NETWORKS OF GAUSSIAN CHANNELS WITH APPLICATIONS TO FEEDBACK SYSTEMS

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Abstract

This paper discusses networks (directed graphs) having one input node, one output node, and an arbitrary number of intermediate nodes, whose branches are noisy communications channels, in which the input to each channel appears at its output corrupted by additive Gaussian noise. Each branch is labeled by a non-negative real parameter which specified how noisy it is. A branch originating at a node has as input a linear combination of the outputs of the branches terminating at that node.

The channel capacity of such a network is defined. Its value is bounded in terms of branch parameter values and procedures for computing values for general networks are described. Explicit solutions are given for the class D_0 which includes series-parallel and simple bridge networks and all other networks having r paths, b branches, and v nodes with $r = b - v + 2$, and for the class D_1 of networks which is inductively defined to include D_0 and all networks obtained by replacing a branch of a network in D_1 by a network in D_1.

The general results are applied to the particular networks which arise from the decomposition of a simple feedback system into successive forward and reverse (feedback) channels. When the feedback channels are noiseless, the capacities of the forward channels are shown to add. Some explicit expressions and some bounds are given for the case of noisy feedback channels.

Networks of Gaussian Channels with Applications to Feedback Systems

PETER ELIAS, FELLOW, IEEE

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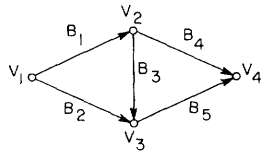
Introduction

The min-cut max-flow theorem[1]-[3] gives the capacity of a network made up of branches of given capacity. It applies to networks of noisy communications channels if the assumption is made that arbitrarily large delays and arbitrarily complex encoding and decoding operations may take place at each interior node.

This paper presents the theory of networks of another kind of channel—a channel with additive Gaussian noise, for which the only operation which takes place at a node is linear combination of the arriving signal and noise voltages, with no significant delay and no decoding or recoding.

The Problem

Consider the Class D of two-terminal networks like that shown in Fig. 1, in which there are no cycles, each of the b branches B_i is directed, and each branch lies on one of the r paths R_i which go from the input terminal on the left to the output terminal on the right. A signal voltage e_0 of mean-square value P_0 (the *signal power*) is applied to the input terminal, node V_1 at the left. At each interior node, the output (signal plus noise) of each branch B_i arriving at the node is given a (positive or negative) weight, the *branch transmission* t_i, and the resulting linear combination of signal and noise voltages is supplied as input to all the branches leaving that node.



PATHS $\begin{cases} R_1 = B_1B_4 \\ R_2 = B_1B_3B_5 \\ R_3 = B_2B_5 \end{cases}$ PATH TRANSMISSIONS $\begin{cases} T_1 = t_1t_4 \\ T_2 = t_1t_3t_5 \\ T_3 = t_2t_5 \end{cases}$

$$\|G_{ij}\| = \left\| \begin{matrix} (1+N_1)(1+N_4) & (1+N_1) & 1 \\ (1+N_1) & (1+N_1)(1+N_3)(1+N_5) & (1+N_5) \\ 1 & (1+N_5) & (1+N_2)(1+N_5) \end{matrix} \right\|$$

Fig. 1. A network in Class D.

Each branch B_i adds to its input voltage e_i a Gaussian noise voltage n_i whose mean-square value (the *noise power*) is a constant N_i (the *noise-to-signal power ratio*, also called the *parameter* of the branch) times the mean-square value P_i (the *input power*) of its input voltage. The noise voltage in each branch is statistically independent of the noise voltages in other branches and of the signal voltage:

$$\overline{e_i^2} = P_i, \quad 0 \leqq i \leqq b; \quad \overline{n_i^2} = N_iP_i, \quad 1 \leqq i \leqq b \qquad \overline{n_in_j} = 0, \quad i \neq j; \quad \overline{n_ie_0} = 0. \tag{1}$$

Since the branch input voltage and its noise are uncorrelated, the mean-square value of the branch output voltage (the *output power*) is just

$$\overline{(e_i + n_i)^2} = \overline{e_i^2} + \overline{n_i^2} = P_i + N_iP_i = P_i(1 + N_i). \tag{2}$$

The power output of each branch generator depends on the power level at its input, and thus on the power level of the signal and of all other noise generators which affect its input power, as well as on the values of the branch transmissions. However, once the power levels of the signal and of all noises and the values of the branch transmissions are fixed, the network is linear. The final output at the right-hand output terminal V_v is a linear combination of the b branch noise generator voltages and the signal voltage e_0. We constrain the values of the branch transmissions t_i by requiring that the coefficient of e_0 in this sum be unity.

The network is equivalent to a single branch (noisy channel) of the same kind as the component branches,

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since the linear combination of the b branch noise voltages which appears in the output is a Gaussian noise voltage, and the overall action of the two-terminal network is to receive an input signal and to produce at its output the input signal plus an independent Gaussian noise. The ratio of output noise power to signal power, N_{b+1}, is a function of the branch transmissions as well as the parameters N_i of the network branches. The *optimum noise-to-signal power* of the network, N_{opt}, is defined as the minimum value of N_{b+1} which can be obtained by varying the branch transmissions.

The problem is to find N_{opt} as a function of the given N_i.

Series and Parallel Networks

To express the results most simply in important special cases it is convenient to associate with each branch, not only the parameter N_i, but the *signal-to-noise ratio*,

$$S_i = 1/N_i, \tag{3}$$

and the *capacity* per use of the channel,

$$C_i = \tfrac{1}{2} \log (1 + S_i). \tag{4}$$

Equivalent quantities are defined for the network: S_{opt} is the maximum signal-to-noise ratio attainable by varying the branch transmissions, and C_{opt} is the largest channel capacity so attainable.

We can then state three results.

Series Networks

A network in D in which all b branches are in series has N_{opt} given by

$$1 + N_{opt} = \prod_{i=1}^{b} (1 + N_i). \tag{5}$$

Parallel Networks

A network in D in which all b branches are in parallel has S_{opt} given by

$$S_{opt} = \sum_{i=1}^{b} S_i. \tag{6}$$

Duality

Given two channels of capacities C_1 and C_2. Let the optimum capacity of the network consisting of the two channels in series be C_s. Let the optimum capacity of the two channels connected in parallel be C_p. Then

$$C_1 + C_2 = C_s + C_p. \tag{7}$$

The result on series networks expressed by (5) does not seem to have been published. The result for parallel branches expressed by (6) is known as optimum diversity combining, or the ratio squarer[4],[5] and was discovered independently of the general theory. Both follow directly from the general results following. The duality relationship of (7) follows directly from (4), (5), and (6), and also seems not to have been published. We have

$$\begin{aligned}
C_s &= \tfrac{1}{2} \log \left(1 + \frac{1}{N_s}\right) \\
&= \tfrac{1}{2} \log \left(1 + \frac{1}{(1 + N_1)(1 + N_2) - 1}\right) \\
&= \tfrac{1}{2} \log \left(\frac{(1 + N_1)(1 + N_2)}{N_1 + N_2 + N_1N_2}\right) \\
&= \tfrac{1}{2} \log \left(\frac{(1 + S_1)(1 + S_2)}{1 + S_1 + S_2}\right) \\
&= \tfrac{1}{2} \log (1 + S_1)(1 + S_2) - \tfrac{1}{2} \log (1 + S_1 + S_2) \\
&= C_1 + C_2 - C_p.
\end{aligned}$$

Equation (7), incidentally, also holds for other pairs of channels, such as two binary symmetric channels with different crossover probabilities p_1 and p_2, or a binary symmetric channel in series with a binary erasure channel and in parallel with it. However, the interpretation of parallel channels is different in those cases; it involves having the receiver observe the outputs of both channels when a common input symbol is applied to both. Since the output symbols of the two channels cannot be combined into an input symbol for the same kind of channel without loss of information, there is no tidy network theory for such channels and we discuss them no further.

Feedback Networks

The next results apply to a subset F of networks in D which represent a dissection in space of the time sequence of forward and return signal flows encountered in a feedback system, as shown in Figs. 2 and 3. The transmitter applies a signal voltage to the input node V_1. It proceeds over a noisy branch B_1 to node V_2 at the receiver. The receiver sends it back over B_2 to V_3 at the transmitter. The transmitter forms a linear combination of the original signal and the noisy version of it received at V_3 and transmits it over B_3 to V_4. In Fig. 2 the receiver then takes a linear combination of the outputs at V_2 and V_4 as the output voltage. In Fig. 3 the process continues. In both figures, and for all nets in F, the branches on the left connecting odd-numbered nodes and the branches on the right connecting even-numbered nodes are noiseless. They serve only to provide linear combinations of previously received values for the next transmission and to provide the requisite delay. Odd-numbered branches, from odd to even nodes, are called *forward* channels; even numbered branches, from even to odd nodes, are *feedback* channels.

The Uniform Delay Property

In practice, delays will be introduced by the forward and feedback channels. In order to avoid having signal voltage samples applied at different times getting mixed up at intermediated nodes, we assume that the noiseless branches on the left and the right have delays selected so as to give the network the *uniform delay property* that all paths connecting any two nodes have the same delay. Therefore, at any node only one signal sample and one sample of the output of each earlier noise generator will

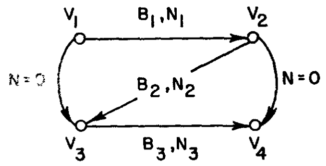

Fig. 2. A network in Class F for $k = 2$.

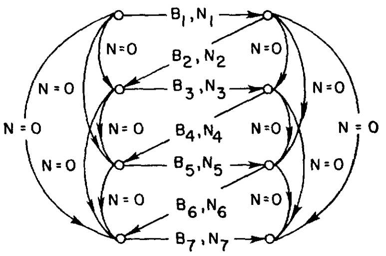

Fig. 3. A network in Class F for $k = 4$.

arrive at a given time over different paths. This can be accomplished for any network in F, or indeed in D, if an initial set of delay values d_i are given for the branches B_i, by increasing some of them in the following fashion. Assign a delay value to each node V_i equal to the maximum delay obtained by adding the delay values of the branches along each path from V_1 to V_i. Then assign to B_i the new delay value d_i' which is the difference between the delay values of its terminal and initial nodes, $d_i' \geqq d_i$.

We will henceforth assume that this process has been carried out for all networks in F or D, and that all have the uniform delay property. It is then not necessary to keep track of the delay values of networks or branches. We now state results for feedback networks.

Noiseless Feedback: For a network in F, if all feedback branches are noiseless, and the k forward branches have capacities C_{2j-1}, $1 \leqq j \leqq k$, then the optimum capacity of the network is given by

$$C_{\text{opt}} = \sum_{j=1}^{k} C_{2j-1} \tag{8}$$

and the optimum signal-to-noise ratio S_{opt} by

$$1 + S_{\text{opt}} = \prod_{j=1}^{k} (1 + S_{2j-1}). \tag{9}$$

In particular, if

$$S = \sum_{j=1}^{k} S_{2j-1}$$

is fixed, but an arbitrarily large k is available, we have

$$\begin{aligned} 1 + S_{\text{opt}} &= \lim_{k\to\infty} \prod_{j=1}^{k} \left(1 + \frac{S}{k}\right) = e^S, \\ S_{\text{opt}} &= e^S - 1. \end{aligned} \tag{10}$$

Noisy Feedback, $k = 2$: For a network in F with two noisy forward channels B_1 and B_3, and one noisy feedback channel B_2, the optimum signal-to-noise ratio is

$$S_{\text{opt}} = S_1 + S_3 + \frac{S_1 S_2 S_3}{(1 + S_1)(1 + S_3) + S_2}. \tag{11}$$

Unfortunately, a general formula like (11) for a net in F with $k > 2$ is not available, although the computation of S_{opt} for any particular case is a straightforward numerical analysis problem. However, we do have some inequalities which hold for all nets in F and which provide some insight.

Noisy Feedback, General Case: For a network in F with k noisy forward branches B_{2j-1}, $1 \leqq j \leqq k$, and $k - 1$ noisy feedback branches B_{2j}, $1 \leqq j \leqq k - 1$, the optimum signal to noise ratio S_{opt} is bounded by

$$S_{\text{opt}} \geqq \sum_{j=1}^{k} S_{2j-1} \tag{12}$$

$$1 + S_{\text{opt}} \leqq \prod_{j=1}^{k} (1 + S_{2j-1}) \tag{13}$$

$$S_{\text{opt}} \leqq \sum_{j=1}^{2k-1} S_j. \tag{14}$$

If signal-to-noise ratio costs c_1 per unit for forward channels and c_2 per unit for feedback channels, so that the total cost for a network in F is

$$c = c_1 \sum_{j=1}^{k} S_{2j-1} + c_2 \sum_{j=1}^{k-1} S_{2j},$$

then for sufficiently large S_{opt}, the cost per unit of S_{opt} may be made arbitrarily close to c_2:

$$\frac{c}{S_{\text{opt}}} \leqq c_2(1 + \delta). \tag{15}$$

The results for noiseless feedback and for noisy feedback with $k = 2$ were published by the author some time ago.[6],[7] Schalkwijk and Kailath have recently investigated the noiseless case from the point of view of error probability for the transmission of discrete messages.[8]–[10] Turin[14] has also dealt with a closely related question. The noiseless feedback results of (8) and (9) are remarkable, since they permit the transmission of a continuous signal of fixed bandwidth over a noisy channel at a rate equal to channel capacity, no matter how large the bandwidth of the forward channel. No coding or decoding is needed, provided that a noiseless feedback channel is available. Furthermore, they do so without introducing any of the discontinuities which must occur when a continuous signal is mapped onto a space of higher dimensionality—discontinuities which were pointed out by Shannon[11] and Kotelnikov,[12] and have recently been discussed by Wozencraft and Jacobs.[13] Equation (10) implies that a signal-to-noise ratio of 10 in bandwidth W is equivalent to a signal-to-noise ratio of $e^{10} - 1$, or about 22 000 if the available forward channel is wideband and has white noise, and a noiseless feedback channel is available (see the literature,[6],[7] for further discussion).

The inequality (12) follows from the parallel network result of (6). The result of setting all feedback channel transmissions at zero and using the forward channels in parallel gives the right side of (12). The optimum choice of branch transmissions must do at least as well. The second inequality, (13), says that noise in the feedback channels does not help; the right side is just the noiseless feedback result of (9). It is a consequence of a more general result which will be given, and which shows that increasing N_i in any branch cannot decrease N_{opt}. The third result, (14), says that, given a choice, it is better to use signal-to-noise ratio in the forward rather than the feedback channels. The total S_{opt} attainable by feedback is less than would be attained by taking all of the feedback channels, turning them around, and using them in parallel with the forward channels, which gives the right side of (14) by (6). This will also be derived later. The final result, (15), shows why feedback is interesting even if it does not do as well as the same amount of signal-to-noise ratio in the forward direction, by (14). Signal-to-noise ratio in the feedback direction may be cheaper, as when a satellite is communicating to Earth, and if it is, it is possible by means of feedback to buy forward signal-to-noise ratio at the same cost, if one wants enough of it. Equation (15) is a direct consequence of (11). It is necessary only to choose S_1 equal to S_3, and S_2 so large that it is possible to have $S_1 \ll S_2$ and $S_1^2 \gg S_2$ at the same time. For $k > 2$ the result will be of the same character, but better, i.e., a smaller δ will do. Or a smaller amount of S_{opt} can be bought at the same unit cost—but the absence of a formula makes the demonstration harder.

General Results

To state and prove the theorem from which the above results follow we need some further definitions. For each pair of paths R_i, R_j from V_1 to V_v in a network in D, we define G_{ij} as a product which contains one factor $(1 + N_k)$ for each branch B_k which lies in both paths; if R_i and R_j share no branches, $G_{ij} = 1$. Formally, if we treat the symbol R_i as denoting the set of branches which are contained in the ith path, then $R_i \cap R_j$ is the set of branches which the two paths have in common, and

$$G_{ij} = \prod_{k:B_k \epsilon R_i \cap R_j} (1 + N_k) \qquad (16)$$
$$= 1 \quad \text{for} \quad R_i \cap R_j \text{ empty}.$$

We also define the *path transmission* T_i of path R_i as the product of the branch transmissions t_k for those branches which lie on R_i:

$$T_i = \prod_{k:B_k \epsilon R_i} t_k. \qquad (17)$$

The *network transmission* $T_{0,b+1}$ is the sum of all path transmissions. By the assumption made in the discussion following (2), the branch transmissions t_k are constrained so that the network transmission, which is the coefficient of the signal voltage e_0 in the output, is unity.

$$T_{0,b+1} = \sum_{i=1}^{r} T_i = 1. \qquad (18)$$

Theorem

For any network in D, we have

$$1 + N_{opt} = \min_{t_k} \left\{ \sum_{i=1}^{r} \sum_{j=1}^{r} G_{ij} T_i T_j \right\} \geqq 1 \Big/ \left\{ \sum_{i=1}^{r} \sum_{j=1}^{r} G_{ij}^{-1} \right\} \qquad (19)$$

and

$$S_{opt} = 1/\min_{t_k} \left\{ \sum_{i=1}^{r} \sum_{j=1}^{r} (G_{ij} - 1) T_i T_j \right\}$$
$$\leqq \sum_{i=1}^{r} \sum_{j=1}^{r} [G_{ij} - 1]^{-1}, \qquad (20)$$

where the T_i are given in terms of the t_k by (17) and are subject to the constraint (18), and G_{ij}^{-1} and $[G_{ij} - 1]^{-1}$ are elements of the inverses of the matrices $||G_{ij}||$ and $||G_{ij} - 1||$. The inverses of $||G_{ij}||$ and $||G_{ij} - 1||$ always exist unless there is at least one noiseless path from input to output, so that some $G_{ii} = 1$. In this case $N_{opt} = 0$ and $S_{opt} = \infty$. These values are attained by setting $T_i = 1$ and all other $T_j = 0$, $j \neq i$.

Equality holds on the right in (19) and (20) for networks in the set D_0, which includes any network in D with r paths, b branches, and v nodes for which

$$r = b - v + 2, \qquad (21)$$

and for networks in the set D_1 which includes the networks in D_0 and, inductively, any network which is constructed from a network in D_1 by replacing any branch by another network in D_1.

Note that D_0 contains simple series networks, for which $r = 1$ and $b = v - 1$, and simple parallel networks, for which $r = b$ and $v = 2$. D_1 therefore contains all series-parallel networks, but it contains others as well—for example, the (topologically equivalent) networks of Figs. 1 and 2, for which $b = 5$, $v = 4$, and $r = 3$, but not the network of Fig. 3, for which $b = 11$, $v = 6$, and $r = 8$, or any network in F with $k > 2$.

Proof: For the proof we need one more definition. T_{ij}, the *transmission from branch i to branch j*, is just the network transmission as defined in (18) for the subnetwork consisting of branch i and all other branches which lie on some directed path which goes through branch i to the initial node of branch j. (Thus, B_i is included in the subnetwork, but B_j is not; and t_i is a common factor of all of the terms in the sum T_{ij}.) If there are no paths through B_i and B_j, or if B_j precedes B_i on such a path, then $T_{ij} = 0$. $T_{0,j}$ is the transmission of a subnetwork with input node V_1 and output node the initial node of B_j, and $T_{i,b+1}$ is the transmission of the subnetwork of paths through branch i to the output node V_v.

We now derive an expression for P_{b+1}, the output power of the network. By the statistical independence of the noise voltage generators from one another and from the signal source, the output power at the right-hand node is the sum of the powers transmitted to that node by these

$b + 1$ separate sources. The source in branch i contributes an amount of power equal to its generated power $P_i N_i$ times the square of the transmission from B_i to the output. Thus,

$$P_{b+1} = \sum_{i=0}^{b} P_i N_i T_{i,b+1}^2 = P_0(1 + N_{b+1}), \qquad (22)$$

where the right-most equality follows from the fact that by the constraint of (18), (2) holds for $i = b + 1$, and where the signal power contributed to the output is represented in the sum by the term for $i = 0$, with $N_0 = 1$ and $T_{0,b+1} = 1$.

Similarly the input power to any branch B_i may be expressed as the sum of the contributions of the generators which lie to its left:

$$P_i = \sum_{k=0}^{i-1} P_k N_k T_{ki}^2. \qquad (23)$$

Here we have assumed that the branches are numbered in an order such that if B_i precedes B_j on some directed path, $i < j$.

By successive substitution of (23) into (22) and in the resulting expressions, the subscripts on the P's appearing on the right can all be reduced to zero. The result is a sum of terms, all of which have P_0 as a factor. There is one term for each of the 2^b subsets W_m of the b branches which has the property that all of the branches in W_m are included in a path from input to output, i.e., that there is an integer f with $R_f \supseteq W_m$. If W_m is such a subset, say $W_m = (B_i, B_j, B_k)$ with $i < j < k$, then the corresponding term is

$$P_0 T_{0i}^2 N_i T_{ij}^2 N_j T_{jk}^2 N_k T_{k,b+1}^2 = P_0(T_{0i} T_{ij} T_{jk} T_{k,b+1})^2 N_i N_j N_k. \qquad (24)$$

The product of the transmission terms which appears on the right is just the sum of the transmissions of all paths from input to output which include all three of the branches B_i, B_j, B_k. If there are no such paths, then one or more of the T_{ij} in (24) will vanish. Thus the output power is expressed in terms of the path transmissions T_i and the branch parameters N_i. Dividing through by P_0 gives an expression for $1 + N_{b+1}$

$$1 + N_{b+1} = \sum_{k=0}^{2^b-1} \{ \sum_{i:R_i \supseteq W_k} T_i \}^2 \prod_{j:B_j \varepsilon W_k} N_j, \qquad (25)$$

where W_0 is the null set, for which the product is taken to be 1. The sum is also 1 for $k = 0$, since it is just the square of the network transmission of (18). Thus excluding the term for $k = 0$ gives an expression for N_{b+1} as a sum of products of positive terms, which is monotone nondecreasing in each N_j. We thus have proved Lemma 1.

Lemma 1

For any given set of path transmissions T_i, the network noise-to-signal ratio N_{b+1} is a monotone nondecreasing function of each branch noise-to-signal ratio N_j.

This lemma provides the proof of (13), which was referred to previously.

We have also proved that N_{b+1} can vanish for a nonvanishing set of path transmissions only if there is some path R_i along which every branch is noiseless, so that setting $T_i = 1$ and $T_j = 0$, $j \neq i$ gives a right-hand side in (25) in which only the term for W_0 remains. The matrix $||G_{ij} - 1||$ will be singular if, and only if, there is such a noiseless path since it will then map the transmission vector $\underline{T}$ with $T_i = 1$ and $T_j = 0$, $j \neq i$ into the null vector. The matrix $||G_{ij}||$ can be singular only under the same circumstances, but may not be even when noiseless paths exist.

We next show the equivalence of the right side of (25) to the quadratic form.

$$\sum_{i=1}^{r} \sum_{j=1}^{r} G_{ij} T_i T_j, \qquad (26)$$

where the T_i are still subject to the constraint (18). Substituting into (26) the definition (16) of G_{ij} gives

$$\sum_{i=1}^{r} \sum_{j=1}^{r} T_i T_j \{ \prod_{m:B_m \varepsilon R_i \cap R_j} (1 + N_m) \}. \qquad (27)$$

Expanding the product gives

$$\sum_{i=1}^{r} \sum_{j=1}^{r} T_i T_j \sum_{k:W_k \subseteq R_i \cap R_j} \{ \prod_{m:B_m \varepsilon W_k} N_m \}. \qquad (28)$$

Inverting the order of summation to sum over all W_k,

$$\sum_{k=0}^{2^b-1} \{ \sum_{i,j:W_k \subseteq R_i \cap R_j} T_i T_j \} \{ \prod_{m:B_m \varepsilon W_k} N_m \}. \qquad (29)$$

We then recognize that the parentheses enclose a term which is just the square of the sum of T_i over the i for which W_k is included in R_i:

$$\sum_{k=0}^{2^b-1} \{ \sum_{i:W_k \subseteq R_i} T_i \}^2 \prod_{m:B_m \varepsilon W_k} N_m, \qquad (30)$$

which is just the right side of (25).

We have thus proved that for T_i constrained by (18),

$$1 + N_{b+1} = \sum_{i=1}^{r} \sum_{j=1}^{r} G_{ij} T_i T_j. \qquad (31)$$

Squaring (18) gives

$$1 = 1^2 = \left\{ \sum_{i=1}^{r} T_i \right\}^2 = \sum_{i=1}^{r} \sum_{j=1}^{r} T_i T_j, \qquad (32)$$

and subtracting (32) from (31) gives

$$N_{b+1} = \sum_{i=1}^{r} \sum_{j=1}^{r} [G_{ij} - 1] T_i T_j, \qquad (33)$$

or

$$S_{b+1} = 1 \Big/ \sum_{i=1}^{r} \sum_{j=1}^{r} (G_{ij} - 1) T_i T_j. \qquad (34)$$

Now N_{opt}, by definition, is the minimum value of N_{b+1} as the branch transmissions are varied, and S_{opt}

is its reciprocal. We have therefore proved the first part of the theorem: namely, the equalities on the left in (19) and (20).

To obtain the inequalities on the right in (19) and (20), we minimize (31) and (33) by varying the path transmissions T_i independently, subject only to the constraint imposed by (18). The additional constraints imposed by the topology of the network and by (17), which expresses the T_i in terms of the real independent variables t_k, are ignored. The results are lower bounds to the minima which (31) and (33) can actually attain in the network.

Using a Lagrange multiplier $2M$, we set the derivative of

$$\sum_{i=1}^{r} \sum_{j=1}^{r} G_{ij}T_iT_j - 2M \sum_{i=1}^{r} T_i \tag{35}$$

with respect to T_i equal to zero. This gives

$$\sum_{j=1}^{r} G_{ij}T_j = M, \qquad 1 \leqq j \leqq r. \tag{36}$$

Using the minimizing T_j which satisfy (36), we multiply by T_i and sum, using the constraint of (18) and attaining a lower bound to $1 + N_{\text{opt}}$:

$$\sum_{i=1}^{r} \sum_{j=1}^{r} G_{ij}T_iT_j = M \sum_{i=1}^{r} T_i = M \leqq 1 + N_{\text{opt}}. \tag{37}$$

Solving (36) for the minimizing T_i gives

$$T_j = M \sum_{i=1}^{r} G_{ij}^{-1}. \tag{38}$$

Summing on j and using (18),

$$1 = \sum_{j=1}^{r} T_j = M \sum_{i=1}^{r} \sum_{j=1}^{r} G_{ij}^{-1}, \tag{39}$$

or from (37),

$$1 + N_{\text{opt}} \geqq M = 1 \Big/ \left\{ \sum_{i=1}^{r} \sum_{j=1}^{r} G_{ij}^{-1} \right\}. \tag{40}$$

This completes the proof of (19) in the theorem. The derivation of (20) is strictly parallel and will be omitted. It remains only to prove the assertions made for networks in D_0 and in D_1. To prove that equality holds on the right in (19) and (20) for networks in D_0, it is necessary to show that for such networks it is possible to vary path transmissions independently by varying branch transmissions. In fact we prove a stronger result.

Lemma 2

A network in D which has $r = b - v + 2$ has a cutset of r branches each of which is included in just one path. Removal of this cutset divides the network into two parts: a tree connected to V_1 (which may reduce to V_1 alone), and a tree connected to V_v (which may reduce to V_v alone).

Given Lemma 2, we can set the r transmissions of the branches in the cutset as the r desired path transmissions and set the transmissions of all other branches equal to unity.

To prove Lemma 2, assign weights to nodes and branches from the left, assigning weight 1 to node V_1 and then assigning to each branch the weight of its initial node and to each node the sum of the weights of its incoming branches. With this assignment the weight of a node or a branch is clearly the number of routes from the input node V_1 to that node or branch.

Choose from each of the r paths the right-most branch of weight 1. This set of branches, c in number, is a cutset, since it interrupts each path. We have $c \leqq r$: $c = r$ if, and only if, no branch is selected more than once.

Deleting the cutset of c branches divides the network into two parts, M_1 connected to V_1 and M_2 connected to V_v. M_1, which contains b_1 branches and v_1 nodes, is a tree, since it is connected and since all of its nodes are of weight 1, so that there is only one path from V_1 to each node. Thus $b_1 = v_1 - 1$, as for any tree.

M_2 is connected to V_v and thus includes at least one tree. Let one of the trees included in M_2 have b_2 branches and v_2 nodes, with $b_2 = v_2 - 1$. Then there are two possible situations. i) M_2 is a tree. In that case $r = b - v + 2$. Or ii) M_2 is larger than a tree, and includes b_3 branches beyond the b_2 branches in a tree which it includes. In that case $r > b - v + 2$. We will prove the labelled statements.

i) If M_2 is a tree, then $b = c + b_1 + b_2 = c + (v_1 - 1) + (v_2 - 1) = c + v - 2$, or $c = b - v + 2$. Since each branch in the cutset connects two trees, it completes just one path, so the number of paths $r = c$, and $r = b - v + 2$.

Q.E.D.$_i$

ii) If M_2 contains b_3 branches beyond those contained in a tree, then $b = c + b_1 + b_2 + b_3 = c + (v_1 - 1) + (v_2 - 1) + b_3 = c + v - 2 + b_3$, or

$$c = b - v + 2 - b_3. \tag{41}$$

Now each branch among the b_3 has weight $\geqq 2$ by construction, so it lies on at least two paths. Without these b_3 branches, V_v has weight at least c, since the c branches in the cutset have weight 1 each and are connected to V_v. Adding each of the b_3 additional branches adds a weight $\geqq 2$ to V_v, since each of them is connected to V_v through the tree included in M_2. Thus the total weight r of V_v is $r \geqq c + 2b_3$. Combining this with (41) gives

$$r \geqq c + 2b_3 = b - v + 2 + b_3 > b - v + 2. \tag{42}$$

Q.E.D.$_{ii}$

For a network M which is in D but not in D_0, $r > b - v + 2$; and it is impossible to independently vary the path transmissions. For $b - v + 2$ is the cyclomatic number of the graph M' obtained from M by adding a branch B_{b+1} directed from V_v to V_1, and is thus the maximum number of linearly independent cycles in a graph-theoretic sense. Thus the set of r cycles in M', each of which consists of a path R_i from V_1 to V_v followed by the branch B_{b+1} from V_v to V_1 are linearly dependent in the graph-theory sense. Therefore, so is the set of the paths themselves in M.

The linear dependence of the R_i implies, by taking logarithms in (17), one or more linear relations between the logarithms of the path transmissions $\log T_i$, leading to constraints of the form

$$\log T_i + \log T_j = \log T_m + \log T_n, \quad \text{or} \quad T_i T_j = T_m T_n \tag{43}$$

and no selection of values for the branch transmissions t_k can provide independent control of all path transmissions.

It may still be possible to achieve equality in (19) and (20) for a network in D which is not in D_0, however, if the optimizing values of the path transmissions happen to satisfy the additional constraints of the form (43) imposed by the network topology. This happens in particular for the networks which are in D_1 but not in D_0.

Lemma 3

Given a network M in D, and a network M' in D_0. Let M'' be constructed by replacing branch B_i in M by the network M'. Then the value of the parameter N''_{opt} of M'' will be the same as the value of the parameter N_{opt} of M if the latter is evaluated using the parameter value N'_{opt} of M' for B_i. The path transmissions obtained in computing N''_{opt} will lead to the same set of transmissions for the subnetwork M' as are obtained directly in the computation of N'_{opt}.

The network M'' is equivalent to the network M with some value of the parameter N_j for branch B_j by the argument following (2), i.e., the subnetwork M' is equivalent to *some* noisy branch B_i, and the only question is what its parameter value is. The optimum set of path transmissions for M'' must lead to the same transmissions inside M' as does the direct optimization of M'. Any other choice would give a larger value to the parameter of M'' by Lemma 1.

Lemma 3 completes the proof of the theorem. Lemma 2 covers networks in D_0 and Lemma 3 justifies the extension of the results to networks in D_1. More practically, it permits the solution of network problems of large order by local reductions—the combining of series or parallel branches, etc.—which greatly reduces the computation. Unfortunately the other tool used for the local reduction of resistive networks—the star-mesh transformation—cannot be used for Gaussian channels, since it leads to transformed branches which have correlated generators. This takes us outside of our present model. Networks with correlated noise present problems which are discussed briefly in a later section.

Proof of Earlier Results

The result of (5) follows from the theorem by noting that for a series network $r = 1$, and $\|G_{ij}\| = \|G_{ii}\|$. Thus,

$$G_{ii} = \prod_{i=1}^{b} (1 + N_i) = 1/G_{ii}^{-1}. \tag{44}$$

Equation (6) follows by noting that for a parallel network, $r = b$ and $\|G_{ij} - 1\|$ is diagonal with elements $G_{ii} = N_i$, so that

$$[G_{11} - 1]^{-1} = S_i, \qquad \sum_{i=1}^{b} \sum_{j=1}^{b} [G_{ij} - 1]^{-1} = \sum_{i=1}^{b} S_i. \tag{45}$$

Equation (11) follows from the evaluation of (20) for the network of Fig. 1. Equation (9) follows by letting S_2 approach infinity in (11), for $k = 2$. For larger k, the first three branches are combined into an equivalent forward branch of capacity $C_1 + C_2$ and it is combined with the next noisy forward branch and the next noiseless feedback branch in the same way, etc.

Equations (12) and (13) have already been justified. Equation (14) follows by throwing away all but the linear terms, i.e., terms having a single N_i as a factor, in (33). By (25) this reduces the right side and provides a lower bound to N_{opt} or an upper bound to S_{opt}. The resulting equations are those for a set of resistors—the noisy branches—with resistance $= N_i$, all in parallel—both the forward and the feedback branches—with the noiseless branches acting as short circuits at the two ends and the conductances $S_i = 1/N_i$ adding.

Reduction of Another Problem to the Above

A more general problem concerning networks of Gaussian channels can be reduced to the previous results. Consider the class of two-terminal networks as in D (mentioned previously), but in which each node may supply a different linear combination of the voltages on its incoming branches to each outgoing branch. This model still leaves the operation at the node simple and linear, and provides an increased number of independently controllable path transmissions. Thus it enlarges the class of networks for which explicit solution is possible and for which equality holds in (19) and (20).

As an example, the network shown in Fig. 4 consisting of five vertices connected by four branches forming a directed path from V_1 to V_2 to V_3 to V_4 to V_5, with three additional branches from V_1 to V_3, V_3 to V_5, and V_2 to V_4 has $b = 7$, $v = 5$, and $r = 5$, and is thus not in D_0: it has no two-terminal subnetworks, and is thus not in D_1.

The reduction to the former case replaces each node V_j which has $I_j > 1$ incoming branches and $O_j > 1$ outgoing branches by I_j nodes at each of which one of the incoming branches arrives and O_j nodes from each of which one of the outgoing branches leaves, together with I_jO_j noiseless branches connecting each of the I_j arrival nodes to each of the O_j departure nodes. The added noiseless branches permit the formation of the desired different linear combinations of input branch voltages for each output branch. In the case of the five-node network already described, replacing V_3 by 4 nodes and 4 branches, as shown in Fig. 5, adds 3 nodes, 4 branches and no paths. Thus $b - v + 2 = 7 + 4 - (5 + 3) + 2 = 5 = r$, and the resulting net is in D_0.

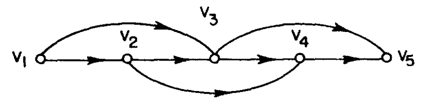

Fig. 4. A network not in Class D_1.

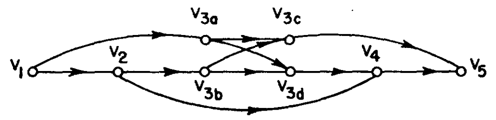

Fig. 5. A reduction of the network of Fig. 4 to a network in Class D_1.

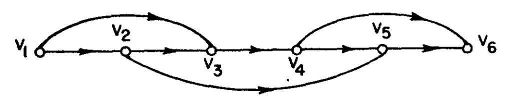

Fig. 6. A network not in Class D_1 which cannot be reduced.

The simplest network which cannot be expanded by the above substitution, has no two-terminal subnetworks, and is not in D_0, is shown in Fig. 6. It consists of six nodes V_1 to V_6 connected in order by five branches, with three additional branches from V_1 to V_3, V_4 to V_6, and V_2 to V_5.

Unfortunately the additional control provided by the change in rules provides no help for networks in F, which remain outside D_0 for $k > 2$.

Networks with Correlated Noise

One can consider networks in which $\overline{n_i n_j} \neq 0$, although the noise and the signal remain uncorrelated. For parallel branches, if we take $\overline{n_i n_j} = G_{ij}$ and T_i as the branch transmission subject to the constraint of (18), then minimizing the mean-square value of the sum

$$\overline{\left[\sum_i T_i(e_0 + n_i)\right]^2} \tag{46}$$

leads to precisely the result of the theorem, with equality in (19) and (20), by precisely (35) to (40). In fact, the proof of the theorem may be taken as a proof that the voltages transmitted to the output node V_e by the different paths R_i have the average product matrix $\|G_{ij}\|$.

For series branches the situation is different, however. Correlated series branches do not commute unless their parameter values are equal. Even the validity of the branch model breaks down. The definition in (1) of the added branch noise power $\overline{n_i^2} = P_i N_i$ is valid as a model of a physical channel so long as the channel is always used at the maximum possible input power. This is always advantageous when branch noises are uncorrelated, so the restriction is not felt in the optimization problem of the theorem. However a more realistic model of a physical channel has an input power limit P_i, and adds a noise of power $N_i P_i$ to any input signal whose power is $\leqq P_i$. With correlated noise in series branches it will sometimes be advantageous to use less than the maximum input power to a branch. No analog to Lemma 1 holds.

As an example consider two identical channels in series. Each accepts inputs of power $\leqq 2$ and adds to them the *same* noise voltage n, of power $\overline{n^2} = 1$. If we apply only 1 unit of signal power to the first channel, invert its output using a branch transmission of -1, and apply the result to the second channel, the output of the second channel has no noise voltage, and therefore an infinite signal-noise ratio. If, however, we apply 2 units of signal power to the first channel, and scale its output voltage by $-\sqrt{2/3}$ to provide 2 units of input power to the second channel, we cannot get a signal-noise power ratio at the output of the second channel which is better than $4/(5 - 2\sqrt{6}) \cong 40$.

Further Bounds on Networks in F

The open questions of greatest interest for applications concern networks in F with $k > 2$ and with noisy feedback. In a feedback system it is reasonable to assume that the transmitter has a limited amount of signal-to-noise ratio S_{odd} available, and that the receiver has a limited amount S_{even}, given by

$$\begin{aligned} S_{\text{odd}} &= \sum_{j=1}^{k} S_{2j-1} \\ S_{\text{even}} &= \sum_{j=1}^{k-1} S_{2j}, \end{aligned} \tag{47}$$

and that they are free to allocate their limited resources between the different forward and feedback channels in the way which maximizes the resulting S_{opt} of F. This freedom may even extend to deciding how large k should be, if the available forward and feedback channels have infinite bandwidth.

In the case of noiseless feedback $k = \infty$ is best and gives the result of (10). When the feedback is noisy, evaluating what S_{opt} the best division of limited power gives and how S_{opt} depends on k involves a great deal of numerical solutions of linear equations subject to constraints of the form of (43). Even evaluating the upper bound to S_{opt} of (20) is not easy. Lower bounds to S_{opt} which are more meaningful than that of (12) can be computed, however, by making use of iteration of networks for which $k = 2$, as shown in Fig. 7.

For the first level network, we assume that the two forward branches have equal signal-to-noise ratio, since this maximizes S_{opt} in (11), for fixed S_{odd}. Denoting their common signal-to-noise ratio as S_1, the feedback branch as S_2, and the resulting S_{opt} as S_3, we have from (11)

$$S_3 = 2S_1 + \frac{S_1^2 S_2}{(1 + S_1)^2 + S_2}. \tag{48}$$

We now consider the second-level network to consist of two forward branches of ratio S_3 and a feedback branch of ratio S_4. The resulting S_{opt} is denoted by S_5, and we have for the kth level

$$\begin{aligned} S_{2k+1} &= 2S_{2k-1} + \frac{S_{2k-1}^2 S_{2k}}{(1 + S_{2k-1})^2 + S_{2k}} \\ S_{\text{odd}} &= 2^k S_1 \\ S_{\text{even}} &= S_{2k} + 2S_{2(k-1)} + \cdots + 2^{k-1} S_2. \end{aligned} \tag{49}$$

For this network the optimum allocation of S_{odd} among the 2^k forward branches has already been made, and each

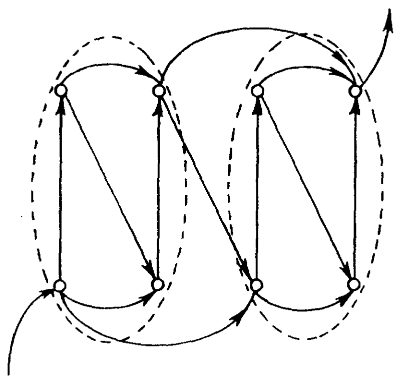

Fig. 7. An iteration of networks in Class F for which $k = 2$.

receives an equal amount. S_{even} is divided unequally, however, with more for higher-numbered branches, in the optimum case. The optimum allocation can be determined by solving (49) for S_{2k}:

$$S_{2k} = \frac{S_{2k+1} - 2S_{2k-1}}{1 - \dfrac{1 + S_{2k+1}}{(1 + S_{2k-1})^2}}. \tag{50}$$

Now differentiating $S_{2k} + 2S_{2k-2}$ with respect to S_{2k-1} for fixed S_{2k+1} and S_{2k-3} and setting the result equal to zero gives

$$\frac{S_{2k-3}^2(1 + S_{2k-3})^2}{[(1 + S_{2k-3})^2 - (1 + S_{2k-1})]^2} = (1 + S_{2k-1}) \frac{(1 + S_{2k-1})^3 - (1 + S_{2k+1})(2 - 3S_{2k-1}) + (1 + S_{2k+1})^2}{[(1 + S_{2k-1})^2 - (1 + S_{2k+1})]^2}. \tag{51}$$

For given S_{2k-3} and S_{2k-1}, this equation is quadratic in S_{2k+1}, and solving it enables us to start with a desired S_1 and S_3 and to generate S_{2k+1} for any k. Alternatively, we may fix S_{2k+1} and S_{2k-1} and solve for S_{2k-3}. Taking the positive square root of each side of (51) gives a quadratic in S_{2k-3} and we can proceed from given values of S_{2k+1} and S_{2k-1} down to S_1. In either case the resulting set of values is optimum in the sense that by keeping the end points fixed, and fixing k, any other division of S_{odd} will take more of it. Choosing all combinations of values for, e.g., S_1 and $S_3 > 2S_1$, generates the full set of optimum curves.

The result, unfortunately, must be displayed as a set of curves rather than an equation. A much weaker lower bound to S_{opt} can be given as an equation. Although it is not the best strategy, we may pick a division of S_{even} which gives us a fixed c such that

$$1 + S_j = c(1 + S_{j-2})^2, \quad \text{odd } j. \tag{52}$$

Then from (50),

$$S_{2k} = \frac{S_{2k+1} - 2S_{2k-1}}{1 - c} \tag{53}$$

and from (49) and (52),

$$S_{\text{even}} = \frac{S_{2k+1} - 2^k S_1}{1 - c} \leqq \frac{S_{\text{opt}} - S_{\text{odd}}}{1 - c}, \tag{54}$$

since $S_{2k+1} \leqq S_{\text{opt}}$. We also have from repeated application of (52)

$$c(1 + S_{\text{opt}}) \geqq c(1 + S_{2k+1}) = c^{2^k}(1 + 2^{-k} S_{\text{odd}})^{2^k}. \tag{55}$$

Together, (54) and (55) provide a useful analytic lower bound to S_{opt}.

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13. ABSTRACT

Abstract—This paper discusses networks (directed graphs) having one input node, one output node, and an arbitrary number of intermediate nodes, whose branches are noisy communications channels, in which the input to each channel appears at its output corrupted by additive Gaussian noise. Each branch is labeled by a non-negative real parameter which specified how noisy it is. A branch originating at a node has as input a linear combination of the outputs of the branches terminating at that node.

The channel capacity of such a network is defined. Its value is bounded in terms of branch parameter values and procedures for computing values for general networks are described. Explicit solutions are given for the class D_0 which includes series-parallel and simple bridge networks and all other networks having r paths, b branches, and v nodes with $r = b - v + 2$, and for the class D_1 of networks which is inductively defined to include D_0 and all networks obtained by replacing a branch of a network in D_1 by a network in D_1.

The general results are applied to the particular networks which arise from the decomposition of a simple feedback system into successive forward and reverse (feedback) channels. When the feedback channels are noiseless, the capacities of the forward channels are shown to add. Some explicit expressions and some bounds are given for the case of noisy feedback channels.

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