$ plot is still triple-valued in the outer region as well, but very slightly. The increased scale for $\tilde{\Omega} = 2.5$ can be seen in Figure 4-4.

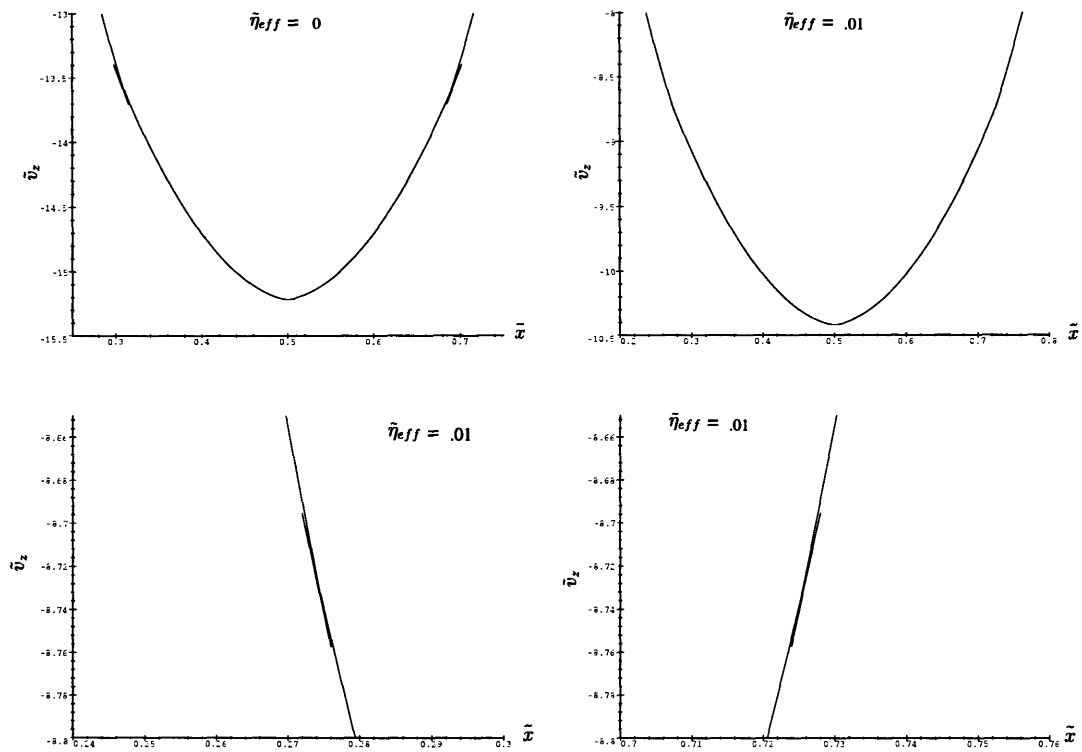


Figure 4-4: Increased scaling of flow velocity profiles for $\tilde{\Omega} = 2.5$ shows that both $\tilde{\eta}_{eff} = 0$ and $\tilde{\eta}_{eff} = .01$ are still multi-valued in the outer regions.

Then, as the frequency continues to increase, the profiles become single-valued for all values of effective viscosity. Further increasing the frequency simply increases the peak value of the flow velocity curve slightly. When the frequency reaches $\tilde{\Omega} = 6$, we see that $\tilde{\eta}_{eff} = 0$ has become triple-valued again. This time, it is also triple-valued in a small region around $\tilde{x} = 0.5$ which is in the center of the duct. At $\tilde{\Omega} = 10$, the range in $\tilde{x}$ that is triple-valued has increased. It is difficult to see whether $\tilde{\eta}_{eff} = .01$

has become multi-valued. Looking at the spin velocity profile for this frequency and effective viscosity in Figure 4-2 (f), it seems possible it is triple-valued in a very small range of $\tilde{x}$. Increasing the scale allows this behavior to be seen. Figure 4-5 shows the multi-valued behavior for $\tilde{\eta}_{eff} = 0$ and $\tilde{\eta}_{eff} = .01$ for a frequency of $\tilde{\Omega} = 10$.

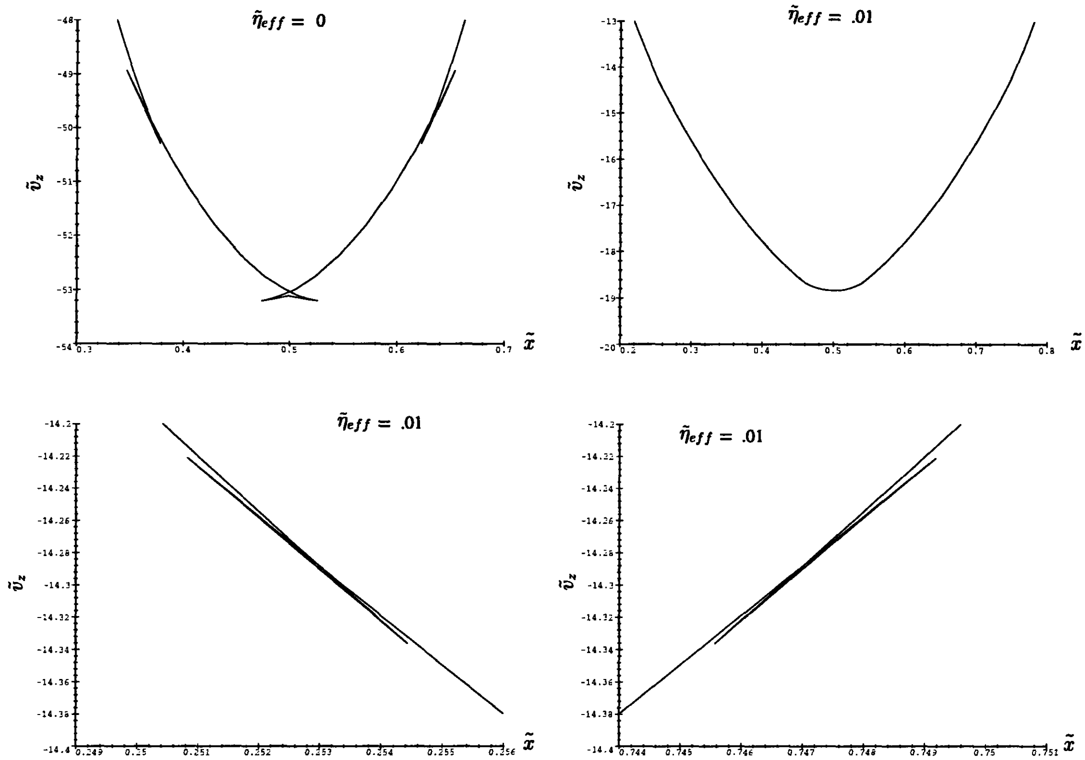


Figure 4-5: Increased scaling of flow velocity profiles for $\tilde{\Omega} = 10$ shows that $\tilde{\eta}_{eff} = .01$ is multi-valued in the outer regions, while $\tilde{\eta}_{eff} = 0$ is multi-valued in the outer and center regions.

It is possible to extrapolate the trend of the flow velocity profiles as the frequency is further increased. The range of $\tilde{x}$ that is multi-valued around $\tilde{x} = 0.5$ will continue to increase with larger frequency. As well, the non-zero effective viscosities will become multi-valued again, first in the outer regions, then in the middle regions.

4.2 Curves to the Left of the $\tilde{\eta}_{eff} = 0$ Curve

The curves to the left of the $\tilde{\eta}_{eff} = 0$ demarcation curve in Figure 4-1 are plots of both positive and negative values of effective viscosity. All of these curves follow a horizontal asymptote of $\tilde{\Omega} = 2$. The left-most curves are positive effective viscosity, and as the value of the effective viscosity is increased, the curves continue to the right with smaller and smaller spacing until a limit curve of $\tilde{\eta}_{eff} = +\infty$ is reached. In addition, as the value of the negative viscosity curves continues to decrease, the curves continue to the left with smaller and smaller spacing until they reach the same limit curve from the other side with $\tilde{\eta}_{eff} = -\infty$.

4.2.1 Spin Velocity $\tilde{\omega}_y(\tilde{x})$ Profiles

The same method of plotting the spin velocity profiles in section 4.1 was used for the profiles to the left of the $\tilde{\eta}_{eff} = 0$ demarcation curve. This time, the second column of viscosities in Table 4.1 represent the positive effective viscosity curves to the left of the demarcation curve and are repeated in Table 4.3. In addition, the viscosities of the negative effective viscosity curves were calculated with the same program found in Appendix D and are also listed in Table 4.3. There is only one real value of $\tilde{\zeta}$ for each effective viscosity even though the equation is second-order. The second root is complex.

$\tilde{\Omega} = 2.05$		$\tilde{\Omega} = 2.5$		$\tilde{\Omega} = 3.2126$	
$\tilde{\eta}_{eff}$	$\tilde{\zeta}$	$\tilde{\eta}_{eff}$	$\tilde{\zeta}$	$\tilde{\eta}_{eff}$	$\tilde{\zeta}$
0	.0024	0	.0151	0	.0195
.01	.0010	.01	.0035	.01	.0038
.025	.0011	.025	.0055	.025	.0064
.05	.0012	.05	.0064	.05	.0079
.1	.0012	.1	.0070	.1	.0088
-.1	.0012	-.1	.0081	-.1	.0107
-.05	.0012	-.05	.0087	-.05	.0116
-.025	.0012	-.025	.0097	-.025	.0131
-.01	.0013	-.01	.0116	-.01	.0157

$\tilde{\Omega} = 4.5$		$\tilde{\Omega} = 6.0$		$\tilde{\Omega} = 10.0$	
$\tilde{\eta}_{eff}$	$\tilde{\zeta}$	$\tilde{\eta}_{eff}$	$\tilde{\zeta}$	$\tilde{\eta}_{eff}$	$\tilde{\zeta}$
0	.0158	0	.0108	0	.0046
.01	.0035	.01	.0030	.01	.0018
.025	.0056	.025	.0043	.025	.0021
.05	.0067	.05	.0048	.05	.0022
.1	.0073	.1	.0051	.1	.0022
-.1	.0085	-.1	.0057	-.1	.0023
-.05	.0091	-.05	.0060	-.05	.0024
-.025	.0102	-.025	.0065	-.025	.0025
-.01	.0122	-.01	.0078	-.01	.0028

Table 4.3: Matlab results of calculating $\tilde{\zeta}$ given some $\tilde{\Omega}$ and $\tilde{\eta}_{eff}$ for the curves of positive and negative effective viscosity to the left of the $\tilde{\eta}_{eff} = 0$ curve.

Because $\tilde{\omega}_y$ is the variable ranged in the parametric plot, the values of $\tilde{\omega}_y(\tilde{x} = 0) = \tilde{\omega}_0$ were calculated with the same Mathematica program that calculated the values in Table 4.2 using Eq. (3.16). The appropriate values of $\tilde{\Omega}$ and $\tilde{\zeta}$ from Table 4.3 were used. The file is found in Appendix D.

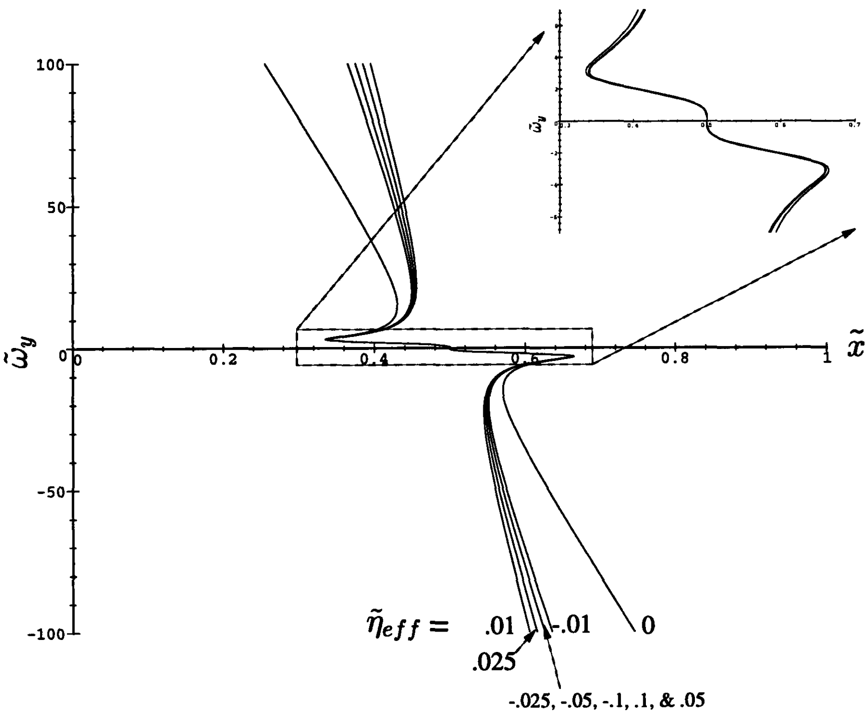
$\tilde{\Omega} = 2.05$		$\tilde{\Omega} = 2.5$		$\tilde{\Omega} = 3.2126$	
$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$
.0024	207.328	.0151	32.0739	.0195	24.58
.0010	498.998	.0035	141.85	.0038	130.571
.0011	453.543	.0055	89.8971	.0064	77.1101
.0012	415.664	.0064	77.1108	.0079	62.2721
.0012	415.664	.0070	70.4129	.0088	55.7965
.0012	415.664	.0081	60.7099	.0107	45.7014
.0012	415.664	.0087	56.4512	.0116	42.073
.0012	415.664	.0097	50.5237	.0131	37.1324
.0013	383.613	.0116	42.0754	.0157	30.8019
$\tilde{\Omega} = 4.5$		$\tilde{\Omega} = 6.0$		$\tilde{\Omega} = 10.0$	
$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$	$\tilde{\zeta}$	$\tilde{\omega}_0$
.0158	30.5884	.0108	45.2551	.0046	107.677
.0035	141.849	.0030	165.659	.0018	276.773
.0056	88.2716	.0043	115.268	.0021	237.089
.0067	73.6092	.0048	103.153	.0022	226.266
.0073	67.4735	.0051	97.0249	.0022	226.266
.0085	57.7996	.0057	86.7027	.0023	216.385
.0091	53.9189	.0060	82.3156	.0024	207.326
.0102	49.9891	.0065	75.9033	.0025	198.992
.0122	39.9445	.0078	63.077	.0026	177.563

Table 4.4: Mathematica results of calculating spin velocity $\tilde{\omega}_0$ at the $\tilde{x} = 0$ planar duct wall for the positive and negative effective viscosity curves to the left of the $\tilde{\eta}_{eff} = 0$ curve.

These values of $\tilde{\Omega}, \tilde{\zeta}$, and $\tilde{\omega}_0$ in Tables 4.3 and 4.4 were used in Maple to plot the spin velocity profiles which are seen in Figure 4-6.

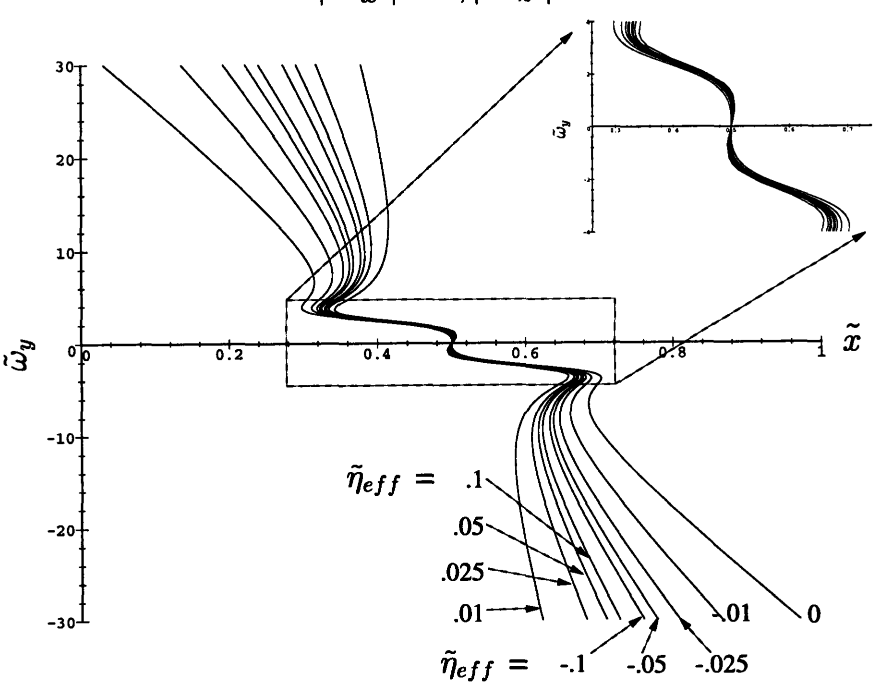
Figure 4-6: Spin velocity spatial distributions for various values of $\tilde{\Omega}$ and $\tilde{\eta}_{eff}$ to the left of the $\tilde{\eta}_{eff} = 0$ curve. a) $\tilde{\Omega} = 2.05$, b) $\tilde{\Omega} = 2.5$, c) $\tilde{\Omega} = 3.2126$, d) $\tilde{\Omega} = 4.5$, e) $\tilde{\Omega} = 6.0$, f) $\tilde{\Omega} = 10.0$.

$| \tilde{B}_x |= 0, | \tilde{H}_z |= 1$



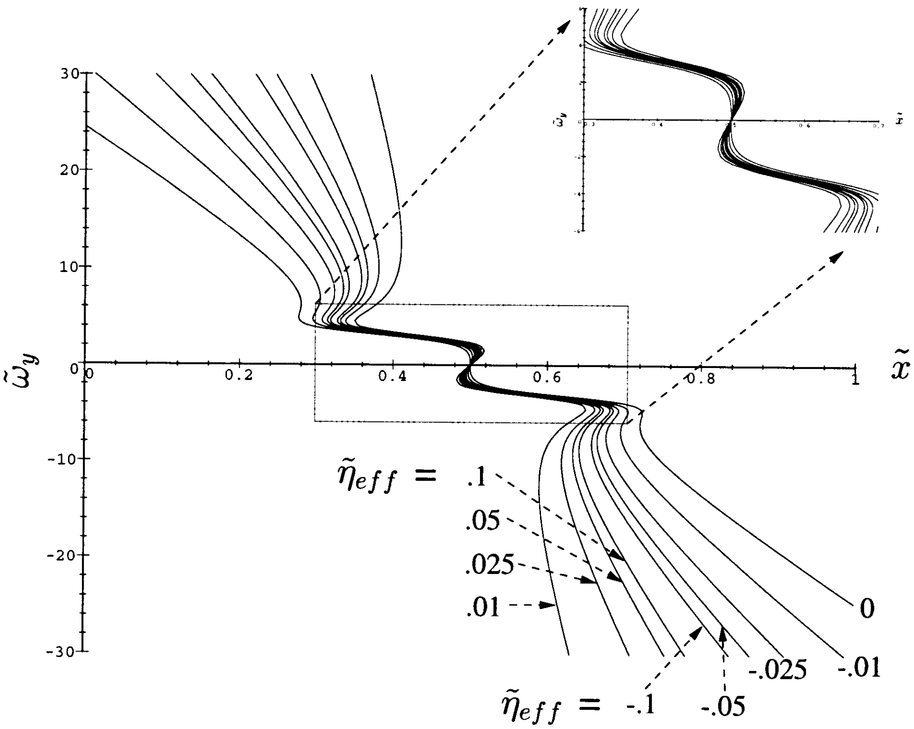
a: $\tilde{\Omega} = 2.05$

$| \tilde{B}_x |= 0, | \tilde{H}_z |= 1$



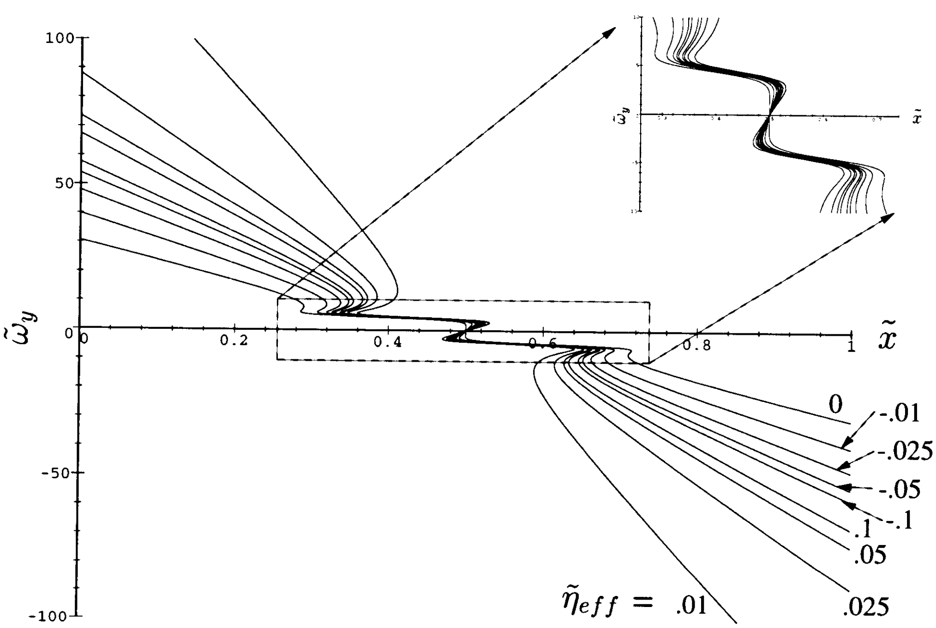
b: $\tilde{\Omega} = 2.5$

$$| \tilde{B}_x | = 0, | \tilde{H}_z | = 1$$

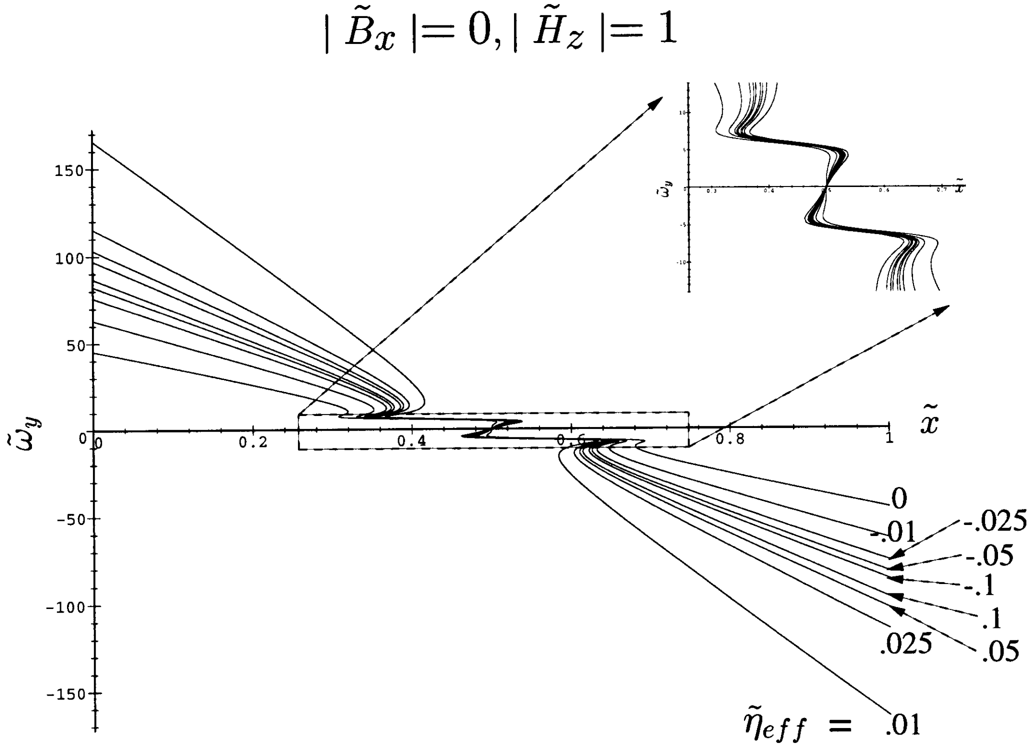


c: $\tilde{\mathbf{\Omega}} = \mathbf{3.2126}$

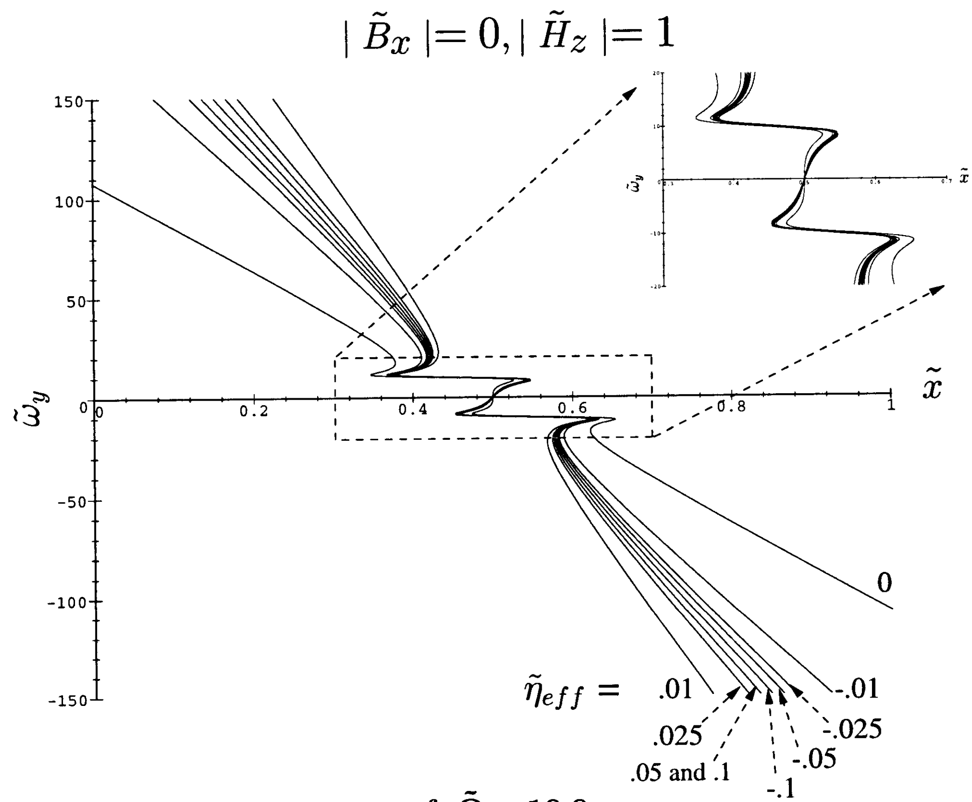
$$| \tilde{B}_x | = 0, | \tilde{H}_z | = 1$$



d: $\tilde{\mathbf{\Omega}} = \mathbf{4.5}$



e: $\tilde{\mathbf{\Omega}} = \mathbf{6.0}$



f: $\tilde{\mathbf{\Omega}} = \mathbf{10.0}$

All of these profiles are multi-valued in $\tilde{\omega}_y$. The first profile, $\tilde{\Omega} = 2.05$, is multi-valued in two regions. But a slight increase in $\tilde{\Omega}$ causes most of the effective viscosity curves to be multi-valued in four regions. Here it is $\tilde{\eta}_{eff} = 0$ that is last to become multi-valued in the middle regions. The positive effective viscosities are first to become triple-valued in the middle regions. Unlike the similar curves in Chapter 3 for the transverse magnetic fields, the $\tilde{\eta}_{eff} = 0$ curve remains multi-valued in its outer regions. Among the frequencies examined, the profiles for $\tilde{\Omega} = 2.5$ and greater have effective viscosity curves that are triple-valued in the middle regions near $\tilde{x} = 0.5$, and double-valued in the outer regions. It is not until $\tilde{\Omega} = 6.0$ that all effective viscosity curves are triple-valued in the middle regions.

The region around $\tilde{x} = 0.5$ is most interesting since the small spin velocity approximation is valid. The curves of positive effective viscosity become triple-valued at significantly lower frequencies than the positive effective viscosity curves that fall to the right of the $\tilde{\eta}_{eff} = 0$ demarcation curve of Figure 4-1. The negative effective viscosity curves follow the same trend as the positive effective viscosity curves that fall to the left of the $\tilde{\eta}_{eff} = 0$ curve. These curves are spaced the same way the curves in Figure 4-1 are spaced. That is, the positive effective viscosity curves approach a limit curve at $\tilde{\eta}_{eff} = +\infty$, which is the same limit curve of the negative effective viscosity curves at $\tilde{\eta}_{eff} = -\infty$.

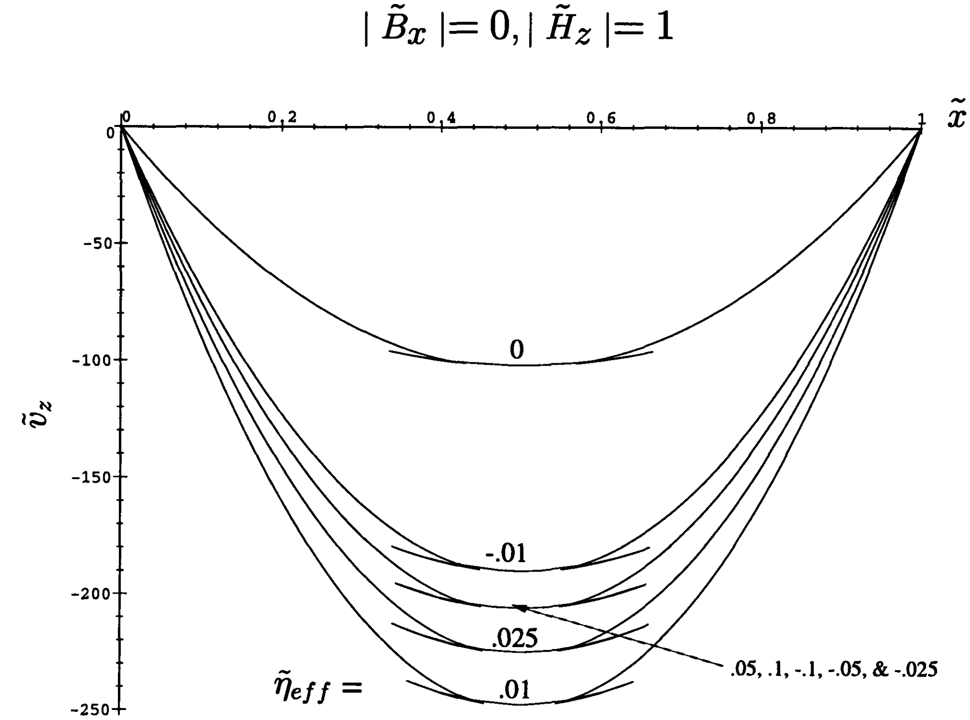
The trend as the frequency increases is for the middle triple-valued regions to spread out in $\tilde{x}$. All of the outer regions remain double-valued for all values of frequency, although the range in $\tilde{x}$ decreases at first, then increases again. With this trend, it is possible that the curves become quadruple-valued for frequencies greater than the examined $\tilde{\Omega}$, as the range of multi-values in the middle spreads to the outer regions of multi-values. Also, looking at the $\tilde{\eta}_{eff} = 0$ curve in Figure 4-6 (f) for $\tilde{\Omega} = 10$, we see that the zig-zag behavior in a very small region around $\tilde{x} = 0.5$ becomes more pronounced. Similar to the curves in the transverse magnetic field cases of Chapter 3, it is possible for larger frequencies that the value of $\tilde{\omega}_y$ have 5 values in this small region. If this trend does continue, for very large frequencies, the spin velocity profiles will become vertical curves, zig-zagging in the central region between $\tilde{x} \approx 0.4$ and $\tilde{x} \approx 0.6$ with multiple-values of $\tilde{\omega}_y$ for a given $\tilde{x}$, including large values of $\tilde{\omega}_y$.

4.2.2 Flow Velocity $\tilde{v}_z(\tilde{x})$ Profiles

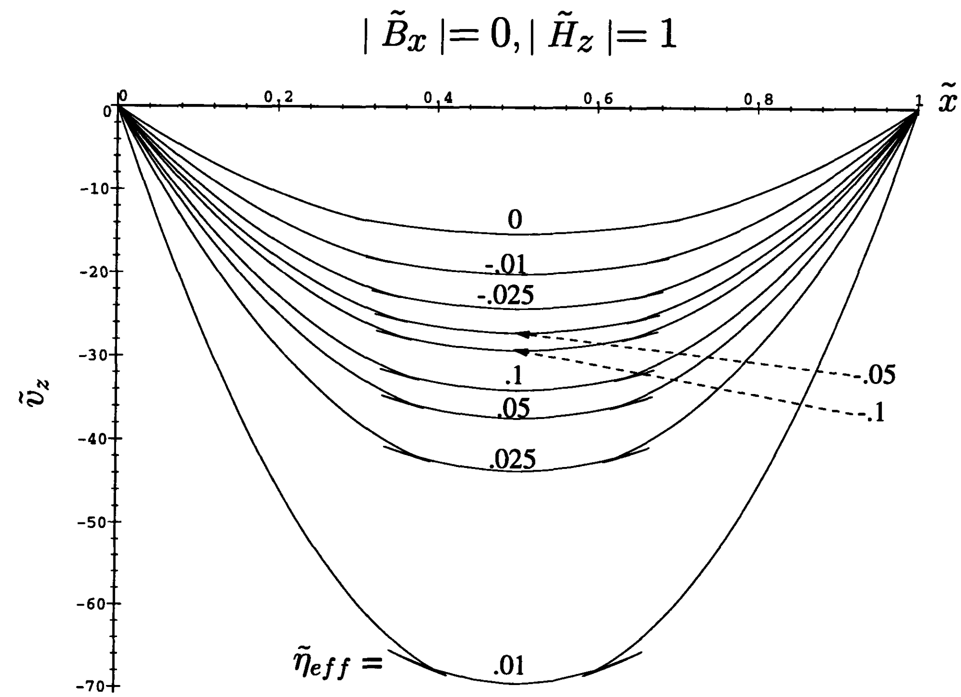
Analogous to section 4.1.2, Eqs. (2.36) - (2.38) were used to plot the flow velocity profiles, seen in Figure 4-7, for the effective viscosity curves that fall to the left of the $\tilde{\eta}_{eff} = 0$ demarcation curve. The appropriate values of $\tilde{\Omega}$, $\tilde{\zeta}$, and $\tilde{\omega}_0$ from Tables 4.3 and 4.4 were used.

Similar to the flow velocity profiles for the effective viscosity curves to the right of the $\tilde{\eta}_{eff} = 0$ demarcation curve, these profiles show that the flow velocity is large in the center of the duct for very small frequencies. As the frequency increases to the value $\tilde{\Omega} = 3.2126$, which is the value where the positive and negative roots meet in Figure 4-1, the peak value of the flow velocity decreases significantly. Then, increasing the frequency further causes the flow velocity to become larger in the center of the duct.

Figure 4-7: Linear flow velocity distributions for various values of $\tilde{\Omega}$ and $\tilde{\eta}_{eff}$ that fall to the left of the $\tilde{\eta}_{eff} = 0$ demarcation curve. a) $\tilde{\Omega} = 2.05$, b) $\tilde{\Omega} = 2.5$, c) $\tilde{\Omega} = 3.2126$, d) $\tilde{\Omega} = 4.5$, e) $\tilde{\Omega} = 6.0$, f) $\tilde{\Omega} = 10.0$.

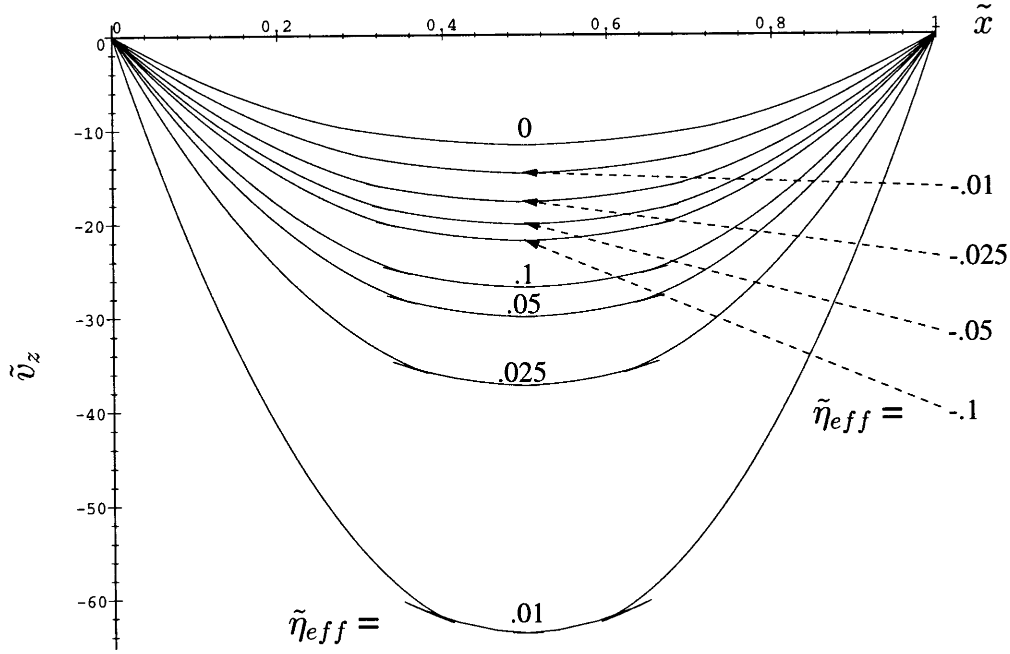


a: $\tilde{\Omega} = 2.05$



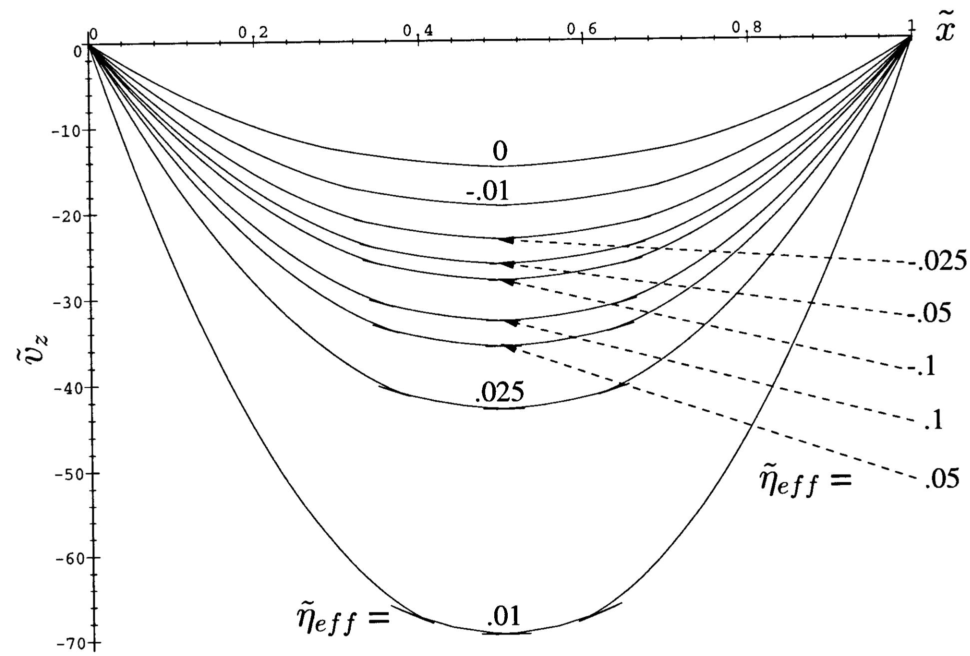
b: $\tilde{\Omega} = 2.5$

$| \tilde{B}_x | = 0, | \tilde{H}_z | = 1$

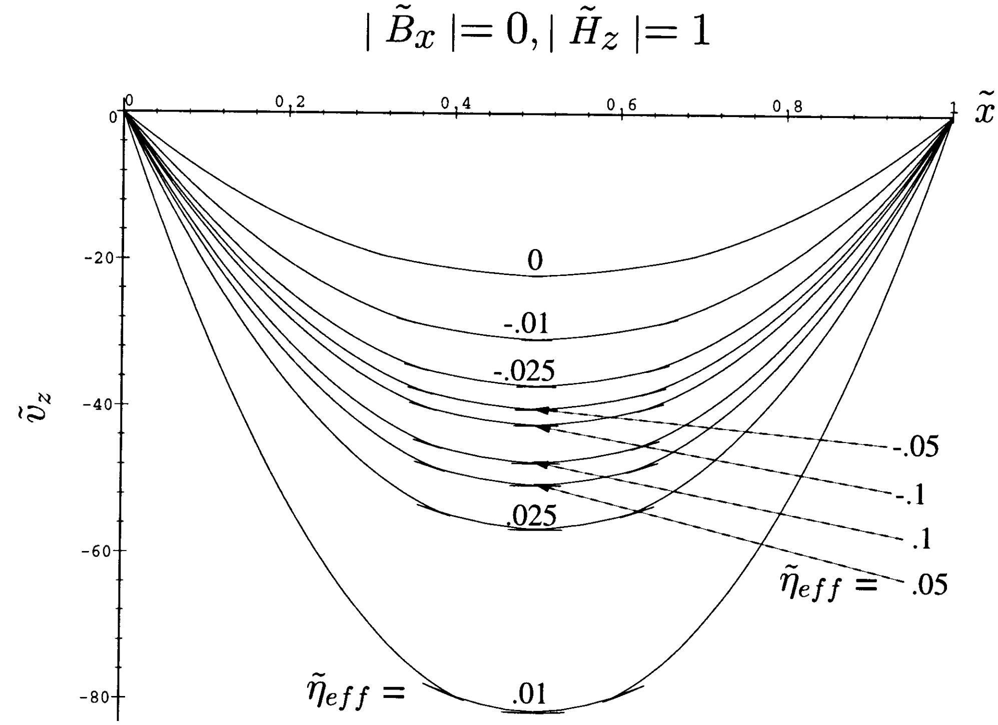


c: $\tilde{\mathbf{\Omega}} = \mathbf{3.2126}$

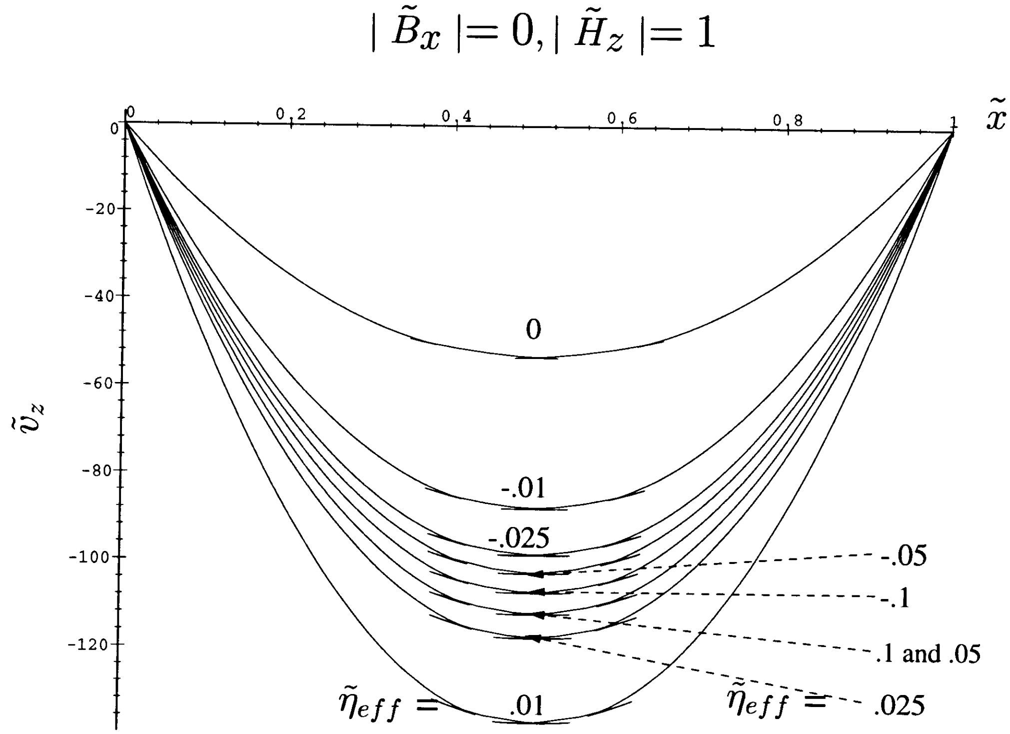
$| \tilde{B}_x | = 0, | \tilde{H}_z | = 1$



d: $\tilde{\mathbf{\Omega}} = \mathbf{4.5}$



e: $\tilde{\mathbf{\Omega}} = \mathbf{6.0}$

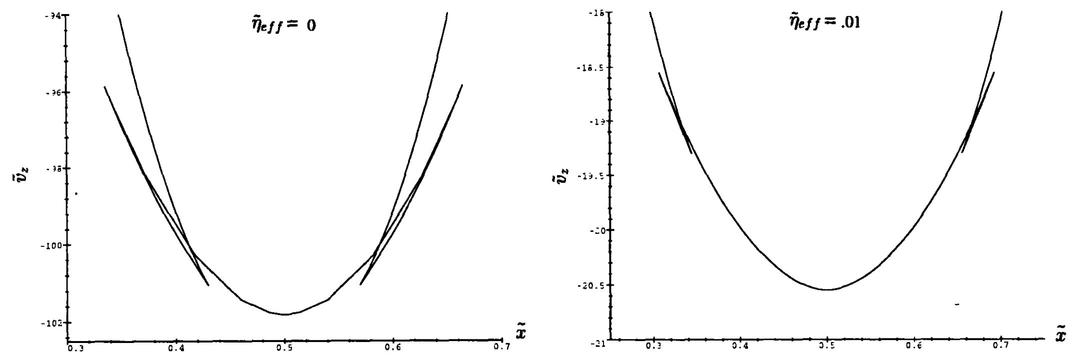


f: $\tilde{\mathbf{\Omega}} = \mathbf{10.0}$

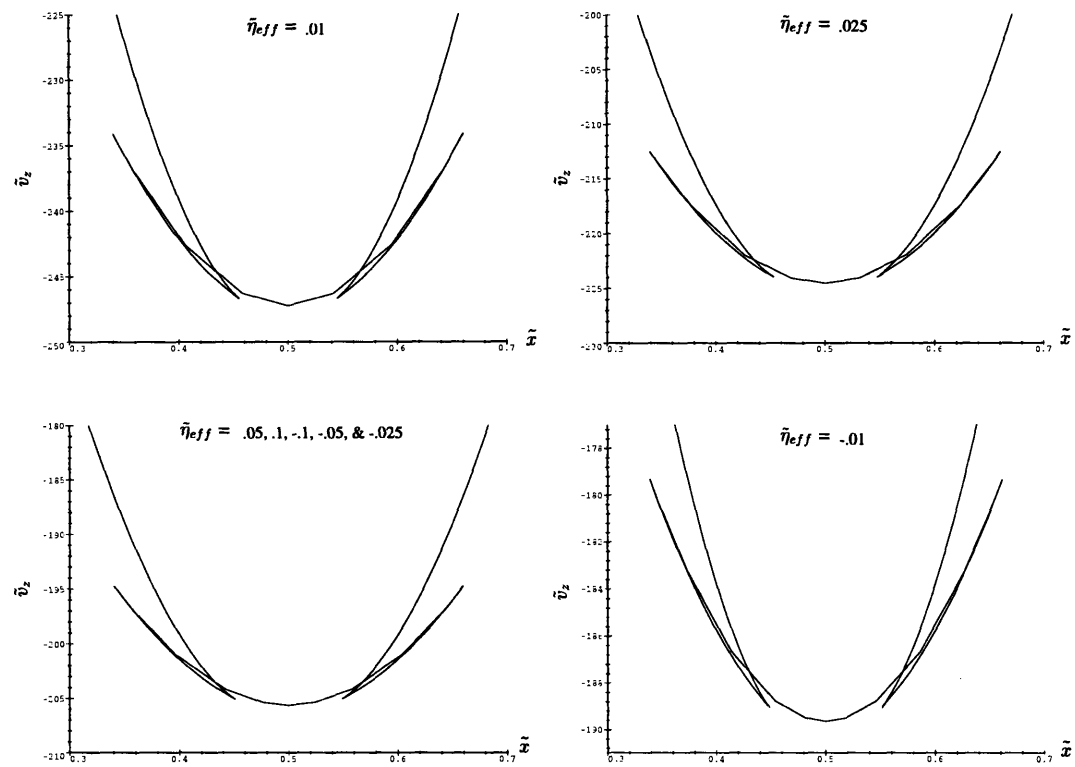
These figures show how a change in frequency affects the profiles of different effective viscosities relative to each other. For a small frequency of $\tilde{\Omega} = 2.05$, the curves of the positive and negative effective viscosities are grouped together without much distinction between the two signs. In fact, $\tilde{\eta}_{eff} = \{.05, .1, -.1, -.05, and -.025\}$ are the same curve since their values of $\tilde{\zeta}$ and $\tilde{\omega}_0$ are the same, to the significant digits produced by Matlab. The positive and negative effective viscosity curves are set apart from the $\tilde{\eta}_{eff} = 0$ curve. Then, as the frequency is increased, the curves begin to group closer to the $\tilde{\eta}_{eff} = 0$ curve, while the $\tilde{\eta}_{eff} = .01$ curve becomes more distinct from the others. At a frequency of $\tilde{\Omega} = 3.2126$, the negative effective viscosity curves are closer to the $\tilde{\eta}_{eff} = 0$ curve, and a distinction between the positive and negative effective viscosity curves can be seen. They follow the same trend as seen in Figure 4-1 where the $\tilde{\eta}_{eff} = +\infty$ limit curve is the same as the $\tilde{\eta}_{eff} = -\infty$ limit curve. As the frequency is increased further, the trend reverses again, and the positive and negative effective viscosity curves are grouped together again, this time separate from both $\tilde{\eta}_{eff} = 0$ and $\tilde{\eta}_{eff} = .01$.

Again, the region around $\tilde{x} = 0.5$ is most interesting since the small spin velocity limit is valid there. As the above described transition from small frequencies to large frequencies happens, there is a change in shape of this central region. Because of the scaling used in Figure 4-7, it is difficult to see the behavior of the curves in this region. It was necessary to examine them more closely, which is done in Figures 4-8 through 4-10. However, it is clear from Figure 4-7 (d), (e), and (f) that all the positive and negative effective viscosities profiles for $\tilde{\Omega} = 4.5, 6.0,$ and 10.0 are triple-valued in the center of the duct.

All of the effective viscosity curves for $\tilde{\Omega} = 2.05$ look essentially the same. Figure 4-8 shows that for $\tilde{\Omega} = 2.05$, there are no multiple values around the region where $\tilde{x} = 0.5$.



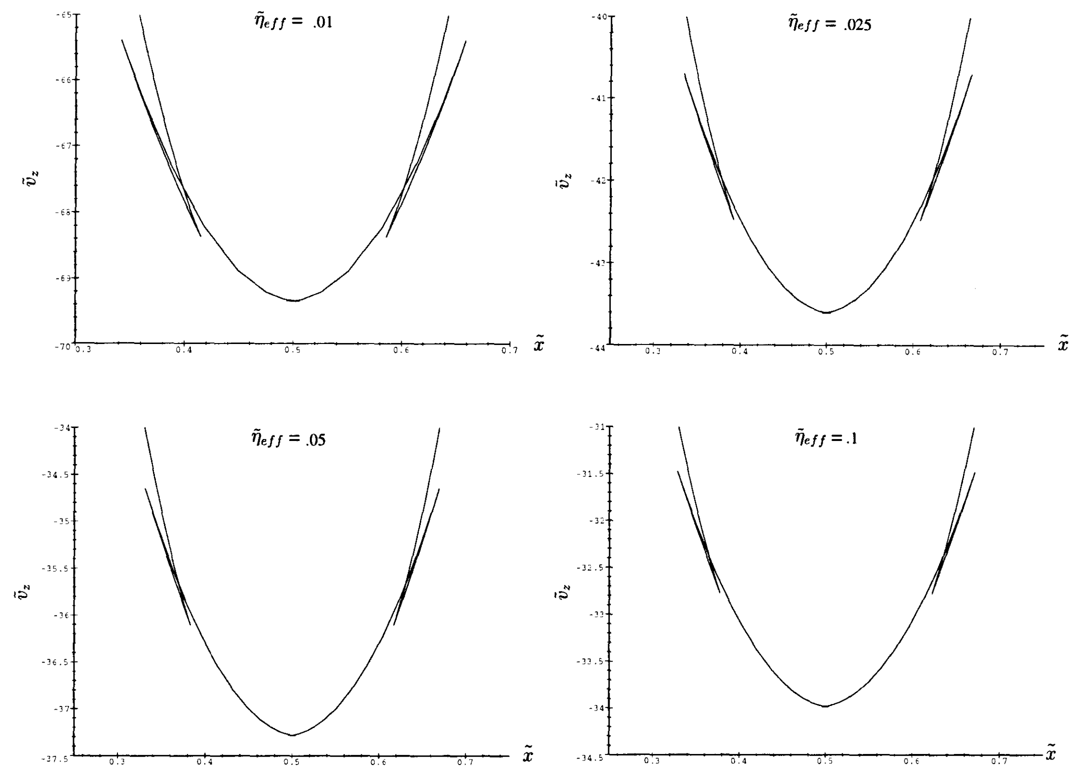
a: $\tilde{\boldsymbol{\Omega}} = \mathbf{2.05}$ positive effective viscosities.



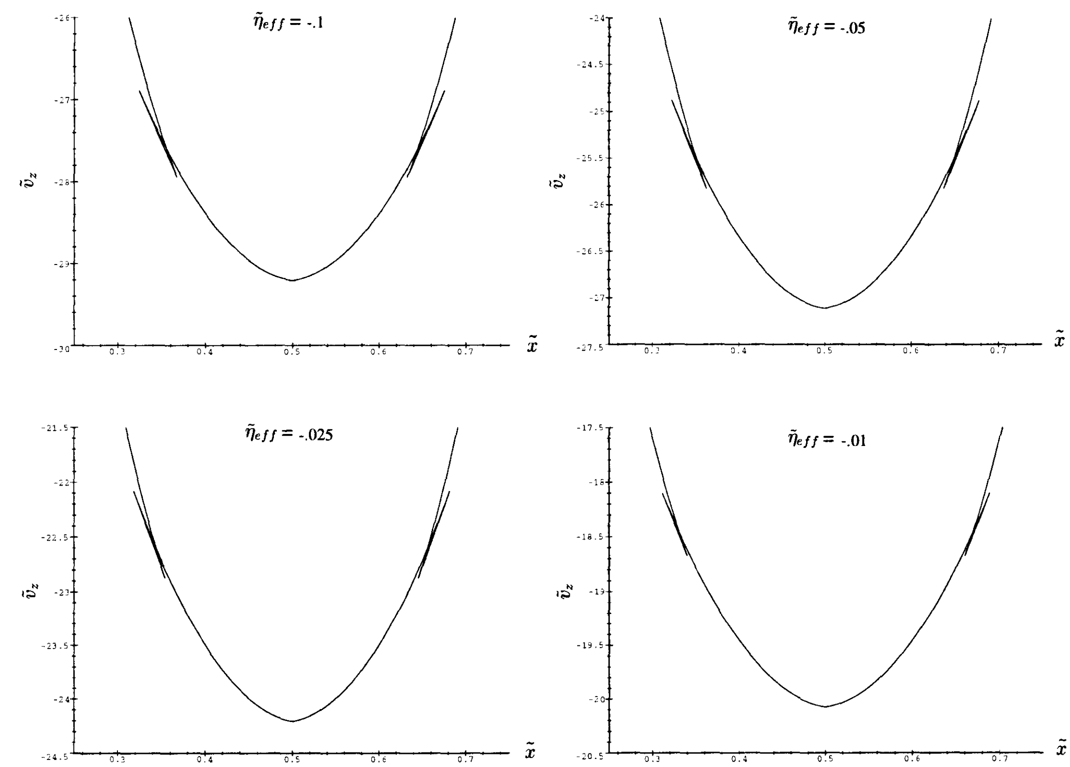
b: $\tilde{\boldsymbol{\Omega}} = \mathbf{2.05}$ negative effective viscosities.

Figure 4-8: $\tilde{\Omega} = 2.05$ curves of Figure 4-7 (a) separated and increased to see the multi-valued behavior of the curves.

Simply looking at Figure 4-7 (b), it seems as if there are no multiple-values in this region as well. However, by separating the curves and increasing the scale, we see that almost every effective viscosity curve is triple-valued over a very small range in $\tilde{x}$ around $\tilde{x} = 0.5$. Figure 4-9 shows the individual curves for $\tilde{\Omega} = 2.5$. The larger positive effective viscosities have a smaller region of $\tilde{x}$ that is triple-valued. The region of $\tilde{x}$ that is triple-valued for the negative effective viscosities is almost indistinguishable.



a: $\tilde{\mathbf{\Omega}} = \mathbf{2.5}$ positive effective viscosities.

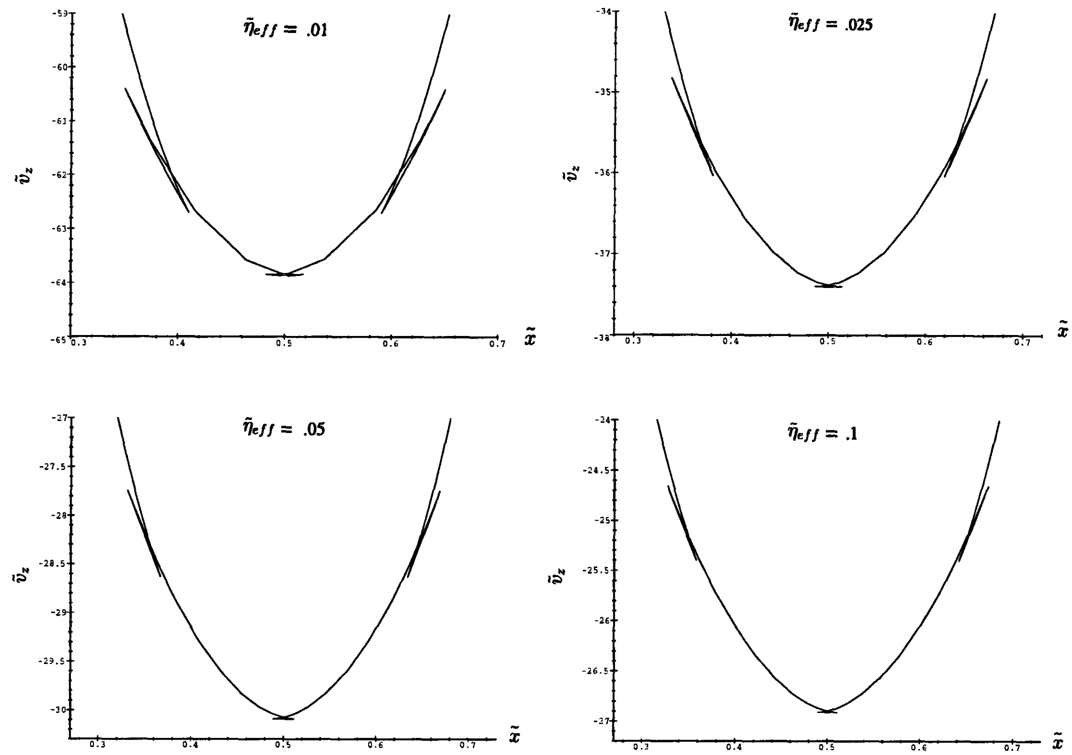


b: $\tilde{\mathbf{\Omega}} = \mathbf{2.5}$ negative effective viscosities.

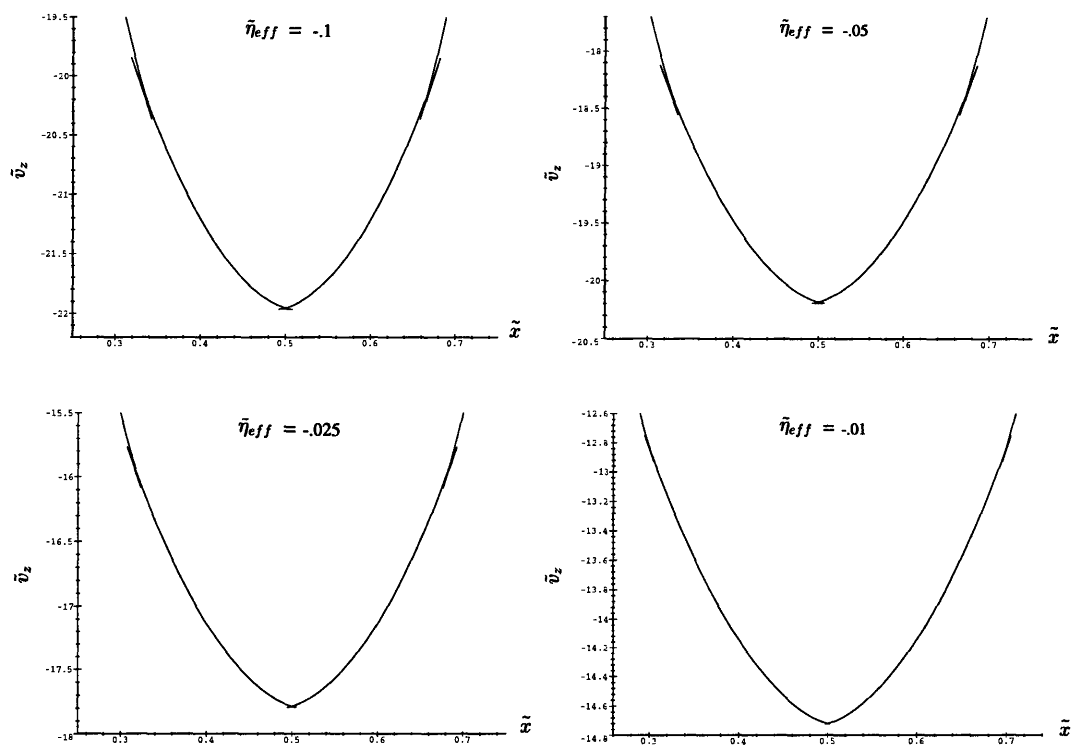
Figure 4-9: $\tilde{\Omega} = 2.5$ curves of Figure 4-7 (b) separated and increased to see the multi-valued behavior of the curves.

Further increasing the frequency to $\tilde{\Omega} = 3.2126$ shows convincingly that all positive and negative effective viscosity curves are multi-valued in the region around $\tilde{x} = 0.5$. The range over $\tilde{x}$ that is triple-valued is now beginning to increase. This is seen in Figure 4-10.

As the frequency increases to $\tilde{\Omega} = 4.5$, then to $\tilde{\Omega} = 6.0$, and finally to $\tilde{\Omega} = 10.0$, as seen in Figure 4-7 (d), (e), and (f), the range over $\tilde{x}$ that is triple-valued in the middle region continues to increase. It increases from $\tilde{\eta}_{eff} = 0$ to $\tilde{\eta}_{eff} = .01$ in the order that the effective viscosity curves range in Figure 4-1: $\tilde{\eta}_{eff} = \{0, -.01, -.025, -.05, -.1, .1, .05, .025, .01\}$. If the trend continues, it is possible that the inner regions around $\tilde{x} = 0.5$ will overlap the outer multi-valued regions given a frequency larger than $\tilde{\Omega} = 10$ or a smaller effective viscosity than $\tilde{\eta}_{eff} = .01$ but larger than 0.



a: $\tilde{\boldsymbol{\Omega}} = \mathbf{3.2126}$ positive effective viscosities.



b: $\tilde{\boldsymbol{\Omega}} = \mathbf{3.2126}$ negative effective viscosities.

Figure 4-10: $\tilde{\Omega} = 3.2126$ curves of Figure 4-7 (c) separated and increased to see the multi-valued behavior of the curves.

Chapter 5

Summary and Conclusions

5.1 Significant Results

The current primary applications of ferromagnetic fluids use DC magnetic fields. The motion of ferromagnetic fluids in DC magnetic fields is simpler to solve because the fluid magnetization is collinear with the magnetic field. The motion of ferrofluids in alternating or traveling wave magnetic fields is complicated by a body torque on the fluid. This torque results from the fluid friction causing the magnetization to lag the alternating or traveling magnetic field. Contrary to intuition, earlier work has shown that when the fluid spin velocity is small, the electromechanical coupling between the magnetic field and flow can lead to an effective dynamic viscosity, $\tilde{\eta}_{eff}$, that can be made zero or negative as a function of the magnetic field strength, direction, and frequency. When this effective viscosity is negative, the fluid pumps backwards which means it flows opposite to the direction the magnetic field travels.

In the small spin velocity approximation ($\tilde{\omega}_y \ll 1$), both the flow and spin velocities of the ferrofluid depend on the effective viscosity. Mathematically, a singularity occurs in both the flow and spin velocities with zero spin viscosity, which causes these velocities to become infinite. This, however, violates the small spin velocity approximation. This thesis explored positive, zero, and negative effective viscosities to determine if the change in sign of the effective viscosity is the cause of the observed backward pumping phenomena. To do so, the governing one dimensional fluid and field equations in the viscous dominated limit were numerically solved without further approximation within a planar duct. This allows fluid inertia to be neglected in order to closer study the behavior of the ferrofluid under conditions of predicted infinite velocity.

Unlike most past work which considered fluid pumping due to applied rotating or traveling magnetic fields, this thesis found that with alternating applied magnetic fields either perpendicular or parallel to the duct axis, time average flow and spin velocities result. The fluid spatial profiles have multi-valued regions where at one spatial position there can be more than one allowed flow and spin velocity.

5.2 Zero Spin-Viscosity in a Planar Duct

To simplify the governing equations, one dimensional magnetic field conditions were studied of a ferrofluid pumping in a planar duct. The equations of fluid flow in this planar duct that were solved included two magnetic field orientations:

- an imposed uniform magnetic field component transverse to the duct axis, $|\tilde{B}_x| = 1,\ |\tilde{H}_z| = 0$
- an imposed uniform magnetic field component along the duct axis, $|\tilde{B}_x| = 0,\ |\tilde{H}_z| = 1$

For each of these cases, curves of non-dimensional frequency $\tilde{\Omega}$ as a function of non-dimensional viscosity $\tilde{\zeta}$ were plotted for nine values of non-dimensional effective viscosity $\tilde{\eta}_{eff}$ (Figures 3-1 and 3-2). Because $\tilde{\Omega}(\tilde{\zeta})$ is a biquadratic 4th order equation, there are four possible roots. For each case, the positive effective viscosities had four positive real roots, which when plotted, look like two general curves. The negative effective viscosities, however, had only two positive real roots which look like one general curve when plotted. A demarcation curve of $\tilde{\eta}_{eff} = 0$ was chosen, and the effective viscosity curves to the right of this demarcation curve were studied separately from the curves to the left.

The curves to the right of the demarcation curve were positive effective viscosity curves. As the value of the effective viscosity increased, the curves continued to the right with increasing viscosity $\tilde{\zeta}$. The curves to the left of the demarcation curve were both positive and negative effective viscosity curves. The left-most curves are positive effective viscosity, while the negative effective viscosity curves lay between the positive effective viscosity curves and the demarcation curve of $\tilde{\eta}_{eff} = 0$. Most interestingly in this region, as the value of positive effective viscosity increases, a limit curve of $\tilde{\eta}_{eff} = +\infty$ is reached. As the value of negative effective viscosity decreases to $\tilde{\eta}_{eff} = -\infty$, this same limit curve is reached from the other side.

At low frequency, the value of $\tilde{\zeta}$ is small as well. For each effective viscosity curve, increasing the frequency causes the viscosity $\tilde{\zeta}$ to increase as well. The viscosity $\tilde{\zeta}$ continues to increase until the frequency at which the positive and negative roots meet. (This value of frequency is $\tilde{\Omega} = 2.0412$ for the transverse magnetic field case and $\tilde{\Omega} = 3.2126$ for the tangential magnetic field case.) Further increasing the frequency above this value decreases the value of the viscosity $\tilde{\zeta}$.

5.2.1 Spin Velocity $\tilde{\omega}_y(\tilde{x})$ Profiles

The behavior of the spin velocity profiles is similar between the two cases of transverse magnetic fields and tangential magnetic fields. There can be four regions that are multi-valued in these profiles. There are two middle regions around $\tilde{x} = 0.5$ which is the center of the duct. There are two outer regions as well. For both cases studied, the outer regions are multi-valued for lower values of viscosity $\tilde{\zeta}$. Since the regions around $\tilde{x} = 0.5$ are where the small spin velocity approximations are valid, they are more interesting. These middle regions become multi-valued when the frequency $\tilde{\Omega}$ is increased.

All multi-valued behavior in the spin velocity profiles is exaggerated when examining the effective viscosity curves that fall to the left of the $\tilde{\eta}_{eff} = 0$ demarcation curve. The smaller the value of viscosity $\tilde{\zeta}$, the larger the range in $\tilde{x}$ in which the profiles can be multi-valued in the outer regions. The larger the value of frequency $\tilde{\Omega}$, the larger the range in $\tilde{x}$ in which the profiles can be multi-valued in the middle regions near the center of the duct.

When examining the effective visocity curves to the left of the demarcation curve, increasing the frequency $\tilde{\Omega}$, which decreases $\tilde{\zeta}$, causes the middle regions of the spin velocity profiles to approach the outer regions. It is possible that these regions overlap for larger frequencies. It is also possible that a new middle region of multiple values with a very small range in $\tilde{x}$ is formed for larger frequencies.

5.2.2 Flow Velocity $\tilde{v}_z(\tilde{x})$ Profiles

The behavior of the flow velocity profiles between the transverse magnetic field case and the tangential magnetic field case is also similar. There can be four regions that are multi-valued in these profiles. Two regions are in the center of the duct around $\tilde{x} = 0.5$, and two regions are further out. Since the regions in the middle of the duct are where the small spin velocity approximations are valid, they are more interesting. These middle regions become multi-valued when the frequency $\tilde{\Omega}$ is increased.

All multi-valued behavior in the flow velocity profiles is exaggerated for the effective viscosity curves that fall to the left of the $\tilde{\eta}_{eff} = 0$ demarcation curve. The smaller the value of $\tilde{\zeta}$, the larger the range in $\tilde{x}$ in which the profiles can be multi-valued in the outer regions. The larger the value of frequency $\tilde{\Omega}$, the larger the range in $\tilde{x}$ in which the profiles can be multi-valued in the middle regions near the center of the duct.

When examining the effective visocity curves to the left of the demarcation curve, increasing the frequency $\tilde{\Omega}$, which decreases $\tilde{\zeta}$, causes the middle regions of the flow velocity profiles to approach the outer regions. It is possible that these regions overlap for larger frequencies.

The trends of multiple values in the flow velocity profiles as frequency is increased are the same as the trends of multiple values in the spin velocity profiles. Sometimes the scaling of the figures makes the multiple-values over a very small range in $\tilde{x}$ difficult to view. The corresponding profile for the flow or spin velocity can be looked at for the same values of $\tilde{\Omega}$ and $\tilde{\eta}_{eff}$ to determine whether there is multi-valued behavior.

5.3 Future Work

5.3.1 Rotating Uniform Magnetic Field

The work done in this thesis for $\tilde{\eta}' = 0$ considers only two of four possible cases. The other two cases that should be considered are magnetic fields at an angle to the duct axis with $\tilde{H}_z$ and $\tilde{B}_x$ in phase and rotating uniform magnetic fields produced by both a transverse and a tangential magnetic field component that are not in phase:

$$\tilde{B}_x = 1,\ \tilde{H}_z = 1 \qquad \text{Field at constant angle} \tag{5.1}$$

$$\tilde{B}_x = 1,\ \tilde{H}_z = i \qquad \text{Rotating Field} \tag{5.2}$$

Some preliminary work has been done in parallel with the work presented in this thesis. With the same assumptions in this thesis, that $\tilde{\eta} = \tilde{\zeta}$ and $\chi_0 = 1$, we repeat Eq. (3.2) here:

$$\alpha = \frac{2\tilde{\zeta}(\tilde{\zeta} - \tilde{\eta}_{eff})}{2\tilde{\zeta} - \tilde{\eta}_{eff}}. \tag{5.3}$$

Specifically for a rotating uniform magnetic field, only the magnitudes are needed when plotting $\tilde{\Omega}(\tilde{\zeta})$. Thus, using Eq. (5.1) and (5.2) in Eq. (2.41) yields

$$\alpha = \frac{\chi_0}{2}\frac{\left[(\tilde{\Omega}^2 - 1) + \left[\tilde{\Omega}^2 - (1+\chi_0)^2\right]\right]}{(1+\chi_0+\tilde{\Omega}^2)^2 + \chi_0^2\tilde{\Omega}^2}. \tag{5.4}$$

By setting Eqs. (5.3) and (5.4) equal to each other, $\tilde{\Omega}$ can be solved for in terms of $\tilde{\zeta}$ and $\tilde{\eta}_{eff}$. This will allow plots of $\tilde{\Omega}(\tilde{\zeta})$ to be made for different values of effective viscosity. The result is a 4th order equation in $\tilde{\Omega}$.

$$\tilde{\Omega}^4 + \tilde{\Omega}^2\left[2(1+\chi_0) + \chi_0^2 - \frac{\chi_0}{\alpha}\right] + \left[(1+\chi_0)^2 + (1+\chi_0)^2\frac{\chi_0}{2\alpha} + \frac{\chi_0}{2\alpha}\right] = 0 \tag{5.5}$$

where α, defined in Eq. (5.3), depends on $\tilde{\zeta}$ and $\tilde{\eta}_{eff}$.

To plot $\tilde{\Omega}$ as a function of $\tilde{\zeta}$ for different values of $\tilde{\eta}_{eff}$, the quadratic formula was used to find $\tilde{\Omega}^2$, and then the square root was taken

$$\tilde{\Omega}^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{5.6}$$

where

$$\begin{aligned} a &= 1 && (5.7) \\ b &= 2(1+\chi_0) + \chi_0^2 - \frac{\chi_0}{\alpha} && (5.8) \\ c &= (1+\chi_0)^2 + (1+\chi_0)^2 \frac{\chi_0}{2\alpha} + \frac{\chi_0}{2\alpha}. && (5.9) \end{aligned}$$

Once $\tilde{\Omega}^2$ was calculated, only those values that were both positive and real were plotted.

So the case studies would be similar, the values of $\tilde{\eta}_{eff}$ were chosen to be

$$\tilde{\eta}_{eff} = \{-.1;\ -.05;\ -.025;\ -.01;\ 0;\ .01;\ .025;\ .05;\ .1\}. \tag{5.10}$$

These values allow the plotted curves to be well spaced and show the trend of change as $\tilde{\eta}_{eff}$ goes from negative values, through zero, and becomes positive. The resulting plot can be seen in Figure 5-1. We see that this figure is similar to Figures 3-1 and 4-1. This plot was made using a Matlab script file, which can be found in Appendix C.

To continue the work, six values of frequency would be chosen, including the frequency $\tilde{\Omega} = 2.6955$ which is where the positive and negative roots meet. By choosing values of $\tilde{\Omega}$, the corresponding values of $\tilde{\zeta}$ can be calculated. Both values will be necessary for plotting the flow and spin velocity profiles. The Matlab script file in Appendix D can be used for this calculation. The script file was made by solving for Eq. (5.5) for $\tilde{\zeta}$ instead of $\tilde{\Omega}$.

$$\begin{aligned} &2\tilde{\zeta}^2\Big[(1+\chi_0+\tilde{\Omega}^2)^2 + \chi_0^2\tilde{\Omega}^2\Big] \\ &- \tilde{\zeta}\Big[2\tilde{\eta}_{eff}\Big[(1+\chi_0+\tilde{\Omega}^2)^2 + \chi_0^2\tilde{\Omega}^2\Big] - \chi_0\Big[(\tilde{\Omega}^2-1) + [\tilde{\Omega}^2 - (1+\chi_0)^2]\Big]\Big] \\ &+ \Big[\tilde{\eta}_{eff}\frac{\chi_0}{2}\Big[(\tilde{\Omega}^2-1) + [\tilde{\Omega}^2 - (1+\chi_0)^2]\Big]\Big] = 0 \end{aligned} \tag{5.11}$$

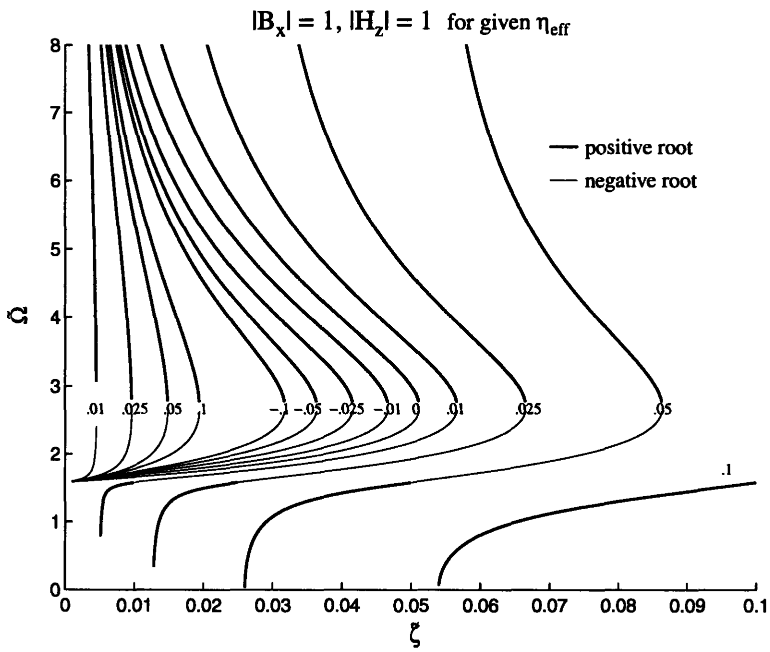


Figure 5-1: Frequency $\tilde{\Omega}$ as a function of viscosity $\tilde{\zeta}$ for nine values of $\tilde{\eta}_{eff}$. The bold lines represent the positive roots of the quadratic, and the plain lines represent the negative roots.

To solve for $\tilde{\zeta}$, the quadratic equation is used.

Again, a parametric plot would be used to make the $\tilde{\omega}_y(\tilde{x})$ and $\tilde{v}_z(\tilde{x})$ profiles. The spin velocity $\tilde{\omega}_y$ would be the variable ranged in this parametric plot, so the values must be known at the boundaries. To do this, the Maple command from Eq. (3.16)

$$Solve[\tilde{x}[\tilde{\omega}_0, \tilde{\Omega}, \chi_0, \tilde{\zeta}, \frac{\partial \tilde{p}'}{\partial \tilde{z}}] == 0, \tilde{\omega}_0] \tag{5.12}$$

which means "Solve for $\tilde{\omega}_0$ when $\tilde{x} = 0$."

There is a difference here compared to the two cases studied in this thesis. The non-dimensional position function, $\tilde{x}$, was defined by Eq. (2.34) and repeated here:

$$\tilde{x} = \frac{-\tilde{\eta}}{\frac{\partial \tilde{p}'}{\partial \tilde{z}}}\left[\int \left(1 - \frac{\tilde{\zeta} + \tilde{\eta}}{2\tilde{\zeta}\tilde{\eta}} \frac{d\langle \tilde{T}_y \rangle}{d\tilde{\omega}_y}\right) d\tilde{\omega}_y + C\right] \tag{5.13}$$

In the transverse-only and tangential-only magnetic field cases, the value of C could be solved for analytically by recognizing that the torque density was zero when $\tilde{\omega}_y$

was zero and that $\tilde{\omega}_y$ is an odd function around $\tilde{x} = 0.5$ which meant $\tilde{\omega}_y(\tilde{x} = 0.5) = 0$. This is no longer the case. For a rotating uniform magnetic field, this constant of integration C can no longer be solved for in closed form. It must be calculated for each changed parameter $\tilde{\Omega}$, $\tilde{\zeta}$ and $\tilde{\eta}_{eff}$. Once this is done, the process for plotting the spin and flow velocity profiles used in this thesis can be used.

5.3.2 Non-Zero Spin-Viscosity ($\tilde{\eta}' \neq 0$) Solutions

These profiles do not represent the true physical motion of ferrofluids in a planar duct since it has been assumed that the spin viscosity is zero ($\tilde{\eta}' = 0$). This was done to reduce a fourth order system to a second order system. Assuming the spin viscosity is zero allows the spin velocities to be non-zero at the duct walls ($\tilde{x} = 0,\ \tilde{x} = 1$).

It is anticipated that the multi-valued nature of the flow and spin velocity profiles will be smoothed out by removing the $\tilde{\eta}' = 0$ assumption so that single-valued solutions exist. That is, at a given position there should be only one flow velocity and one spin velocity solution. When $\tilde{\eta}' \neq 0$, Eqs. (2.23) - (2.24) remain a 4th order system, and both the flow and spin velocities must be zero at the $\tilde{x} = 0$ and $\tilde{x} = 1$ boundaries. These equations must be numerically integrated by the Runge-Kutta method. This method, however, cannot directly solve two-point boundary-valued problems with boundary conditions at both boundaries. It requires specifications of the functions and their derivatives at one boundary. Therefore, one must specify not only that

$$\tilde{v}_z(\tilde{x} = 0) = 0; \quad \tilde{\omega}_y(\tilde{x} = 0) = 0 \tag{5.14}$$

but guess the values of the derivatives at $\tilde{x} = 0$:

$$D_1 = \frac{d\tilde{v}_z}{d\tilde{x}}\Big|_{\tilde{x}=0}, \quad D_2 = \frac{d\tilde{\omega}_y}{d\tilde{x}}\Big|_{\tilde{x}=0}. \tag{5.15}$$

Then, Newton's method must be used to find the best values of D_1 and D_2 such that numerically integrating the flow and spin velocity equations causes $\tilde{v}_z(\tilde{x} = 1) = 0$ and $\tilde{\omega}_y(\tilde{x} = 1) = 0$.

The profiles of $\tilde{\omega}_y(\tilde{x})$ and $\tilde{v}_z(\tilde{x})$ from this method would accurately describe the motion of the ferrofluids.

Appendix A

Maple Files for Parametric Plots

A.1 Transverse Magnetic Field Only

```
> denomin:=(wy, Om, chi) -> ((wy^2 - Om^2 + 1 + chi)^2 + (2 + chi)^2*Om^2);

                                     2     2            2             2   2
      denomin := (wy,Om,chi) -> (wy   - Om   + 1 + chi)   + (2 + chi)   Om
-------------------------------------------------------------------------------
> t1:=(wy,Om,chi,Bx,Hz) -> -wy*(abs(Bx)^2*(wy^2 - Om^2 + 1) + abs(Hz)^2*(wy^2
- Om^2 + (1 + chi)^2));

    t1 := (wy,Om,chi,Bx,Hz) ->

                       2    2     2                2    2     2              2
       - wy (abs(Bx)   (wy   - Om   + 1) + abs(Hz)   (wy   - Om   + (1 + chi) ))
-------------------------------------------------------------------------------
> t2:=(wy,Om,chi,Bx,Hz) -> (chi*(wy^2 - Om^2) + I*Om*(wy^2 - Om^2 - 1 - chi))*
Hz*conjugate(Bx);

      t2 := (wy,Om,chi,Bx,Hz) ->

                 2     2             2     2
         (chi (wy   - Om ) + I Om (wy   - Om   - 1 - chi)) Hz conjugate(Bx)
-------------------------------------------------------------------------------
> t3:=(wy,Om,chi,Bx,Hz) -> (chi*(wy^2 - Om^2) - I*Om*(wy^2 - Om^2 - 1 - chi))*
conjugate(Hz)*Bx;

      t3 := (wy,Om,chi,Bx,Hz) ->

                 2     2             2     2
         (chi (wy   - Om ) - I Om (wy   - Om   - 1 - chi)) conjugate(Hz) Bx
-------------------------------------------------------------------------------
> t:=(wy,Om,chi,Bx,Hz)-> .5*chi*(t1(wy,Om,chi,Bx,Hz) + t2(wy,Om,chi,Bx,Hz) +
t3(wy,Om,chi,Bx,Hz))/denomin(wy,Om,chi);
```

```
t := (wy,Om,chi,Bx,Hz) -> .5 chi

  (t1(wy, Om, chi, Bx, Hz) + t2(wy, Om, chi, Bx, Hz) + t3(wy, Om, chi, Bx, Hz))

   /denomin(wy, Om, chi)
--------------------------------------------------------------------------------
> dt:=(wy,Om,chi,Bx,Hz)->diff(t(wy,Om,chi,Bx,Hz),wy);

          dt := (wy,Om,chi,Bx,Hz) -> diff(t(wy, Om, chi, Bx, Hz), wy)
--------------------------------------------------------------------------------
> alphacrit:=(zeta,eta)->2*zeta*eta/(zeta+eta);

                                                   zeta eta
                     alphacrit := (zeta,eta) -> 2 ----------
                                                  zeta + eta
--------------------------------------------------------------------------------
> x:=(wy,Om,chi,zeta,dpdz)->-(integrate(1-dt(wy,Om,chi,1,0)/alphacrit(zeta,
zeta),wy) - .5*dpdz/zeta)*(zeta/dpdz);

        x := (wy,Om,chi,zeta,dpdz) ->

             /               dt(wy, Om, chi, 1, 0)           dpdz\
             |integrate(1 - ---------------------, wy) - .5 ----| zeta
             \               alphacrit(zeta, zeta)           zeta/
           - -----------------------------------------------------------
                                       dpdz
--------------------------------------------------------------------------------
> dwydx:=(wy,Om,chi,Bx,Hz,zeta,eta,dpdz)-> (-dpdz/eta)/(1 - dt(wy,Om,chi,Bx,Hz)
/alphacrit(zeta,eta));

                                                                  dpdz
dwydx := (wy,Om,chi,Bx,Hz,zeta,eta,dpdz) -> - ---------------------------------
                                                  /    dt(wy, Om, chi, Bx, Hz)\
                                              eta |1 - -----------------------|
                                                  \      alphacrit(zeta, eta) /
--------------------------------------------------------------------------------
> dwydx(wy,Om,chi,Bx,Hz,zeta,eta,dpdz);

                                           2    2     2
  - dpdz/(eta (1 - 1/2 (.5 chi (- abs(Bx)  (wy  - Om  + 1)

                 2    2     2             2                2              2
     - abs(Hz)  (wy  - Om  + (1 + chi) ) - wy (2 abs(Bx)  wy + 2 abs(Hz)  wy)

     + (2 chi wy + 2 I Om wy) Hz conjugate(Bx)

     + (2 chi wy - 2 I Om wy) conjugate(Hz) Bx)

       /     2    2            2           2  2
      /  ((wy  - Om  + 1 + chi)  + (2 + chi)  Om ) - 2.0 chi (
     /

                   2    2     2                2    2     2            2
```

```
     - wy (abs(Bx)  (wy   - Om  + 1) + abs(Hz)  (wy   - Om  + (1 + chi) ))

                2     2              2     2
      + (chi (wy   - Om ) + I Om (wy   - Om   - 1 - chi)) Hz conjugate(Bx)

                2     2              2     2
      + (chi (wy   - Om ) - I Om (wy   - Om   - 1 - chi)) conjugate(Hz) Bx)

        2     2                 /     2     2            2           2   2 2
     (wy   - Om   + 1 + chi) wy  /  ((wy   - Om   + 1 + chi)  + (2 + chi)  Om ) )
                               /

     (zeta + eta)/(zeta eta)))
--------------------------------------------------------------------------------
> dvzdwy:=(wy,Om,zeta)-> (-2*wy + t(wy,Om,1,1,0)/zeta)/dwydx(wy,Om,1,1,0,zeta,
zeta,1);

                                                   t(wy, Om, 1, 1, 0)
                                          - 2 wy + ------------------
                                                          zeta
       dvzdwy := (wy,Om,zeta) -> ------------------------------------
                                    dwydx(wy, Om, 1, 1, 0, zeta, zeta, 1)
--------------------------------------------------------------------------------
> dvzdwy(wy,Om,zeta);

  /                    2     2      \
  |              wy (wy   - Om   + 1)|
- |- 2 wy - .5 ------------------| zeta
  \                   %1 zeta       /

  /           2     2              2          2     2     2          2     2      \
  |         wy   - Om   + 1       wy        wy   (wy   - Om   + 1) (wy   - Om   + 2)|
  |     - .5 ------------- - 1.0 --- + 2.0 ----------------------------------|
  |               %1              %1                             2                |
  |                                                             %1                |
  |1 - ------------------------------------------------------------------|
  \                                  zeta                                      /

                                 2     2     2       2
%1 :=                         (wy   - Om   + 2)  + 9 Om
--------------------------------------------------------------------------------
> f:=(wy,Om,zeta,dpdz) -> -((integrate((t(wy,Om,1,1,0)/zeta - 2*wy) * (1 -
dt(wy,Om,1,1,0)/alphacrit(zeta,zeta)),wy))/(dpdz/zeta));

f := (wy,Om,zeta,dpdz) ->

             /         t(wy, Om, 1, 1, 0)\ /     dt(wy, Om, 1, 1, 0) \
  integrate(|- 2 wy + ------------------| |1 - ---------------------|, wy) zeta
             \                zeta       / \   alphacrit(zeta, zeta)/
- ---------------------------------------------------------------------------
                                     dpdz
--------------------------------------------------------------------------------
> f(wy,Om,zeta,dpdz);
```

```
                                              2                      2
        1.       .2500000000                 wy                     Om
 - (- --------- + ----------- - .1250000000 -------- - 1.250000000 ---------
         2  2          2                       2                      2  2
     zeta  %1     zeta  %1                 zeta  %1              zeta  %1

                     2         4           2
          2         wy        Om          Om
     - 1. wy  + 3. ------- + ------- + 5. -------
                   zeta %1   zeta %1     zeta %1

                                             2        2
                                         2. wy  - 2. Om  + 4.
                  arctan(.1666666667 --------------------)
                                                2 1/2
                                              (Om )              4.
     - .08333333333 ---------------------------------------- + -------
                                             2 1/2               zeta %1
                                     zeta (Om )

                    2  2                            2  2
                  Om  wy                ln(%1)    Om  wy
     + 1.375000000 --------- + .1250000000 ------ - 1. -------
                     2  2                   zeta      zeta %1
                  zeta  %1

                      4                        2
                    Om                        wy
     - .2500000000 --------- - .6250000000 ---------) zeta/dpdz
                      2  2                   2  2
                  zeta  %1               zeta  %1

                 4        2  2        2      4        2
%1 :=          wy  - 2. wy  Om  + 4. wy  + Om  + 5. Om  + 4.
--------------------------------------------------------------------------
>
> #OMEGA == 2.0412;
>
--------------------------------------------------------------------------
> # Eta_eff = 0  -->  zeta = 0, .0375;
>vz0:=(wy)->f(wy,2.0412,.0375,1)-eval(subs(w0=12.2371,f(w0,2.0412,.0375,1)));
--------------------------------------------------------------------------
> # Eta_eff = .01  -->  zeta = .0434 and .0043;
>vz01a:=(wy)->f(wy,2.0412,.0043,1)-eval(subs(w0=115.27,f(w0,2.0412,.0043,1)));

vz01a :=

wy -> f(wy, 2.0412, .0043, 1)-eval(subs(w0 = 115.27, f(w0, 2.0412, .0043, 1)))
>vz01b:=(wy)->f(wy,2.0412,.0434,1)-eval(subs(w0=10.4053,f(w0,2.0412,.0434,1)));

 vz01b := wy ->

   f(wy, 2.0412, .0434, 1)-eval(subs(w0 = 10.4053, f(w0, 2.0412, .0434, 1)))
--------------------------------------------------------------------------
```

```
> #Eta_eff = .025 --> zeta = .0545 and .0086;
>vz025a:=(wy)->f(wy,2.0412,.0545,1)-eval(subs(w0=8.02088,f(w0,2.0412,.0545,1)));

 vz025a := wy ->

    f(wy, 2.0412, .0545, 1)-eval(subs(w0 = 8.02088, f(w0, 2.0412, .0545, 1)))
>vz025b:=(wy)->f(wy,2.0412,.0086,1)-eval(subs(w0=57.1213,f(w0,2.0412,.0086,1)));

 vz025b := wy ->

    f(wy, 2.0412, .0086, 1)-eval(subs(w0 = 57.1213, f(w0, 2.0412, .0086, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .05 --> zeta = .0764 and .0123;
>vz05a:=(wy)->f(wy,2.0412,.0764,1)-eval(subs(w0=5.33242,f(w0,2.0412,.0764,1)));

 vz05a := wy ->

    f(wy, 2.0412, .0764, 1)-eval(subs(w0 = 5.33242, f(w0, 2.0412, .0764, 1)))
>vz05b:=(wy)->f(wy,2.0412,.0123,1)-eval(subs(w0=39.6237,f(w0,2.0412,.0123,1)));

 vz05b := wy ->

    f(wy, 2.0412, .0123, 1)-eval(subs(w0 = 39.6237, f(w0, 2.0412, .0123, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .1 --> zeta = .1249 and .0150;
>vz1a:=(wy)->f(wy,2.0412,.1249,1)-eval(subs(w0=3.13429,f(w0,2.0412,.1249,1)));

 vz1a := wy ->

    f(wy, 2.0412, .1249, 1)-eval(subs(w0 = 3.13429, f(w0, 2.0412, .1249, 1)))
>vz1b:=(wy)->f(wy,2.0412,.0150,1)-eval(subs(w0=32.3002,f(w0,2.0412,.0150,1)));

 vz1b := wy ->

    f(wy, 2.0412, .0150, 1)-eval(subs(w0 = 32.3002, f(w0, 2.0412, .0150, 1)))
--------------------------------------------------------------------------------
>
>
> #Eta_eff = -.01 --> zeta = .0330;
>vz01m:=(wy)->f(wy,2.0412,.0330,1)-eval(subs(w0=14.0693,f(w0,2.0412,.0330,1)));

 vz01m := wy ->

    f(wy, 2.0412, .0330, 1)-eval(subs(w0 = 14.0693, f(w0, 2.0412, .0330, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.025 --> zeta = .0284;
>vz025m:=(wy)->f(wy,2.0412,.0284,1)-eval(subs(w0=16.537,f(w0,2.0412,.0284,1)));

vz025m :=

wy -> f(wy, 2.0412, .0284, 1)-eval(subs(w0 = 16.537, f(w0, 2.0412, .0284, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.05 --> zeta = .0246;
>vz05m:=(wy)->f(wy,2.0412,.0246,1)-eval(subs(w0=19.2673,f(w0,2.0412,.0246,1)));
```

```
 vz05m := wy ->

    f(wy, 2.0412, .0246, 1)-eval(subs(w0 = 19.2673, f(w0, 2.0412, .0246, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.1 --> zeta = .0217;
>vz1m:=(wy)->f(wy,2.0412,.0217,1)-eval(subs(w0=21.9914,f(w0,2.0412,.0217,1)));

 vz1m := wy ->

    f(wy, 2.0412, .0217, 1)-eval(subs(w0 = 21.9914, f(w0, 2.0412, .0217, 1)))
--------------------------------------------------------------------------------
> plot({[x(wy,2.0412,1,.1249,1),wy,wy=-3.13429..3.13429],[x(wy,2.0412,1,.0764,
1),wy,wy=-5.33242..5.33242],[x(wy,2.0412,1,.0545,1),wy,wy=-8.02088..8.02088],
[x(wy,2.0412,1,.0434,1),wy,wy=-10.4053..10.4053],[x(wy,2.0412,1,.0375,1),wy,
wy=-12.2371..12.2371]},0..1,-13..13,numpoints=600, resolution=600, color=black,
 thickness=1);
> plot({[x(wy,2.0412,1,.0043,1),wy,wy=-115.27..115.27],[x(wy,2.0412,1,.0086,1),
wy,wy=-57.1213..57.1213],[x(wy,2.0412,1,.0123,1),wy,wy=-39.6237..39.6237],
[x(wy,2.0412,1,.0150,1),wy,wy=-32.3002..32.3002],[x(wy,2.0412,1,.0330,1),wy,
wy=-14.0693..14.0693],[x(wy,2.0412,1,.0284,1),wy,wy=-16.537..16.537],[x(wy,
2.0412,1,.0246,1),wy,wy=-19.2673..19.2673],[x(wy,2.0412,1,.0217,1),wy,
wy=-21.9914..21.9914],[x(wy,2.0412,1,.0375,1),wy,wy=-12.2371..12.2371]},0..1,
-30..30,numpoints=600,resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,2.0412,1,.1249,1),vz1a(wy),wy=-3.13429..3.13429],[x(wy,2.0412,1,
.0764,1),vz05a(wy),wy=-5.33242..5.33242],[x(wy,2.0412,1,.0545,1),vz025a(wy),
wy=-8.02088..8.02088],[x(wy,2.0412,1,.0434,1),vz01b(wy),wy=-10.4053..10.4053],
[x(wy,2.0412,1,.0375,1),vz0(wy),wy=-12.2371..12.2371]},0..1,numpoints=600,
resolution=600, color=black, thickness=1);
> plot({[x(wy,2.0412,1,.0043,1),vz01a(wy),wy=-115.27..115.27],[x(wy,2.0412,1,
.0086,1),vz025b(wy),wy=-57.1213..57.1213],[x(wy,2.0412,1,.0123,1),vz05b(wy),
wy=-39.6237..39.6237],[x(wy,2.0412,1,.0150,1),vz1b(wy),wy=-32.3002..32.3002],
[x(wy,2.0412,1,.0330,1),vz01m(wy),wy=-14.0693..14.0693],[x(wy,2.0412,1,.0284,
1),vz025m(wy),wy=-16.537..16.537],[x(wy,2.0412,1,.0246,1),vz05m(wy),
wy=-19.2673..19.2673],[x(wy,2.0412,1,.0217,1),vz1m(wy),wy=-21.9914..21.9914],
[x(wy,2.0412,1,.0375,1),vz0(wy),wy=-12.2371..12.2371]},0..1,-60..-12.5,
numpoints=600,resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
>
> #OMEGA == 1.05
>
--------------------------------------------------------------------------------
> #Eta_eff = 0 --> zeta = .0048;
>vz0:=(wy)->f(wy,1.05,.0048,1)-eval(subs(w0=103.157,f(w0,1.05,.0048,1)));

vz0 :=

   wy -> f(wy, 1.05, .0048, 1)-eval(subs(w0 = 103.157, f(w0, 1.05, .0048, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .01 --> zeta = .0140; .0017;
>vz01a:=(wy) -> f(wy,1.05,.0140,1)-eval(subs(w0=34.6863,f(w0,1.05,.0140,1)));

vz01a :=
```

```
        wy -> f(wy, 1.05, .0140, 1)-eval(subs(w0 = 34.6863, f(w0, 1.05, .0140, 1)))
>vz01b:=(wy) -> f(wy,1.05,.0017,1)-eval(subs(w0=293.114,f(w0,1.05,.0017,1)));

vz01b :=

        wy -> f(wy, 1.05, .0017, 1)-eval(subs(w0 = 293.114, f(w0, 1.05, .0017, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .025 --> zeta = .0301; .0020;
>vz025a:=(wy) -> f(wy,1.05,.0301,1)-eval(subs(w0=15.5517,f(w0,1.05,.0301,1)));

vz025a :=

        wy -> f(wy, 1.05, .0301, 1)-eval(subs(w0 = 15.5517, f(w0, 1.05, .0301, 1)))
>vz025b:=(wy) -> f(wy,1.05,.0020,1)-eval(subs(w0=248.996,f(w0,1.05,.0020,1)));

vz025b :=

        wy -> f(wy, 1.05, .0020, 1)-eval(subs(w0 = 248.996, f(w0, 1.05, .0020, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .05 --> zeta = .0574, .0021;
>vz05a:=(wy) -> f(wy,1.05,.0574,1)-eval(subs(w0=7.60533,f(w0,1.05,.0574,1)));

vz05a :=

        wy -> f(wy, 1.05, .0574, 1)-eval(subs(w0 = 7.60533, f(w0, 1.05, .0574, 1)))
>vz05b:=(wy) -> f(wy,1.05,.0021,1)-eval(subs(w0=237.091,f(w0,1.05,.0021,1)));

vz05b :=

        wy -> f(wy, 1.05, .0021, 1)-eval(subs(w0 = 237.091, f(w0, 1.05, .0021, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .1 --> zeta = .1120; .0021;
>vz1a:=(wy) -> f(wy,1.05,.1120,1)-eval(subs(w0=3.41289,f(w0,1.05,.1120,1)));

vz1a :=

        wy -> f(wy, 1.05, .1120, 1)-eval(subs(w0 = 3.41289, f(w0, 1.05, .1120, 1)))
>vz1b:=(wy) -> f(wy,1.05,.0021,1)-eval(subs(w0=237.091,f(w0,1.05,.0021,1)));

vz1b :=

        wy -> f(wy, 1.05, .0021, 1)-eval(subs(w0 = 237.091, f(w0, 1.05, .0021, 1)))
--------------------------------------------------------------------------------
>
>
> #Eta_eff = -.01 --> zeta = .0027;
>vzm01:=(wy) -> f(wy,1.05,.0027,1)-eval(subs(w0=184.18,f(w0,1.05,.0027,1)));

vzm01 :=

        wy -> f(wy, 1.05, .0027, 1)-eval(subs(w0 = 184.18, f(w0, 1.05, .0027, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.025 --> zeta = .0024;
```

```
>vzm025:=(wy) -> f(wy,1.05,.0024,1)-eval(subs(w0=207.329,f(w0,1.05,.0024,1)));

vzm025 :=

   wy -> f(wy, 1.05, .0024, 1)-eval(subs(w0 = 207.329, f(w0, 1.05, .0024, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.05 --> zeta = .0023;
>vzm05:=(wy) -> f(wy,1.05,.0023,1)-eval(subs(w0=216.387,f(w0,1.05,.0023,1)));

vzm05 :=

   wy -> f(wy, 1.05, .0023, 1)-eval(subs(w0 = 216.387, f(w0, 1.05, .0023, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.1 --> zeta = .0022;
>vzm1:=(wy) -> f(wy,1.05,.0022,1)-eval(subs(w0=226.268,f(w0,1.05,.0022,1)));

vzm1 :=

   wy -> f(wy, 1.05, .0022, 1)-eval(subs(w0 = 226.268, f(w0, 1.05, .0022, 1)))
--------------------------------------------------------------------------------
> plot({[x(wy,1.05,1,.0048,1),wy,wy=-103.157..103.157],[x(wy,1.05,1,.0140,1),
wy,wy=-34.6863..34.6863],[x(wy,1.05,1,.0301,1),wy,wy=-15.5517..15.5517],[x(wy,
1.05,1,.0574,1),wy,wy=-7.60533..7.60533],[x(wy,1.05,1,.1120,1),wy,
wy=-3.41289..3.41289]},0..1,-13..13,numpoints=600, resolution=600, color=
black, thickness=1);
> plot({[x(wy,1.05,1,.0048,1),wy,wy=-103.157..103.157],[x(wy,1.05,1,.0017,1),
wy,wy=-293.114..293.114],[x(wy,1.05,1,.0020,1),wy,wy=-248.996..248.996],[x(wy,
1.05,1,.0021,1),wy,wy=-237.091..237.091],[x(wy,1.05,1,.0027,1),wy,wy=-184.18..
184.18],[x(wy,1.05,1,.0024,1),wy,wy=-207.329..207.329],[x(wy,1.05,1,.0023,1),
wy,wy=-216.387..216.387],[x(wy,1.05,1,.0022,1),wy,wy=-226.268..226.268]},.45..
0.55,-6..6,numpoints=600,resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,1.05,1,.1120,1),vz1a(wy),wy=-3.41289..3.41289],[x(wy,1.05,1,
.0574,1),vz05a(wy),wy=-7.60533..7.60533],[x(wy,1.05,1,.0301,1),vz025a(wy),
wy=-15.5517..15.5517],[x(wy,1.05,1,.0140,1),vz01a(wy),wy=-34.6863..34.6863],
[x(wy,1.05,1,.0048,1),vz0(wy),wy=-103.157..103.157]},0..1,numpoints=600,
resolution=600, color=black, thickness=1);
> plot([x(wy,1.05,1,.0140,1),vz01a(wy),wy=-34.6863..34.6863],.66..0.67,
-15.5..-15.1,numpoints=1000,resolution=1000,color=black,thickness=1);
> plot({[x(wy,1.05,1,.0017,1),vz01b(wy),wy=-293.114..293.114],[x(wy,1.05,1,.
0020,1),vz025b(wy),wy=-248.996..248.996],[x(wy,1.05,1,.0021,1),vz05b(wy),
wy=-237.091..237.091],[x(wy,1.05,1,.0027,1),vzm01(wy),wy=-184.18..184.18],
[x(wy,1.05,1,.0024,1),vzm025(wy),wy=-207.329..207.329],[x(wy,1.05,1,.0023,1),
vzm05(wy),wy=-216.387..216.387],[x(wy,1.05,1,.0022,1),vzm1(wy),wy=-226.268..
226.268],[x(wy,1.05,1,.0048,1),vz0(wy),wy=-103.157..103.157]},0..1,numpoints=
600,resolution=600,color=black,thickness=1);
> plot([x(wy,1.05,1,.0022,1),vzm1(wy),wy=-226.268..226.268],.3..0.7, -113..
-105,numpoints=1000,resolution=1000,color=black,thickness=1);
--------------------------------------------------------------------------------
>
> #OMEGA == 1.3;
>
--------------------------------------------------------------------------------
> #Eta_eff = 0 --> zeta = .0225;
```

```
>vz0:=(wy)->f(wy,1.3,.0225,1)-eval(subs(w0=21.176,f(w0,1.3,.0225,1)));

 vz0 :=

    wy -> f(wy, 1.3, .0225, 1)-eval(subs(w0 = 21.176, f(w0, 1.3, .0225, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .01 --> zeta = .0294, .0038;
>vz01a:=(wy) -> f(wy,1.3,.0294,1)-eval(subs(w0=15.946,f(w0,1.3,.0294,1)));

 vz01a :=

    wy -> f(wy, 1.3, .0294, 1)-eval(subs(w0 = 15.946, f(w0, 1.3, .0294, 1)))
>vz01b:=(wy) -> f(wy,1.3,.0038,1)-eval(subs(w0=130.571,f(w0,1.3,.0038,1)));

 vz01b :=

    wy -> f(wy, 1.3, .0038, 1)-eval(subs(w0 = 130.571, f(w0, 1.3, .0038, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .025 --> zeta = .0426, .0066;
>vz025a:=(wy) -> f(wy,1.3,.0426,1)-eval(subs(w0=10.6488,f(w0,1.3,.0426,1)));

 vz025a :=

    wy -> f(wy, 1.3, .0426, 1)-eval(subs(w0 = 10.6488, f(w0, 1.3, .0426, 1)))
>vz025b:=(wy) -> f(wy,1.3,.0066,1)-eval(subs(w0=74.7443,f(w0,1.3,.0066,1)));

 vz025b :=

    wy -> f(wy, 1.3, .0066, 1)-eval(subs(w0 = 74.7443, f(w0, 1.3, .0066, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .05 --> zeta = .0675, .0084;
>vz05a:=(wy) -> f(wy,1.3,.0675,1)-eval(subs(w0=6.27691,f(w0,1.3,.0675,1)));

 vz05a :=

    wy -> f(wy, 1.3, .0675, 1)-eval(subs(w0 = 6.27691, f(w0, 1.3, .0675, 1)))
>vz05b:=(wy) -> f(wy,1.3,.0084,1)-eval(subs(w0=58.5068,f(w0,1.3,.0084,1)));

 vz05b :=

    wy -> f(wy, 1.3, .0084, 1)-eval(subs(w0 = 58.5068, f(w0, 1.3, .0084, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .1 --> zeta = .1197, .0094;
>vz1a:=(wy) -> f(wy,1.3,.1197,1)-eval(subs(w0=3.16523,f(w0,1.3,.1197,1)));

 vz1a :=

    wy -> f(wy, 1.3, .1197, 1)-eval(subs(w0 = 3.16523, f(w0, 1.3, .1197, 1)))
>vz1b:=(wy) -> f(wy,1.3,.0094,1)-eval(subs(w0=52.1724,f(w0,1.3,.0094,1)));

 vz1b :=

    wy -> f(wy, 1.3, .0094, 1)-eval(subs(w0 = 52.1724, f(w0, 1.3, .0094, 1)))
--------------------------------------------------------------------------------
```

```
>
>
> #Eta_eff = -.01 --> zeta = .0181;
>vzm01:=(wy) -> f(wy,1.3,.0181,1)-eval(subs(w0=26.5873,f(w0,1.3,.0181,1)));

 vzm01 :=

    wy -> f(wy, 1.3, .0181, 1)-eval(subs(w0 = 26.5873, f(w0, 1.3, .0181, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.025 --> zeta = .0149;
>vzm025:=(wy) -> f(wy,1.3,.0149,1)-eval(subs(w0=32.5267,f(w0,1.3,.0149,1)));

 vzm025 :=

    wy -> f(wy, 1.3, .0149, 1)-eval(subs(w0 = 32.5267, f(w0, 1.3, .0149, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.05 --> zeta = .0129;
>vzm05:=(wy) -> f(wy,1.3,.0129,1)-eval(subs(w0=37.7334,f(w0,1.3,.0129,1)));

 vzm05 :=

    wy -> f(wy, 1.3, .0129, 1)-eval(subs(w0 = 37.7334, f(w0, 1.3, .0129, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.1 --> zeta = .0118;
>vzm1:=(wy) -> f(wy,1.3,.0118,1)-eval(subs(w0=41.3489,f(w0,1.3,.0118,1)));

 vzm1 :=

    wy -> f(wy, 1.3, .0118, 1)-eval(subs(w0 = 41.3489, f(w0, 1.3, .0118, 1)))
--------------------------------------------------------------------------------
> plot({[x(wy,1.3,1,.0225,1),wy,wy=-21.176..21.176],[x(wy,1.3,1,.0294,1),wy,
wy=-15.946..15.946],[x(wy,1.3,1,.0426,1),wy,wy=-10.6488..10.6488],[x(wy,1.3,1,
.0675,1),wy,wy=-6.27691..6.27691],[x(wy,1.3,1,.1197,1),wy,wy=-3.16523..
3.16523]},0..1,-13..13,numpoints=600, resolution=600, color=black, thickness=
1);
> plot({[x(wy,1.3,1,.0225,1),wy,wy=-21.176..21.176],[x(wy,1.3,1,.0038,1),wy,
wy=-130.571..130.571],[x(wy,1.3,1,.0066,1),wy,wy=-74.7443..74.7443],[x(wy,1.3,
1,.0084,1),wy,wy=-58.5068..58.5068],[x(wy,1.3,1,.0094,1),wy,wy=-52.1724..
52.1724],[x(wy,1.3,1,.0181,1),wy,wy=-26.5873..26.5873],[x(wy,1.3,1,.0149,1),
wy,wy=-32.5267..32.5267],[x(wy,1.3,1,.0129,1),wy,wy=-37.7334..37.7334],[x(wy,
1.3,1,.0118,1),wy,wy=-41.3489..41.3489]},.3..0.7,-5..5,numpoints=600,
resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,1.3,1,.1197,1),vz1a(wy),wy=-3.16523..3.16523],[x(wy,1.3,1,.0675,
1),vz05a(wy),wy=-6.27691..6.27691],[x(wy,1.3,1,.0426,1),vz025a(wy),
wy=-10.6488..10.6488],[x(wy,1.3,1,.0294,1),vz01a(wy),wy=-15.946..15.946],
[x(wy,1.3,1,.0225,1),vz0(wy),wy=-21.176..21.176]},0..1,numpoints=600,
resolution=600, color=black, thickness=1);
> plot({[x(wy,1.3,1,.0038,1),vz01b(wy),wy=-130.571..130.571],[x(wy,1.3,1,
.0066,1),vz025b(wy),wy=-74.7443..74.7443],[x(wy,1.3,1,.0084,1),vz05b(wy),
wy=-58.5068..58.5068],[x(wy,1.3,1,.0094,1),vz1b(wy),wy=-52.1724..52.1724],
[x(wy,1.3,1,.0181,1),vzm01(wy),wy=-26.5873..26.5873],[x(wy,1.3,1,.0149,1),
vzm025(wy),wy=-32.5267..32.5267],[x(wy,1.3,1,.0129,1),vzm05(wy),wy=-37.7334..
37.7334],[x(wy,1.3,1,.0118,1),vzm1(wy),wy=-41.3489..41.3489],[x(wy,1.3,1,
```

```
.0225,1),vz0(wy),wy=-21.176..21.176]},0..1,numpoints=600,resolution=600,color=
black,thickness=1);
> plot([x(wy,1.3,1,.0181,1),vzm01(wy),wy=-26.5873..26.5873],.31..0.315,-11.4..
-11.3,numpoints=1000,resolution=1000,color=black,thickness=1);
--------------------------------------------------------------------------------
>
> #OMEGA == 5;
>
--------------------------------------------------------------------------------
> #Eta_eff = 0 --> zeta = .0159;
>vz0:=(wy)->f(wy,5,.0159,1)-eval(subs(w0=30.3867,f(w0,5,.0159,1)));

 vz0 := wy -> f(wy, 5, .0159, 1)-eval(subs(w0 = 30.3867, f(w0, 5, .0159, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .01 --> zeta = .0224, .0036;
>vz01a:=(wy) -> f(wy,5,.0224,1)-eval(subs(w0=21.2167,f(w0,5,.0224,1)));

vz01a := wy -> f(wy, 5, .0224, 1)-eval(subs(w0 = 21.2167, f(w0, 5, .0224, 1)))
>vz01b:=(wy) -> f(wy,5,.0036,1)-eval(subs(w0=137.88,f(w0,5,.0036,1)));

 vz01b := wy -> f(wy, 5, .0036, 1)-eval(subs(w0 = 137.88, f(w0, 5, .0036, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .025 --> zeta = .0353, .0056;
>vz025a:=(wy) -> f(wy,5,.0353,1)-eval(subs(w0=12.9145,f(w0,5,.0353,1)));

   vz025a :=

      wy -> f(wy, 5, .0353, 1)-eval(subs(w0 = 12.9145, f(w0, 5, .0353, 1)))
>vz025b:=(wy) -> f(wy,5,.0056,1)-eval(subs(w0=88.2714,f(w0,5,.0056,1)));

   vz025b :=

      wy -> f(wy, 5, .0056, 1)-eval(subs(w0 = 88.2714, f(w0, 5, .0056, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .05 --> zeta = .0593, .0067;
>vz05a:=(wy) -> f(wy,5,.0593,1)-eval(subs(w0=6.76095,f(w0,5,.0593,1)));

vz05a := wy -> f(wy, 5, .0593, 1)-eval(subs(w0 = 6.76095, f(w0, 5, .0593, 1)))
>vz05b:=(wy) -> f(wy,5,.0067,1)-eval(subs(w0=73.6089,f(w0,5,.0067,1)));

vz05b := wy -> f(wy, 5, .0067, 1)-eval(subs(w0 = 73.6089, f(w0, 5, .0067, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .1 --> zeta = .1087, .0073;
>vz1a:=(wy) -> f(wy,5,.1087,1)-eval(subs(w0=4.74432,f(w0,5,.1087,1)));

 vz1a := wy -> f(wy, 5, .1087, 1)-eval(subs(w0 = 4.74432, f(w0, 5, .1087, 1)))
>vz1b:=(wy) -> f(wy,5,.0073,1)-eval(subs(w0=67.4731,f(w0,5,.0073,1)));

 vz1b := wy -> f(wy, 5, .0073, 1)-eval(subs(w0 = 67.4731, f(w0, 5, .0073, 1)))
--------------------------------------------------------------------------------
>
>
> #Eta_eff = -.01 --> zeta = .0123;
>vzm01:=(wy) -> f(wy,5,.0123,1)-eval(subs(w0=39.6096,f(w0,5,.0123,1)));
```

```
vzm01 := wy -> f(wy, 5, .0123, 1)-eval(subs(w0 = 39.6096, f(w0, 5, .0123, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.025 --> zeta = .0103;
>vzm025:=(wy) -> f(wy,5,.0103,1)-eval(subs(w0=47.512,f(w0,5,.0103,1)));

vzm025 := wy -> f(wy, 5, .0103, 1)-eval(subs(w0 = 47.512, f(w0, 5, .0103, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.05 --> zeta = .0092;
>vzm05:=(wy) -> f(wy,5,.0092,1)-eval(subs(w0=53.3206,f(w0,5,.0092,1)));

vzm05 := wy -> f(wy, 5, .0092, 1)-eval(subs(w0 = 53.3206, f(w0, 5, .0092, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.1 --> zeta = .0086;
>vzm1:=(wy) -> f(wy,5,.0086,1)-eval(subs(w0=57.1147,f(w0,5,.0086,1)));

 vzm1 := wy -> f(wy, 5, .0086, 1)-eval(subs(w0 = 57.1147, f(w0, 5, .0086, 1)))
--------------------------------------------------------------------------------
> plot({[x(wy,5,1,.0159,1),wy,wy=-30.3867..30.3867],[x(wy,5,1,.0224,1),wy,
wy=-21.2167..21.2167],[x(wy,5,1,.0353,1),wy,wy=-12.9145..12.9145],[x(wy,5,1,
.0593,1),wy,wy=-6.76095..6.76095],[x(wy,5,1,.1087,1),wy,wy=-4.74432..4.74432]},
0..1,-13..13,numpoints=600, resolution=600, color=black, thickness=1);
> plot({[x(wy,5,1,.0159,1),wy,wy=-30.3867..30.3867],[x(wy,5,1,.0036,1),wy,
wy=-137.88..137.88],[x(wy,5,1,.0056,1),wy,wy=-88.2714..88.2714],[x(wy,5,1,
.0067,1),wy,wy=-73.6089..73.6089],[x(wy,5,1,.0073,1),wy,wy=-67.4731..67.4731],
[x(wy,5,1,.0123,1),wy,wy=-39.6096..39.6096],[x(wy,5,1,.0103,1),wy,wy=-47.512..
47.512],[x(wy,5,1,.0092,1),wy,wy=-53.3206..53.3206],[x(wy,5,1,.0086,1),wy,
wy=-57.1147..57.1147]},.3..0.7,-8..8,numpoints=600,resolution=600,color=black,
thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,5,1,.1087,1),vz1a(wy),wy=-4.74432..4.74432],[x(wy,5,1,.0593,1),
vz05a(wy),wy=-6.76095..6.76095],[x(wy,5,1,.0353,1),vz025a(wy),wy=-12.9145..
12.9145],[x(wy,5,1,.0224,1),vz01a(wy),wy=-21.2167..21.2167],[x(wy,5,1,.0159,1),
vz0(wy),wy=-30.3867..30.3867]},0..1,numpoints=600, resolution=600, color=black,
 thickness=1);
> plot([x(wy,5,1,.0224,1),vz01a(wy),wy=-21.2167..21.2167],.3..0.7,-11..-10,
numpoints=1000,resolution=1000,color=black,thickness=1);
> plot({[x(wy,5,1,.0036,1),vz01b(wy),wy=-137.88..137.88],[x(wy,5,1,.0056,1),
vz025b(wy),wy=-88.2714..88.2714],[x(wy,5,1,.0067,1),vz05b(wy),wy=-73.6089..
73.6089],[x(wy,5,1,.0073,1),vz1b(wy),wy=-67.4731..67.4731],[x(wy,5,1,.0123,1),
vzm01(wy),wy=-39.6096..39.6096],[x(wy,5,1,.0103,1),vzm025(wy),wy=-47.512..
47.512],[x(wy,5,1,.0092,1),vzm05(wy),wy=-53.3206..53.3206],[x(wy,5,1,.0086,1),
vzm1(wy),wy=-57.1147..57.1147],[x(wy,5,1,.0159,1),vz0(wy),wy=-30.3867..
30.3867]},.25..0.75,-70..-62,numpoints=600,resolution=600,color=black,
thickness=1);
> plot([x(wy,5,1,.0086,1),vzm1(wy),wy=-57.1147..57.1147],.3..0.7,-28.5..-25,
numpoints=1000,resolution=1000,color=black,thickness=1);
--------------------------------------------------------------------------------
>
> #OMEGA == 3.25;
>
--------------------------------------------------------------------------------
> #Eta_eff = 0 --> zeta = .0284;
>vz0:=(wy)->f(wy,3.25,.0284,1)-eval(subs(w0=16.5104,f(w0,3.25,.0284,1)));
```

```
vz0 :=

   wy -> f(wy, 3.25, .0284, 1)-eval(subs(w0 = 16.5104, f(w0, 3.25, .0284, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .01 --> zeta = .0343, .0041;
>vz01a:=(wy) -> f(wy,3.25,.0343,1)-eval(subs(w0=13.4498,f(w0,3.25,.0343,1)));

vz01a :=

   wy -> f(wy, 3.25, .0343, 1)-eval(subs(w0 = 13.4498, f(w0, 3.25, .0343, 1)))
>vz01b:=(wy) -> f(wy,3.25,.0041,1)-eval(subs(w0=120.942,f(w0,3.25,.0041,1)));

vz01b :=

   wy -> f(wy, 3.25, .0041, 1)-eval(subs(w0 = 120.942, f(w0, 3.25, .0041, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .025 --> zeta = .0458, .0078;
>vz025a:=(wy) -> f(wy,3.25,.0458,1)-eval(subs(w0=9.71053,f(w0,3.25,.0458,1)));

vz025a :=

   wy -> f(wy, 3.25, .0458, 1)-eval(subs(w0 = 9.71053, f(w0, 3.25, .0458, 1)))
>vz025b:=(wy) -> f(wy,3.25,.0078,1)-eval(subs(w0=63.0845,f(w0,3.25,.0078,1)));

vz025b :=

   wy -> f(wy, 3.25, .0078, 1)-eval(subs(w0 = 63.0845, f(w0, 3.25, .0078, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .05 --> zeta = .0683, .0104;
>vz05a:=(wy) -> f(wy,3.25,.0683,1)-eval(subs(w0=5.93777,f(w0,3.25,.0683,1)));

vz05a :=

   wy -> f(wy, 3.25, .0683, 1)-eval(subs(w0 = 5.93777, f(w0, 3.25, .0683, 1)))
>vz05b:=(wy) -> f(wy,3.25,.0104,1)-eval(subs(w0=47.0517,f(w0,3.25,.0104,1)));

vz05b :=

   wy -> f(wy, 3.25, .0104, 1)-eval(subs(w0 = 47.0517, f(w0, 3.25, .0104, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .1 --> zeta = .1168, .0122;
>vz1a:=(wy) -> f(wy,3.25,.1168,1)-eval(subs(w0=3.72385,f(w0,3.25,.1168,1)));

vz1a :=

   wy -> f(wy, 3.25, .1168, 1)-eval(subs(w0 = 3.72385, f(w0, 3.25, .1168, 1)))
>vz1b:=(wy) -> f(wy,3.25,.0122,1)-eval(subs(w0=39.953,f(w0,3.25,.0122,1)));

vz1b :=

    wy -> f(wy, 3.25, .0122, 1)-eval(subs(w0 = 39.953, f(w0, 3.25, .0122, 1)))
--------------------------------------------------------------------------------
>
```

```
>
> #Eta_eff = -.01 --> zeta = .0242;
>vzm01:=(wy) -> f(wy,3.25,.0242,1)-eval(subs(w0=19.5858,f(w0,3.25,.0242,1)));

vzm01 :=

   wy -> f(wy, 3.25, .0242, 1)-eval(subs(w0 = 19.5858, f(w0, 3.25, .0242, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.025 --> zeta = .0205;
>vzm025:=(wy) -> f(wy,3.25,.0205,1)-eval(subs(w0=23.3304,f(w0,3.25,.0205,1)));

vzm025 :=

   wy -> f(wy, 3.25, .0205, 1)-eval(subs(w0 = 23.3304, f(w0, 3.25, .0205, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.05 --> zeta = .0179;
>vzm05:=(wy) -> f(wy,3.25,.0179,1)-eval(subs(w0=26.8831,f(w0,3.25,.0179,1)));

vzm05 :=

   wy -> f(wy, 3.25, .0179, 1)-eval(subs(w0 = 26.8831, f(w0, 3.25, .0179, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.1 --> zeta = .0161;
>vzm1:=(wy) -> f(wy,3.25,.0161,1)-eval(subs(w0=30.0125,f(w0,3.25,.0161,1)));

vzm1 :=

   wy -> f(wy, 3.25, .0161, 1)-eval(subs(w0 = 30.0125, f(w0, 3.25, .0161, 1)))
--------------------------------------------------------------------------------
> plot({[x(wy,3.25,1,.0284,1),wy,wy=-16.5104..16.5104],[x(wy,3.25,1,.0343,1),
wy,wy=-13.4498..13.4498],[x(wy,3.25,1,.0458,1),wy,wy=-9.71053..9.71053],[x(wy,
3.25,1,.0683,1),wy,wy=-5.93777..5.93777],[x(wy,3.25,1,.1168,1),wy,wy=-3.72385..
3.72385]},0..1,-13..13,numpoints=600, resolution=600, color=black, thickness=
1);
> plot({[x(wy,3.25,1,.0284,1),wy,wy=-16.5104..16.5104],[x(wy,3.25,1,.0041,1),
wy,wy=-120.942..120.942],[x(wy,3.25,1,.0078,1),wy,wy=-63.0845..63.0845],
[x(wy,3.25,1,.0104,1),wy,wy=-47.0517..47.0517],[x(wy,3.25,1,.0122,1),wy,
wy=-39.953..39.953],[x(wy,3.25,1,.0242,1),wy,wy=-19.5858..19.5858],
[x(wy,3.25,1,.0205,1),wy,wy=-23.3304..23.3304],[x(wy,3.25,1,.0179,1),wy,
wy=-26.8831..26.8831],[x(wy,3.25,1,.0161,1),wy,wy=-30.0125..30.0125]},.3..0.7,
-5..5,numpoints=600,resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,3.25,1,.1168,1),vz1a(wy),wy=-3.72385..3.72385],[x(wy,3.25,1,
.0683,1),vz05a(wy),wy=-5.93777..5.93777],[x(wy,3.25,1,.0458,1),vz025a(wy),
wy=-9.71053..9.71053],[x(wy,3.25,1,.0343,1),vz01a(wy),wy=-13.4498..13.4498],
[x(wy,3.25,1,.0284,1),vz0(wy),wy=-16.5104..16.5104]},0..1,numpoints=600,
resolution=600, color=black, thickness=1);
> plot([x(wy,3.25,1,.0343,1),vz01a(wy),wy=-13.4498..13.4498],.3..0.7,-7..-6,
numpoints=1000,resolution=1000,color=black,thickness=1);
> plot({[x(wy,3.25,1,.0041,1),vz01b(wy),wy=-120.942..120.942],[x(wy,3.25,1,
.0078,1),vz025b(wy),wy=-63.0845..63.0845],[x(wy,3.25,1,.0104,1),vz05b(wy),
wy=-47.0517..47.0517],[x(wy,3.25,1,.0122,1),vz1b(wy),wy=-39.953..39.953],
[x(wy,3.25,1,.0242,1),vzm01(wy),wy=-19.5858..19.5858],[x(wy,3.25,1,.0205,1),
vzm025(wy),wy=-23.3304..23.3304],[x(wy,3.25,1,.0179,1),vzm05(wy),wy=-26.8831..
```

```
26.8831],[x(wy,3.25,1,.0161,1),vzm1(wy),wy=-30.0125..30.0125],[x(wy,3.25,1,
.0284,1),vz0(wy),wy=-16.5104..16.5104]},0..1,numpoints=600,resolution=600,
color=black,thickness=1);
> plot([x(wy,3.25,1,.0161,1),vzm1(wy),wy=-30.0125..30.0125],.3..0.7,-14.8..
-13,numpoints=1000,resolution=1000,color=black,thickness=1);
--------------------------------------------------------------------------------
>
> #OMEGA == 10;
>
--------------------------------------------------------------------------------
> #Eta_eff = 0 --> zeta = .0047;
>vz0:=(wy)->f(wy,10,.0047,1)-eval(subs(w0=105.364,f(w0,10,.0047,1)));

vz0 := wy -> f(wy, 10, .0047, 1)-eval(subs(w0 = 105.364, f(w0, 10, .0047, 1)))
>vz0temp:=(wy)->f(wy,30,.00055187,1)-eval(subs(w0=905.008,f(w0,30,.00055187,1)));

 vz0temp := wy ->

    f(wy, 30, .00055187, 1)-eval(subs(w0 = 905.008, f(w0, 30, .00055187, 1)))
> plot([x(wy,30,1,.00055187,1),wy,wy=-905.008..905.008],0..1,-100..100);
> plot([x(wy,30,1,.00055187,1),vz0temp(wy),wy=-905.008..905.008],.25..0.75,
-470..-420);
--------------------------------------------------------------------------------
> #Eta_eff = .01 --> zeta = .0129, .0018;
>vz01a:=(wy) -> f(wy,10,.0129,1)-eval(subs(w0=37.6553,f(w0,10,.0129,1)));

  vz01a :=

     wy -> f(wy, 10, .0129, 1)-eval(subs(w0 = 37.6553, f(w0, 10, .0129, 1)))
>vz01b:=(wy) -> f(wy,10,.0018,1)-eval(subs(w0=276.773,f(w0,10,.0018,1)));

  vz01b :=

     wy -> f(wy, 10, .0018, 1)-eval(subs(w0 = 276.773, f(w0, 10, .0018, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .025 --> zeta = .0276, .0021;
>vz025a:=(wy) -> f(wy,10,.0276,1)-eval(subs(w0=16.4497,f(w0,10,.0276,1)));

  vz025a :=

     wy -> f(wy, 10, .0276, 1)-eval(subs(w0 = 16.4497, f(w0, 10, .0276, 1)))
>vz025b:=(wy) -> f(wy,10,.0021,1)-eval(subs(w0=237.089,f(w0,10,.0021,1)));

  vz025b :=

     wy -> f(wy, 10, .0021, 1)-eval(subs(w0 = 237.089, f(w0, 10, .0021, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .05 --> zeta = .0525, .0022;
>vz05a:=(wy) -> f(wy,10,.0525,1)-eval(subs(w0=9.80991,f(w0,10,.0525,1)));

  vz05a :=

     wy -> f(wy, 10, .0525, 1)-eval(subs(w0 = 9.80991, f(w0, 10, .0525, 1)))
>vz05b:=(wy) -> f(wy,10,.0022,1)-eval(subs(w0=226.266,f(w0,10,.0022,1)));
```

```
  vz05b :=

     wy -> f(wy, 10, .0022, 1)-eval(subs(w0 = 226.266, f(w0, 10, .0022, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .1 --> zeta = .1024, .0023;
>vz1a:=(wy) -> f(wy,10,.1024,1)-eval(subs(w0=5.18971,f(w0,10,.1024,1)));

  vz1a :=

     wy -> f(wy, 10, .1024, 1)-eval(subs(w0 = 5.18971, f(w0, 10, .1024, 1)))
>vz1b:=(wy) -> f(wy,10,.0023,1)-eval(subs(w0=216.385,f(w0,10,.0023,1)));

  vz1b :=

     wy -> f(wy, 10, .0023, 1)-eval(subs(w0 = 216.385, f(w0, 10, .0023, 1)))
--------------------------------------------------------------------------------
>
>
> #Eta_eff  = -.01 --> zeta = .0029;
>vzm01:=(wy) -> f(wy,10,.0029,1)-eval(subs(w0=171.405,f(w0,10,.0029,1)));

  vzm01 :=

     wy -> f(wy, 10, .0029, 1)-eval(subs(w0 = 171.405, f(w0, 10, .0029, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.025 --> zeta = .0026;
>vzm025:=(wy) -> f(wy,10,.0026,1)-eval(subs(w0=191.3,f(w0,10,.0026,1)));

   vzm025 :=

      wy -> f(wy, 10, .0026, 1)-eval(subs(w0 = 191.3, f(w0, 10, .0026, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.05 --> zeta = .0025;
>vzm05:=(wy) -> f(wy,10,.0025,1)-eval(subs(w0=198.992,f(w0,10,.0025,1)));

  vzm05 :=

     wy -> f(wy, 10, .0025, 1)-eval(subs(w0 = 198.992, f(w0, 10, .0025, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.1 --> zeta = .0024;
>vzm1:=(wy) -> f(wy,10,.0024,1)-eval(subs(w0=207.326,f(w0,10,.0024,1)));

  vzm1 :=

     wy -> f(wy, 10, .0024, 1)-eval(subs(w0 = 207.326, f(w0, 10, .0024, 1)))
--------------------------------------------------------------------------------
> plot({[x(wy,10,1,.0047,1),wy,wy=-105.364..105.364],[x(wy,10,1,.0129,1),wy,
wy=-37.6553..37.6553],[x(wy,10,1,.0276,1),wy,wy=-16.4497..16.4497],[x(wy,10,1,
.0525,1),wy,wy=-9.80991..9.80991],[x(wy,10,1,.1024,1),wy,wy=-5.18971..
5.18971]},0..1,-30..30,numpoints=600, resolution=600, color=black, thickness=
1);
> plot({[x(wy,10,1,.0047,1),wy,wy=-105.364..105.364],[x(wy,10,1,.0018,1),wy,
wy=-276.773..276.773],[x(wy,10,1,.0021,1),wy,wy=-237.089..237.089],
```

```
[x(wy,10,1,.0022,1),wy,wy=-226.266..226.266],[x(wy,10,1,.0023,1),wy,
wy=-216.385..216.385],[x(wy,10,1,.0029,1),wy,wy=-171.405..171.405],
[x(wy,10,1,.0026,1),wy,wy=-191.3..191.3],[x(wy,10,1,.0025,1),wy,wy=-198.992..
198.992],[x(wy,10,1,.0024,1),wy,wy=-207.326..207.326]},.3..0.7,-12..12,
numpoints=600,resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,10,1,.1024,1),vz1a(wy),wy=-5.18971..5.18971],[x(wy,10,1,.0525,1),
vz05a(wy),wy=-9.80991..9.80991],[x(wy,10,1,.0276,1),vz025a(wy),wy=-16.4497..
16.4497],[x(wy,10,1,.0129,1),vz01a(wy),wy=-37.6553..37.6553],[x(wy,10,1,.0047,
1),vz0(wy),wy=-105.364..105.364]},0..1,numpoints=600, resolution=600, color=
black, thickness=1);
> plot([x(wy,10,1,.0129,1),vz01a(wy),wy=-37.6553..37.6553],0..1,numpoints=
1000,resolution=1000,color=black,thickness=1);
> plot({[x(wy,10,1,.0018,1),vz01b(wy),wy=-276.773..276.773],[x(wy,10,1,.0021,
1),vz025b(wy),wy=-237.089..237.089],[x(wy,10,1,.0022,1),vz05b(wy),
wy=-226.266..226.266],[x(wy,10,1,.0023,1),vz1b(wy),wy=-216.385..216.385],
[x(wy,10,1,.0029,1),vzm01(wy),wy=-171.405..171.405],[x(wy,10,1,.0026,1),
vzm025(wy),wy=-191.3..191.3],[x(wy,10,1,.0025,1),vzm05(wy),wy=-198.992..
198.992],[x(wy,10,1,.0024,1),vzm1(wy),wy=-207.326..207.326],[x(wy,10,1,.0047,
1),vz0(wy),wy=-105.364..105.364]},0..1,numpoints=600,resolution=600,color=
black,thickness=1);
> plot([x(wy,10,1,.0024,1),vzm1(wy),wy=-207.326..207.326],.3..0.7,-104..-96,
numpoints=1000,resolution=1000,color=black,thickness=1);
--------------------------------------------------------------------------------
>
```

A.2 Tangential Magnetic Field Only

```
> denomin:=(wy, Om, chi) -> ((wy^2 - Om^2 + 1 + chi)^2 + (2 + chi)^2*Om^2);

                                      2     2            2             2   2
       denomin := (wy,Om,chi) -> (wy   - Om   + 1 + chi)   + (2 + chi)   Om
--------------------------------------------------------------------------------
> t1:=(wy,Om,chi,Bx,Hz) -> -wy*(abs(Bx)^2*(wy^2 - Om^2 + 1) + abs(Hz)^2*(wy^2
- Om^2 + (1 + chi)^2));

     t1 := (wy,Om,chi,Bx,Hz) ->

                   2     2     2                2     2     2            2
        - wy (abs(Bx)   (wy   - Om   + 1) + abs(Hz)   (wy   - Om   + (1 + chi) ))
--------------------------------------------------------------------------------
> t2:=(wy,Om,chi,Bx,Hz) -> (chi*(wy^2 - Om^2) + I*Om*(wy^2 - Om^2 - 1 - chi))*
Hz*conjugate(Bx);

      t2 := (wy,Om,chi,Bx,Hz) ->

                 2     2              2     2
         (chi (wy   - Om ) + I Om (wy   - Om   - 1 - chi)) Hz conjugate(Bx)
--------------------------------------------------------------------------------
> t3:=(wy,Om,chi,Bx,Hz) -> (chi*(wy^2 - Om^2) - I*Om*(wy^2 - Om^2 - 1 - chi))*
conjugate(Hz)*Bx;

      t3 := (wy,Om,chi,Bx,Hz) ->

                 2     2              2     2
         (chi (wy   - Om ) - I Om (wy   - Om   - 1 - chi)) conjugate(Hz) Bx
--------------------------------------------------------------------------------
> t:=(wy,Om,chi,Bx,Hz)-> .5*chi*(t1(wy,Om,chi,Bx,Hz) + t2(wy,Om,chi,Bx,Hz) +
t3(wy,Om,chi,Bx,Hz))/denomin(wy,Om,chi);

t := (wy,Om,chi,Bx,Hz) -> .5 chi

  (t1(wy, Om, chi, Bx, Hz) + t2(wy, Om, chi, Bx, Hz) + t3(wy, Om, chi, Bx, Hz))

   /denomin(wy, Om, chi)
--------------------------------------------------------------------------------
> dt:=(wy,Om,chi,Bx,Hz)->diff(t(wy,Om,chi,Bx,Hz),wy);

          dt := (wy,Om,chi,Bx,Hz) -> diff(t(wy, Om, chi, Bx, Hz), wy)
--------------------------------------------------------------------------------
> alphacrit:=(zeta,eta)->2*zeta*eta/(zeta+eta);

                                                 zeta eta
                    alphacrit := (zeta,eta) -> 2 ----------
                                                 zeta + eta
--------------------------------------------------------------------------------
> x:=(wy,Om,chi,zeta,dpdz)->-(integrate(1-dt(wy,Om,chi,0,1)/alphacrit(zeta,
zeta),wy) - .5*dpdz/zeta)*(zeta/dpdz);

        x := (wy,Om,chi,zeta,dpdz) ->
```

```
            /                dt(wy, Om, chi, 0, 1)            dpdz\
            |integrate(1 - ---------------------, wy) - .5 ----| zeta
            \                alphacrit(zeta, zeta)            zeta/
          - ----------------------------------------------------------
                                        dpdz
------------------------------------------------------------------------------
> dwydx:=(wy,Om,chi,Bx,Hz,zeta,eta,dpdz)-> (-dpdz/eta)/(1 - dt(wy,Om,chi,Bx,Hz)
/alphacrit(zeta,eta));

                                                                  dpdz
dwydx := (wy,Om,chi,Bx,Hz,zeta,eta,dpdz) -> - ----------------------------------
                                                    /    dt(wy, Om, chi, Bx, Hz)\
                                                eta |1 - -----------------------|
                                                    \      alphacrit(zeta, eta)  /
------------------------------------------------------------------------------
> dwydx(wy,Om,chi,Bx,Hz,zeta,eta,dpdz);

                                                       2    2     2
  - dpdz/(eta (1 - 1/2 (.5 chi (- abs(Bx)  (wy  - Om  + 1)

                   2    2     2             2                 2              2
       - abs(Hz)  (wy  - Om  + (1 + chi) ) - wy (2 abs(Bx)  wy + 2 abs(Hz)  wy)

       + (2 chi wy + 2 I Om wy) Hz conjugate(Bx)

       + (2 chi wy - 2 I Om wy) conjugate(Hz) Bx)

         /     2     2           2             2   2
        /  ((wy  - Om  + 1 + chi)  + (2 + chi)  Om ) - 2.0 chi (
       /

                  2    2     2                  2    2     2              2
      - wy (abs(Bx)  (wy  - Om  + 1) + abs(Hz)  (wy  - Om  + (1 + chi) ))

                2     2             2     2
       + (chi (wy  - Om ) + I Om (wy  - Om  - 1 - chi)) Hz conjugate(Bx)

                2     2             2     2
       + (chi (wy  - Om ) - I Om (wy  - Om  - 1 - chi)) conjugate(Hz) Bx)

        2     2                  /     2     2           2             2   2 2
      (wy  - Om  + 1 + chi) wy  /  ((wy  - Om  + 1 + chi)  + (2 + chi)  Om ) )
                               /

      (zeta + eta)/(zeta eta)))
------------------------------------------------------------------------------
> dvzdwy:=(wy,Om,zeta)-> (-2*wy + t(wy,Om,1,0,1)/zeta)/dwydx(wy,Om,1,0,1,zeta,
zeta,1);

                                                     t(wy, Om, 1, 0, 1)
                                           - 2 wy + ------------------
                                                           zeta
        dvzdwy := (wy,Om,zeta) -> ------------------------------------
```

```
                              dwydx(wy, 0m, 1, 0, 1, zeta, zeta, 1)
-----------------------------------------------------------------------------
> dvzdwy(wy,0m,zeta);

  /                  2     2     \
  |            wy (wy  - 0m  + 4)|
- |- 2 wy - .5 ------------------| zeta
  \                 %1 zeta      /

  /            2     2              2          2     2     2          2     2    \
  |          wy  - 0m  + 4        wy         wy  (wy  - 0m  + 4) (wy  - 0m  + 2)|
  |     - .5 ------------ - 1.0 --- + 2.0 ----------------------------------|
  |               %1                %1                               2          |
  |                                                               %1           |
  |1 - --------------------------------------------------------------------|
  \                                  zeta                                       /

                                   2     2     2       2
%1 :=                         (wy  - 0m  + 2)  + 9 0m
-----------------------------------------------------------------------------
> f:=(wy,0m,zeta,dpdz) -> -((integrate((t(wy,0m,1,0,1)/zeta - 2*wy) * (1 -
dt(wy,0m,1,0,1)/alphacrit(zeta,zeta)),wy))/(dpdz/zeta));

f := (wy,0m,zeta,dpdz) ->

            /         t(wy, 0m, 1, 0, 1)\ /     dt(wy, 0m, 1, 0, 1) \
  integrate(|- 2 wy + ------------------| |1 - ---------------------|, wy) zeta
            \                zeta       / \    alphacrit(zeta, zeta)/
- -----------------------------------------------------------------------------
                                       dpdz
-----------------------------------------------------------------------------
> f(wy,0m,zeta,dpdz);

                                          2        2
                                   2. wy  - 2. 0m  + 4.
                  arctan(.1666666667 --------------------)
                                                2 1/2
                                            (0m )              2.
   - (.1666666667 ---------------------------------------- + ---------
                                        2 1/2                     2   2
                                 zeta (0m )                  zeta  %1

                                      2                       2
       .5000000000                  wy                      0m
     - ----------- + .5000000000 --------- + 2.500000000 ---------
              2                       2   2                   2   2
        zeta  %1                  zeta  %1               zeta  %1

                     4                     2          4             2
                   0m                    wy         0m            0m
     + .5000000000 --------- - .1250000000 -------- + ------- + 5. -------
                      2   2                    2      zeta %1      zeta %1
                  zeta  %1                 zeta  %1
```

```
                    2   2           2   2
                  Om  wy          Om  wy        4.           2
     + .6250000000 --------- - 1. ------- + ------- - 1. wy
                     2   2        zeta %1   zeta %1
                  zeta  %1

                  ln(%1)
     + .1250000000 ------) zeta/dpdz
                   zeta

              4        2   2         2     4        2
%1 :=       wy  - 2. wy  Om  + 4. wy  + Om  + 5. Om  + 4.
--------------------------------------------------------------------------
--------------------------------------------------------------------------
--------------------------------------------------------------------------
> #OMEGA == 3.2126
--------------------------------------------------------------------------
--------------------------------------------------------------------------
> #Eta_eff = 0 --> zeta = .0195;
> vz0:=(wy)->f(wy,3.2126, .0195,1)-eval(subs(w0=24.58,f(w0,3.2126,.0195,1)));

vz0 :=

wy -> f(wy, 3.2126, .0195, 1) - eval(subs(w0 = 24.58, f(w0, 3.2126, .0195, 1)))
> plot([x(wy,3.2126,1,.0195,1),wy,wy=-24.58..24.58]);
> plot([x(wy,3.2126,1,.0195,1),vz0(wy),wy=-24.58..24.58]);
--------------------------------------------------------------------------
> #Eta_eff = .01 --> zeta = .0257, .0038;
>vz01a:=(wy)->f(wy,3.2126,.0257,1)-eval(subs(w0=18.3633,f(w0,3.2126,.0257,1)));

 vz01a := wy ->

    f(wy, 3.2126, .0257, 1) - eval(subs(w0 = 18.3633, f(w0, 3.2126, .0257, 1)))
>vz01b:=(wy)->f(wy,3.2126,.0038,1)-eval(subs(w0=130.571,f(w0,3.2126,.0038,1)));

 vz01b := wy ->

    f(wy, 3.2126, .0038, 1) - eval(subs(w0 = 130.571, f(w0, 3.2126, .0038, 1)))
--------------------------------------------------------------------------
> #Eta_eff=.025 --> zeta = .0381, .0064;
>vz025a:=(wy)->f(wy,3.2126,.0381,1)-eval(subs(w0=11.9454,f(w0,3.2126,.0381,1)));

 vz025a := wy ->

    f(wy, 3.2126, .0381, 1) - eval(subs(w0 = 11.9454, f(w0, 3.2126, .0381, 1)))
> plot([x(wy,3.2126,1,.0381,1),vz025a(wy),wy=-11.9454..11.9454]);
>vz025b:=(wy)->f(wy,3.2126,.0064,1)-eval(subs(w0=77.1101,f(w0,3.2126,.0064,1)));

 vz025b := wy ->

    f(wy, 3.2126, .0064, 1) - eval(subs(w0 = 77.1101, f(w0, 3.2126, .0064, 1)))
--------------------------------------------------------------------------
> #Eta_eff=.05 --> zeta = .0616, .0079;
>vz05a:=(wy)->f(wy,3.2126,.0616,1)-eval(subs(w0=6.63099,f(w0,3.2126,.0616,1)));
```

```
 vz05a := wy ->

    f(wy, 3.2126, .0616, 1) - eval(subs(w0 = 6.63099, f(w0, 3.2126, .0616, 1)))
>vz05b:=(wy)->f(wy,3.2126,.0079,1)-eval(subs(w0=62.2721,f(w0,3.2126,.0079,1)));

 vz05b := wy ->

    f(wy, 3.2126, .0079, 1) - eval(subs(w0 = 62.2721, f(w0, 3.2126, .0079, 1)))
--------------------------------------------------------------------------------
> #Eta_eff=.1 --> zeta = .1107, .0088;
>vz1a:=(wy)->f(wy,3.2126,.1107,1)-eval(subs(w0=3.58458,f(w0,3.2126,.1107,1)));

 vz1a := wy ->

    f(wy, 3.2126, .1107, 1) - eval(subs(w0 = 3.58458, f(w0, 3.2126, .1107, 1)))
>vz1b:=(wy)->f(wy,3.2126,.0088,1)-eval(subs(w0=55.7965,f(w0,3.2126,.0088,1)));

 vz1b := wy ->

    f(wy, 3.2126, .0088, 1) - eval(subs(w0 = 55.7965, f(w0, 3.2126, .0088, 1)))
--------------------------------------------------------------------------------
> #Eta_eff=-.01 --> zeta = .0157;
>vzm01:=(wy)->f(wy,3.2126,.0157,1)-eval(subs(w0=30.8019,f(w0,3.2126,.0157,1)));

 vzm01 := wy ->

    f(wy, 3.2126, .0157, 1) - eval(subs(w0 = 30.8019, f(w0, 3.2126, .0157, 1)))
--------------------------------------------------------------------------------
> #Eta_eff=-.025 --> zeta = .0131;
>vzm025:=(wy)->f(wy,3.2126,.0131,1)-eval(subs(w0=37.1324,f(w0,3.2126,.0131,1)));

 vzm025 := wy ->

    f(wy, 3.2126, .0131, 1) - eval(subs(w0 = 37.1324, f(w0, 3.2126, .0131, 1)))
--------------------------------------------------------------------------------
> #Eta_eff=-.05 --> zeta = .0116;
>vzm05:=(wy)->f(wy,3.2126,.0116,1)-eval(subs(w0=42.073,f(w0,3.2126,.0116,1)));

vzm05 :=

wy -> f(wy, 3.2126, .0116, 1) - eval(subs(w0=42.073, f(w0, 3.2126, .0116, 1)))
--------------------------------------------------------------------------------
> #Eta_eff=-.1 --> zeta = .0107;
>vzm1:=(wy)->f(wy,3.2126,.0107,1)-eval(subs(w0=45.7014,f(w0,3.2126,.0107,1)));

 vzm1 := wy ->

    f(wy, 3.2126, .0107, 1) - eval(subs(w0 = 45.7014, f(w0, 3.2126, .0107, 1)))
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> plot({[x(wy,3.2126,1,.0195,1),wy,wy=-24.58..24.58],[x(wy,3.2126,1,.0257,1),
wy,wy=-18.3633..18.3633],[x(wy,3.2126,1,.0381,1),wy,wy=-11.9454..11.9454],
[x(wy,3.2126,1,.0616,1),wy,wy=-6.63099..6.63099],[x(wy,3.2126,1,.1107,1),wy,
```

```
wy=-3.58458..3.58458]},0..1,numpoints=600,resolution=600,color=black,thickness
=1);
--------------------------------------------------------------------------------
> plot({[x(wy,3.2126,1,.0195,1),wy,wy=-24.58..24.58],[x(wy,3.2126,1,.0038,1),
wy,wy=-130.571..130.571],[x(wy,3.2126,1,.0064,1),wy,wy=-77.1101..77.1101],
[x(wy,3.2126,1,.0079,1),wy,wy=-62.2721..62.2721],[x(wy,3.2126,1,.0088,1),wy,
wy=-55.7965..55.7965],[x(wy,3.2126,1,.0157,1),wy,wy=-30.8019..30.8019],[x(wy,
3.2126,1,.0131,1),wy,wy=-37.1324..37.1324],[x(wy,3.2126,1,.0116,1),wy,
wy=-42.073..42.073],[x(wy,3.2126,1,.0107,1),wy,wy=-45.7014..45.7014]},0..1,
-40..40,numpoints=600,resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,3.2126,1,.0195,1),vz0(wy),wy=-24.58..24.58],[x(wy,3.2126,1,
.0257,1),vz01a(wy),wy=-18.3633..18.3633],[x(wy,3.2126,1,.0381,1),vz025a(wy),
wy=-11.9454..11.9454],[x(wy,3.2126,1,.0616,1),vz05a(wy),wy=-6.63099..6.63099],
[x(wy,3.2126,1,.1107,1),vz1a(wy),wy=-3.58458..3.58458]},0..1,numpoints=600,
resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,3.2126,1,.0195,1),vz0(wy),wy=-24.58..24.58],[x(wy,3.2126,1,.0038,
1),vz01b(wy),wy=-130.571..130.571],[x(wy,3.2126,1,.0064,1),vz025b(wy),
wy=-77.1101..77.1101],[x(wy,3.2126,1,.0079,1),vz05b(wy),wy=-62.2721..62.2721],
[x(wy,3.2126,1,.0088,1),vz1b(wy),wy=-55.7965..55.7965],[x(wy,3.2126,1,.0157,1),
vzm01(wy),wy=-30.8019..30.8019],[x(wy,3.2126,1,.0131,1),vzm025(wy),
wy=-37.1324..37.1324],[x(wy,3.2126,1,.0116,1),vzm05(wy),wy=-42.073..42.073],
[x(wy,3.2126,1,.0107,1),vzm1(wy),wy=-45.7014..45.7014]},0..1,numpoints=600,
resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot([x(wy,3.2126,1,.0107,1),vzm1(wy),wy=-45.7014..45.7014],.25..0.75,
-22.2..-19.5,numpoints=600,resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> #OMEGA == 2.05;
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> #Eta_eff = 0 --> zeta = .0024;
>vz0:=(wy)->f(wy,2.05, .0024,1)-eval(subs(w0=207.328,f(w0,2.05,.0024,1)));

vz0 :=

   wy -> f(wy, 2.05, .0024, 1) - eval(subs(w0=207.328, f(w0, 2.05, .0024, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .01 --> zeta = .0113, .0010;
>vz01a:=(wy) -> f(wy,2.05,.0113,1) - eval(subs(w0=43.2218,f(w0,2.05,.0113,1)));

vz01a :=

   wy -> f(wy, 2.05, .0113, 1) - eval(subs(w0=43.2218, f(w0, 2.05, .0113, 1)))
>vz01b:=(wy) -> f(wy,2.05,.0010,1)-eval(subs(w0=498.998,f(w0,2.05,.0010,1)));

vz01b :=

   wy -> f(wy, 2.05, .0010, 1)-eval(subs(w0=498.998, f(w0, 2.05, .0010, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .025 --> zeta = .0262, .0011;
```

```
>vz025a:=(wy) -> f(wy,2.05,.0262,1)-eval(subs(w0=18.0109,f(w0,2.05,.0262,1)));

vz025a :=

   wy -> f(wy, 2.05, .0262, 1)-eval(subs(w0=18.0109, f(w0, 2.05, .0262, 1)))
>vz025b:=(wy) -> f(wy,2.05,.0011,1)-eval(subs(w0=453.543,f(w0,2.05,.0011,1)));

vz025b :=

   wy -> f(wy, 2.05, .0011, 1)-eval(subs(w0=453.543, f(w0, 2.05, .0011, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .05 --> zeta = .0512, .0012;
>vz05a:=(wy) -> f(wy,2.05,.0512,1)-eval(subs(w0=8.56627,f(w0,2.05,.0512,1)));

vz05a :=

   wy -> f(wy, 2.05, .0512, 1)-eval(subs(w0=8.56627, f(w0, 2.05, .0512, 1)))
>vz05b:=(wy) -> f(wy,2.05,.0012,1)-eval(subs(w0=415.664,f(w0,2.05,.0012,1)));

vz05b :=

   wy -> f(wy, 2.05, .0012, 1)-eval(subs(w0=415.664, f(w0, 2.05, .0012, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .1 --> zeta = .1012, .0012;
>vz1a:=(wy) -> f(wy,2.05,.1012,1)-eval(subs(w0=3.42359,f(w0,2.05,.1012,1)));

vz1a :=

   wy -> f(wy, 2.05, .1012, 1)-eval(subs(w0=3.42359, f(w0, 2.05, .1012, 1)))
>vz1b:=(wy) -> f(wy,2.05,.0012,1)-eval(subs(w0=415.664,f(w0,2.05,.0012,1)));

vz1b :=

   wy -> f(wy, 2.05, .0012, 1)-eval(subs(w0=415.664, f(w0, 2.05, .0012, 1)))
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> #Eta_eff = -.01 --> zeta = .0013;
>vzm01:=(wy) -> f(wy,2.05,.0013,1)-eval(subs(w0=383.613,f(w0,2.05,.0013,1)));

vzm01 :=

   wy -> f(wy, 2.05, .0013, 1)-eval(subs(w0=383.613, f(w0, 2.05, .0013, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.025, -.05, -.1 --> zeta = .0012;
>vzm025:=(wy) -> f(wy,2.05,.0012,1)-eval(subs(w0=415.664,f(w0,2.05,.0012,1)));

vzm025 :=

   wy -> f(wy, 2.05, .0012, 1)-eval(subs(w0=415.664, f(w0, 2.05, .0012, 1)))
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> plot({[x(wy,2.05,1,.0024,1),wy,wy=-207.328..207.328],[x(wy,2.05,1,.0113,1),
wy,wy=-43.2218..43.2218],[x(wy,2.05,1,.0262,1),wy,wy=-18.0109..18.0109],
```

```
[x(wy,2.05,1,.0512,1),wy,wy=-8.56627..8.56627],[x(wy,2.05,1,.1012,1),wy,wy=
-3.42359..3.42359]},0..1,-30..30,numpoints=600,resolution=600,color=black,
thickness=1);
> plot({[x(wy,2.05,1,.0024,1),wy,wy=-207.328..207.328],[x(wy,2.05,1,.0010,1),
wy,wy=-498.998..498.998],[x(wy,2.05,1,.0011,1),wy,wy=-453.543..453.543],
[x(wy,2.05,1,.0012,1),wy,wy=-415.664..415.664],[x(wy,2.05,1,.0013,1),wy,
wy=-383.613..383.613]},.3..0.7,-7..7,numpoints=1000,resolution=1000,color=
black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,2.05,1,.0024,1),vz0(wy),wy=-207.328..207.328],[x(wy,2.05,1,.0113,
1),vz01a(wy),wy=-43.2218..43.2218],[x(wy,2.05,1,.0262,1),vz025a(wy),
wy=-18.0109..18.0109],[x(wy,2.05,1,.0512,1),vz05a(wy),wy=-8.56627..9.56627],
[x(wy,2.05,1,.1012,1),vz1a(wy),wy=-3.42359..3.42359]},0..1,numpoints=600,
resolution=600,color=black,thickness=1);
> plot([x(wy,2.05,1,.0262,1),vz025a(wy),wy=-18.0109..18.0109],.2..0.8,
-8.5..-6.5,numpoints=1000,resolution=1000,color=black,thickness=1);
> plot({[x(wy,2.05,1,.0024,1),vz0(wy),wy=-207.328..207.328],[x(wy,2.05,1,
.0010,1),vz01b(wy),wy=-498.994..498.994],[x(wy,2.05,1,.0011,1),vz025b(wy),
wy=-453.543..453.543],[x(wy,2.05,1,.0012,1),vz05b(wy),wy=-415.664..415.664],
[x(wy,2.05,1,.0013,1),vzm01(wy),wy=-383.613..383.613]},0..1,numpoints=600,
resolution=600,color=black,thickness=1);
> plot([x(wy,2.05,1,.0013,1),vzm01(wy),wy=-383.613..383.613],.3..0.7,-191..
-177,numpoints=1000,resolution=1000,color=black,thickness=1);
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> #OMEGA == 2.5;
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> #Eta_eff = 0 --> zeta = .0151;
>vz0:=(wy)->f(wy,2.5, .0151,1)-eval(subs(w0=32.0739,f(w0,2.5,.0151,1)));

 vz0 :=

    wy -> f(wy, 2.5, .0151, 1)-eval(subs(w0=32.0739, f(w0, 2.5, .0151, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .01 --> zeta = .0216, .0035;
>vz01a:=(wy) -> f(wy,2.5,.0216,1)-eval(subs(w0=22.0868,f(w0,2.5,.0216,1)));

 vz01a :=

    wy -> f(wy, 2.5, .0216, 1)-eval(subs(w0=22.0868, f(w0, 2.5, .0216, 1)))
>vz01b:=(wy) -> f(wy,2.5,.0035,1)-eval(subs(w0=141.85,f(w0,2.5,.0035,1)));

 vz01b :=

    wy -> f(wy, 2.5, .0035, 1)-eval(subs(w0=141.85, f(w0, 2.5, .0035, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .025 --> zeta = .0347, .0055;
>vz025a:=(wy) -> f(wy,2.5,.0347,1)-eval(subs(w0=13.2873,f(w0,2.5,.0347,1)));

 vz025a :=

    wy -> f(wy, 2.5, .0347, 1)-eval(subs(w0=13.2873, f(w0, 2.5, .0347, 1)))
```

```
>vz025b:=(wy) -> f(wy,2.5,.0055,1)-eval(subs(w0=89.8971,f(w0,2.5,.0055,1)));

 vz025b :=

    wy -> f(wy, 2.5, .0055, 1)-eval(subs(w0=89.8971, f(w0, 2.5, .0055, 1)))
-------------------------------------------------------------------------------
> #Eta_eff = .05 --> zeta = .0587, .0064;
>vz05a:=(wy) -> f(wy,2.5,.0587,1)-eval(subs(w0=7.20961,f(w0,2.5,.0587,1)));

 vz05a :=

    wy -> f(wy, 2.5, .0587, 1)-eval(subs(w0=7.20961, f(w0, 2.5, .0587, 1)))
>vz05b:=(wy) -> f(wy,2.5,.0064,1)-eval(subs(w0=77.1108,f(w0,2.5,.0064,1)));

 vz05b :=

    wy -> f(wy, 2.5, .0064, 1)-eval(subs(w0=77.1108, f(w0, 2.5, .0064, 1)))
-------------------------------------------------------------------------------
> #Eta_eff = .1 --> zeta = .1081, .0070;
>vz1a:=(wy) -> f(wy,2.5,.1081,1)-eval(subs(w0=3.30883,f(w0,2.5,.1081,1)));

 vz1a :=

    wy -> f(wy, 2.5, .1081, 1)-eval(subs(w0=3.30883, f(w0, 2.5, .1081, 1)))
>vz1b:=(wy) -> f(wy,2.5,.0070,1)-eval(subs(w0=70.4129,f(w0,2.5,.0070,1)));

 vz1b :=

    wy -> f(wy, 2.5, .0070, 1)-eval(subs(w0=70.4129, f(w0, 2.5, .0070, 1)))
-------------------------------------------------------------------------------
-------------------------------------------------------------------------------
> #Eta_eff = -.01 --> zeta = .0116;
>vzm01:=(wy) -> f(wy,2.5,.0116,1)-eval(subs(w0=42.0754,f(w0,2.5,.0116,1)));

 vzm01 :=

    wy -> f(wy, 2.5, .0116, 1)-eval(subs(w0=42.0754, f(w0, 2.5, .0116, 1)))
-------------------------------------------------------------------------------
> #Eta_eff = -.025 --> zeta = .0097;
>vzm025:=(wy) -> f(wy,2.5,.0097,1)-eval(subs(w0=50.5237,f(w0,2.5,.0097,1)));

 vzm025 :=

    wy -> f(wy, 2.5, .0097, 1)-eval(subs(w0=50.5237, f(w0, 2.5, .0097, 1)))
-------------------------------------------------------------------------------
> #Eta_eff = -.05 --> zeta = .0087;
>vzm05:=(wy) -> f(wy,2.5,.0087,1)-eval(subs(w0=56.4512,f(w0,2.5,.0087,1)));

 vzm05 :=

    wy -> f(wy, 2.5, .0087, 1)-eval(subs(w0=56.4512, f(w0, 2.5, .0087, 1)))
-------------------------------------------------------------------------------
> #Eta_eff = -.1 --> zeta = .0081;
>vzm1:=(wy) -> f(wy,2.5,.0081,1)-eval(subs(w0=60.7099,f(w0,2.5,.0081,1)));
```

```
 vzm1 :=

    wy -> f(wy, 2.5, .0081, 1)-eval(subs(w0=60.7099, f(w0, 2.5, .0081, 1)))
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> plot({[x(wy,2.5,1,.0151,1),wy,wy=-32.0739..32.0739],[x(wy,2.5,1,.0216,1),wy,
wy=-22.0868..22.0868],[x(wy,2.5,1,.0347,1),wy,wy=-13.2873..13.2873],[x(wy,2.5,
1,.0587,1),wy,wy=-7.20961..7.20961],[x(wy,2.5,1,.1081,1),wy,wy=-3.30883..
3.30883]},0..1,numpoints=600,resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,2.5,1,.0151,1),wy,wy=-32.0739..32.0739],[x(wy,2.5,1,.0035,1),wy,
wy=-141.85..141.85],[x(wy,2.5,1,.0055,1),wy,wy=-89.8971..89.8971],[x(wy,2.5,
1,.0064,1),wy,wy=-77.1108..77.1108],[x(wy,2.5,1,.0070,1),wy,wy=-70.4129..
70.4129],[x(wy,2.5,1,.0116,1),wy,wy=-42.0754..42.0754],[x(wy,2.5,1,.0097,1),
wy,wy=-50.5237..50.5237],[x(wy,2.5,1,.0087,1),wy,wy=-56.4512..56.4512],
[x(wy,2.5,1,.0081,1),wy,wy=-60.7099..60.7099]},.26..0.74,-4..4,numpoints=600,
resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,2.5,1,.0151,1),vz0(wy),wy=-32.0739..32.0739],[x(wy,2.5,1,.0216,
1),vz01a(wy),wy=-22.0868..22.0868],[x(wy,2.5,1,.0347,1),vz025a(wy),
wy=-13.2873..13.2873],[x(wy,2.5,1,.0587,1),vz05a(wy),wy=-7.20961..7.20961],
[x(wy,2.5,1,.1081,1),vz1a(wy),wy=-3.30883..3.30883]},0..1,numpoints=600,
resolution=600,color=black,thickness=1);
> plot([x(wy,2.5,1,.1081,1),vz1a(wy),wy=-3.30883..3.30883],0..1,numpoints=
1000,resolution=1000,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,2.5,1,.0151,1),vz0(wy),wy=-32.0739..32.0739],[x(wy,2.5,1,.0035,
1),vz01b(wy),wy=-141.85..141.85],[x(wy,2.5,1,.0055,1),vz025b(wy),wy=-89.8971..
89.8971],[x(wy,2.5,1,.0064,1),vz05b(wy),wy=-77.1108..77.1108],[x(wy,2.5,1,.
0070,1),vz1b(wy),wy=-70.4129..70.4129],[x(wy,2.5,1,.0116,1),vzm01(wy),
wy=-42.0754..42.0754],[x(wy,2.5,1,.0097,1),vzm025(wy),wy=-50.5237..50.5237],
[x(wy,2.5,1,.0087,1),vzm05(wy),wy=-56.4512..56.4512],[x(wy,2.5,1,.0081,1),
vzm1(wy),wy=-60.7099..60.7099]},0..1,numpoints=600,resolution=600,color=black,
thickness=1);
> plot([x(wy,2.5,1,.0081,1),vzm1(wy),wy=-60.7099..60.7099],.25..0.75,-30..-26,
numpoints=1000,resolution=1000,color=black,thickness=1);
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> #OMEGA == 4.5;
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> #Eta_eff = 0 --> zeta = .0158;
>vz0:=(wy)->f(wy,4.5, .0158,1)-eval(subs(w0=30.5884,f(w0,4.5,.0158,1)));

 vz0 :=

    wy -> f(wy, 4.5, .0158, 1)-eval(subs(w0=30.5884, f(w0, 4.5, .0158, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .01 --> zeta = .0222, .0035;
>vz01a:=(wy) -> f(wy,4.5,.0222,1)-eval(subs(w0=21.4238,f(w0,4.5,.0222,1)));
```

```
 vz01a :=

    wy -> f(wy, 4.5, .0222, 1)-eval(subs(w0=21.4238, f(w0, 4.5, .0222, 1)))
>vz01b:=(wy) -> f(wy,4.5,.0035,1)-eval(subs(w0=141.849,f(w0,4.5,.0035,1)));

 vz01b :=

    wy -> f(wy, 4.5, .0035, 1)-eval(subs(w0=141.849, f(w0, 4.5, .0035, 1)))
-------------------------------------------------------------------------------
> #Eta_eff = .025 --> zeta = .0352, .0056;
>vz025a:=(wy) -> f(wy,4.5,.0352,1)-eval(subs(w0=12.9697,f(w0,4.5,.0352,1)));

 vz025a :=

    wy -> f(wy, 4.5, .0352, 1)-eval(subs(w0=12.9697, f(w0, 4.5, .0352, 1)))
>vz025b:=(wy) -> f(wy,4.5,.0056,1)-eval(subs(w0=88.2716,f(w0,4.5,.0056,1)));

 vz025b :=

    wy -> f(wy, 4.5, .0056, 1)-eval(subs(w0=88.2716, f(w0, 4.5, .0056, 1)))
-------------------------------------------------------------------------------
> #Eta_eff = .05 --> zeta = .0591, .0067;
>vz05a:=(wy) -> f(wy,4.5,.0591,1)-eval(subs(w0=6.61044,f(w0,4.5,.0591,1)));

 vz05a :=

    wy -> f(wy, 4.5, .0591, 1)-eval(subs(w0=6.61044, f(w0, 4.5, .0591, 1)))
>vz05b:=(wy) -> f(wy,4.5,.0067,1)-eval(subs(w0=73.6092,f(w0,4.5,.0067,1)));

 vz05b :=

    wy -> f(wy, 4.5, .0067, 1)-eval(subs(w0=73.6092, f(w0, 4.5, .0067, 1)))
-------------------------------------------------------------------------------
> #Eta_eff = .1 --> zeta = .1085, .0073;
>vz1a:=(wy) -> f(wy,4.5,.1085,1)-eval(subs(w0=4.33254,f(w0,4.5,.1085,1)));

 vz1a :=

    wy -> f(wy, 4.5, .1085, 1)-eval(subs(w0=4.33254, f(w0, 4.5, .1085, 1)))
>vz1b:=(wy) -> f(wy,4.5,.0073,1)-eval(subs(w0=67.4735,f(w0,4.5,.0073,1)));

 vz1b :=

    wy -> f(wy, 4.5, .0073, 1)-eval(subs(w0=67.4735, f(w0, 4.5, .0073, 1)))
-------------------------------------------------------------------------------
-------------------------------------------------------------------------------
> #Eta_eff = -.01 --> zeta = .0122;
>vzm01:=(wy) -> f(wy,4.5,.0122,1)-eval(subs(w0=39.9445,f(w0,4.5,.0122,1)));

 vzm01 :=

    wy -> f(wy, 4.5, .0122, 1)-eval(subs(w0=39.9445, f(w0, 4.5, .0122, 1)))
-------------------------------------------------------------------------------
> #Eta_eff = -.025 --> zeta = .0102;
```

```
>vzm025:=(wy) ->f(wy,4.5,.0102,1)-eval(subs(w0=47.9891,f(w0,4.5,.0102,1)));

 vzm025 :=

    wy -> f(wy, 4.5, .0102, 1)-eval(subs(w0=47.9891, f(w0, 4.5, .0102, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.05 --> zeta = .0091;
>vzm05:=(wy) -> f(wy,4.5,.0091,1)-eval(subs(w0=53.9189,f(w0,4.5,.0091,1)));

 vzm05 :=

    wy -> f(wy, 4.5, .0091, 1)-eval(subs(w0=53.9189, f(w0, 4.5, .0091, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.1 --> zeta = .0085;
>vzm1:=(wy) -> f(wy,4.5,.0085,1)-eval(subs(w0=57.7996,f(w0,4.5,.0085,1)));

 vzm1 :=

    wy -> f(wy, 4.5, .0085, 1)-eval(subs(w0=57.7996, f(w0, 4.5, .0085, 1)))
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> plot({[x(wy,4.5,1,.0158,1),wy,wy=-30.5884..30.5884],[x(wy,4.5,1,.0222,1),wy,
wy=-21.4238..21.4238],[x(wy,4.5,1,.0352,1),wy,wy=-12.9697..12.9697],[x(wy,4.5,
1,.0591,1),wy,wy=-6.61044..6.61044],[x(wy,4.5,1,.1085,1),wy,wy=-4.33254..
4.33254]},0..1,numpoints=600,resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,4.5,1,.0158,1),wy,wy=-30.5884..30.5884],[x(wy,4.5,1,.0035,1),wy,
wy=-141.849..141.849],[x(wy,4.5,1,.0056,1),wy,wy=-88.2716..88.2716],[x(wy,4.5,
1,.0067,1),wy,wy=-73.6092..73.6092],[x(wy,4.5,1,.0073,1),wy,wy=-67.4735..
67.4735],[x(wy,4.5,1,.0122,1),wy,wy=-39.9445..39.9445],[x(wy,4.5,1,.0102,1),
wy,wy=-49.9891..49.9891],[x(wy,4.5,1,.0091,1),wy,wy=-53.9189..53.9189],[x(wy,
4.5,1,.0085,1),wy,wy=-57.7996..57.7996]},.26..0.74,-10..10,numpoints=600,
resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,4.5,1,.0158,1),vz0(wy),wy=-30.5884..30.5884],[x(wy,4.5,1,.0222,
1),vz01a(wy),wy=-21.4238..21.4238],[x(wy,4.5,1,.0352,1),vz025a(wy),
wy=-12.9697..12.9697],[x(wy,4.5,1,.0591,1),vz05a(wy),wy=-6.61044..6.61044],
[x(wy,4.5,1,.1085,1),vz1a(wy),wy=-4.33254..4.33254]},0..1,numpoints=600,
resolution=600,color=black,thickness=1);
> plot([x(wy,4.5,1,.0222,1),vz01a(wy),wy=-21.4238..21.4238],.45..0.55,-12..
-10,numpoints=1000,resolution=1000,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,4.5,1,.0158,1),vz0(wy),wy=-30.5884..30.5884],[x(wy,4.5,1,.0035,
1),vz01b(wy),wy=-141.849..141.849],[x(wy,4.5,1,.0056,1),vz025b(wy),
wy=-88.2716..88.2716],[x(wy,4.5,1,.0067,1),vz05b(wy),wy=-73.6092..73.6092],
[x(wy,4.5,1,.0073,1),vz1b(wy),wy=-67.4735..67.4735],[x(wy,4.5,1,.0122,1),
vzm01(wy),wy=-39.9445..39.9445],[x(wy,4.5,1,.0102,1),vzm025(wy),wy=-47.9891..
47.9891],[x(wy,4.5,1,.0091,1),vzm05(wy),wy=-53.9189..53.9189],[x(wy,4.5,1,
.0085,1),vzm1(wy),wy=-57.7996..57.7996]},0..1,numpoints=600,resolution=600,
color=black,thickness=1);
> plot([x(wy,4.5,1,.0085,1),vzm1(wy),wy=-57.7996..57.7996],.25..0.75,-29..-24,
numpoints=1000,resolution=1000,color=black,thickness=1);
--------------------------------------------------------------------------------
```

```
---------------------------------------------------------------------------
---------------------------------------------------------------------------
> #OMEGA == 6.0;
---------------------------------------------------------------------------
---------------------------------------------------------------------------
> #Eta_eff = 0 --> zeta = .0108;
>vz0:=(wy)->f(wy,6, .0108,1)-eval(subs(w0=45.2551,f(w0,6,.0108,1)));

 vz0 := wy -> f(wy, 6, .0108, 1)-eval(subs(w0=45.2551, f(w0, 6, .0108, 1)))
---------------------------------------------------------------------------
> #Eta_eff = .01 --> zeta = .0178, .0030;
>vz01a:=(wy) -> f(wy,6,.0178,1)-eval(subs(w0=26.996,f(w0,6,.0178,1)));

 vz01a := wy -> f(wy, 6, .0178, 1)-eval(subs(w0=26.996, f(w0, 6, .0178, 1)))
>vz01b:=(wy) -> f(wy,6,.0030,1)-eval(subs(w0=165.659,f(w0,6,.0030,1)));

vz01b := wy -> f(wy, 6, .0030, 1)-eval(subs(w0=165.659, f(w0, 6, .0030, 1)))
---------------------------------------------------------------------------
> #Eta_eff = .025 --> zeta = .0315, .0043;
>vz025a:=(wy) -> f(wy,6,.0315,1)-eval(subs(w0=14.5751,f(w0,6,.0315,1)));

   vz025a :=

      wy -> f(wy, 6, .0315, 1)-eval(subs(w0=14.5751, f(w0, 6, .0315, 1)))
>vz025b:=(wy) -> f(wy,6,.0043,1)-eval(subs(w0=115.268,f(w0,6,.0043,1)));

   vz025b :=

      wy -> f(wy, 6, .0043, 1)-eval(subs(w0=115.268, f(w0, 6, .0043, 1)))
---------------------------------------------------------------------------
> #Eta_eff = .05 --> zeta = .0560, .0048;
>vz05a:=(wy) -> f(wy,6,.0560,1)-eval(subs(w0=6.99514,f(w0,6,.0560,1)));

vz05a := wy -> f(wy, 6, .0560, 1)-eval(subs(w0=6.99514, f(w0, 6, .0560, 1)))
>vz05b:=(wy) -> f(wy,6,.0048,1)-eval(subs(w0=103.153,f(w0,6,.0048,1)));

vz05b := wy -> f(wy, 6, .0048, 1)-eval(subs(w0=103.153, f(w0, 6, .0048, 1)))
---------------------------------------------------------------------------
> #Eta_eff = .1 --> zeta = .1057, .0051;
>vz1a:=(wy) -> f(wy,6,.1057,1)-eval(subs(w0=5.10211,f(w0,6,.1057,1)));

 vz1a := wy -> f(wy, 6, .1057, 1)-eval(subs(w0=5.10211, f(w0, 6, .1057, 1)))
>vz1b:=(wy) -> f(wy,6,.0051,1)-eval(subs(w0=97.0249,f(w0,6,.0051,1)));

 vz1b := wy -> f(wy, 6, .0051, 1)-eval(subs(w0=97.0249, f(w0, 6, .0051, 1)))
---------------------------------------------------------------------------
---------------------------------------------------------------------------
> #Eta_eff = -.01 --> zeta = .0078;
>vzm01:=(wy) -> f(wy,6,.0078,1)-eval(subs(w0=63.077,f(w0,6,.0078,1)));

 vzm01 := wy -> f(wy, 6, .0078, 1)-eval(subs(w0=63.077, f(w0, 6, .0078, 1)))
---------------------------------------------------------------------------
> #Eta_eff = -.025 --> zeta = .0065;
>vzm025:=(wy) -> f(wy,6,.0065,1)-eval(subs(w0=75.9033,f(w0,6,.0065,1)));
```

```
   vzm025 :=

      wy -> f(wy, 6, .0065, 1)-eval(subs(w0=75.9033, f(w0, 6, .0065, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.05 --> zeta = .0060;
>vzm05:=(wy) -> f(wy,6,.0060,1)-eval(subs(w0=82.3156,f(w0,6,.0060,1)));

vzm05 := wy -> f(wy, 6, .0060, 1)-eval(subs(w0=82.3156, f(w0, 6, .0060, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.1 --> zeta = .0057;
>vzm1:=(wy) -> f(wy,6,.0057,1)-eval(subs(w0=86.7027,f(w0,6,.0057,1)));

 vzm1 := wy -> f(wy, 6, .0057, 1)-eval(subs(w0=86.7027, f(w0, 6, .0057, 1)))
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> plot({[x(wy,6,1,.0108,1),wy,wy=-45.2551..45.2551],[x(wy,6,1,.0178,1),wy,
wy=-26.996..26.996],[x(wy,6,1,.0315,1),wy,wy=-14.5751..14.5751],[x(wy,6,1,
.0560,1),wy,wy=-6.99514..6.99514],[x(wy,6,1,.1057,1),wy,wy=-5.10211..5.10211]},
0..1,numpoints=600,resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,6,1,.0108,1),wy,wy=-45.2551..45.2551],[x(wy,6,1,.0030,1),wy,
wy=-165.659..165.659],[x(wy,6,1,.0043,1),wy,wy=-115.268..115.268],[x(wy,6,1,
.0048,1),wy,wy=-103.153..103.153],[x(wy,6,1,.0051,1),wy,wy=-97.0249..97.0249],
[x(wy,6,1,.0078,1),wy,wy=-63.077..63.077],[x(wy,6,1,.0065,1),wy,wy=-75.9033..
75.9033],[x(wy,6,1,.0060,1),wy,wy=-82.3156..82.3156],[x(wy,6,1,.0057,1),wy,
wy=-86.7027..86.7027]},.26..0.74, -14..14,numpoints=600,resolution=600,color=
black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,6,1,.0108,1),vz0(wy),wy=-45.2551..45.2551],[x(wy,6,1,.0178,1),
vz01a(wy),wy=-26.996..26.996],[x(wy,6,1,.0315,1),vz025a(wy),wy=-14.5751..
14.5751],[x(wy,6,1,.0560,1),vz05a(wy),wy=-6.99514..6.99514],[x(wy,6,1,.1057,1),
vz1a(wy),wy=-5.10211..5.10211]},0..1,numpoints=600,resolution=600,color=black,
thickness=1);
> plot([x(wy,6,1,.0108,1),vz0(wy),wy=-45.2551..45.2551],0..1,numpoints=1000,
resolution=1000,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,6,1,.0108,1),vz0(wy),wy=-45.2551..45.2551],[x(wy,6,1,.0030,1),
vz01b(wy),wy=-165.659..165.659],[x(wy,6,1,.0043,1),vz025b(wy),wy=-115.268..
115.268],[x(wy,6,1,.0048,1),vz05b(wy),wy=-103.153..103.153],[x(wy,6,1,.0051,1),
vz1b(wy),wy=-97.0249..97.0249],[x(wy,6,1,.0078,1),vzm01(wy),wy=-63.077..
63.077],[x(wy,6,1,.0065,1),vzm025(wy),wy=-75.9033..75.9033],[x(wy,6,1,.0060,1),
vzm05(wy),wy=-82.3156..82.3156],[x(wy,6,1,.0057,1),vzm1(wy),wy=-86.7027..
86.7027]},0..1,numpoints=600,resolution=600,color=black,thickness=1);
> plot([x(wy,6,1,.0057,1),vzm1(wy),wy=-86.7027..86.7027],.25..0.75,-43..-39,
numpoints=1000,resolution=1000,color=black,thickness=1);
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> #OMEGA == 10;
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> #Eta_eff = 0 --> zeta = .0046;
```

```
>vz0:=(wy)->f(wy,10, .0046,1)-eval(subs(w0=107.677,f(w0,10,.0046,1)));

vz0 := wy -> f(wy, 10, .0046, 1)-eval(subs(w0=107.677, f(w0, 10, .0046, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .01 --> zeta = .0128, .0018;
>vz01a:=(wy) -> f(wy,10, .0128,1)-eval(subs(w0=37.9572,f(w0,10,.0128,1)));

  vz01a :=

     wy -> f(wy, 10, .0128, 1)-eval(subs(w0=37.9572, f(w0, 10, .0128, 1)))
>vz01b:=(wy) -> f(wy,10,.0018,1)-eval(subs(w0=276.773,f(w0,10,.0018,1)));

  vz01b :=

     wy -> f(wy, 10, .0018, 1)-eval(subs(w0=276.773, f(w0, 10, .0018, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .025 --> zeta = .0275, .0021;
>vz025a:=(wy) -> f(wy,10,.0275,1)-eval(subs(w0=16.4881,f(w0,10,.0275,1)));

  vz025a :=

     wy -> f(wy, 10, .0275, 1)-eval(subs(w0=16.4881, f(w0, 10, .0275, 1)))
>vz025b:=(wy) -> f(wy,10,.0021,1)-eval(subs(w0=237.089,f(w0,10,.0021,1)));

  vz025b :=

     wy -> f(wy, 10, .0021, 1)-eval(subs(w0=237.089, f(w0, 10, .0021, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .05 --> zeta = .0524, .0022;
>vz05a:=(wy) -> f(wy,10,.0524,1)-eval(subs(w0=9.71204,f(w0,10,.0524,1)));

  vz05a :=

     wy -> f(wy, 10, .0524, 1)-eval(subs(w0=9.71204, f(w0, 10, .0524, 1)))
>vz05b:=(wy) -> f(wy,10,.0022,1)-eval(subs(w0=226.266,f(w0,10,.0022,1)));

  vz05b :=

     wy -> f(wy, 10, .0022, 1)-eval(subs(w0=226.266, f(w0, 10, .0022, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = .1 --> zeta = .1023, .0022;
>vz1a:=(wy) -> f(wy,10,.1023,1)-eval(subs(w0=5.18128,f(w0,10,.1023,1)));

  vz1a :=

     wy -> f(wy, 10, .1023, 1)-eval(subs(w0=5.18128, f(w0, 10, .1023, 1)))
>vz1b:=(wy) -> f(wy,10,.0022,1)-eval(subs(w0=226.266,f(w0,10,.0022,1)));

  vz1b :=

     wy -> f(wy, 10, .0022, 1)-eval(subs(w0=226.266, f(w0, 10, .0022, 1)))
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> #Eta_eff = -.01 --> zeta = .0028;
```

```
>vzm01:=(wy) -> f(wy,10,.0028,1)-eval(subs(w0=177.563,f(w0,10,.0028,1)));

  vzm01 :=

     wy -> f(wy, 10, .0028, 1)-eval(subs(w0=177.563, f(w0, 10, .0028, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.025 --> zeta = .0025;
>vzm025:=(wy) -> f(wy,10,.0025,1)-eval(subs(w0=198.992,f(w0,10,.0025,1)));

  vzm025 :=

     wy -> f(wy, 10, .0025, 1)-eval(subs(w0=198.992, f(w0, 10, .0025, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.05 --> zeta = .0024;
>vzm05:=(wy) -> f(wy,10,.0024,1)-eval(subs(w0=207.326,f(w0,10,.0024,1)));

  vzm05 :=

     wy -> f(wy, 10, .0024, 1)-eval(subs(w0=207.326, f(w0, 10, .0024, 1)))
--------------------------------------------------------------------------------
> #Eta_eff = -.1 --> zeta = .0023;
>vzm1:=(wy) -> f(wy,10,.0023,1)-eval(subs(w0=216.385,f(w0,10,.0023,1)));

  vzm1 :=

     wy -> f(wy, 10, .0023, 1)-eval(subs(w0=216.385, f(w0, 10, .0023, 1)))
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
--------------------------------------------------------------------------------
> plot({[x(wy,10,1,.0046,1),wy,wy=-107.677..107.677],[x(wy,10,1,.0128,1),wy,
wy=-37.9572..37.9572],[x(wy,10,1,.0275,1),wy,wy=-16.4881..16.4881],[x(wy,10,
1,.0524,1),wy,wy=-9.71204..9.71204],[x(wy,10,1,.1023,1),wy,wy=-5.18128..
5.18128]},0..1,-50..50,numpoints=600,resolution=600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,10,1,.0046,1),wy,wy=-107.677..107.677],[x(wy,10,1,.0018,1),wy,
wy=-276.773..276.773],[x(wy,10,1,.0021,1),wy,wy=-237.089..237.089],[x(wy,10,1,
.0022,1),wy,wy=-226.266..226.266],[x(wy,10,1,.0022,1),wy,wy=-226.266..226.266],
[x(wy,10,1,.0028,1),wy,wy=-177.563..177.563],[x(wy,10,1,.0025,1),wy,
wy=-198.992..198.992],[x(wy,10,1,.0024,1),wy,wy=-207.326..207.326],[x(wy,10,1,
.0023,1),wy,wy=-216.385..216.385]},.3..0.7,-20..20,numpoints=600,resolution=
600,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,10,1,.0046,1),vz0(wy),wy=-107.677..107.677],[x(wy,10,1,.0128,1),
vz01a(wy),wy=-37.9572..37.9572],[x(wy,10,1,.0275,1),vz025a(wy),wy=-16.4881..
16.4881],[x(wy,10,1,.0524,1),vz05a(wy),wy=-9.71204..9.71204],[x(wy,10,1,.1023,
1),vz1a(wy),wy=-5.18128..5.18128]},0..1,numpoints=600,resolution=600,color=
black,thickness=1);
> plot([x(wy,10,1,.0128,1),vz01a(wy),wy=-37.9572..37.9572],.2..0.8,-20..-13,
numpoints=1000,resolution=1000,color=black,thickness=1);
--------------------------------------------------------------------------------
> plot({[x(wy,10,1,.0046,1),vz0(wy),wy=-107.677..107.677],[x(wy,10,1,.0018,1),
vz01b(wy),wy=-276.773..276.773],[x(wy,10,1,.0021,1),vz025b(wy),
wy=-237.089..237.089],[x(wy,10,1,.0022,1),vz05b(wy),wy=-226.266..226.266],
[x(wy,10,1,.0022,1),vz1b(wy),wy=-226.266..226.266],[x(wy,10,1,.0028,1),
```

```
vzm01(wy),wy=-177.563..177.563],[x(wy,10,1,.0025,1),vzm025(wy),wy=-198.992..
198.992],[x(wy,10,1,.0024,1),vzm05(wy),wy=-207.326..207.326],[x(wy,10,1,
.0023,1),vzm1(wy),wy=-216.385..216.385]},0..1,numpoints=600,resolution=600,
color=black,thickness=1);
> plot([x(wy,10,1,.0023,1),vzm1(wy),wy=-216.385..216.385],.3..0.7,-108..-100,
numpoints=1000,resolution=1000,color=black,thickness=1);
--------------------------------------------------------------------------------
>
```

Appendix B

Mathematica Programs Calculating Boundary Spin Velocity $\tilde{\omega}_0$

B.1 Transverse Field Only, $|\tilde{B}_x| = 1,\ |\tilde{H}_z| = 0$

```
denom[wy_,Om_,chi_] := ((wy^2 - Om^2 + 1 + chi)^2 +
(2 + chi)^2*Om^2)

t1[wy_,Om_,chi_,Bx_,Hz_] := -wy*(Abs[Bx]^2*(wy^2 - Om^2 + 1) +
Abs[Hz]^2*(wy^2 - Om^2 + (1 + chi)^2))

t2[wy_,Om_,chi_,Bx_,Hz_] := (chi*(wy^2 - Om^2) + I*Om*
(wy^2 - Om^2 - 1 - chi))*Hz*Conjugate[Bx]

t3[wy_,Om_,chi_,Bx_,Hz_] := (chi*(wy^2 - Om^2) - I*Om*
(wy^2 - Om^2 - 1 - chi))*Conjugate[Hz]*Bx

t[wy_,Om_,chi_,Bx_,Hz_] := .5*chi*(t1[wy,Om,chi,Bx,Hz] +
t2[wy,Om,chi,Bx,Hz] + t3[wy,Om,chi,Bx,Hz]) /
denom[wy,Om,chi]

dt[wy_,Om_,chi_,Bx_,Hz_] := D[t[wyp,Om,chi,Bx,Hz],wyp]/.wyp->wy

alphacrit[zeta_,eta_] := 2*zeta*eta/(zeta + eta)

x[wy_,Om_,chi_,zeta_,dpdz_] = -(Integrate[1-dt[wyp,Om,chi,1,0]/
alphacrit[zeta,zeta],wyp]  -
.5*dpdz/zeta)*(zeta/dpdz) /.wyp->wy
```

```
        0.5 dpdz                           2
-(((wy - -------- + (2. chi wy - 2. chi Om  wy +
          zeta

                 3
          2. chi wy ) /

                           2       2           2
         (4 (1 + 2 chi + chi  + 2 Om  + 2 chi Om  +

                2   2     4       2           2
            chi  Om  + Om  + 2 wy  + 2 chi wy  -

                 2   2     4
            2 Om  wy  + wy ) zeta)) zeta) / dpdz)
```

```
Solve[x[w0,2.0412,1,.0375,1]==0,w0]

{{w0 -> -1.87816 - 1.43639 I}, {w0 -> -1.87816 + 1.43639 I},

  {w0 -> 2.42626 - 1.52846 I}, {w0 -> 2.42626 + 1.52846 I},

  {w0 -> 12.2371}}

Solve[x[w0,2.0412,1,.0434,1]==0,w0]

{{w0 -> -1.88133 - 1.43979 I}, {w0 -> -1.88133 + 1.43979 I},

  {w0 -> 2.43906 - 1.54095 I}, {w0 -> 2.43906 + 1.54095 I},

  {w0 -> 10.4053}}

Solve[x[w0,2.0412,1,.0043,1] == 0, w0]

{{w0 -> -1.86046 - 1.41316 I}, {w0 -> -1.86046 + 1.41316 I},

  {w0 -> 2.36488 - 1.48488 I}, {w0 -> 2.36488 + 1.48488 I},

  {w0 -> 115.27}}

Solve[x[w0,2.0412,1,.0545,1] == 0, w0]

{{w0 -> -1.88727 - 1.44568 I}, {w0 -> -1.88727 + 1.44568 I},

  {w0 -> 2.46399 - 1.57086 I}, {w0 -> 2.46399 + 1.57086 I},

  {w0 -> 8.02088}}

Solve[x[w0,2.0412,1,.0086,1] == 0 ,w0]

{{w0 -> -1.8627 - 1.41659 I}, {w0 -> -1.8627 + 1.41659 I},

  {w0 -> 2.37179 - 1.48875 I}, {w0 -> 2.37179 + 1.48875 I},
```

```
  {w0 -> 57.1213}}

Solve[x[w0,2.0412,1,.0764,1] == 0, w0]

{{w0 -> -1.89868 - 1.4555 I}, {w0 -> -1.89868 + 1.4555 I},

  {w0 -> 2.50472 - 1.66546 I}, {w0 -> 2.50472 + 1.66546 I},

  {w0 -> 5.33242}}

Solve[x[w0,2.0412,1,.0123,1] == 0, w0]

{{w0 -> -1.86464 - 1.41943 I}, {w0 -> -1.86464 + 1.41943 I},

  {w0 -> 2.37797 - 1.4924 I}, {w0 -> 2.37797 + 1.4924 I},

  {w0 -> 39.6237}}

Solve[x[w0,2.0412,1,.1249,1] == 0, w0]

{{w0 -> -1.92173 - 1.47061 I}, {w0 -> -1.92173 + 1.47061 I},

  {w0 -> 2.35618 - 1.91081 I}, {w0 -> 2.35618 + 1.91081 I},

  {w0 -> 3.13429}}

Solve[x[w0,2.0412,1,.0150,1] == 0, w0]

{{w0 -> -1.86607 - 1.42145 I}, {w0 -> -1.86607 + 1.42145 I},

  {w0 -> 2.38262 - 1.49527 I}, {w0 -> 2.38262 + 1.49527 I},

  {w0 -> 32.3002}}

Solve[x[w0,2.0412,1,.0330,1] == 0, w0]

{{w0 -> -1.87573 - 1.43365 I}, {w0 -> -1.87573 + 1.43365 I},

  {w0 -> 2.41684 - 1.5202 I}, {w0 -> 2.41684 + 1.5202 I},

  {w0 -> 14.0693}}

Solve[x[w0,2.0412,1,.0284,1] == 0, w0]

{{w0 -> -1.87325 - 1.43073 I}, {w0 -> -1.87325 + 1.43073 I},

  {w0 -> 2.40756 - 1.51271 I}, {w0 -> 2.40756 + 1.51271 I},

  {w0 -> 16.537}}

Solve[x[w0,2.0412,1,.0246,1] == 0, w0]

{{w0 -> -1.8712 - 1.42822 I}, {w0 -> -1.8712 + 1.42822 I},
```

```
  {w0 -> 2.40017 - 1.50716 I}, {w0 -> 2.40017 + 1.50716 I},
  {w0 -> 19.2673}}
Solve[x[w0,2.0412,1,.0217,1] == 0, w0]
{{w0 -> -1.86965 - 1.42624 I}, {w0 -> -1.86965 + 1.42624 I},
  {w0 -> 2.3947 - 1.50326 I}, {w0 -> 2.3947 + 1.50326 I},
  {w0 -> 21.9914}}


Solve[x[w0,1.05,1,.0048,1] == 0, w0]
{{w0 -> -0.888014 - 1.34937 I},
  {w0 -> -0.888014 + 1.34937 I},
  {w0 -> 1.39282 - 1.48715 I}, {w0 -> 1.39282 + 1.48715 I},
  {w0 -> 103.157}}
Solve[x[w0,1.05,1,.0140,1] == 0, w0]
{{w0 -> -0.888185 - 1.35496 I},
  {w0 -> -0.888185 + 1.35496 I}, {w0 -> 1.40219 - 1.4974 I},
  {w0 -> 1.40219 + 1.4974 I}, {w0 -> 34.6863}}
Solve[x[w0,1.05,1,.0017,1] == 0, w0]
{{w0 -> -0.888008 - 1.34747 I},
  {w0 -> -0.888008 + 1.34747 I}, {w0 -> 1.38971 - 1.484 I},
  {w0 -> 1.38971 + 1.484 I}, {w0 -> 293.114}}
Solve[x[w0,1.05,1,.0301,1] == 0, w0]
{{w0 -> -0.889015 - 1.36447 I},
  {w0 -> -0.889015 + 1.36447 I},
  {w0 -> 1.41883 - 1.51908 I}, {w0 -> 1.41883 + 1.51908 I},
  {w0 -> 15.5517}}
Solve[x[w0,1.05,1,.0020,1] == 0, w0]
```

```
{{w0 -> -0.888007 - 1.34766 I},

  {w0 -> -0.888007 + 1.34766 I}, {w0 -> 1.39001 - 1.4843 I},

  {w0 -> 1.39001 + 1.4843 I}, {w0 -> 248.996}}

Solve[x[w0,1.05,1,.0574,1] == 0, w0]

{{w0 -> -0.891832 - 1.37972 I},

  {w0 -> -0.891832 + 1.37972 I},

  {w0 -> 1.44457 - 1.57028 I}, {w0 -> 1.44457 + 1.57028 I},

  {w0 -> 7.60533}}

Solve[x[w0,1.05,1,.0021,1] == 0, w0]

{{w0 -> -0.888007 - 1.34772 I},

  {w0 -> -0.888007 + 1.34772 I}, {w0 -> 1.39011 - 1.4844 I},

  {w0 -> 1.39011 + 1.4844 I}, {w0 -> 237.091}}

Solve[x[w0,1.05,1,.1120,1] == 0, w0]

{{w0 -> -0.901696 - 1.40623 I},

  {w0 -> -0.901696 + 1.40623 I},

  {w0 -> 1.42739 - 1.72954 I}, {w0 -> 1.42739 + 1.72954 I},

  {w0 -> 3.41289}}

Solve[x[w0,1.05,1,.0021,1] == 0, w0]

{{w0 -> -0.888007 - 1.34772 I},

  {w0 -> -0.888007 + 1.34772 I}, {w0 -> 1.39011 - 1.4844 I},

  {w0 -> 1.39011 + 1.4844 I}, {w0 -> 237.091}}

Solve[x[w0,1.05,1,.0027,1] == 0,w0]

{{w0 -> -0.888007 - 1.34809 I},

  {w0 -> -0.888007 + 1.34809 I}, {w0 -> 1.39071 - 1.485 I},

  {w0 -> 1.39071 + 1.485 I}, {w0 -> 184.18}}

Solve[x[w0,1.05,1,.0024,1] == 0, w0]

{{w0 -> -0.888007 - 1.3479 I}, {w0 -> -0.888007 + 1.3479 I},
```

```
  {w0 -> 1.39041 - 1.4847 I}, {w0 -> 1.39041 + 1.4847 I},
  {w0 -> 207.329}}
Solve[x[w0,1.05,1,.0023,1] == 0, w0]
{{w0 -> -0.888007 - 1.34784 I},
  {w0 -> -0.888007 + 1.34784 I}, {w0 -> 1.39031 - 1.4846 I},
  {w0 -> 1.39031 + 1.4846 I}, {w0 -> 216.387}}
Solve[x[w0,1.05,1,.0022,1] == 0, w0]
{{w0 -> -0.888007 - 1.34778 I},
  {w0 -> -0.888007 + 1.34778 I}, {w0 -> 1.39021 - 1.4845 I},
  {w0 -> 1.39021 + 1.4845 I}, {w0 -> 226.268}}


Solve[x[w0,1.5,1,.0308,1] == 0, w0]
{{w0 -> -1.33573 - 1.40384 I}, {w0 -> -1.33573 + 1.40384 I},
  {w0 -> 1.86893 - 1.51558 I}, {w0 -> 1.86893 + 1.51558 I},
  {w0 -> 15.1674}}
Solve[x[w0,1.5,1,.0371,1] == 0,w0]
{{w0 -> -1.33769 - 1.40771 I}, {w0 -> -1.33769 + 1.40771 I},
  {w0 -> 1.87816 - 1.52597 I}, {w0 -> 1.87816 + 1.52597 I},
  {w0 -> 12.3961}}
Solve[x[w0,1.5,1,.0041,1] == 0, w0]
{{w0 -> -1.32837 - 1.38576 I}, {w0 -> -1.32837 + 1.38576 I},
  {w0 -> 1.83251 - 1.48332 I}, {w0 -> 1.83251 + 1.48332 I},
  {w0 -> 120.943}}
Solve[x[w0,1.5,1,.0492,1] == 0, w0]
{{w0 -> -1.34164 - 1.41475 I}, {w0 -> -1.34164 + 1.41475 I},
  {w0 -> 1.89601 - 1.55008 I}, {w0 -> 1.89601 + 1.55008 I},
  {w0 -> 9.05384}}
```

```
Solve[x[w0,1.5,1,.0078,1] == 0, w0]

{{w0 -> -1.32928 - 1.38842 I}, {w0 -> -1.32928 + 1.38842 I},

  {w0 -> 1.83724 - 1.48688 I}, {w0 -> 1.83724 + 1.48688 I},

  {w0 -> 63.0867}}

Solve[x[w0,1.5,1,.0726,1] == 0,w0]

{{w0 -> -1.34976 - 1.42687 I}, {w0 -> -1.34976 + 1.42687 I},

  {w0 -> 1.92522 - 1.61717 I}, {w0 -> 1.92522 + 1.61717 I},

  {w0 -> 5.73612}}

Solve[x[w0,1.5,1,.0106,1] == 0, w0]

{{w0 -> -1.33 - 1.39041 I}, {w0 -> -1.33 + 1.39041 I},

  {w0 -> 1.84089 - 1.48975 I}, {w0 -> 1.84089 + 1.48975 I},

  {w0 -> 46.148}}

Solve[x[w0,1.5,1,.1232,1] == 0, w0]

{{w0 -> -1.36819 - 1.44701 I}, {w0 -> -1.36819 + 1.44701 I},

  {w0 -> 1.85583 - 1.81606 I}, {w0 -> 1.85583 + 1.81606 I},

  {w0 -> 3.08315}}

Solve[x[w0,1.5,1,.0125,1] == 0, w0]

{{w0 -> -1.3305 - 1.39173 I}, {w0 -> -1.3305 + 1.39173 I},

  {w0 -> 1.8434 - 1.49177 I}, {w0 -> 1.8434 + 1.49177 I},

  {w0 -> 38.9742}}


Solve[x[w0,1.5,1,.0262,1] == 0, w0]

{{w0 -> -1.33435 - 1.40091 I}, {w0 -> -1.33435 + 1.40091 I},

  {w0 -> 1.86231 - 1.50878 I}, {w0 -> 1.86231 + 1.50878 I},

  {w0 -> 18.028}}

Solve[x[w0,1.5,1,.0220,1] == 0, w0]

{{w0 -> -1.33312 - 1.39817 I}, {w0 -> -1.33312 + 1.39817 I},
```

```
  {w0 -> 1.85637 - 1.50308 I}, {w0 -> 1.85637 + 1.50308 I},

  {w0 -> 21.6808}}

Solve[x[w0,1.5,1,.0189,1] == 0, w0]

{{w0 -> -1.33224 - 1.39611 I}, {w0 -> -1.33224 + 1.39611 I},

  {w0 -> 1.85206 - 1.49916 I}, {w0 -> 1.85206 + 1.49916 I},

  {w0 -> 25.4154}}

Solve[x[w0,1.5,1,.0169,1] == 0, w0]

{{w0 -> -1.33169 - 1.39476 I}, {w0 -> -1.33169 + 1.39476 I},

  {w0 -> 1.84932 - 1.49676 I}, {w0 -> 1.84932 + 1.49676 I},

  {w0 -> 28.5505}}



Solve[x[w0,1.3,1,.0225,1] == 0,w0]

{{w0 -> -1.13528 - 1.38367 I}, {w0 -> -1.13528 + 1.38367 I},

  {w0 -> 1.65841 - 1.5046 I}, {w0 -> 1.65841 + 1.5046 I},

  {w0 -> 21.176}}

Solve[x[w0,1.3,1,.0294,1]== 0,w0]

{{w0 -> -1.13663 - 1.38799 I}, {w0 -> -1.13663 + 1.38799 I},

  {w0 -> 1.66703 - 1.51431 I}, {w0 -> 1.66703 + 1.51431 I},

  {w0 -> 15.946}}

Solve[x[w0,1.3,1,.0038,1]== 0,w0]

{{w0 -> -1.13224 - 1.37133 I}, {w0 -> -1.13224 + 1.37133 I},

  {w0 -> 1.63606 - 1.48342 I}, {w0 -> 1.63606 + 1.48342 I},

  {w0 -> 130.571}}

Solve[x[w0,1.3,1,.0426,1]== 0,w0]

{{w0 -> -1.13951 - 1.3959 I}, {w0 -> -1.13951 + 1.3959 I},

  {w0 -> 1.68363 - 1.53656 I}, {w0 -> 1.68363 + 1.53656 I},
```

```
  {w0 -> 10.6488}}

Solve[x[w0,1.3,1,.0066,1]== 0,w0]

{{w0 -> -1.13263 - 1.37324 I}, {w0 -> -1.13263 + 1.37324 I},

  {w0 -> 1.63929 - 1.48618 I}, {w0 -> 1.63929 + 1.48618 I},

  {w0 -> 74.7443}}

Solve[x[w0,1.3,1,.0675,1]== 0,w0]

{{w0 -> -1.14584 - 1.40949 I}, {w0 -> -1.14584 + 1.40949 I},

  {w0 -> 1.71109 - 1.59576 I}, {w0 -> 1.71109 + 1.59576 I},

  {w0 -> 6.27691}}

Solve[x[w0,1.3,1,.0084,1]== 0,w0]

{{w0 -> -1.13289 - 1.37445 I}, {w0 -> -1.13289 + 1.37445 I},

  {w0 -> 1.64139 - 1.48803 I}, {w0 -> 1.64139 + 1.48803 I},

  {w0 -> 58.5068}}

Solve[x[w0,1.3,1,.1197,1]== 0,w0]

{{w0 -> -1.16143 - 1.43242 I}, {w0 -> -1.16143 + 1.43242 I},

  {w0 -> 1.66737 - 1.77751 I}, {w0 -> 1.66737 + 1.77751 I},

  {w0 -> 3.16523}}

Solve[x[w0,1.3,1,.0094,1]== 0,w0]

{{w0 -> -1.13305 - 1.37512 I}, {w0 -> -1.13305 + 1.37512 I},

  {w0 -> 1.64257 - 1.48908 I}, {w0 -> 1.64257 + 1.48908 I},

  {w0 -> 52.1724}}

Solve[x[w0,1.3,1,.0181,1]== 0,w0]

{{w0 -> -1.13448 - 1.38085 I}, {w0 -> -1.13448 + 1.38085 I},

  {w0 -> 1.653 - 1.49899 I}, {w0 -> 1.653 + 1.49899 I},

  {w0 -> 26.5873}}

Solve[x[w0,1.3,1,.0149,1]== 0,w0]

{{w0 -> -1.13393 - 1.37876 I}, {w0 -> -1.13393 + 1.37876 I},
```

```
  {w0 -> 1.64912 - 1.49517 I}, {w0 -> 1.64912 + 1.49517 I},
  {w0 -> 32.5267}}
Solve[x[w0,1.3,1,.0129,1]== 0,w0]
{{w0 -> -1.1336 - 1.37745 I}, {w0 -> -1.1336 + 1.37745 I},
  {w0 -> 1.64672 - 1.49289 I}, {w0 -> 1.64672 + 1.49289 I},
  {w0 -> 37.7334}}
Solve[x[w0,1.3,1,.0118,1]== 0,w0]
{{w0 -> -1.13342 - 1.37672 I}, {w0 -> -1.13342 + 1.37672 I},
  {w0 -> 1.64541 - 1.49167 I}, {w0 -> 1.64541 + 1.49167 I},
  {w0 -> 41.3489}}



Solve[x[w0,5.0,1,.0159,1] == 0, w0]
{{w0 -> -4.81375 - 1.47147 I}, {w0 -> -4.81375 + 1.47147 I},
  {w0 -> 5.34367 - 1.49718 I}, {w0 -> 5.34367 + 1.49718 I},
  {w0 -> 30.3867}}
Solve[x[w0,5.0,1,.0224,1] == 0, w0]
{{w0 -> -4.82388 - 1.47599 I}, {w0 -> -4.82388 + 1.47599 I},
  {w0 -> 5.37626 - 1.50573 I}, {w0 -> 5.37626 + 1.50573 I},
  {w0 -> 21.2167}}
Solve[x[w0,5.0,1,.0036,1] == 0, w0]
{{w0 -> -4.79207 - 1.46032 I}, {w0 -> -4.79207 + 1.46032 I},
  {w0 -> 5.29631 - 1.4876 I}, {w0 -> 5.29631 + 1.4876 I},
  {w0 -> 137.88}}
Solve[x[w0,5.0,1,.0353,1] == 0, w0]
```

```
{{w0 -> -4.84166 - 1.48295 I}, {w0 -> -4.84166 + 1.48295 I},
  {w0 -> 5.46654 - 1.5391 I}, {w0 -> 5.46654 + 1.5391 I},
  {w0 -> 12.9145}}
Solve[x[w0,5.0,1,.0056,1] == 0, w0]
{{w0 -> -4.79583 - 1.4624 I}, {w0 -> -4.79583 + 1.4624 I},
  {w0 -> 5.30301 - 1.48876 I}, {w0 -> 5.30301 + 1.48876 I},
  {w0 -> 88.2714}}
Solve[x[w0,5.0,1,.0593,1] == 0, w0]
{{w0 -> -4.86812 - 1.49113 I}, {w0 -> -4.86812 + 1.49113 I},
  {w0 -> 5.7035 - 1.93525 I}, {w0 -> 5.7035 + 1.93525 I},
  {w0 -> 6.76095}}
Solve[x[w0,5.0,1,.0067,1] == 0, w0]
{{w0 -> -4.79786 - 1.4635 I}, {w0 -> -4.79786 + 1.4635 I},
  {w0 -> 5.30684 - 1.48945 I}, {w0 -> 5.30684 + 1.48945 I},
  {w0 -> 73.6089}}
Solve[x[w0,5.0,1,.1087,1] == 0, w0]
{{w0 -> -4.90479 - 1.49845 I}, {w0 -> -4.90479 + 1.49845 I},
  {w0 -> 4.74432}, {w0 -> 4.83254 - 2.10714 I},
  {w0 -> 4.83254 + 2.10714 I}}
Solve[x[w0,5.0,1,.0073,1] == 0, w0]
{{w0 -> -4.79896 - 1.46409 I}, {w0 -> -4.79896 + 1.46409 I},
  {w0 -> 5.30898 - 1.48984 I}, {w0 -> 5.30898 + 1.48984 I},
  {w0 -> 67.4731}}
Solve[x[w0,5.0,1,.0123,1] == 0, w0]
{{w0 -> -4.80776 - 1.46859 I}, {w0 -> -4.80776 + 1.46859 I},
  {w0 -> 5.32814 - 1.49368 I}, {w0 -> 5.32814 + 1.49368 I},
  {w0 -> 39.6096}}
```

```
Solve[x[w0,5.0,1,.0103,1] == 0, w0]

{{w0 -> -4.80431 - 1.46686 I}, {w0 -> -4.80431 + 1.46686 I},

  {w0 -> 5.32017 - 1.49202 I}, {w0 -> 5.32017 + 1.49202 I},

  {w0 -> 47.512}}

Solve[x[w0,5.0,1,.0092,1] == 0, w0]

{{w0 -> -4.80237 - 1.46587 I}, {w0 -> -4.80237 + 1.46587 I},

  {w0 -> 5.31596 - 1.49118 I}, {w0 -> 5.31596 + 1.49118 I},

  {w0 -> 53.3206}}

Solve[x[w0,5.0,1,.0086,1] == 0, w0]

{{w0 -> -4.8013 - 1.46532 I}, {w0 -> -4.8013 + 1.46532 I},

  {w0 -> 5.31372 - 1.49074 I}, {w0 -> 5.31372 + 1.49074 I},

  {w0 -> 57.1147}}
```

```
Solve[x[w0,3.25,1,.0284,1] == 0, w0]

{{w0 -> -3.07932 - 1.46206 I}, {w0 -> -3.07932 + 1.46206 I},

  {w0 -> 3.62695 - 1.51563 I}, {w0 -> 3.62695 + 1.51563 I},

  {w0 -> 16.5104}}

Solve[x[w0,3.25,1,.0343,1] == 0, w0]

{{w0 -> -3.08487 - 1.46553 I}, {w0 -> -3.08487 + 1.46553 I},

  {w0 -> 3.6486 - 1.52753 I}, {w0 -> 3.6486 + 1.52753 I},

  {w0 -> 13.4498}}

Solve[x[w0,3.25,1,.0041,1] == 0, w0]

{{w0 -> -3.05429 - 1.44343 I}, {w0 -> -3.05429 + 1.44343 I},

  {w0 -> 3.55873 - 1.48756 I}, {w0 -> 3.55873 + 1.48756 I},

  {w0 -> 120.942}}

Solve[x[w0,3.25,1,.0458,1] == 0, w0]
```

```
{{w0 -> -3.09508 - 1.47137 I}, {w0 -> -3.09508 + 1.47137 I},
  {w0 -> 3.69833 - 1.56225 I}, {w0 -> 3.69833 + 1.56225 I},
  {w0 -> 9.71053}}
Solve[x[w0,3.25,1,.0078,1] == 0, w0]
{{w0 -> -3.05833 - 1.44679 I}, {w0 -> -3.05833 + 1.44679 I},
  {w0 -> 3.56736 - 1.49041 I}, {w0 -> 3.56736 + 1.49041 I},
  {w0 -> 63.0845}}
Solve[x[w0,3.25,1,.0683,1] == 0, w0]
{{w0 -> -3.11288 - 1.47997 I}, {w0 -> -3.11288 + 1.47997 I},
  {w0 -> 3.80432 - 1.73229 I}, {w0 -> 3.80432 + 1.73229 I},
  {w0 -> 5.93777}}
Solve[x[w0,3.25,1,.0104,1] == 0, w0]
{{w0 -> -3.06111 - 1.44903 I}, {w0 -> -3.06111 + 1.44903 I},
  {w0 -> 3.57375 - 1.49264 I}, {w0 -> 3.57375 + 1.49264 I},
  {w0 -> 47.0517}}
Solve[x[w0,3.25,1,.1168,1] == 0, w0]
{{w0 -> -3.14301 - 1.49049 I}, {w0 -> -3.14301 + 1.49049 I},
  {w0 -> 3.4215 - 2.07128 I}, {w0 -> 3.4215 + 2.07128 I},
  {w0 -> 3.72385}}
Solve[x[w0,3.25,1,.0122,1] == 0, w0]
{{w0 -> -3.06302 - 1.45052 I}, {w0 -> -3.06302 + 1.45052 I},
  {w0 -> 3.57834 - 1.49431 I}, {w0 -> 3.57834 + 1.49431 I},
  {w0 -> 39.953}}
Solve[x[w0,3.25,1,.0242,1] == 0, w0]
{{w0 -> -3.07525 - 1.45937 I}, {w0 -> -3.07525 + 1.45937 I},
  {w0 -> 3.61294 - 1.50876 I}, {w0 -> 3.61294 + 1.50876 I},
  {w0 -> 19.5858}}
```

```
Solve[x[w0,3.25,1,.0205,1] == 0, w0]

{{w0 -> -3.07157 - 1.45684 I}, {w0 -> -3.07157 + 1.45684 I},

  {w0 -> 3.60147 - 1.50359 I}, {w0 -> 3.60147 + 1.50359 I},

  {w0 -> 23.3304}}

Solve[x[w0,3.25,1,.0179,1] == 0, w0]

{{w0 -> -3.06894 - 1.45495 I}, {w0 -> -3.06894 + 1.45495 I},

  {w0 -> 3.59386 - 1.50037 I}, {w0 -> 3.59386 + 1.50037 I},

  {w0 -> 26.8831}}

Solve[x[w0,3.25,1,.0161,1] == 0, w0]

{{w0 -> -3.06709 - 1.4536 I}, {w0 -> -3.06709 + 1.4536 I},

  {w0 -> 3.58879 - 1.49832 I}, {w0 -> 3.58879 + 1.49832 I},

  {w0 -> 30.0125}}
```

```
Solve[x[w0,10,1,.0047,1] == 0, w0]

{{w0 -> -9.78799 - 1.47621 I}, {w0 -> -9.78799 + 1.47621 I},

  {w0 -> 10.2973 - 1.4845 I}, {w0 -> 10.2973 + 1.4845 I},

  {w0 -> 105.364}}

Solve[x[w0,10,1,.0129,1] == 0, w0]

{{w0 -> -9.81587 - 1.48437 I}, {w0 -> -9.81587 + 1.48437 I},

  {w0 -> 10.368 - 1.48655 I}, {w0 -> 10.368 + 1.48655 I},

  {w0 -> 37.6553}}

Solve[x[w0,10,1,.0018,1] == 0, w0]

{{w0 -> -9.77628 - 1.47229 I}, {w0 -> -9.77628 + 1.47229 I},

  {w0 -> 10.2787 - 1.48431 I}, {w0 -> 10.2787 + 1.48431 I},

  {w0 -> 276.773}}
```

```
Solve[x[w0,10,1,.0276,1] == 0, w0]

{{w0 -> -9.85213 - 1.49256 I}, {w0 -> -9.85213 + 1.49256 I},

  {w0 -> 10.6853 - 1.52634 I}, {w0 -> 10.6853 + 1.52634 I},

  {w0 -> 16.4497}}

Solve[x[w0,10,1,.0021,1] == 0, w0]

{{w0 -> -9.77754 - 1.47272 I}, {w0 -> -9.77754 + 1.47272 I},

  {w0 -> 10.2805 - 1.48432 I}, {w0 -> 10.2805 + 1.48432 I},

  {w0 -> 237.089}}

Solve[x[w0,10,1,.0525,1] == 0, w0]

{{w0 -> -9.89071 - 1.49839 I}, {w0 -> -9.89071 + 1.49839 I},

  {w0 -> 9.74765 - 2.62435 I}, {w0 -> 9.74765 + 2.62435 I},

  {w0 -> 9.80991}}

Solve[x[w0,10,1,.0022,1] == 0, w0]

{{w0 -> -9.77796 - 1.47287 I}, {w0 -> -9.77796 + 1.47287 I},

  {w0 -> 10.2811 - 1.48433 I}, {w0 -> 10.2811 + 1.48433 I},

  {w0 -> 226.266}}

Solve[x[w0,10,1,.1024,1] == 0, w0]

{{w0 -> -9.93058 - 1.50139 I}, {w0 -> -9.93058 + 1.50139 I},

  {w0 -> 5.18971}, {w0 -> 9.77713 - 1.54371 I},

  {w0 -> 9.77713 + 1.54371 I}}

Solve[x[w0,10,1,.0023,1] == 0, w0]

{{w0 -> -9.77838 - 1.47301 I}, {w0 -> -9.77838 + 1.47301 I},

  {w0 -> 10.2817 - 1.48433 I}, {w0 -> 10.2817 + 1.48433 I},

  {w0 -> 216.385}}

Solve[x[w0,10,1,.0029,1] == 0, w0]

{{w0 -> -9.78085 - 1.47385 I}, {w0 -> -9.78085 + 1.47385 I},

  {w0 -> 10.2855 - 1.48436 I}, {w0 -> 10.2855 + 1.48436 I},
```

```
  {w0 -> 171.405}}

Solve[x[w0,10,1,.0026,1] == 0, w0]

{{w0 -> -9.77962 - 1.47344 I}, {w0 -> -9.77962 + 1.47344 I},

  {w0 -> 10.2836 - 1.48434 I}, {w0 -> 10.2836 + 1.48434 I},

  {w0 -> 191.3}}

Solve[x[w0,10,1,.0025,1] == 0, w0]

{{w0 -> -9.77921 - 1.4733 I}, {w0 -> -9.77921 + 1.4733 I},

  {w0 -> 10.283 - 1.48434 I}, {w0 -> 10.283 + 1.48434 I},

  {w0 -> 198.992}}

Solve[x[w0,10,1,.0024,1] == 0, w0]

{{w0 -> -9.77879 - 1.47315 I}, {w0 -> -9.77879 + 1.47315 I},

  {w0 -> 10.2824 - 1.48433 I}, {w0 -> 10.2824 + 1.48433 I},

  {w0 -> 207.326}}


Solve[x[w0,30,1,.00055187,1] == 0, w0]

{{w0 -> -29.764 - 1.47807 I}, {w0 -> -29.764 + 1.47807 I},

  {w0 -> 30.2651 - 1.48026 I}, {w0 -> 30.2651 + 1.48026 I},

  {w0 -> 905.008}}
```

B.2 Tangential Field Only, $|\tilde{B}_x| = 0$, $|\tilde{H}_z| = 1$

```
denom[wy_,Om_,chi_] := ((wy^2 - Om^2 + 1 + chi)^2 +
(2 + chi)^2*Om^2)

t1[wy_,Om_,chi_,Bx_,Hz_] := -wy*(Abs[Bx]^2*(wy^2 - Om^2 + 1) +
Abs[Hz]^2*(wy^2 - Om^2 + (1 + chi)^2))

t2[wy_,Om_,chi_,Bx_,Hz_] := (chi*(wy^2 - Om^2) + I*Om*
(wy^2 - Om^2 - 1 - chi))*Hz*Conjugate[Bx]

t3[wy_,Om_,chi_,Bx_,Hz_] := (chi*(wy^2 - Om^2) - I*Om*
(wy^2 - Om^2 - 1 - chi))*Conjugate[Hz]*Bx

t[wy_,Om_,chi_,Bx_,Hz_] := .5*chi*(t1[wy,Om,chi,Bx,Hz] +
t2[wy,Om,chi,Bx,Hz] + t3[wy,Om,chi,Bx,Hz]) /
denom[wy,Om,chi]

dt[wy_,Om_,chi_,Bx_,Hz_] := D[t[wyp,Om,chi,Bx,Hz],wyp]/.wyp->wy

alphacrit[zeta_,eta_] := 2*zeta*eta/(zeta + eta)

x[wy_,Om_,chi_,zeta_,dpdz_] = -(Integrate[1-dt[wyp,Om,chi,0,1]/
alphacrit[zeta,zeta],wyp]  -
.5*dpdz/zeta)*(zeta/dpdz) /.wyp->wy

         0.5 dpdz                        2           3
-(((wy - -------- + (2. chi wy + 4. chi  wy + 2. chi  wy -
           zeta

                  2             3
           2. chi Om  wy + 2. chi wy ) /

                             2       2             2
         (4 (1 + 2 chi + chi  + 2 Om  + 2 chi Om  +

               2   2     4       2            2
             chi  Om  + Om  + 2 wy  + 2 chi wy  -

                2   2    4
             2 Om  wy  + wy ) zeta)) zeta) / dpdz)

Solve[x[w0,3.2126,1,.0195,1]==0,w0]

{{w0 -> -3.03477 - 1.52174 I}, {w0 -> -3.03477 + 1.52174 I},

  {w0 -> 3.56528 - 1.40085 I}, {w0 -> 3.56528 + 1.40085 I},

  {w0 -> 24.58}}

Solve[x[w0,3.2126,1,.0257,1]==0,w0]

{{w0 -> -3.04176 - 1.52396 I}, {w0 -> -3.04176 + 1.52396 I},
```

```
  {w0 -> 3.58773 - 1.40272 I}, {w0 -> 3.58773 + 1.40272 I},

  {w0 -> 18.3633}}

Solve[x[w0,3.2126,1,.0038,1]==0,w0]

{{w0 -> -3.01556 - 1.51426 I}, {w0 -> -3.01556 + 1.51426 I},

  {w0 -> 3.51973 - 1.39983 I}, {w0 -> 3.51973 + 1.39983 I},

  {w0 -> 130.571}}

Solve[x[w0,3.2126,1,.0381,1]==0,w0]

{{w0 -> -3.05477 - 1.52742 I}, {w0 -> -3.05477 + 1.52742 I},

  {w0 -> 3.64374 - 1.41153 I}, {w0 -> 3.64374 + 1.41153 I},

  {w0 -> 11.9454}}

Solve[x[w0,3.2126,1,.0064,1]==0,w0]

{{w0 -> -3.0189 - 1.51571 I}, {w0 -> -3.0189 + 1.51571 I},

  {w0 -> 3.52636 - 1.39975 I}, {w0 -> 3.52636 + 1.39975 I},

  {w0 -> 77.1101}}

Solve[x[w0,3.2126,1,.0616,1]==0,w0]

{{w0 -> -3.07615 - 1.53127 I}, {w0 -> -3.07615 + 1.53127 I},

  {w0 -> 3.8191 - 1.49058 I}, {w0 -> 3.8191 + 1.49058 I},

  {w0 -> 6.63099}}

Solve[x[w0,3.2126,1,.0079,1]==0,w0]

{{w0 -> -3.02079 - 1.5165 I}, {w0 -> -3.02079 + 1.5165 I},

  {w0 -> 3.53033 - 1.39974 I}, {w0 -> 3.53033 + 1.39974 I},

  {w0 -> 62.2721}}

Solve[x[w0,3.2126,1,.1107,1]==0,w0]

{{w0 -> -3.1098 - 1.53296 I}, {w0 -> -3.1098 + 1.53296 I},

  {w0 -> 3.57587 - 2.05112 I}, {w0 -> 3.57587 + 2.05112 I},

  {w0 -> 3.58458}}

Solve[x[w0,3.2126,1,.0088,1]==0,w0]
```

```
{{w0 -> -3.02192 - 1.51697 I}, {w0 -> -3.02192 + 1.51697 I},

  {w0 -> 3.53276 - 1.39974 I}, {w0 -> 3.53276 + 1.39974 I},

  {w0 -> 55.7965}}

Solve[x[w0,3.2126,1,.0157,1]==0,w0]

{{w0 -> -3.03032 - 1.52019 I}, {w0 -> -3.03032 + 1.52019 I},

  {w0 -> 3.55292 - 1.40021 I}, {w0 -> 3.55292 + 1.40021 I},

  {w0 -> 30.8019}}

Solve[x[w0,3.2126,1,.0131,1]==0,w0]

{{w0 -> -3.02721 - 1.51904 I}, {w0 -> -3.02721 + 1.51904 I},

  {w0 -> 3.545 - 1.39994 I}, {w0 -> 3.545 + 1.39994 I},

  {w0 -> 37.1324}}

Solve[x[w0,3.2126,1,.0116,1]==0,w0]

{{w0 -> -3.02538 - 1.51834 I}, {w0 -> -3.02538 + 1.51834 I},

  {w0 -> 3.54061 - 1.39984 I}, {w0 -> 3.54061 + 1.39984 I},

  {w0 -> 42.073}}

Solve[x[w0,3.2126,1,.0107,1]==0,w0]

{{w0 -> -3.02428 - 1.51791 I}, {w0 -> -3.02428 + 1.51791 I},

  {w0 -> 3.53804 - 1.39979 I}, {w0 -> 3.53804 + 1.39979 I},

  {w0 -> 45.7014}}
```

```
Solve[x[w0,2.05,1,.0024,1] == 0, w0]

{{w0 -> -1.86264 - 1.52323 I}, {w0 -> -1.86264 + 1.52323 I},

  {w0 -> 2.36511 - 1.34634 I}, {w0 -> 2.36511 + 1.34634 I},

  {w0 -> 207.328}}

Solve[x[w0,2.05,1,.0113,1] == 0, w0]
```

```
{{w0 -> -1.87014 - 1.52759 I}, {w0 -> -1.87014 + 1.52759 I},
  {w0 -> 2.38316 - 1.34638 I}, {w0 -> 2.38316 + 1.34638 I},
  {w0 -> 43.2218}}
Solve[x[w0,2.05,1,.0010,1] == 0, w0]
{{w0 -> -1.86145 - 1.52249 I}, {w0 -> -1.86145 + 1.52249 I},
  {w0 -> 2.36246 - 1.34639 I}, {w0 -> 2.36246 + 1.34639 I},
  {w0 -> 498.998}}
Solve[x[w0,2.05,1,.0262,1] == 0, w0]
{{w0 -> -1.88236 - 1.53359 I}, {w0 -> -1.88236 + 1.53359 I},
  {w0 -> 2.41889 - 1.34869 I}, {w0 -> 2.41889 + 1.34869 I},
  {w0 -> 18.0109}}
Solve[x[w0,2.05,1,.0011,1] == 0, w0]
{{w0 -> -1.86153 - 1.52254 I}, {w0 -> -1.86153 + 1.52254 I},
  {w0 -> 2.36265 - 1.34639 I}, {w0 -> 2.36265 + 1.34639 I},
  {w0 -> 453.543}}
Solve[x[w0,2.05,1,.0512,1] == 0, w0]
{{w0 -> -1.90157 - 1.54061 I}, {w0 -> -1.90157 + 1.54061 I},
  {w0 -> 2.50124 - 1.36606 I}, {w0 -> 2.50124 + 1.36606 I},
  {w0 -> 8.56627}}
Solve[x[w0,2.05,1,.0012,1] == 0, w0]
{{w0 -> -1.86162 - 1.52259 I}, {w0 -> -1.86162 + 1.52259 I},
  {w0 -> 2.36284 - 1.34639 I}, {w0 -> 2.36284 + 1.34639 I},
  {w0 -> 415.664}}
Solve[x[w0,2.05,1,.1012,1] == 0, w0]
{{w0 -> -1.93417 - 1.54657 I}, {w0 -> -1.93417 + 1.54657 I},
  {w0 -> 2.69273 - 1.67056 I}, {w0 -> 2.69273 + 1.67056 I},
  {w0 -> 3.42359}}
```

```
Solve[x[w0,2.05,1,.0013,1] == 0, w0]

{{w0 -> -1.8617 - 1.52265 I}, {w0 -> -1.8617 + 1.52265 I},

  {w0 -> 2.36302 - 1.34638 I}, {w0 -> 2.36302 + 1.34638 I},

  {w0 -> 383.613}}
```

```
Solve[x[w0,2.5,1,.0151,1] == 0, w0]

{{w0 -> -2.3206 - 1.52454 I}, {w0 -> -2.3206 + 1.52454 I},

  {w0 -> 2.83992 - 1.37339 I}, {w0 -> 2.83992 + 1.37339 I},

  {w0 -> 32.0739}}

Solve[x[w0,2.5,1,.0216,1] == 0, w0]

{{w0 -> -2.32679 - 1.52717 I}, {w0 -> -2.32679 + 1.52717 I},

  {w0 -> 2.85748 - 1.37449 I}, {w0 -> 2.85748 + 1.37449 I},

  {w0 -> 22.0868}}

Solve[x[w0,2.5,1,.0035,1] == 0, w0]

{{w0 -> -2.30911 - 1.51892 I}, {w0 -> -2.30911 + 1.51892 I},

  {w0 -> 2.81282 - 1.37296 I}, {w0 -> 2.81282 + 1.37296 I},

  {w0 -> 141.85}}

Solve[x[w0,2.5,1,.0347,1] == 0, w0]

{{w0 -> -2.33868 - 1.53149 I}, {w0 -> -2.33868 + 1.53149 I},

  {w0 -> 2.89966 - 1.37987 I}, {w0 -> 2.89966 + 1.37987 I},

  {w0 -> 13.2873}}

Solve[x[w0,2.5,1,.0055,1] == 0, w0]

{{w0 -> -2.31113 - 1.51998 I}, {w0 -> -2.31113 + 1.51998 I},

  {w0 -> 2.81715 - 1.37293 I}, {w0 -> 2.81715 + 1.37293 I},

  {w0 -> 89.8971}}
```

```
Solve[x[w0,2.5,1,.0587,1] == 0, w0]

{{w0 -> -2.35837 - 1.53665 I}, {w0 -> -2.35837 + 1.53665 I},

  {w0 -> 3.01251 - 1.41626 I}, {w0 -> 3.01251 + 1.41626 I},

  {w0 -> 7.20961}}

Solve[x[w0,2.5,1,.0064,1] == 0, w0]

{{w0 -> -2.31203 - 1.52045 I}, {w0 -> -2.31203 + 1.52045 I},

  {w0 -> 2.81914 - 1.37292 I}, {w0 -> 2.81914 + 1.37292 I},

  {w0 -> 77.1108}}

Solve[x[w0,2.5,1,.1081,1] == 0, w0]

{{w0 -> -2.39095 - 1.54021 I}, {w0 -> -2.39095 + 1.54021 I},

  {w0 -> 3.04921 - 1.88272 I}, {w0 -> 3.04921 + 1.88272 I},

  {w0 -> 3.30883}}

Solve[x[w0,2.5,1,.0070,1] == 0, w0]

{{w0 -> -2.31263 - 1.52075 I}, {w0 -> -2.31263 + 1.52075 I},

  {w0 -> 2.82049 - 1.37293 I}, {w0 -> 2.82049 + 1.37293 I},

  {w0 -> 70.4129}}

Solve[x[w0,2.5,1,.0116,1] == 0, w0]

{{w0 -> -2.31719 - 1.52298 I}, {w0 -> -2.31719 + 1.52298 I},

  {w0 -> 2.83121 - 1.37309 I}, {w0 -> 2.83121 + 1.37309 I},

  {w0 -> 42.0754}}

Solve[x[w0,2.5,1,.0097,1] == 0, w0]

{{w0 -> -2.31532 - 1.52208 I}, {w0 -> -2.31532 + 1.52208 I},

  {w0 -> 2.82669 - 1.37299 I}, {w0 -> 2.82669 + 1.37299 I},

  {w0 -> 50.5237}}

Solve[x[w0,2.5,1,.0087,1] == 0, w0]

{{w0 -> -2.31433 - 1.5216 I}, {w0 -> -2.31433 + 1.5216 I},

  {w0 -> 2.82436 - 1.37296 I}, {w0 -> 2.82436 + 1.37296 I},
```

```
  {w0 -> 56.4512}}

Solve[x[w0,2.5,1,.0081,1] == 0, w0]

{{w0 -> -2.31373 - 1.5213 I}, {w0 -> -2.31373 + 1.5213 I},

  {w0 -> 2.82298 - 1.37294 I}, {w0 -> 2.82298 + 1.37294 I},

  {w0 -> 60.7099}}




Solve[x[w0,4.5,1,.0158,1] == 0, w0]

{{w0 -> -4.31493 - 1.51498 I}, {w0 -> -4.31493 + 1.51498 I},

  {w0 -> 4.84354 - 1.42533 I}, {w0 -> 4.84354 + 1.42533 I},

  {w0 -> 30.5884}}

Solve[x[w0,4.5,1,.0222,1] == 0, w0]

{{w0 -> -4.32463 - 1.51742 I}, {w0 -> -4.32463 + 1.51742 I},

  {w0 -> 4.87401 - 1.42659 I}, {w0 -> 4.87401 + 1.42659 I},

  {w0 -> 21.4238}}

Solve[x[w0,4.5,1,.0035,1] == 0, w0]

{{w0 -> -4.29402 - 1.50847 I}, {w0 -> -4.29402 + 1.50847 I},

  {w0 -> 4.79808 - 1.42572 I}, {w0 -> 4.79808 + 1.42572 I},

  {w0 -> 141.849}}

Solve[x[w0,4.5,1,.0352,1] == 0, w0]

{{w0 -> -4.34216 - 1.52092 I}, {w0 -> -4.34216 + 1.52092 I},

  {w0 -> 4.95959 - 1.43696 I}, {w0 -> 4.95959 + 1.43696 I},

  {w0 -> 12.9697}}

Solve[x[w0,4.5,1,.0056,1] == 0, w0]

{{w0 -> -4.29782 - 1.50978 I}, {w0 -> -4.29782 + 1.50978 I},

  {w0 -> 4.80488 - 1.42549 I}, {w0 -> 4.80488 + 1.42549 I},
```

```
  {w0 -> 88.2716}}

Solve[x[w0,4.5,1,.0591,1] == 0, w0]

{{w0 -> -4.36818 - 1.52403 I}, {w0 -> -4.36818 + 1.52403 I},

  {w0 -> 5.29308 - 1.67223 I}, {w0 -> 5.29308 + 1.67223 I},

  {w0 -> 6.61044}}

Solve[x[w0,4.5,1,.0067,1] == 0, w0]

{{w0 -> -4.29977 - 1.51043 I}, {w0 -> -4.29977 + 1.51043 I},

  {w0 -> 4.80858 - 1.42539 I}, {w0 -> 4.80858 + 1.42539 I},

  {w0 -> 73.6092}}

Solve[x[w0,4.5,1,.1085,1] == 0, w0]

{{w0 -> -4.40487 - 1.52435 I}, {w0 -> -4.40487 + 1.52435 I},

  {w0 -> 4.33254}, {w0 -> 4.54274 - 2.14273 I},

  {w0 -> 4.54274 + 2.14273 I}}

Solve[x[w0,4.5,1,.0073,1] == 0, w0]

{{w0 -> -4.30082 - 1.51078 I}, {w0 -> -4.30082 + 1.51078 I},

  {w0 -> 4.81064 - 1.42534 I}, {w0 -> 4.81064 + 1.42534 I},

  {w0 -> 67.4735}}

Solve[x[w0,4.5,1,.0122,1] == 0, w0]

{{w0 -> -4.30913 - 1.51335 I}, {w0 -> -4.30913 + 1.51335 I},

  {w0 -> 4.8287 - 1.42516 I}, {w0 -> 4.8287 + 1.42516 I},

  {w0 -> 39.9445}}

Solve[x[w0,4.5,1,.0102,1] == 0, w0]

{{w0 -> -4.3058 - 1.51235 I}, {w0 -> -4.3058 + 1.51235 I},

  {w0 -> 4.82105 - 1.42519 I}, {w0 -> 4.82105 + 1.42519 I},

  {w0 -> 47.9891}}

Solve[x[w0,4.5,1,.0091,1] == 0, w0]

{{w0 -> -4.30393 - 1.51178 I}, {w0 -> -4.30393 + 1.51178 I},
```

```
  {w0 -> 4.81701 - 1.42523 I}, {w0 -> 4.81701 + 1.42523 I},

  {w0 -> 53.9189}}

Solve[x[w0,4.5,1,.0085,1] == 0, w0]

{{w0 -> -4.3029 - 1.51145 I}, {w0 -> -4.3029 + 1.51145 I},

  {w0 -> 4.81485 - 1.42526 I}, {w0 -> 4.81485 + 1.42526 I},

  {w0 -> 57.7996}}
```

```
Solve[x[w0,6,1,.0108,1] == 0, w0]

{{w0 -> -5.8051 - 1.50864 I}, {w0 -> -5.8051 + 1.50864 I},

  {w0 -> 6.32572 - 1.43919 I}, {w0 -> 6.32572 + 1.43919 I},

  {w0 -> 45.2551}}

Solve[x[w0,6,1,.0178,1] == 0, w0]

{{w0 -> -5.81932 - 1.5118 I}, {w0 -> -5.81932 + 1.5118 I},

  {w0 -> 6.36625 - 1.43883 I}, {w0 -> 6.36625 + 1.43883 I},

  {w0 -> 26.996}}

Solve[x[w0,6,1,.0030,1] == 0, w0]

{{w0 -> -5.78704 - 1.50374 I}, {w0 -> -5.78704 + 1.50374 I},

  {w0 -> 6.29074 - 1.44073 I}, {w0 -> 6.29074 + 1.44073 I},

  {w0 -> 165.659}}

Solve[x[w0,6,1,.0315,1] == 0, w0]

{{w0 -> -5.84273 - 1.51569 I}, {w0 -> -5.84273 + 1.51569 I},

  {w0 -> 6.49167 - 1.44771 I}, {w0 -> 6.49167 + 1.44771 I},

  {w0 -> 14.5751}}

Solve[x[w0,6,1,.0043,1] == 0, w0]

{{w0 -> -5.79023 - 1.50468 I}, {w0 -> -5.79023 + 1.50468 I},
```

```
  {w0 -> 6.29599 - 1.44043 I}, {w0 -> 6.29599 + 1.44043 I},

  {w0 -> 115.268}}

Solve[x[w0,6,1,.0560,1] == 0, w0]

{{w0 -> -5.87393 - 1.51844 I}, {w0 -> -5.87393 + 1.51844 I},

  {w0 -> 6.84064 - 2.12761 I}, {w0 -> 6.84064 + 2.12761 I},

  {w0 -> 6.99514}}

Solve[x[w0,6,1,.0048,1] == 0, w0]

{{w0 -> -5.79144 - 1.50503 I}, {w0 -> -5.79144 + 1.50503 I},

  {w0 -> 6.29806 - 1.44032 I}, {w0 -> 6.29806 + 1.44032 I},

  {w0 -> 103.153}}

Solve[x[w0,6,1,.1057,1] == 0, w0]

{{w0 -> -5.91304 - 1.51802 I}, {w0 -> -5.91304 + 1.51802 I},

  {w0 -> 5.10211}, {w0 -> 5.72717 - 2.0045 I},

  {w0 -> 5.72717 + 2.0045 I}}

Solve[x[w0,6,1,.0051,1] == 0, w0]

{{w0 -> -5.79215 - 1.50523 I}, {w0 -> -5.79215 + 1.50523 I},

  {w0 -> 6.29932 - 1.44025 I}, {w0 -> 6.29932 + 1.44025 I},

  {w0 -> 97.0249}}

Solve[x[w0,6,1,.0078,1] == 0, w0]

{{w0 -> -5.79845 - 1.50696 I}, {w0 -> -5.79845 + 1.50696 I},

  {w0 -> 6.31121 - 1.4397 I}, {w0 -> 6.31121 + 1.4397 I},

  {w0 -> 63.077}}

Solve[x[w0,6,1,.0065,1] == 0, w0]

{{w0 -> -5.79546 - 1.50615 I}, {w0 -> -5.79546 + 1.50615 I},

  {w0 -> 6.30536 - 1.43995 I}, {w0 -> 6.30536 + 1.43995 I},

  {w0 -> 75.9033}}

Solve[x[w0,6,1,.0060,1] == 0, w0]
```

```
{{w0 -> -5.79429 - 1.50583 I}, {w0 -> -5.79429 + 1.50583 I},
  {w0 -> 6.30317 - 1.44006 I}, {w0 -> 6.30317 + 1.44006 I},
  {w0 -> 82.3156}}
Solve[x[w0,6,1,.0057,1] == 0, w0]
{{w0 -> -5.79358 - 1.50563 I}, {w0 -> -5.79358 + 1.50563 I},
  {w0 -> 6.30187 - 1.44012 I}, {w0 -> 6.30187 + 1.44012 I},
  {w0 -> 86.7027}}
```

```
Solve[x[w0,10,1,.0046,1] == 0, w0]
{{w0 -> -9.78781 - 1.49874 I}, {w0 -> -9.78781 + 1.49874 I},
  {w0 -> 10.2969 - 1.45544 I}, {w0 -> 10.2969 + 1.45544 I},
  {w0 -> 107.677}}
Solve[x[w0,10,1,.0128,1] == 0, w0]
{{w0 -> -9.81611 - 1.50406 I}, {w0 -> -9.81611 + 1.50406 I},
  {w0 -> 10.3688 - 1.44981 I}, {w0 -> 10.3688 + 1.44981 I},
  {w0 -> 37.9572}}
Solve[x[w0,10,1,.0018,1] == 0, w0]
{{w0 -> -9.77631 - 1.49611 I}, {w0 -> -9.77631 + 1.49611 I},
  {w0 -> 10.2788 - 1.45716 I}, {w0 -> 10.2788 + 1.45716 I},
  {w0 -> 276.773}}
Solve[x[w0,10,1,.0275,1] == 0, w0]
{{w0 -> -9.85274 - 1.50859 I}, {w0 -> -9.85274 + 1.50859 I},
  {w0 -> 10.6996 - 1.45037 I}, {w0 -> 10.6996 + 1.45037 I},
  {w0 -> 16.4881}}
Solve[x[w0,10,1,.0021,1] == 0, w0]
```

```
{{w0 -> -9.7776 - 1.49642 I}, {w0 -> -9.7776 + 1.49642 I},

  {w0 -> 10.2806 - 1.45698 I}, {w0 -> 10.2806 + 1.45698 I},

  {w0 -> 237.089}}

Solve[x[w0,10,1,.0524,1] == 0, w0]

{{w0 -> -9.89151 - 1.51053 I}, {w0 -> -9.89151 + 1.51053 I},

  {w0 -> 9.71204}, {w0 -> 9.80648 - 2.62799 I},

  {w0 -> 9.80648 + 2.62799 I}}

Solve[x[w0,10,1,.0022,1] == 0, w0]

{{w0 -> -9.77802 - 1.49652 I}, {w0 -> -9.77802 + 1.49652 I},

  {w0 -> 10.2812 - 1.45692 I}, {w0 -> 10.2812 + 1.45692 I},

  {w0 -> 226.266}}

Solve[x[w0,10,1,.1023,1] == 0, w0]

{{w0 -> -9.93135 - 1.50953 I}, {w0 -> -9.93135 + 1.50953 I},

  {w0 -> 5.18128}, {w0 -> 9.7845 - 1.56704 I},

  {w0 -> 9.7845 + 1.56704 I}}

Solve[x[w0,10,1,.0022,1] == 0, w0]

{{w0 -> -9.77802 - 1.49652 I}, {w0 -> -9.77802 + 1.49652 I},

  {w0 -> 10.2812 - 1.45692 I}, {w0 -> 10.2812 + 1.45692 I},

  {w0 -> 226.266}}

Solve[x[w0,10,1,.0028,1] == 0, w0]

{{w0 -> -9.78054 - 1.49711 I}, {w0 -> -9.78054 + 1.49711 I},

  {w0 -> 10.285 - 1.45656 I}, {w0 -> 10.285 + 1.45656 I},

  {w0 -> 177.563}}

Solve[x[w0,10,1,.0025,1] == 0, w0]

{{w0 -> -9.77929 - 1.49682 I}, {w0 -> -9.77929 + 1.49682 I},

  {w0 -> 10.2831 - 1.45674 I}, {w0 -> 10.2831 + 1.45674 I},

  {w0 -> 198.992}}
```

```
Solve[x[w0,10,1,.0024,1] == 0, w0]

{{w0 -> -9.77887 - 1.49672 I}, {w0 -> -9.77887 + 1.49672 I},

  {w0 -> 10.2825 - 1.4568 I}, {w0 -> 10.2825 + 1.4568 I},

  {w0 -> 207.326}}

Solve[x[w0,10,1,.0023,1] == 0, w0]

{{w0 -> -9.77844 - 1.49662 I}, {w0 -> -9.77844 + 1.49662 I},

  {w0 -> 10.2818 - 1.45686 I}, {w0 -> 10.2818 + 1.45686 I},

  {w0 -> 216.385}}
```

Appendix C

Matlab Script for plot of $\tilde{\Omega}$ versus $\tilde{\zeta}$

C.1 Figure 3-1 where $|\tilde{B}_x| = 1,\ |\tilde{H}_z| = 0$.

```
function [x_p, x_m, Om_p_p, Om_p_m, Om_m_p, Om_m_m ] = zeta_omega10

% This is a plot of Omega versus zeta for the given case.
% It is for both the positive and negative root of the quadratic.

% CASE:  |Bx| == 1; |Hz| == 0

global chi
chi = 1;

clf
hold on
axis([0 .081 0 5]);

M = moviein(9);

for i = 1:9
  a = 0;
  b = 0;
  c = 0;
  Om_sq_p = 0;
  Om_sq_m = 0;
  x_p = 0;
  x_m = 0;
  Om_p_p = 0;
  Om_p_m = 0;
  Om_m_p = 0;
  Om_m_m = 0;

  v = [-.1 -.05 -.025 -.01 0 .01 .025 .05 .1];
  eta_eff = v(i);
```

```
% eta_eff = input('What value for eta_eff?  ');
% chi = input('What value for chi?  ');
  counter = 0;

  for zeta = 0.001:0.00001:.13
    counter = counter + 1;

    a = 2*zeta*(zeta - eta_eff);
    b = (chi^2 + 2*chi + 2)*(2*zeta)*(zeta - eta_eff) - (chi/2)*(2*zeta - eta_eff);
    c = (chi^2 + 2*chi + 1)*2*zeta*(zeta - eta_eff) + (chi/2)*(2*zeta - eta_eff);

    Om_sq_p = (-b + sqrt(b^2 - 4*a*c))/(2*a);
    Om_sq_m = (-b - sqrt(b^2 - 4*a*c))/(2*a);

    if ((Om_sq_p > 0) & (imag(Om_sq_p) == 0) & (Om_sq_p ~= 0))
x_p(counter) = zeta;
Om_p_p(counter) = sqrt(Om_sq_p);
    else
x_p(counter) = zeta;
Om_p_p(counter) = NaN;
    end
    if ((Om_sq_m > 0) & (imag(Om_sq_m) == 0) & (Om_sq_m ~= 0))
x_m(counter) = zeta;
Om_m_p(counter) = sqrt(Om_sq_m);
    else
x_m(counter) = zeta;
Om_m_p(counter) = NaN;
    end
  end

  h = plot(x_p, Om_p_p, 'y-');
  get (h);
  set (h, 'LineWidth', 1.2);
  hold on
  j = plot(x_m, Om_m_p, 'g-');
  get (j);
  set (j, 'LineWidth', .3);

  M(:,i) = getframe;
end

movie(M)

hold on
count = 0;
  for xgo = .065:.0001:.069
count = count + 1;
xtry(count) = xgo;
ytry1(count) = 4.5;
ytry2(count) = 4.0;
  end
htry1 = plot(xtry, ytry1, 'y-');
htry2 = plot(xtry, ytry2, 'g-');
get (htry1);
```

```
set (htry1, 'LineWidth', 1.2);
hold on
get (htry2);
set (htry2, 'LineWidth', .3);
hold on



stitle('\18\times |B_x| = 1, |H_z| = 0  \15\times for given \eta_{eff}');

sxlabel('\down{10} \15\times \zeta\Tilde')
sylabel('\up{10} \15\times \Omega\Tilde')

stext(.07, 4.5, '\14\times + root');
stext(.07, 4.0, '\14\times - root');

stext(.076, 0.73, '\10\times .1');
stext(.0735, 2.0, '\10\times .05');
stext(.051, 2.0, '\10\times .025');
stext(.042, 2.0, '\10\times .01');
stext(.0375, 2.0, '\10\times 0');
stext(.0323, 2.0, '\10\times -.01');
stext(.0262, 2.0, '\10\times -.025');
stext(.02215, 2.0, '\10\times -.05');
stext(.01865, 2.0, '\10\times -.1');
stext(.0154, 2.0, '\10\time .1');
stext(.0114, 2.0, '\10\time .05');
stext(.0064, 2.0, '\10\time .025');
stext(.0024, 2.0, '\10\time .01');

grid on
```

C.2 Figure 4-1 where $|\tilde{B}_x| = 0,\ |\tilde{H}_z| = 1.$

```
function [x_p, x_m, Om_p_p, Om_p_m, Om_m_p, Om_m_m ] = zeta_omega01

% This is a plot of Omega versus zeta for the given case.
% It is for both the positive and negative root of the quadratic.

% CASE:  |Bx| == 0; |Hz| == 1

global chi
chi = 1;

clf
hold on
axis([0 .066 0 6]);

M = moviein(8);

for i = 1:8
  a = 0;
  b = 0;
  c = 0;
  Om_sq_p = 0;
  Om_sq_m = 0;
  x_p = 0;
  x_m = 0;
  Om_p_p = 0;
  Om_p_m = 0;
  Om_m_p = 0;
  Om_m_m = 0;

  v = [-.1 -.025 -.01 0 .01 .025 .05 .1];
  eta_eff = v(i);
%  eta_eff = input('What value for eta_eff?  ');
%  chi = input('What value for chi?  ');
  counter = 0;

  for zeta = 0.001:0.00005:.066
    counter = counter + 1;

    alpha = (2*zeta*(zeta - eta_eff))/(2*zeta - eta_eff);

    a = 1;
    b = 2 + 2*chi + chi^2 - chi/(2*alpha);
    c = chi^2 + 2*chi + 1 + (chi/(2*alpha))*(1 + chi)^2;

    Om_sq_p = (-b + sqrt(b^2 - 4*a*c))/(2*a);
    Om_sq_m = (-b - sqrt(b^2 - 4*a*c))/(2*a);

    if ((Om_sq_p > 0) & (imag(Om_sq_p) == 0) & (Om_sq_p ~= 0))
x_p(counter) = zeta;
Om_p_p(counter) = sqrt(Om_sq_p);
    else
```

```
x_p(counter) = zeta;
Om_p_p(counter) = NaN;
    end
    if ((Om_sq_m > 0) & (imag(Om_sq_m) == 0) & (Om_sq_m ~= 0))
x_m(counter) = zeta;
Om_m_p(counter) = sqrt(Om_sq_m);
    else
x_m(counter) = zeta;
Om_m_p(counter) = NaN;
    end
  end

  h = plot(x_p, Om_p_p, 'y-');
  get (h);
  set (h, 'LineWidth', 1.2);
  hold on
  j = plot(x_m, Om_m_p, 'g-');
  get (j);
  set (j, 'LineWidth', .3);

  M(:,i) = getframe;
end

movie(M)


stitle('\18\times |B_x| = 0, |H_z| = 1  \15\times for given \eta_{eff}');

sxlabel('\down{10} \15\times \zeta\Tilde')
sylabel('\up{10} \15\times \Omega\Tilde')

stext(.04, 5.0, '\14\times \bold{bold} \normal line pos. root');
stext(.04, 4.5, '\14\times \light{light} \normal line neg. root');

stext(.062, 1.1, '\10\times .1');
stext(.06, 3.18, '\10\times .05');
stext(.037, 3.18, '\10\times .025');
stext(.0232, 3.18, '\10\times .01');
stext(.019, 3.18, '\10\times 0');
stext(.014, 3.18, '\10\times -.01');
stext(.0107, 3.4, '\10\times -.025');
%stext(.0095, 3.18, '\10\times -.05');
stext(.0093, 3.18, '\10\times -.1');
stext(.0085, 3.4, '\10\time .1');
stext(.0065, 3.09, '\10\time .05');
stext(.0047, 3.4, '\10\time .025');
stext(.0024, 3.18, '\10\time .01');

grid on;
```

C.3 Figure 5-1 where $|\tilde{B}_x| = 1$, $|\tilde{H}_z| = 1$.

```
function [x_p, x_m, Om_p_p, Om_p_m, Om_m_p, Om_m_m ] = zeta_omega11

% This is a plot of Omega versus zeta for the given case.
% It is for both the positive and negative root of the quadratic.

% CASE:  |Bx| == 1; |Hz| == 1

global chi
chi = 1;

clf
hold on
axis([0 .1 0 8]);

M = moviein(9);

for i = 1:9
  a = 0;
  b = 0;
  c = 0;
  Om_sq_p = 0;
  Om_sq_m = 0;
  x_p = 0;
  x_m = 0;
  Om_p_p = 0;
  Om_p_m = 0;
  Om_m_p = 0;
  Om_m_m = 0;

  v = [-.1 -.05 -.025 -.01 0 .01 .025 .05 .1];
  eta_eff = v(i);
%  eta_eff = input('What value for eta_eff?  ');
%  chi = input('What value for chi?  ');
  counter = 0;

  for zeta = 0.001:0.0001:.1
    counter = counter + 1;

    alpha = (2*zeta*(zeta - eta_eff))/(2*zeta - eta_eff);

    a = 1;
    b = 2*(1 + chi) + chi^2 - chi/(alpha);
    c = chi^2 + 2*chi + 1 + chi/(2*alpha) + (chi/(2*alpha))*(1 + chi)^2;

    Om_sq_p = (-b + sqrt(b^2 - 4*a*c))/(2*a);
    Om_sq_m = (-b - sqrt(b^2 - 4*a*c))/(2*a);

    if ((Om_sq_p > 0) & (imag(Om_sq_p) == 0) & (Om_sq_p ~= 0))
x_p(counter) = zeta;
Om_p_p(counter) = sqrt(Om_sq_p);
    else
```

```
x_p(counter) = zeta;
Om_p_p(counter) = NaN;
    end
    if ((Om_sq_m > 0) & (imag(Om_sq_m) == 0) & (Om_sq_m ~= 0))
x_m(counter) = zeta;
Om_m_p(counter) = sqrt(Om_sq_m);
    else
x_m(counter) = zeta;
Om_m_p(counter) = NaN;
    end
  end

  h = plot(x_p, Om_p_p, 'y-');
  get (h);
  set (h, 'LineWidth', 1.2);
  hold on
  j = plot(x_m, Om_m_p, 'g-');
  get (j);
  set (j, 'LineWidth', .3);

  M(:,i) = getframe;
end

movie(M)

hold on
count = 0;
  for xgo = 0.07:.0001:.074
count = count + 1;
xtry(count) = xgo;
ytry1(count) = 6.5;
ytry2(count) = 6.0;
  end
htry1 = plot(xtry, ytry1, 'y-');
htry2 = plot(xtry, ytry2, 'g-');
get (htry1);
set (htry1, 'LineWidth', 1.2);
hold on
get (htry2);
set (htry2, 'LineWidth', .3);
hold on



stitle('\18\times |B_x| = 1, |H_z| = 1  \15\times for given \eta_{eff}');

sxlabel('\down{10} \15\times \zeta\Tilde')
sylabel('\up{10} \15\times \Omega\Tilde')

stext(.075, 6.5, '\14\times positive root');
stext(.075, 6.0, '\14\times negative root');

stext(.095, 1.8, '\10\times .1');
stext(.085, 2.7, '\10\times .05');
```

```
stext(.065, 2.7, '\10\times .025');
stext(.055, 2.7, '\10\times .01');
stext(.05, 2.7, '\10\times 0');
stext(.0445, 2.7, '\10\times -.01');
stext(.038, 2.7, '\10\times -.025');
stext(.033, 2.7, '\10\times -.05');
stext(.0295, 2.7, '\10\times -.1');
stext(.019, 2.7, '\10\time .1');
stext(.014, 2.7, '\10\time .05');
stext(.008, 2.7, '\10\time .025');
stext(.003, 2.7, '\10\time .01');

grid on
```

Appendix D

Matlab Script Calculating $\tilde{\zeta}$ given $\tilde{\Omega}$ and $\tilde{\eta}_{eff}$

NOTE...for all scripts, a value of $\chi_0 = 1$ was used.

D.1 Finding $\tilde{\zeta}$ if $|\tilde{B}_x| = 1,\ |\tilde{H}_z| = 0$

```
function [] = calc

% Given a certain eta_effective and a certain Omega, what is zeta?
%  (chose calculating zeta because Omega is more controlable?)

% For CASE |Bx| = 1,  |Hz| = 0;

Om = input('What value for Omega?  ');
eta_eff = input('What value for eta_effective?  ');
chi = input('What value for chi?  ');

a = 2*((1 + chi + Om^2)^2 + chi^2*Om^2);
b = -(2*eta_eff*((1 + chi + Om^2)^2 + chi^2 + Om^2) + chi*(Om^2 - 1));
c = (chi/2)*eta_eff*(Om^2 - 1);

zeta_p = (-b + sqrt(b^2 - 4*a*c))/(2*a);
zeta_m = (-b - sqrt(b^2 - 4*a*c))/(2*a);

if ((imag(zeta_p) == 0) & (zeta_p >= 0))
zeta_p
else
'zeta (plus root) was not positive, real'
end

if ((imag(zeta_m) == 0) & (zeta_m >= 0))
```

```
zeta_m
else
'zeta (minus root) was not postive, real'
end
```

D.2 Finding $\tilde{\zeta}$ if $|\tilde{B}_x| = 0,\ |\tilde{H}_z| = 1$

```
function [] = calc

% Given a certain eta_effective and a certain Omega, what is zeta?
%  (chose calculating zeta because Omega is more controlable?)

% For CASE |Bx| = 0,  |Hz| = 1;

Om = input('What value for Omega?  ');
eta_eff = input('What value for eta_effective?  ');
chi = input('What value for chi?  ');

a = 2*((1 + chi + Om^2)^2 + chi^2*Om^2);
b = -(2*eta_eff*((1 + chi + Om^2)^2 + chi^2*Om^2) + chi*(Om^2 - (1 + chi)^2));
c = (chi/2)*eta_eff*(Om^2 - (1 + chi)^2);

zeta_p = (-b + sqrt(b^2 - 4*a*c))/(2*a);
zeta_m = (-b - sqrt(b^2 - 4*a*c))/(2*a);

if ((imag(zeta_p) == 0) & (zeta_p >= 0))
zeta_p
else
'zeta (plus root) was not positive, real'
end

if ((imag(zeta_m) == 0) & (zeta_m >= 0))
zeta_m
else
'zeta (minus root) was not postive, real'
end
```

D.3 Finding $\tilde{\zeta}$ if $|\tilde{B}_x| = 1, \ |\tilde{H}_z| = 1$

```
function [] = calc

% Given a certain eta_effective and a certain Omega, what is zeta?
%  (chose calculating zeta because Omega is more controlable?)

% For CASE |Bx| = 1,  |Hz| = 1;

Om = input('What value for Omega?  ');
eta_eff = input('What value for eta_effective?  ');
chi = input('What value for chi?  ');

a = 2*((1+chi+Om^2)^2 + chi^2*Om^2);
b = -2*eta_eff*((1+chi+Om^2)^2 + chi^2*Om^2) - chi*((Om^2-1) + (Om^2 - (1+chi)^2));
c = (chi/2)*eta_eff*((Om^2-1) + (Om^2 - (1+chi)^2));

zeta_p = (-b + sqrt(b^2 - 4*a*c))/(2*a);
zeta_m = (-b - sqrt(b^2 - 4*a*c))/(2*a);

if ((imag(zeta_p) == 0) & (zeta_p >= 0))
zeta_p
else
'zeta (plus root) was not positive, real'
end

if ((imag(zeta_m) == 0) & (zeta_m >= 0))
zeta_m
else
'zeta (minus root) was not postive, real'
end
```

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