ESSAYS ON INTERNATIONAL ASSET PRICING

by

RENÉ MARCEL STULZ

Lic. sc. écon., University of Neuchâtel, Switzerland
(1975)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS OF THE DEGREE OF
DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1980

Signature of Author ____________________
May 19, 1980

Certified by ____________________
Thesis Supervisor

Accepted by ____________________
Chairman, Department Committee

ESSAYS ON INTERNATIONAL ASSET PRICING

by

RENÉ MARCEL STULZ

Submitted to the Department of Economics on May 19, 1980 in partial fulfillment of the requirements for the degree of Doctor of philosophy

ABSTRACT

This dissertation consists of three essays devoted to the question of how asset prices are determined in open economies.

In the first essay, we construct an intertemporal equilibrium model of international asset pricing which does not depend on the assumption that naive purchasing power parity - the version of purchasing power parity which assumes that the exchange rate always changes to offset exactly changes in the domestic price level - holds. It is shown that deviations from naive purchasing power parity affect asset demands and asset prices. If naive purchasing power parity holds or if markets are complete in an Arrow-Debreu sense, equilibrium asset prices for domestic risky assets can be obtained without taking into account the relationships among asset markets located in different countries. In general, the prices of domestic risky assets which are not redundant can not be obtained without taking those relationships into account. Given a definition of world real consumption and real expected returns, it is shown that the real expected excess return of an asset is an increasing function of the covariance of that asset with world real consumption. World real consumption does not, in general, correspond to a basket of commodities which can be consumed by all investors, as some goods are non-traded, at least in the short-run.

In the second essay, we look at the effects of barriers to international investment on asset demands and asset prices. Barriers to international investment are assumed to take the form of a proportional tax. The tax takes a different value for long holdings of foreign assets and for short holdings of foreign assets. It is shown that all investors in the domestic country will hold the same portfolio of risky assets, which will differ, in general, from the world portfolio of risky assets and from the market portfolio of domestic risky assets. In general, domestic investors will not have nonzero holdings of all foreign assets. Assets which are not held in nonzero amounts by investors in one country are called non-traded assets. It is shown that assets with large betas

with respect to the world market portfolio of risky assets are likely to have positive alphas.

The third essay is devoted to the question of how international trade affects asset demands and asset prices. Within a simple model of complete specialization, it is shown that domestic net holdings of foreign assets are a decreasing function of the domestic expenditure elasticity of imports. If relative prices are the only state variables, it will be true that all investors hold common stocks in identical proportions. Finally, it is shown that, *ceteris paribus*, the difference between the forward rate and the future expected spot rate is a decreasing function of the domestic expenditure elasticity of imports and an increasing function of the foreign expenditure elasticity of imports of the domestic good.

Thesis Supervisor: Fischer Black

ACKNOWLEDGEMENTS

It is difficult to find a good way to express my appreciation to all those who, in one way or another, helped me in the course of my studies at M.I.T. I wish however to thank first Fisher Black, who never tired of giving me extremely useful advice. Stan Fischer read carefully the various drafts of this dissertation and made many useful comments. Don Lessard is guilty of having introduced me to international finance, as it is viewed from the floor of E-52. He also gave me a lot of useful advice while I was writing this dissertation.

The dissertation is the last hurdle in the graduate program. I wish to thank Professor Bishop, without whose help I would never have crossed the first hurdle. I am also grateful to Professor Juvet, without whose encouragements I would never have left Switzerland to study abroad.

I have been lucky to have a most cheerful and helpful office mate, Bob Cumby, whom I wish to thank.

My studies at M.I.T. were financed in a most generous way by the Swiss National Research Fund.

My wife, Pat Reagan, read all the drafts of my thesis and made a lot of useful comments. She also helped in numerous ways during the last four years, while she was pursuing the same course of study.

This dissertation is dedicated to my parents, without whose sacrifices and encouragements I would never have written it.

TABLE OF CONTENTS

PART I

A MODEL OF INTERNATIONAL ASSET PRICING

Section I: Introduction

This paper presents a model of international asset pricing which requires neither that a change in the domestic price level is exactly offset by a change in the exchange rate, nor that the exchange rate dynamics be given exogenenously. In the following, we adopt the terminology suggested by Holmes (1967) and define as naive purchasing power parity the version of purchasing power parity which states that a change in a broad-based price index is exactly offset by a change in the exchange rate.[1] The major results of the paper can be summarized as follows:

1. Optimal portfolios of risky assets in open economies

   If naive purchasing power parity does not hold and markets are incomplete in some well-defined sense, investors will not hold the world market portfolio of risky assets, because they want to hedge against unanticipated deviations from naive purchasing power parity. Foreign nominal bonds and foreign index bonds - if they exist - will in general belong in an optimal portfolio which provides a hedge against unanticipated deviations from naive purchasing power parity. Conditions under which investors will hold the world market portfolio of risky assets are derived.

---

1 The argument for the use of broad indices has been stated most clearly in Frenkel (1978). Frenkel (1978), Lee (1976) and Officer (1976) offer recent reviews of the history of the purchasing power parity theory. Holmes (1967) looks at the various interpretations of Cassel's work.

2. Pricing of assets which belong in the world market portfolio of risky assets

Given appropriate definitions of the world real expected excess return on a risky asset and world real consumption, it is shown that the world real expected excess return on a risky asset is equal to the ratio of the covariance of that asset with world real consumption and the covariance of a reference portfolio with world real consumption, multiplied by the world real expected excess return of that reference portfolio. In general, no investor can buy one unit of world real consumption. It is shown that if naive purchasing power parity holds, real consumption will be perfectly correlated across countries if the world economy has complete markets in the sense of permitting an unconstrained Pareto-optimum.

3. Pricing of bonds

There will be a risk premium on the forward rate if the exchange rate (domestic price of foreign currency) is positively correlated with world real consumption. Because deviations from naive purchasing power parity imply relative price changes, in general world real consumption will be correlated with unanticipated deviations from naive purchasing power parity. The pricing of index-bonds is also considered. In the presence of deviations from naive purchasing, there can be an index-bond in each country, and those index-bonds will not be perfect substitutes. The model does not require the presence of outside assets with a safe nominal return to yield a risk premium on the forward rate and the risk premium could be zero in the presence of such outside assets.

The present paper focuses on the implications of the model for finance theory. In Stulz (1980), we discuss the implications of the model for macro-economics. Although most models of international asset pricing assume that naive purchasing power parity holds exactly, in the remainder of this section, we shall argue that naive purchasing power parity should hold exactly only in very exceptional circumstances which are not relevant for modern finance theory, and briefly review the literature on international asset pricing.

The empirical evidence on naive purchasing power parity is mixed. The best results have been obtained when one country suffers from hyperinflation, whereas the other does not. Otherwise, naive purchasing power parity does not hold exactly, at least in the short run.[2] Several explanations for the existence of deviations from naive purchasing power parity have been advanced. Cassel himself thought that the exchange rate was not always an equilibrium exchange rate. Most of the explanations of why there are deviations from naive purchasing power parity appear in macroeconomic models which do not

---

2 Officer (1976), Genberg (1978), Kohlhagen (1978) review the recent empirical literature.

show how the assumed aggregate relationships work at a microeconomic level.[3]

One possible explanation for the existence of deviations from naive purchasing power parity is extremely simple, has a sound foundation in general equilibrium theory and is perfectly consistent with the assumption that goods markets are efficient. Naive purchasing power parity theory is a theory about the relationship between changes in the exchange rate and changes in the ratio of a domestic and a foreign price index. If relative prices of commodities change over time, unless the weights of the domestic and foreign price indices are the same, the ratio of the two price indices will change even if there is no inflation in the sense of a proportional change in all money prices of a country. If all money prices in a country double, one would expect the exchange rate to fall to the point where its value is half of its former value. When some money prices rise and some fall, changes in a price index may reflect more changes in relative prices than pure inflation.

The proposition that changes in relative prices of different commodities at the same location can prevent naive purchasing power parity from holding has two possible interpretations. One interpretation has a long intellectual tradition, centered around the contributions of Viner, Balassa and Samuelson, and deals with the long run

---

[3] The macroeconomic literature has been reviewed by Dornbusch (1978), Bilson (1979), Isard (1978), Schadler (1977). See Stockman (1978) for the major exception to our statement.

biases implied by the use of various price indices.[4] Another interpretation, which states that those relative price changes matter for shorter periods of time, does not have such a tradition. This is not surprising. Historically, naive purchasing power parity theory generated much interest when monetary authorities wanted to choose a new parity for their currency after a period of monetary disorder. In such circumstances, relative prices should not have mattered very much.

For the proposition that relative price changes matter even over relatively short periods to be empirically relevant, it is needed that those relative prices which matter are the relative prices of commodities at one location rather than the relative prices of identical commodities at different locations. Some recent work, in particular Stockman (1978), has shown that there were indeed significant changes in relative prices in the recent past.[5] If commodity markets are not efficient with respect to changes in the exchange rate, changes in relative prices of different commodities at one location could obviously be swamped by exchange rate induced changes in relative prices of identical commodities at different locations. In the following, it is assumed that commodity markets are always in equilibrium and that all investors have rational expectations. If there are no costs to international arbitrage - i.e. commodities can move freely, instantaneously and at zero cost (there are no tariffs and no transportation

---

4 See Balassa (1964), Viner (1937) and Samuelson (1948). Empirical tests are reported in Officer (1976). Both Genberg (1978) and Frankel (1978) estimate time trends in deviations from PPP.

5 See also Genberg (1978).

costs) - the law of one price always holds exactly when commodity markets are in equilibrium. Studies which compare commodities which are identical except for their location do not show significant inefficiencies. Unfortunately, most studies compare commodities that differ in more than their location.[6] It is true that the law of one price requires assumptions which are too simplistic to offer a realistic view of the world for most purposes, but we do not believe that this is the case for our purpose.

Our discussion suggests three possible assumptions about the nature of changes in money prices. One assumption could be that only pure monetary inflation matters. In this case, naive purchasing power parity always holds exactly. Such an assumption makes sense if relative price effects are swamped by the effects of pure inflation. We have claimed that such an assumption does not make sense if used to describe the current period of flexible exchange rates. However, many models of international asset pricing use that assumption. Both Kouri (1977) and Fama and Farber (1979) have models in which there is only one good, which is internationally traded, and in which, consequently, naive purchasing power parity holds trivially. Grauer, Litzenberger and Stehle (1976) have many goods whose relative price is stochastic and which all are traded, but they make assumptions about the utility function of investors which ensure that naive purchasing power parity holds.

---

6 Genberg (1975) provides results for identical goods. Isard (1977), Kravis and Lipsey (1978), Dunn (1970) are among the best studies which reject the law of one price.

A second possible assumption is that only relative price changes matter. Such an approach does not make much sense either for the current period of floating exchange rates, because everybody acknowledges that naive purchasing power parity explains a significant fraction of the exchange rate changes which occurred in the recent past. This assumption has been used, but in a peculiar way, in a highly influential and innovative contribution by Solnik (1973). In Solnik's model, there are as many consumed goods as there are countries and none of those goods are traded. Consumers in each country consume only one good which is consumed nowhere else. The exchange rate for each pair of currencies changes stochastically through time, whereas the domestic price of the good in each country is one. An increase in the exchange rate (domestic price of the foreign currency) means an increase in the domestic purchasing power of foreign assets. However, foreign assets held by domestic residents have a domestic purchasing power only if those assets allow domestic investors to buy goods which can ultimately increase their consumption. There are no such goods in Solnik's model. It is incorrect to say that the assumption that no consumption good is traded in Solnik's model does not matter, because if goods are traded, Solnik's model will not hold in general, as the results of this paper show.[7]

---

[7] Solnik (1978) argues that his paper could be reinterpreted in such a way that the exchange rate would be the relative price of commodity baskets. In that case, as will be seen in this paper, Solnik's model can hold only for a very special utility function, which would be even more restrictive than the utility function used by Grauer, Litzenberger and Stehle (1977). Both Fama and Farber (1979) and Grauer, Litzenberger and Stehle (1977) criticize Solnik (1973) on the grounds that his model involves money illusion. Dumas (1977) argued that Solnik's results have nothing to do with money illusion.

For Solnik's model to be compatible with rational expectations, it

The third possible assumption is that both relative price changes and pure inflation are important. This assumption is the most appealing. Unfortunately, there is no model of international asset pricing which uses that assumption to derive general pricing relationships for assets.[8]

This paper aims at filling this gap. Many economists are likely to believe that of the three possible assumptions about the nature of changes in nominal prices, the assumption which states that both relative price changes and pure inflation matter is the most relevant for the current period of floating exchange rates. However, the use of that assumption is also theoretically very appealing, because it turns out that models satisfying the first and second assumptions are just polar cases of a model using the third assumption.

In the second section of this paper, we present the dynamics used for commodity and asset prices. Care is taken to show the economic implications of those assumptions. Section 3 presents the derivation of the asset demand functions. As the derivation of asset demand

---

is necessary the risk premium on foreign bonds stays constant through time. However, for the risk premium on foreign bonds to stay constant through time, it is required that net foreign investment does not change through time, which itself implies that new foreign investment must be identically equal to zero. The distribution of the exchange rate can not be arbitrary, because otherwise, there is no guarantee that net foreign investment will always be equal to zero if there are trade flows. For a discussion of the conditions under which Merton's model (see Merton (1973)) is compatible with rational expectations, see Hellwig (1977).

8 Kouri (1976) has a model in which deviations from purchasing power parity imply hedging demands for bonds. The assets of his model are nominal bonds and a forward contract, and the model is developed for a very restrictive utility function. Kouri's model assumes deviations from the law of one price, but does not derive their equilibrium implications. Kouri focuses his discussion on default risk. See also Heckerman (1973), Kouri and de Macedo (1979) and Wihlborg (1979).

function is well-known,[9] we adopt a highly compact notation in most of the section. The section derives however in detail the excess returns used throughout the paper, introduces money and discusses the problem of inverting the variance-covariance matrix of excess returns. Section 4 is devoted entirely to a discussion of some economic implications of the asset demand functions. Most of the section discusses the conditions under which common stocks are held in identical proportions by (a) all domestic investors and (b) by all investors. The remainder of the section gives the conditions under which the non-stochastic part of the return of index-bonds will be the same across countries and discusses the demand for money. In Section 5, the equilibrium relationships among expected asset returns are derived. The crucial problem of Section 5 is how to aggregate domestic and foreign demand functions for assets in the most general case. It must be noted that the most general case allows for (a) departures from naive purchasing power parity, (b) an indeterminate number of state variables, and (c) non-zero covariances between risky assets and the exchange rate. Whereas in Solnik (1973), only demand functions for the case in which (a) and (c) hold are derived, and the equilibrium relationships for that case are not given, because "The resulting risk pricing relations are rather complex and

---

9 See Merton (1971), Merton (1973), Breeden (1978), Breeden (1979). Cox, Ingersoll and Ross (1978) have a more formal presentation and discuss the role of state variables. Fischer (1975) introduces commodity prices and index-bonds.

and not intuitively appealing,"[10] no such problem arises here. Section 6 discusses in detail the equilibrium relationships. Implications for the pricing of various assets are drawn; the model is presented with various simplifying assumptions; and the correlation of business cycles across countries is discussed. Concluding remarks are presented in Section 7. Throughout the paper, we assume that there are no barriers to international investment.[11] This is not because we believe that such barriers do not exist, but because introducing barriers to international investment would not have significant implications for the issues which are discussed in this paper.

---

[10] See Solnik (1973), p. 102.

[11] Black (1974) introduced the concept of barriers to international investment. Adler and Dumas (1976) and Kouri (1976) have looked at default risk. See also Stulz (1980b).

Section 2: Asset and commodity price dynamics

In this section, we describe the characteristics of the various assets an investor can hold in his portfolio and explain how commodity prices change through time.

2.1. The exchange rate and commodity prices.

We assume that the path of domestic commodity prices is described by the following stochastic differential equation:[12]

$$\frac{dP_i}{P_i} = \mu_{P_i}(\underline{s}, t)dt + \sigma_{P_i}(\underline{s}, t)dz_{P_i} \qquad i = 1, \ldots, K \tag{2.1}$$

where P_i is the price in domestic currency of the i-th commodity, $\underline{s}$ is a S x 1 vector of state variables and dz_{P_i} is a Wiener process. It is assumed that there is only one foreign country, and asterisks are used to designate prices of commodities and assets of the country. There are K^* commodities in the foreign country, and K^* does not need to be equal to K.

Later on, we will specify some of the state variables which belong to the vector $\underline{s}$. The functional relationships between the S state variables and the mean and variance of the rate of change of domestic commodity prices will not be made more precise. It will always be assumed that P_i is the equilibrium price of commodity i, but we will

---

[12] Merton (1978) gives an excellent introduction for economists of those equations. See also Brock (1974) and the references in Brock (1974) and Merton (1978). For a more mathematical exposition, see for instance Arnold (1974) or Friedman (1975).

never look at the supply of good i. Consequently, (2.1) has to be viewed as a very general reduced form equation for the price of good i. The only important restriction, for our purposes, on the path of good i given by (2.1) is that, in non-mathematical terms, that path has to be smooth. This restriction is important because it prevents our model from being used to study a world in which exchange rates are fixed most of the time and change only by a discrete amount when they change.[13] If there are no obstacles to international commodity arbitrage, in the sense that arbitrage can always be made instantaneously and at zero costs, then for homogeneous, traded, commodities the law of one price must hold exactly. Let the domestic price of one unit of foreign money be e. In that case the law of one price states that:

$$P_j(t) = e(t)P_j^*(t) \qquad j \in T(t) \qquad (2.2)$$

where $P_j^*(t)$ is the price in foreign money of good j and T(t) is the set of traded goods at time t; by convention, the T first goods of K and K^* are the traded goods. The set of traded goods does not need to be constant, but it is assumed that the set of consumed goods in each country is constant. In this paper, it will <u>always</u> be assumed that (2.2) holds for <u>all</u> traded goods at time t. Clearly, in the real world spatial arbitrage of commodities is neither instantanenous nor costless. This assumption is an important simplifying assumption. In the conclusion, we will discuss its implications for our results.

---

[13] Kouri (1976) used a Poisson process to model discrete changes in the exchange rate.

The exchange rate follows an equation similar to (2.1):

$$\frac{de}{e} = \mu_e(\underline{s}, t)dt + \sigma_e(\underline{s}, t)dz_e \tag{2.3}$$

(2.3) precludes jumps in the exchange rate. Using (2.1) through (2.3) and Ito's Lemma, we get a relative version of the law of one price:

$$\frac{dP_j}{P_j} = \frac{de}{e} + \frac{dP_j^*}{P_j^*} + \frac{de}{e} \cdot \frac{dP_j^*}{P_j^*} \tag{2.4}$$

where $(de/e)(dP_j^*/P_j^*)$ is the covariance between e and P_j^*. It must be noted that the expected rate of change of the domestic price of good j is ($\rho_{a,b}$ is the correlation coefficient between a and b):

$$E_t\left\{\frac{dP_j}{P_j}\right\} = \mu_e(\underline{s}, t)dt + \sigma_{P_j^*}(\underline{s}, t)dt + \rho_{e,P_j^*}\sigma_e\sigma_{P_j^*}dt \tag{2.5}$$

which is different from the sum of the expected rate of change of the exchange rate and of the foreign price of commodity j.

Now, let P be a domestic price index and P^* be a foreign price index. It is assumed that P is defined by:

$$P(t) = \prod_{i=i}^{K} P_i(t)^{a_i(t)} \tag{2.6}$$

For the moment, the weights of the index P can be anything, although the reader may want to think of them as expenditure shares for the domestic country. P^* can be defined in the same way, but there is no reason for the weights of P^* to be the same as those of P. Naive purchasing power parity theory states that:

$$P = eP^* \tag{2.7}$$

In empirical studies on naive purchasing power parity, P has sometimes been the consumer price indes, the wholesale price indes, the price index of non-traded goods or other possible indices. Observable price indices are constructed so that empirical studies have to use the relative version of naive purchasing power parity, which is:

$$\frac{de}{e} = \frac{dP}{P} - \frac{dP^*}{P} - \frac{dP.dP^*}{P.P^*} + (\frac{dP^*}{P^*})^2 \tag{2.8}$$

The covariance and variance terms of (2.8) show that in general it is not strictly true that when naive purchasing power parity holds the rate of change of the exchange rate is equal to the difference between the rate of change of the domestic price level minus the rate of change of the foreign price level.

In general, the fact that (2.8) holds does not imply that (2.4) holds. When (2.4) holds, (2.8) will hold if either (a) $a_i = a_i^*$, $\forall\ i \in T$, $a_i = a_i^* = 0$, $\forall\ i \notin T$, or (b) $dP_i/P_i = dP_j/P_j$, $\forall$ i, j. If price indices reflect consumption patterns exactly, then (a) describes the case of world economy in which all investors consume goods in identical proportions, whereas (b) describes the case of a world economy in which prices change only because of pure inflation of the case of a world economy in which there is only one good which is traded.[14] Note that both for (a) and (b) we assume that indices P and P^* correspond to indices which would be used to test naive

---

[14] The case described by (a) corresponds to Grauer, Litzenberger and Stehle (1976); the case described by (b) corresponds to Kouri (1977) and Fama and Farber (1979). We assume, in our statement, that the price indices in each country reflect the consumption vector of that country in some way. Otherwise, the case in which two price indices with identical weights are compared is not interesting.

purchasing power parity. As neither (a) nor (b) need to hold in our model, it follows that in general naive purchasing power parity does not need to hold in our model.

(2.4) applies only to traded goods whereas (2.8) does not need to. Our model allows for the existence of non-traded goods, but our results do not depend at all on whether or not there are non-traded goods. Note also that the empirically observed fact that deviations from naive purchasing power parity tend to be to some extent self-correcting could easily be taken into account in this model. Some evidence presented in Fama and Schwert (1979) suggests that relative prices often seem to move more in the short run than in the long run. While there is a good economic reason why big changes in relative prices should be followed by changes in the opposite direction - the short run supply curve is generally less elastic than the long run supply curve - there is however no reason why each particular relative price should follow a random walk with zero drift. We, therefore, want it to be possible for our model to accommodate secular deviations from naive purchasing power parity.

2.2. Asset price dynamics.

The price dynamics for all financial assets are now described. It is assumed that for each asset the returns accrue in the form of capital gains. Dividends distributed continuously could be accommodated without problems in this model. It is also assumed that there are no transaction costs and that markets are in equilibrium all the time. Finally, it is assumed that there are no barriers to international investment.

In each country, there are four types of assets. There are common stocks, futures contracts, bonds which promise the payment of a certain amount of money at a certain date, and money. To restrict the menu of available assets, we will assume that all futures contracts and bonds are of instantaneous maturity. This means that we will not be able, for instance, to study the term structure of forward premia in this model.

There are n stocks which are traded only on the domestic stock market, whereas the other stocks are traded on the foreign stock market. In the following, stocks traded only on the domestic stock market will be called domestic stocks and it will be assumed that these stocks follow a stochastic differential equation of the form:

$$\frac{dI_i}{I_i} = \mu_{I_i}(\underline{s}, t)dt + \sigma_{I_i}(\underline{s}, t)dz_{I_i} \qquad i = 1, \ldots, n \tag{2.10}$$

where I_i is the price in domestic money of common stock i. As no dividends are paid, it is not clear what is implied by the fact that a stock is traded on a domestic stock market, except that with barriers to international investment, à la Black (1974), those stocks *may* become less attractive to foreigners. The fact that a stock is traded on the domestic stock market certainly does not imply anything about the operations of the firm which issued that stock.[15] It is perfectly possible for that firm to own foreign plants, to have foreign nominal debts, and so on. As a sizeable number of firms do have foreign operations, one would like to avoid any assumption which

---

[15] For a discussion of the value of the firm with international operations, see Lessard (1978).

reduces the influence of those operations on stock prices. There is clearly no good economic reason why an unanticipated change in the exchange rate should always have no effect on the price in domestic money of each stock. One would also expect that in some cases the variance of the domestic price of a stock would depend on the exchange rate. After all, if a firm has many liabilities expressed in foreign money and none in domestic money, one would not expect the distribution of its cash-flow to stay constant if the exchange rate changes.

In the same vein, our model allows for the effects changes in the money prices of various commodities can have on the price of a stock. If a firm has a big inventory of one particular commodity and the money price of that commodity increases, one would expect the price of the stock of that firm to stay constant if nothing else changes. Changes in money prices can also affect the variance of a particular stock. It must finally be noted that dI_i/I_i can be uncorrelated with the price level defined by the index P, but that at the same time $\mu_{I_i}(\underline{s}, t)$ can be correlated with the rate of inflation. If the rate of inflation changes, one would expect the rate of change of the exchange rate to change and the nominal interest rate to change. Obviously, if the nominal rate of interest changes, one would expect all nominal expected rates of return on risky assets to change also. Considering that for many pairs of currencies, the interest rate differential has switched signs several times over the past few years, the fact that our model allows for such changes is a useful empirical feature.

There are n^* foreign common stocks, and of course n^* does not need to be equal to n. I_j^*, which is the price in foreign money of the j-th common stock, follows a differential equation of the same form as (2.10). Let D_j^I be the domestic price of the j-th foreign stock. Using the law of one price for assets, we can get the dynamics for D_j^I:

$$\frac{dD_j^I}{D_j^I} = \frac{dI_j^*}{I_j^*} + \frac{de}{e} + \frac{de}{e} \quad \frac{dI_j^*}{I_j^*} \tag{2.11}$$

It is important to note that in general the expected return in domestic money of a foreign stock depends positively on the covariance between that stock and the exchange rate.

In the domestic country, there are N futures contracts available. Futures contracts have zero value (see Black (1976)). It is therefore useful to use indexed bonds instead of futures contracts. By indexed bond, we mean a bond whose return is indexed on one or a group of state variables. An indexed bond does not need to be indexed on a state variable which is the money price of one or a group of commodities. It can be indexed on other state variables. One of those indexed bonds could be indexed on P. If a bond indexed on P exists, by convention we will write its price H_1 and use the contraction index-bond to designate it. As indexed bonds perform exactly the same role as futures contracts, we will use indifferently the concepts of futures contract and indexed bond. The reader should remember, however, that what we call a futures contract does not correspond exactly to a real world futures contract. Let H_j be the price of such an indexed bond. The dynamics for H_j are given by:

$$\frac{dH_j}{H_j} = \mu_{H_j}(\underline{s}, t)dt + \sigma_{H_j}(\underline{s}, t)dz_{H_j} \qquad i = 1, \ldots, N \tag{2.12}$$

There are N^* foreign futures contracts available, and an equation like (2.13) applies to those contracts. D_j^H is the price in domestic currency of the j-th foreign indexed-bond.

In each country, there is one bond which yields a safe nominal rate of return. If B is the price of the domestic nominal bond in domestic currency, we can write:

$$\frac{dB}{B} = \mu_B(\underline{s}, t)dt \qquad (2.13)$$

Note that μ_B, which is the certain rate of return on the nominal bond, depends on the state variables and time. This implies that, although the rate of return on this period's nominal bond is certain, the certain rate of return on next period's nominal bond is unknown this period. B^* is the price in foreign money of the foreign nominal bond, and D^B is its price in a domestic money.

It is assumed that domestic investors hold only domestic money and that money does not yield a nominal return. These two assumptions are for pure convenience. They are justified by the fact that the main reason for which money appears in our model is to allow us to compare our results with alternative models of international asset pricing which include money among the available assets.

Section 3: Asset demand functions

This section describes how the asset demand functions are obtained. We focus on the aspect of the optimization problem of investors which do not appear either in Merton (1973) or in earlier models of international asset pricing using the same techniques. Most of the equations in this section will be written in a very compact form.

3.1. The accumulation equation.

Let W be the nominal wealth of a representative investor. w_{I_i}, $w_{I_i^*}$, w_{H_i}, $w_{H_i^*}$, w'_{B^*} and w_B are the proportions of wealth of the representative investor invested respectively in the i-th domestic stock, foreign stock, domestic indexed bond, foreign indexed bond, foreign nominal bond and domestic nominal bond. The proportion of his wealth the representative investor holds in domestic money is w_M. The stock budget constraint for the representative investor is:

$$w_B + w'_{B^*} + \Sigma w_{I_i} + \Sigma w_{I_i^*} + \Sigma w_{H_i} + \Sigma w_{H_i^*} + w_M = 1 \tag{3.1}$$

The flow budget constraint, giving the change in nominal wealth, is:

$$dW = \Sigma w_{I_i} \frac{dI_i}{I_i} W + \Sigma w_{I_i^*} \frac{dD_i^I}{D_i^I} W + \Sigma w_{H_i} \frac{dH_i}{H_i} W + \Sigma w_{H_i^*} \frac{dD_i^H}{D_i^H} W \tag{3.2}$$

$$+ w'_{B^*} \frac{dD^B}{D^B} W + w_B \frac{dB}{B} W - \Sigma P_i c_i dt$$

where c_i is the <u>rate</u> of consumption of commodity i. Using (3.1) to substitute for w_B in (3.2) yields (we write $w_{B^*} = \Sigma w_{I_i^*} + \Sigma w_{H_i^*} + w'_B$):

$$dW = \sum_i^n w_{I_i}\left(\frac{dI_i}{I_i} - \frac{dB}{B}\right)W + \sum_i^n w_{I_i^*}\left(\frac{dI_i^*}{I_i^*} + \frac{de\,dI_i^*}{eI_i^*} - \frac{dB^*}{B^*}\right)W$$

$$\sum_i^n w_{H_i}\left(\frac{dH_i}{H_i} - \frac{dB}{dB}\right)W + \sum_i^n w_{H_i^*}\left(\frac{dH_i^*}{H_i^*} + \frac{de\,dH_i^*}{eH_i^*} - \frac{dB^*}{B^*}\right)W$$

$$w_{B^*}\left(\frac{dB^*}{B^*} + \frac{de}{e} - \frac{dB}{B}\right)W - w_M\frac{dB}{B}W + \frac{dB}{B}W - \sum_i^n P_i c_i dt \qquad (3.3)$$

In compact form, we can rewrite (3.3) as:

$$dW = \sum_i^n w_{A_i}\frac{dA_i}{A_i}W - w_M\frac{dB}{B}W + \frac{dB}{B}W - \sum_i^n P_i c_i dt \qquad (3.4)$$

From now on, the expressions in parentheses of (3.3), which are written as dA_i/A_i in (3.4), will be called the nominal excess returns for domestic investors and dA_i/A_i is the nominal excess return on the i-th risky asset. Whereas the proportion of wealth held in money does not appear in (3.2), it does appear once w_B has been substituted out. Note that the Z assets correspond to the sum $(n + n^* + N + N^* + 1)$, and are numbered continuously in the order in which they appear in that summation. There are Z risky assets for domestic investors. The Z-th risky asset is the foreign nominal bond. There are Z risky assets for foreign investors. All the risky assets for foreign investors are the same as those for domestic investors except the Z-th of those assets which for the foreign investors is the domestic bond.

3.2. The Bellman function.

The representative investor maximizes an expected utility function which can be written as:

$$E_{t_0}\left\{\int_{t_0}^{t_T} u(c_1(t), \ldots, c_K(t), L(t), t)dt + B(W(t_T), \underline{s}(t), t_T)\right\}$$

where u( ) is twice differentiable and strictly concave in c_i's. B( ) is strictly concave in $W(t_T)$, and is the investor's request function. E_{t_0} is the expectation operator conditional on the information available at time t_0.

As in Fama and Farber (1979) and Kouri (1977), money is held because it is more convenient for investors to buy goods with money than to barter.[16] In our model, the analysis is made easier if we think of money as an instrument which decreases time spent shopping and treat time spent shopping like time spent working. Let the time it takes the representative individual to shop be:

$$L = L(w_M W, \Sigma P_i c_i) \qquad L_1 < 0,\ L_2 > 0 \tag{3.5}$$

For analytical convenience, we assume that:

$$L(0, \Sigma P_i c_i) = \infty \tag{3.5a}$$

We also assume that:

$$u_L < 0 \qquad u_{LL} > 0 \tag{3.5b}$$

It must be noted that (3.5), (3.5a) and (3.5b) guarantee that every investor will hold some money. Equation (3.5) implies a crucial assumption, which is that money does not make it any easier to buy financial assets; it is as easy, for instance, to buy stock with a bond as with money. This assumption is not realistic, but it simplifies everything considerably.

---

[16] See Fischer (1974).

We can now write the so-called Bellman function for the representative investor:

$$J(W(t_0), \underline{s}(t_0), t_0) = \text{Max } E_{t_0} \left\{ \int_{t_0}^{t_T} u(c_1(t), \ldots, c_K(t), \right.$$

$$\left. L(t), t)dt + B(W(t_T), \underline{s}(t_T), t_T) \right\} \tag{3.6}$$

Taking a Taylor-series expansion of (3.6) around $J(W(t_0), \underline{s}(t_0), t_0)$ and neglecting higher order terms yields:

$$0 = \text{Max } E_{t_0} \left\{ u(c_1(t_0), \ldots, c_K(t_0), L(t_0), t_0)dt + J_W dW \right.$$

$$\frac{1}{2}J_{WW}(dW)^2 + \frac{1}{2}\underline{J}_{WS}(dW.\underline{dS}) + \frac{1}{2}(\underline{dS}.dW)\underline{J}_{WS} + \frac{1}{2}\Sigma\Sigma J_{S_i S_j} dS_i dS_j$$

$$\left. + \underline{J}'_S \underline{dS} + J_t \right\} \tag{3.7}$$

where $\underline{J}_{WS}$ is an 1 x S vector and $\underline{dS}$ an S x 1 vector. In the following, we underline a vector once and a matrix twice. Note that in (3.6) the utility function is a function of non-stochastic elements. The ratio in which any pair of commodities will be consumed will result in:

$$u_{c_i}/u_{c_j} = P_i/P_j \qquad \forall\ i, j \tag{3.8}$$

The number of units of money held will depend on the wealth of the investor directly as:

$$u_L L_1 = \mu_B \tag{3.9}$$

It follows from (3.8) that given $\Sigma P_i c_i$, all the c_i's are determined. Consequently, we can solve first for $\Sigma P_i c_i$, which we write C, and later on choose the optimal c_i's, which must satisfy (3.8) and (3.9). Of course, the utility an investor derives from C depends on the vector of money prices. We define the indirect utility function of consumption expenditures as:[17]

$$U(C(t), P_1(t), \; . \; . \; . \; , P_K(t), L(t), t) =$$

$$\underset{c_i}{\text{Max}} \; u(c_1(t), \; . \; . \; . \; , c_K(t), L(t), t) \text{ s.t. } \Sigma P_i c_i = C \tag{3.9}$$

and substitute that function in (3.7). Passing the expectation operator through in (3.7) then yields, after dividing by dt:

$$U(C(t_o), P_1(t_0), \; . \; . \; . \; , P_K(t_0), L(t_0), t_o) + J_W \underline{w}_a' \underline{\mu}_a W - J_W w_M \mu_B W -$$

$$J_W C + \frac{1}{2} J_{WW} \underline{w}_a' \underline{\underline{V}}_{aa} \underline{w}_a W^2 + \frac{1}{2} \underline{w}_a' \underline{\underline{V}}_{as} \underline{J}_{SW} W + \frac{1}{2} \underline{J}_{WS} \underline{\underline{V}}_{sa} \underline{w}_a W + \; . \; . \; . = 0 \tag{3.10}$$

where only the terms which contain control variables have been written. The matrix $\underline{\underline{V}}_{aa}$ is the variance-covariance matrix of asset returns. The matrix $\underline{\underline{V}}_{as}$ is the covariance matrix of asset returns with state variables. $\underline{\mu}_a$ is the vector of excess expected returns on risky assets. Note that the returns used here are those of (3.4).

3.3. Demand functions for assets.

There is no stochastic element in (3.10) and that equation is unconstrained. The first-order conditions for a maximum can be obtained using ordinary calculus.

We can obtain a first-order condition for money holdings in terms

---

[17] We follow Breeden (1979).

of the indirect utility function of consumption expenditures:

$$U_L L_{w_M} W = J_W \mu_B W \tag{3.11}$$

The first-order condition for consumption expenditures is:

$$U_C = J_W \tag{3.12}$$

Where U_C is the total derivative of (3.9) with respect to consumption. Substituting (3.12) in (3.11) gives us:

$$U_L L_{w_M} = U_C \mu_B \tag{3.13}$$

The demand for money implied by (3.13) will be discussed in Section 4. However, it must be noticed that (3.13) is slightly different from the first-order condition Fama and Farber (1979) get, as in their case μ_B divided by $(1 + \mu_B)$, because here investors consume continuously, whereas in their model they consume at discrete intervals.

The first-order conditions for risky assets can be written as:

$$J_W \underline{\mu}_a W + J_{WW} \underline{\underline{V}}_{aa} \underline{w}_a W^2 + \underline{\underline{V}}_{as} \underline{J}_{SW} W = 0 \tag{3.14}$$

To solve (3.14) for $\underline{w}_a$, we need $\underline{\underline{V}}_{aa}$ to be non-singular. $\underline{\underline{V}}_{aa}$ will have rank Z if the payoff of no asset can be reproduced by a linear combination of the payoff of other assets. Clearly, if there are both futures contracts available for each individual commodity and index-bonds <u>à la</u> Fischer (1975), it will not be possible to invert $\underline{\underline{V}}_{aa}$. Similarly, as will be shown in Section 4, if there are commodities for which no futures contract can be made, and if there is one index-bond <u>à la</u> Fischer in each country, $\underline{\underline{V}}_{aa}$ will not be invertible if naive purchasing power parity holds. We will consequently assume that Z is the number of risky assets whose payoff can not be reproduced by a linear combination of the payoff of other assets. We can then solve for $\underline{w}_a$:

$$\underline{w}_a W = \underline{\underline{V}}_{aa}^{-1}\left(\frac{-J_W}{J_{WW}}\right)\underline{\mu}_a + \underline{\underline{V}}_{aa}^{-1}\underline{\underline{V}}_{as}\left(\frac{-\underline{J}_{WS}}{J_{WW}}\right) \tag{3.15}$$

The properties of (3.15) will be discussed at length in the next section.

Section 4: The economic implications of asset demand functions

In this section, the economic implications of asset demand functions are discussed. In the introduction to this paper, we explained that three assumptions on the dynamics of money prices were possible. This section is mainly concerned with the question of how asset demand functions, in a model which assumes that relative price changes matter differ from asset demand functions in models which assume that only pure inflation matters. The first part of this section discusses the demand functions for common stocks. The second part of this section discusses the demand functions for other assets.

4.1. Common stocks.

We now have two groups of risky assets: common stocks and the other risky assets. $\underline{w}_I$ is the vector of demands for common stocks. It is composed of the first n + n* elements of $\underline{w}_a$. The vector $\underline{w}_R$ is composed of the last Z - n - n* elements of $\underline{w}_a$. The first element of $\underline{w}_R$ is the domestic index-bond, when there is such a bond. It can be shown that:

$$\underline{w}_I = \left\{ \underline{\underline{V}}_{II} - \underline{\underline{V}}_{IR}\underline{\underline{V}}_{RR}^{-1}\underline{\underline{V}}_{RI} \right\}^{-1} \left\{ \left(\frac{-J_W}{J_{WW}W}\right)\left(\underline{\mu}_{a_I} - \underline{\underline{V}}_{IR}\underline{\underline{V}}_{RR}^{-1}\underline{\mu}_{a_R}\right) + \left(\underline{\underline{V}}_{IS} - \underline{\underline{V}}_{IR}\underline{\underline{V}}_{RR}^{-1}\underline{\underline{V}}_{RS}\right)\left(\frac{-\underline{J}_{WS}}{J_{WW}W}\right) \right\} \qquad (4.1)$$

where $\underline{\underline{V}}_{II}$ is the variance-covariance matrix of the excess returns of common stocks, $\underline{\underline{V}}_{RR}$ is the variance-covariance matrix of the excess returns of risky assets which are not common stocks, $\underline{\underline{V}}_{IR}$ is the covariance matrix of excess common stock returns with the excess returns of other risky assets. $\underline{\mu}_{a_I}$ is composed of the first n + n* elements of $\underline{\mu}_a$, whereas $\underline{\mu}_{a_R}$ is composed of the other elements of $\underline{\mu}_a$. $\underline{\underline{V}}_{IS}$ is the covariance matrix of the excess returns of common stocks with state variables. Looking at (4.1), it should be clear that the mutual funds theorems presented in earlier work on international asset pricing will not hold in this model. Indeed, unless the vector $\underline{J}_{WS}$ has only zeros for all investors, there is no way that an investor will be indifferent between holding the market portfolio of common stocks or his optimal portfolio of common stocks. This result should not be surprising, as it holds in all models which introduce state variables, starting from Merton (1973). It is however important to notice that in our model, the mere fact that naive purchasing power parity does not hold is enough, for general utility functions, to break down traditional two - or three-funds theorems, but it is not enough to void earlier results that investors would hold all their stocks in the form of a mutual fund which would be the market portfolio. The remainder of this part of this section is devoted to a discussion of the conditions under which investors or all investors will or will not hold all their stocks in the form of a common mutual fund. The results are arranged in propositions and corollaries to facilitate the exposition.

> Proposition 1. If (a) there are no futures markets for commodities and (b) naive purchasing power parity does not hold, then investors who have an indirect utility function of consumption expenditures which does not imply both a relative risk aversion equal to one and constant expenditure shares, will in general use common stocks to

hedge against the effects of relative price changes.

We assume that the first K state variables are the logarithms of commodity prices. Breeden (1978) has shown that by differentiating the first-order conditions for a maximum of the indirect utility function of consumption expenditures, one could obtain:[18]

$$\frac{-\underline{J}_{WS}}{J_{WW}} = \frac{-\underline{C}_S}{C_W} + \begin{Bmatrix} \dfrac{C\underline{\alpha}}{C_W} - \dfrac{T}{C_W}\underline{m} \\ \underline{0} \end{Bmatrix}$$

where $\underline{\alpha}$ is the vector of average expenditure shares of the investor, $\underline{m}$ is the vector of marginal expenditure shares, T is the absolute risk tolerance of the investor, which is $-U_C/U_{CC}$. $\underline{C}_S$ and C_W are derivatives of the consumption function of the investor, which we write:

$$C = C(W, \underline{s}, t)$$

Hedging demands in an international setting have been studied in detail in Stulz (1979b). We are here concerned solely with the existence of those demands.[19]

Clearly, if some common stocks are correlated with commodity prices, $\underline{\underline{V}}_{IS}$ will have some nonzero elements. As there are no futures contracts, $\underline{\underline{V}}_{RS}$ will have one row, which gives the covariances between the excess return of the foreign nominal bond and the state variables. Consequently:

$$\underline{\underline{V}}_{IS} - \underline{V}_{Ie}\frac{1}{\sigma_e^2}\underline{V}_{eS} \neq 0 \qquad \text{if some } \rho_{I_i S_j} \neq \rho_{I_i e}\rho_{eS_j}$$

---

[18] We need to assume that the marginal utility of consumption is not affected by a change in L.

[19] Note that by hedging demands, we mean hedging demands à la Merton (1973), and not the more natural concept of differential demands for risky assets which would keep the indirect utility function of wealth constant if an unanticipated change in state variables occurs. See Breeden (1978) for this distinction.

It follows that if an investor has no hedging demands, this has to be caused by the fact that that investor has a utility function such that the terms of $\underline{J}_{WS}$ are equal to zero at least for those multiplied by a nonzero term in the demand functions for common stocks. If:

$$C(t)\underline{\alpha}(t) \neq T(t)\underline{m}(t)$$

there will be some nonzero terms in $\underline{J}_{WS}$.

Proposition 1 offers a result which is clearly related to the fact that naive purchasing power parity does not hold. When naive purchasing power parity does not hold, the demand functions for assets can be considerably different from what they are when it holds. In general, one would expect at least some stocks to be correlated with some money prices. This implies that if investors do not have hedging demands for stocks, one reason within this model should be that condition (a) of Proposition 1 does not hold. There are obviously futures markets in the real world.

> Proposition 2. If for each state variable correlated with a common stock, there exists a futures market, then all domestic investors will hold common stocks in identical proportions, whether naive purchasing power parity holds or not.

Let F be the number of state variables which are correlated with common stocks. We can choose to renumber the N + N* + 1 risky assets which are not common stocks, so that the first F of those assets are futures contracts on the state variables which are correlated with common stocks. In this case, the first F columns of $\underline{\underline{V}}_{RS}$ will be exactly equal to the first F columns of $\underline{\underline{V}}_{RR}$. Each row of $\underline{\underline{V}}_{RS}$ which has a nonzero element in the first F columns must have zeroes in the last N + N* + 1 - F columns. If a row of $\underline{\underline{V}}_{RS}$ has a nonzero element in any of the N + N* + 1 - F last columns, the column of $\underline{\underline{V}}_{IR}$ which has the same number as that row will have only zeroes. Let Q be the number of other risky assets which are correlated with state variables not correlated with common stocks and number the risky assets which are not common stocks so that those Q assets are the last Q risky assets. We now know that the last Q columns of $\underline{\underline{V}}_{IR}$ will have only zeroes. By assumption, $\underline{\underline{V}}_{RR}$ is block diagonal. The upper left-hand corner block has dimension (N + N* + 1 - Q) x (N + N* + 1 - Q), whereas the lower right hand corner block has dimension Q x Q. The product $\underline{\underline{V}}_{IR}\underline{\underline{V}}_{RR}^{-1}$ is a matrix of dimension (n + n*) x (N + N* + 1) which has zeroes everywhere except in a square submatrix of dimension (N + N* + 1 - Q) x (N + N* + 1 - Q) in the upper left-hand corner. It follows that the only elements of $\underline{\underline{V}}_{RS}$ which will not be multiplied by zeroes are those which belong to the (N + N* + 1 - Q) first rows of that matrix. By assumption, the nonzero elements of the first (N + N* + 1 - Q) rows of $\underline{\underline{V}}_{RS}$ belong to the F first columns and correspond to the upper left-hand corner (N + N* + 1 - Q) x F submatrix of $\underline{\underline{V}}_{RR}$. By the definition of the inverse of matrix, the product $\underline{\underline{V}}_{RR}^{-1}\underline{\underline{V}}_{RS}$ yields a matrix which in its first N + N* + 1 - Q rows will be composed only of zero elements except for an F x F identity matrix in the upper left-hand corner. The nonzero elements of $\underline{\underline{V}}_{IR}$ will multiply an identity matrix of dimension F x F. As the first F futures contracts are perfectly correlated with the state variables with which common stocks are correlated, $\underline{\underline{V}}_{IR}\underline{\underline{V}}_{RR}^{-1}\underline{\underline{V}}_{RS}$ is equal to $\underline{\underline{V}}_{IS}$. Obviously, $\underline{\underline{V}}_{IS}$ cancels out and $\underline{J}_{WS}$ is multiplied by a matrix of zeroes.

It is important to notice that investors will hold stocks in identical proportions if naive purchasing power parity does not hold even when markets are not complete. What is needed for investors to hold common stocks in identical proportions is simply that whatever hedging they could do through the stock market, they could do it better elsewhere.

There is a particular extension of Proposition 2 which is interesting. It is presented in the following Corollary. Proposition 2 did not require the existence of index-bonds. Obviously, if there was a common stock correlated with each commodity, Proposition 2 required enough futures contracts for index-bonds to have a payoff which could be duplicated through those futures contracts. However, there is a case in which there are two index-bonds and two nominal bonds, at least, which also yields the result that all domestic investors hold common stocks in identical proportions.

<u>Corollary 2.1.</u> If (a) there is a futures contract for each state variable j such that $j > k$, (b) each domestic investor i has constant expenditure shates such that:

$$\underline{\alpha} = \underline{m} = \lambda^i \underline{a} + (1 - \lambda^i)\underline{a}^* \qquad 0 \leq \lambda^i \leq 1$$

where $\underline{a}$ is the vector of weights of the price index used for the domestic index-bond and $\underline{a}^*$ the vector of weights for the foreign index-bond, and (c) $C^i(W, \underline{s}, t)$ has its first K derivatives with respect to state variables equal to zero for each domestic investor, then all domestic investors will hold stocks in identical proportions, if the two index-bonds exist.

It is enough to show that the investor does not need common stocks to hedge against unfavorable changes in commodity prices. As for state variables which are not commodity prices, the argument would be a mere repetition of the argument in Proposition 2, we look only at the case in which stocks are not correlated with state variables which are not commodity prices. If e is not a state variable, note that:

$$V_{IS}\binom{\alpha}{\underline{0}} = \lambda^{i}\underline{V}_{IP} + (1 + \lambda^{i})\underline{V}_{IP*} + (1 - \lambda^{i})\underline{V}_{IE}$$

where $\underline{0}$ is an (S - K) x 1 vector of zeroes, and $\underline{V}_{IP}$ is a (n + n*) x 1 vector of covariances of common stocks with the domestic price index. Let F be the futures contracts correlated with commodity prices or the exchange rate, including the two index-bonds. Number the futures contracts so that the first F futures contracts will be those correlated with commodity prices and the exchange rate. By assumption, only the first F columns of $\underline{\underline{V}}_{IR}$ have nonzero elements. This means that we can neglect the last S - F rows of $\underline{\underline{V}}_{RR}^{-1}$ and $\underline{\underline{V}}_{RS}$ as they will have only zeroes once multiplied by $\underline{\underline{V}}_{IR}$. Note also that, by assumption, $\underline{\underline{V}}_{RR}$ will be block diagonal and that the upper left-hand corner block will have dimension F x F. Finally, $\underline{\underline{V}}_{RS}$ will have zeroes in the last S - K elements of its first F rows. In that case:

$$\underline{\underline{V}}_{RS}^{F}\binom{\alpha}{\underline{0}} = \lambda^{i}\underline{V}_{RP}^{F} + (1 - \lambda^{i})\underline{V}_{RP*}^{F} + (1 - \lambda^{i})\underline{V}_{Re}^{F}$$

Choose the first three risky bonds to be respectively the domestic index-bonds, the foreign index-bond and the foreign nominal bond. It follows that $\underline{V}_{RP}$ is the first column of $\underline{\underline{V}}_{RR}$, $\underline{V}_{RP*}$ the second column and $\underline{V}_{Re}$ the third column, but, by assumption, $\underline{V}_{RP}$ is just $\underline{V}^{F}_{RP}$ augmented by N + N* + 1 - F rows of zeroes. $\underline{\underline{V}}^{-1}_{RR}$ times the right-hand side of the last expression will yield a vector of zeroes except for λ^{i} in its first row, $(1 - \lambda^{i})$ in its second row and $(1 - \lambda^{i})$ in its third row. Clearly, $\underline{\underline{V}}_{IR}\underline{\underline{V}}^{-1}_{RR}\underline{\underline{V}}_{RS}\binom{\alpha}{\underline{0}}$ will be equal to $\underline{\underline{V}}_{IS}\binom{\alpha}{\underline{0}}$. This implies that stocks will not be used as hedges. Note that if the exchange rate is a state variable, it is still true that investors can create a portfolio which is perfectly correlated with their price index which does not contain stocks. It is also true that they can hold a risky asset which is perfectly correlated with the exchange rate.

It follows from these results that the fact that naive purchasing power parity does not hold introduces the possibility that investors will hold different portfolios of stocks. However, it does not follow from the fact that naive purchasing power parity does not hold that not all investors will hold identical portfolios of stocks.

Proposition 2 and its Corollary apply to domestic investors. An important question is to know under which conditions <u>all</u> investors in the world will hold an identical portfolio of common stocks. Clearly, if foreign investors do not need stocks to hedge against unfavorable changes in state variables, they will all hold the same portfolio of stocks. The condition under which stocks will not be useful for hedging purposes for foreign investors will be that there exists "enough" futures contracts, which means that investors can construct portfolios of risky assets, which are not common stocks, such that for each state variable correlated with at least one common stock, the portfolio of risky assets which is the most correlated with that state variable does not contain any common stock. However, because the exchange rate enters the vector of excess returns of common stocks for domestic investors through the covariances of the excess returns of foreign common stocks with the exchange rate, the question of whether or not all investors will hold identical portfolios of common stocks has to be examined.

> Proposition 3. If Proposition 2 or its Corollary hold, then all investors in the world will hold stocks in identical proportions.

Proposition 2 and its corollary imply that stocks will not be used for hedging purposes. Note now that a typical element of $\underline{\underline{V}}_{II}$ is:

$$\left(\frac{dI_i}{I_i} - \frac{dB}{B}\right)\left(\frac{dI_i^*}{I_i^*} + \frac{de}{e} \quad \frac{dI_i^*}{I_i^*} - \frac{dB^*}{B^*}\right)$$

By Ito's Lemma, the covariance term in the second parenthesis disappears when the multiplication of the two parentheses is performed. Consequently, a typical element of $\underline{\underline{V}}_{II}$ is just the product of the excess returns of two stocks, each excess return being expressed in the currency of the country in which the stock is issued. Clearly, $\underline{\underline{V}}_{IR}$ depends on the numeraire currency through its last column, which is multiplied by (-1) when the numeraire currency is changed. However, the product $\underline{\underline{V}}_{IR}\underline{\underline{V}}_{RR}^{-1}\underline{\underline{V}}_{RI}$ does not depend on the numeraire currency. Note simply that a change in the numeraire currency changes the sign of the last column and the last row of $\underline{\underline{V}}_{RR}^{-1}$, the right-hand lower corner element of $\underline{\underline{V}}_{RR}^{-1}$

being unchanged. We now have shown that the first bracketed term of (4.1) does not depend on the numeraire currency. Note now that:

$$\underline{\mu}_I = \underline{\mu}_I^* + \underline{V}_{Ie}$$

where $\underline{V}_{Ie}$ is the vector of covariances between the exchange rate and common stocks. But:

$$\underline{\underline{V}}_{IR}(\underline{\underline{V}}_{RR})^{-1}\underline{\mu}_R = \underline{\underline{V}}^*_{IR}(\underline{\underline{V}}^*_{RR})^{-1}(\underline{\mu}^*_R - \underline{V}^*_{Re})$$

But the vector $\underline{V}^*_{Re}$ is the last column of $\underline{\underline{V}}^*_{RR}$. It has to be true that $(\underline{\underline{V}}^*_{RR})^{-1}\underline{V}^*_{Re}$ is equal to a column vector with zeroes everywhere except in its last row, which should have its element equal to one. Hence:

$$-\underline{\underline{V}}^*_{IR}(\underline{\underline{V}}^*_{RR})^{-1}\underline{\underline{V}}^*_{Re} = \underline{V}_{Ie}$$

As in the demand functions for stocks the left-hand term of that expression is multiplied by (-1), $\underline{V}_{Ie}$ cancels out with the identical vector which is used to transform $\underline{\mu}_I$ into $\underline{\mu}^*_I$. It is important to notice that the covariances between the exchange rate and the excess returns on common stocks do not prevent investors across the world from being indifferent between holding either the world portfolio of common stocks or common stocks individually.

4.2. The demands for other risky assets.

We now turn to the other risky assets and to the demand for money. It can easily be shown that the demand functions for other risky assets can be written:

$$\underline{w}_R = \left\{ \underline{\underline{V}}_{RR} - \underline{\underline{V}}_{RI}\underline{\underline{V}}_{II}^{-1}\underline{\underline{V}}_{IR} \right\}^{-1} \left\{ \left(\frac{-J_W}{J_{WW}W}\right)\left(\underline{\mu}_{a_R} - \underline{\underline{V}}_{RI}\underline{\underline{V}}_{II}^{-1}\underline{\mu}_{a_I}\right) \right.$$

$$\left. + \left(\underline{\underline{V}}_{RS} - \underline{\underline{V}}_{RI}\underline{\underline{V}}_{II}^{-1}\underline{\underline{V}}_{IS}\right)\left(\frac{-\underline{J}_{WS}}{J_{WW}W}\right) \right\} \tag{4.2}$$

Obviously, the existence of hedging demand functions is not a very interesting question for the other risky assets, as most of them, or all of them, are futures on state variables! One interesting question however is when is the domestic index-bond a perfect substitute for the foreign index-bond. In other words, when is the real rate of interest the same everywhere?

<u>Proposition 4.</u> Suppose that the index-bonds are such that $a_i = a_i^*$, for all i's. In that case, and in that case only, the index-bonds are perfect substitutes.

For a domestic investor, the excess return on a foreign index-bond over a domestic index bond is:

$$\frac{dA_z}{A_z} = \frac{dH^*}{H^*} + \frac{de}{e} - \frac{dH}{H}$$

where H* is the price in foreign money of a foreign index-bond. By substitution:

$$\frac{dA_z}{A_z} = \Sigma a_i \frac{dP_i^*}{P_i^*} + \frac{de}{e} - \Sigma a_i \frac{dP_i}{P_i} + \mu_H^* - \mu_H$$

The law of one price implies that all the stochastic terms in this last equation cancel out. It follows that the condition required by Proposition 4 implies that the excess real returns of the foreign index-bond is a safe real return. However, the excess real return on an asset corresponds to the case in which an investor does not make any net investment. As by arbitrage it is not possible to make an investment which costs nothing and yields something for sure, $\mu_H = \mu_{H*}$.

Proposition 4 implies that, for the price indices used for domestic and foreign index-bonds, naive purchasing power parity holds. If naive purchasing power parity does not hold, hence $a_i \neq a_i^*$, then the result does not hold, as is seen by:

$$\frac{dH}{H} - E\left(\frac{dH}{H}\right) = \frac{d(eH^*)}{eH^*} - E\left(\frac{d(eH^*)}{eH^*}\right) + \Sigma(a_i - a_i^*)\left(\frac{dP_i}{P_i} - E\left(\frac{dP_i}{P_i}\right)\right)$$

If index-bonds were index-bonds with respect to identical bundles of commodities, the real rates on those bonds would have to be equal. A blanket assertion that "In open, frictionless capital markets, the risk-free real rate of interest must be the same in all countries,"[20] is either wrong or misleading. If naive purchasing power parity holds, the assertion is correct; when naive purchasing power parity does not hold, the correct assertion is that the real rate of interest is, at a point in time, the same for all investors who have the same average and marginal expenditure shares. As, in general, index-bonds are going to be indexed on price indices which reflect the consumption patterns of investors of the country in which they are issued, index-bonds will have different real rates of return when naive purchasing power parity does not hold, in the sense that the part of the return of index-bonds which does not depend on the change in a price index will not be

---

[20] See Fama and Farber (1979), p. 644.

the same for all index-bonds.

We now turn to the demand for money. First, we look at an example; then we show that the Fama-Farber result on the demand for purchasing power risks does not need to hold. Let the function giving the time it takes to buy C be

$$L = C^{1+q}(w_M W)^{-q} \qquad q > 0$$

Taking the first-order conditions for consumption and money holdings yields:

$$U_1 + U_2(1 + q)C^q(w_M W)^{-q} = J_W$$

$$-U_2 q(w_M)^{-(q+1)}C^{1+q} = J_W \mu_B$$

We can solve those first-order conditions to get:

$$\frac{-U_2 q(w_M W)^{-(q+1)}C^{(1+q)}}{U_1 + U_2(1 + q)C^q(w_M W)^{-q}} = \mu_B$$

The money demand function implicit in those first-order conditions does not depend on wealth directly, in the sense that given U_1, U_2, q and C, a change in W will not affect $w_M W$. The money demand function increases with C and falls with μ_B.

Fama and Farber (1979) find that "Through his holdings of nominal bonds denominated in different monies, an investor ends up with the same fraction of the total purchasing power risk of the money supply of every country."[21] Our example just confirmed that nothing in our

---

[21] Id., p. 645.

model differentiates the money demand side from what it is in Fama and Farber, in the sense that the technical difference in formulating the demand for money really does not matter when money demand is concerned. However, that is not true on the demand side for nominal bonds. In our model, the proportions in which domestic investors hold foreign nominal bonds and any other risky asset depend on the derivative of the consumption function with respect to the state variables and on the covariance of the various assets with the state variables. There is no reason why all investors should hold foreign nominal bonds and any other risky assets in equal proportions, provided those assets can be used to hedge against changes in state variables for which there is no futures contract available.

We can give a simple example. Suppose that relative price changes are not correlated with changes in P and P*. In that case, if P and P* have different weights, it necessarily follows that e will be correlated with some money price. It is possible for P and P* to each follow a path which is nonstochastic; a particular case of a nonstochastic path would be the case in which both P and P* are constant. By assumption, we have:

$$\frac{dP_i}{P_i} \cdot \frac{dP}{P} = \frac{dP^*_i}{P^*_i} \cdot \frac{dP^*}{P^*} = 0 \qquad \text{for all } i\text{'s } T = K = K^*$$

By substituting P and P*, we get:

$$\frac{dP_i}{P_i} \; \sum_i \frac{dP_i}{P_i}(a_i - a^*_i) - \frac{de}{e}\sum_i a^*_i \frac{dP_i}{P_i} = - \frac{dP^*_i}{P^*_i} \cdot \frac{de}{e} \quad \text{for all } i\text{'s}$$

The covariance of both money prices with the exchange rate can be zero only if $a_i = a^*_i$, for all i's. That is a necessary condition for naive purchasing power parity to hold if prices of commodities are stochastic. Clearly, if the exchange rate is correlated with some money prices, the foreign nominal bond can be used as a hedge against adverse unanticipated changes in some money prices. As P and P* are, by assumption, independent of the relative prices, it is not the case that the hedging function of the foreign nominal bond is duplicated by either one of the index-bonds. The investors who need to hold positive quantities of the foreign nominal bond for hedging purposes will hold more of those nominal bonds in their portfolio than investors who hold those bonds because they hold the market portfolio of risky assets as they would do in the economy described by Fama and Farber. Furthermore, even if there is no money supply in the sense of Fama and Farber, it is still the case that investors who need foreign nominal bonds for hedging purposes will want to hold nominal bonds and will be willing to pay a premium to get them.

Section 5: Equilibrium relationships

This section develops the equilibrium relationships among prices of assets whose nominal returns are risky in at least one country. In our model, money is the only asset which is not risky in that sense.

5.1. Asset demand functions and consumption.

For the domestic investor i, the asset demand functions written in compact form using the Breeden decomposition of Section 4 are:

$$\underline{w}^i_a W^i = \frac{T^i}{C^i_W} \underline{\underline{V}}^{-1}_{aa} \underline{\mu}^D_a - \underline{\underline{V}}^{-1}_{aa} \underline{\underline{V}}_{as} \left\{ \frac{\underline{C}^i_S}{C^i_W} - \left(\frac{C^i \underline{\alpha}^i}{C^i_W} - \frac{T^i \underline{m}^i}{C^i_W} \right) \right\} \tag{5.1}$$

Following Breeden (1979), (4.1) can be rewritten with $\underline{\mu}^D_a$ on the l.h.s.:

$$\underline{\mu}^D_a = (T^i)^{-1} \underline{\underline{V}}_{aa} \underline{w}^i_a W^i - (T^i)^{-1} \underline{\underline{V}}_{as} \left\{ \frac{\underline{C}^i_S}{C^i_W} - \left(\frac{C^i \underline{\alpha}^i}{C^i_W} - \frac{T^i \underline{m}^i}{C^i_W} \right) \right\} \tag{5.2}$$

Note now that the covariance of risky assets with consumption is:

$$V_{aC_i} = C^i_W dW^i \left(\frac{d\underline{A}}{\underline{A}}\right) + \underline{dS}\left(\frac{d\underline{A}}{\underline{A}}\right)' \underline{C}^i_S = C^i_W \underline{\underline{V}}_{aa} \underline{w}^i_a W^i + \underline{\underline{V}}_{as} C^i_S \tag{5.3}$$

By substitution, after adding up across domestic investors, we get:

$$\underline{\mu}^D_a - \underline{\underline{V}}_{aP} \underline{m}^D = \left\{ \underline{V}_{aC_D} - C_D \underline{\underline{V}}_{aP} \underline{\alpha}^D \right\} (T^D)^{-1} \tag{5.4}$$

where $\underline{\underline{V}}_{aP}$ is the matrix of covariances between asset prices and domestic commodity prices; and:

$$\underline{\alpha}^D = \frac{1}{C_D} \Sigma C^i \underline{\alpha}^i \tag{5.5}$$

$$\underline{m}^D = \frac{1}{T_D} \Sigma \underline{m}^i T^i \tag{5.6}$$

Let P_m and P_A be two price indices of the form of (2.6). The weights of the first price index are a consumption weighted average of the domestic investors' average expenditure shares. The weights of the second index are a risk tolerance weighted average of marginal expenditure shares. We can rewrite (5.6) as:

$$\underline{\mu}_a^D - \underline{V}_{aP_m} = (T^D)^{-1}(\underline{V}_{aC_D} - C_D\underline{V}_{aP_A}) \tag{5.7}$$

If we had a closed economy, we now would have an equilibrium equation _à la_ Breeden (1979).

5.2. Equilibrium asset pricing and foreign consumption.

Everything we did in 5.1. could be done for investors in the foreign country. Aggregating over foreign investors, we get:

$$T^F\underline{\mu}_a^F = \underline{V}_{aC_F^*} - C_F^*\underline{V}_{aP_A^*} + T^F\underline{V}_{aP_m^*} \tag{5.8}$$

However, we can not aggregate (5.7) and (5.8), because to get (5.7) we aggregated domestic consumption in domestic currency, whereas to get (5.8) we aggregated foreign consumption in foreign currency. We need to transform (5.8) in a way which allows us to aggregate the result with (5.7). First, we multiply (5.7) by e on both sides. Now, we look at the elements of (5.8) separately. A typical element of $\underline{V}_{aC_F^*}$ is:

$$\frac{dA_i^F}{A_i^F}dC_F^*$$

The superscript F in dA_i^F/A_i^F is necessary, because the nominal excess return of an asset abroad will differ from the nominal excess return at home by some terms which have been described in Section 4. We have covariance or a variance is taken. Consequently, if we had:

$$\frac{dA_i^F}{A_i^F} d(eC^F)$$

we could add it to the same element in $\underline{V}_{aC_D}$. If C_F is foreign consumption in domestic money, we can write, using Ito's Lemma:

$$\underline{V}_{aC_F} = \frac{dA_i^F}{A_i^F} \frac{dC_F}{e} - \frac{dA_i^F}{A_i^F} \frac{de}{e} \frac{C_F}{e} = \frac{1}{e}(\underline{V}_{aC_F} - \underline{V}_{ae} C_F) \tag{5.9}$$

By the same token:

$$C_F \underline{V}_{aP_A^*} = \frac{1}{e}(C_F \underline{V}_{a(eP_A^*)} - C_F \underline{V}_{ae}) \tag{5.10}$$

$$T^F \underline{V}_{aP_m^*} = eT^F \frac{1}{e}(\underline{V}_{a(eP_m^*)} - \underline{V}_{ae}) \tag{5.11}$$

Substitution in (5.8) equations (5.9) to (5.11) after having multiplied (5.8) by e yields:

$$\underline{\mu}_a^F - \underline{V}_{a(eP_m^*)} + \underline{V}_{ae} = (eT^F)^{-1}(\underline{V}_{aC_F} - C_F \underline{V}_{a(eP_A^*)}) \tag{5.12}$$

5.3. World aggregation.

Now, we could add up (5.12) to (5.7). However, we would get two different vectors of expected excess returns which would not add up. Note that a representative element of $\underline{\mu}_a^D$ differs from $\underline{\mu}_a^F$ in the following way:

$$\mu_{a_i}^D - \mu_{a_i}^F = \rho_{ea_i} \sigma_{a_i} \sigma_e \qquad \forall\ i$$

It should be clear that the element on the r.h.s. is just $V_{a_i e}$. It follows that $\underline{\mu}_a^F = \underline{\mu}_a^D - \underline{V}_{ae}$. The vector $\underline{V}_{ae}$ in (5.11) cancels out. We then get:

$$\underline{\mu}_a^D - \underline{V}_{aP_{m(W)}} = (T^W)^{-1}(\underline{V}_{aC_W} - C_W\underline{V}_{aP_{A(W)}}) \tag{5.13}$$

where $P_{m(W)}$ designates the world price index whose weights are a risk tolerance weighted average of marginal expenditure shares and $P_{A(W)}$ designates the world price index whose weights are a consumption expenditures weighted average of average expenditure shares. (5.13) is the fundamental asset pricing equation of this paper. It must be noted that the equation gives the equilibrium expected returns as viewed in the domestic country. That equation holds no matter what the state variables are and no matter how many futures markets there are. (5.13) holds whether purchasing power parity holds or not. The next section is devoted to a detailed discussion of the economic implications of the pricing equation (5.13).

Section 6: Economic implications of equilibrium relationships

In Section 5, we obtained a general equation for the pricing of risky assets. In this section, we discuss the economic implications of that equation. First, we look at the pricing of various assets in the most general case. Secondly, we look at how the general equation is transformed when various simplifying assumptions are made. Finally, we discuss the implications of the general equation for the relationships between aggregate consumption among different countries.

6.1. Pricing of various assets.

Premultiplying (5.13) by $\underline{w}_a^{S'}$, which is the vector of supplies of the various assets, expressed as fractions of world wealth in domestic money, we get:

$$T^W \mu_M^D = V_{MC_W} + C_W V_{MP_{A(W)}} - T^W V_{MP_{m(W)}} \tag{6.1}$$

where M stands for world market portfolio of risky assets. Note that μ_M^D is the expected excess return on the world market portfolio of risky assets. We can solve (6.1) for T^W, and substitute that solution in (5.13) to obtain:

$$\underline{\mu}_A^D - \underline{V}_{aP_{m(W)}} = \left\{\frac{\underline{V}_{aC_W} - C_W \underline{V}_{aP_{A(W)}}}{V_{MC_W} - C_W V_{MP_{A(W)}}}\right\} \left[\mu_M^D - V_{MP_{A(W)}}\right] \tag{6.2}$$

However, by using Ito's Lemma it follows that:

$$V_{a_i C_W} - C_W V_{a_i P_{A(W)}} = C_{WR} V_{a_i C_{WR}} \tag{6.4}$$

where C_{WR} is world consumption divided by $P_{A(W)}$, which is a price index of weighted average expenditure shares. Let:

$$\frac{V_{a_i C_{RW}}}{V_{MC_{RW}}} = \frac{\beta_{iC}}{\beta_{MC}} \tag{6.4}$$

Then we can write:

$$\mu_a^D - V_{aP_{m(W)}} = \frac{\beta_{aC}}{\beta_{MC}} \left[\mu_M^D - V_{MP_{m(W)}}\right] \tag{6.5}$$

(6.5) tells us that the excess return of a risky asset depends on (a) the covariance of the excess return of that asset with world consumption divided by the world price index which would be accurate if investors had constant expenditure shares, divided by the covariance of the excess return of the world market portfolio with the same price index, (b) the excess return on the world market portfolio, (c) the covariance of the excess return of that asset and of the excess return of the world portfolio of risky assets with a risk tolerance weighted marginal expenditure shares price index.

In the next part of this section, we will discuss in detail the role of the various price indices. A subject of key interest in models of international asset pricing is how bonds - both nominal bonds and index-bonds - are priced. Asset Z is the asset whose payoff corresponds to the excess return of the foreign nominal bond for a domestic resident or to the excess return of a domestic bond plus σ_e^2 for a foreign resident. β_{ZC} is simply β_{eC}. This yields the following result for the forward premium:

$$\mu_B - \mu_{B*} = \mu_e - V_{eP_{m(W)}} - \frac{\beta_{eC}}{\beta_{MC}} \left[\mu_M^D - V_{MP_{m(W)}}\right] \tag{6.6}$$

The key point to (6.6) is stated in the following proposition.

> Proposition 5. Ceteris paribus, an increase in the covariance between the exchange rate e and world real consumption, will translate itself into an increase in the differential between the foreign nominal rate of interest and the domestic nominal rate of interest.

The risk premium of the forward rate, as given by (6.6), can be different from zero when other models argue it should be zero and can be equal to zero when other models claim it should be different from zero. It has been argued that if there are so-called outside assets, then there will be a risk premium.[22] In our model, so-called outside assets can increase or fall without changing the risk premium. In general, in a truly intertemporal model, changes in today's stock of outside assets will affect the rate of consumption of the representative investor through its effect on the path of the government flow budget constraint. It follows that whereas the existence of a stock of so-called outside assets ensures that those assets will be correlated with the world market portfolio, there is no guarantee that they will be correlated with aggregate real consumption, in the sense that an increase in the stock of outside assets can be accompanied by an increase in the expected tax liability of investors in a way which just cancels off the effect of that increase in outside assets! Of course, few economists would expect the effects of an

---

[22] See Kouri (1976), Kouri (1977) and Fama and Farber (1979). That result is also obtained in Frankel (1978).

increase in outside assets to cancel each other exactly for all investors. However, in a model with uncertainty, the ultimate outcome of a change in the stock of outside assets is going to depend, among other things, on the risk aversion coefficient of the investors and how complete markets are.

The models of Kouri (1977) and Fama and Farber (1979) are not compatible with the existence of effects of a change in the stock of outside assets on the tax liability of investors, unless investors do not have rational expectations. It follows that for the models of Kouri (1977) and Fama and Farber (1979) to be truly equilibrium intertemporal models, extremely strong assumptions have to be made about the distribution of changes in the stocks of outside assets. In our model, independently of whether naive purchasing power parity holds or not, more realistic monetary policies have complex effects which do not allow to say that <u>necessarily</u> the existence of so-called outside assets will create a risk premium.

The reason for which it is possible to have a risk premium without outside assets in our model is more obvious. If a state variable is correlated with the exchange rate, there will be some investors who will want to hold foreign nominal bonds to hedge against unanticipated changes in that state variable. We have seen at the end of Section 4 that if P and P* change independently of changes in relative prices, e would be correlated with some money prices. This is enough to create the possibility of a risk premium.

We now turn to the excess rate of return on index-bonds. Clearly, what we said about the premium on the nominal bonds did not depend on whether or not index-bonds exist, except for the fact that the consumption path depends on the assets available. However, we study the pricing of index-bonds only when nominal bonds are available. In that case, the required excess return on the domestic index-bond is:

$$\mu_H - \mu_B = V_{PP_{m(W)}} + \frac{\beta_{PC}}{\beta_{MC}} \left[\mu_M^D - V_{MP_{m(w)}}\right] \tag{6.7}$$

The risk premium on index-bonds depends on the covariance between the price index on which those bonds are indexed and world real consumption. As naive purchasing power parity does not need to hold in our model, it is perfectly possible for β_P to be different from β_{eP*}, which shows that the real rate of return of the domestic index bond can be different from the real rate of return of the foreign index-bond. The difference between the expected nominal rates of return of the index-bonds are:

$$\begin{aligned} \mu_H - \mu_{H*} &= \mu_e - V_{eP_{m(W)}} - \frac{\beta_{eC}}{\beta_{MC}} \left[\mu_M^D - V_{MP_{m(W)}}\right] \\ &+ \frac{(\beta_P - \beta_{P*})}{\beta_{MC}} \left[\mu_M^D - V_{MP_{m(W)}}\right] - V_{(eP*)P_{m(W)}} \end{aligned} \tag{6.8}$$

If naive purchasing power parity holds, the terms involving beta coefficients cancel out and $V_{PP_{m(W)}} - V_{(eP*)P_{m(W)}}$ cancels out. We can then write:

$$\mu_H - \mu_{H*} = \mu_P - \mu_{P*} - V_{PP*} - V_{eP} + V_{P*P*}$$

However, using Ito's Lemma:

$$V_{PP*} - V_{eP} = V_{P*P*} - V_{PP}$$

This yields finally:

$$\mu_H - \mu_P + \sigma_P^2 = \mu_{H*} - \mu_{P*} + \sigma_{P*}^2$$

The l.h.s. of that equation is the real rate of return at home of the domestic index-bond, whereas the r.h.s. is the real return abroad of the foreign index-bond. Clearly, when naive purchasing power parity does not hold:

$$\beta_{eC} \neq \beta_{PC} - \beta_{P*C}$$

if P, P* and e are correlated with real world aggregate consumption.

6.2. Alternative assumptions.

<u>Proposition 6.</u> Suppose there are Q assets whose returns are not correlated with state variables. If (a) naive purchasing power parity holds, or (b) none of those Q assets are correlated with e, then:

$$\mu_i = \beta_i \mu_Q = \frac{\beta_{iC}}{\beta_{QC}} \mu_Q \qquad \forall\ i \in Q \tag{6.9}$$

where:

$$Q = \sum_{i \in Q} w^S_{da_i} \frac{dA_i}{A_i}$$

$$\beta_i = \mathrm{Cov}(A_i, Q)/\mathrm{Var}(Q)$$

$$\beta_{iC} = \mathrm{Cov}(A_i, C)$$

$$\beta_{QC} = \mathrm{Cov}(Q, C)$$

Note that the weights of portfolio Q do not need to sum to one and that μ_i is the expected excess return of the i-th asset belonging to Q and μ_Q is the expected excess return on the portfolio of assets belonging to Q. When some assets belonging to Q are correlated with the exchange rate, then (6.9) is an equation which gives real expected excess returns. This can happen, because of the conditions of Proposition 6, only if naive purchasing power parity holds. When naive purchasing power parity does not hold, (6.9) gives nominal expected returns.

Proposition 6 is important, because it allows one to understand under which conditions a straightforward extension of the Sharpe-Lintner asset pricing model holds in this paper.[23] Naive purchasing power parity or the assumption about the correlations of the assets in Q with the exchange rate guarantees that the relevant returns are the same for all investors. It is then easy to divide the assets into two groups, those correlated with state variables and those which are not. Once this is done, an equation like (4.1) can be written; after eliminating the terms equal to zero and using the results of Section 5, it is easy to verify that Proposition 6 holds. Assumptions (a) and (b) eliminate the covariance terms in μ_i involving the exchange rate.

Proposition 7. If $\mu^{DR}_{a_i}$ is the real excess return on asset i in

[23] See Sharpe (1964) and Lintner (1965). A simple extension of the Sharpe-Lintner model holds for earlier models as far as the pricing of common stocks is concerned.

terms of the price index $P_{m(W)}$ in domestic currency, then:

$$\underline{\mu}_a^{DR} = \frac{\beta_{aC}}{MC} \mu_M^{DR} \tag{6.10}$$

Proposition 7 gives the asset pricing equation in its most compact form. Note simply that:

$$d\left(\frac{I_i}{P_{m(W)}}\right) \frac{P_{m(W)}}{I_i} - d\left(\frac{B}{P_{m(W)}}\right) \frac{P_{m(W)}}{B} = \frac{dI_i}{I_i} - V_{a_i P_{m(W)}} - \frac{dB}{B}$$

Starting from nominal demand functions for assets for individual investors, Proposition 7 shows that we actually ended up with pricing relationships which involve real rates of return and real covariances. However, the pricing equation (6.10) involves the use of a price index which is a risk aversion weighted marginal expenditure shares price index. Whereas this does not imply that the price index can not be observed, cases in which the risk aversion coefficients can be eliminated are useful. The risk aversion coefficient can clearly be eliminated if the marginal expenditure shares are the same for all investors. This case has been studied by Breeden (1979) for a closed economy. When two economies are considered, the hypothesis that marginal expenditure shares will be identical among all investors of <u>one</u> country but different among countries slightly more attractive. In that case, the pricing equation becomes:

$$\underline{\mu}_a^D - \frac{T^D}{T^W}\underline{V}_{aP_m} - \frac{T^F}{T^W}(\underline{V}_{aP_{m*}} - \underline{V}_{ae}) =$$

$$\frac{\beta_{aC}}{\beta_{MC}} \left[\mu_M^D - \frac{T^D}{T^W} V_{MP_m} - \frac{T^F}{T^W}(V_{MP_{m*}} - V_{Me})\right]$$

where P_{m*} is the marginal expenditure shares index of foreign investors in foreign currency and P_m is the marginal expenditure shares index of domestic investors in domestic currency.

Ceteris paribus, an increase in the covariance of an asset's nominal return with the price index $P_{A(W)}$ which is the index of average expenditure shares, is going to decrease the required expected excess return of that asset. However, at the same time, an increase in the covariance of that asset with the marginal expenditure shares price index, ceteris paribus, will increase the required expected excess return on that asset. The two indices have opposite effects on the required nominal excess return of an asset. Investors value an asset because it is correlated with real consumption. An asset whose price is highly correlated with $P_{A(W)}$ can not in general be highly correlated with real consumption. At the same time, investors are concerned with the real value of the expected return of an asset. That real value depends on what they are going to buy with one more dollars to spend, and is given for society as a whole by the marginal expenditure shares price index. When there are many commodities whose price varies stochastically over time, the average and marginal purchasing power of one unit of money varies across investors. The price indices used in our pricing equations are pure theoretical constructions. There may well exist no investor whose price indices are those of the world as a whole. The commodities which have positive weights in the price indices of one investor may even have zero weights in the price indices of another investor. Note finally that an interesting assumption is $T^D = T^F$. In that case, the risk aversion coefficients dissappear and one needs only to know the marginal expenditure shares and average expenditure shares indices for each country.

6.3. Correlation of consumption across countries.

Breeden (1979) shows that for a closed economy with one consumption good and complete markets, consumption across investors will be perfectly correlated. We can get a result equivalent to the result of Breeden. The result can be formulated in a way which shows a fundamental difference between a world in which naive purchasing power parity holds and a world in which it does not hold.

> Propostion 8. If markets are complete, in the sense that an unconstrained Pareto-optimal equilibrium is achieved, then:
>
> (1) If naive purchasing power parity holds, all investors will have their consumption expenditures evaluated in a common currency perfectly correlated.
>
> (2) If naive purchasing power parity does not hold, only investors who have identical utility functions and identical wealth, will have perfectly correlated consumption.

note immediately that the concept of complete markets used here implies that Proposition 3 holds, but that the number of markets required by Proposition 8 will in general be greater than the number of markets required by Proposition 3. In Appendix I we show that the covariance between the consumption of a domestic investor i and the consumption of a foreign investor j, expressed in domestic currency, is:

$$\text{Cov}(C_i, eC_j^*) = e\left[T_i\underline{\mu}_S - \underline{\underline{V}}_{SS}\binom{\underline{m}_i}{\underline{0}}T_i\right]' \underline{\underline{V}}_{SS}^{-1}\left[T_j^*\underline{\mu}_S^* - \underline{\underline{V}}_{SS}\binom{\underline{m}}{\underline{0}}T_j^*\right]$$

$$+ eC_i\binom{\underline{\alpha}_i}{\underline{0}}' \left[T_j^*\underline{\mu}_S^* - \underline{\underline{V}}_{SS}\binom{\underline{m}_j^*}{\underline{0}}T_j^*\right] + T_i\left[\underline{\mu}_S\right.$$

$$\left. - \underline{\underline{V}}_{SS}\binom{\underline{m}_i}{\underline{0}}T_i\right]' eC^*\binom{\underline{\alpha}_j^*}{\underline{0}} + C_i\binom{\underline{\alpha}_i}{\underline{0}}'\underline{\underline{V}}_{SS}\binom{\underline{\alpha}_j^*}{\underline{0}}eC_j^*$$

The variance of the consumption of the domestic investor is easily obtained by setting $e = 1$, and $T_j^* = T_i$, $\underline{\mu}_S^* = \underline{\mu}_S$, $\underline{m}_j^* = \underline{m}_i$, $\underline{\alpha}_j^* = \underline{\alpha}_i$ and $C_j^* = C_i$. In the same way, it is possible to obtain the variance of the consumption of the foreign investor. If naive purchasing power parity holds, then $\underline{\underline{V}}_{SS}\binom{\underline{m}_i}{\underline{0}}$ is the covariance of the state variables with the price index of the domestic investor or of the foreign investor in domestic money, as $\underline{m}_i = \underline{m}_j^* = \underline{\alpha}_i = \underline{\alpha}_j^*$ in that case. We can eliminate the price level from $\text{Cov}(C_i, eC_j^*)$ by choosing to look at the covariance between real consumption. Looking at real consumption and real returns, we get:

$$\text{Cov}(C_i^R, C_j^R) = T_iT_j^*\underline{\mu}_S'V_{SS}^{-1}\underline{\mu}_S$$

In that case:

$$\sigma^2_{C_i^R} = T_i^2\underline{\mu}_S'\underline{\underline{V}}_{SS}^{-1}\underline{\mu}_S$$

$$\sigma^2_{C_j^R} = T_j^2\underline{\mu}_S'\underline{\underline{V}}_{SS}^{-1}\underline{\mu}_S$$

Hence:

$$\frac{\text{Cov}(C_i^R, C_j^R)}{\sqrt{\sigma^2_{C_i^R}}\sqrt{\sigma^2_{C_i^R}}} = 1$$

Note that it does not matter why naive purchasing power parity holds. In other words, it is possible to have many goods whose relative price changes stochastically over time, as long as all investors consume those goods in fixed proportions.

$Cov(D_i, eC^*_j)$ depends on the individual investor i and j only through T_i, eT^*_j, $\underline{m}_i$, $\underline{m}^*_j$, $\underline{\alpha}_i$, $\underline{\alpha}^*_j$, C_i, eC^*_j. Our assumption sets all those variables equal among investors. In that case, $Cov(C_i, eC^*_j)$ is simply σ^2_C. It is shown in Appendix I that this is indeed a necessary and sufficient condition for the consumptions of two investors to be perfectly correlated.

Note that the result about the covariance of consumption when naive purchasing power parity holds should not be too surprizing. When naive purchasing power parity holds, real returns on risky assets are the same for all investors in the world. Investors hedged against unanticipated changes in state variables which are not relative prices will have perfectly correlated unanticipated changes in real wealth, as they all have the same price index.

Section 7: The results in perspective

If naive purchasing power parity holds, it can be argued that international asset pricing is irrelevant. We have seen that if naive purchasing power parity holds, consumption across countries will be perfectly correlated. Using that fact, we can write:

$$\underline{\mu}_a^D - \underline{V}_{aP_{m(W)}} = \frac{\underline{\beta}_{aC_D}}{\beta_{M_D C_D}} \left[\mu^D_{M_D} - V_{M_D P_{m(W)}} \right] \tag{7.1}$$

where the subscript D stands for domestic. M_D designates the market portfolio of risky assets of the domestic economy. Note that if naive purchasing power parity always holds, for any asset i, $\beta_{a_i C_D}/\beta_{M_D C_D} = \beta_{a_i C}/\beta_{M_D C}$. $\beta_{a_i C_D}$ is the consumption beta of the i-th asset, but here consumption is domestic real consumption. Because naive purchasing power parity holds, the real return at home of a risky asset is the same as the real return abroad of the same risky asset. It follows that $\underline{V}_{aP_{m(W)}}$ is equal to $\underline{V}_{aP_m}$, where P_m is a domestic price index. Of course, naive purchasing power parity also implies that the domestic price index which uses marginal expenditure shares is identical to the price index which uses average expenditure shares. When naive purchasing power parity holds, the relationship (6.11) holds and that relationship corresponds exactly to the relationship obtained by Breeden (1979).

When naive purchasing power parity does not hold, international asset pricing plays a crucial role. Indeed, when naive purchasing power parity does not hold, equilibrium expected returns on risky assets can not, in general, be obtained without using foreign data.

The only exception to our statement is the case in which markets are complete in an Arrow-Debreu sense.[24] If markets are complete, the equilibrium expected returns on domestic risky assets can be recovered from the prices of the primitive securities which span the state space. Clearly, for operational purposes, markets are not complete in an Arrow-Debreu sense, because it is not possible to observe the prices of primitive securities which span the state space.

If naive purchasing power parity does not hold, the deviations from naive purchasing power parity will affect the equilibrium expected returns of risky assets. Deviations from naive purchasing power parity will create differential demands for risky assets, because investors want to hedge against unanticipated deviations from naive purchasing power parity. With the model presented in this paper, the nature of deviations from naive purchasing power parity does not matter for the pricing relationships we derive, as long as we can assume that all prices are equilibrium prices. For some purposes, this generality should be useful. For other purposes, it would clearly be useful to know more about how various types of exchange rate and commodity price dynamics affect equilibrium expected returns. In the remainder of this section, we do not intend to study equilibrium expected returns for a particular class of exchange rate and commodity price dynamics, but rather to show that our model puts less restrictions on the nature of deviations from naive purchasing power parity than it might appear.

---

[24] I thank Sandy Grossman for a useful suggestion related to this point.

Until now, we had two types of goods: traded commodities and non-traded commodities. What is relevant, for the existence of deviations from naive purchasing power parity, is not whether or not a commodity will <u>always</u> be non-traded. All that is needed is that some goods can not be traded instantaneously. Whenever instantaneous arbitrage is not possible for commodity i, it is possible that a time t, $P_i(t) \neq e(t)P^*_i(t)$. The nature of the deviations from naive purchasing power parity will impose restrictions on how much $P_i(t)$ can differ from $e(t)P^*_i(t)$. However, for our model, the expected path of commodity prices does not matter, as long as it is an equilibrium path, in the sense that it does not affect the nature of the pricing relationships we obtain. It follows that it does not matter whether or not investors expect that at some date T, $P_i(T) = e(T)P^*_i(T)$. However, $P_i(T) = e(T)P^*_i(T)$ could be obtained in two different ways: (1) Investors expect factors of production to move; (2) Investors expect commodity i to move. There is no good reason to state that, for a given good i, producers can not direct their shipments in such a way that they expect $P_i(T) = e(T)P^*_i(T)$ to hold. Roll (1979) seems to provide some good empirical arguments for a world which would look like that, with T being close to t. Our model does not require that a commodity which is not instantaneously traded will never be traded. It should be clear by now that differences in tastes are not essential for our analysis, in the sense that all investors could have the same utility function, except for the fact that the commodities they consume have different <u>locations</u>, and (6.11) would still not hold. For international asset pricing to be relevant, it is enough to claim that commodity arbitrage, in the sense of a riskless transaction, is not possible for all commodities. If for most commodities, instantaneous arbitrage is not possible, one would think that the role trade flows play in reducing deviations of P_i from eP_i^* for all non-instantaneously traded commodities, would affect asset prices in significant ways. Future research should explore the effects of trade flows on asset prices in a general equilibrium setting.

Appendix I

We drop subscripts i and j. Because of Ito's Lemma and the assumption of complete markets:

$$\mathrm{Cov}(ec^*,c) = c_w \underline{w}_S' \underline{\underline{V}}_{SS} \underline{w}_S^* \, WW^* ec^*_w + ec_w \underline{w}_S' \underline{\underline{V}}_{SS} \underline{C}_S^*$$

$$+ ec^*_w \underline{w}_S^* \underline{\underline{V}}_{SS} \underline{c}_S W^* + \underline{C}_S' \underline{\underline{V}}_{SS} \underline{C}_s^* \, e \tag{I.1}$$

where $\underline{\underline{V}}_{SS}$ is the variance-covariance matrix of state variables and $\underline{w}_S$ is a vector of proportions of wealth invested in assets perfectly correlated with state variables. Note that:

$$c_w \underline{w}_S' \underline{\underline{V}}_{SS} \underline{\underline{V}}_{SS}^{-1} \underline{\underline{V}}_{SS} \underline{w}_S^* ec^*_w WW^* = [T\underline{\mu}_S - \underline{\underline{V}}_{SS} \underline{c}_S + \underline{\underline{V}}_{SS} c \left(\frac{\alpha}{\underline{0}}\right) - \underline{\underline{V}}_{SS} \left(\frac{m}{\underline{0}}\right) T] \cdot$$

$$\underline{\underline{V}}_{SS}^{-1} [eT^* \underline{\mu}_S^* - \underline{\underline{V}}_{SS} e\underline{c}_S^* + \underline{\underline{V}}_{SS} ec^* \left(\frac{\alpha}{\underline{0}}\right) - \underline{\underline{V}}_{SS} \left(\frac{m^*}{\underline{0}}\right) eT^*] \tag{I.2}$$

$$\underline{c}_S' \underline{\underline{V}}_{SS} \underline{w}_S^* ec^*_w W^* = \underline{c}_S' [eT^* \underline{}_S - \underline{\underline{V}}_{SS} e\underline{c}_S^* + \underline{\underline{V}}_{SS} ec^* \left(\frac{\alpha}{\underline{0}}\right)$$

$$- \underline{\underline{V}}_{SS} \left(\frac{m^*}{\underline{0}}\right) T^* e] . \tag{I.3}$$

where $\underline{\mu}_S$ is the vector of excess returns for a domestic investor of the S assets which are each perfectly correlated with one state variable. From (I.1), I.2) and (I.3) it is straightforward to obtain the expression on top of page . For consumption in domestic currency to be perfectly correlated, the expression on top of page has to be equal to the product of the standard deviation of the consumption of each investor. Let $\underline{\mu}_S^R$ be the real excess returns on the assets perfectly correlated with state variables for the domestic investor, whereas $\underline{\mu}_{S*}^R$ are the real returns for a foreign investor. We need:

$$\left\{ e\underline{\mu}_S^R \underline{\underline{V}}_{SS}^{-1} \underline{\mu}_{S*}^R \; TT* + eC\left(\underline{\overset{\alpha}{0}}\right)' \underline{\mu}_{S*}^R \; T* \right.$$

$$\left. + \; eC*\left(\frac{\alpha}{\underline{0}}\right)' \underline{\mu}_S^R \; T + eC\left(\underline{\overset{\alpha}{0}}\right)' \underline{\underline{V}}_{SS}\left(\frac{\alpha*}{\underline{0}}\right) C*^{\;2} \right\}$$

$$= \left\{ \underline{\mu}_S^R \underline{\underline{V}}_{SS}^{-1} \underline{\mu}_S^R \; TT* + 2C\left(\frac{\alpha}{\underline{0}}\right)' \; \underline{\mu}_S^R \; + \right.$$

$$\left. C\left(\frac{\alpha}{\underline{0}}\right)' \underline{\underline{V}}_{SS}\left(\frac{\alpha}{\underline{0}}\right) C \right\} \left\{ \underline{\mu}_{S*}^R \underline{\underline{V}}_{SS}^{-1} \; \underline{\mu}_{S*}^R \; T*T*e^2 \right.$$

$$\left. + \; 2e^2 C*\left(\frac{\alpha*}{\underline{0}}\right)' \underline{\mu}_{S*}^R + e^2 C*\left(\frac{\alpha*}{\underline{0}}\right)' \underline{\underline{V}}_{SS}\left(\frac{\alpha*}{\underline{0}}\right) C* \right\} \qquad \text{(I.4)}$$

From (I.4) it is straightforward to obtain the conditions of our proposition.

REFERENCES

Adler, M. and B. Dumas, 1975, "Optimal International Acquisitions," Journal of Finance, Vol. 30, 1, 1-19.

Adler, M. and B. Dumas, 1976, "Portfolio Choice and the Demand for Forward Exchange," American Economic Review, 66, 332-339.

Arnold, L., 1974, Stochastic differential equations, Wiley & Sons.

Blassa, B., 1964, "The Purchasing Power Parity Doctrine: A Reappraisal," Journal of Political Economy, 72, 584-596.

Bilson, O., "Recent developments in monetary models of exchange rate determination," IMF Staff Papers, 26, 201-223.

Black, F., 1974, "International Capital Market Equilibrium with Investment Barriers," Journal of Financial Economics, 1, 337-352.

Black, F., 1976, "The Pricing of Commodity Contracts," Journal of Financial Economics, 3, 167-179.

Breeden, D.T., 1978, "The Role of Futures Markets in a Continuous-Time Model," mimeo.

Breeden, D.T., 1978, "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities," Journal of Financial Economics, 7, 265-296.

Brock, W.A., "Introduction to Stochastic Calculus: A User's Manual," mimeo.

Cox, J.C., Ingersoll J.E. and S.A. Ross, 1978, "A Theory of the Term Structure of Interest Rates," mimeo.

Dornbusch, R., 1979, "Monetary Policy under Exchange Rate Flexibility," in Managed Exchange Rate Flexibility, Conference volume No. 20, Federal Reserve Bank of Boston, Boston.

Dumas, B., 1977, Discussion, Journal of Finance, 38, 512-515.

Dunn, R.M., 1970, "Flexible Exchange Rates and Oligopoly Pricing: A Study of Canadian Markets," Journal of Political Economy, 78 140-151.

Fama, F.F. and G.W. Schwert, 1979, "Inflation, Interest and Relative Prices," Journal of Business, 52, 183-209.

Fama, E.F. and A. Farber, 1979, "Money, Bonds and Foreign Exchange," American Economic Review, 69, 639-649.

Fischer, S., 1974, "Money and the Production Function," Economic Inquiry, 12, 517-533.

Fischer, S., 1975, "The Demand for Index-Bonds," Journal of Political Economy, 83, 509-534.

Frankel, J., 1978, "Exchange Rates Since Floating: Theory and Evidence," Unpublished Dissertation, M.I.T.

Frenkel, J.A., 1978, "A Monetary Approach to the Exchange Rate: Doctrinal Aspects and Empirical Evidence," in J.A. Frenkel and H.G. Johnson, The Economics of Exchange Rates, Wesley, Reading, Mass.

Friedman, A., 1975, Stochastic differential equations and applications, Vol. 1, Academic Press, New York.

Genberg, H., 1975, World Inflation and the Small Open Economy, eds., Swedisch Industrial Publications, Stockholm.

Genberg, H., 1978, "Purchasing Power Parity under Fixed and Flexible Exchange Rates," Journal of International Economics, 8, 247-276.

Grauer, F.L.A., R.H. Litzenberger and R.E. Stehle, 1976, "Sharing Rules and Equilibrium in an International Capital Market under Uncertainty," Journal of Financial Economics, 3, 233-256.

Hellwig, M., 1977, "On the validity of the intertemporal capital asset pricing model," CORE Discussion Paper, No. 7744.

Holmes, J.M., 1967, "The Purchasing Power Parity Theory: In Defence of Gustav Cassel as a Modern Theorist," Journal of Political Economy, 75, 686-695.

Isard, P., 1977, "How Far Can we Push the 'Law of One Price'," American Economic Review, 67, 942-948.

Isard, P., 1978, "Exchange Rate Determination: A Survey of Popular Views and Recent Models," Princeton Studies in International Finance, No. 42,

Kohlhagen, S.M., 1978, "The Behavior of Foreign Exchange Markets - A Critical Survey of the Empirical Literature," New York University Series in Finance and Economics, New York University Press, New York.

Kouri, P.K., 1976, "The Determinants of the Forward Premium," mimeo., Institute for International Economic Studies, Stockholm.

Kouri, P.K., 1977, "International Investment and Interest Rate Linkages under Flexible Exchange Rates," in R.Z. Aliber, ed., The Political Economy of Monetary Reform, New York.

Kouri, P.K. and B. de Macedo, 1978, "Exchange rates and the international adjustment process," Brookings Papers, 1, 111-158.

Kravis, I.B. and Robert E. Lipsey, "Price Behavior in the Light of Balance of Payments Theories," Journal of International Economics, 8, 193-246.

Lee, M.A., 1976, Purchasing Power Parity, Marcel Dekker, Inc., New York.

Lessard, D.R., 1979, "Evaluating Foreign Projects: An Adjusted Present Value Approach," in D.R. Lessard, ed., International Financial Management, Gorham and Lamont.

Lintner, J., 1965, "Security Prices, Risk, and Maximal Gains from Diversification," Journal of Finance, 20, 587-615.

Merton, R.C., 1971, "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," Journal of Economic Theory, 3, 373-413.

Merton, R.C., 1973, "An Intertemporal Capital Asset Pricing Model," Econometrica, 41, 867-887.

Merton, R.C., 1978, "On the Mathematics and Economic Assumptions of Continuous-Time Models," mimeo., Massachusetts Institute of Technology.

Officer, L., 1976, "The Purchasing Power Parity Theory of Exchange Rates: A Review Article," IMF Staff Papers, 23, 1-61.

Roll, R., 1979, "Violations of purchasing power parity and their implications for efficient international commodity markets," in Sarnat, M. and Szego G., International finance and trade, Vol. I, Ballinger, Cambridge, Mass.

Samuelson, P.A., 1948, "Disparity in Postwar Exchange Rates," in S. Harris, ed., Foreign Economic Policy for the United States, Harvard University Press, Cambridge,

Schadler, S., 1977, "Sources of Exchange Rate Variability: Theory and Empirical Evidence," IMF Staff Papers, 24, 253-296.

Sharpe, W.F., 1964, "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," Journal of Finance, 19, 425-442.

Solnik, B., 1973, European Capital Markets, Lexington Books, Lexington.

Solnik, B., "International Parity Conditions and Exchange Risk," Journal of Banking and Finance, 2, 281-293.

Stockman, A.C., "A Theory of Exchange Rate Determination," UCLA Working Paper, June.

Stockman, A.C., "International Relative Prices under Fixed and Flexible Exchange Rate Systems," UCLA Working Paper, September.

Stulz, R.M., 1979b, "Terms of Trade and International Investment," mimeo.

Stulz, R.M., 1980, "The forward rate and macroeconomics," in progress.

Stulz, R., 1980b, "On the effects of barriers to international investment," mimeo.

Viner, J., 1937, Studies in the Theory of International Trade, Harper and Row, New York.

Wihlborg, C., 1978, "Currency risks in international financial markets," Princeton studies in international finance, No. 44.

PART II

ON THE EFFECTS OF BARRIERS TO INTERNATIONAL INVESTMENT

Section 1: Introduction

The question which lies at the heart of international finance is: Does the fact that investors live in different countries mean that they face different investment opportunity sets? Most of the recent literature in international finance has dealt with the fact that countries have different monies.[1] Whereas we do not intend to minimize the importance of the issues associated with the concept of exchange rate risk, one must be aware that the fact that different countries use different monies is not the only raison d'être of international finance.

Some authors in the past have assumed that investors cannot buy foreign securities.[2] They did not claim that such an assumption is an exact representation of reality, but that in fact the difficulties associated with holding foreign securities are often important enough to make that assumption more accurate than the opposite assumption of no barriers to international investment. If barriers to international investment are significant, one would suspect that they are likely to play a larger role in determining an investor's portfolio than the fact that countries use different monies.

It is obviously not true that asset markets are completely segmented across countries. One can however find numerous facts which

---

[1] See the discussion and references in Stulz (1980).

[2] For instance, Adler and Dumas (1975). Stapleton and Subrahmanyam (1977) have a numeric example of a world in which there is incomplete segmentation, where the barriers to international investment are of a type which could generate complete segmentation.

show that there are barriers to international investment. Whereas reality seems to be in that grey area between complete segmentation and no segmentation at all, most asset pricing models are concerned with the extreme case of no barriers to international investment. The most important exception is the model of Black (1974).

The Black model assumes a tax on the value of holdings of foreign assets of an investor. That tax is viewed as a representation of barriers to international investment. He assumes that negative holdings of a particular risky asset mean a negative tax. As the tax increases, it is true that investors hold less foreign assets, but it may become optimal for them to sell foreign assets short in large amounts. It follows that complete segmentation of asset markets is not a limiting case of the Black model. Because short-sales are difficult even in domestic economies, one would suspect that it should be even more difficult to hold foreign assets short. Last, investors seem to hold small amounts of foreign assets, in the sense that their absolute position in each particular foreign risk asset is small, rather than a portfolio of foreign assets whose value is small, but which may contain large amounts of foreign securities held short.

In the present paper, the critical feature of the Black model which produces the results we just summarized is removed. This allows us to obtain a model of barriers to international investment which includes as a special case the case of complete segmentation. The Black model is a special case of our model. As in the Black model, barriers to international investment are called "taxes". It should be clear that what we have in mind is that, whereas taxes as such can be barriers to international investment, barriers to international investment can take a number of forms, some of them being non-pecuniary, and proportional taxes are a way to represent them.[3]

Because exchange risks are irrelevant for our argument, we use a framework in which exchange rates do not appear at all. Introducing exchange rates in our model would <u>not</u> change our results, in the sense that the effect of barriers to international investment would be the same as it is in this paper, as long as those barriers are of a type which in the limit can produce complete segmentation. Formally, the portfolios which obtain in this paper would obtain in a world in which there is only one good, in which there are neither transportation costs nor tariffs and in which there is a safe real bond in each country.

Finally, the model presented here is not a general equilibrium model, in the sense that the barriers to international investment are given and no attempt is made to explain those barriers. To the extent that those barriers correspond to taxes, no attempt is made to explain how the revenue from those taxes is spent.

[3] Clearly, not all barriers to international investment take the form of proportional costs. One would expect costs associated with increasing one's knowledge of foreign capital markets to be essentially "lump sum" costs.

Section 2: Optimal Portfolios

There are n^d risky assets in the domestic country and n^f risky assets in the foreign country. $E(\underline{\tilde{R}})$ is the $(n^d + n^f) \times 1$ vector of expected returns on risky assets. The first n^f elements of $E(\underline{\tilde{R}})$ correspond to the expected returns of domestic risky assets. The set D includes all domestic risky assets, whereas the set F includes all foreign risky assets. The matrix $\underline{\underline{V}}_{II}$ is the $(n^d + n^f) \times (n^d + n^f)$ variance-covariance matrix of the returns of risky assets, which is assumed to be non-singular. In the following, a capital letter underlined once indicates a vector. If $i \in F$, a domestic investor has to pay θ_i^{dl} on each dollar of asset i he holds long and θ_i^{ds} on each dollar of asset i he holds short. If $i \in D$, θ_i^{fl} and θ_i^{fs} are respectively the tax on long holdings and the tax on short holdings of security i for a foreign investor. It is assumed that, $\forall i$, $\theta_i^{dl} \geq \theta_i^{ds}$, $\theta_i^{fl} \geq \theta_i^{fs}$. A domestic investor does not have to pay a tax on his holdings of domestic assets. Let $\underline{w}_I^k$ be the vector of fractions of the k-th investor's wealth invested long in each risky asset, whereas $\underline{v}_I^k$ is the vector of fractions of investor k's wealth invested short in each risky asset. By definition, $\underline{w}_I^k \geq \underline{0}$, $\underline{v}_I^k \geq \underline{0}$, where $\underline{0}$ is an $(n^d + n^f) \times 1$ vector of zeros. If $k \in d$, investor k is a domestic investor, whereas if $k \in f$, he is a foreign investor. R^d is the domestic short-term interest rate, whereas R^f is the foreign short-term interest rate. θ_r^{dl} is the amount a domestic investor has to pay on each dollar he lends to a foreign investor, whereas θ_r^{ds} is the amount he has to pay on each dollar he borrows from a foreign investor. θ_r^{fl} and θ_r^{fs} are the taxes a foreign investor pays respectively on his loans to and borrowings from domestic investors. If international lending or borrowing occurs, either some domestic investors borrow abroad and none lend abroad, or some domestic investors lend abroad and none borrow abroad. Interest rates will always be such that if the domestic country borrows abroad, a domestic investor is indifferent between borrowing from a domestic investor and borrowing from a foreign investor; whereas if the domestic country lends abroad, a domestic investor is indifferent between lending to a domestic investor or lending to a foreign investor.

It is assumed that each investor's preferences can be represented by a utility function which depends positively on the espected returns of his portfolio and negatively on the variance of his portfolio. It follows that the optimal portfolio of investor k, k ε d, can be obtained by solving:

$$\text{Min } \frac{1}{2}(\underline{w}_I^k - \underline{v}_I^k)\underline{\underline{V}}_{II}(\underline{w}_I^k - \underline{v}_I^k)$$

so that:

$$\underline{w}_I^k \geq \underline{0} \tag{1}$$

$$\underline{v}_I^k \geq \underline{0} \tag{2}$$

$$\underline{w}_I^{k\prime}E(\underline{\tilde{R}}) - \underline{v}_I^{k\prime}E(\underline{\tilde{R}}) - \underline{w}_I^{k\prime}\underline{\theta}^{dl} - \underline{v}_I^k\underline{\theta}^{ds} + (1 - w_I^{k\prime}\underline{e} + v_I^{k\prime}\underline{e})R^d$$
$$\geq E(\tilde{R}^k) \tag{3}$$

where $\underline{e}$ is a $(n^d + n^f) \times 1$ vector of ones, $\underline{\theta}^{dl}$ is a $(n^d + n^f) \times 1$ vector of tax rates for domestic investors on long holdings, whereas $\underline{\theta}^{ds}$ is a $(n^d + n^f) \times 1$ vector of tax rates on short holdings. $\underline{w}_I^{k\prime}$ is the transpose of $\underline{w}_I^k$. $E(\tilde{R}^k)$ is given. It is assumed that for $k \in d$, $E(\tilde{R}^k) > R^d$, and for $k \in f$, $E(\tilde{R}^k) > R^f$.

With the assumptions we have made, there is a unique vector, $(\underline{w}_I^{ko} - \underline{v}_I^{ko})$ which is a solution to investor's optimization problem.[4] Let L^k be the Lagrangian function associated with the optimization problem of the k-th investor and let λ^k to be the multiplier associated with (3). For a portfolio to be optimal, it has to satisfy (1)-(3) and:

$$\frac{\partial L^k}{\partial \underline{w}_I^k} = \underline{\underline{V}}_{II}(\underline{w}_I^k - \underline{v}_I^k) - \lambda^k \left\{ E(\underline{\tilde{R}}) - \underline{e}R^d - \underline{\theta}^{dl} \right\} \geq 0 \quad (4)$$

$$\frac{\partial L^k}{\partial \underline{v}_I^k} = \underline{\underline{V}}_{II}(\underline{w}_I^k - \underline{v}_I^k) + \lambda_k \left\{ E(\underline{\tilde{R}}) - \underline{e}R^d + \underline{\theta}^{ds} \right\} \geq 0 \quad (5)$$

$$\underline{w}_I^{k\prime} \frac{\partial L^k}{\partial w_I^k} = 0 \quad (6)$$

$$\underline{v}_I^{k\prime} \frac{\partial L^k}{\partial \underline{v}_I^k} = 0 \quad (7)$$

Suppose there exists an asset i such that for that asset the inequality

---

[4] Note that the problem is very similar to the problem faced by the literature dealing with transaction costs, except that here the initial allocation does not matter. Smith and Milne (1979) derive equilibrium relationships in a model with transaction costs. Our problem, as described by the maximization problem and constraints (1)-(3), is the same problem as the one Smith and Milne would face if (1) investors have no initial allocation in securities and (2) some assets have zero transaction costs. For a discussion of optimal portfolios in the presence of transaction costs and references to the literature, see Abrams and Karmarker (1980).

holds strictly both in (4) and in (5). Inspection of (4)-(7) shows that there exists a positive number E such that a portfolio which is optimal for $E(\tilde{R}_i)$ is also optimal for $E(\tilde{R}_i) \pm E$. From (4)-(7), it is clear that an increase of E or a decrease of E in the expected return of the i-th asset will not affect the investor's holdings of that asset, which are equal to zero. In the following, we will call any asset which, for investor k, will be such that asset (4) and (5) hold with strict inequality, a <u>non-traded</u> asset for investor k.[5]

(4) and (5) can be rearranged in the following way:

$$\lambda^k \left\{ E(\underline{\tilde{R}}) - \underline{e}R^d + \underline{\theta}^{ds} \right\} \geq \underline{\underline{V}}_{II}(\underline{w}_I^k - \underline{v}_I^k) \geq \lambda^k \left\{ E(\underline{\tilde{R}}) - \underline{e}R^d - \underline{\theta}^{dl} \right\} \tag{8}$$

Inspection of (8) shows that a necessary condition for the existence of non-traded assets for investor k, k ε d, is that $-\underline{\theta}^{ds} \neq \underline{\theta}^{dl}$. It will immediately be recognized: (1) that if k ε d, all assets i ε D will be traded for investor k, and (2) that the Black model is the only possible model with $\theta_i^{dl} > 0$, for some i, which does not satisfy the necessary condition for the existence of non-traded assets for investor k.

---

5 Note that the literature on transaction costs has non-traded assets, but they differ across investors and generally belong to his initial allocation, which means that investors hold them, but do not trade them, whereas here investors do not hold them and do not trade them.

Adding together (6) and (7) yields:

$$- (\underline{w}_I^k - \underline{v}_I^k)'\underline{\underline{V}}_{II}(\underline{w}_I^k - \underline{v}_I^k) + \lambda^k \{(\underline{w}_I^k - \underline{v}_I^k)'E(\underline{\tilde{R}}) - \underline{w}_I^{k}{}'\underline{\theta}^{dl}$$

$$+ \underline{v}_I^{k}{}'\underline{\theta}^{ds} + (\underline{v}_I^k - \underline{w}_I^k)'\underline{e}R^d\} = 0 \tag{9}$$

Define λ_u^k so that:

$$\frac{\lambda_u^k}{\lambda^k} = \frac{1}{(\underline{w}_I^k - \underline{v}_I^k)'\underline{e}} \tag{10}$$

Because $E(\tilde{R}_k) > R^d$, $0 < \lambda_u^k < \infty$. If we multiply the first-order conditions by λ_u^k/λ^k, they still hold. However, the portfolio (λ_u^k/λ^k) $(\underline{w}_I^k - \underline{v}_I^k)$ has weights summing up to one. Let the expected return on that portfolio be $E(\tilde{R}^d)$. At the optimum, (3) holds with equality. Choose an investor k^u, so that investor requires an expected return on his portfolio of $E(\tilde{R}^d) = E(\tilde{R}^{k^u})$. If that investor buys risky assets in proportions $(\lambda_u^k/\lambda^k)(\underline{w}_I^k - \underline{v}_I^k)$, his portfolio will satisfy the first-order conditions (4) - (7). It is easily shown that portfolio (λ_u^k/λ^k) $(\underline{w}_I^k - \underline{v}_I^k)$ is a feasible portfolio for investor k^u. If investor k^u does not hold portfolio $(\lambda_u^k/\lambda^k)(\underline{w}_I^k - \underline{v}_I^k)$, this implies that the solution to the investor's optimization problem is not unique, but this case has been ruled out by our assumptions. It follows that all domestic investors hold the same portfolio of risky assets up to a scalar multiplication. Let the portfolio $(\lambda_u^k/\lambda^k)(\underline{w}_I^k - \underline{v}_I^k)$ be written $\underline{w}_I^d - \underline{v}_I^d$, and as the weights of that portfolio sum up to one, it can be interpreted as a mutual fund totally invested in risky assets. Each domestic investor k is indifferent between holding $(\underline{w}_I^k - \underline{v}_I^k)$ plus the safe asset or a combination of the safe asset and mutual fund $(\underline{w}_I^d - \underline{v}_I^d)$, which we write M_d. The same analysis could be replicated for foreign investors and would yield a mutual fund M_f. Because the investment opportunity set of foreign investors differs from the investment opportunity set of domestic investors, one would expect M_f to be different from M_d. The differences between M_f and M_d will be explored in Section 4.

Section 3: Equilibrium Relationships

In this section, equilibrium relationships which are comparable with those obtained by Black (1974) are derived.

(4) and (5) can be rewritten:

$$\frac{\lambda^k}{\lambda_u^k} W^k \underline{\underline{V}}_{II}^{-1} (\underline{w}_I^d - \underline{v}_I^d) - \lambda^k W^k \{E(\underline{\tilde{R}}) - \underline{e}R^d - \underline{\theta}^{dl}\} \geq \underline{0} \tag{11}$$

$$-\frac{\lambda^k}{\lambda_u^k} W^k \underline{\underline{V}}_{II} (\underline{w}_I^d - \underline{v}_I^d) + \lambda^k W^k \{E(\underline{\tilde{R}}) - \underline{e}R^d + \underline{\theta}^{ds}\} \geq \underline{0} \tag{12}$$

Relationships (11) and (12) hold for all domestic investors. Interchanging superscripts d and f in (11) and (12) yields relationships (11') and (12'), which are not written out, and which hold for foreign investors. Using (11) and (11') to aggregate across all investors yeilds:

$$\sum_{k \in d} \frac{\lambda^k}{\lambda_u^k} W^k \underline{\underline{V}}_{II} (\underline{w}_I^d - \underline{v}_I^d) + \sum_{k \in f} \frac{\lambda^k}{\lambda_u^k} W^k \underline{\underline{V}}_{II} (\underline{w}_I^f - \underline{v}_I^f) \geq$$

$$\sum_{k \in d} \lambda^k W^k \{E(\underline{\tilde{R}}) - \underline{e}R^d - \underline{\theta}^{dl}\} + \sum_{k \in f} \lambda^k W^k \{E(\underline{\tilde{R}})$$

$$- \underline{e}R^f - \underline{\theta}^{fl}\} \tag{13}$$

Define:

$$W^W = \sum_{k \in d} W^k + \sum_{k \in f} W^k \tag{14}$$

$$\lambda^d = \sum_{k \in d} \lambda^k W^k / W^W \tag{15}$$

$$\lambda^f = \sum_{k \varepsilon f} \lambda^k W^k / W^W \tag{16}$$

$$\lambda^m = \lambda^d + \lambda^f \tag{17}$$

$$\overline{R} = \frac{\lambda^d}{\lambda^m} R^d + \frac{\lambda^f}{\lambda^m} R^f \tag{18}$$

$$\underline{\theta}^1 = \frac{\lambda^d}{\lambda^m} \underline{\theta}^{d1} + \frac{\lambda^f}{\lambda^m} \underline{\theta}^{f1} \tag{19}$$

$$\underline{\theta}^s = \frac{\lambda^d}{\lambda^m} \underline{\theta}^{ds} + \frac{\lambda^d}{\lambda^m} \underline{\theta}^{fs} \tag{20}$$

Using those definitions, we can rewrite (13) as:

$$\underline{\underline{V}}_{II} \underline{w}_I^S \leq \lambda^m \{E(\tilde{\underline{R}}) - \underline{e}\overline{R} - \underline{\theta}^1\} \tag{21}$$

where $\underline{w}_I^S$ is the vector of supplies of risky assets expressed as proportions of world wealth. From (12) and (12'), we can get:

$$\underline{\underline{V}}_{II} \underline{w}_I^S \leq \lambda^m \{E(\tilde{\underline{R}}) - \underline{e}\overline{R} + \underline{\theta}^s\} \tag{22}$$

Define now:

$$\theta^{m1} = \underline{w}_I^{S\prime} \underline{\theta}^1 \tag{23}$$

$$\theta^{ms} = \underline{w}_I^{S\prime} \underline{\theta}^s \tag{24}$$

$$\sigma_m^2 = \underline{w}_I^{S\prime} \underline{\underline{V}}_{II} \underline{w}_I^S \tag{25}$$

Premultiply (21) and (22) by $\underline{w}_I^{S\prime}$. From the resulting expressions, we get:

$$\frac{E(\tilde{R}_m) - \overline{R} + \theta^{ms}}{\sigma_m^2} \geq \frac{1}{\lambda^m} \geq \frac{E(\tilde{R}_m) - \overline{R} - \theta^{1s}}{\sigma_m^2} \tag{26}$$

where $E(\tilde{R}_m)$ is the expected return on the world market portfolio of risky assets. It follows from (21), (22) and (26) that at the equilibrium we will have:

$$\beta^m \{E(\tilde{R}_m) - \bar{R}\} + \underline{\theta}^1 + \underline{\beta}^m \theta^{ms} \underline{\geq} E(\tilde{\underline{R}}) - \underline{e}\bar{R} \underline{\geq}$$

$$\{E(\tilde{R}_m) - \bar{R}\} - \underline{\theta}^s - \underline{\beta}^m \theta^{ml} \qquad (27)$$

where β_i^m is equal to $\mathrm{Cov}(\tilde{R}_i, \tilde{R}_m)/\sigma_m^2$. Before we go any further, we want to remind the reader that, in this model, the world market portfolio is not an efficient portfolio when tax rates are different from zero. A trivial example which shows why the world market portfolio is not an efficient portfolio is the case in which tax rates are infinite. In that case, no investor would ever hold foreign assets. The return on the world market portfolio for any investor would be minus infinity!

Note immediately that whenever $\underline{\theta}^1 = -\underline{\theta}^s$, (27) holds with equality and corresponds exactly to the pricing equation of the Black model. (27) shows that for the case in which we are most interested, which occurs whenever $\underline{\theta}^1 \neq -\underline{\theta}^s$, additional information is required if one wants to define the required excess expected rates of return uniquely.

Define:

$$\alpha_i = E(\tilde{R}_i) - \bar{R} - \beta_i^m \{E(\tilde{R}_m) - \bar{R}\} \qquad (28)$$

Substituting (28) in (27) yields:

$$\underline{\theta}^1 + \underline{\beta}^m \theta^{ms} \underline{\geq} \underline{\alpha} \underline{\geq} - \underline{\theta}^s - \underline{\beta}^m \theta^{ml} \qquad (29)$$

where $\underline{\alpha}$ is the $(n^d + n^f) \times 1$ vector of alphas. It follows from (29) that the deviations of before-tax required excess expected returns on risky assets from the before-tax required excess expected returns as predicted by the CAPM will be bounded from above and from below. Furthermore, if we know the tax rates and $\underline{\theta}^d$ and $\underline{\theta}^f$, we can predict the range within which those deviations will lie. That range will, if all assets are taxed equally and $\lambda^d = \lambda^f$, be a strictly increasing function of beta. If $\lambda^d \neq \lambda^f$, and if $\theta_i^{dl} = \theta^{dl}\forall i$, $\theta_i^{ds} = \theta^{ds}\forall i$, $\theta_i^{fl} = \theta^{fl}\forall i$, $\theta_i^{fs} = \theta^{fs}\forall i$, then the range within which the alphas must lie will be increasing in beta for <u>all</u> domestic assets, and will be increasing in beta for <u>all</u> foreign assets. However, in this case, it would be possible to have assets i and j, $\beta_i^m > \beta_j^m$, $i \in D$, $j \in F$, so that the range of α_i would be smaller than the range of α_j. As long as short sales carry a negative tax, even if $\theta_i^{ds} < \theta_i^{dl}$, $\theta_j^{fs} < \theta_j^{fl}$, it is possible to say unambiguously that assets with "large" betas will have negative alphas. Furthermore, this result holds as long as θ^{ms} is negative, which can occur even if in some countries short sales do not carry a negative tax. However, when short sales do not carry a negative tax "on average", the model does not allow us to predict that results similar to those of the Black model will obtain. Figure 1 shows the range of alphas for the case $\lambda_d = \lambda_f$, $\theta^{dl} = \theta^{ds} = \theta^{fl} = \theta^{fs}$. It must be noticed that as long as θ^{ml} is not too large, the lower bound of the required excess expected return will be increasing with beta, which is the case shown on the figure. Note also that (29) does not prevent the alphas from being a linear function of beta.

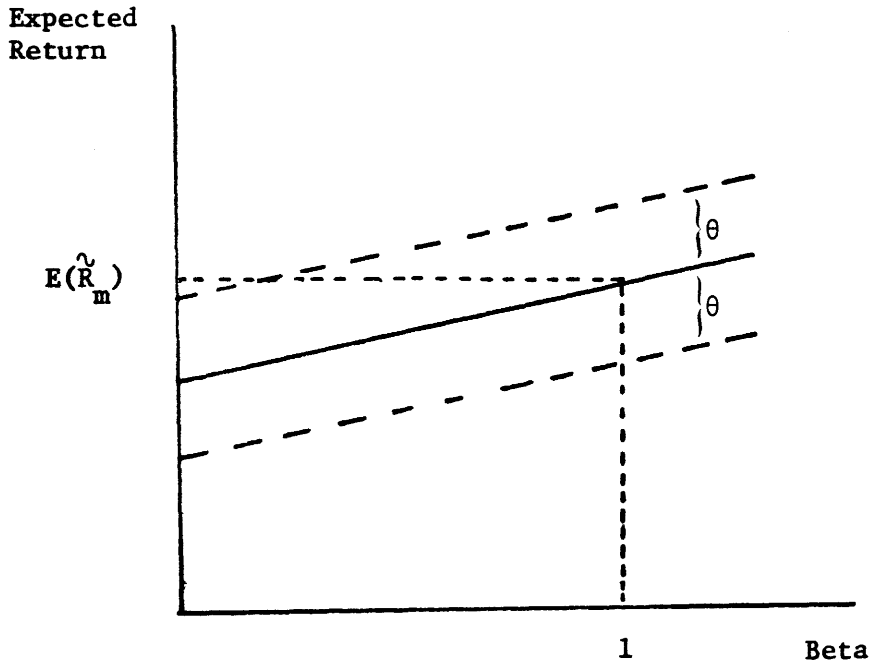

Figure 1

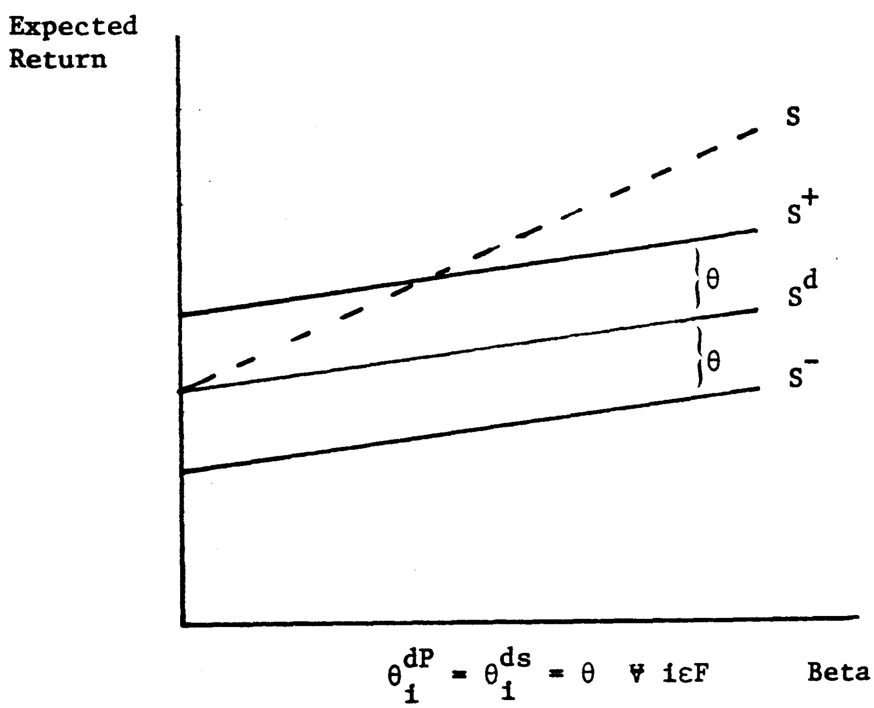

Figure 2

(28) has an important implication about the effect of differences in barriers to international investment across countries. Suppose there exists a country which is such that no investor who invests in that country would have to pay a tax on his investments. The range of alpha for a security of this country will be smaller than the range of alpha for any other security with the same beta but issued in another country. Furthermore, there will be no investor in the world for whom a security of that country will be a non-traded security. If $\theta^{ms} = 0$, all alphas for the securities of that country with respect to the world pre-tax security market line would be negative. As it seems easy for foreigners to invest in the U.S., the U.S. market portfolio could perform worse on average than a portfolio of foreign securities with an identical beta with respect to the world market portfolio as the U.S. market portfolio - on paper only, because in the real world, there would be barriers to international investment for U.S. investors who want to invest in foreign countries.

Alternatively, we can look at the pricing of assets with respect to the domestic portfolio. Define a_d to be the proportion of domestic wealth invested in risky assets. By definition:

$$a^d = \sum_{k \, \varepsilon \, d} \frac{\lambda^k W^k}{\lambda_u^k W^d} \qquad (30)$$

where W^d is domestic wealth. Define also:

$$\lambda_u^d = \sum_{k \, \varepsilon \, d} \frac{\lambda^k W^k}{a^d W^d} \qquad (31)$$

Aggregating (11) and (12) across investors, and using (30) and (31), we get:

$$\lambda_u^d \{E(R) - \underline{e}R^d + \underline{\theta}^{ds}\} \geq \underline{\underline{V}}_{II}(\underline{w}_I^d - \underline{v}_I^d) \geq \lambda_u^d \{E(R) - \underline{e}R^d - \underline{\theta}^{dl}\} \tag{32}$$

Premultiply (11) by $\underline{w}_I^d$, and (12) by $\underline{v}_I^d$, and add (11) and (12) after aggregating across investors. Define σ_d^2 as the variance of the return on mutual fund M_d, and θ^d as the total tax rate on mutual fund M_d. We get:

$$\sigma_d^2 = \lambda_u^d \ \{E(R_d) - R^d - \theta^d\} \tag{33}$$

It follows that the price of risk on the domestic mutual fund is uniquely defined by (33). Substitute (33) in (32) and rearrange to get:

$$\underline{\beta}^d \ \{E(R_d) - R^d\} + \underline{\theta}^{dl} - \underline{\beta}^d\theta^d \geq E(R) - \underline{e}R^d \geq \underline{\beta}^d \{E(R_d) - R_d\} - \underline{\theta}^{ds} - \underline{\beta}^d\theta^d \tag{34}$$

It follows from (34) that the return on all domestic assets is uniquely defined, as:

$$E(\tilde{R}_i) - R^d = \beta_i^d \ \{E(\tilde{R}_d) - R^d\} - \beta_i^d\theta^d \qquad i \in D \tag{35}$$

Note however that β_i^d, which is equal to $\mathrm{Cov}(R_i,R_d)/\sigma_d^2$, is defined with respect to the portfolio of risky assets of the domestic country, and not with respect to the market portfolio of the domestic country, which is a market value weighted portfolio of domestic securities. A similar relationship holds for foreign risky assets. It follows that if we know the stochastic properties of the mutual funds M_d and M_f, we can determinate the required expected excess returns on all risky assets. From (35), we can say that with respect to the portfolio of risky assets of the domestic country, domestic assets will have negative alphas if the total tax paid on the domestic portfolio of risky assets is positive. Figure 2 shows the range of alphas as given by (34) and (35). The dotted line S is the pre-tax security market line obtained from CAPM using M_d as the market portfolio. The line S^d corresponds to the after-tax security market line for domestic risky assets, as given by (35). It has a less steep slope than S. S^+ and S^- indicate respectively the highest and lowest boundaries for the after-tax required expected returns for foreign assets.

Section 4: Comparing Optimal Portfolios

The mutual funds M_d and M_f differ, up to a scalar multiplication, by a portfolio $\underline{h}$ whose weights sum up to zero. The present section characterizes $\underline{h}$ and discusses the existence and characteristics of non-traded assets.

Note that portfolios $\underline{h}^d$ and $\underline{h}^f$ have to exist so that:

$$\underline{w}_I^S = \underline{w}_I^d - \underline{v}_I^d + \underline{h}^d \tag{36}$$

$$\underline{w}_I^S = \underline{w}_I^f - \underline{v}_I^f + \underline{h}^f \tag{37}$$

Premultiply (36) and (37) by $\underline{e}'$ and it follows immediately that the weights of portfolios $\underline{h}^d$ and $\underline{h}^f$ sum up to zero for each portfolio. From the equilibrium condition for the markets for risky assets, we know that:

$$\sum_{k \varepsilon d} \frac{\lambda^k}{\lambda_u^k} W^k (\underline{w}_I^d - \underline{v}_I^d) + \sum_{k \varepsilon f} \frac{\lambda^k}{\lambda_u^k} W^k (\underline{w}_I^f - \underline{v}_I^f) = \underline{w}_I^S W^W \tag{38}$$

Substituting (36) and (37) in (38) yields:

$$\sum_{k \varepsilon d} \frac{\lambda^k}{\lambda_u^k} W^k \underline{h}^d = - \sum_{k \varepsilon f} \frac{\lambda^k}{\lambda_u^k} W^k \underline{h}^f \tag{39}$$

Let $\underline{h}^d = \underline{h}$. (39) implies that $\underline{h}^f$ is $-\underline{h}$ times a strictly positive scalar.

To characterize $\underline{h}$, let us first rewrite (5) in the following form:

$$\underline{\underline{V}}_{II}(\underline{w}_I^k - \underline{v}_I^k) = \lambda^k \{E(\underline{\tilde{R}}) - \underline{e}R^d - \underline{\theta}^{d1}\} + \lambda^k \underline{Q}^k \tag{40}$$

At the optimum, the elements of the vector $\underline{Q}^k$ are non-negative and satisfy:

$$0 \leq Q_i^k \leq \theta_i^{d1} + \theta_i^{s1} \qquad \forall\ i \in F \tag{41}$$

$$Q_i^k = 0 \quad \forall\ i \in D \tag{42}$$

Knowing the vector $\underline{Q}^k$, we can solve (40) for the vector $(\underline{w}_I^k - \underline{v}_I^k)$ by matrix inversion to get:

$$(\underline{w}_I^k - \underline{v}_I^k) = \lambda^k \underline{\underline{V}}_{II}^{-1} \{E(\underline{\tilde{R}}) - \underline{e}R^d - \underline{\theta}^{d1} + \underline{Q}^k\} \ \forall\ k \in d \tag{43}$$

A similar relationship holds for all $k \in f$. Multiply (43) by W^k and aggregate across all investors to get:

$$\underline{w}_I^S = \sum_{k \in d} \frac{\lambda^k W^k}{W^W} \underline{\underline{V}}_{II}^{-1} \{E(\underline{\tilde{R}}) - \underline{e}R^d - \underline{\theta}^{d1} + \underline{Q}^k\} + \sum_{k \in f} \frac{\lambda^k W^k}{W^W} \underline{\underline{V}}_{II}^{-1} \{E(\underline{\tilde{R}}) - \underline{e}R^f - \underline{\theta}^{f1} + \underline{Q}^k\} \tag{44}$$

Premultiplying (44) by $\underline{\underline{V}}_{II}$ and using the definitions of Section 3, we get:

$$\underline{\underline{V}}_{II}\underline{w}_I^S = \lambda_m \{E(\underline{\tilde{R}}) - \underline{e}\bar{R} - \underline{\theta}^1 + \underline{Q}\} \tag{45}$$

where:

$$\underline{Q} = \sum_{k \in d} \frac{\lambda^k W^k}{\lambda_m W^W} \underline{Q}^k + \sum_{k \in f} \frac{\lambda^k W^k}{\lambda_m W^W} \underline{Q}^k \tag{46}$$

Premultiply (45) by $\underline{w}_I^{S}$, to get:

$$\sigma_m^2 = \lambda_m \{E(\tilde{R}_m) - \bar{R} - \theta^m + Q^m\} \tag{47}$$

where:

$$Q^m = \underline{w}_I^{S\prime}\underline{Q} \tag{48}$$

Using (47) to substitute λ_m in (45) yields:

$$\beta^m \{E(\tilde{R}_m) - \bar{R} - \theta^m + Q^m\} + E(\underline{\tilde{R}}) - \underline{e}\bar{R} - \underline{\theta}^1 + \underline{Q} \tag{49}$$

For asset j to be non-traded, it is required that $Q_j > 0$. It follows from inspection of (49) that, for a given beta, a non-traded asset will have a higher alpha than a traded asset with the same beta. As, for the moment, we know nothing about the vector $\underline{Q}$, we concentrate our attention on the implications of (49) for traded assets. Every traded asset j which is held long in both countries has $Q_j = 0$. Every traded asset j which is held short in one country will have either $Q_j = (\lambda^d/\lambda_m)(\theta_j^{dl} + \theta_j^{ds})$, if asset j is a foreign asset, or $Q_j = (\lambda^f/\lambda_m)(\theta_j^{fl} + \theta_j^{fs})$. It follows that, for traded assets, (49) corresponds exactly to the world after-tax expected excess return of riskly assets. (49) states that those after-tax expected excess returns are linear in beta.

If all traded assets are held long in all countries, then $Q_j = 0$ for every traded asset j. In that case, the following relationship holds for all traded assets:

$$\alpha_j = \theta_j^1 - \beta_j^m(\theta^m - Q^m) \tag{50}$$

Note that (50) will always hold at least for all traded assets held long in both countries. A necessary and sufficient condition for all traded assets with "large" betas to have negative alphas is $\theta^m < Q^m$. Because some traded assets could be held short, it is not true that all assets with "low" betas will have positive alphas; only those assets with "low" betas which are held long and traded internationally will have positive alphas.

Let i and j be two traded assets chosen so that $\beta_i^m > \beta_j^m$. If alpha is an increasing function of beta, then:

$$\frac{\beta_i^m}{\beta_j^m}\{E(\tilde{R}_j) - \overline{R}\} \quad - \quad \{E(\tilde{R}_i) - \overline{R}\} > 0$$

However, from (49), for traded assets held long, we know that:

$$\frac{\beta_i^m}{\beta_j^m}\{E(\tilde{R}_j) - \overline{R}\} \quad - \quad \{E(\tilde{R}_i) - \overline{R}\} = \frac{\beta_i^m}{\beta_j^m}\theta_j^1 - \theta_i^1$$

But we know that $\theta_j^1 > 0$, and $\theta_j^1 > 0$. It follows that a necessary and sufficient condition for alpha to be a declining function of beta is, for all traded assets i and j:

$$\frac{\beta_i^m}{\beta_j^m} > \frac{\theta_i^1}{\theta_j^1}$$

Clearly, for arbitrary tax rates, it is not possible to obtain any general result, except that a necessary condition for traded asset i to have a larger alpha that traded asset j is that asset i is taxed at a rate which exceeds the rate at which asset j is taxed.

We now restrict ourself to a world in which barriers to international investment are the same for all assets of a given country. In that case, $\theta^1_i = \theta^1_j$, for all $i \in D$, $j \in D$, and $\theta^1_i = \theta^1_j$, for all $i \in F$, $j \in F$. It follows immediately that the alphas of domestic traded assets held long by foreigners will be a decreasing function of beta, and that the alphas of foreign traded assets held long by domestic investors will be a decreasing function of beta. Furthermore, for any two traded assets i and j, α_i will be smaller than α_j provided that β^m_i/β^m_u is large enough. If $\theta^1_i > \theta^1_j$, for $i \in D$, $j \in F$, there will be some betas such that for two assets if, β^m_i/β^m_j is small, but larger than one, $\alpha_i > \alpha_j$.

Note now from (49) that for any beta, an asset which is traded and held long will have the highest possible alpha. It follows that (50) gives the highest possible value of alpha for all assets and that highest possible value of alpha is a declining function of beta for all assets of a given country. Because some assets will have lower betas than those predicted by (50), the security market line, as given by (49), will intersect with the security market line of the CAPM, at a beta which larger than one. Note that from (50) we also get the lowest possible value of alpha which is simply the highest possible value of alpha minus two times the tax rate on assets held short.

Non-traded assets will not be on the after-tax security market line which is implied by (50). In the remainder of this section, we will characterize - at least to some extent - non-traded assets and show that the existence of non-traded assets does not affect our discussion about assets with "large" betas, because all assets with "large" betas will be traded. To characterize non-traded assets, we first need to obtain the mutual fund M_d. Using (49), (40) can be rewritten as:

$$\underline{w}_I^d - \underline{v}_I^d = \lambda_u^d \underline{w}_I^S \frac{E(\tilde{R})_m - \overline{R} - \theta^m + Q^m}{\sigma_m^2} + \lambda_u^d V_{II}^{-1} \left\{ \underline{e}\overline{R} + \underline{\theta}^1 - \underline{Q} - \underline{e}R^d - \underline{\theta}^{dl} + \underline{Q}_u^d \right\} \qquad (51)$$

(51) has a form which is easy to recognize. If $\underline{Q} = 0$, $\underline{Q}_u^d = 0$, then (51) has the same form as the equivalent relationship in the Black model. It follows that if $\underline{Q} = \underline{0}$ and $\underline{Q}_u^d = \underline{0}$, exactly the same results as in the Black model would appear to hold. However, this is illusory, because $\underline{Q} = 0$ and $\underline{Q}_u^d = \underline{0}$ imply that no investor sells any asset short, unless $\underline{\theta}^{ds} = -\underline{\theta}^{dl}$ and $\underline{\theta}^{fs} = -\underline{\theta}^{fl}$. Furthermore, whereas in the Black model, it can be stated that the investor is indifferent between holding his portfolio of risky assets or a combination of mutual funds such that each mutual fund has well understood properties, in this model, any convex combination of the mutual funds which appear in the Black model, or any other mutual funds, would be <u>inefficient</u>, unless each mutual fund has non-negative investment proportions in all assets when the world portfolio of risky assets is one of those mutual funds, or all mutual funds have investment proportions of the same sign for all assets, whenever simultaneously selling short and holding long a foreign asset involves a sure loss. An additional problem is that, in general, $\underline{Q} \geq \underline{0}$, $\underline{Q}_u^d \geq \underline{0}$. We have found no way to explain either $\underline{Q}$ or

$\underset{\sim}{Q}_u^d$ in terms of a portfolio strategy. Clearly, if we know which assets are non-traded, we can solve for $\underset{\sim}{Q}$, which implies solving for $\underset{\sim}{Q}_u^d$ and $\underset{\sim}{Q}_u^f$. The result which can be obtained by using that approach is that if the covariance of an asset with the market value weighted portfolio of non-traded assets is large, that asset has to be traded. The obvious problem with that approach is that it requires a list of all non-traded assets.

Another approach is more useful. As we are not interested in the question of why interest rates can differ across countries, let us assume without loss of generality that $R = R^d = R^f$. This simplification does not affect our results, as long as taxes on international borrowing and lending do not exceed comparable taxes on any risky asset. The case of taxes on lending and borrowing which exceed comparable taxes on risky assets is not discussed in this paper.[6] Let asset j be non-traded for domestic investors. It follows that:

$$E(\tilde{R}_j) - R + \theta_j^{ds} > \frac{1}{\lambda_u^d} \operatorname{Cov}(\tilde{R}_j, \tilde{R}_d) > E(\tilde{R}_j) - R - \theta_j^{dl} \qquad (52)$$

For more simplicity, again without loss of generality, assume that $\theta_j^{ds} = \theta_j^{dl} = \theta$. Clearly, asset j has to be held positively by foreign investors. It follows that:

---

[6] Clearly, if taxes on international borrowing and lending are high enough compared to other taxes, investors may want to borrow abroad by selling short foreign stocks. I thank Fischer Black for this point.

$$\frac{1}{\lambda_u^f}\mathrm{Cov}(\tilde{R}_j,\tilde{R}_f) = E(\tilde{R}_j) - R \tag{53}$$

Substitute (53) and (52) to get:

$$\theta > \frac{1}{\lambda_u^d}\mathrm{Cov}(\tilde{R}_j,\tilde{R}_d) - \frac{1}{\lambda_u^f}\mathrm{Cov}(\tilde{R}_j,\tilde{R}_f) > -\theta \tag{54}$$

Now, use (36) and (37) to get:

$$\theta > \left(\frac{1}{\lambda_u^d} - \frac{1}{\lambda_u^f}\right)\mathrm{Cov}(\tilde{R}_j,\tilde{R}_m) + \left(\frac{1}{\lambda_u^d} + \frac{\lambda^d}{\lambda^f}\frac{1}{\lambda_u^d}\right)\mathrm{Cov}(\tilde{R}_i,\tilde{R}_h) > -\Theta \tag{55}$$

From (49) and (51), we get:

$$\underline{\underline{V}}_{II}\underline{h} = (1 - \lambda_u^d/\lambda_m)\underline{\underline{V}}_{II}\underline{w}_I^S + \lambda_u^d(\underline{\theta}^{d1} - \underline{\theta}^1 + \underline{K} - \underline{Q}_u^d) \tag{56}$$

Clearly, $\mathrm{Cov}(\tilde{R}_i,\tilde{R}_h)$ is a function of β_i^m, a constant, and Q_i^d. As long as (55) can be rewritten in such a way that $\mathrm{Cov}(\tilde{R}_i,\tilde{R}_m)$ is multiplied by a non-zero element, it will be the case that assets with "large" betas will be traded, because each element of $\underline{Q}$ and $\underline{Q}_u^d$ is bounded from above and below. Note that that result is not surprizing, in the sense that from (54) it is clear that an asset which would be correlated with no other asset, and hence which would have a negligible covariance with mutual funds M_d and M_f, would definitely be non-traded, because $\theta > 0 > -\theta$, for $\theta > 0$. (55) can be rewritten so that $\mathrm{Cov}(\tilde{R}_j,\tilde{R}_m)$ is multiplied by:

$$H = \left(\frac{\lambda_u^f - \lambda_u^d}{\lambda_u^f\lambda_u^d}\right) + \left(\frac{\lambda^d + \lambda^f}{\lambda^f\lambda_u^d}\right)\left(1 - \frac{\lambda_u^d}{\lambda_m}\right) \tag{57}$$

It can be shown that a necessary condition for $H = 0$ is that $\lambda_u^d = \lambda_u^f$, which implies that $\lambda_u^d = \lambda_m$. The reader can verify that the necessary condition for $H = 0$ implies that the portfolio $\underline{h}$ has zero variance, which is possible only if all investors in the world hold the market portfolio of risky assets and which will occur only if there are no barriers to international investment.

We now have shown that assets with "large" betas will always be traded. To conclude this section, it may be useful to give a simple example of a world with non-traded assets. Suppose all investors are the same and each country is a carbon copy of the other country, except for the fact that risky assets in one country are not correlated with risky assets in the other country. There is a tax θ on holding foreign assets either short or long. Finally, $\underline{\underline{V}}_{II}$ is diagonal. Clearly, in such a framework, for given equilibrium rates of return, all assets whose before-tax excess expected return is smaller than θ will not be traded. Take risky asset i, which is a domestic asset. For that asset not to be traded, it is necessary that:

$$0 > \lambda \frac{1}{\sigma_i^2}(E(\tilde{R}_i) - \theta - R) \tag{58}$$

Because of the symmetry, λ is the same everywhere. Note that we take symmetry to imply that the variance-covariance matrix of domestic assets is the same as the variance-covariance matrix of foreign assets. As $W^d = W^f$, the market for i will be in equilibrium with (58) holding when:

$$2w^S_{I_i} = \frac{1}{\sigma_i^2}(E(\tilde{R}_i^*) - R)$$

By substitution, i will have an equilibrium expected return compatible with the assumption of non-trading if:

$$0 > 2w^S_{I_i} - \lambda(1/\sigma_i^2)\theta$$

Clearly, there will always be a θ such that this will obtain. Note that in a diagonal model, all the assets with the lowest betas will be non-traded and those assets will have a higher expected return than if they were traded. In other words, before-tax, all low beta assets will have positive alphas.

Section 5: Some Operational Results

The Black model had a rather useful result for empirical purposes: low beta assets have high alphas. The result of the Black model holds for the world portfolio of risky assets and for portfolios of domestic and foreign investors. The efficient portfolio of risky assets for a domestic investor is not observable, unless one knows exactly which barriers to international investment he faces. Clearly, if the efficient portfolio of risky assets for domestic investors is observable, there is no problem in testing whether the model presented here holds or not. It follows that, in general, results which pertain to the portfolio of risky assets held by investors in one country will not be testable.[7] Note however that if a linear relationship is found between the expected excess returns of domestic risky assets and the betas of those assets with respect to the market portfolio of domestic risky assets, that relationship would show that the market portfolio of risky assets of the domestic country is efficient. Clearly, the only case in which such a result would hold is if barriers to international investment are large enough to prevent domestic investors from investing abroad, unless of course the world portfolio of common stocks is perfectly correlated with the domestic portfolio of common stocks. If barriers to international investment are large enough to prevent

[7] See Roll (1978) for an excellent discussion of the implications for empirical research of the fact that efficient portfolios are not observable.

domestic investors from investing abroad, then (35) reduces to the CAPM with respect to the domestic market portfolio of risky assets. Any test of the CAPM which cannot reject the existence of a linear relationship between expected returns and beta using a portfolio of domestic assets as the market portfolio is a test which either rejects the hypothesis that there are no barriers to international investment (which would also imply that assets are not effectively taxed) or accepts the hypothesis that the world portfolio of risky assets is perfectly correlated with the domestic portfolio used.

Let us now turn to the asset pricing relationships which hold for the world as a whole. If there exists non-traded assets at all, it should be possible to create a risky asset which has a beta of one, but whose expected excess return is larger than the expected excess return on the world market portfolio of risky assets. The obvious problem with testing that hypothesis is that an estimate of the beta of an asset is not, in general, equal to the true beta of that asset, which is unknown. If we did know the true betas or risky assets, the fact that some assets with betas equal to or greater than one have positive alphas would be a strong indication of the presence of non-taded assets.

Finally, it should be stressed that barriers to international investment can help to explain why low beta assets with respect to the U.S. market portfolio of risky assets have high alphas. Suppose that $\underline{\underline{V}}_{II}$, the variance-covariance matrix of risky assets, has some rows in which there is only one element. Take asset j to be an American risky asset such that the j-th row of $\underline{\underline{V}}_{II}$ has only one non-zero element.

If barriers to international investment exist, one would expect asset j to be non-traded. This means that asset j would have to be held only by U.S. investors. Expected returns on risky assets would have to be such that American investors would want to hold all the supply of asset j, but not all the supply of asset k, which has a high beta with respect to the U.S. market portfolio. Clearly, this can be achieved if, with respect to the U.S. portfolio, low beta assets have high alphas.

REFERENCES

Abrams, R.A. and U.S. Karmarkar, "Optimal Multiperiod Investment - Consumption Policies," Econometrica, Volume 48, Number 2 (March 1980), pp. 333-354.

Adler, M. and B. Dumas, "Optimal International Acquisitions," Journal of Finance, Volume 30, Number 1 (March 1975), pp. 1-19.

Black, F., "International Capital Market Equilibrium with Investment Barriers," Journal of Financial Economics, Volume 1, Number 3 (December 1974), pp. 337-352.

Roll, R., "A Critique of the Asset Pricing Theory's Tests; Part I: On Past and Potential Testability of the Theory," Journal of Financial Economics, Volume 4, Number 2 (March 1977), pp. 129-176.

Smith, C.W. and F.A. Milne, "Capital Asset Pricing with Proportional Transaction Costs," mimeo, 1979.

Stapleton, R.C. and N.G. Subrahmanyam, "Market Imperfections, Capital Market Equilibrium and Corporation Finance," Journal of Finance, Volume 32, Number 2 (May 1977), pp. 307-319.

Stulz R.M., "A Model of International Asset Pricing," mimeo, 1980.

PART III

INTERNATIONAL TRADE AND THE DEMAND OF RISKY ASSETS

Section 1: Introduction

How does international trade affect a country's holdings of risky assets? The proponents of the portfolio approach to international investment generally argue that, if there are no barriers to international investment, investors will want to diversify their holdings of risky assets and the diversification motive will cause them to hold foreign assets. Because the returns on foreign assets are not perfectly correlated with those of domestic assets,[1] it is true that an investor who adds foreign assets to his portfolio of domestic assets should obtain a more efficient portfolio in mean-variance space. Within such an approach, international trade does not matter, in the sense that a change in the pattern of trade of a country will not affect that country's holdings of risky assets.

Several recent papers have explored the nature of exchange rate risks[2] and shown that the nature of those risks depends on which assumptions are made about the nature of international trade. If all investors consume the same basket of commodities and the price of that basket, when expressed in a given currency, is the same everywhere in the world, the real rate of return on an asset is the same for all investors, and in that sense exchange risk is irrelevant. If, on the other hand, changes in the exchange rate reflect mainly changes in

---

[1] A useful review of the literature on international diversification is Lessard (1979).

[2] See Solnik (1973), Heckerman (1973), Grauer, Litzenberger and Stehle (1976), Kouri (1976), Kouri (1977), Adler and Dumas (1977), Fama and Farber (1979), Frankel (1979), Solnik (1979) and Stulz (1980). Grauer, Litzenberger and Stehle, Kouri (1977), Fama and Farber and Frankel have models in which investors consume only one good (it may be a composite good). Consequently, their models do not exhibit risk _a la_ Solnik or terms of trade risk, as _real_ returns on all assets are the same for all investors.

the exchange rate reflect mainly changes in the terms of trade, then investors may want to bear exchange rate risks because changes in the exchange rate are correlated with the price of their imports. In a world in which there are many commodities and in which the exchange rate is correlated with the terms of trade, there should be a relationship between a country's imports and that country's holdings of risky assets. Unfortunately, the existing literature does not have much to say about that relationship.[3]

The present paper explores the relationship between a country's international trade and that country's holdings of risky assets within a real model, in which money has no role whatsoever. It is shown that in a world of complete specialization:

1. If the only state variables are relative prices, all investors will hold common stocks in identical proportions.
2. If there are no futures contracts, a country's optimal net holdings of foreign assets are a decreasing function of that country's expenditure elasticity of imports.
3. If there are future contracts, the extent to which investors bear terms of trade risks is a decreasing function of their expenditure elasticity of imports.

---

[3] The exceptions are mainly Heckerman (1973) and Kouri (1977). Heckerman, in a model in which investors maximize an approximation of real wealth, shows heuristically that investors may want to bear exchange risks without being rewarded by a larger expected return. Kouri is concerned with a model in which there exists stochastic deviations from purchasing power parity and default risk on foreign bonds. With the assumptions used in the present paper substituted in Kouri's model, the risk premium he derives would be equal to zero.

4. If the only state variables are the initial endowments of (n-1) countries and the initial endowment in n commodities, markets are complete with only stocks and bonds for one class of production functions.
5. The risk premium which corresponds to the difference between the forward rate and the expected future spot rate is a function of the expenditure elasticities of imports of the domestic country and of the foreign countries. Precisely, the forward rate will, *ceteris paribus*, be a decreasing function of the domestic country's aggregate expenditure elasticity of imports and an increasing function of the expenditure elasticity of imports of foreign countries for the good produced by the domestic country.

As this paper is devoted to the question of how a country's pattern of trade affects its holdings of risky assets, some very strong simplifying assumptions are made to guarantee that changes in asset prices will not affect the pattern of trade of the various countries, in the sense that it will not be possible for a country to import a good for some level of domestic wealth and export it for some other level of domestic wealth. In particular, it is assumed throughout the paper that each country is completely specialized in the production of a single good. This assumption is clearly unrealistic, but it allows the derivation of sharper results and simplifies the analysis considerably. It implies that in the present paper terms of trade, exchange rates and relative prices can be used indifferently. It must be stressed that many factors are relevant in determining a country's holdings of foreign assets.[4] Whereas this paper explains how international trade affects a country's holdings of risky assets, the framework used is too simple to explain why investors in general do not diversify their holdings of stocks internationally as much as would be required if they held a mean-variance efficient portfolio of risky assets.

In Section 2, the assumptions which characterize the model are presented and asset demand functions are derived. In Section 3, the asset demand functions are studied under the assumption of given expected excess returns for all assets. In Section 4, the expected excess returns on bonds are obtained and used to characterize the equilibrium holdings of bonds, for a particular case of our model. In Section 5, the limitations of the approach used in the paper are briefly discussed.

---

[4] For a discussion of some of those factors, see for instance Black (1978) and Dufey (1976).

Section 2: The model

In this section, the asset demand functions for a representative investor are derived. First, the commodity and price dynamics are presented. Using these dynamics, the flow budget equation for the investor is then derived. Finally, the flow budget equation is used to solve for the asset demand functions which maximize the intertemporal expected utility function of the investor.

1.1. Commodity price dynamics.

Each country is assumed to produce only one commodity. It is also assumed that there are as many commodities as there are countries. In each country, the produced good is the numeraire. Instantaneous arbitrage of commodities is possible at zero costs, because there are no transportation costs or tariffs. Let $P_i^j(t)$ be the price in country j of good i at date t. $P_i^j(t)$ is the number of units of good i which can be obtained in exchange for one unit of good j. Under these assumptions, the law of one price holds for commodities:

$$P_i^k(t) = P_i^j(t)/P_k^j(t) \qquad \forall\ i, j, k, t \tag{2.1}$$

By definition:

$$P_i^i(t) = 1 \qquad \forall\ i, t \tag{2.2}$$

Nonstochastic deviations from the law of one price could easily be introduced in this paper. More complex deviations form the law of one price would however affect our results more seriously, as will be explained in Section 5.

Relative prices in country j follow a stochastic differential equation[1] which we write as:

$$\frac{dP_i^j}{P_i^j} = \mu_{P_i^j}(\underline{S}, t)dt + \sigma_{P_i^j}(\underline{S}, t)dz_{P_i^j} \qquad \forall\ i \tag{2.3}$$

$\mu_{P_i^j}$ is the expected instantaneous rate of change of the price P_i^j. The expected rate of change of P_i^j depends on time and on an S x 1 vector of state variables $\underline{S}$. In the following, it will be assumed that investors have rational expectations and homogeneous beliefs. $\sigma_{P_i^j}$ is the instantaneous standard deviation of the rate of change of P_i^j. $\sigma_{P_i^j}$ depends also on a vector of state variables and time. Finally, $dz_{P_i^j}$ is a Wiener process.

Equation (2.3) gives a reduced-form equation for the rate of change of P_i^j. The vector of $\underline{S}$ includes all the variables, except time t, which can affect the distribution of the rate of change of the relative prices. The dynamics of the state variables are given by:

$$dS_i = \mu_{S_i}(\underline{S}, t)dt + \sigma_{S_i}(\underline{S}, t)dz_{S_i} \qquad \forall\ i \tag{2.4}$$

There are n countries. The dynamics for the relative prices in any country k ≠ j can be obtained by differentiating (2.1) and substituting (2.2) and (2.3) in the solution.

---

[1] Merton (1978) derives such stochastic differential equations using only elementary probability concepts. For an alternative derivation see Arnold (1974).

1.2. Asset price dynamics.

Assets can move freely across countries.[2] There are no transaction costs. The income of each asset accrues in the form of capital gains. There are no taxes. Assets are traded only at equilibrium prices.

For the moment, it is assumed that in each country only two assets are issued. The first asset is a safe bond in terms of the numeraire of that country. Let B_j be the safe bond in the j-th country. The dynamics for B_j are:

$$\frac{dB_j}{B_j} = \mu_{B_j}(\underline{S}, t)dt \tag{2.5}$$

At date t, the instantaneous rate of return on the j-th bond is certain. It is however possible for the safe rate of return to change through time as both the state variables on which it depends and time change. Note immediately that in country k the j-th bond is not a safe asset. The price of the j-th bond in country k is simply $P_j^k B_j$. By Ito's Lemma:

$$\frac{d(P_j^k B_j)}{P_j^k B_j} = \frac{dP_j^k}{P_j^k} + \frac{dB_j}{B_j} \tag{2.6}$$

In country k, the price of the j-th bond is perfectly correlated with the price of the j-th commodity.

In each country, there exists also one common stock. The total value of the common stocks issued in one country corresponds to the value of the industry of that country in terms of the domestic numeraire.

---

2 Black (1974), Adler and Dumas (1975), Adler and Dumas (1976), Kouri (1977), Stulz (1979a), Stulz (1980b), look at the valuation effects of various barriers to international investment.

It could be assumed that each firm in an industry issues common stocks separately. Nothing would change in the remainder of the paper, except that the notation would be considerably heavier. It is assumed that production is uncertain and takes time. In other words, the firm will purchase inputs at time t'. At time t' + k, the output will be sold. The quantity of output which can be produced with inputs purchased at time t' depends on the realization of random variables between t' and t' + k. This is enough to guarantee that the value of the stock of an industry changes through time stochastically. The common stock in country j can be exchanged at zero cost at any point in time for $I_j(t)$ units of commodity j. The dynamics for I_j are given by:

$$\frac{dI_j}{I_j} = \mu_{I_j}(\underline{S}, t)dt + \sigma_{I_j}(\underline{S}, t)dz_{I_j} \tag{2.7}$$

μ_{I_j} is the expected rate of change of I_j and is a function of time and of the vector of state variables. σ_{I_j} is the instantaneous standard deviation of the rate of change of I_j and is a function of time and the state variables. The rate of change of the j-th stock in terms of the numeraire of country k is obtained by using Ito's Lemma:

$$\frac{d(P_j^k I_j)}{P_j^k I_j} = \frac{dI_j}{I_j} + \frac{dP_j^k}{P_j^k} + \frac{dP_j^k}{P_j^k} \cdot \frac{dI_j}{I_j} \tag{2.8}$$

In general, the covariance term in (8) will be different from zero.

1.3. Budget equations.

Let W^{ki} be the wealth of the i-th investor of country k in terms of the numeraire of country k. $w_{I_j}^{ki}$ is the fraction of his wealth that investor holds in the j-th common stock, whereas $n^{ki}_{B_j}$ is the fraction of his wealth he holds in the j-th bond. The stock budget constraint for that investor is:

$$\sum_j^n n^{ki}_{B_j} + \sum_j^n w^{ki}_{I_j} = 1 \tag{2.9}$$

The flow budget constraint is:

$$\sum_j^n n^{ki}_{B_j}\left(\frac{dB_j}{B_j} + \frac{dP^k_j}{P^k_j}\right)W^{ki} + \sum_j^n w^{ki}_{I_j}\left(\frac{dI_j}{I_j} + \frac{dI_j}{I_j}\cdot\frac{dP^k_j}{P^k_j} + \frac{dP^k_j}{P^k_j}\right)W^{ki}$$

$$- \sum_j^n P^k_j c^{ki}_j dt = dW^{ki} \tag{2.10}$$

where c^{ki}_j is the investor's consumption of the j-th commodity. Using (2.9), $n^{ki}_{B_i}$ can be eliminated from (2.10). If we set $w^{ki}_{B_j} = w^{ki}_{I_j} + n^{ki}_{B_j}$, (2.10) can be rewritten as:

$$\sum_{j\neq i}^n w^{ki}_{B_j}\left(\frac{dB_j}{B_j} + \frac{dB_j}{P^k_j} - \frac{dB_i}{B_i}\right)W^{ki} + \sum_j^n w^{ki}_{I_j}\left(\frac{dI_j}{I_j} + \frac{dI_j}{I_j}\cdot\frac{dP^k_j}{P^k_j} - \frac{dB_j}{B_j}\right)W^{ki}$$

$$+ \frac{dB_i}{B_i} - \sum_j^n P^k_j c^{ki}_j dt = dW^{ki} \tag{2.11}$$

The expressions in parentheses in (2.11) give the excess returns of the various assets in the form in which they will be used in the remainder of the paper.

1.4. Optimization.

The representative investor has a Von Neumann-Morgenstern intertemporal utility function, which will be assumed to have all the required properties to guarantee that his optimization problem has a unique solution. In particular, it is assumed that at each point in time his utility function is twice differentiable and strictly concave in the c_j's. The intertemporal expected utility function can be written as:

$$E_{t_0}\left[\int_{t_0}^{t_{ki}} u^{ki}(\underline{c}^{ki}(t),\ t)dt + B^{ki}(W^{ki}(t_{ki}),\ \underline{S}(t_{ki}),\ t_{ki})\right]$$

where $B^{ki}(W^{ki}(t_{ki}),\ \underline{S}(t_{ki}),\ t_{ki})$ is the bequest function of the representative investor and is assumed to be concave in wealth. $\underline{c}^{ki}(t)$ is the vector formed by the c_j^{ki}'s. Let:

$$J^{ki}(W^{ki}(t_0),\ \underline{S}(t_0),\ t_0) = \max E_{t_0}\left[\int_{t_0}^{t_{ki}} u^{ki}(\underline{c}^{ki}(t),\ t)dt\right.$$

$$\left. + B^{ki}(W^{ki}(t_{ki}),\ \underline{S}(t_{ki}),\ t_{ki})\right] \tag{2.12}$$

The function defined by (2.12) is usually called the value function.[3] The first-order conditions for the optimization problem of the investor are:

$$J_W^{ki}P_j^k = u^{ki}_{c_j^{ki}} \qquad \forall\ j \tag{2.13}$$

$$J_w^{ki}\underline{\mu}_B^k W^{ki} + J_{WW}^{ki}(\underline{\underline{V}}_{PP}^k\underline{w}_B^{ki} + \underline{\underline{V}}_{PI}^k\underline{w}_I^{ki})(W^{ki})^2 + \underline{\underline{V}}_{PS}^k\underline{J}_{WS}^{ki}W^{ki} = 0 \tag{2.14}$$

$$J_W^{ki}\underline{\mu}_I W^{ki} + J_{WW}^{ki}(\underline{\underline{V}}_{II}\underline{w}_I^{ki} + \underline{\underline{V}}_{IP}^k\underline{w}_B^{ki})(W^{ki})^2 + \underline{\underline{V}}_{IS}^k\underline{J}_{WS}^{ki}W^{ki} = 0 \tag{2.14}$$

where $\underline{\mu}_B^k$ is the vector of excess expected returns on bonds when the numeraire k is used and has dimension (n - 1) x 1. Vectors are

---

3 For the general method used here to derive asset demand functions, see for example Merton (1971) or Fischer (1975).

underlined once and matrices twice in the following. When a matrix or a vector has a superscript, that superscript denotes the numeraire used. $\underline{\underline{V}}^k_{PP}$ is the variance-covariance matrix of the excess returns of bonds and has dimension (n - 1) x (n - 1). $\underline{\underline{V}}^k_{PI}$ is the (n - 1) x n matrix of covariances of excess returns on bonds with excess returns on stocks. $\underline{\underline{V}}^k_{PS}$ is the (n - 1) x S matrix of covariances of excess returns on bonds with the state variables. $\underline{\underline{V}}_{II}$ is the n x n variance-covariance matrix of stock returns. Lastly, $\underline{\mu}^k_I$ is the vector of expected excess returns on common stocks. $\underline{w}^{ki}_I$ and $\underline{w}^{ki}_B$ are respectively the n x 1 vector of common stock holdings and the (n - 1) x 1 vector of net holdings of risky bonds of investor ki. By looking at (2.11), it can be seen that, although the excess returns on common stocks depend on the numeraire, the variance-covariance matrix of excess returns on common stocks does not depend on the numeraire used by investor ki.

The first-order conditions can be used to get the following asset demand functions:

$$\underline{w}^{ki}_B = (\underline{\underline{V}}^k_{PP} - \underline{\underline{V}}^k_{PI}\underline{\underline{V}}^{-1}_{II}\underline{\underline{V}}^k_{IP})^{-1}\left\{\left(\frac{-J^{ki}_W}{J^{ki}_{WW}W^{ki}}\right)(\underline{\mu}^k_B - \underline{\underline{V}}^k_{PI}\underline{\underline{V}}^{-1}_{II}\underline{\mu}^k_I) + (\underline{\underline{V}}^k_{PS} - \underline{\underline{V}}^k_{PI}\underline{\underline{V}}^{-1}_{II}\underline{\underline{V}}^k_{IS})\left(\frac{-\underline{J}^{ki}_{WS}}{J^{ki}_{WW}W^{ki}}\right)\right\} \tag{2.16}$$

$$\underline{w}^{ki}_I = (\underline{\underline{V}}_{II} - \underline{\underline{V}}^k_{IP}(\underline{\underline{V}}^k_{PP})^{-1}\underline{\underline{V}}^k_{PI})^{-1}\left\{\left(\frac{-J^{ki}_W}{J^{ki}_{WW}W^{ki}}\right)(\underline{\mu}^k_I - \underline{\underline{V}}^k_{IP}(V^k_{PP})^{-1}\underline{\mu}^k_B) + (\underline{\underline{V}}^k_{IS} - \underline{\underline{V}}^k_{IP}(\underline{\underline{V}}^k_{PP})^{-1}\underline{\underline{V}}^k_{PS})\left(\frac{-\underline{J}^{ki}_{WS}}{J^{ki}_{WW}W^{ki}}\right)\right\} \tag{2.17}$$

These demand functions will be discussed in detail in the following section.

Section 3: Economic Implications

In Section 2, a model was constructed and used to derive asset demand functions. In this section, the economic implications of the asset demand functions derived from that model are discussed. The first three parts of this section deal with a world in which there are no futures contracts (which does not imply that markets are in some sense incomplete), whereas the last part of this section deals with the case in which there exist both futures contracts and bonds indexed on state variables.

2.1. The demand for common stocks.

The demand functions for assets obtained in Section 2 have two terms; one term depends on a scalar, $(J^{ki}_{W}/J^{ki}_{WW}W^{ki})$, whereas the other term depends on a vector, $(\underline{J}^{ki}_{WS}/J^{ki}_{WW}W^{ki})$. The part of the asset demand functions which depends on the scalar will be called the mean-variance efficient demand for assets, whereas the other part will be called, following Merton (1973), the hedging demand for assets.

Let ${}_{M}\underline{w}^{k}_{I}$ be the vector of mean-variance efficient demands for common stocks. It can be shown that the ratio of any two elements of ${}_{M}\underline{w}^{k}_{I}$ is the result of an optimization program which maximizes the expected return of a portfolio of risky assets for a given variance.

> <u>Proposition 1.</u> The ratio of any two elements of ${}_{M}\underline{w}^{k}_{I}$ does not depend on the numeraire an investor uses.

A proof of this proposition is given in Appendix I. Proposition 1 should not be surprizing, as the investment opportunity set is the same for all investors in the world economy.

Clearly, if the demand for risky assets was simply ${}_{M}\underline{w}^{k}_{I}$, then investors would hold the world portfolio of risky assets. However, the model incorporates hedging demands for both stocks and bonds which depend on each investor's utility function.

> <u>Proposition 2.</u> Investors will not hold stocks to hedge against unfavorable changes in relative prices.

Note that $\underline{J}^{ki}_{WS}$ is multiplied by:

$$\underline{\underline{V}}^{k}_{IS} - \underline{\underline{V}}^{k}_{IP}(\underline{\underline{V}}^{k}_{PP})^{-1}\underline{\underline{V}}^{k}_{PS} \qquad \text{(A)}$$

The state variables are numbered so that the first n-1 state variables are relative prices. Let S_C designate those state variables which are not relative prices. (A) can be rewritten as:

$$(\underline{\underline{V}}^{k}_{IP} \quad \underline{\underline{V}}^{k}_{IS_C}) - \underline{\underline{V}}^{k}_{IP}(\underline{\underline{V}}^{k}_{PP})^{-1}(\underline{\underline{V}}^{k}_{PP} \quad \underline{\underline{V}}_{PS_C})$$

From the definition of the inverse of a matrix, this expression can be rewritten as:

$$(\underline{\underline{V}}^{k}_{IP} \quad \underline{\underline{V}}^{k}_{IS_C}) - \underline{\underline{V}}^{k}_{IP}(\underline{\underline{I}} \quad \underline{\underline{V}}^{k}_{PP})^{-1}\underline{\underline{V}}_{PS_C})$$

It follows naturally that:

$$(\underline{\underline{0}} \quad \underline{\underline{V}}^{k}_{IS_C}) - \underline{\underline{V}}^{k}_{IP}(\underline{\underline{0}} \quad (\underline{\underline{V}}^{k}_{PP})^{-1}\underline{\underline{V}}_{PS_C})$$

This last expression proves our result, as it implies that the n-1 first elements of the vector $\underline{J}^{k}_{WS}$ are multiplied by zero.

It is necessary to stress the fact that investors may want to hold a hedging portfolio of common stocks to hedge against other state variables than relative prices.

2.2. The demand for bonds.

The determinants of the demand for bonds can be divided into three parts. Let ${}_{M}\underline{w}_{B}^{ki}$ be the mean variance efficient demand for bonds; ${}_{H}\underline{w}_{B}^{ki}$ is the vector of demands for bonds as relative price hedges and ${}_{C}\underline{w}_{B}^{ki}$ is the vector of demands for bonds as hedges against state variables which are not relative prices. The reader can easily verify that for each state variable $i \in S_C$, there will be a portfolio of risky assets which will have the highest correlation with that state variable. An investor's demand for bonds to hedge against state variable i will be the weights of the bonds in that portfolio multiplied by a scalar. Those results have been proved in a different context in Merton (1973, 1978). In this section, however, we are interested in ${}_{H}\underline{w}_{B}^{ki}$, which until the end of this section we will simply call the vector of hedging demands for bonds. The reader can easily verify that that vector can be obtained by assuming that state variable $i \in S_C$, $\forall$ i, are not correlated with relative prices. This assumption is made for convenience until the end of this section. It has no effect whatsoever on our results. ${}_{M}\underline{w}_{B}^{ki}$ is obtained by choosing the portfolio of risky assets in numeraire k which has the highest mean for a given variance. It is important to notice that, because a bond which is risky with respect to numeraire k will not be risky with respect to the numeraire of the country in which it is issued, the bonds included in the vector ${}_{M}\underline{w}_{B}^{ki}$ will not be the same as the bonds included in vector ${}_{M}\underline{w}_{B}^{qi}$, which will be the mean-variance efficient portfolio of bonds of an investor in country q.

Proposition 3. Hedging demands for bonds are:

$$\underline{H}^{w}_{B}{}^{ki} = \left(\frac{-\underline{J}^{ki}_{WP}}{J^{ki}_{WW}W^{ki}}\right) = \frac{-\underline{C}^{ki}_{P}}{C^{ki}_{W}W^{ki}} + \frac{C^{ki}\underline{\alpha}^{ki}}{C^{ki}_{W}W^{ki}} - \frac{T^{ki}\cdot\underline{m}^{ki}}{C^{ki}_{W}W^{ki}}$$

First, we look at the first equality. Note that the hedging demands for bonds can be written:

$$\underline{H}^{w}_{B}{}^{ki} = \left[\underline{\underline{V}}^{k}_{PP} - \underline{\underline{V}}^{k}_{PI}\underline{\underline{V}}^{-1}_{II}\underline{\underline{V}}^{k}_{IP}\right]^{-1}\left[\underline{\underline{V}}^{k}_{PS} - \underline{\underline{V}}^{k}_{PI}\underline{\underline{V}}^{-1}_{II}\underline{\underline{V}}^{k}_{IS}\right]\left(\frac{-\underline{J}^{ki}_{WS}}{J^{ki}_{WW}W^{ki}}\right) \tag{a}$$

Note that $\underline{\underline{V}}^{k}_{PS} = [\underline{\underline{V}}^{k}_{PP} \quad \underline{\underline{0}}]$, where $\underline{\underline{0}}$ is here an (n - 1) x (S - n + 1) null matrix. $\underline{\underline{V}}^{k}_{PI}$ has zeroes in its last (n - n') columns. The last (n - n') columns of $\underline{\underline{V}}^{k}_{PI}$, when $\underline{\underline{V}}^{k}_{PI}$ multiples $\underline{\underline{V}}^{-1}_{II}$, make the last (n - n') columns of the resulting matrix equal to zero, because of the fact that $\underline{\underline{V}}_{II}$ is block diagonal and its lower right-hand corner block has dimension (n - n') x (n - n'). The last (n - n') columns of $\underline{\underline{V}}^{k}_{PI}\underline{\underline{V}}^{-1}_{II}$ multiply the elements of $\underline{\underline{V}}^{k}_{IS}$ which are not correlated with relative prices. This allows us to rewrite $\underline{\underline{V}}^{k}_{PI}\underline{\underline{V}}^{-1}_{II}\underline{\underline{V}}^{k}_{IS}$ as $[\underline{\underline{V}}^{k}_{PI}\underline{\underline{V}}^{-1}_{II}]\ \underline{\underline{0}}$. Hence, the first bracketed term of (a) multiplies its inverse augmented by (S - n + 1) columns of zeroes. The product of the two bracketed terms is a matrix which is an identity matrix of dimension (n - 1) x (n - 1) augmented by (s - n + 1) columns of zeroes. But:

$$[\underline{\underline{I}} \quad \underline{\underline{0}}]\ \underline{J}^{ki}_{WS} = \underline{J}^{ki}_{WP}$$

if $\underline{\underline{I}}$ is that identity matrix.

The second equality follows from Breeden (1979). Define U(C, $\underline{P}$, t) as the indirect utility function of consumption expenditures. The vector $\underline{P}$ is a vector of logarithms of relative prices. By differentiation:

$$\left(\frac{J^{ki}_{\underline{W}P}}{J^{ki}_{WW}W^{ki}}\right) = \frac{C^{ki}_{\underline{P}}}{C^{ki}_{W}W^{ki}} + \frac{U^{ki}_{\underline{C}P}}{U^{ki}_{CC}C^{ki}_{W}W^{ki}} \tag{b}$$

By differentiating the indirect utility function, it is possible to establish, as in Breeden (1979), the equality of the second term on the right-hand side of (b) with the terms in the parenthesis on the right-hand side of the expression given in Proposition 3. T^{ki} is the absolute risk tolerance of investor ki and is equal to $-(U^{ki}_{C}/U^{ki}_{CC}) \cdot \underline{\alpha}^{ki}$ is the investor's vector of average expenditure shares, and $\underline{m}^{ki}$ is the investor's vector of marginal expenditure shares; both vectors of expenditure shares contain only expenditure shares for foreign commodities.

Proposition 3 shows that the hedging demands for bonds depend exclusively on the consumption function of the investor, his risk tolerance and his vector of marginal and average expenditure shares. The first equality in Proposition 3 allowed us to show that the variance-covariance matrix of asset returns does not affect the determination of hedging demands for bonds. The second equality allowed us to make those hedging demands meaningful. Note that $(-C^{ki}_{P^k_j} + C^{ki}\alpha^{ki}_j)$ corresponds to the variation in consumption expenditures necessary to keep utility constant following a change in P^k_j. The last term of the hedging demands for bonds is negative, as it is the product of the absolute risk tolerance of an investor multiplied by the vector of marginal expenditure shares for foreign goods, divided by the derivative of the consumption function of the investor with respect to wealth. *Ceteris* paribus, an investor's marginal expenditure share for a foreign good will decrease his demand for the bond issued in the country in which that good is produced, whereas the opposite effect will take place if his average expenditure share for that good is increased. There is no reason for the hedging demands for all bonds to be positive for an investor. If an investor has a negative hedging demand for a particular bond, one would expect him to have a high elasticity of expenditure for the good which serves as a numeraire in the country which that bond is issued. Another way of stating that fact is that investors are more likely to hedge against unanticipated changes in the price of "necessities" than against unanticipated changes in the price of "luxuries."

Note that if (18) does not hold, investors will also want to hold foreign bonds (in either positive or negative amounts) to hedge against unanticipated changes in state variables which are not terms of trade.

2.3. Foreign investment.

Until now, we have looked at the asset demand functions of a representative investor. Here, the investments of country k in country j are considered:

> Proposition 4. Given (A) the distribution of asset returns, (B) the aggregate consumption function of country k and (C) the aggregate risk tolerance of country k, the value of the investments of country k in country j is a decreasing function of the expenditure elasticity of country k's imports of consumption goods from country j.

Let $C^K(W^K, \underline{S}, t)$ be the aggregate consumption function of country k. By definition:

$$C^K(W^K, \underline{S}, t) = \sum_{i=i}^{N^K} C^{ki}(W^{ki}, \underline{S}\ t) \qquad \text{(c)}$$

where N^K is the number of investors in country k. Define:

$$C^K\underline{\alpha}^K = \Sigma\ C^{ki}\underline{\alpha}^{ki} \qquad C^K = \Sigma\ C^{ki} \qquad \text{(d)}$$

$$T^K\underline{m}^K = \Sigma\ T^{ki}\underline{m}^{ki} \qquad T^K = \Sigma\ T^{ki} \qquad \text{(e)}$$

$\underline{\alpha}^K$ is an expenditure weighted average of average expenditure shares, whereas $\underline{m}^K$ is a risk tolerance weighted average of marginal expenditure shares. It can be shown that $\underline{m}^K$ is the vector of marginal expenditure shares for country K. Finally, define:

$$W^K = \Sigma W^{ki} \qquad \text{(f)}$$

We can, using (c) - (f), add up the hedging demands for bonds of all investors of economy K get:

$$\underline{H}^{W^K}_{B}W^K = - \frac{\underline{C}^K_P}{C^K_W} + \frac{C^K\underline{\alpha}^K}{C^K_W} - \frac{T^K\underline{m}^K}{C^K_W} \qquad \text{(g)}$$

Look now at the j-th row of (g):

$$H^{W^K}_{B_j} W^K = - \frac{C^K_{P_j}}{C^K_W} + \frac{C^K\alpha_j}{C^K_W} - \frac{T^K m_j}{C^K_W}$$

But $C^K\alpha^K_j$ is the consumption of the j-th good in country k. As the j-th good is not produced in country k, $C^K\alpha^K_j$ is equal to the value of the imports of consumption goods produced in country j in country k. The expenditure elasticity of those imports is simply m^K_j/α^K_j. If

$C^K(W^K, \underline{S}, t)$ and T^K do not change, an increase in m_j^K/α_j^K decreases ${}_H w_{B_j}^K$ for a given distribution of asset returns.

If $w_{B_j}^K W^K$ is the value of the investments of country k in country j, we have proved Proposition 4, because by assumption the mean-variance efficient portfolio of country j is constant. Note that for investor ki, the value of his investments in country j is the sum of his stock holdings in country j plus his holdings of bonds in country j, i.e. $w_{I_j}^{ki} + n_{B_j}^{ki} = w_{B_j}^{ki}$. Consequently, adding $w_{B_j}^{ki}$ over all i's yields the value of the investments of country k in country j.

What Proposition 4 says is that, in equilibrium, a country invests more in a country from which its imports have a high expenditure elasticity. Clearly, the value of a country's investments in a foreign country can be negative, as we have placed no restriction whatsoever on borrowings. In this model, the higher the investment of one country in another country, the more investors in that country are hedging against unanticipated unfavorable changes in the price of the commodity produced by the other country.

Proposition 4 has an important corollary. Whereas Proposition 4 explains why country K has a higher investment in country j than in country q when α_j^K/m_j^K is bigger than α_q^K/m_q^K, the corollary explains the ratio of the value of a country's foreign investments to its total wealth.

Corollary. If $w_F^K W^K$ is the value of the foreign investments of country K, then w_F^K is, given conditions (A) - (C) of Proposition IV, a decreasing function of the consumption elasticity of imports of country K.

To show why the corollary holds, from the vector ${}_{H}w^{K}_{B}W^{K}$ which is the vector of foreign investments for hedging purposes of country K. Let $\underline{e}$ be a (n - 1) x 1 vector of ones. Then:

$$
{}_{H}w^{K}_{F}W^{K} = \underline{e}' \cdot {}_{H}w^{K}_{B}W^{K}
$$

is the foreign hedging investment of country K. If ${}_{H}w^{K}_{F}$ is a decreasing function of the expenditure elasticity of imports, then the corollary is correct, because the mean-variance efficient portfolio of bonds of country K does not depend on either marginal or average propensities to consume foreign goods. Note now:

$$
{}_{H}w^{K}_{F}W^{K} = -\frac{\underline{e}'C^{K}_{P}}{C^{K}_{W}} + \frac{C^{K}\alpha^{K}_{M}}{C^{K}_{W}} - \frac{T^{K}m^{K}_{M}}{C^{K}_{W}} \tag{h}
$$

where α^{K}_{M} is the average expenditure share of imports of country K and m^{K}_{M} is the marginal expenditure share of imports of country K. (h) has exactly the same form as the equation giving the j-th row of (g) and consequently, because of the argument used in Proposition 4, the corrollary has to be true.

The corollary is important because it focuses on a country's aggregate holdings of foreign assets. If the expenditure elasticity of imports is high, a country will want to hold a negative amount of foreign assets. As foreign countries may also want to hold a negative amount of domestic assets, it should be stressed that a high expenditure elasticity of imports has no implication for the question of whether or not a country is borrowing abroad in the sense that its capital account would show a net debt with respect to foreigners.

3.4. Futures contracts and foreign bonds.

In the preceeding analysis, we assumed that the only assets available were bonds indexed on state variables and stocks. Suppose now that futures contracts are introduced in the model. The payoff of each futures contract can be exactly replicated by a portfolio of bonds which involves zero net investment. It follows that an investor will be indifferent between holding a futures contract and a portfolio of bonds which replicates the payoff of that futures contract. In a world in which a complete set of futures contracts on commodities exists, the n - 1 first elements of the vector $\underline{w}_B^{ki}$ do not correspond to the net demands for foreign bonds of the i-th investor of country K. For instance, $\underline{w}_{B_j}^{ki}$ is not the fraction of the wealth of the i-th investor of country k invested in the bond denominated in the numeraire of the j-th country, but is the net demand of that investor for assets perfectly correlated with the relative price of the j-th commodity. As futures contracts have zero value, the sum $\sum_{i \neq k} w_{B_j}^{ki} W^{ki}$ does not necessarily correspond to the value of the investments of the k-th country in country j.

With a complete set of futures contracts on commodities, the results of this section explain the relative price risks investors want to take. In this case, a more general model would be needed to explain in which form those relative price risks will be born. Unless each country buys a share in the world portfolio of common stocks whose value is equal to that country's net wealth, some bonds will be traded internationally. Clearly, whether a country borrows abroad to invest in stocks will depend on that country's aggregate risk tolerance.

Whereas our model always explains which relative price risks investors choose to bear, it does not always explain the value of their foreign investments. To know whether our model has useful implications about the value of a country's foreign investments even when there exists a full set of futures contracts on commodities, it would be needed to know to which extent investors share terms of trade risks <u>internationally</u> using bonds and to which extent they share those risks <u>internationally</u> using futures contracts. Suppose that futures contracts are mainly used to share risks <u>within</u> countries. In that case, our results about the determinants of the value of a country's foreign investments still hold.

Our point can be stated in a different way. Suppose all investors in a country are the same. In the absence of futures contracts, we can determine the value of that country's net holdings of foreign assets if we know that country's aggregate consumption function, its expenditure elasticity of imports and its aggregate risk tolerance. If futures contracts exist, the model does not tell us the value of a country's holdings of foreign assets. That value, in this case, is indeterminate, because an investor is indifferent between holding foreign bonds or futures contracts, but the futures contracts have zero value.

Section 4: Equilibrium Implications

Until now, we have studied asset demand functions for given expected excess returns on risky assets. In this section, we make some additional assumptions which allow us to study the holdings of risky assets at the equilibrium. Because, as we have seen in Section 3, the demands for bonds are the sum of the hedging demands for bonds and of the mean-variance efficient demands for bonds, a more restrictive model can help us to understand when observed holdings of bonds will be of the same sign as the investors' demands for bonds as relative price hedges.

4.1 Constant production opportunity set.

Each country is completely specialized in the production of a single good and it is assumed that it never pays to store commodities. Production of good i involves combining a quantity of good i and a technology. The technology is freely available for all investors of country i and has constant returns to scale. The number of investors in each country is large. Investors in country i will, at the equilibrium, invest the available stock of commodity i, which we write K_i, and will issue common stocks to share the risks of their investment with foreign investors. The production risks investors of country i take are given by the solution of the optimization problem of Section 2. $K_i(t)$ is assumed to satisfy the following differential equation:

$$dK_i(t) = [K_i(t)\mu_{K_i}(\underline{S},t) - C_i^W(\underline{S},t)]dt + K_i(t)\sigma_{K_i}(\underline{S},t)dz_{K_i} \quad \forall\, i \qquad (18)$$

where $C_i^W(\underline{S},t)$ is world aggregate consumption of commodity i evaluated in the numeraire of country i. With perfect competition, the value of the stock of commodity i invested in the production of commodity i must be equal to the value of the industry producing commodity i, and consequently we can write $K_i(t) = I_i(t)$. The return on an investment in industry i can be written:

$$\frac{dI_i}{I_i} = \mu_{K_i}(\underline{S},t)ds + \sigma_{K_i}(\underline{S},t)dz_{K_i} \quad \forall\, i \tag{19}$$

If the production opportunity set is stationary, $\mu_{K_i}(\underline{S},t)$ and $\sigma_{K_i}(\underline{S},t)$ are constants, which we can simply write μ_{K_i} and σ_{K_i}. All investors of country j are identical, $\forall$ j, infinitely lived and have an expected intertemporal utility function which can be written:

$$E\left[\int_{t_0}^{\infty} e^{-\gamma^i t} U^i(\underline{c}^i(t))dt\right] \tag{20}$$

where the vector $\underline{c}^i(t)$ is the vector of consumption rates for the various goods for investor i at time t. Clearly, at time t the endowment of the world economy in the various goods is exogeneous. Let $\underline{K}$ be the n x 1 vector of K_i's. The endowment of the various countries is also exogeneous. Let $\underline{W}$ be the vector of endowments of n-1 countries, so that W_i is the endowment of country i. Clearly, if we know $\underline{K}$, $\underline{W}$ and a price vector $\underline{P}^d$, we can obtain the initial endowment of the n-th country. Given $\underline{K}$ and $\underline{W}$, relative prices and interest rates are endogeneous:

$$\mu_B = \mu_B(\underline{K},\underline{W}) \tag{21}$$

$$\underline{P}^d = \underline{P}^d(\underline{K},\underline{W}) \tag{22}$$

It is assumed that interest rates and relative prices are given by twice differentiable functions which have bounded third derivatives. With that assumption, Ito's Lemma can be used to obtain the dynamics of relative prices and interest rates in terms of the 2n-1 state variables.

It is required that the vector of state variables, $\underline{S}(t)$, be such that if at time t+k, $\underline{S}(t+k) = \underline{S}(t)$, all endogeneous variables will have the same value. It is rather obvious why the stocks of the various commodities are exogeneous. However, if we want investors to differ across countries, the price of a commodity will not depend only on the stock of that commodity available. Suppose that there are only two groups of investors. Some investors consume very much of commodity i, whereas the others don't. Given the stocks of the various commodities, the relative prices of the commodities will obviously depend, given the utility functions, on the wealth of the investors who have a high expenditure share for good i relative to those investors who don't.

For the equilibrium to be a rational expectations equilibrium, it is necessary that the functions which determinate relative prices and interest rates used by investors correspond to the true functions which determinate relative prices and interest rates. If all investors were the same, we could solve for those functions.[8] In this paper, we take the existence of a rational expectations equilibrium for granted, as we are only interested in the demands for risky assets.

Proposition 5. Given that (a) the production opportunity set is constant, (b) all investors of country i are identical, $\forall$ i, and have the utility function given by (20), (c) interest rates and relative prices are differentiable functions of the state variables, and (d) each country issues common stocks and bonds in its own numeraire, the world economy has complete markets in the Arrow-Debreu sense.

The state variables are $\underline{K}$ and $\underline{W}$. Note that common stock i has a return which is perfectly correlated with the change in K_i. Indeed, we know that:

$$\frac{dK_i}{K_i} = [\mu_{K_i} - \frac{C_i(\underline{K},\underline{W})}{K_i}] \, dt + \sigma_{K_i} dz_{K_i}$$

However,

$$\frac{dK_i}{K_i} + \frac{C_i(\underline{K},\underline{W})}{K_i} dt = \frac{dI_i}{I_i}$$

Hence, by Ito's Lemma:

$$\frac{dK_i dI_i}{K_i \; I_i} = \sigma^2_{K_i}$$

---

8 See Cox, Ingersoll and Ross (1978). Richard and Sundaresan (1979) have a model with several commodities, but all investors are identical.

Clearly, there exists no asset which is perfectly correlated with W_i. Investors can however manufacture such an asset. Because the functions which give the relative prices and interest rates are differentiable in the state variables, we can inverse the function which gives the relative prices to get:

$$\underline{W} = \underline{W}(\underline{P}^d, \underline{K})$$

We have just shown that there exists an asset perfectly correlated with each element of $\underline{K}$ and we know that there exists assets perfectly correlated with $\underline{P}^d$. It follows that investors can form a portfolio of common stocks and bonds which is perfectly correlated with state variable $\underline{W}$.

Given Proposition 5 and the results of Section 3, it immediately follows that:

> Corollary II. Investors will hold only common stocks to hedge against unanticipated changes in $\underline{K}$ and will hold only bonds to hedge against unanticipated changes in $\underline{P}^d$.

4.2. Equilibrium relationships.

We now derive the equilibrium expected rates of return on bonds. For the sake of simplicity, we assume that the only relevant state variables for the investor are relative prices. The case in which the vector $\underline{K}$ matters independently would not alter our discussion significantly.

Equilibrium obtains on the market for bond i when:

$$\sum_{k\neq i} \sum_{j}^{N_i} n_{B_k}^{ij} W^{ij} + \sum_{k\neq i} \sum_{j}^{N_k} n_{B_i}^{kj} P_k^i W^{kj} = 0 \qquad (23)$$

Note that W^{kj} is the wealth, evaluated in numeraire of country k, of the j-th investor of country k. Condition (23) can be rewritten as:

$$\sum_{j}^{N_i} W^{ij} - \sum_{k\neq i} \sum_{j}^{N_k} w_{I_i}^{kj} P_k^i W^{kj} = \sum_{k\neq i} \sum_{j}^{N_i} w_{B_k}^{ij} W^{ij} - \sum_{k\neq i} \sum_{j}^{N_k} w_{B_i}^{kj} P_k^i W^{kj} \qquad (24)$$

The left-hand side of (24) is equal to the net international investment position of the i-th country. The right-hand side of (23) is equal to the value of the net demand for foreign bonds of country i, minus the value, in the numeraire of country i, of the net demand of other countries for the bond of country i.

When the market for each stock is in equilibrium, a sufficient condition for (23) to hold is for the market for each bond j, $j\neq i$, to be in equilibrium. Let v_i be the net international investment position of country i expressed as a fraction of world wealth computed in the numeraire of country i. $\underline{v}^*$ is an (n-1) x 1 vector whose elements correspond to the net investment position of countries different from country i, expressed as fractions of world wealth. $\underline{\underline{H}}^k$ is an (n-1) x (n-1) matrix which is an identity matrix in which the k-th column has been replaced by a vector which has minus one in each row. The condition which ensures that all bond markets are in equilibrium can be written as:

$$\sum_{k\neq i} (\underline{\underline{H}}^k)' P^i_k (\sum_j^{N_k} \underline{W}_B^{kj} w^{kj}) + \sum_j^{N_i} \underline{w}_B^{ij} W^{ij} + \underline{v}^* W^W = 0 \tag{25}$$

where W^W is world wealth evaluated in the numeraire of country i and $(\underline{\underline{H}}^k)'$ is the transpose of $\underline{\underline{H}}^k$. For (23) to obtain, assets have to be numbered across countries so that, for all q, the q-th bond in each vector $\underline{w}_B^{kj}$, for all k≠q, is the bond of the q-th country, for investors of country i. For all investors of the b-th country, the q-th bond is the bond of the i-th country.

The demand for bonds, as defined by equation (16), can be substituted in equation (25). In Appendix 2, that substitution is performed and the vector of excess returns on foreign bonds for investors of country is derived from equation (25). Define:

$$T_R^W = -(\sum_k \sum_j^{N_k} P^i_k \frac{J_W^{kj} \lambda^{kj}}{J_{WW}^{kj} P^i_k W^{kj}})$$

$$\lambda^{kj} = \frac{P^i_k W^{kj}}{W^W}$$

Using these definitions, we can write the vector of excess expected returns on foreign bonds for country i as:

$$\underline{\mu}_B^i = (T_R^W)^{-1} (-\underline{\underline{V}}_{PP}^i \underline{v}^* + \underline{\underline{V}}_{PI}^i \underline{w}_I^S - \underline{N}^k) + (T_R^W)^{-1} \underline{\underline{V}}_{PP}^i \underline{h}^* \tag{26}$$

where $\underline{N}^k$ is a term defined in Appendix 2, $\underline{w}_I^S$ is a vector of ratios of the value of the supply, expressed in numeraire of country i, of the various common stocks, divided by world net wealth W^W, and $\underline{h}^*$ has as a representative element:

$$h^*_v = \sum_{v} \sum_{j}^{N_v} w_{B_j}^{vk} \lambda^{vj} - \sum_{k \neq v} \sum_{j}^{N_k} w_{B_v}^{kj} \lambda^{kj}$$

It follows that h^*_v corresponds to the hedging demand for foreign bonds by investors of country v expressed as a fraction of world wealth minus the hedging demand for bonds of country v by foreign investors expressed as a fraction of world wealth. The reader can verify that if $\underline{h}^*$ has only zeroes, (26) yields risk premia which have the same interpretation as those generated by a model <u>à la</u> Solnik generalized so as to include non-zero covariances between stocks and domestic returns of foreign bonds. Strictly speaking, (26) will reduce to Solnik's vector of risk premia if investors have logarithmic utility functions and if stocks are not correlated with terms of trade. With those restrictive assumptions, investors would hedge their portfolio of common stocks against exchange risks by taking a short position in each foreign currency for an amount equal to their long position in the stock of the country in which the currency is issued. However, looking back at the first part of this section, the restrictive assumptions we have just mentioned imply that relative prices are not correlated with the stock of goods available! With those restrictive assumptions, if a country has a net wealth larger than the value of its industry, it will want to borrow abroad more than it wants to lend to foreigners at home. This phenomenon will, <u>ceteris paribus</u>, increase the expected excess return of foreign bonds with respect to the domestic bond. Note that the change in the excess expected returns can be brought about merely by a change in the expected rate of change of the terms of trade and does not necessarily imply a change in interest rates. In the remainder of this section, we are interested in the effect of hedging demands for bonds on the excess return of foreign bonds with respect to the domestic bonds. To simplify the discussion, we assume that all the excess expected return of foreign bonds is due to the hedging demands for bonds. Consequently, we assume that the risk premia on bonds which would obtain in a generalized Solnik model are all equal to zero.

The holdings of the bond of the j-th country by investors of country i can be written as:

$$w^i_{B_j} W^i = \left(\frac{-J^i_W}{J^i_{WW}}\right)(T^W_R)^{-1}\left[\sum_{k\neq j} P^i_k h^k_j - \sum_v P^i_j h^j_v\right] + h^i_j + \ldots$$

where h^k_j is the hedging demand of country k for the bond of country j expressed in the numeraire of country k. The terms which have been written out implicitly contain the assumption that the expected excess return on the j-th foreign bond for domestic residents is the expected excess return for residents of the j-th country on the domestic bond, multiplied by minus one. It is well-known that this is not true, but the terms which correspond to the difference between those expected returns have been left out (that difference is often called the Siegel paradox, see Siegel (1972)). The term in brackets corresponds to the world's hedging demands for the bond of country j minus country j's hedging demand for foreign bonds. If the term in parentheses is small, country i's holdings of bonds of country j will depend mainly on its hedging demand for the bond of country j. Suppose that all investors have an expenditure elasticity for the good produced in country j sufficiently high to guarantee that they have negative hedging demands for bonds issued in country j. This fact implies that investors of country j should have a low expenditure elasticity of imports. Provided that the expenditure leasticity of imports is low enough, investors of country j will have a positive aggregate hedging demand for foreign bonds. In this case, the term in brackets will be the sum of two negative terms. Suppose alternatively that investors always have a high expenditure elasticity for foreign goods. Provided that the expenditure elasticity is high enough, the term in brackets will be the sum of two negative terms. Suppose alternatively that investors always have a high expenditure elasticity for foreign goods. Provided that the expenditure elasticity is high enough, the term in brackets will be the sum of one positive and one negative term. It follows that one would expect the mean-variance demand for foreign bonds to be large in absolute value when tastes across countries are very similar.

To gain a better understanding of the determinants of country i's demand for bonds, suppose further that there is only one investor in each country and that for each investor k there exists an approximation to his value function which we can write:

$$J^k(W^k, \underline{P}^k, t) = A^k \ln \frac{W^k}{P^k} + B^k$$

where:

$$P^k = (1 - \frac{W^k}{G^k})P^k + \frac{W^k}{G^k}P^k_m$$

$$P^k = \prod_j (P^k_j)^{\alpha^k_j} \qquad \sum_j \alpha^k_j = 1$$

$$P^k_m = \prod_j (P^k_j)^{m^k_j} \qquad \sum_j m^k_j = 1$$

By definition, W^k is the wealth of investor k in the numeraire of country k. $\underline{P}^k$ is a vector of logarithms of relative prices in country k. A^k, B^k and G^k are unspecified functions such that A^k and G^k depend neither on relative prices nor on wealth, whereas B^k does not depend on wealth. If the investor had a logarithmic utility function, his value function would, with the assumptions of this section, take the functional form we assume. Otherwise, if some $\alpha^k_j \neq m^k_j$, there is no well-behaved utility function which yields such a value function for all levels of wealth. If $\alpha^k_j = m^k_j$, for all j's, investor k has unitary relative risk tolerance. In that case, investor k will have no hedging demands for foreign bonds. Suppose now that $\alpha^k_k \neq m^k_j$, for some j. It is easy to show that:

$$h^k_j = \frac{W^k P^k_m (\frac{W^k}{G^k})}{P^k_W W^k + P^k}(\alpha^k_j - m^k_j)$$

where P^k_W is the derivative of P^k with respect to wealth W^k.

The value function of investor k depends on his real wealth. The investor consumes a basket of commodities which is a combination of two different baskets. As a wealth of the investor increases, he consumes more of one basket and less of the other one. If $\alpha_j^k > m_j^k$, the expenditure elasticity of investor k for good j is smaller than one. It follows that, for this example, the sign of the hedging demands for bonds depends only on the expenditure elasticities.

It is often argued that the expenditure elasticities for imports are larger than one.[9] If that is true, clearly all investors, in the model considered, would have negative hedging demands for foreign bonds and hence would want to reverse-hedge their imports. By substituting the hedging demands obtained from our example in the equilibrium condition for bonds, it can be shown that (a) if the terms due to the Siegel Paradox are neglected, all investors can hold negative amounts of foreign bonds (or equivalently be short in futures on foreign commodities) and (b) if expenditure elasticities for imports are different from one, it is possible for investors to have substantial hedging demands, in absolute value, for foreign bonds, with no investor having substantial holdings of foreign bonds, in absolute value, in his mean-variance efficient portfolio of risky assets. The fact that investors want to hedge, or reverse-hedge, their imports does not, within our model, imply the existence of large risk premia, in absolute value, on foreign bonds.

---

[9] See for instance Magee and Houthakker (1969) for such a claim.

Section 5: Conclusion

In this paper, the relationship between a country's international trade and that country's holdings of risky assets has been studied. It has been shown that how much exchange or terms of trade risks investors want to bear is a function of their expenditure elasticity of imports. It is neither true, in general, that investors will systematically avoid bearing exchange or terms of trade risks for which they are not rewarded by having a higher expected rate of return, nor is it true that investors will hold a real-bond portfolio with shares proportional to their average expenditure shares. When investors do not trade futures contracts internationally, the paper also has very strong implications about the composition of a country's holdings of assets issued in foreign countries.

To get the results obtained in this paper, many strong assumptions were made. It is not difficult to relax the assumption that only one good is produced in each country, as shown in Stulz (1979), but it is difficult to relax the assumption that if one good is produced in one country, it is produced nowhere else. Clearly, the way to relax the assumption of complete specialization in production is to construct a general equilibrium model in which the production decision is endogeneous. Some forms of taxes and equilibrium deviations from the law of one price could also be introduced in the model.

Numerous papers have been written to show the advantages of international diversification of portfolios.[10] The present paper

[10] For a theoretical discussion which neglects the point of our paragraph, see Subrahmanyam (1975).

points out an advantage of international trade of assets which has been neglected. In the absence of future markets on commodities, the existence of international trade of assets allows investors to hedge against terms of trade risks. In other words, if investors in an economy which does not trade assets internationally do not have available a complete set of markets, it is possible that those investors may face a complete set of markets once international trade in assets is allowed without an increase in the number of assets which exist in the world economy. Of course, this result is heavily dependent on the assumption that each country denominates its safe bond in a different numeraire and on the assumption of the law of one price. The law of one price could be relaxed. However, if deviations from the law of one price are stochastic, a bond issued in country j does not promise the payment of a certain quantity of the good of country j in country q. Further reserach should explain why countries have different numeraires.[11]

[11] For a discussion of the choice of a numeraire in open economies, see Stulz (1980).

APPENDIX I

We have to prove that if the numeraire is changed, each element of M^{ki}_{wI} stays the same. We know already that $\underline{\underline{V}}_{II}$ does not depend on the numeraire. Now, we look at:

$$\underline{\underline{V}}^{k}_{IP}(\underline{\underline{V}}^{k}_{PP})^{-1}\underline{\underline{V}}^{k}_{PI}$$

Clearly, $\underline{\underline{V}}^{k}_{IP}$ depends on the numeraire. Suppose we choose to use numeraire v instead of numeraire k. Choose a representative element of $\underline{\underline{V}}^{k}_{PP}$, for instance the i-th element of the j-th row:

$$V^{k}_{P_iP_j} = \frac{dP^{k}_{j}}{P^{k}_{j}}\,\frac{dP^{k}_{i}}{P^{k}_{i}}$$

By the law of one price:

$$V^{v}_{P_jP_i} = V^{k}_{P_jP_i} - V^{k}_{P_jP_v} - V^{k}_{P_iP_v} + V^{k}_{P_vP_v}$$

To get the matrix $\underline{\underline{V}}^{v}_{PP}$ from $\underline{\underline{V}}^{k}_{PP}$ (note that in $\underline{\underline{V}}^{v}_{PP}$ assets are renumbered so that asset v becomes asset k) we need to subtract, from the matrix $\underline{\underline{V}}^{k}_{PP}$, the v-th row from every other row and the v-th column from every other column, change the sign of the v-th column and then change the sign of the v-th row. Let $\underline{\underline{H}}^{v}$ be an identity matrix in which the v-th column has been replaced by a column of (-1)'s. Then:

$$\underline{\underline{V}}^{v}_{PP} = \underline{\underline{H}}^{v}\underline{\underline{V}}^{k}_{PP}(\underline{\underline{H}}^{v})^{-1}$$

$$\underline{\underline{V}}^{v}_{PI} = \underline{\underline{H}}^{v}\underline{\underline{V}}^{k}_{PI}$$

$$\underline{\underline{V}}^{v}_{IP} = \underline{\underline{V}}^{v}_{IP}\underline{\underline{H}}^{v}$$

It follows that:

$$\underline{\underline{V}}^{v}_{IP}(\underline{\underline{V}}^{v}_{PP})^{-1}\underline{\underline{V}}^{v}_{PI} = \underline{\underline{V}}^{k}_{IP}\underline{\underline{H}}^{v\prime}(\underline{\underline{H}}^{v\prime})^{-1}(\underline{\underline{V}}^{k}_{PP})^{-1}\underline{\underline{H}}^{v}\underline{\underline{V}}^{k}_{PI} = \underline{\underline{V}}^{k}_{IP}(\underline{\underline{V}}^{k}_{PP})^{-1}\underline{\underline{V}}^{k}_{PI}$$

We now have proved that the first bracketed term stays the same when the numeraire changes.

Note now that:

$$\underline{\mu}^{k}_{I} - \underline{\underline{V}}^{k}_{IP}(\underline{\underline{V}}^{k}_{PP})^{-1}\underline{\mu}^{k}_{B} = \underline{\mu}_{I} + \underline{V}_{I.P(k)} - \underline{\underline{V}}_{IP}\underline{\underline{H}}^{v\prime}(\underline{\underline{H}}^{v\prime})^{-1}(\underline{\underline{V}}^{k}_{PP})^{-1}$$

$$\cdot\ (H^{v})^{-1}H^{v}\underline{\mu}^{k}_{B} = \underline{\mu}_{I} + \underline{V}_{I.P(k)} - \underline{\underline{V}}^{v}_{IP}(\underline{\underline{V}}^{v}_{PP})^{-1}T^{v}\underline{\mu}^{k}_{B}$$

where $\underline{\mu}_{I}$ does not depend on the numeraire and $\underline{V}_{I.P(k)}$ has as a typical element $\mathrm{Cov}(I_i, P^k_i)$. But, for $j \neq v$:

$$\mu^{k}_{B_j} = \mu^{b}_{B_j} - \mu^{v}_{B_k} - V_{P^v_j . P^v_k} + V_{P^k_v . P^k_v}$$

and for $j = v$:

$$\mu^{k}_{B} = -\mu^{k}_{B_j} - V_{P^v_j . P^v_k} + V_{P^k_v . P^k_v}$$

It follows that:

$$\underline{\underline{H}}^{v}\mu^{k}_{B} = \mu^{v}_{B} - \underline{V}_{P(v).P^v_k}$$

The vector $\underline{V}_{P(v).P^v_k}$ is the v-th column of $\underline{\underline{V}}^{v}_{PP}$. Hence, $(\underline{\underline{V}}^{v}_{PP})^{-1}\underline{V}_{P(v).P^v_k}$ yields an (n-1) x 1 column vector which has zero everywhere except one in its v-th row. $\underline{\underline{V}}_{IP}^{v}$ times that vector yields $\underline{V}_{I.P_v^v}$. But the v-th column of $\underline{\underline{V}}_{IP}^{v}$ is simply (-1) times the k-th column of $\underline{\underline{V}}_{IP}^{k}$. We then get:

$$\underline{V}_{I.P(k)} - \underline{\underline{V}}_{IP}^{v}(\underline{\underline{V}}_{PP}^{v})^{-1}\underline{\mu}_B^{v} + \underline{V}_{I.P_v^v} = \underline{V}_{I.P(v)} - \underline{\underline{V}}_{IP}^{v}(\underline{\underline{V}}_{PP}^{v})^{-1}\underline{\mu}_B^{v}$$

To see this, note that $\underline{V}_{I.P(k)}$ is an n x 1 column vector with zero in its k-th row. Choose the i-th row; by Ito's Lemma:

$$\mathrm{Cov}(I_i, P_i^k) = \mathrm{Cov}(I_i, P_i^v) - \mathrm{Cov}(I_i, P_k^v)$$

for $i \neq k$, $i \neq v$, the second covariance on the right-hand side is the term $V_{I_i.P_k^v}$ multiplied by minus one. The v-th term will have on the right-hand side simply $-\mathrm{Cov}(I_v, P_k^v)$, and it will cancel out because it is $-\underline{V}_{I_v.P_k^v}$. Finally, $\mathrm{Cov}(I_k, P_k^k)$ is equal to zero, and the k-th row of $\underline{V}_{I.P(v)}$ is simply $\underline{V}_{I_k.P_k^v}$. This completes the proof.

APPENDIX II

Let $\underline{w}_I^s$ be the vector of proportions of world wealth supplied in the form of each common stock. From Appendix I, it is trivial to show that:

$$(T_R^W)^{-1}(\underline{\underline{V}}_{II}^d - \underline{\underline{V}}_{IP}^d(\underline{\underline{V}}_{PP}^d)^{-1}\underline{\underline{V}}_{PI}^d)\underline{w}_I^s + \underline{\underline{V}}_{IP}^d(\underline{\underline{V}}_{PP}^d)^{-1}\underline{\mu}_B^d = \underline{\mu}_I^d$$

Substituting (21) in (23) yields:

$$\underline{v}^*W^W = -\sum_{k\neq i}(\underline{\underline{H}}^k)'P_k^d(\underline{\underline{V}}_{PP}^k - \underline{\underline{V}}_{PI}^k\underline{\underline{V}}_{II}^{-1}\underline{\underline{V}}_{IP}^k)^{-1}\Big(\sum_j^{N_k}\frac{-J_W^{kj}}{J_{WW}^{kj}}\Big)(\underline{\mu}_B^k - \underline{\underline{V}}_{PI}^k\underline{\underline{V}}_{II}^{-1}\underline{\mu}_I^k) + \ldots$$

$$= -\sum_{k\neq i}(\underline{\underline{V}}_{PP}^d - \underline{\underline{V}}_{PI}^d\underline{\underline{V}}_{II}^{-1}\underline{\underline{V}}_{IP}^d)^{-1}\Big(\sum_j\frac{-J_W^{kj}P_d^kW^k}{J_{WW}^{kj}W^{kj}}\cdot\frac{W^W}{W^W}\Big)(\underline{\underline{H}}^k\underline{\mu}_B^k -$$

$$\underline{\underline{H}}^k\underline{\underline{V}}_{PI}^d\underline{\underline{V}}_{II}^{-1}(\underline{\mu}_I^d - \underline{V}_{I.P_k^d})) + \ldots$$

Substituting $\underline{\mu}_I^d$ in the last line yields:

$$\underline{v}^* = -(\underline{\underline{V}}_{PP}^d - \underline{\underline{V}}_{PI}^d\underline{\underline{V}}_{II}^{-1}\underline{\underline{V}}_{IP}^d)^{-1}\sum_{k\neq d}\Big(\sum_j\frac{-J_W^{kj}\lambda^{kj}}{J_{WW}^{kj}W^{kj}}\Big)(\underline{V}^d_{P.P_k^d} + \underline{\underline{V}}_{PI}^d\underline{\underline{V}}_{II}^{-1}\underline{V}_{I.P_k^d})$$

$$- T_R^W\underline{\underline{V}}_{PP}^{-1}\underline{\mu}_B^d + (\underline{\underline{V}}_{PP}^d)^{-1}\underline{\underline{V}}_{PI}^d\underline{w}_I^S - \sum_{k\neq d}\sum_j\frac{1}{W^W}(\underline{\underline{H}}^k,h^{kj}) - \sum_j\frac{1}{W^W}\underline{h}^{dj} \qquad \text{(A)}$$

Define the first term on the right-hand side as $\underline{N}$, and define:

$$\underline{h}^d = -\sum_{k\neq d}\sum_j\frac{1}{W^W}(\underline{\underline{H}}^{k'}h^{kj}) - \sum_j\frac{1}{W^W}\underline{h}^{dj} = -\sum_{k\neq d}\sum_j\underline{\underline{H}}^{k'}\underline{w}_H^{kj}\lambda^{kj} - \sum_j\frac{1}{W^W}\underline{w}_H^{dj}\lambda^{dj}$$

Substituting those definitions in (A) and then inverting (A) yields the result in the text.

REFERENCES

ADLER, Michael and Bernard DUMAS, 1975, "Optimal International Acquisitions", Journal of Finance, 30, 1-19.

ADLER, Michael and Bernard DUMAS, 1976, "Portfolio Choice and the Demand for Forward Exchange", American Economic Review, 66, 332-339.

ARNOLD, Ludwig, 1974, Stochastic Differential Equations, (Wiley & Sons).

BLACK, Fischer, 1974, "International Capital Market Equilibrium with Investment Barriers", Journal of Financial Economics, 1, 337-352.

BLACK, Fischer, 1978, "The Ins and Outs of Foreign Investment", Financial Analysts Journal, May-June, 1-7.

BREEDEN, Douglas T., 1978, "The Role of Futures Markets in a Continuous-time Model", mimeo.

BREEDEN, Douglas T., 1979, "An Intertemporal Model with Stochastic Consumption and Investment Opportunities", Journal of Financial Economics, 7, 265-296.

COX, John C., John E. INGERSOLL, and Stephen A. ROSS, 1978, "A Theory of the Term Structure of Interest Rates", mimeo.

DUFEY, Gunter, 1976, "The Structure of Private Foreign Investment", mimeo.

FAMA, Eugene F. and André FARBER, 1979, "Money, Bonds and Foreign Exchange", American Economic Review, 69, 639-649.

FISCHER, Stanley, 1975, "The Demand for Index-bonds", Journal of Political Economy, 83, 509-534.

FRANKEL, Jeff, 1978, "Exchange Rates Since Floating: Theory and Evidence", unpublished dissertation, M.I.T.

GRAUER, Frederick L. A., Robert H. LITZENBERGER and Robert E. STEHLE, 1976, "Sharing Rules and Equilibrium in an International Capital Market Under Uncertainty", Journal of Financial Economics, 3, 233-256.

HECKERMAN, Donald G., 1973, "On the Effects of Exchange Risk", Journal of International Economics, 3, 379-387.

HOURHAKKER, H.S. and Stephen P. MAGEE, 1969, "Income and Price Elasticities in World Trade", Review of Economics and Statistics, 51, 111-125.

KOURI, Pentti K., 1976, "The Determinants of the Forward Premium", mimeo, Institute for International Studies, Stockholm.

KOURI, Pentti K., 1977, "International Investment and Interest Rate Linkages Under Flexible Exchange Rates", in Robert Z. Aliber, ed., The Political Economy of Monetary Reform, New York.

LESSARD, Donald R., 1979, "International Diversification", forthcoming in The Investment Manager's Handbook, Dow-Jones Irwin, New York.

MERTON, Robert C., 1971, "Optimum Consumption and Portfolio Rules in a Continuous-time Model", Journal of Economic Theory, 3, 373-413.

MERTON, Robert C., 1978, "On the Mathematics and Economic Assumptions of Continuous-time Models", mimeo., Massachusetts Institute of Technology.

MERTON, Robert C., 1978, "On the Microeconomic Theory of Investment Under Uncertainty", in K.J. Arrow and M.O. Intriligator (eds.), Handbook of Mathematical Economics, North-Holland (Amsterdam).

RICHARD, Scott F. and M. SUNDARESAN, 1979, "A Continuous Time Equilibrium Model of Commodity Prices in a Multigood Model", mimeo.

SOLNIK, Bruno, 1973, European Capital Markets, (Lexington Books).

SOLNIK, Bruno, 1977, "Testing International Asset Pricing: Some Pessimistic Views", Journal of Finance, 33, 503-512.

STULZ, René M., 1979, "Terms of Trade and International Investment", mimeo.

STULZ, René M., 1980, "A Model of International Asset Pricing", mimeo.

STULZ, René M., 1980b, "On the Effects of Barriers to International Investment", mimeo.

SUBRAHMANYAM, Marty G., 1975, "On the Optimality of International Capital Market Integration", Journal of Financial Economics, 2, 3-28.