An Approach to the Preliminary Design of Controlled Structures

by

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Abstract

Current trends in flexible space structures often place many flexible modes of the spacecraft inside the bandwidth of active controllers required to meet pointing and alignment requirements. To properly design these structures, the presence of active control must be taken into account. The current approach to this problem has been to optimize the structure and the control of the system simultaneously. However, this methodology suffers from the fact that numerical optimization provides the engineer with very little insight into the problem. This insight is crucial in the early stages of design of the controlled structure. Even in cases where numerical optimization is to be used, it is necessary to have a basic understanding of the problem in order to properly select the design variables.

This work seeks to rectify this problem. Five mechanisms for improving the performance of a controlled structure were identified. These were disturbability, controllability, observability, open-loop dynamics, and robustness. These terms described the infuence of the disturbance on the system, the influence of the control on the system, the influence of the system on the performance output, the effects of natural frequency and damping, and the effects of poor modelling.

A series of simple problems were solved which show how the relative improtance of each of these quantities changes as the problem definition changes for different types of disturbances, performance outputs and control levels. The analysis leads to a set of design rules which should be useful in preliminary design. The gradients are found for arbitrary systems and are broken down into subgradients so that the relative importance of the five mechanisms can be tracked in more complex system optimizations. The design rules, in conjunction with the insight obtained from the subgradients are used to interpret optimization results in this thesis and other research.

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Contents

Chapter 1

Introduction to Controlled Structure Optimization

Lately, there has been a great deal of interest in methodologies which can be used to design the structural and control subsystems of large space structures simultaneously. Traditionally, the design of these subsystems has been performed separately, with the control design occurring long after the structural design has been completed. This method worked quite well when structures were smaller and relatively stiff. Most, if not all, of the flexible modes of these spacecraft were well outside of the bandwidth of the controller, hence the structure and control design did not dramatically interact.

Many proposed spacecraft do not have this property. Some of the most notorious of these are the great observatories [1]. These are the successors of Hubble. They are large spacecraft ($\sim 10 - 100$m) which must support one or more telescopes or radio antennae. The large size of these structures coupled with constraints on weight due to launch capabilities gives them very low fundamental frequencies (< 1.0Hz). Increasingly precise pointing and alignment requirements demand large control bandwidths. The net result is that many structural modes lie inside the control bandwidth and hence must be controlled.

This ties the control design so intimately to the structural design, that it is not at all clear what makes up a good controlled structure. One is faced with either designing the structure to meet design objectives directly, or designing it to increase the effectiveness of the control system. Often, it will be impossible to follow both of these approaches simultaneously and the optimal controlled structure will represent a compromise between the two. The first obvious step toward alleviating this problem is to use a computer to search over some space of control and structural designs for an optimal controlled structure. This is known as controlled structure optimization. There have been many investigations into this problem recently. Numerous formulations and solutions of the controlled structure problem have been suggested. Unfortunately, even when the problem is well posed, and the solution is very efficient, the answer to even the simplest problems often defies physical understanding.

This understanding of the results of optimization is critical. In the preliminary design of a structure, there are so many decisions to be made that the use of a computer program for optimization would be infeasible. Numeric optimization requires that the problem be fairly narrowly defined. In defining such a problem, certain basic assumptions must be made which, once made, are no longer subject to scrutiny under the context of a control structures optimization problem. The design process must be sufficiently advanced, that the bulk of engineering decisions remaining are basically the sizing and positioning of structural and control elements. A computer program can tell you where the best place to put an actuator is, or how large to make the battens in a truss, but you cannot ask it to design an optimal spacecraft.

The main goal of this thesis is to gain insight into what features of a controlled structure should drive its design. The approach taken here begins with the formulation and solution of the dynamic performance costs associated with some very simple controlled structures. Detailed analysis of the solutions to these problems will give insight into the controlled structure problem which can be applied very early in the design process. Ultimately, one would like to use this insight as a guide throughout most of the design of a controlled structure. Numerical optimization would then be used only in the very last stages of design to obtain the last bit of performance possible.

The remainder of this chapter is composed of two sections. The first reviews and organizes the literature on controlled structure optimization. The second examines some of the optimal designs obtained by other researchers in their examples and several possible mechanisms by which these designs improve the performance of the system will be suggested. This will provide a starting point for the work conducted in later chapters.

Chapter Two goes on to a more rigorous definition of the controlled structure problems to be discussed, and gives formulae necessary for the evaluation of the cost and its gradient for a given design vector. Chapter Three introduces the concept of the typical section, a very simple controlled structure, and uses it to examine some fundamental issues in control/structure interaction. Design rules of thumb suitable for use in preliminary design are formulated based on the typical sections. Chapter Four presents a beam model which will be used to validate the design rules from Chapter Three. Also in Chapter Four, issues which the typical sections could not address (such as the interaction of several modes with a controller) will be investigated.

1.1 Literature Review

The purpose of the literature review is three–fold. First, it is intended to acquaint the reader with the work that has preceded this thesis and organize it into a useful form. Second, it will provide the basis for the selection of the problem formulations used in the rest of this thesis. There are many problem formulations, and it would be prohibitive to study all of them. And third, it will show the necessity for this work.

Before continuing, it is necessary to make some definitions. This will simplify the ensuing discussion. In all of the work covered here, the plant is always assumed to be a finite dimensional, linear, time–invariant structure. At least some of the design variables are structural parameters and will therefore affect not only the closed loop, but also the open loop dynamics of the system. The equation of motion for the structure can therefore always be expressed as:

$$M(\alpha)\ddot{r}(t) + D(\alpha)\dot{r}(t) + K(\alpha)r(t) = F(\alpha)u(t) + v(t) \tag{1.1}$$

where α is a vector of design parameters, $r(t)$ is a vector of physical or modal displacements, $u(t)$ is a vector of control forces, and $v(t)$ is a vector of disturbance forces which may or may not be included in the problem. Often, it is simpler to express these equations in state space form:

$$\underbrace{\begin{bmatrix} \dot{r}(t) \\ \ddot{r}(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}}_{A(\alpha)} \underbrace{\begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix}}_{B(\alpha)} u(t) + \begin{bmatrix} 0 \\ v(t) \end{bmatrix} \tag{1.2}$$

In some formulations, the controller must rely on sensors for knowledge of the system:

$$y(t) = C(\alpha)x(t) + w(t) \tag{1.3}$$

where $w(t)$ is noise which might corrupt the sensor output.

The open loop eigenvalues, λ_i^{ol}, and eigenvectors, ϕ_i^{ol}, are the solutions of the equation:

$$A\phi_i^{\text{ol}} = \lambda_i^{\text{ol}}\phi_i^{\text{ol}} \tag{1.4}$$

For convenience, it will be assumed that eigenvalues are always ordered by increasing magnitude. In instances where the controller is static feedback of the sensed output (i.e. $u = -C_c y$), the closed loop eigenvalues and eigenvectors are the solutions of:

$$\underbrace{[A - BC_cC]}_{A_{cl}} \ \phi_i^{\text{cl}} = \lambda_i^{\text{cl}}\phi_i^{\text{cl}} \tag{1.5}$$

where C_c is the matrix of feedback gains. Any eigenvalue can be expressed as the sum of a real and imaginary part:

$$\lambda_i = \sigma_i + i\omega_i \tag{1.6}$$

where $i = \sqrt{-1}$ and the damping ratio is defined to be:

$$\zeta_i = \frac{\sigma_i}{\sqrt{\sigma_i^2 + \omega_i^2}} \tag{1.7}$$

With these definitions, it is now possible to look at some of the work done in controlled structure optimization.

There are three basic stages in the controlled structure optimization problem. First, one must clearly define the problem requiring optimization. Second, the problem must be solved. And third, the solution should be analyzed to verify that it is a reasonable design, and also to find ways of changing the problem formulation to get better designs. These three stages will be addressed one at a time in the ensuing sections.

1.1.1 Problem Formulations

Any optimization problem will have three basic components—a design vector, a cost, and constraints. The design vector in controlled structure optimizations includes structural and control parameters which can be varied during the design process. The structural parameters can be anything including, but not limited to, structural dimensions, actuator/sensor placement, and non–structural masses. The control parameters can be such things as the gains in direct output feedback, or weighting values used in the cost to compute LQR/LQG control. The problem is greatly simplified if it is assumed that the design variable can be varied continuously. Although, one can think of design variables which can only take on integer values (such as the number of sensors and/or actuators used by the controller), their inclusion is beyond the scope of this thesis. The reader is referred to work done by Sepulveda and Schmidt [2] for a treatment of these types of design variables.

The cost is a function which maps every allowable design vector to a real number: the cost. The cost indicates the "goodness" of a design. By convention, lower values of the cost indicate better designs. The goal of the optimization is to find a design vector which minimizes this cost.

The constraints define the space of allowable designs. Basically, there are two types. The first types of constraints are side bounds on the elements of the design vector.

$$\alpha_i^l \leq \alpha_i \leq \alpha_i^u \tag{1.8}$$

This prevents obtaining impossible or unrealistic solutions. Any designs that are not within these limits are usually meaningless. For example, if one of the structural parameters is the magnitude of a lumped mass, it would be important to place a lower bound on it to prevent attempts at evaluating designs with negative mass. Because this type of constraint is placed on individual elements of the design vector directly, it can be thought of as a low level constraint. This type of constraint is present in all of the examples in the next section, but it will only be mentioned when it is of significance.

The other types of constraints are higher level constraints on the design vector as a whole and have the form:

$$f(\alpha) \leq f^u \tag{1.9}$$

where $f(\alpha)$ is another cost. Such a constraint may be used to keep the total mass of the system below some level. Designs which violate this constraint are not necessarily impossible, instead, they simply don't satisfy some design requirement. In Reference [3], it is shown that if $\alpha^\star$ is the design which optimizes the problem:

$$\begin{array}{ll} \text{Minimize} & f_1(\alpha) \\ \text{with constraints} & f_i(\alpha) \leq f_i^u \quad i = 2, 3, \ldots \end{array} \tag{1.10}$$

then there exists some set of weighting parameters c_i such that $\alpha^\star$ also optimizes the combined cost :

$$f(\alpha) = \sum_i c_i f_i(\alpha) \tag{1.11}$$

This indicates that constraining costs or forming new cost functionals which are a weighted sum of others are equivalent ways of dealing with competing objectives. For the remainder of this discussion, no distinction will be made between the two.

In the field of controlled structures, there seems to be at least one problem formulation for each researcher. However, the problem formulations all have the same basic structure. They are composed of five parts—the structure definition, the control definition, the disturbance, static performance metrics, and dynamic performance metrics. With the exception of the structure definition, there are only a handful of choices used for these parts. The next sections address these parts individually and discuss how they appear in the literature.

Structure Definition

The structure definition is a description of the structure and its associated structural design variables. There is not a great deal to be said about the structural definition at this point. Naturally, every problem formulation that uses a different structure will have a different structure definition. Some of the most popular structures used as examples in the literature however are beams or simple trusses. These are systems that are just complex enough to demonstrate various optimization formulations and algorithms. The structural parameters varied in the optimization procedures are almost universally related to the sizing and placement of structural elements and control actuators and sensors.

Disturbances

The disturbance is what creates the need for a control system. There are four disturbance types which appear in the literature. The first is a simple prescribed initial condition of the system.

$$x(0,\alpha) = x_0(\alpha) \tag{1.12}$$

For this type of disturbance, one goal of the control system would be to bring the state of the system to zero. The initial condition, $x_0(\alpha)$,can be a function of the design vector. This happens most commonly when the initial condition is a displacement of the system resulting from the application of a prescribed loading against its stiffness:

$$r_0(\alpha) = K^{-1}(\alpha) f_0 \tag{1.13}$$

Recall that $x^T = \left[r_0^T \; \dot{r}_0^T\right]$ Initial conditions dependent on the design vector are used by Belvin and Park, Salame et. al., and Miller and Shim [4–8].

A second type of disturbance is also specified as an initial displacement, except that it is usually independent of the design vector. This is when the desire is to execute a slew maneuver. The object is to move the system from some prespecified initial state to a prespecified final state. Typically, the initial state is a rigid body displacement of the system, and the desired final state is simply zero. These kinds of problems are examined by Hale et. al. and Messac et. al. [9–11].

The third type of disturbance used is zero–mean Gaussian White Noise. In these problems, use is made of the disturbance vector $v(t)$ in Equation 1.2. The covariance of the disturbance vector is given by:

$$E\left[v(t)v(\tau)^T\right] = V(\alpha)\delta(t-\tau) \tag{1.14}$$

where δ is the Dirac–delta function. This type of disturbance is used most often with LQR/LQG controllers.

The last type of disturbance is a prespecified, time–varying disturbance force. This type of disturbance has been suggested in two forms. The first form assumes that the disturbance is a sum of harmonics:

$$v(t) = \sum_i v_i sin(\Omega_i t + \phi_i) \tag{1.15}$$

This is what is used by Thomas, Lust, and Schmit [12]. The other form of time varying disturbance is the set of forces that would be exerted on a body if a slew were to be performed using a bang–bang controller for the rigid body modes.

$$v(t) = \begin{cases} v & 0 \le t < \frac{t_f}{2} \\ -v & \frac{t_f}{2} \le t \le t_f \end{cases} \tag{1.16}$$

Table 1.1: Research Into Controlled Structure Optimization

Reference	Disturbance					Static Metric		Dynamic Metric				Control				Solution Method			
	Nonlnitial Disp.		White Noise	Slew	Time Var. Force	Mass	F-Norm of G	Eigenvalue	Quadratic	Magnitude	Robustness	LQR/LQG	Output Feedback	Filtered Feedback	Postive Real	Gradient	Multi-level Dec.	Approximation	Brute Force
[3,34]			X			X			X		X	X				X			
[4]		X				X			X			X				X			
[5]		X				X			X			X							
[6]		X				X			X			X				X			
[7,8]		X				X		X	X			X				X			
[9,10]				X		X			X			X				X			
[11]				X		X			X			X				X			
[12]					X	X				X			X					X	
[13]	X					X	X	X					X			X			
[14]			X			X			X			X							X
[15,17-20]	X					X	X	X				X				X			
[16]	X					X	X	X			X	X				X			
[21]	X						X	X				X					X		
[22]					X	X			X	X		X				X			
[23-26]			X			X			X							X			
[27]					X	X	X	X		X			X						X
[28]			X			X			X				X	X	X				
[29]	X					X		X			X		X						
[30]	X							X					X						
[31]	X					X						X							
[32]	X						X	X					X			X			
[33]			X			X			X			X							
[35]			X						X			X					X		

Table 1.1 lists the papers covered in this chapter, and indicates which types of disturbances were used. Also listed are the static metrics, dynamic metrics, control definitions and solution methods used. The purpose of this table is to give the reader an overview of what type of work has been done in this field. It is clear from this table, that a great deal of work has been done in defining different types of controlled structures problems and developing algorithms for their solution.

Control Definition

The control definition is a description of the type of control that is to be used on the plant. There are four types which are commonly used. The first type is the Linear Quadratic Regulator or Linear Quadratic Gaussian (LQR/LQG). For deterministic disturbances, the control must be LQR. In that case, the control specification is simply: "select the control $u(t)$ such that the cost,

$$J_c = x_f^T Q_f x_f + \int_0^{t_f} \left(x^T(t) Q x(t) + u^T(t) R u(t) \right) dt \tag{1.17}$$

is minimized". The matrices, Q and Q_f, must be symmetric and positive semidefinite, while the matrix R must be symmetric and positive definite. The infinite horizon LQR control minimizes the cost functional:

$$J_c = \int_0^{\infty} \left(x^T(t) Q x(t) + u^T(t) R u(t) \right) dt \tag{1.18}$$

If the disturbance is Gaussian White Noise, then the cost to be minimized by the control is:

$$J_c = \lim_{t \to \infty} E \left[x^T(t) Q x(t) + u^T(t) R u(t) \right] \tag{1.19}$$

Determining the control for this problem is identical to the infinite horizon LQR problem above, when the full state is available to compute the control. The chief difference between the solutions obtained for the finite and infinite horizon LQR is that if the control is expressed as a gain matrix multiplying the state vector,

$$u(t) = -C_c(t) x(t) \tag{1.20}$$

then the feedback matrix $C_c(t)$ is constant for the infinite horizon LQR, and time–varying for the finite horizon problem. These types of controllers are very popular because modern control theory makes the computation of the multi-input, multi-output (MIMO) optimal control relatively easy, especially in the infinite horizon problem. In this case, the control gains are static and can be found from the solution of an algebraic Ricatti equation.

LQG control is used when the disturbance is Gaussian White Noise and the only knowledge the controller has about the system comes from sensors which are also corrupted by Gaussian White Noise. The goal of the controller is still to optimize the cost for the stochastic LQR given above (Equation 1.19). This type of control is considered by Milman et. al. and Salama et. al. [3,5] , but it is never actually used in an example.

The next type of controller is direct output feedback. The control law is simply a constant feedback gain matrix which multiplies the output vector:

$$u(t) = -C_c y(t) \tag{1.21}$$

where the gains in the control matrix C_c are included as design variables.

This type of controller is used most often when the goal of optimization is to reduce some performance metric other than those used for the LQR/LQG controllers. (e.g. Reference [13]) In those cases, one cannot use modern control theory to efficiently compute the optimal control gains. Placing the control gains in the design vector permits the optimal feedback to be computed numerically.

Similar to direct output feedback is filtered output feedback. In this case, the control law is described by the state space equation:

$$\begin{aligned} \dot{z}(t) &= A_c z(t) + B_c y(t) \\ u(t) &= C_c z(t) \end{aligned} \tag{1.22}$$

where the order of the control state space equation is less than or equal to that of the structural state space equation. One would like to include all of the elements of the control matrices as design variables. However, it has been shown that there are an infinite number of combinations of control gains which will produce controllers with the same dynamic response. Hence, there are often an infinite number of optimal controllers. Slater [14] gives a method where the number of degrees of freedom one has in controller selection is sufficiently reduced, that the expression for any controller is unique.

The last type of controller is a special case of the direct output feedback and filtered output feedback controllers. It is called positive real feedback. Stated simply, positive real feedback controllers are either dynamic or static controllers which are defined to be incapable of adding energy to the system. The simplest example of this type of controller is collocated velocity feedback. The advantage of these controllers is that no matter how poorly the dynamics of the system have been modelled, these controllers cannot destabilize it. Hence, they are very robust. The down side of their use is that positive–realness represents an additional constraint on the controller, hence they may not be as efficient as optimal LQR/LQG. In other words, without the positive–real constraint, optimal LQR/LQG will find the control which will produce the absolute minimum performance cost. Because this controller gives the greatest reduction in the performance cost, any constraints which force one to use a different controller by definition cannot give the same reduction. The fifth column of Table 1.1 shows how these various controllers are used in the literature.

Static Metric

The static metric is one of the types of costs used in the literature. Its chief characteristic is that its computation is based solely on quantities which do not depend on the dynamic behavior of the structure and controller. By far, the most common static metric is the mass of the system. This is a natural choice due to the cost (in dollars) of boosting mass into orbit. Constraining or including this metric in the cost will limit the overall mass of the optimal structure. A subset of these designs are constant mass designs. This constraint is practical for systems affiliated with a dedicated launcher with a fixed payload capacity.

Another static metric used is the Frobenius norm of the feedback gain matrix:

$$J = \mathrm{tr}\left\{C_c^T R C_c\right\} \tag{1.23}$$

where the gain matrix, C_c must be constant, and the weighting matrix, R must be symmetric and positive definite. A more massive or stiffer structure will usually require larger control forces to meet dynamic requirements. Hence larger structures will need larger control gains. Therefore, this metric also tends to limit the mass and/or stiffness of the structure.

The last static metric considered is the static deflection due to some prescribed loading:

$$J = c^T K^{-1} f \tag{1.24}$$

where f is the load vector and c is a vector which maps the static deformation shape onto an output. Unlike the previous two cases, this type of metric will prevent the design of structures which are too flimsy to satisfy mission requirements.

The appearance of the three static metrics in the literature is shown in the third column of Table 1.1.

Dynamic Metric

Just as the static metric measures static quantities in the system, the dynamic metric is a measure of the dynamic behavior. These are quantities which depend on the time response of the controlled structure to one of the disturbances mentioned above.

The simplest dynamic metrics are those which are based on the closed loop eigenvalues of the system. Basically, there are only three of this type which appear in the literature.

$$\begin{aligned} J_1 &= -\omega_i^{\mathrm{cl}} \\ J_2 &= \sigma_i^{\mathrm{cl}} \\ J_3 &= -\zeta_i^{\mathrm{cl}} \end{aligned} \tag{1.25}$$

The negative signs are placed on J_1 and J_3 by convention to convert maximization of frequency or damping ratio into a minimization problem. The attractive feature of formulations of this type is that the disturbance does not need to be defined explicitly. Also, this tends to be one of the least expensive dynamic metrics to compute. As an example, computation of quadratic performance metrics (see below) will at best require the eigenvalue decomposition of a Hamiltonian matrix of order $2n$ (where n is the size of the state vector). The above costs however only require the eigenvalue decomposition of a matrix of half that order.

The next dynamic metric does require a disturbance. It is the quadratic performance metric, and it has three basic forms—two which are used with displacement and slew disturbances:

$$
\begin{aligned}
J &= x_f^T Q_f x_f + \int_0^{t_f} \left(x^T Q x + u^T R u\right) dt \\
J &= \int_0^{\infty} \left(x^T Q x + u^T R u\right) dt
\end{aligned}
\tag{1.26}
$$

and one which is used with Gaussian White Noise disturbances:

$$
J = \lim_{t \to \infty} E\left[x^T Q x + u^T R u\right] \tag{1.27}
$$

These metrics are identical to the costs used for the LQR/LQG controllers. It is important to understand that in this context, minimizing these costs is a global objective of the optimization. Both the structure *and* the control are designed to minimize this metric. When these costs appeared in the previous section for the LQR/LQG controllers, they were a local objective which the control had to minimize for a given structure. There was no requirement that this be the actual dynamic performance objective for the controlled structure. In fact, there are quite a few papers where the local objective used to design the control is not the same as the global objective used for the overall design of the controlled structure [7, 8, 15–22].

The last dynamic metric considers the maximum absolute value of some output of the system:

$$
J = \max_t \left|c^T x_t\right| \tag{1.28}
$$

where c is a vector which maps the state onto the output. This type of metric is used exclusively with time–varying deterministic disturbances in the literature (Table 1.1).

It is now necessary to state on which problem formulations this work will focus. It would be prohibitive to examine the results obtained from every problem formulation. Also, some problem formulations do not capture all of the facets of the design problem.

Consider the formulations which use the eigenvalues of the closed loop system as the only performance metric. These formulations pay good attention to the temporal behavior of a system, but they completely ignore the spatial behavior. Tailoring the eigenvectors of a system can be very important for minimizing the effects of disturbances on performance or improving the performance of the controller. The importance of this is stressed by Messac *et. al.* [11], and is demonstrated in an example of theirs which reappears below. Miller and Shim [6] also note the dependence of many optimal designs on the disturbances. This implies that the computational efficiency gained by using only eigenvalue–based dynamic metrics comes at a high price precisely because the influence of eigenvectors and disturbances was sacrificed.

The quadratic costs are very popular in control theory simply because they can be efficiently optimized. Although they do not always reflect, exactly, the quantities of interest in the problem (e.g. maximum controller output), one can usually obtain a system with the desired behavior by adjusting the penalty matrices. For this reason, quadratic costs can be a very good approximation. If one thinks of the controlled structure optimization as an extension of optimal control theory to include structural parameters as well as control gains, then it makes little sense to further complicate the problem by doing away with a cost which simplifies control design. The remainder of this work will deal exclusively with this as a dynamic metric. Also, only initial displacement and stochastic disturbances will be considered because optimal control theory was formulated around these.

Along similar lines, the only static metric used will be system mass. This has the dual advantage that it is one of the main quantities of interest in spacecraft design, and it can usually be expressed as a weighted sum of the structural parameters. This will be very useful for numeric optimizations which will have to be performed.

1.1.2 Solution Methods

Once one has the problem defined, the next step is to solve it. This is usually accomplished through the use of a computer program. The purpose of this section is to acquaint the reader with some of the numerical techniques which are being used in optimizing controlled structures. Table 1.1 shows some of the different methods used by other researchers.

By far the most popular techniques use gradient optimization. This method uses the gradient of the cost to find successively better designs. This gradient is either computed analytically, or numerically using finite difference techniques. There are a great many gradient–based optimization algorithms including Newton's method, modified Newton's method, Quasi–Newton methods, and conjugate gradient methods. The reader is referred to reference [36] for a good description of these algorithms.

Multi–level decomposition seeks to reduce the computational effort required for optimization by the use of several sub–optimizations. These sub–optimizations iterate over the design variables to find a design which optimizes some internal criterion. For example, one choice of sub–optimization objectives might be to increase the fundamental frequency of a structure. Several of these sub-optimizations are performed simultaneously each using a different subset of the design variables. At a higher level, there is an algorithm which coordinates the sub–optimizations to find the optimal design. This parallel processing can be performed on several processors at once, and hence can significantly reduce solution time.

Often, to reduce the computational effort, the costs and local constraints are linearized. This approximate system is then optimized with an additional constraint on how "far" the new design can be from the old one. The linearizations are recomputed at the new design and the process is repeated. This method limits the number of times computationally expensive non–linear functions must be evaluated and is known as sequential approximation.

Sometimes when there are only one or two parameters, one can adopt a brute force approach. The idea is to compute the cost for a grid of points inside the design space. The values of the cost form a curve if only one design parameter is used, and a surface if two are used. It is then trivial to pick the minimum off of the curve or surface by inspection. The advantage of this somewhat computationally expensive technique is that in addition to obtaining an optimal solution, one gains knowledge of the behavior of the cost over the design space.

1.1.3 Result Analysis

By far, the most common conclusion reached in the current literature on controlled structure optimization is that the methods employed do in fact produce optimal structures for the problems defined. Where some space is devoted to discussing results, there are rarely enough examples worked out to state anything conclusive. Given the current state of controlled structure optimization, this is to be expected. The majority of the effort in the controlled structure community has been devoted to stating the problem and solving it. These are formidable tasks. Only now is this field sufficiently mature that it is possible to start the exhaustive analysis which will be necessary to gain insight into the solutions.

The remainder of this thesis is directed at attempting to understand some of the results of controlled structure optimization. This chapter concludes by establishing a firm starting point through detailed analysis of examples which exist in the literature. The next section is a discussion of some of the approaches one can use in improving the performance of a controlled structure. The emphasis is on various techniques for physically accomplishing some of these approaches. Examples used by several researchers in controlled structure optimization are presented and their solutions analyzed. The idea is to note how these approaches appear in the solutions of these problems.

1.2 The Approaches to Improved Performance

There are two steps in solving the controlled structure problem. The first step is to determine what the important features of the problem are and how they should be changed to improve performance. The second step is to determine how these changes might be accomplished through changes in the physical design. To date, both of these steps have been combined into a single optimization step, thus sacrificing insight into the problem. The first part of this section begins to address the "what" of the problem. In subsequent subsections, various examples are presented which hint at the "how" of the problem.

There are five natural ways one can improve the performance of a controlled structure. Simply put, one can reduce the effect of the disturbance on the system, decrease the effect of the system on the output, increase the effect of the control on the system, improve the dynamic response of the open loop system, or increase the robustness of the closed loop system. For convenience, these methods of improving controlled performance will be called: reduction of structural disturbability, reduction of output observability, increase of controllability, improvement of open loop response, and improvement of robustness. These approaches are shown at the top of Table 1.2. The columns of the table list some specific techniques with which these goals might be accomplished. In other words, the top row lists what should be done while the subsequent rows indicate how it might be done.

Disturbability can be reduced in a number of ways. The most obvious is to simply remove the disturbance. For example, if there is an antenna on the spacecraft which must be slewed to maintain communications, then the motion of that antenna can introduce disturbances into the structure at inopportune times. One might consider doing away completely with the slewing antenna if it is feasible (e.g. replace it with a phased array).

If removal of the noise source is prohibitive (if not impossible) then the next thing one might try is actively or passively isolating the disturbance from the structure. If

Table 1.2: Approaches to Improving Controlled Structure Performance

Reduce Disturbability	Reduce Observability	Increase Controllability	Improve Open-Loop Response	Increase Robustness
Remove disturbance				
Active isolation	Active isolation			Use area-averaging sensors
Passive isolation	Passive isolation	Remove damping	Add damping	Add damping
Stiffen system against disturbance forces		Soften system	Stiffen system	Increase gap between modelled and unmodelled frequencies
Position nodes near disturbances	Position nodes near output point	Position anti-nodes near actuators / sensors		Position nodes of unmodelled modes near actuators / sensors
Position disturbances near nodes	Position output point near nodes	Position actuators / sensors near anti-nodes		Position actuators / sensors near nodes of unmodelled modes

the disturbance cannot be removed, and isolation is not an option (e.g. it would make little sense to isolate the attitude control system from the structure inside the attitude control band) then the only recourse in disturbance reduction lies in modifying the structure directly. The idea here is to either move the disturbance to places where the structure has little motion or move places where there is little motion to the disturbance locations. For broad band disturbances, the former will be very difficult to accomplish. In a structure with many modes, nodes will be scattered all over, and points where many modes have nodes will be rare. However, it is possible to design a structure in such a way that this does in fact happen. An example of how this might be accomplished is to place lumped masses at the point where the disturbance enters the structure. Nodes of all of the modes of the structure will move toward these points.

One special type of disturbance on the structure is the control actuators themselves. The controller can disturb modes which initially had no error while trying to correct errors in other modes. The most notable example of this occurs in slew maneuvers. In these cases, the disturbance is basically an initial displacement of a rigid body mode from some desired final state. The control forces required to correct this error will excite the initially quiet flexible modes. One approach to fixing this problem are to attempt to make the flexible modes as uncontrollable as possible in the same way one would attempt to make them undisturbable. Another approach is to shape the commands to the actuators in such a way that the slew is accomplished while putting a minimum amount of energy into the flexible modes (Reference [37]).

These same tricks work in reverse for observability if the goal is to reduce the motion of the structure at isolated points (e.g. pathlength control used in interferometry only cares about the positions of mirrors in the light path).

To improve the controllability of a structure, one would like to move the sensors and actuators of the control system to positions where the structure has the most motion or vice verse. This would work well if the overall goal were to quiet the structure. Another thing one might try is to place the control where the disturbance enters the structure or where the motion of the structure needs to be reduced. This would ensure that the modes that were the most strongly controlled were also the most disturbable or observable. Finally, to maximize the controllability of a structure, one would like to make it as soft and underdamped as possible. This will reduce the amount of force needed by the actuators to move the structure. It is of interest to note that the changes suggested here for improving controllability are exactly the opposite of those used to decrease disturbability. The reason is that for the former, one is attempting to make the structure more sensitive to an applied control load, while for the latter, one is attempting to make it less sensitive to an applied disturbance load.

Primarily, the control system is employed to close the gap between the performance of the open loop system and the desired performance of the closed loop system. Naturally, improving the open loop response of the system will narrow the gap and simplify the problem. Basically, the idea is to add stiffness to the open loop system to make it faster, and damping to reduce ringing.

The last area one might aim at in improving controlled structure performance is robustness. First and foremost, one of the best ways of improving the robustness of a structure is to add damping. This has been shown to make the system less sensitive to parametric uncertainty, and also reduce the possibility of destabilizing modes which were not modelled in the control design. There will always be some of these as most structures have an infinite number of modes.

The next thing one might try is to make the unmodelled modes as uncontrollable, undisturbable, and unobservable as possible using the methods mentioned above. This will essentially decouple the modelled from the unmodelled system. A novel approach to this has been suggested by Collins [38]. Instead of being placed at a point, the sensor is distributed over a portion of the structure. It is possible to design these area-averaging sensors in such a way that they are inherently less sensitive to the higher frequency, unmodelled modes without sacrificing phase margin in the transition regime and therefore performance in the controlled regime.

Still another approach might be to specify that a portion of the control system be positive real. As mentioned before, this kind of control is very robust. In many respects, one can think of it as "electric damping" since it is theoretically possible to implement these designs passively.

The next section presents some examples used in the literature along with their solutions and some of the more interesting conclusions of their creators. Where necessary, further analysis of the problem is performed to help clarify the solution. This author then adds his own insights into the problem.

1.2.1 Example 1: Cantilevered beam of Belvin and Park

Belvin and Park [4] work out an example on an idealized beam (Figure 1.1). The beam is cantilevered at the root and pinned at the tip. The beam consists of ten Timoshenko beam elements of equal length and width. The design parameters in this problem are the thicknesses of the elements. There are five transverse force actuators located at the positions shown in the figure, and the control is full state feedback.

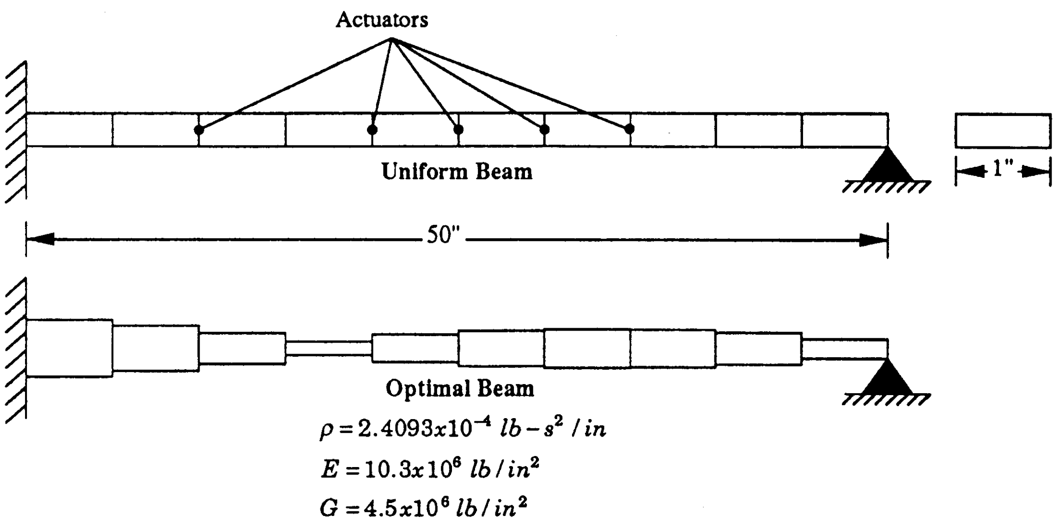

Figure 1.1: Cantilevered–Pinned Beam used by Belvin and Park

The disturbance is an initial velocity and displacement error corresponding to the peak response due to a step force given by:

$$f = M^{1/2}T\gamma \tag{1.29}$$

where T is the mass normalized modal transformation matrix and γ is an arbitrarily selected vector of ones. Thus, the disturbance force effects all modes of the system equally.

The design goal is to minimize the quadratic cost:

$$J = \int_0^\infty \left(x^T Q x + u^T R u\right) dt \tag{1.30}$$

with the state and control penalty matrices selected to penalize the system energy and static control work.

$$Q = \begin{bmatrix} \gamma_r K & 0 \\ 0 & \gamma_{\dot r} M \end{bmatrix} \qquad R = F^T K^{-1} F \tag{1.31}$$

where F is the control input matrix. The constants γ_r and $\gamma_{\dot r}$ are arbitrary in this problem. The mass of the system is held fixed, and all of the element thicknesses are allowed to vary under the constraint that they remain above a small, non–zero, minimum value.

The lower part of Figure 1.1 shows the optimum design found for this problem which was found to be independent of γ_r and $\gamma_{\dot{r}}$. The distribution of material is strikingly similar to what one would expect the bending strain distribution to be for the first mode of the uniform beam. Belvin and Park were actually able to prove that for this problem formulation, the cost is inversely proportional to the cube of the natural frequency of the first mode when the number of actuators is equal to the number of modes retained in the design model. Clearly, this inverse cube relationship prefers stiffening of the first mode over all other mechanisms for improving controlled performance. This type of answer is very encouraging. One would like to have the cost associated with a mode of the structure go down as the frequency of the mode goes up. This will allow one to truncate the plant model with good confidence because the higher frequency modes will not participate very strongly in the cost.

In Table 1.2, there were two approaches which could involve the technique of stiffening the system. The first is to stiffen the system against the disturbance. In their work, Belvin and Park show that the peak response used as the initial condition was inversely proportional to the stiffness of the system. This means that the influence of the disturbance in the cost in inversely proportional to the fourth power of the frequency. It is clear from this that the the stiffening in this example is, in fact, needed to reduce the disturbability.

1.2.2 Example 2: Truss example of Miller and Shim

A truss example appears in the work done by Miller and Shim [6]. Their structure is a ten bar, two–dimensional truss (Figure 1.2). The truss members are modelled as bar elements with pinned joints, and the design variables are their cross–sectional areas. There are lumped masses located at each node. These masses are fixed and are large enough that the mass of the rest of the structure can be ignored in the dynamic response. There are four actuators, one at each free node, capable of exerting vertical forces. The control is full state feedback.

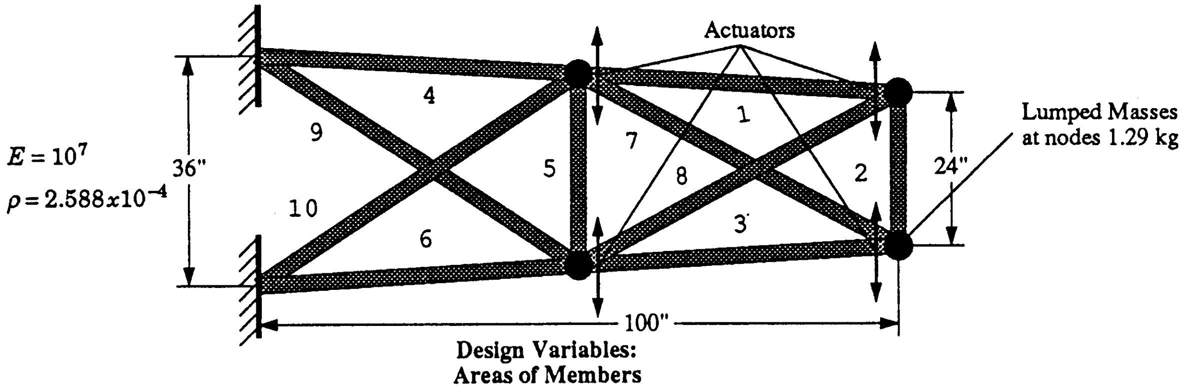

Figure 1.2: Ten bar truss used by Miller and Shim

The objective of the control and structural design is to minimize the cost:

$$J = q_1 W + q_2 \int_0^\infty \left(x^T Q x + u^T R u \right) dt \tag{1.32}$$

where W is the weight of the structure, and the state and control penalty matrices penalize system energy and static control work.

$$Q = \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} \qquad R = F^T K^{-1} F \tag{1.33}$$

The weighting parameters q_1 and q_2 were selected to achieve a "sufficient" reduction in the dynamic performance cost and weight in the structure from a nominal, uniform structure.

The initial conditions for this problem correspond to the static deflection of the truss due to a prescribed loading which was instantaneously removed. Two cases were examined. In the first case, the loading was an equal upward force at each free node of the truss. In the second case, the loading was again equal forces at each node, but the loading at the inner two nodes was downward and not upward. The first loading was selected to excite primarily the first mode of the structure, while the second was selected to excite the second mode.

The free structural parameters in this problem are the cross sectional areas of the members. Figures 1.3 and 1.4 show the optimal cross sectional areas obtained by Miller

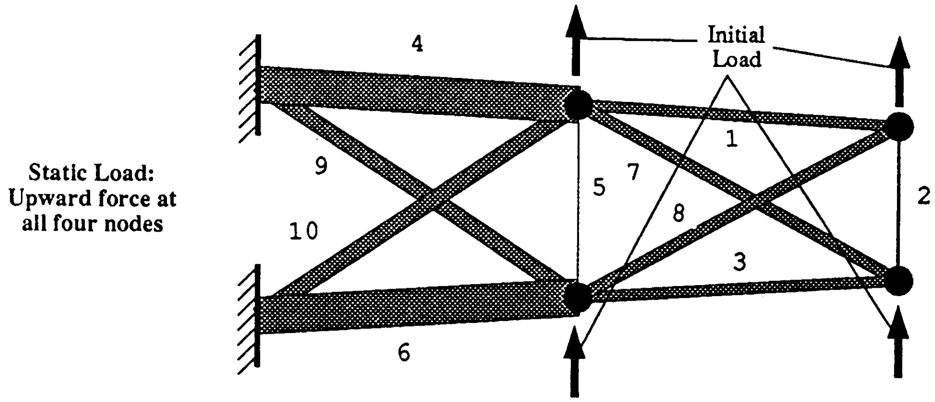

Figure 1.3: Optimal design for load case 1

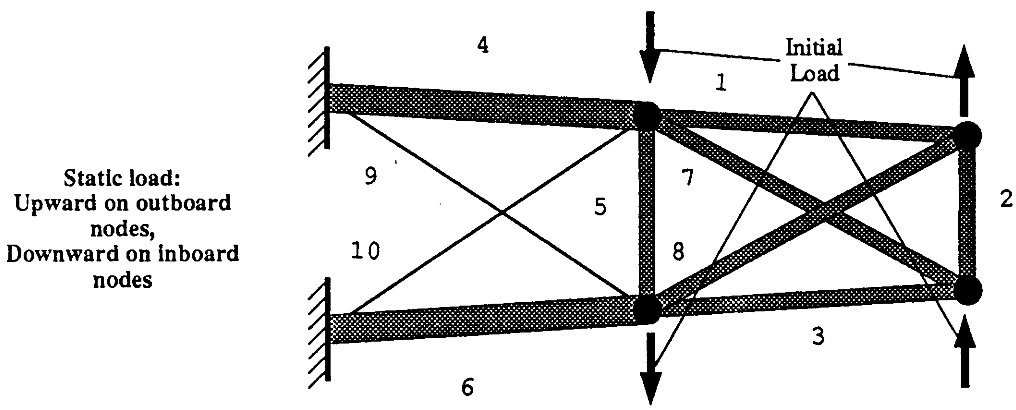

Figure 1.4: Optimal design for load case 2

and Shim for the two static load cases. Notice that in their optimal design for the second load case, there is a significant amount of material in the battens (members 2 and 5). Because these would normally be low stain areas for the problem described, it was decided to redo the optimizations for both load cases. The results obtained by this author are compared to those of Miller and Shim in Figures 1.5 and 1.6. There is close agreement between the designs obtained by Miller and Shim and those obtained here for the first load case. In the second design, it was found that the cross sectional areas of the battens went to the minimum side constraints. Miller and Shim reported that they were having difficulty with the penalty functions used to meet the constraints. Most likely, this was the source of the discrepancy.

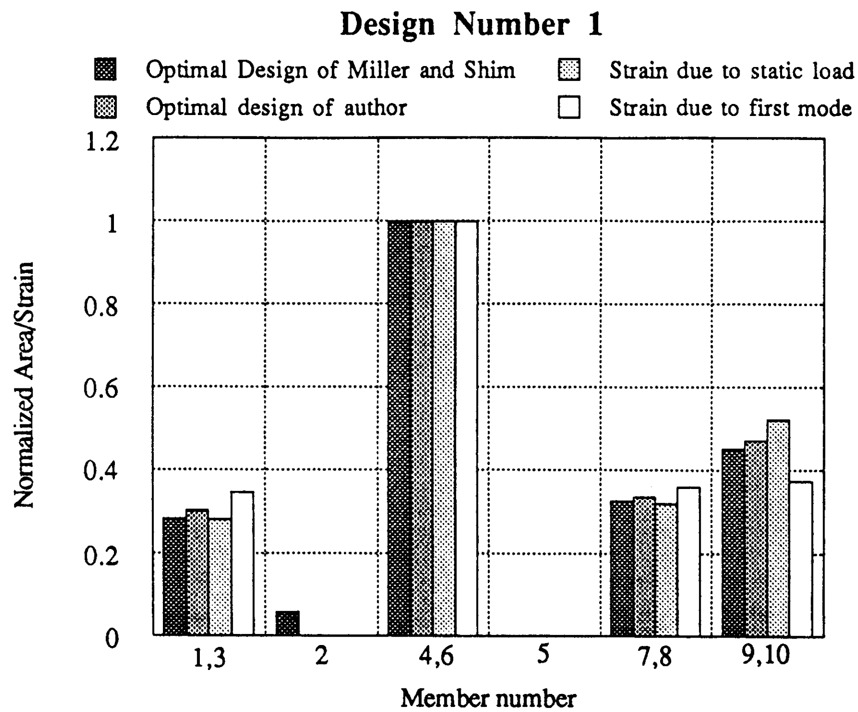

Figure 1.5: Optimal designs of Miller and Shim and this author for first load case, and strain due to static load and first mode.

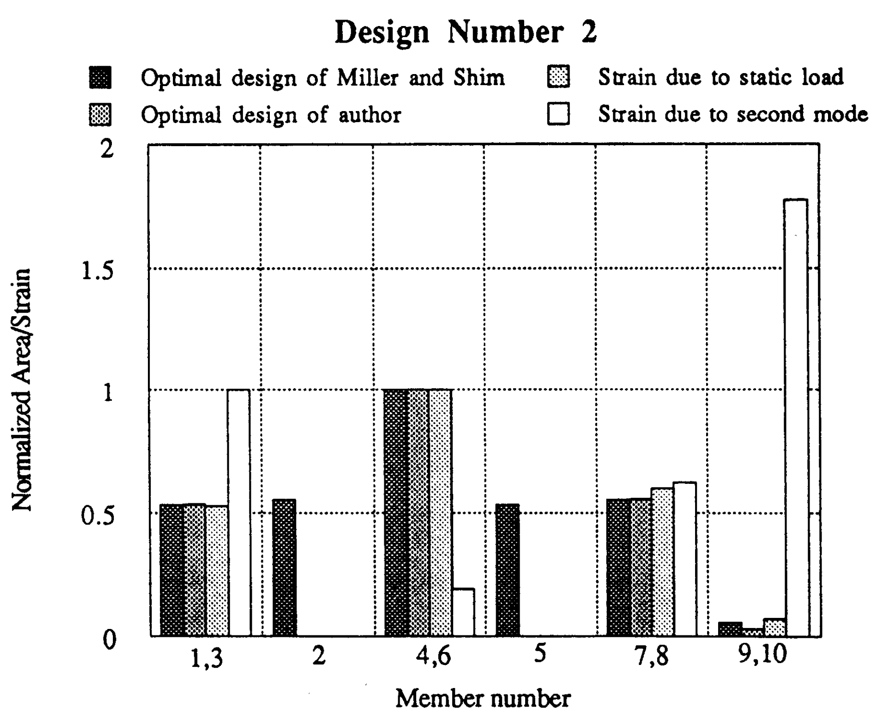

Figure 1.6: Optimal designs of Miller and Shim and this author for second load case, and strain due to static load and second mode.

Although Miller and Shim do not go into it, there is a very interesting explanation for why the designs obtained are optimal. Because of the size of the lumped masses at the nodes, increasing member size does very little to change the mass matrix, hence the purpose of added material in this problem is to stiffen the system. As mentioned above, one might stiffen the system to speed up the open loop dynamics or reduce the sensitivity of the system to the disturbance forces. Included in Figures 1.5 and 1.6 are the strains induced in the members of a uniform structure due to the static loadings and also due the the first and second mode shapes. Remember, the first load case was selected by Miller and Shim to excite the first mode, while the second was selected to excite the second mode.

Comparison of the first optimal design with the static and modal strains is inconclusive. There is good agreement for all three. In the second case, the modal strain is much larger than the static strain in members 9 and 10 and smaller in members 4 and 6. The optimal design, however, agrees very closely with the static strain, hence one can be reasonably certain that the goal of stiffening the system is solely an attempt at reducing the influence of the disturbance.

1.2.3 Example 3: Beam example of Onoda and Haftka

Onoda and Haftka [23–26] use a beam–like structure to demonstrate their optimization algorithm. The upper part of Figure 1.7 shows a depiction of their structure. The disturbance is assumed to be a stochastic force acting along the entire length of the structure:

$$p(x,t) = \sqrt{3}(x/L)f_w(t) \qquad -L \leq x \leq L \tag{1.34}$$

where $f_w(t)$ is Gaussian White Noise. Because this disturbance is asymmetric and it is correlated over the entire length of the structure, only the asymmetric modes of this symmetric structure need to be considered. A Bernoulli–Euler beam consisting of five finite elements used in their analysis is shown in the lower part of the figure.

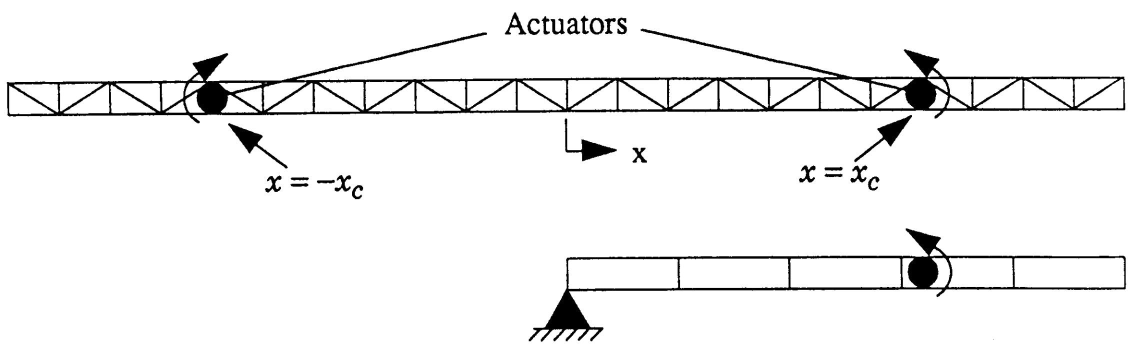

Figure 1.7: Beam–like spacecraft of Onoda and Haftka

Two types of controllers were used—direct output feedback and full state feedback. However, attention here will be restricted to the full state feedback case as the output feedback case did not produce any designs significantly different from the full state feedback case.

The design variables were the cross–sectional areas of the finite elements and the position of a torque actuator. The stiffness and mass of each element is assumed to be proportional to its area.

The design objective in this case was to minimize a weighted sum of the the mass of the structure and the control effort:

$$J = q_1 W + q_2 \int_0^\infty u^2 dt \tag{1.35}$$

The weighting parameters q_1 and q_2 were selected according to assumptions regarding the mass of the controller as a function of the control effort.

The performance of the system was constrained:

$$J_p = \int_0^\infty x^T Q x dt \leq J_p^u \tag{1.36}$$

The matrix Q was selected to penalize the mean square displacement along the beam:

$$x^T Q x = \frac{1}{L} \int_0^L y^2 d\chi \tag{1.37}$$

where y is the vertical displacement of the beam as a function of the spatial coordinate χ.

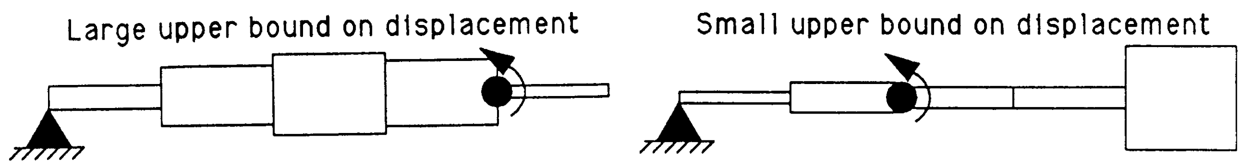

Figure 1.8: Optimal designs for Haftka and Onoda's beam problem.

Figure 1.8 shows the two types of optimal designs found by Onoda and Haftka. The first one corresponds to the case when J_p^u had a modest value (Expensive control). The similarity between this shape and the strain distribution for the first mode of this system makes it clear that the objective is simply to stiffen the first mode. The shape of the disturbance is identical to the mode shape for the rigid body mode and hence, it will be orthogonal to the other modes. Hence, the stiffening of the first mode is not meant to reduce the influence of the external disturbance. Instead however, it appears that its purpose is to keep the actuator from disturbing this mode as it attempts to correct the error induced in the first mode. As J_p^u is made smaller (tighter constraint on controlled performance) the shape approaches the second one shown in the figure. Onoda and Haftka suggest that placing mass at the end is an attempt to reduce the disturbability of the system, as this is where the disturbance is largest. This surmise is most likely correct, as the system has little strain at that point, and therefore, it cannot be there for stiffening.

1.2.4 Example 4: Compression rod example of Messac *et. al.*

Messac, Truner, and Soosaar [11] use a finite element rod (Figure 1.9) as an example of controlled structure optimization where the disturbance is a slew maneuver. The rod is composed of twenty rod finite elements of equal length. The structural design variables are the cross-sectional areas of the elements. In one case, an extra lumped mass was included at the right end of the rod. In the other case, this mass was omitted.

The actuator in this problem is a force actuator at the left end of the rod which acts along the rod's axis. The goal is to translate the rod from an initial rigid body

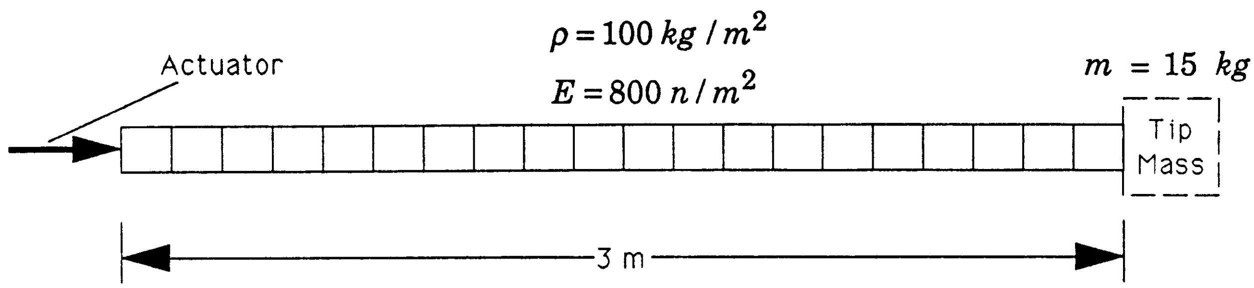

Figure 1.9: Compression rod of Messac *et.al.*

displacement, to a final displacement of zero. The cost for this problem was:

$$J = \frac{1}{2}x_f^T S x_f + \frac{1}{2}\int_0^{t_f} \left(x^T Q x + u^2\right) dt \tag{1.38}$$

where x_f is the state vector at time $t = t_f$ and the matrices Q and S are selected to penalize the sum of the squares of the nodal displacements. Also, S was selected to be approximately nine orders of magnitude larger than Q to ensure that the final error was very small.

The control $u(t)$ is always optimal. Note that because the cost functional is over a finite time, the control cannot be expressed as a time–invariant gain matrix multiplying the state. However, optimal control theory does make it possible to compute the control and evaluate the cost for any reasonable vector of structural parameters.

The structural design parameters in this problem were the cross sectional areas of the finite elements, and the total mass of the structure was held constant. Two different designs were obtained without and with the tip mass (Figure 1.10). In both, the first modal frequency was substantially increased. However, the two designs seem to be mirror images of each other. Messac *et al* provide a good explanation for the optimal design without the tip mass. Figure 1.11 shows the displacement eigenvectors for the optimal bar and a uniform bar of equal mass. The displacement in the first four flexible modes at the actuator has been substantially decreased. In this slew maneuver, the initial displacement is all in the rigid body mode. The flexible modes are initially undisturbed.

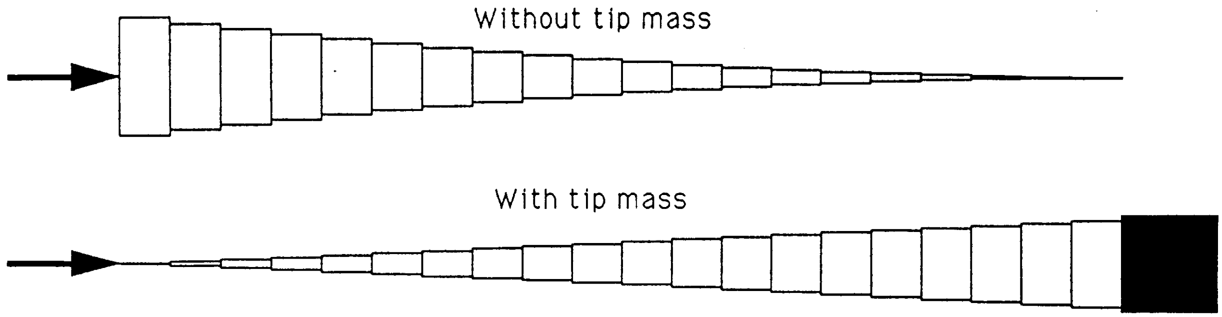

Figure 1.10: Optimal designs for compression rod

As the controller acts to correct this, it will cause disturbances in the other modes which will have to be controlled out. Reducing the displacement in the first four flexible modes at the actuator will reduce this. Messac et al point out that it is this type of improvement that eigenvalue optimization will miss.

This same argument can be applied to the case with the tip mass to show that the goal is to make the flexible modes more controllable.

1.2.5 Example 5: Cantilevered beam example of Milman *et. al.*

Milman, Salama, Scheid, Bruna, and Gibson use several examples to illustrated their homotopy algorithm. The first consists of a cantilevered beam composed of three Bernoulli–Euler finite elements (Figure 1.12). Each element has a circular cross section, and the structural variables are the cross–sectional area of each element. The control is full state feedback acting through a transverse force actuator located at the tip of the beam. The disturbance acting on the system is a pressure wave modelled as three uncorrelated force impulses located at the free nodes of the beam.

The design goal is to find a combination of structural parameters which will optimize a composite cost function based on the structural mass and the performance of the

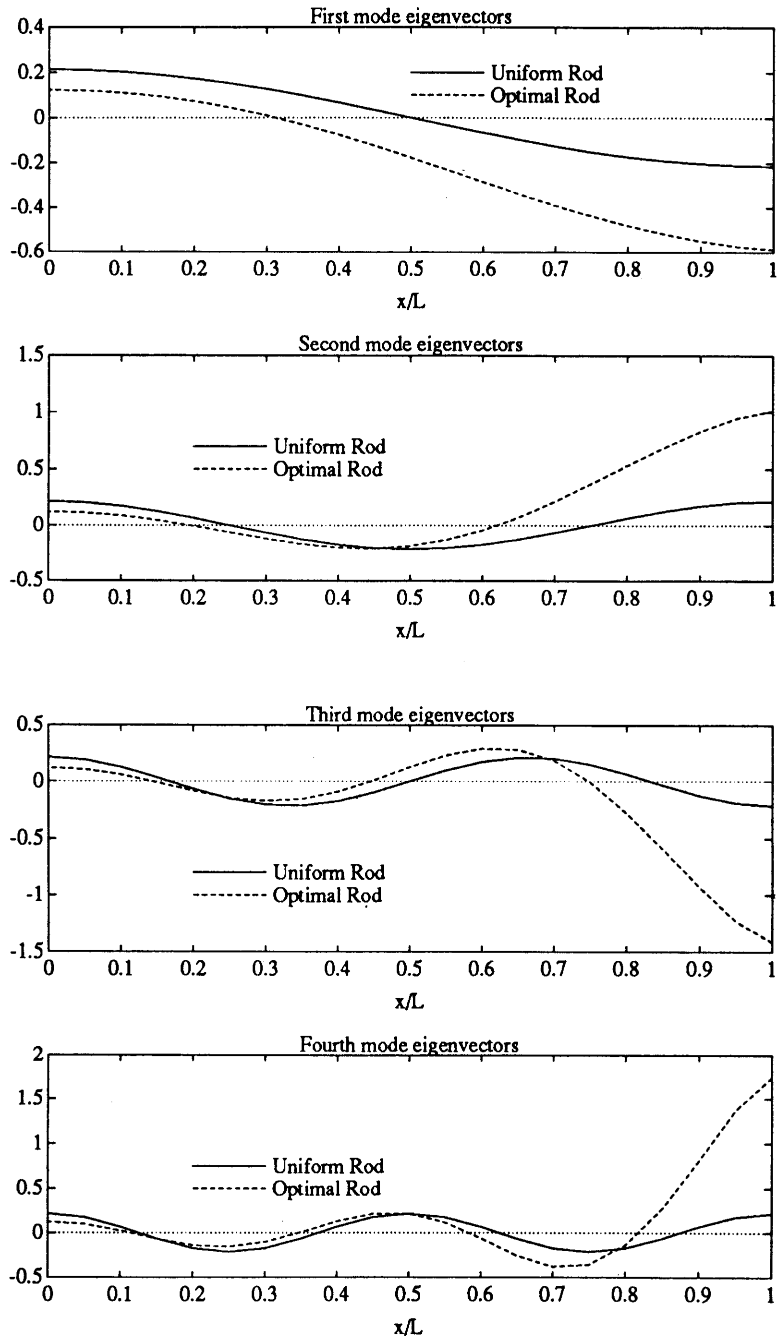

Figure 1.11: Eigenvectors and eigenvalues for the uniform and optimal compression rods (no tip mass).

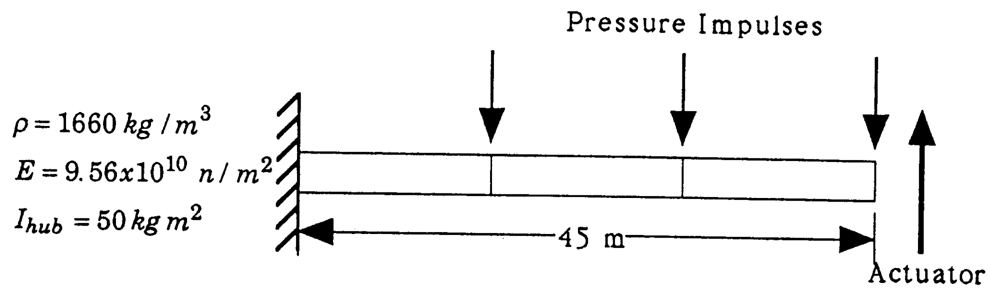

Figure 1.12: Cantilevered beam of Milman *et.al*

system when LQR control is used:

$$\begin{aligned} J_\lambda &= (1-\lambda)J_s + \lambda J_c \\ J_s &= \sum_{i=1}^{3} \rho l_i A_i \\ J_c &= \int_0^\infty (r^T Q_r r + \dot{r}^T Q_{\dot{r}} \dot{r} + u^T R u) dt \end{aligned} \tag{1.39}$$

where l_i, A_i, and ρ are the length, cross-sectional area and density of element i, and r, $\dot{r}$, and u are the displacement, velocity, and control vectors for the system.

The matrices Q_r and $Q_{\dot{r}}$ are arbitrarily defined to be $100K$, and $100M$ respectively, where M and K are the mass and stiffness matrices of the system. This type of state weighting penalizes the total energy in the system. The matrix, R, weights the control effort and is defined to be 10^{-4}.

The parameter, λ, was varied form zero to unity and the composite cost, J_λ, was optimized at each point. This generates the family of designs which represent optimal trade–offs between performance and mass.

The same basic shape was obtained for all values of λ. The top half of Figure 1.13 depicts the optimal design for the structure with a mass of 466 kg ($\lambda = .99$). Milman *et. al.* reason that the control force at the tip of the beam makes the closed loop modes similar to those of a cantilevered–pinned beam. The similarity between their optimal design and the first strain mode shape of a cantilevered–pinned beam leads them to

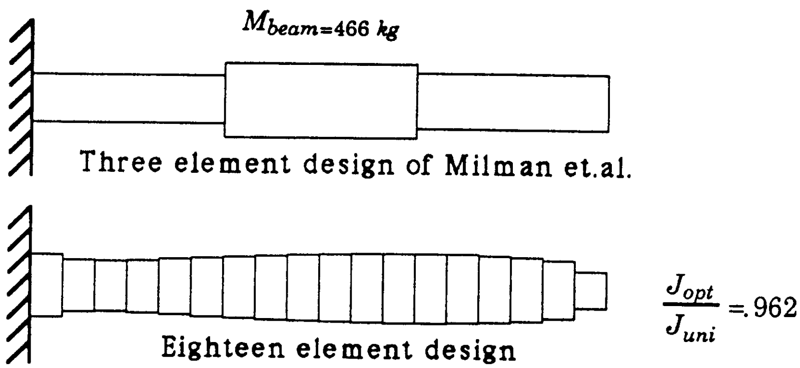

Figure 1.13: Optimal designs for beam problem

conclude that the algorithm is stiffening the first closed loop mode of the system. The author repeated this optimization with an eighteen element beam and allowed lumped masses at the free nodes. The result is also shown in the figure. The lumped masses are all set to zero, and the structure is largest where one would expect the greatest strain for the first closed loop mode, hence this would seem to confirm their conclusion.

There is another interpretation of these results however. Basically, one section of the beam is being stiffened while another section near the root of the beam is being softened. It is entirely possible that this design is a compromise between enhanced controllability and reduced disturbability. The softening at the root will provide a sort of hinge. This will give the controller better authority over all of the modes. On the other hand, raising the stiffness of the beam away from the root (where there is no disturbance) will reduce disturbability.

1.2.6 Example 6: Hub–beam example of Milman *et. al.*

Another example consists of a rigid hub with a flexible appendage (Figure 1.14). The appendage is a three–finite element Bernoulli–Euler beam. Each element has a rectangular cross section and all of the elements are assumed to have an equal width. The design variables in this problem are the nodal depths of the elements at the four nodes. The depth of each element varies linearly from one end to the other (unlike in the previous

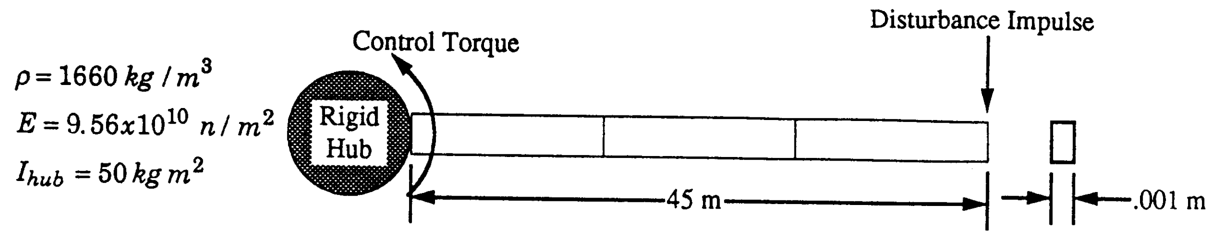

Figure 1.14: Hub–beam model of Milman *et. al.*

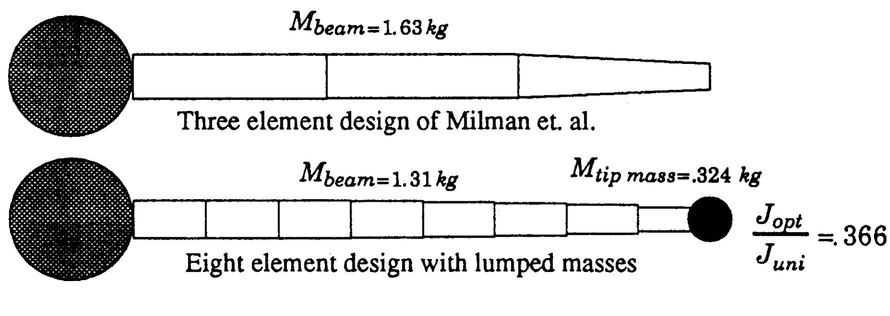

Figure 1.15: Optimal designs for a light beam ($\lambda small$).

case where it was held constant). The controller is full state feedback through a torque actuator located at the hub, and the disturbance is a transverse impulse at the tip of the beam. The design objective is to minimize the cost as in the previous example, except that the state penalty matrix is redefined to penalize tip displacement.

The optimal shapes for low and high values of λ were strikingly different in this problem (Figures 1.15 and 1.16). Milman et al. suggest that the buildup of material at the center of the beam for low values of λ is an attempt to stiffen it, and the buildup of mass at the end for high values of λ is an attempt to provide an inertial force to counteract the disturbance. Unfortunately, this model is too coarse to leave these as much mores than suspicions. This author repeated these optimizations with an eight element beam and also included lumped masses at each free node. The sizes of these masses were included as design variables. These designs are also shown in the figures.

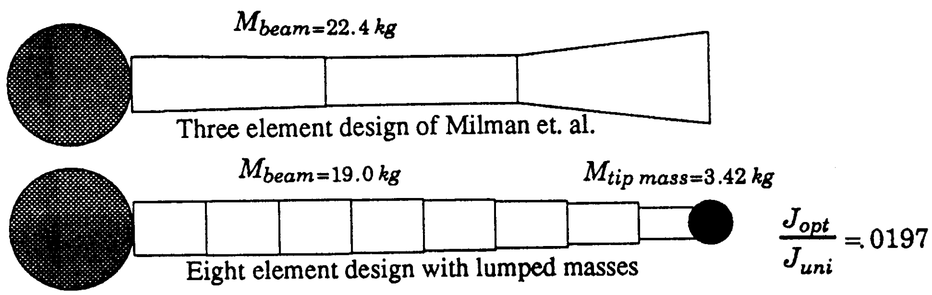

Figure 1.16: Optimal designs for a heavy beam ($\lambda large$).

Instead of being different, the designs for the larger models are similar. All but the tip lumped mass are set to zero, and the beam elements are clearly being adjusted to stiffen the first mode of the system. The difference between the shapes of the thickness distributions can be accounted for. For small values of λ, the optimal beam is very light compared to the inertia of the hub. The beam acts clamped at that end. This causes more strain at the root hence more material is placed there. As λ is increased, the hub inertia becomes less compared to the inertia of the beam. The hub end begins to look pinned instead of clamped. Material is moved from the root toward the center of the beam as the relative strain at the hub goes down.

The major difference between the two designs is the amount of effort spent diminishing the disturbance versus stiffening the system. This problem would be a good candidate for multilevel decomposition. The upper level design variable could be the size of the tip mass and the amount of mass available to stiffen the first mode. The lower level design problem would be to stiffen the first mode given the tip mass and the amount of material for stiffening.

1.3 Summary

This chapter reviewed various approaches to the controlled structure optimization problem. It was found that the problem formulations used most frequently could be divided into five separate parts: a structural model, a control model, disturbances, static metrics, and dynamic metrics. These were discussed separately. As a compromise between using a realistic problem formulation and keeping cost computations low, it was decided to limit further discussion to linear structures with optimal controllers and either initial condition or white noise disturbances. Similarly, the static metric was selected to be the mass of the system, and the dynamic metric was chosen to be the standard quadratic costs used in LQR/LQG formulations. A brief summary of the solution techniques used to solve these problems was then given.

It was noted that numeric optimization can be used only on very narrowly defined problems. The need for insight into what makes an optimal controlled structure was discussed. This insight is indispensable in the preliminary design of the controlled structure because it is this insight that allows the numerical problem to be properly defined. Currently the best source for this insight lies in the examples worked out in the literature. Several of the more illustrative of these were presented along with some interesting speculations about how a structure should be modified to improve controlled performance.

These examples however are too sparse to give more than glimpses into the controlled structure problem. The conclusions reached are very example–specific and cannot be generalized with confidence. The work in this thesis approaches this dilemma in two ways. First, in Chapter Three, typical sections are used to obtain fundamental relationships between the performance costs and modal parameters for a single mode system. Also, a two mode system is used to investigate the need for damping when unmodelled modes are present. Chapter Four uses a beam problem to expand this investigation. Solution of several optimization problems with different types of disturbances, weighting matrices, and boundary conditions will give confidence in generalizing the results from the typical sections to higher order systems. Use of the beam will also permit investigation of issues which are difficult to investigate on simple systems, such as the effect of several modes interacting with a controller, and the relative benefit of adding damping, mass, or stiffness to the structure.

Before continuing however, Chapter Two rigorously defines the controlled structure optimization problem. Algebraic formulae for the cost functionals and gradients are presented. Also, a special form of the gradient is developed which can give useful insight into how a change in the structure can physically affect the performance.

Chapter 2

Definition of Controlled Structure Problem, Cost, and Associated Gradients

Chapter Three will conclude with some rules of thumb for the design of controlled structures. These rules are based on analysis of some very rudimentary systems. It is hoped that the behavior of the typical sections will be sufficiently similar to the behavior of much more complicated systems to make these rules useful in preliminary design. To demonstrate their validity, it is necessary to optimize a more complex controlled structure. Comparing the results of optimization on this system with those predicted by the design rules will find flaws in the design rules and/or give confidence in their use. In particular, the typical sections are good for studying the temporal behavior, but one would also like to understand the spatial behavior of the problem.

Furthermore, study of how the control design and structural design interact on a more complex system is desireable for another reason. It makes it possible to study some of the tradeoffs involved in adjusting modal properties such as stiffness, controllability, or disturbability. In the typical section problems, the means by which modal parameters might be adjusted and the relative cost for their adjustment are ignored. A change in a physical structure will generally influence all of the modal parameters of all of the modes to some degree. A model based on this structure will show how some parameters must be improved at the expense of others when optimization is performed.

Chapter Four describes a two dimensional Bernoulli-Euler beam and optimizes it for several different problem formulations. This model represents a good tradeoff between simplicity (for making analysis more tractable) and complexity (for verifying design rules). Unfortunately, this system is too large to compute performance costs in closed form. This will be true of any system which has more than one or two modes, because computing the optimal control will require a solution of polynomial equations with order greater than five. In general, these equations do not have analytical solutions.

The only alternative is to perform the optimizations numerically. Numeric optimization requires a precisely defined set of design variables and costs. The goal of this chapter is to present the definitions for the controlled structures problems which will be dealt with in the remainder of this thesis. There are two major parts. The first gives detailed formulations for several different types of controlled structure problems. Costs are given for open loop response of a system, response of a perfectly modelled system with LQR/LQG control, and response of a system with LQR/LQG control and unmodelled dynamics. The next part of the chapter gives a method for computing the gradients of these costs . Also, a special form of the gradient is developed which is useful in understanding what is physically being done to a controlled structure through optimization.

2.1 Problem Formulation

Every optimization problem has three basic components; a design vector, a cost, and constraints. The design vector consists of the real valued structural and control parameters which can be varied to obtain an optimal design. In a well defined problem, these elements can be varied independently. As an example, in the beam problem to come in

Chapter Four, one might choose as design parameters the thicknesses of the beam at several points. The design vector α would then be:

$$\alpha = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} \tag{2.1}$$

where t_1 through t_n correspond to the thicknesses at n points on the beam.

The cost maps the design vector onto the real axis in the form of a scalar cost for a specified domain of allowable designs, D:

$$J(\cdot) : \alpha \rightarrow \Re, \qquad \forall \alpha \in D, J(\alpha) \geq 0 \tag{2.2}$$

This cost measures the "goodness" of a design. Better designs will yield lower values of the cost. Hence, the optimization problem consists of finding a design inside the domain of the design vector defined by the constraints which minimizes the cost.

The domain of the design vector is defined by the constraints. In general, these can be nonlinear, however, in this thesis, only the linear case is considered:

$$D = \{\alpha \in \Re^n : \quad A_\alpha^T \alpha - b_\alpha \geq 0\} \tag{2.3}$$

Each column of A_α and each element of b_α correspond to a single linear constraint. There are methods for handling linear constraints which cannot be used in the nonlinear case, which greatly simplify analysis. Linear constraints appear frequently in controlled structure problems. For example, it would be impossible for the beam thicknesses in the design vector described above to be negative. In that case, the constraint matrices would be:

$$A_\alpha = I_{n \times n} \qquad b_\alpha = 0_{n \times 1} \tag{2.4}$$

Where $I_{n \times n}$ is the identity matrix of order n and $0_{n \times 1}$ is a column of n zeros.

In a controlled structure problem, the cost is often expressed as a weighted sum of a structural cost, J_s, and a control cost, J_c [3]:

$$J(\alpha) = (1 - \lambda) J_s(\alpha) + \lambda J_c(\alpha) \quad 0 \leq \lambda \leq 1 \tag{2.5}$$

The control cost penalizes motion of the system and the control effort expended to minimize that motion. The structural cost penalizes the structural parameters and is usually related to the total mass of the structure. Designs found by optimizing $J(\alpha)$ with values of λ near zero will yield light structures with poor controlled performance, and optimized λ near unity will yield heavy structures with better controlled performance.

Define $\alpha^\star$ as the familiy of designs generated by optimizing the above cost for all values of the weighting parameter λ.

$$\alpha^\star = \{\alpha_0 : J(\alpha_0) = \min_{\alpha \in D} J(\alpha) \quad \forall \lambda : 0 \leq \lambda \leq 1\} \tag{2.6}$$

This design family represents all of the optimum tradeoffs between structural weight and performance. A typical design strategy would be to generate this family of designs, and then select the one which best met the design criteria. It can be shown that this same design family can be generated by constraining the structural cost and optimizing the performance cost [3]:

$$\alpha^\star = \{\alpha_0 : J_c(\alpha_0) = \inf_{\alpha \in D, J_s(\alpha) \leq J_s^\star} J_c(\alpha) \quad \forall J_s^\star : 0 \leq J_s^\star\} \tag{2.7}$$

or, constraining the performance cost and optimizing the structural cost:

$$\alpha^\star = \{\alpha_0 : J_s(\alpha_0) = \inf_{\alpha \in D, J_c(\alpha) \leq J_c^\star} J_s(\alpha) \quad \forall J_c^\star : 0 \leq J_c^\star\} \tag{2.8}$$

Hence, these three problem formulations are equivalent. However, a formulation where one prespecifies the maximum structural or performance cost has the advantage that both of these quantities have more physical significance than the weighting paramter λ. The disadvantage is that constraining the performance or structural cost can result in a non-linear constraint. Methods for treating non-linear constraints tend to be extremely expensive computationally [36], hence the unconstrained formulation is preferred when one is interested in the solution of the most general controlled structure problems.

If, however, the structural cost consists of only structural mass, and if the design parameters are selected carefully, then it is possible to state the structural cost as a weighted sum of the design parameters. In this case, a constraint on the structural cost would be linear, and would not cause any great computational burden. This will always be the case for the problems discussed in the next chapter, and the optimization problems considered will be assumed to have the structure of Equation 2.7.

All of the systems to be discussed here are linear time invariant and can be described by a state space equation:

$$\dot{x}(t) = Ax(t) + Bu(t) + v(t) \tag{2.9}$$

where $x(t)$ is a state vector for the system, $u(t)$ is the control vector and $v(t)$ is a disturbance vector. The performance cost for optimal control problems involving systems like this have two basic forms. The first assumes that there is no disturbance ($v(t) = 0$) and the system has some initial state, $x(0)$ which may be deterministic or stochastic:

$$J_1 = E\left[\int_0^\infty \left\{x^T(t)Qx(t) + u^T(t)Ru(t)\right\} dt\right] \qquad E\left[x(0)x^T(0)\right] = S \tag{2.10}$$

The matrices Q and R are constant (for a given design vector) and symmetric. They penalize the state of the system and the control effort respectively. The second form of the problem assumes that the disturbance is Gaussian White Noise:

$$J_2 = \lim_{t\to\infty} E\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] \qquad E\left[v(t)v^T(\tau)\right] = S\delta(t-\tau) \tag{2.11}$$

It is a property of linear, time-invariant systems that if the transfer function from the state to the control can also be expressed as a linear, time-invariant system, and if the matrices S in Equations 2.10 and 2.11 are numerically identical, then the costs J_1 and J_2 are equal [39]. For the remainder of this chapter, only the stochastic performance cost J_2 will be explicitly stated, and it will be implicit that the results obtained will also be valid for the cost J_1.

To summarize, the controlled structure optimization problem will be to find the design vector α_0 which minimizes the cost, $J_c(\alpha)$, subject to a set of linear constraints.

$$J_c(\alpha_0) = \inf_{\alpha\in D} J_c(\alpha), \qquad D = \left\{\alpha : A_\alpha^T \alpha - b_\alpha \geq 0\right\} \tag{2.12}$$

It is assumed that a constraint on the structural cost has been included in D, and the performance cost is defined in terms of the steady state response of the system to Gaussian White Noise. The rest of this section presents some specific controlled structure problems often encountered and their associated costs.

2.1.1 Open loop optimization

The simplest controlled structure optimization problem occurs when there is no controller present at all. The primary reason for this type of optimization is that the optimal cost represents an upper bound on the cost when a controller is included in the problem. The addition of control should not impair performance. If the problem is redone with a controller in place, then by comparing the resulting performance with the optimum open loop performance, the benefit of adding active control will be apparent. This is important for preliminary design when one needs to know if the benefit gained from the controller justifies its implementation.

If the system is linear, time-invariant, its equations of motion can be expressed by a state space equation:

$$\dot{x}(t) = A(\alpha)x(t) + v(t) \qquad E\left[v(t)v^T(\tau)\right] = S(\alpha)\delta(t-\tau) \tag{2.13}$$

and the open loop performance cost is:

$$J_c(\alpha) = \lim_{t\to\infty} E\left[x^T(t)Q(\alpha)x(t)\right] \tag{2.14}$$

This problem has a simple solution [39]. If the open loop system is stable, then the cost can be readily computed through the solution of a Lyapunov Equation.

$$J(\alpha) = \mathrm{tr}\{PS(\alpha)\} \qquad L = PA(\alpha) + A^T(\alpha)P + Q(\alpha) = 0 \tag{2.15}$$

where the subscript c on the control cost has been dropped.

Note that in this problem, the dynamic feedback matrix, the disturbance matrix, and the state penalty matrix are all functions of the design vector. This is the most general statement of the problem. Throughout this section, it wll be assumed that all system matrices and penalty matrices are functions of the design vector. Specific examples of how the design vector can influence these matrices are given in Chapter Four.

2.1.2 LQR optimization

If a system has an active controller, then its equation of motion can be written as:

$$\dot{x}(t) = A(\alpha)x(t) + B(\alpha)u(t) + v(t) \qquad E\left[v(t)v^T(\tau)\right] = S(\alpha)\delta(t-\tau) \tag{2.16}$$

where u is a control vector. The cost is augmented to include a penalty on the control effort.

$$J(\alpha) = \lim_{t\to\infty}\left[x^T(t)Q(\alpha)x(t) + u^T(t)R(\alpha)u(t)\right] \tag{2.17}$$

If the matrix $R(\alpha)$ is positive definite, and the matrix $Q(\alpha)$ is positive semi-definite, and all modes are controllable from the actuators and observeable from the sensors, then there exists a control law which will give the optimal performance for a given design vector. The solution is known as the Optimal Linear Quadratic Regulator (LQR). If P is the solution of the Ricatti Equation:

$$L = PA(\alpha) + A^T(\alpha)P + Q(\alpha) - PB(\alpha)R^{-1}(\alpha)B^T(\alpha)P = 0 \tag{2.18}$$

then the optimal control and corresponding optimal cost are:

$$u(t) = -R^{-1}(\alpha)B^T(\alpha)Px(t) \qquad J(\alpha) = \mathrm{tr}\{PS(\alpha)\} \tag{2.19}$$

Generally, it will be impossible to implement this type of control. The full or exact state is not usually available for computation of the control. Instead, knowledge of the state is obtained through an estimator driven by sensors which can be corrupted by noise. Furthermore, there will always be some disagreement between the model of the system and the actual system. Hence, the control law above will be sub-optimal and the actual performance cost may be higher.

This type of analysis is useful, however, because it does generally predict a lower bound on the performance cost of the system. For all other controllers, the performance cost will lie somewhere between the cost for this system and the cost for the open loop system. The remainder of this section gives some methods for computing the performance cost when the measurements and/or models are imperfect.

2.1.3 LQG Optimization

When the state vector is not available for computation of the control, then one must rely on sensors which measure a subset of the states. The system is described by the equation:

$$\begin{aligned} \dot{x}(t) &= A(\alpha)x(t) + B(\alpha)u(t) + v(t) \qquad & E\left[v(t)v^T(\tau)\right] &= S(\alpha)\delta(t-\tau) \\ y(t) &= C(\alpha)x(t) + w(t) & E\left[w(t)w^T(\tau)\right] &= W(\alpha)\delta(t-\tau) \end{aligned} \tag{2.20}$$

where the vector y represents the output of the sensors, and the vector w represents noise in the sensor output. If the performance cost is still given by Equation 2.17, and the sensor noise is modelled as Gaussian White Noise, then one has the Linear Quadratic Gaussian problem (LQG). The optimal control law can be expressed as the output of a linear system driven by the output of the sensors. If P_1 and P_2 are the solutions of the Ricatti Equations:

$$\begin{aligned} L_1 &= P_1A(\alpha) + A^T(\alpha)P_1 + Q(\alpha) - P_1B(\alpha)R^{-1}(\alpha)B^T(\alpha)P_1 = 0 \\ L_2 &= P_2A^T(\alpha) + A(\alpha)P_2 + S(\alpha) - P_2C^T(\alpha)W^{-1}(\alpha)C(\alpha)P_2 = 0 \end{aligned} \tag{2.21}$$

then the optimal controller is:

$$\begin{aligned} \dot{x}_c(t) &= (A - BR^{-1}B^TP_1 - P_2C^TW^{-1}C)x_c(t) + P_2C^TW^{-1}y(t) \\ u(t) &= -R^{-1}B^TP_1x_c(t) \end{aligned} \tag{2.22}$$

and the corresponding optimal cost is:

$$J(\alpha) = \operatorname{tr}\left[P_1S + P_2P_1BR^{-1}B^TP_1\right] \tag{2.23}$$

Note that the system and penalty matrices are no longer being expressed explicitly as functions of the design vector. This was done for clarity, as the equations in the following sections become somewhat more complicated.

2.1.4 Imperfectly modelled systems

In the previous cases, the equations of motion are assumed to capture the dynamics of the physical system perfectly. Unfortunately, this is rarely the case and the control system must be designed with stability and performance robustness in mind. Traditionally, the approach to this problem has been to use two models of the system — a design model and cost:

$$\begin{aligned} \dot{x}_d(t) &= A_d x_d(t) + B_d u(t) + v_d(t) & E\left[v_d(t)v_d^T(\tau)\right] &= S_d\delta(t-\tau) \\ y(t) &= C_d x(t)_d + w_d(t) & E\left[w_d(t)w_d^T(\tau)\right] &= W_d\delta(t-\tau) \end{aligned} \tag{2.24}$$

$$J_d(\alpha) = \lim_{t\to\infty}\left[x_d^T(t)Q_d x_d(t) + u^T(t)R_d u(t)\right] \tag{2.25}$$

and an evaluation model and cost:

$$\begin{aligned} \dot{x}_e(t) &= A_e x_e(t) + B_e u(t) + v_e(t) & E\left[v_e(t)v_e^T(\tau)\right] &= S_e\delta(t-\tau) \\ y(t) &= C_e x(t)_e + w_e(t) & E\left[w_e(t)w_e^T(\tau)\right] &= W_e\delta(t-\tau) \end{aligned} \tag{2.26}$$

$$J(\alpha) = \lim_{t\to\infty}\left[x_e^T(t)Q_e x_e(t) + u^T(t)R_e u(t)\right] \tag{2.27}$$

The control system is designed based on the design model, but the evaluation model represents the dynamics of the actual system better than the design model (but still not perfectly). The evaluation model can include parameter errors, or unmodelled dynamics. In the work done in Chapter Three, only the effect of unmodelled modes is considered. The definitions and cost presented here, however would not change if other modelling errors were used. If the performance and stability of the evaluation model with the controller from the design model is satisfactory, then the controller is considered robust enough to work on the actual system.

Because the controller for the design model will be suboptimal for the evalution model, the LQR and LQG controllers are of diminished importance. However, they do have the property that as the design model becomes more similar to the evaluation model, these controllers approach the optimum. In this thesis, these will be the only type of controllers considered, even though there are more robust controllers which might be used in practice.

If the evaluation model has noise free sensors ($w_e = 0$), the number of sensors is equal to the number of states in the design model, and all modes of the design model are observable, then the state of the design model can be reconstructed statically from the outputs and an LQR controller can be used. If P_1 is the solution of the Ricatti Equation:

$$L_1 = P_1 A_d + A_d^T P_1 + Q_d - P_1 B_d R_d^{-1} B_d^T P_1 = 0 \tag{2.28}$$

Then the control law based on the sensor output is:

$$u(t) = -R_d^{-1} B_d^T P_1 C_d^{-1} y(t) \tag{2.29}$$

Impinging this control on the evaluation model produces the closed loop system:

$$\dot{x}_e(t) = A_{\text{cl}} x_e(t) + v_e(t) \qquad A_{\text{cl}} \equiv A_e - B_e R_d^{-1} B_d^T P_1 C_d^{-1} C_e \tag{2.30}$$

The LQR and LQG solutions will produce controllers which are guaranteed to stabilize the design model. However, there are no such guarantees about stabilizing the evaluation model. Therefore, it is necessary to check if the eigenvalues for the above closed loop system all lie in the left half plane. Unless this is true, the system is unstable. This is an extremely undesireable feature, and the cost associated with such designs should be defined to be infinite. If the closed loop evaluation model is stable however, then the cost for the LQR problem can be found from the associated Lyapunov equation:

$$\begin{aligned} J(\alpha) &= \operatorname{tr}\{P_2 S\} \\ L_2 &= P_2 A_{\text{cl}} + A_{\text{cl}}^T P_2 + Q_{\text{cl}} = 0 \\ Q_{\text{cl}} &\equiv Q_e + C_e^T C_d^{-T} P_1 B_d R_d^{-1} R_e R_d^{-1} B_d^T P_1 C_d^{-1} C_e \end{aligned} \tag{2.31}$$

Similarly for the LQG problem, if P_1 and P_2 are the solutions of the equations:

$$
\begin{aligned}
L_1 &= P_1 A_d + A_d^T P_1 + Q_d - P_1 B_d R_d^{-1} B_d^T P_1 = 0 \\
L_2 &= P_2 A_d^T + A_d P_2 + S_d - P_2 C_d^T W_d^{-1} C_d P_2 = 0
\end{aligned} \tag{2.32}
$$

then the optimum controller based on the design model is given by:

$$
\begin{aligned}
\dot{x}_c(t) &= A_c x_c(t) + B_c y(t) \\
u(t) &= -C_c x_c(t)
\end{aligned} \tag{2.33}
$$

$$
B_c \equiv P_2 C_d^T W_d^{-1} \qquad C_c \equiv R_d^{-1} B_d^T P_1 \qquad A_c \equiv A_d - B_d C_c - B_c C_d
$$

The equation of motion for the closed loop system which results when this controller is used on the evaluation model is:

$$
\begin{bmatrix} \dot{x}_e(t) \\ \dot{x}_c(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A_e & -B_e C_c \\ B_c C_e & A_c \end{bmatrix}}_{A_{\text{cl}}} \begin{bmatrix} x_e(t) \\ x_c(t) \end{bmatrix} + \underbrace{\begin{bmatrix} v_e(t) \\ B_c w_e(t) \end{bmatrix}}_{v_{\text{cl}}} \tag{2.34}
$$

If the above system is stable, then the performance cost for the evaluation model is found from the solution of the following Lyapunov equation:

$$
\begin{aligned}
&L_3 = P_3 A_{\text{cl}} + A_{\text{cl}}^T P_3 + Q_{\text{cl}} = 0 \quad J(\alpha) = \text{tr}\{P_3 S_{\text{cl}}\} \\
&Q_{\text{cl}} \equiv \begin{bmatrix} Q_e & 0 \\ 0 & C_c^T R_e C_c \end{bmatrix} \qquad S_{\text{cl}} \equiv \begin{bmatrix} S_e & 0 \\ 0 & B_c W_e B_c^T \end{bmatrix}
\end{aligned} \tag{2.35}
$$

These are all of the costs which will be needed in Chapter Four. While there are an infinite number of other performance metrics one might use in evaluating a controlled structure, these costs are representative of what is most commonly used.

All of these costs have the same basic form. First, one must solve one or several Lyapunov and/or Ricatti equations, and then combine the solutions to find a performance cost. Efficient numerical algorithms already exist for solving these equations and they will not be discussed here. When it is necessary to solve these equations in Chapter Four, the MATLAB subroutines LQR and LYAP [40] will be used.

The performance costs have been defined in a manner such that they can be readily calculated for a given design vector. It is now possible to use a computer program which searches over the domain of allowable designs for an optimum design. The next section develops the equations necessary to find the gradients of these functions. This will facilitate numeric optimization.

2.2 Gradients

The search for the optimum of a multivariate function can be greatly aided by the availability of the gradient of the function. The decision to use the gradient in a numerical search is motivated by the complexity of the equations one must solve to compute it. All of the costs in the previous section have a gradient which can be computed with approximately the same effort needed to evaluate the cost itself. This gives a good savings over the effort required to compute the gradient numerically using a finite difference approach. This section uses Lagrange Multiplier methods to find expressions for these gradients. This technique for finding the gradients in these problems was used by Milman *et.al.* [3] for perfectly modelled systems with LQR/LQG controllers and is presented here. This work extends the technique to the imperfectly modelled cases.

2.2.1 Lagrange multipliers

All of the functions of the previous section could be expressed in the form:

$$J(\alpha) = F(\alpha, P_1, P_2, \ldots) \tag{2.36}$$

subject to the constraints:

$$L_1(\alpha, P_1, P_2, \ldots) = 0, \qquad L_2(\alpha, P_1, P_2, \ldots) = 0, \ldots \tag{2.37}$$

where all of the constraints are symmetric, matrix equations. It is possible to obtain equations which can be solved to find the gradient of J using the method of Lagrange

Multipliers. First, the functions and constraints are assembled into a single expression, $J^\star$, called the Lagrangian.

$$J^\star = F(\alpha, P_1, P_2, \dots) + \operatorname{tr}\{H_1 L_1(\alpha, P_1, P_2, \dots)\} + \operatorname{tr}\{H_2 L_2(\alpha, P_1, P_2, \dots)\} + \dots \tag{2.38}$$

The matrices H_i are symmetric and are called the Lagrange Multiplier Matrices.

It can be shown that the derivative of the cost J with respect to a parameter α_i is the same as the derivative of the Lagrangian when the matrices P_i and H_i are selected so as to make the derivative of the Lagrangian with respect to these matrices equal to zero.

$$\frac{\partial J}{\partial \alpha_i} = \left.\frac{\partial J^\star}{\partial \alpha_i}\right|_{\frac{\partial J^\star}{\partial P_1}=0,\frac{\partial J^\star}{\partial P_2}=0,\dots\frac{\partial J^\star}{\partial H_1}=0,\frac{\partial J^\star}{\partial H_2}=0,\dots} \tag{2.39}$$

Setting the derivative of the Lagrangian with respect to the matrices H_i to zero simply recovers the constraint equations. These are Lyapunov and Ricatti equations for which numeric solution techniques exist. Zeroing the derivative of the Lagrangian with respect to the matrices P_i produces another set of matrix equations. It turns out that for the problem formulations under consideration, these equations are Lyapunov equations and can be used to solve for the Lagrange Multiplier Matrices, H_i. Once the matrices P_i and H_i have been computed, it is then a simple excercise to compute the derivative of the performance cost with respect to the parameter α_i from Equation 2.39. The next section shows how this technique can be applied to the problem formulations defined above.

2.2.2 Gradients for specific problems

For the open loop optimization problem, there was only one constraint equation (Equation 2.15). Therefore, the Lagrangian becomes:

$$J^\star = \operatorname{tr}\{PS\} + \operatorname{tr}\left\{H(PA + A^T P + Q)\right\} \tag{2.40}$$

The matrix P has already been computed for the function evaluation. Therefore it is only necessary to compute the matrix H. This is doned by setting the derivative of this expression with respect to P to zero:

$$\frac{\partial J^\star}{\partial P} = HA^T + AH + S = 0 \tag{2.41}$$

This is a Lyapunov equation which can be solved to find H. The gradient of the function is then given by:

$$\frac{\partial J}{\partial \alpha_i} = \mathrm{tr}\left\{P\frac{\partial S}{\partial \alpha_i}\right\} + \mathrm{tr}\left\{H\left(P\frac{\partial A}{\partial \alpha_i} + \frac{\partial A^T}{\partial \alpha_i}P + \frac{\partial Q}{\partial \alpha_i}\right)\right\} \tag{2.42}$$

where P and H are are found from Equations 2.15 and 2.41. One of these scalar equations must be solved for each parameter in the design vector to obtain the entire gradient.

The LQR problem uses the same function as the open loop problem, but it uses a Ricatti equation constraint (Equation 2.18). This gives the Lagrangian a slightly different form. Substituting Equation 2.18 into Equation 2.38 gives:

$$J^\star = \mathrm{tr}\{PS\} + \mathrm{tr}\left\{HPA + HA^TP + HQ - HPBR^{-1}B^TP\right\} \tag{2.43}$$

Setting the derivative with respect to P to zero yields:

$$\begin{aligned}\frac{\partial J^\star}{\partial P} &= HA_{\mathrm{cl}}^T + A_{\mathrm{cl}}H + S = 0 \\ A_{\mathrm{cl}} &\equiv A - BR^{-1}B^TP\end{aligned} \tag{2.44}$$

This is another Lyapunov Equation in H. The gradient of the performance cost is:

$$\frac{\partial J}{\partial \alpha_i} = \mathrm{tr}\left\{P\frac{\partial S}{\partial \alpha_i}\right\} + \mathrm{tr}\left\{H\left(P\frac{\partial A}{\partial \alpha_i} + \frac{\partial A^T}{\partial \alpha_i} + \frac{\partial Q}{\partial \alpha_i} - P\frac{\partial}{\partial \alpha_i}\left(BR^{-1}B^T\right)P\right)\right\} \tag{2.45}$$

Equation 2.41 for the open loop case, and 2.44 for the closed loop case can be recognized as the Lyapunov equations which must be solved to compute the covariance of the state of the system. This makes the Lagrange Multiplier Matrices also the state covariance matrices in these problems.

For the LQG problem, there are two constraint equations (Equations 2.21). Combining these into the Lagrangian gives:

$$
\begin{aligned}
J^{\star} &= \operatorname{tr}\left\{P_1 S + P_2 P_1 B R^{-1} B^T P_1\right\} \\
&+ \operatorname{tr}\left\{H_1 P_1 A + H_1 A^T P_1 + H_1 Q - H_1 P_1 B R^{-1} B^T P_1\right\} \\
&+ \operatorname{tr}\left\{H_2 P_2 A^T + H_2 A P_2 + H_2 S - H_2 P_2 C^T W^{-1} C P_2\right\}
\end{aligned}
\tag{2.46}
$$

Taking the derivative of this expression with respect to P_1 and P_2:

$$
\begin{aligned}
\frac{\partial J^{\star}}{\partial P_1} &= H_1 A_{\text{reg}}^T + A_{\text{reg}} H_1 + S + P_2 P_1 B R^{-1} B^T + B R^{-1} B^T P_1 P_2 = 0 \\
\frac{\partial J^{\star}}{\partial P_2} &= H_2 A_{\text{est}}^T + A_{\text{est}} H_2 + P_1 B R^{-1} B^T P_1 = 0 \\
A_{\text{est}} &\equiv A - P_2 C^T W^{-1} C \qquad A_{\text{reg}} \equiv A - B R^{-1} B^T P_1
\end{aligned}
\tag{2.47}
$$

The matrices A_{reg} and A_{est} represent the dynamics of the closed-loop regulator and estimator respectively. The Lagrange Multiplier matrices can be seen to represent the closed loop covariance matrices for the regulator and estimator. These equations can be solved for H_1 and H_2 which are needed in computing the gradient:

$$
\begin{aligned}
\frac{\partial J}{\partial \alpha_i} &= \operatorname{tr}\left\{P_1 \frac{\partial S}{\partial \alpha_i}\right\} + \operatorname{tr}\left\{P_2 P_1 \frac{\partial}{\partial \alpha_i}(B R^{-1} B^T) P_1\right\} \\
&+ \operatorname{tr}\left\{H_1 \left(P_1 \frac{\partial A}{\partial \alpha_i} + \frac{\partial A^T}{\partial \alpha_i} P + \frac{\partial Q}{\partial \alpha_i} - P_1 \frac{\partial}{\partial \alpha_i}\left(B R^{-1} B^T\right) P_1\right)\right\} \\
&+ \operatorname{tr}\left\{H_2 \left(P_2 \frac{\partial A^T}{\partial \alpha_i} + \frac{\partial A}{\partial \alpha_i} P_2 + \frac{\partial S}{\partial \alpha_i} - P_2 \frac{\partial}{\partial \alpha_i}\left(C W^{-1} C^T\right) P_2\right)\right\}
\end{aligned}
\tag{2.48}
$$

For the poorly modelled LQR and LQG problems, things become somewhat more complicated. For the LQR problem, there are two constraint equations — one Lyapunov, and one Ricatti. Assembling these and the cost into a Lagrangian gives:

$$
\begin{aligned}
J^{\star} &= \operatorname{tr}\{S P_2\} \\
&+ \operatorname{tr}\left\{H_1 P_1 A_d + H_1 A_d^T P_1 + H_1 Q_d - H_1 P_1 B_d R_d^{-1} B_d^T P_1\right\} \\
&+ \operatorname{tr}\left\{H_2 P_2 A_{\text{cl}} + H_2 A_{\text{cl}}^T P_2 + H_2 Q_{\text{cl}}\right\}
\end{aligned}
\tag{2.49}
$$

Taking derivatives with respect to P_2 and P_1 gives Lyapunov equations for H_2 and H_1.

$$\begin{aligned}
\frac{\partial J^\star}{\partial P_2} &= H_2 A_{\text{cl}}^T + A_{\text{cl}} H_2 + S = 0 \\
\frac{\partial J^\star}{\partial P_1} &= H_1 A_{\text{reg}}^T + A_{\text{reg}} H_1 \\
&- (B_d R_d^{-1} B_e^T P_2 H_2 C_e^T C_d^{-T} + C_d^{-1} C_e H_2 P_2 B_e R_d^{-1} B_d^T) \\
&+ C_d^{-1} C_e H_2 C_e^T C_d^{-T} P_1 B_d R_d^{-1} R_e R_d^{-1} B_d^T \\
&+ B_d R_d^{-1} R_e R_d^{-1} B_d^T P_1 C_d^{-1} C_e H_2 C_e^T C_d^{-T} = 0 \qquad (2.50) \\
A_{\text{reg}} &\equiv A_d - B_d R_d^{-1} B_d^T P_1
\end{aligned}$$

Notice that the Lyapunov Equation for H_1 is a function of H_2, hence these two equations are coupled. However, H_1 does not appear in the equation for H_2, and it is possible to solve for H_2 and then H_1 using standard techniques. Once both have been computed, the gradient can be found.

$$\begin{aligned}
\frac{\partial J}{\partial \alpha_i} &= \operatorname{tr}\left\{\frac{\partial S}{\partial \alpha_i} P_2\right\} \\
&+ \operatorname{tr}\left\{H_1\left(P_1 \frac{\partial A_d}{\partial \alpha_i} + \frac{\partial A_d^T}{\partial \alpha_i} P_1 + \frac{\partial Q_d}{\partial \alpha_i} - P_1 \frac{\partial}{\partial \alpha_i}\left(B_d R^{-1} B_d^T\right) P_1\right)\right\} \\
&+ \operatorname{tr}\left\{H_2\left(P_2 \frac{\partial A_{\text{cl}}}{\partial \alpha_i} + \frac{\partial A_{\text{cl}}^T}{\partial \alpha_i} P_2 + \frac{\partial Q_{\text{cl}}}{\partial \alpha_i}\right)\right\} \\
\frac{\partial A_{\text{cl}}}{\partial \alpha_i} &\equiv \frac{\partial A_e}{\partial \alpha_i} - \frac{\partial}{\partial \alpha_i}\left(B_e R_d^{-1} B_d^T\right) P_1 C_d^{-1} C_e - B_e R^{-1} B_d^T P_1 \frac{\partial}{\partial \alpha_i}\left(C_d^{-1} C_e\right) \\
\frac{\partial Q_{\text{cl}}}{\partial \alpha_i} &\equiv \frac{\partial Q_e}{\partial \alpha_i} + \frac{\partial}{\partial \alpha_i}\left(C_e^T C_d^{-T}\right) P_1 B_d R_d^{-1} R_e R_d^{-1} B_d^T P_1 C_d^{-1} C_e \\
&+ C_e^T C_d^{-T} P_1 \frac{\partial}{\partial \alpha_i}\left(B_d R_d^{-1} R_e R_d^{-1} B_d^T\right) P_1 C_d^{-1} C_e \\
&+ C_e^T C_d^{-T} P_1 R_d^{-1} R_e R_d^{-1} B_d^T P_1 \frac{\partial}{\partial \alpha_i}\left(C_d^{-1} C_e\right) \qquad (2.51)
\end{aligned}$$

For the LQG case, the Lagrangian involves three constraints (Equations 2.32 and 2.35).

$$\begin{aligned}
J^\star &= \operatorname{tr}\{P_3 S_{\text{cl}}\} \\
&+ \operatorname{tr}\left\{H_1 P_1 A_d + H_1 A_d^T P_1 + H_1 Q_d - H_1 P_1 B_d R_d^{-1} B_d^T P_1\right\} \\
&+ \operatorname{tr}\left\{H_2 P_2 A_d^T + H_2 A_d P_2 + H_2 S_d - H_2 P_2 C_d^T W_d^{-1} C_d P_2\right\} \\
&+ \operatorname{tr}\left\{H_3 P_3 A_{\text{cl}} + H_3 A_{\text{cl}}^T P_3 + H_3 Q_{\text{cl}}\right\} \qquad (2.52)
\end{aligned}$$

The matrix H_3 can be obtained by solving the Lyapunov equation that results from zeroing the derivative with respect to P_3.

$$\frac{\partial J^*}{\partial P_3} = H_3 A_{\rm cl}^T + A_{\rm cl} H_3 + S_{\rm cl} = 0 \tag{2.53}$$

For the equations to compute H_1 and H_2, it is useful to subdivide the matrices H_3 and P_3.

$$H_3 = \begin{bmatrix} H_{31} & H_{32} \\ H_{32}^T & H_{33} \end{bmatrix} \qquad P_3 = \begin{bmatrix} P_{31} & P_{32} \\ P_{32}^T & P_{33} \end{bmatrix} \tag{2.54}$$

where H_{31} and P_{31} are square matrices of the same order as A_e, and H_{33} and P_{33} are square matrices of the same order as A_d. Plugging these expressions into Equation 2.52 and taking the derivatives with respect to P_1 and P_2 gives Lyapunov Equations which can be solved for H_1 and H_2.

$$\begin{aligned} \frac{\partial J^*}{\partial P_1} &= H_1 A_{\rm reg}^T + A_{\rm reg} H_1 + H_{33} P_1 B_d R_d^{-1} R_e R_d^{-1} B_d^T + B_d R_d^{-1} R_e R_d^{-1} B_d^T P_1 H_{33} \\ &- \begin{bmatrix} H_{32}^T & H_{33} \end{bmatrix} P_3 \begin{bmatrix} B_e B_d^T \\ B_d B_d^T \end{bmatrix} - \begin{bmatrix} B_d B_e^T & B_d B_d^T \end{bmatrix} P_3 \begin{bmatrix} H_{32} \\ H_{33} \end{bmatrix} = 0 \qquad (2.55) \\ \frac{\partial J^*}{\partial P_2} &= H_2 A_{\rm est} + A_{\rm est}^T H_2 + P_{33} P_2 C_d^T W_d^{-1} W_e W_d^{-1} C_d P_2 + P_2 C_d^T W_d^{-1} W_e W_d^{-1} C_d P_2 P_{33} \\ &+ \begin{bmatrix} C_d^T C_e & -C_d^T C_d \end{bmatrix} P_3 \begin{bmatrix} P_{32} \\ P_{33} \end{bmatrix} + \begin{bmatrix} P_{32}^T & P_{33} \end{bmatrix} P_3 \begin{bmatrix} C_e^T C_d \\ -C_d^T C_d \end{bmatrix} = 0 \qquad (2.56) \\ A_{\rm reg} &\equiv A_d - B_d R_d^{-1} B_d^T P_1 \qquad A_{\rm est} \equiv A_d - P_2 C_d^T W_d^{-1} C_d \end{aligned}$$

It is now possible to compute the gradient of the performance cost.

$$\begin{aligned} \frac{\partial J}{\partial \alpha_i} &= \operatorname{tr}\left\{P_3 \frac{\partial S_{\rm cl}}{\partial \alpha_i}\right\} + \operatorname{tr}\left\{H_3 \left(P_3 \frac{\partial A_{\rm cl}}{\partial \alpha_i} + \frac{\partial A_{\rm cl}^T}{\partial \alpha_i} P_3 + \frac{\partial Q_{\rm cl}}{\partial \alpha_i}\right)\right\} \\ &+ \operatorname{tr}\left\{H_1 \left(P_1 \frac{\partial A_d}{\partial \alpha_i} + \frac{\partial A_d^T}{\partial \alpha_i} P_1 + \frac{\partial Q_d}{\partial \alpha_i} + P_1 \frac{\partial}{\partial \alpha_i}\left(B_d R_d^{-1} B_d^T\right) P_1\right)\right\} \\ &+ \operatorname{tr}\left\{H_2 \left(P_2 \frac{\partial A_d^T}{\partial \alpha_i} + \frac{\partial A_d}{\partial \alpha_i} P_2 + \frac{\partial S_d}{\partial \alpha_i} - P_2 \frac{\partial}{\partial \alpha_i}\left(C_d^T W_d^{-1} C_d\right) P_2\right)\right\} \qquad (2.57) \\ \frac{\partial A_{\rm cl}}{\partial \alpha_i} &\equiv \begin{bmatrix} \frac{\partial}{\partial \alpha_i} A_e & -\frac{\partial}{\partial \alpha_i}(B_e R_d^{-1} B_d^T) P_1 \\ P_2 \frac{\partial}{\partial \alpha_i}(C_d^T W_d^{-1} C_e) & \frac{\partial}{\partial \alpha_i} A_d - \frac{\partial}{\partial \alpha_i}(B_d R_d^{-1} B_d^T) P_1 - P_2 \frac{\partial}{\partial \alpha_i}(C_d^T W_d^{-1} C_d) \end{bmatrix} \end{aligned}$$

$$\frac{\partial S_{\rm cl}}{\partial \alpha_i} \equiv \begin{bmatrix} \frac{\partial}{\partial \alpha_i} S_e & 0 \\ 0 & P_2 \frac{\partial}{\partial \alpha_i}(C_d^T W_d^{-1} W_e W_d^{-1} C_d) P_2 \end{bmatrix}$$

$$\frac{\partial Q_{\rm cl}}{\partial \alpha_i} \equiv \begin{bmatrix} \frac{\partial}{\partial \alpha_i} Q_e & 0 \\ 0 & P_1 \frac{\partial}{\partial \alpha_i}(B_d R_d^{-1} R_e R_d^{-1} B_d^T) P_1 \end{bmatrix}$$

An interesting feature of the gradients for these problems is the amount of computational effort involved in their calculation. In each case, one has to solve one additional Lyapunov equation for each Ricatti or Lyapunov equation required to evalute the cost. Then, a scalar derivative equation must be solved for each design parameter to obtain the derivative of the cost with respect to that parameter. In most cases, the bulk of the computational effort is involved in solving the Lyapunov and Ricatti Equations, hence it takes approximately twice as much effort to compute the cost and gradient as it does to compute the cost alone. This computational efficiency favors using minimization algorithms which make direct use of the gradient. To obtain the same information contained in the gradient, one would have to evaluate the function once for every design variable.

The gradient equations thus far have been treated as a necessity for numerical optimization, and in there present form, they do not convey very much insight into the problem. However, it should be noted that the gradient equation for the perfectly modelled LQR problem has a tantalizing form. Basically, this gradient has four components:

$$\begin{aligned} \text{Component 1 :} \quad J_{\alpha,1} &= \operatorname{tr}\left\{H\left(P\frac{\partial A}{\partial \alpha} + \frac{\partial A^T}{\partial \alpha}P\right)\right\} \\ \text{Component 2 :} \quad J_{\alpha,2} &= \operatorname{tr}\left\{P\frac{\partial S}{\partial \alpha}\right\} \\ \text{Compenent 3 :} \quad J_{\alpha,3} &= \operatorname{tr}\left\{H\frac{\partial Q}{\partial \alpha}\right\} \\ \text{Compenent 4} \quad J_{\alpha,4} &= -\operatorname{tr}\left\{HP\frac{\partial}{\partial \alpha}\left(BR^{-1}B^T\right)P\right\} \end{aligned} \tag{2.58}$$

In Chapter Four, knowledge gained from the typical sections will be used to perform a coordinate transformation which will make these quantities agree exactly with four of the five features of the system one might change to improve performance; open loop dyanmics, disturbability, observability, and controllability; respectively.

This completes the toolbox of mathematics and theory which is necessary for optimization. The only thing required to successfully optimize the problems in the next chapters numerically is an algorithm. In this work, numeric optimizations were performed using the CONSTR subroutine in the MATLAB optimization toolbox (Reference [41]). This is a program which performs constrained optimization using a Sequential Quadratic Programming (SQP) method and the BFGS Hessian updating formula.

Chapter 3

Typical Sections

Controlled structure optimization is plagued by a high degree of complexity. A linearly elastic structure can be modelled as a large number of independent modes, the behavior of which are fairly easy to analyze. Unfortunately, the addition of control to this system couples these modes (especially at high control levels) and obscures the relationship between changes in the structure and changes in close-loop performance. A small change in a structure changes the open loop dynamics and the manner in which the controller and disturbances influence the system. The tradeoffs which must be made among these in designing a good structure can be very subtle.

The current approach to this problem has been a brute force attack where all of the design degrees of freedom are given to a computer program which employs a numerical algorithm to search for an optimal design. This method yields correct results, but it suffers from two problems. The computer program cannot tell one why a design is optimal, and the results are often difficult to interpret. Secondly, although the design is optimal for the defined problem, it does not suggest changes in the structure outside of the defined design space which might go a long way toward improving performance.

One solution to this problem is to use controlled structure typical sections. The phrase "typical section" was coined in aeroelasticity. The aeroelastic behavior of a three dimensional aircraft wing is extremely complex and also defies attempts at detailed analysis. A typical section refers to a two dimensinal model of a section of a wing whose behavior is typical of the wing as a whole. Detailed analysis of this section is much easier, yet it can yield results which give insight into the behavior of the entire wing.

This thesis defines the typical section for a controlled structure to be a simple one or two mode spring-mass-dashpot system and its associated controller. These very simple systems capture many of the important features of any controlled structure problem. This simplicity, however, does come at a cost. The simplicity of the aeroelastic typical section was attained by ignoring three-dimensional effects. The simplicity of the single mode controlled structure typical section is attained by ignoring the effects of coupling between modes. In cases were this coupling is of secondary importance, this typical section will be a good predictors of which types of designs will be optimal in more complex systems. The two mode typical section only starts to address the modal coupling case.

The next section defines four representative types of controlled structure problems. The different types of problems are distinguished by their disturbance and their control objective as reflected by the state penalty used in the quadratic cost functional for the problem. This will organize the discussion of results for typical sections analyzed in subsequent sections of this chapter.

The passive elements of a single mass typical section are manipulated to optimize open loop performance and optimal LQR performance. It is possible in these two cases to obtain a closed form expression for the performance cost of this system. The functional dependence of these expressions on the passive parameters will then be explored for each of the four types of controlled structure problems. In the following section, a two mass typical section is presented. This model will be used to explore the effects of attempting to control a system using a controller based on a reduce order design model.

Throughout this chapter, general conclusions will be reached concerning the typical sections. These conclusions will be stated succinctly as "design rules of thumb." At the the end of the chapter, these design rules will be discussed, with particular emphasis on how they might apply to more realistic systems and how they correspond to the results in Chapter one.

3.1 Classification of controlled structure problems

In the previous chapter, there were three matrices of special importance associated with the LQR controlled structure. These were the state penalty matrix, Q, the disturbance matrix, S, and the control penalty matrix, R. These three matrices, along with the matrices describing the open-loop dynamics and controllability of the system (A and B), uniquely determine the control law and controlled performance cost of the system. It would be prohibitive to consider every possible form these matrices might take for even a simple system. Fortunately, there are only a few basic forms of penalty matrices and disturbances commonly used with flexible structures. The most frequently used and realistic of these will be used to establish a small set of controlled structure problems. The discussion of the typical sections in the later sections will be mercifully abbreviated by limiting it to these cases.

Before continuing with this discussion, it is convenient to restate the equations of motion in several forms along with some important equations and relationships. First, all structures will be assumed to have the following matrix equation of motion:

$$M\ddot{r} + C\dot{r} + Kr = Fu \tag{3.1}$$

where r is always a vector of physical displacements in the system. Often, this equation will be used in its state space form:

$$\underbrace{\begin{bmatrix} \dot{r} \\ \ddot{r} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} r \\ \dot{r} \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix}}_{B} u \tag{3.2}$$

There are two formulations of the LQR problem. The first is known as the deterministic problem. For this formulation, it is assumed that with the exception of the forces which produce initial displacements and velocities in the system, there are no other disturbances. The optimal control minimizes the cost functional:

$$J = E\left[\int_0^\infty \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right)\right] \tag{3.3}$$

In this formulation, the covariance of the initial state is required to compute the performance cost. This is captured in the disturbance matrix, S:

$$S = E\left[x(0)x^T(0)\right] \tag{3.4}$$

Note that the use of the word deterministic in this problem refers to the fact that once the initial state is specified, the response of the system for all succeeding time is deterministic. In cases where the initial state is also deterministic, the expectations ($E[]$)in the above equations can be dropped.

The second LQR problem formulation is known as the stochastic problem and assumes that all of the disturbances can be assembled into a single vector of zero-mean, Gaussian white-noise. The optimal control is then selected to minimize the cost functional:

$$J = \lim_{t\to\infty} E\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] \tag{3.5}$$

The disturbance matrix, S, now represents the covariance of the white noise process.

$$S\delta(t-\tau) = E\left[\nu(t)\nu^T(\tau)\right] \tag{3.6}$$

where the equation of motion has been augmented to include the disturbance vector, ν.

$$\dot{x} = Ax + Bu + \nu \tag{3.7}$$

Occasionally, it will be useful to express some relationships in modal form. Modal coordinates, q are mapped onto the physical coordinates, r, by the mass normalized modal transformation matrix, T, which diagonalizes both the mass and stiffness matrix of the system:

$$r = Tq \qquad T^TMT = I \qquad T^TKT = \begin{bmatrix} \omega_1^2 & & \\ & \omega_2^2 & \\ & & \ddots \end{bmatrix} = \Lambda \tag{3.8}$$

Matrices which have been transformed to the modal form will be designated by a subscript q. Thus, M_q would simply be the identity matrix, I. With these expressions in mind, it is now possible to proceed with categorizing the problem formulations.

3.1.1 State penalties

The state penalty matrix Q is required by LQR theory to be symmetric and positive semidefinite. If a structure has n states, then one has $n(n+1)/2$ degrees of freedom available in selecting the state penalty matrix. In practice, these degrees of freedom are never used. Instead, various methods are used to reduce the problem of choosing many values in a matrix to the selection of a handful of parameters. One keeps adjusting these parameters and evaluating the performance of the resulting closed loop system until a controller is obtained which meets the design specifications. This iteration is necessary for design specifications which cannot be expressed quadratically (e.g. maximum bounds on displacements). Ideally, one would like the number of free parameters to be small to simplify the iteration process.

One way of reducing the specification of the state penalty matrix down to two parameters is to penalize a weighted sum of the potential and kinetic energy of the system.

$$Q = \begin{bmatrix} \gamma_K K & 0 \\ 0 & \gamma_M M \end{bmatrix} \tag{3.9}$$

For the purposes of this thesis, this will be called the energy penalty method. The parameter γ_K weights the potential energy in the system and is adjusted to produce a design with a sufficiently rapid response. The parameter γ_M weights the kinetic energy in the system and is adjusted to ensure that the system has acceptable levels of active damping.

This type of penalty is attractive for several reasons. First, there is good physical understanding of what this matrix is penalizing: energy. Second, this matrix has an interesting form when converted into modal form:

$$Q_q = \begin{bmatrix} \gamma_K T^T K T & 0 \\ 0 & \gamma_M T^T M T \end{bmatrix} = \begin{bmatrix} \gamma_K \Lambda & 0 \\ 0 & \gamma_M I \end{bmatrix} \tag{3.10}$$

There is no coupling of the open loop modes through the performance output. This will tend to make the typical section results more reliable when generalized to larger systems. Also, when the number of modes is equal to the number of actuators, all modes are indendently controllable, and the control penalty matrix is selected to penalize static work done by the controller:

$$R = B^T K^{-1} B \tag{3.11}$$

then it is possible to solve for the optimal control law and cost in closed form regardless of the size of the system [4].

There are several drawbacks to this type of penalty. First, motion everywhere on the structure is penalized. This includes portions of the structure whose motion is irrelevant to the design objectives. A second problem is that the penalty on a mode of the structure is proportional to the square of its natural frequency. This means that higher frequency modes will be penalized to a greater degree. This is fine if the actual performance objective is to minimize energy. However, if one really is seeking to reduce amplitude, then systems optimized with this performance metric will have unneeded amounts of control in the higher frequency modes.

Both of these problems can be fixed by using a displacement penalty method. Typically, one is only concerned with the motion of a select number of points. The vector which describes the motion of these points will be called the output vector, e. The physical coordinates are mapped to the output vector by the matrix, N.

$$e = Nr \tag{3.12}$$

The state penalty matrix is arranged to penalize displacements in the output vector:

$$e^T Q_e e = x^T Q x \Rightarrow Q = \begin{bmatrix} N^T Q_e N & 0 \\ 0 & 0 \end{bmatrix} \quad x = \begin{bmatrix} r \\ \dot{r} \end{bmatrix} \tag{3.13}$$

The parameters to be adjusted are contained in the output weighting matrix, Q_e. Usually the output weighting matrix is assumed to be diagonal. One advantage of this type of penalty is that if one of the outputs is too large, then one only needs to increase the corresponding diagonal element of Q_e on the next iteration. Also, this penalty matrix is free of the arbitrary frequency weighting introduced by using the energy penalty matrix. However, this penalty matrix will not in general have the diagonalizing property of the energy penalty matrix.

These two types of state penalties, energy and displacement, will be the representative control objectives used in the four typical section problem classifications.

3.1.2 Disturbances

Modelling the disturbances correctly is not very important in optimal LQR theory. This is because the optimal control law for a given structure is independent of where and how the disturbances enter the structure. The optimal structure, on the other hand, does depend on the nature of the disturbances. Hence greater care must be taken to ensure that these disturbances are modelled realistically. Almost all disturbances in disturbances can usually be shown to originate from external forces or torques. The approach here will be to ultimately specify the disturbance environment for each case as a function of the covariance of some vector of disturbance forces.

For the deterministic LQR problem, the disturbance is an initial state of the system. There are two cases to consider — initial displacement disturbances and initial velocity disturbances.

For the initial displacement disturbance, it is tempting to arbitrarily specify the covariance of the initial displacement, and leave it independent of the design of the structure. However, a more realistic approach is to assume the initial displacement comes about due to some initial static load on the system, v_0, whose covariance is specified [6]:

$$r(0) = K^{-1}Gv_0 \qquad E\left[v_0 v_0^T\right] = V \tag{3.14}$$

The matrix G maps the disturbance forces onto the physical coordinates in the same way that the matrix F maps the control forces as in Equation 3.1. The dependence of the displacement on the stiffness, K captures the effect that stiffer systems should be inherently less sensitive to disturbances. The disturbance matrix for this case is simply:

$$S = \begin{bmatrix} E\left[r(0)r^T(0)\right] & E\left[r(0)\dot{r}^T(0)\right] \\ E\left[\dot{r}(0)r^T(0)\right] & E\left[\dot{r}(0)\dot{r}^T(0)\right] \end{bmatrix} = \begin{bmatrix} K^{-1}GVG^TK^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.15}$$

For similar reasons, an initial velocity disturbance should be specified as the velocity imparted to the system by an impulse vector v_0 with covariance V.

$$\dot{r}(0) = M^{-1}Gv \qquad S = \begin{bmatrix} 0 & 0 \\ 0 & M^{-1}GVG^TM^{-1} \end{bmatrix} \tag{3.16}$$

Moving on to the stochastic LQR problem, one assumes that the white noise disturbance is a vector of forces acting on the system through the matrix, G.

$$M\ddot{r} + C\dot{r} + Kr = Fu + Gv \qquad E\left[v(t)v^T(t)\right] = V\delta(t-\tau) \tag{3.17}$$

In state space form, this equation becomes:

$$\begin{bmatrix} \dot{r} \\ \ddot{r} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{bmatrix} r \\ \dot{r} \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix} u + \begin{bmatrix} 0 \\ M^{-1}G \end{bmatrix} v \tag{3.18}$$

The disturbance matrix can again be specified in terms of the covariance of the disturbance.

$$S\delta(t-\tau) = E\left[\nu(t)\nu^T(\tau)\right] = \begin{bmatrix} 0 & 0 \\ 0 & M^{-1}GVG^TM^{-1} \end{bmatrix} \delta(t-\tau) \tag{3.19}$$

These are all of the types of disturbances which will be considered in the ensuing sections. However, notice that the disturbance matrices for the impulse and the white noise disturbances are identical. This allows a simplification to be made. Instead of three types of disturbances, there are really only two. Define a *displacement disturbance* to be the initial displacement due to a static load, and define a *velocity disturbance* to mean either the impulsive disturbance *or* the white noise disturbance. For the rest of this work, disturbances will be described by these terms.

3.1.3 Control penalty matrices

All of the typical sections presented will have only one actuator. Hence, the control penalty matrix will simply be a scalar. In systems with more actuators, one would attempt to find ways of reducing the problem of selecting a control penalty matrix in a manner similar to that used on the state penalty matrix. For the purpose of the typical sections, this is not necessary, and the control penalty in Equation 3.5 will be left as a simple scalar:

$$R = \rho^2 \tag{3.20}$$

The preceeding discussion is not meant to be an exhaustive description of all possible penalty matrices and disturbance environments. Rather, it is intended to present a representative sampling of how these matrices are used.

In the previous discussion, two types of state penalty, two types of disturbances, and one type of control penalty were presented. This leaves four possible permutations which should be considered. These permutations will be known as the "type" of the controlled structure problem. The four types are arbitrarily defined as:

- Type I: Energy penalty with velocity disturbance
- Type II: Displacement penalty with velocity disturbance
- Type III: Energy penalty with displacement disturbance
- Type IV: Displacement penalty with displacement disturbance

It is helpful to visualize an example for each of these types of problem. A case where the Type I problem might come up is in a flexible space structure where the objective of the control is just vibration suppression and the disturbances are modelled stochastically. If the goal of the control is to reduce the motion of specific points on the structure, then the displacement penalty should be used and one would have the Type II problem. One might need this type of control in interferometry where it is important to control the mounting points for the optics. One is likely to use a displacement disturbance in a slew maneuver where the goal is to translate the system from some initial state to a desired final state. If many points need to be controlled, such as the surface of a slewing antenna, then the energy penalty might be used and one would have a Type III problem. If, on the other hand one is only interested in one or two points, (such as the end effector on a robotic arm), then a displacment penalty should be used and the problem would be Type IV.

The next sections will use these problem types to guide their discussion of the typical sections.

3.2 Single Mass Typical Section

As mentioned before, any linear structure can be modelled as a collection of independent modes. The goal of this first typical section is to study how the controller and a single mode of a structure interact, and also how changes in the structure can improve controlled and uncontrolled performance.

Consider the equation of motion for the i^{th} mode of a structure which has a single actuator. This equation can be written:

$$\ddot{q}_i + 2\zeta_i\dot{q}_i + \omega_i^2 q_i = f_i u \tag{3.21}$$

where ω_i is the natural frequency of the mode, ζ_i is the modal damping, and f_i corre-

Table 3.1: Analogy between typical section paramters and methods for improving controlled performance

Modal parameter	Mechanism
f	Controllability
g	Disturbability
n	Observability
ζ, ω	Open Loop Dynamics

sponds to the degree to which the actuator influences the i^{th} mode.

$$f_i = T_i^T F \tag{3.22}$$

Let the vector T_i be the i^{th} column of the mass normalized modal transformation matrix T. The influence of the disturbance on the mode, and the influence of the mode on the output, e (if a displacement penalty is used) are expressed by:

$$g_i = T_i^T G \qquad n_i = T_i N \tag{3.23}$$

Notice that all but one of the mechanisms to improve controlled performance mentioned in chapter one (increase controllability, decrease disturbability, decrease observability, change open loop dyanamics) now correspond roughly to scalar quantities. (Table 3.1)

The above equations of motion and parameters describe a system consisting of a single mass, spring, dashpot, and some assorted influence parameters. Figure 3.1 shows just such a system. For clarity, the subscripts denoting the i^{th} mode have been dropped. This system has the advantage that there is good physical understanding of all of the parameters.

This simple system will be examined under two separate types of control architectures — open loop, and optimal LQR. The desire is to examine how the sensitivity of the cost to different modal parameters changes as the controller changes. For this analysis, it is

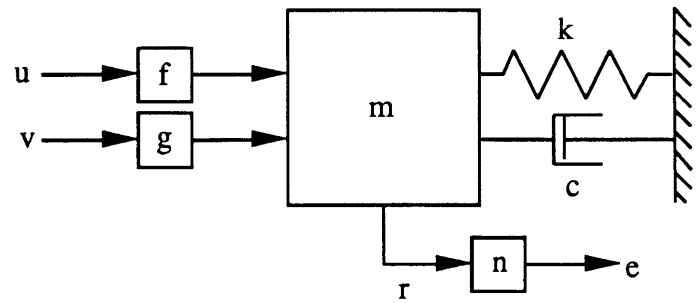

Figure 3.1: Single mass typical section: $m = 1$, $k = \omega^2$, $c = 2\zeta\omega$

useful to recast the equations of motion in modal form.

$$\underbrace{\begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} q \\ \dot{q} \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ f \end{bmatrix}}_{B} u \qquad e = \underbrace{\begin{bmatrix} n & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} q \\ \dot{q} \end{bmatrix}}_{x} \tag{3.24}$$

3.2.1 Open Loop Performance

The simplest approach one might use in obtaining a desired dynamic response from a system is to design the system in such a way that it meets the design requirements without the addition of active control. This approach has the advantages that such a design is very robust (no chance of destabilization) and reliable (no control system to fail). The reasonsthat active control is used at all is that, if done correctly, it allows one to obtain performance that would be impossible to realize passively..

The analysis of the single mass typical section starts with the computation of its open loop performance. This is done for three reasons. First, it establishes the worst performance that the controlled system can have. It is assumed that the addition of control always improves performance. Second, it establishes an initial performance cost which can be compared to the performance cost for the controlled system. This will give a measure of the benefit attained by using the controller. Finally, comparison of this case with the controlled case in the next section will allow one to distinguish how the designs for the controlled and uncontrolled sturctures should differ.

The performance costs for all four of the problem types defined above can be computed simultaneously. The state penalty matrix and disturbance matrix are arbitrarily defined as:

$$Q = \begin{bmatrix} q_d & \\ & q_v \end{bmatrix} \qquad S = \begin{bmatrix} s_d & \\ & s_v \end{bmatrix} \tag{3.25}$$

The open loop performance cost can be computed in terms of the scalar parameters q_d, q_v, s_d, and s_v. Later, appropriate expressions can be substituted for these parameters to obtain the performance cost for the four types of problems.

The goal of optimization is to minimize the cost functional:

$$\begin{aligned} J &= E\left[\int_0^\infty \left(x^T Q x\right) dt\right] \\ \text{or} \quad J &= \lim_{t\to\infty} E\left[x^T Q x\right] \end{aligned} \tag{3.26}$$

Equation 2.15 gives the algebraic solution of this cost functional and can be solved in closed form to compute the open loop performance cost for the single mass typical section:

$$\begin{aligned} J &= s_d \left(q_d \left(\frac{\zeta}{\omega} + \frac{1}{4\zeta\omega} \right) + q_v \frac{\omega}{4\zeta} \right) + s_v \left(q_d \frac{1}{4\zeta\omega^3} + q_v \frac{1}{4\zeta\omega} \right) \\ J &= J_d + J_v \end{aligned} \tag{3.27}$$

The cost has been split into two components — one due to displacement disturbances, and one due to velocity disturbances.

The discussion now turns to the form of this cost for the four types of defined problems.

Type I: Energy penalty with velocity disturbance

For Type I performance, the penalty and disturbance parameters have the form:

$$q_d = \gamma_K K = \gamma_K \omega^2 \qquad q_v = \gamma_M M = \gamma_M \qquad s_d = 0 \qquad s_v = M^{-1} G^T V G M^{-1} = V g^2 \tag{3.28}$$

With these substitutions, the open loop performance cost becomes:

$$J = V\frac{g^2}{4\zeta\omega}(\gamma_K + \gamma_M) \tag{3.29}$$

The above equation confirms what one would expect for this system, namely that increasing the natural frequency or the damping, or decreasing the impact of the disturbance on the mode will improve the open loop performance. However, it is interesting to examine how these changes improve performance. Figure 3.2 shows three sets of plots of the response of this typical section. The top group of plots show the displacement of the mass, q as a function of time. The second group of plots depicts the time history of the potential energy output variable $(\omega q)^2$. The lower group of plots show the time history of the kinetic energy output variable, $\dot{q}^2$ Each group of plots consists of four curves. The solid lines in each group represent the response of a nominal system with $\omega = 1$, $\zeta = .1$, and $g = 1$. The remaining curves depict the change in the reponse which is incurred by increasing or decreasing these values.

The purpose of these plots is to demonstrate the manner in which varying the passive parameters changes the performance of the system. It can be seen in the modal response plot that increasing the damping ratio decreases the time constant for the system. This attenuates the initial error more rapidly, thus reducing the cost as Equation 3.29 indicates.

Increasing the frequency also decreases the time constant and there fore improves performance in the same way as damping. However, notice that the increase frequency in the modal response plot has actually decreased amplitude of the modal response, but in the energy output plots, this decrease dissappears. This is because increasing the freqency in this system makes the system less sensitive to the disturbances (thus decreasing the modal response), but the use of an energy penalty causes the increased frequency to make the ouput more sensitive to the modal response (thus increasing the energy outputs). These two effects exactly cancel each other in this problem.

Finally, Figure 3.2 shows that decreasing the disturbability simply decreases the modal response.

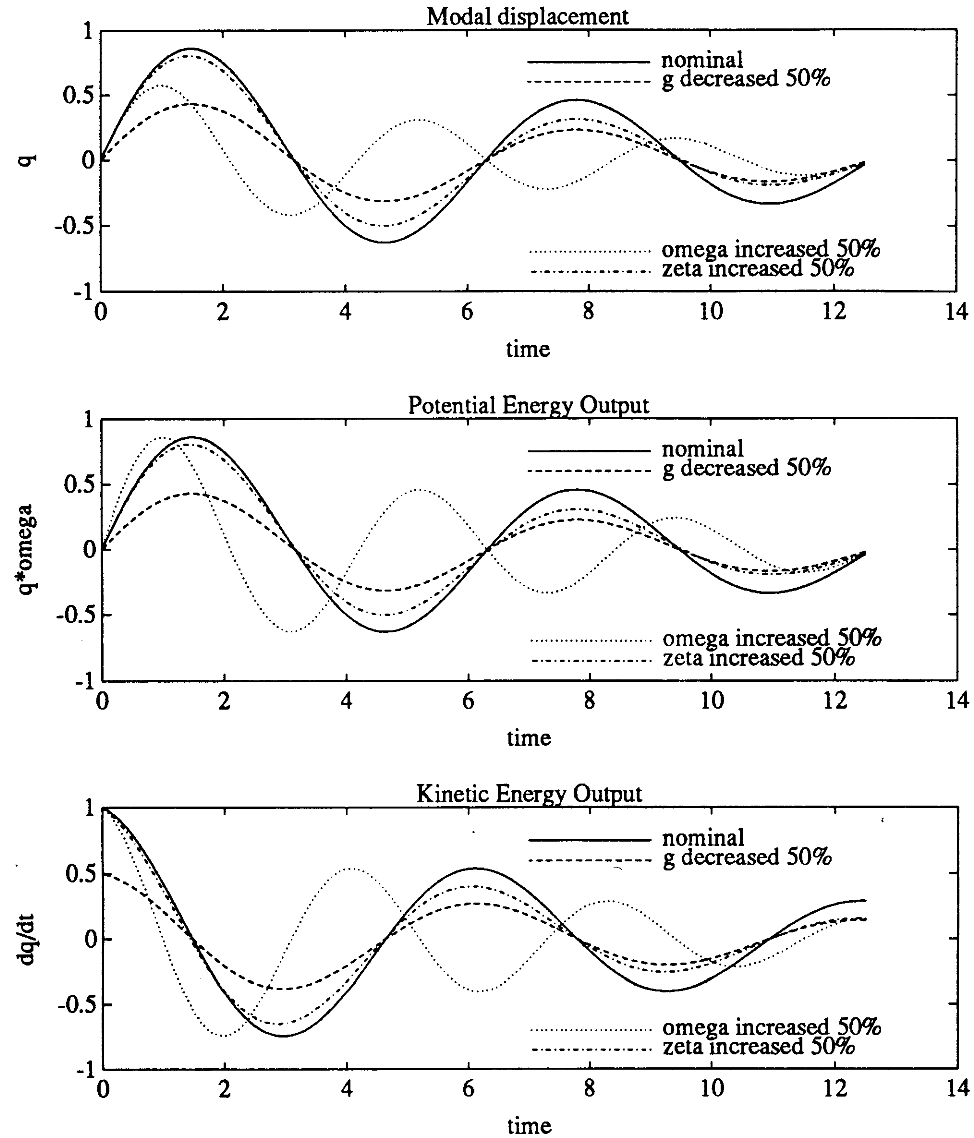

Figure 3.2: Modal displacement and outputs for open loop response of Type I problem. Nominal system: $\omega = 1$, $\zeta = .1$, $g = 1$, $V = 1$, $\gamma_K = 1$, $\gamma_M = 1$

It is tempting to look at Equation 3.29 and claim that "This system is most sensitive to changes in disturbability" due to the proportionality to the square of g. Unfortunately, there is simply not enough information to justify this claim. For example, if it is possible to change the damping in this system with an extremely small change in the structure compared to what it would take to effect a similar change in the disturbability, then one could just as easily claim that "This system is most sensitive to changes in damping". What is unknown is the cost of an incremental chang in damping or disturbability. For the most part, claims like this will have to be postponed until one is discussing a physical system (as in chapter four).

A great deal of use will be made of the derivatives of the costs for the typical sections. However, it is more useful to normalize the derivative by the function value itself:

$$J_\alpha = \frac{\frac{\partial J}{\partial \alpha}}{J} \tag{3.30}$$

where α is some paramter in the cost. This quantity is sometimes called the normalized sensitivity, but here it will be abbreviated to sensitivity.

There are basically two uses for the sensitivities of these costs to changes in modal parameters. The first use concerns the sign of the sensitivity. Regardless of how much one might need to change the structure, Equation 3.29 shows that increasing the modal frequency or damping decreases the cost (negative sensitivity), and increasing the disturbability has an opposite effect (positive sensitivity). The other use of these sensitivities will be observations about how they change relative to one another as the problem type or control architecture is changed. This will provide clues about how the design of a structure should change.

Type II: Displacement penalty with velocity disturbance

The penalty and disturbance parameters have the following form for Type II performance:

$$q_d = q_e n^2 \qquad q_v = 0 \qquad s_d = 0 \qquad s_v = V g^2 \tag{3.31}$$

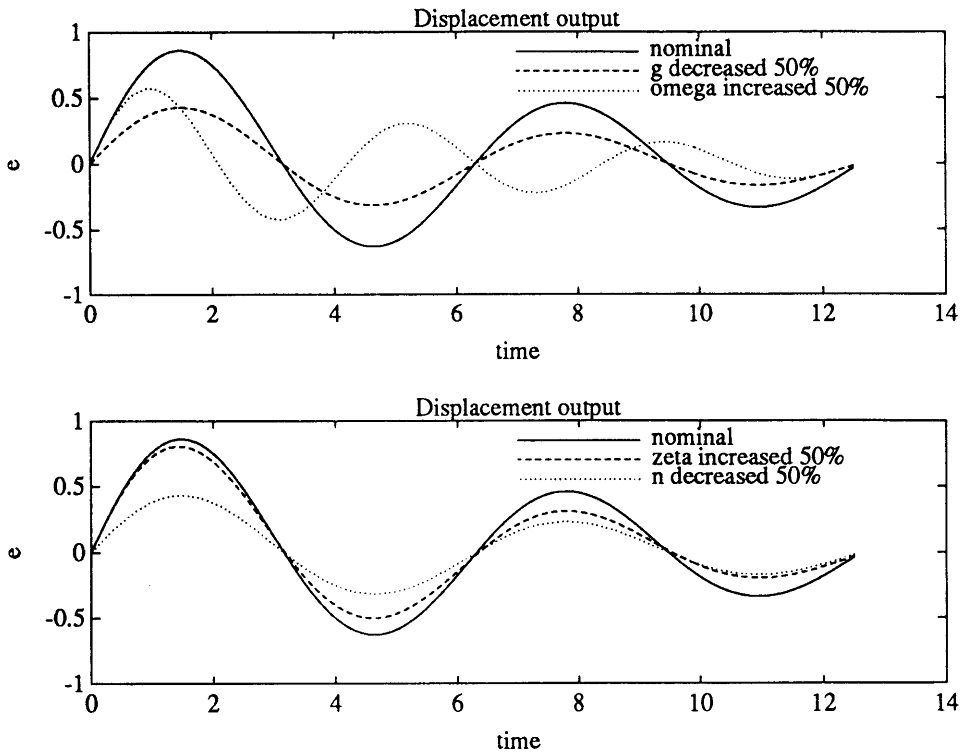

Figure 3.3: Outputs for open loop response of Type II problem. Nominal system: $\omega = 1$, $\zeta = .1$, $g = 1$, $V = 1$, $n = 1$, $q_e = 1$.

where q_e is the scalar which weights the displacement output e. Substituting these into Equation 3.27 gives an open loop performance cost for this problem type.

$$J = V\frac{q_e g^2 n^2}{4\zeta\omega^3} \tag{3.32}$$

There are two differences between this expression and the cost for the Type I problem (Equation 3.29). The first is the presence of the additional modal parameter, n, and the second is the increased sensitivity of the cost to changes in the natural frequency. The senstivity of the cost to modal damping or disturbability has not changed. Figure 3.3 shows time histories of the displacement output variable, e, for a nominal system, ($g = \omega = n = 1$, $\zeta = .1$) and also for changes in the modal parameters. The modal displacement for this system is identical to that of the Type I problem. Notice that changing the influence of the disturbance on the mode, g, or the influence of the mode on the output n have an identical result. As will be shown later, this is only true for the open loop case. In the LQR controlled case, changing the observability will change the control gain matrix, and, therefore, the closed loop dyanmics.

Figure 3.3 also illustrates why this system is more sensitive to frequency. The displacement penalty does not give a higher cost to higher frequency modes. Hence the advantage of a reduced modal response due to increasing the natural freqency of the section is not cancelled by increasing displacement penalty as it was with the energy penalty.

Type III: Energy penalty with displacement disturbance

When the parameters for energy penalty with displacement disturbance are used:

$$q_d = \gamma_K \omega^2 \qquad q_v = \gamma_M \qquad s_d = K^{-1} G V G^T K^{-1} = V \frac{g^2}{\omega^4} \qquad s_v = 0 \tag{3.33}$$

one obtains the following cost for the Type III system with open loop performance:

$$J = V \frac{g^2}{\omega^3} \left(\gamma_K \left(\zeta + \frac{1}{4\zeta} \right) + \frac{\gamma_M}{4\zeta} \right) \tag{3.34}$$

The sensitivity of this cost to disturbability is identical to the previous two cases. In fact, this will be true for all of the problem types and levels of control. This is because the disturbance influence parameter, g, prefilters the disturbance before the control can influence it.

Figure 3.4 shows the modal displacement and outputs for an example of this problem. Again, increasing the frequency attenuates the initial response of the system. However, the effect is greater than it was for the velocity disturbance. The frequency penalty incurred by using the energy penalty is insufficient to cancel this effect, as it was for the Type I problem. This accounts for the higher sensitivity to frequency for using the displacement disturbance.

The influence of damping in this problem is substantially different. Equation 3.34 suggests that large amounts of damping can actually inhibit performance if the potential energy penalty (γ_K) is used. The performance cost can be minimized with respect to

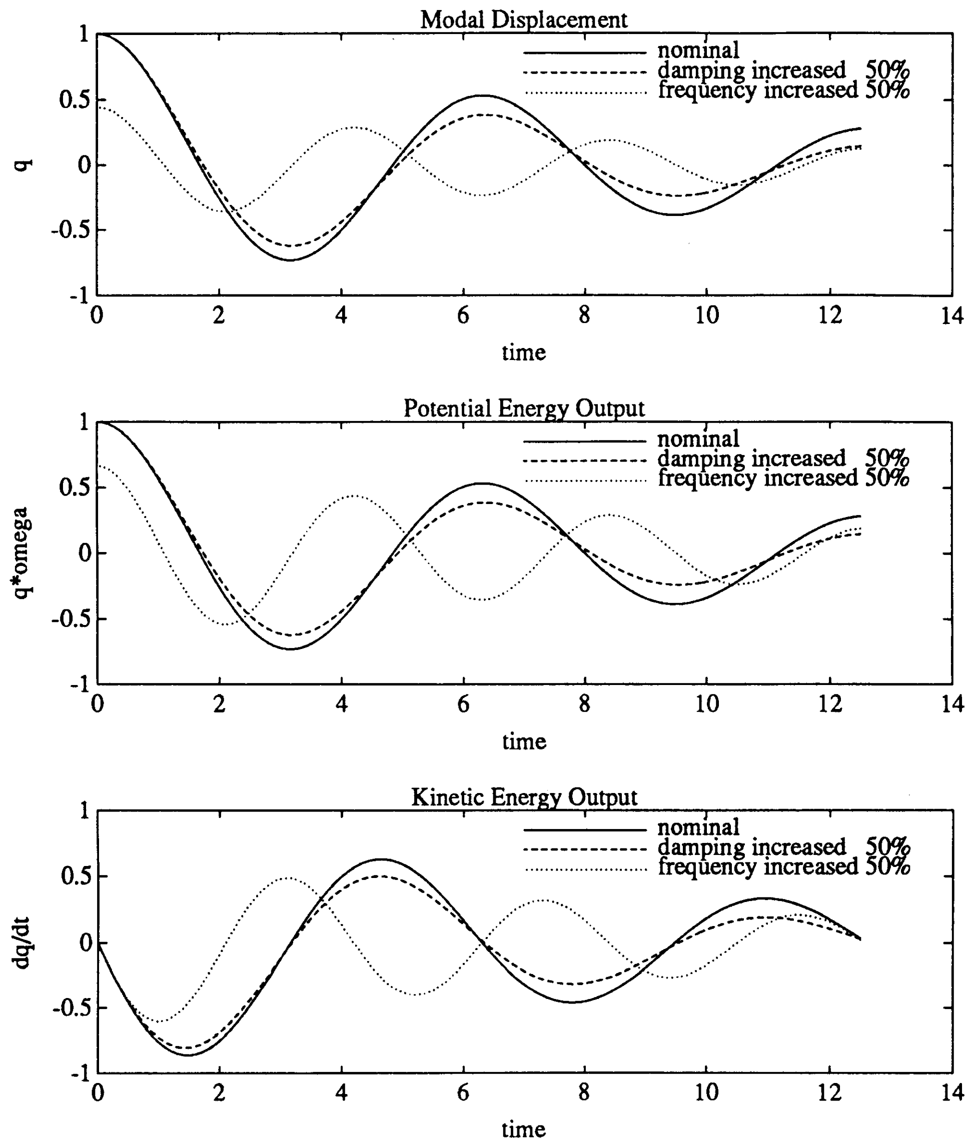

Figure 3.4: Modal displacement and outputs for open loop response of Type III problem. Nominal system: $\omega = 1$, $\zeta = .1$, $g = 1$, $V = 1$, $\gamma_K = 1$, γ_M=1

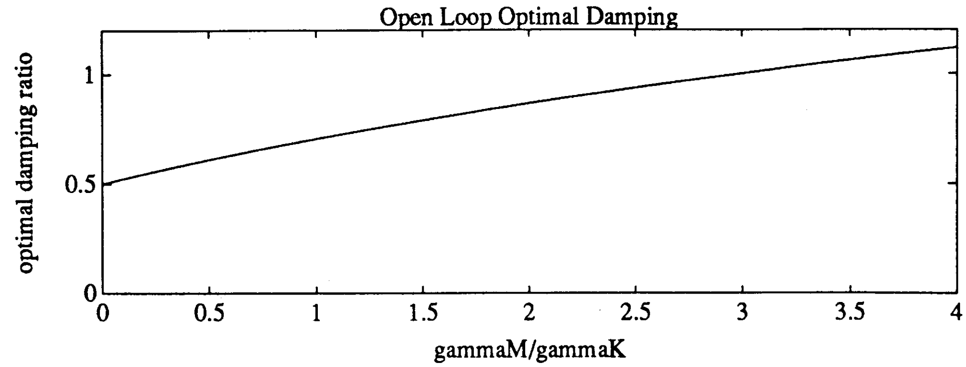

Figure 3.5: Optimal damping ratios for the open loop, Type III problem

the damping ratio to find an optimal damping ratio:

$$\left.\frac{\partial J}{\partial \zeta}\right|_{\zeta_{opt}} = V\frac{g^2}{\omega^3}\left(\gamma_K - \frac{\gamma_K + \gamma_M}{4\zeta_{opt}^2}\right) = 0$$

$$\zeta_{opt} = \frac{1}{2}\sqrt{1 + \frac{\gamma_M}{\gamma_K}} \qquad (3.35)$$

Figure 3.5 shows a plot of this optimal damping versus the ratio γ_M/γ_K. If there is no penalty on the kinetic energy ($\gamma_M = 0$), then the optimal damping is 50%. The optimal value of damping increases monotonically to infinity as the kinetic energy penalty is increased compared to the potential energy penalty. The reason for this is that large amounts of damping make the system sluggish. Sluggishness is good from the perspective of controlling velocity, but it tends to increase the amount of time for which a displacement error exists. This is illustrated by Figure 3.6 which shows the modal response of a typical section with 10%, 50%, and 90% critical damping. The ringing or sluggish decay of the initial displacement error increases the cost for low and high damping levels respectively.

Type IV: Displacement penalty with displacement disturbance

The disturbance and penalty parameters for the Type IV problem are:

$$q_d = q_e n^2 \qquad q_v = 0 \qquad s_d = V\frac{g^2}{\omega^4} \qquad s_v = 0 \qquad (3.36)$$

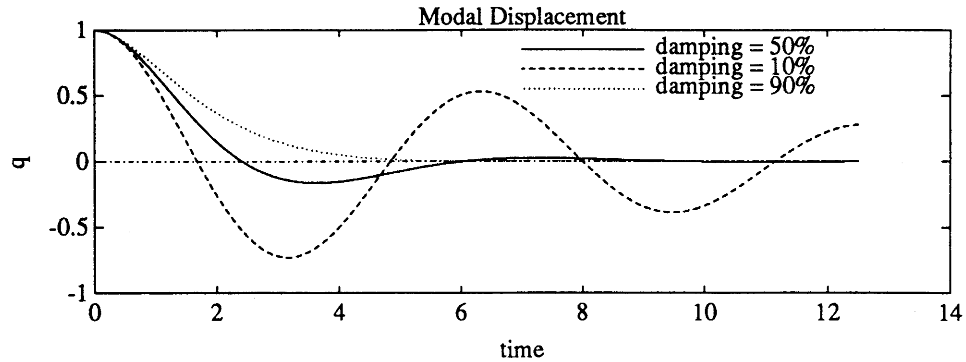

Figure 3.6: Modal response for typical section with displacement disturbance for varying levels of damping

These give rise to an open loop performance cost of:

$$J = V\frac{q_e n^2}{\omega^5}\left(\zeta + \frac{1}{4\zeta}\right) \tag{3.37}$$

for this typical section. This is identical to the potential energy portion of the Type III cost with the exception that the sensitivity to the frequency has been increased by a power of two and the term n^2 now appears in the cost. An identical effect was noted in comparing the Type II cost with the Type I cost. Hence there is really nothing new encountered in this problem other than the fact that the optimal damping level is always 50% (no penalty on velocity).

Conclusions and design rules for open loop typical sections

In the open loop problem, there are several modal parameters which one could imagine changing in a real structure to improve open loop performance. These were the natural frequency, ω, the damping ratio, ζ, the impact of a disturbance on the mode, g, and the influence of the mode on the performance output, n. Table 3.2 summarizes the relationships between the performance cost and these parameters for the four defined problem types.

Several conclusions can be made concerning the role of the modal parameters in these costs. First of all, it can be seen that the parameters g, and n seem to play identical

Table 3.2: Open Loop Performance of Single Mass Typical Section

	Velocity disturbance	Displacement disturbance
Energy Penalty	I: $J_v = V\frac{g^2}{4\zeta\omega}(\gamma_K + \gamma_M)$	III: $J_d = V\frac{g^2}{\omega^3}\left(\gamma_K\left(\zeta + \frac{1}{4\zeta}\right) + \gamma_M\right)$
Displacement Penalty	II: $J_v = Vq_e\frac{g^2n^2}{4\zeta\omega^3}$	IV: $J_d = Vq_e\frac{g^2n^2}{\omega^5}\left(\zeta + \frac{1}{4\zeta}\right)$

roles. In fact, the only difference between these two in the open loop cost is that g scales the disturbance as it enters the system, and n scales the output as it leaves the system. The problem type has no influence on how the cost behaves with respect to these parameters. These points can be combined to yield the first design rule.

Design rule 1 *Disturbance and output isolation reap similar benefits for all of the problem types.*

The next design rule concerns the role of damping in open loop performance. Notice that in those cases where the optimal damping for the system is infinite (Type I, Type II, and Type III when only kinetic energy penalty is used), the sensitivity of the cost with respect to the damping is the same. Furthermore, in cases where optimal damping does exist, the optimal damping is at least 50% when there is no penalty on velocity, and increases to infinity as the velocity penalty is increased relative to the displacement penalty. This is restated in the following design rule:

Design rule 2 *When a displacement penalty is used in the presence of a displacment disuturbance, there exists a finite amount of damping which will give the optimal performance. Otherwise, increased damping always give the same benefit for all problem types.*

The most complex parameter in this problem is frequency. Whereas the other parameters influence the performance costs in a relatively simple manner, there are three ways in which the frequency can influence the cost.

The most obvious way that changes in frequency influence the cost is by changing the speed of the open loop response. This effect is present in all of the problem types. It was noted for the Type I problem that this was the only significant manner in which the frequency influenced the cost. (There were two other ways in which the cost was influenced by the frequency, but they cancelled each other.) Hence, the sensitivity of the cost to frequency due to this effect is the same as the sensitivity of the cost to frequency in the Type I problem.

Design rule 3 *Increasing the natural frequency for a constant damping ratio decreases the time constant in the open loop problem. The performance cost is inversely proportional to frequency due to this effect for all of the defined problem types.*

The next way that frequency can influence the performance cost was encountered with the energy penalties. It was remarked before that for a given modal amplitude, both the potential energy and the kinetic energy were proprotional to the square of the natural frequency. Hence, changes in frequency also affect the observability in these types of problems.

Design rule 4 *In open loop problems which penalize the potential and kinetic energy of the system, increasing the frequency increases the observability in the system. For this effect, the cost is proportional to the square of the frequency.*

Finally, frequency affects the disturbability of the system. For the displacement disturbance, this influence is obvious, because the disturbance matrix was inversely proportional to the fourth power of the frequency (Equation 3.36). This effect is slightly more subtle when a velocity disturbance is used, however. In that case, the disturbance matrix is independent of the frequency (Equation 3.28). However, as was noted in the Type I and Type III problem, the modal response does seem to be inversely proportional to the frequency. This effect is even more clearly illustrated by Figure 3.7, which is the modal response of two systems with different natural frequencies and no damping. Because the cost is proportional to the square of the modal response, one can conclude

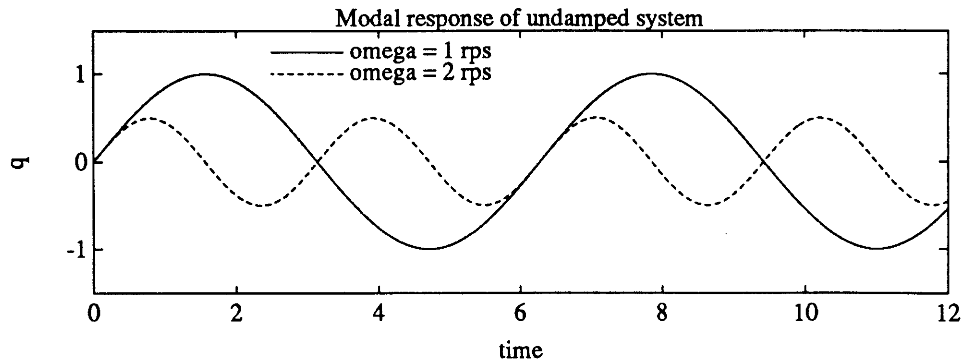

Figure 3.7: Modal response for open loop system with velocity disturbance, $V = 1$, $g = 1$

that the cost is inversely proportional to the square of the frequency for this effect. These relationships between the frequency and the disturbability are summarized in the following design rule:

Design rule 5 *Increasing the frequency of a mode decreases its disturbability in all of the problem types. In problems which use a displacement disturbance, the cost is inversely proportional to the fourth power of the frequency, and in problems which use a velocity disturbance, the cost is inversely proportional to the square of the frequency due to this effect.*

Change of variable

The parameters for the typical section were selected based on a desire to have each one correspond to a single method of improving performance. This desire was illustrated in Table 3.1. Unfortunately, the design rules above show that these parameters do not correspond directly to disturbability, observability, and the open loop time constant. In particular, it was shown that frequency was involved in all of these phenomena, and not just the open loop time constant. From the viewpoint of gaining physical insight into the behavior of controlled structures, this is not a desirable feature.

The expressions in Table 3.2 show how the overall frequency participates in the cost. This is acceptable in the typical section where there is really only one way to change stiffness. However, in a real structure, this is not the case. Recall that in the truss example of Miller and Shim (Example 3), it was shown that the stiffening used to make the structure less disturbable was distinctly different from the stiffening one would use to merely shift modal frequencies.

A natural solution to this problem is to absorb frequency into variables which are related directly to the concepts of observability, controllability, and disturbability. Any remaining appearances of frequency in performance costs will then reflect the role of the open loop time constant in the problem. This is easily accomplished by normalizing velocity in Equation 3.24 by frequency:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix} u \\ \frac{d}{dt}\begin{bmatrix} q \\ \frac{\dot{q}}{\omega} \end{bmatrix} &= \begin{bmatrix} 0 & \omega \\ -\omega & -2\zeta\omega \end{bmatrix}\begin{bmatrix} q \\ \frac{\dot{q}}{\omega} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{f}{\omega} \end{bmatrix} u \\ \dot{\mathcal{X}} &= \mathcal{A}\mathcal{X} + \mathcal{B}u \end{aligned} \tag{3.38}$$

The state penalty and disturbance matrices must be transformed to reflect this change of variable.

- Displacement Penalty

$$Q = \begin{bmatrix} n^2 q_e & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow Q = \begin{bmatrix} n^2 q_e & 0 \\ 0 & 0 \end{bmatrix} \tag{3.39}$$

- Energy Penalty

$$Q = \begin{bmatrix} \gamma_K \omega^2 & 0 \\ 0 & \gamma_M \end{bmatrix} \Rightarrow Q = \begin{bmatrix} \gamma_K \omega^2 & 0 \\ 0 & \gamma_M \omega^2 \end{bmatrix} \tag{3.40}$$

- Displacement Disturbance

$$S = \begin{bmatrix} V\frac{g^2}{\omega^4} & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow S = \begin{bmatrix} V\frac{g^2}{\omega^4} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.41}$$

- Velocity Disturbance

$$S = \begin{bmatrix} 0 & 0 \\ 0 & Vg^2 \end{bmatrix} \Rightarrow S = \begin{bmatrix} 0 & 0 \\ 0 & V\frac{g^2}{\omega^2} \end{bmatrix} \tag{3.42}$$

The disturbance and penalty matrices now have a dependence on frequency that was suggested by the design rules. Notice also that the control matrix, $\mathcal{B}$, also has a frequency dependence. This reflects the fact that a control force on a mode has a larger effect if the mode is softer. These matrices suggest the definition of some new variables.

- Disturbability

$$\mathcal{G} = \begin{cases} \frac{g}{\omega^2} & \text{displacement disturbance} \\ \frac{g}{\omega} & \text{velocity disturbance} \end{cases} \tag{3.43}$$

- Observability

$$\mathcal{N}_d = \begin{cases} n\sqrt{q_e} & \text{displacement penalty} \\ \sqrt{\gamma_K}\omega & \text{energy penalty} \end{cases} \tag{3.44}$$

$$\mathcal{N}_v = \begin{cases} 0 & \text{displacement penalty} \\ \sqrt{\gamma_M}\omega & \text{energy penalty} \end{cases} \tag{3.45}$$

- Controllability

$$\mathcal{F} = \frac{f}{\omega} \tag{3.46}$$

These variables correspond directly to disturbability, controllability, and observability. Using these parameters in the expressions in Table 3.2, one can see that the four problem types now become two problem types — velocity disturbance and displacement disturbance (Table 3.3). The cost for both cases is proportional to response time. This agrees with the design rules.

This concludes the analysis of the open loop behavior of the typical section. In the next section, the performance costs for both of these problem types will be computed in closed form for the case when optimal LQR control is used. By observing how the senstivities change due to the addition of control, one can infer in what ways the open loop design should differ from the closed loop designs for a controlled structure.

Table 3.3: Open Loop Performance of Single Mass Typical Section

Velocity disturbance	**Displacement disturbance**
$J_v = \frac{\mathcal{G}^2}{4\zeta\omega}(\mathcal{N}_d^2 + \mathcal{N}_v^2)$	$J_d = \frac{\mathcal{G}^2}{4\zeta\omega}(\mathcal{N}_d^2(4\zeta^2+1) + \mathcal{N}_v^2)$

3.2.2 Optimal LQR Performance

The control case most often examined in controlled structure problems is the optimal LQR. An analysis of this type assumes that the full state of the system is available for computing the control forces, and also that the system is perfectly modelled. Unfortunately, this is rarely the case. Information of the state is obtained through a limited number of sensors which are often corrupted by noise and no model is ever perfect. However, the advantage of computing the performance cost for this type of controller is that it places a lower bound on the performance cost for all possible controllers. One can think of the the optimal design for an open loop plant and the optimal design for an LQR controlled plant as marking extremes. The optimal designs for the system under any type of control must in a sense "lie" somewhere between these two extremes.

The object of optimization is to minimize the cost functional:

$$\begin{aligned} J &= E\left[\int_0^\infty \left(\mathcal{X}^T Q \mathcal{X} + \rho^2 u^2\right) dt\right] \\ \text{or} \quad J &= \lim_{t\to\infty} E\left[\mathcal{X}^T Q \mathcal{X} + \rho^2 u^2\right] \end{aligned} \tag{3.47}$$

where the disturbances are the same as in the previous section. The equations to solve for the performance cost and control gains were given in chapter two. (Equations 2.18 and 2.19). For the typical section, these equations can be solved in closed form to yield:

- Optimal Control: $u = -C_c\mathcal{X} \qquad \left(\beta_d \equiv \frac{\mathcal{N}_d\mathcal{F}}{\omega\rho} \qquad \beta_v \equiv \frac{\mathcal{N}_v\mathcal{F}}{\omega\rho}\right)$

$$C_c = \frac{\omega}{\mathcal{F}}\left[\ \sqrt{\beta_d^2+1}-1 \quad \sqrt{4\zeta^2+\beta_v^2+2\sqrt{\beta_d^2+1}-2}-2\zeta\ \right] \tag{3.48}$$

- Performance Cost

$$J = \frac{\rho^2\omega}{\mathcal{F}^2}\left(s_d\left(\sqrt{\beta_d^2+1}\sqrt{4\zeta^2+\beta_v^2+2\sqrt{\beta_d^2+1}-2}-2\zeta\right)\right.$$

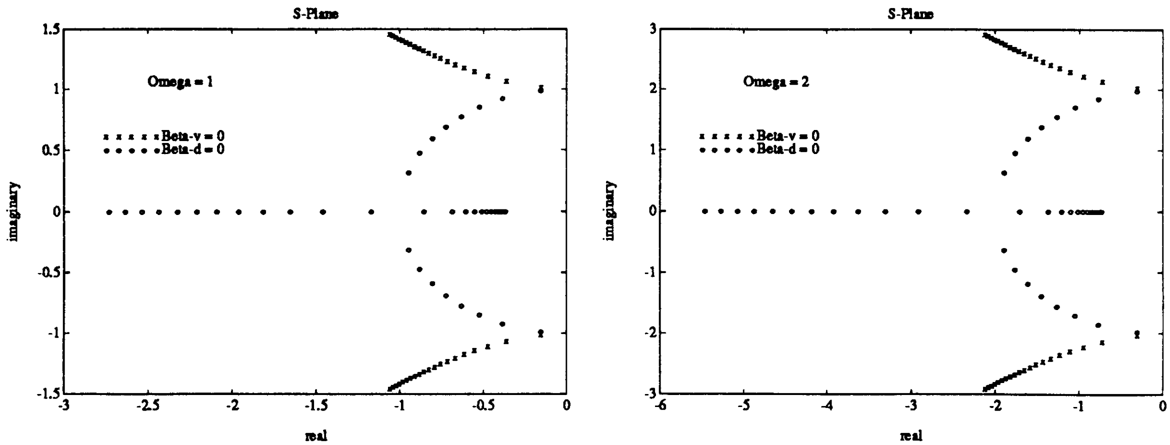

Figure 3.8: Root locus for LQR controlled single mass typical section: β_d^2 and β_v^2 varied from .1 to 10

$$+ \quad s_v \left(\sqrt{4\zeta^2 + \beta_v^2 + 2\sqrt{\beta_d^2 + 1} - 2} - 2\zeta \right) \Bigg) \tag{3.49}$$

The paramters β_d and β_v have special significance. They determine the position of the closed loop poles measured relative to the position of the open loop poles in the S-plane. This is illustrated by figure 3.8 which shows the closed loop poles for two systems. Both plants are undamped and identical with the exception that the natural frequencies of the two differ by a factor of two. The points marked by 'x' in the figures correspond to the root locus obtained by setting β_v to zero and varying β_d (displacment penalty), and the points marked by 'o' correspond to the root locus obtained by setting β_d to zero and varying β_v (velocity penalty). All other designs based upon a mix of these state penalties lie to the left of these bands. The only difference between the plots for the two different plants is the scaling of the axes.

The shapes of these root loci are common for LQR control. When no velocity penalty is used ($\beta_v = 0$), the poles move out to 45° asymptotes as the penalty on displacement is increased relative to the control penalty (increasing β_d). The control gains start out as predominantly velocity feedback to move the poles away from the imaginary axis for small values of β_d implying that the cost is more sensitive to damping thatn frequency at low levels of control effort. More displacement feedback is required as β_d is increased

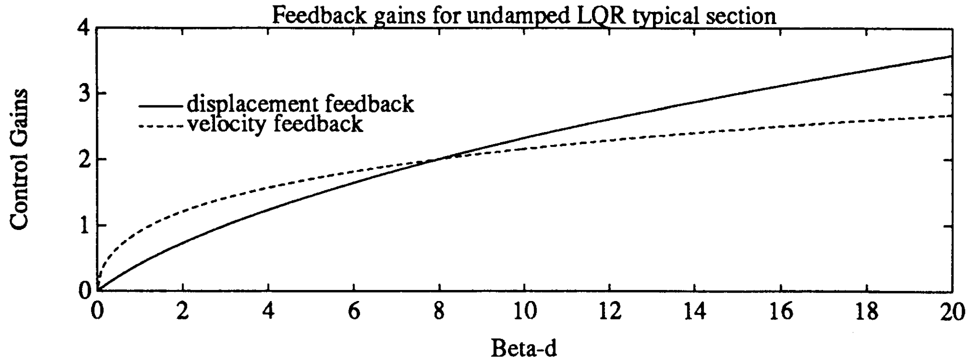

Figure 3.9: Feedback gains for LQR control of single mass typical section: $\omega = 1$, $\zeta = 0$, $\beta_v = 0$

(Figure 3.9). If only velocity is penalized, then the control only attempts to augment the damping in the system. Hence, only velocity feedback is used and the poles move around the perimeter of a circle centered on the origin.

The primary interest in the LQR typical section lies with the behavior of the cost. Not surprisingly, this cost is significantly more complicated than it was in the open loop case. To facilitate analysis, the study of the closed loop cost will be divided into three sections. First, the role of damping will be addressed. Then, the behavior of the cost for low levels of control (expensive control) will be examined. Finally, the behavior of the cost for high levels of control (cheap control) will be studied.

The role of damping

To understand how the importance of damping changes when LQR control is added, it is useful to compute the sensitivity of the performance cost to the damping ratio:

$$\begin{aligned}
\frac{\frac{\partial J_d}{\partial \zeta}}{J_d} &= \frac{-2 + 4\zeta\sqrt{\beta_d^2+1}\,(4\zeta^2+\beta)^{-1/2}}{-2\zeta + \sqrt{\beta_d^2+1}\sqrt{4\zeta^2+\beta}} \\
\frac{\frac{\partial J_v}{\partial \zeta}}{J_v} &= \frac{-2 + 4\zeta\,(4\zeta^2+\beta)^{-1/2}}{-2\zeta + \sqrt{4\zeta^2+\beta}} \\
\beta &\equiv \beta_v^2 + 2\sqrt{\beta_d^2+1} - 2
\end{aligned} \tag{3.50}$$

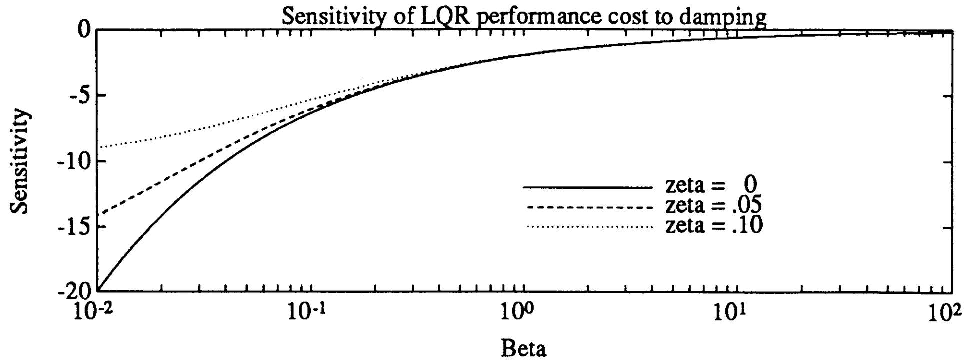

Figure 3.10: Sensitivity of LQR performance cost to passive damping ratio for varying level of control, β. (Velocity disturbance)

The quantity, β, has been used to simplify the above expressions. It is important to note that β is monotonically increasing with both β_d and β_v. Clearly, the sensitivity of the cost to the damping ratio approaches zero as the level of control is increased. This is illustrated using the performance cost for a velocity disturbance in figure 3.10. Notice that regardless of the level of passive damping, the magnitude of the sensitivity always decreases as the control level is increased. This leads to the following design rule:

Design rule 6 *Decreasing the expense of using active control (increasing β) decreases the importance of passive damping to dynamci peforamnce.*

Physically, this makes sense. A large value of β indicates that the control is being used to drastically alter the dynamics of the system. The internal forces due to damping become insignificant when compared to the control forces.

Another interesting feature of the damping appears when a displacement disturbance is used with a displacement penalty. For the open loop case, it was found that this combination led to a finite value of optimal damping. This is also true for the case of LQR control:

$$\left.\frac{\partial J_d}{\partial \zeta}\right|_{\zeta_{opt}} = \frac{\rho^2\omega}{\mathcal{F}^2}\left(-2 + 4\zeta_{opt}\sqrt{\beta_d^2+1}\left(4\zeta_{opt}^2 + \beta_v^2 + 2\sqrt{\beta_d^2+1} - 2\right)^{-1/2}\right) = 0$$

$$\zeta_{opt}^2 = \frac{\beta_v^2 + 2\sqrt{\beta_d^2+1} - 2}{4\beta_d^2} \tag{3.51}$$

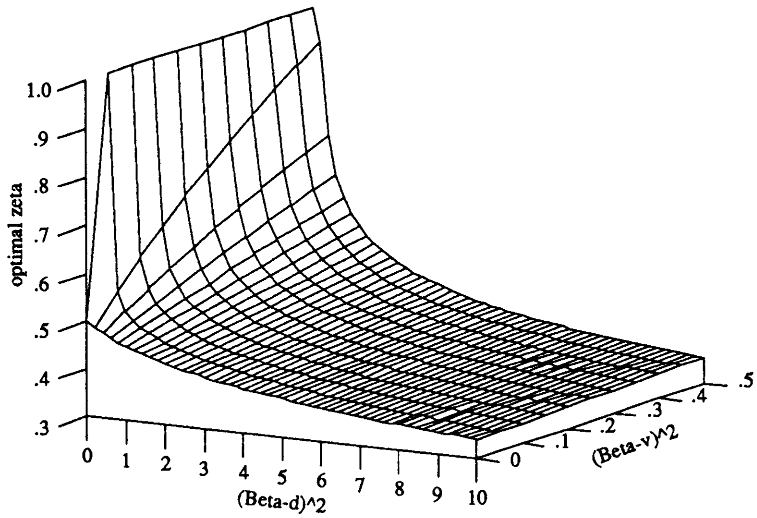

Figure 3.11: Optimal passive damping ratio for LQR controlled typical secition with displacement disturbance and varying levels of displcement control (β_d) and velocity control, (β_v).

This function is shown plotted for various values of β_d and β_v in figure 3.11. Just as in the open loop case, the optimal damping increases as the penalty on velocity is made large relative to the penalty on displacement. An interesting feature of the optimal damping, however, is that it decreases as the displacement penalty, β_d is increased. This decrease is necessary when control forces become larger. In that case, large amounts of damping would cause the system to resist the efforts of the controller instead of aiding them.

In the next two sections, the other parameters in the problem will be examined. This analysis will be simplified if one assumption is made about the damping. In flexible space structures, the largest level of passive damping one can hope to achieve is on the order of 10%. For the rest of this chapter, it will be assumed that passive damping levels exceeding this are impossible.

The behavior of the performance cost can be divided into two regimes — expensive control and cheap control. In modern control theory, expensive control refers to cases for the optimal LQR controller in which the penalty on control is much larger than the penalty on the state. In terms of the variables defined above, expensive control should correspond to small values of β_d and β_v. Conversely, cheap control refers to those cases where the penalty on control is small compared to the state penalty (large β_d or β_v).

Expensive Control

Mathematically, expensive control is defined to be the case when β_d is less than unity. β_v could be included in this definition as well, but it is not necessary. If β_d is less than unity, then the Binomial Expansion Theorem can be used to simplify the performance cost.

$$\begin{aligned} \beta_d \ll 1 \quad &\Rightarrow \quad \sqrt{\beta_d^2+1} \approx 1 + \frac{1}{2}\beta_d^2 \\ J \;&\approx \frac{\rho^2\omega}{\mathcal{F}^2} \left(s_d \left(\left(1+\frac{1}{2}\beta_d^2\right)\sqrt{4\zeta^2+\beta_v^2+\beta_d^2} - 2\zeta \right) \right. \\ &+ \quad \left. s_v \left(\sqrt{4\zeta^2+\beta_v^2+\beta_d^2} - 2\zeta \right) \right) \end{aligned} \tag{3.52}$$

This equation is still slightly too complicated to be useful. However, it is apparent that it can be simplified further by either assuming that the damping ratio is large or small when compared to the quantity:

$$\zeta_{cr} = \frac{1}{2}\sqrt{\beta_v^2 + 2\sqrt{\beta_d^2+1} - 2} \tag{3.53}$$

Heavy and light damping can be conveniently defined:

- Light damping $\quad \zeta \ll \zeta_{cr}$
- Heavy damping $\quad \zeta \gg \zeta_{cr}$

Notice that this is not an absolute measure, rather, the character of the damping (Light or Heavy) is determined from its magnitude relative to the active damping induce by the control.

If one has heavy damping, the Binomial Expansion Theorem can be exploited a second time on the performance cost:

$$\begin{aligned}
\zeta \gg \zeta_{cr} \quad &\Rightarrow \quad \sqrt{4\zeta^2 + \beta_v^2 + \beta_d^2} \approx 2\zeta + \frac{\beta_d^2 + \beta_v^2}{4\zeta} \\
J \quad &\approx \quad \frac{\rho^2\omega}{\mathcal{F}^2}\left(s_d\left(\beta_d^2\left(\zeta + \frac{1}{4\zeta}\right) + \frac{\beta_v^2}{4\zeta}\right) + s_v\left(\frac{\beta_d^2}{4\zeta} + \frac{\beta_v^2}{4\zeta}\right)\right) \\
J_d \quad &= \quad \frac{\mathcal{G}^2}{4\zeta\omega}\left(\mathcal{N}_d^2\left(4\zeta^2 + 1\right) + \frac{\mathcal{N}_v^2}{4\zeta}\right) \qquad J_v = \frac{\mathcal{G}^2}{4\zeta\omega}\left(\mathcal{N}_d^2 + \mathcal{N}_v^2\right) \qquad (3.54)
\end{aligned}$$

This result is identical to the open loop performance. One can conclude that the bulk of the dynamic performance in this case is being obtained passively.

Design rule 7 *For a heavily damped ($\zeta \gg \zeta_{cr}$), lightly controlled ($\beta_d \ll 1$) system, the contribution of the control to improving performance is insignificant when compared to the benefits of the passive damping..*

Alternatively, if the damping is less than the critical level, it drops out of the cost entirely, taking the natural frequency with it:

$$\begin{aligned}
\zeta \ll \zeta_{cr} \quad &\Rightarrow \quad \sqrt{4\zeta^2 + \beta_d^2} - 2\zeta \approx \sqrt{\beta_v^2 + \beta_d^2} \\
J \quad &\approx \quad \frac{s_d\rho^2\omega}{\mathcal{F}^2}\sqrt{\beta_v^2 + \beta_d^2} + \frac{s_v\rho^2\omega}{\mathcal{F}^2}\sqrt{\beta_v^2 + \beta_d^2} \\
J_d \quad &\approx \quad \frac{\mathcal{G}^2\rho}{\mathcal{F}}\sqrt{\mathcal{N}_v^2 + \mathcal{N}_d^2} \qquad J_v \approx \frac{\mathcal{G}^2\rho}{\mathcal{F}}\sqrt{\mathcal{N}_v^2 + \mathcal{N}_d^2} \qquad (3.55)
\end{aligned}$$

Design rule 8 *In a lightly damped system ($\zeta \ll \zeta_{cr}$), the benefits of the passive damping and the natural frequency are insignificant when measured relative to the benefits derived from the active control.*

It is interesting to note that the cost takes identical forms for both the displacement and the velocity disturbances. The reason for this is that a lightly controlled, lightly

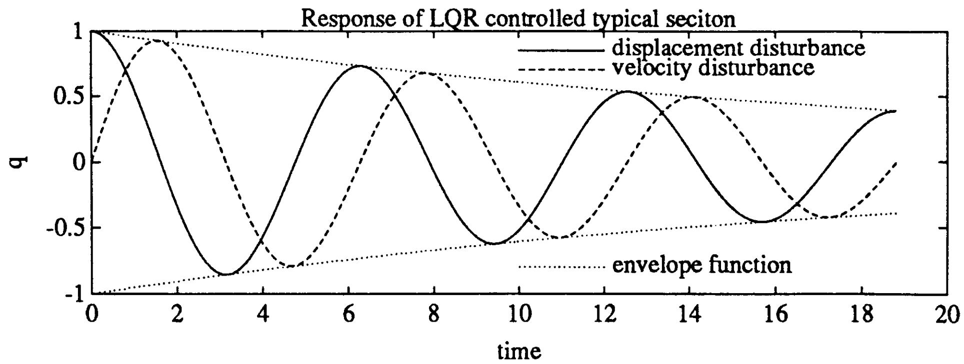

Figure 3.12: Modal response of single mass typical section due to displacment and velocity disturbance: $\omega = 1$, $\zeta = 0$, $\mathcal{G} = 1$, $\mathcal{F} = 1$, $\mathcal{N}_d = .1$, $\mathcal{N}_v = 0$.

damped system will have a highly oscillatory response. In such cases, the performance is largely determined by the envelope function — the decaying exponential which bound the sinusoidal response. The time constant for this exponential is simply the real part of the closed loop poles. Figure 3.12 shows how the envelope functions are always identical for a lightly damped, lightly controlled system with a displacement disturbance, and a velocity disturbance, thus making the corresponding performance costs equal. Similar arguments can be made for the similarity in the behavior of the cost with respect to the displacement and velocity observability.

Design rule 9 *For a lightly damped, lightly controlled system, the sensitivities of the performance cost to observability, disturbability, frequency, and controllability are identical for a velocity and a displacement disturbance and a velocity and a displacement state penalty.*

The final feature of this cost to be observed concerns the sensitivity to observability. This sensitivity has been reduced from the open loop case ($\sqrt{\mathcal{N}_d}$ appears in the cost instead of $\mathcal{N}_d$). This is because as the mode becomes more observable, the control effort also increases, thus blunting its effect on the cost.

Design rule 10 *The sensitivity to observability of a lightly controlled, lightly damped structure is less than that of the same structure open loop.*

In the next section, the behavior of the system as the displacement penalty, β_d exceeds unity is explored.

Cheap Control

The analysis of the LQR controlled typical section now continues with the case of cheap control which was defined in the previous section as occurring when the quantity β_d is much greater than one. This could be extended to included the case for when β_v becomes large, however, an examination of Equation 3.49 shows that large values of β_v do not significantly alter the behavior of the performance cost (other than increasing the critical damping value).

When β_d exceeds unity, the character of the equation does change, however:

$$\begin{aligned} \beta_d \gg 1 \quad &\Rightarrow \quad \sqrt{\beta_d^2+1} \approx \beta_d \\ J \quad &\approx \quad \frac{\rho^2\omega}{\mathcal{F}^2}\left(s_d\beta_d\sqrt{\beta_v^2+\beta_d}+s_v\sqrt{\beta_v^2+\beta_d}\right) \end{aligned} \tag{3.56}$$

Breaking this case down into two subcases, consider first the instance when there is no penalty on velocity ($\beta_v = 0$).

$$J_d \approx \frac{\mathcal{G}^2\sqrt{\rho\mathcal{N}_d^3}}{\sqrt{\mathcal{F}\omega}} \qquad\qquad J_v \approx \frac{\mathcal{G}^2\sqrt{\omega\rho^3\mathcal{N}_d}}{\sqrt{\mathcal{F}^3}} \tag{3.57}$$

This is a surprising result. When one considers a displacement disturbance, moving from expensive to cheap control increases the sensitivity of the cost to observability, decreases the sensitivity to controllability, and introduces a negative sensitivity to frequency (increasing frequency decreases the cost). For a velocity disturbance, the effect is exactly the opposite — the sensitivity to observability decreases, the sensitivity to controllability increases, and a positive sensitivity to frequency is developed. Figure 3.13 helps illustrate this effect. It shows two plots for the performance cost at varying natural frequencies. One corresponds to a displacement disturbance, the other corresponds to a velocity disturbance. At high frequencies, β_d becomes small (Equation 3.48) the cost is

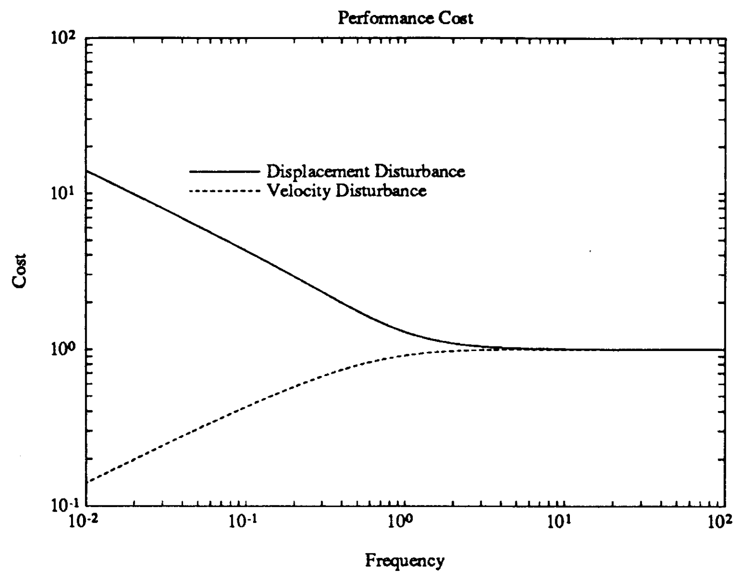

Figure 3.13: LQR Performance Cost: $\mathcal{N}_d = 1$, $\mathcal{N}_v = 0$, $\mathcal{F} = 1$, $\mathcal{G} = 1$

insensitive to changes in frequency, as stated in design rule 8. However, as frequency is decreased, its influences on the performance costs are exactly the opposite of each other.

The reason for these markedly different behaviors is due to the fact that at high levels of control, it is the initial behavior of the system which dominates the performance cost. Figure 3.14 shows the closed loop response of the typical section for a displacement disturbance, a velocity disturbance, and different values of controllability, observability, and frequency. The key difference between the two disturbance types lies in the peak response. With the displacement disturbance, the peak modal response is at time $t = 0$. The only way to reduce it is to reduce the observability, hence observablity's heightened importance. (One could also change the disturbability, but the effect would not be any different from the open loop case). The higher frequency helps by increasing the rate at which the passive response would return the displacement to zero.

For the velocity disturbance, the effectiveness of the controller determines the peak response, hence the increased role of controllability. As for the frequency, a slower system gives the controller more time to act in reducing the peak, thus giving a benefit to reducing the natural frequency.

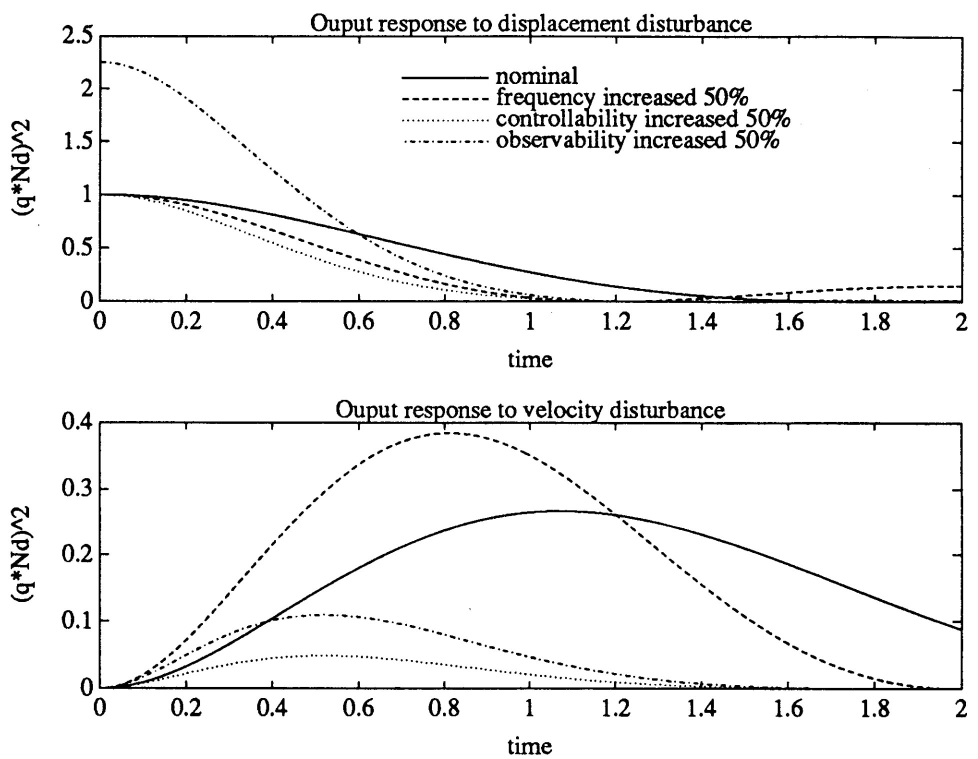

Figure 3.14: Performance output of single mass typical section. Nominal system: $\omega = 1$, $\mathcal{F} = 1$, $\mathcal{N}_d = 1$, $\mathcal{N}_v = 0$, ,$\mathcal{G} = 1$, $\rho^2 = .1$

The above ideas and equations can be used to formulate the following design rules:

Design rule 11 *For cheap control, the sensitivity of the cost to frequency is postive for a velocity disturbance and negative for a displacement disturbance. In both cases, the magnitude of the sensitivity is less than it would be in the open loop case.*

Design rule 12 *For cheap control, the sensitivity of the performance cost to observability for a displacement disturbance is greater than it was in the expensive control case and less than it was in the expensive control case for a velocity disturbance.*

Design rule 13 *The sensitivity of the performance cost to controllability increases for a velocity disturbance and decreases for a displacement disturbance in moving from the expensive control case to the cheap control case.*

The above analysis assumed that the velocity penalty was negligible when compared to the displacement penalty. If the velocity control parameter is significantly larger than the square root of the displacement control parameter, and the displacement control parameter still exceeds unity, the following happens:

$$
\begin{gathered}
1 \ll \beta_d \ll \beta_v^2 \Rightarrow \\
J_d \approx \frac{\mathcal{G}^2 \mathcal{N}_d \mathcal{N}_v}{\omega} \qquad\qquad J_v \approx \frac{\mathcal{G}^2 \rho \mathcal{N}_v}{\mathcal{F}}
\end{gathered} \tag{3.58}
$$

For the velocity disturbance, the result is identical to the expensive control case. However, the result for the displacement disturbance is interesting. The control penalty and the controllability have disappeared from the performance cost. The reason for this is that even if the control penalty is zero, there is a non-zero performance cost associated with the state penalty. Imagine what would have to happen to the system to control displacement perfectly. From an initial displacement, the mass would have to be translated to the origin instantaneously. Unfortunately, this would require an infinite velocity, driving that portion of the cost to infinity. Similarly, an attempt to control velocity perfectly from an initial displacement would require that the mass not move at all, thus driving the displacement portion of the cost to infinity. The expression above represents the best compromise one can hope to achieve.

The foregoing design rules cover almost everything one can learn about controlled structure design from the single mass typical section. In a perfect world, it is all one would need to intelligently design a controlled structure. Unfortunately, it is not a perfect world. The next section explores the consequences of using optimal control on a poorly modelled plant, and how adjustment of certain structural parameters can help alleviate them.

3.3 Spillover Typical Section

The previous section showed how a controller can interact with a single mode of a structure, and how it influenced the dynamic performance of the resulting system. It is possible to use these results to predict in a broad way how the control will influence the modelled modes in an actual, more complex stucture. These results cannot, however, be applied to modes of the structure which exist, but were not included in the control design. Most physical structures are built up of continuous elements such as the members in a truss. The continuous nature of these elements makes it possible for the structure to have an infinite number of modes.

Modern control theory requires a finite dimensional model for the design of the controller. This forces the control designer to ignore all of the modes in a structure above some arbitrary frequency. At best, the presence of unmodelled modes in the structure will impair performance. At worst, they will be destabilized by the controller. This class of problems has been dubbed with the name "Spillover." Referring to the effects of the control spilling over into unmodelled modes.

One way to study some of the issues involved in the problem is to use the two mass typical section shown in Figure 3.15. This simple system has only two modes. The mass on the right is driven by a control force, u, and the position and velocity of the mass on

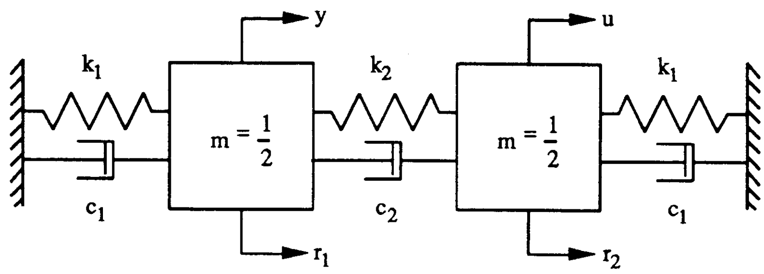

Figure 3.15: Two mass typical section

the left (r and $\dot{r}$) are available for the computation of the control. The idea is to design the controller using information about the first mode and then evaluate the performance of the system resulting when one attempts to use the single mode controller to control the two mode structure.

The equation of motion for this system can be written:

$$\begin{bmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \ddot{r}_1 \\ \ddot{r}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_1 + c_2 \end{bmatrix} \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{3.59}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} \tag{3.60}$$

Transforming this system into modal coordinates yields:

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 2\zeta_1\omega_1 & \\ & 2\zeta_2\omega_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & \\ & \omega_2^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \tag{3.61}$$

A simplification may be made to this system by normalizing time by the fundamental frequency.

$$\tau = \omega_1 t \tag{3.62}$$

The equation of motion then becomes:

$$\begin{bmatrix} q_1'' \\ q_2'' \end{bmatrix} + \begin{bmatrix} 2\zeta_1 & \\ & 2\zeta_2\lambda \end{bmatrix} \begin{bmatrix} q_1' \\ q_2' \end{bmatrix} + \begin{bmatrix} 1 & \\ & \lambda^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mu$$

$$\lambda \equiv \omega_2/\omega_1 \qquad \mu \equiv u/\omega_1^2 \tag{3.63}$$

where prime ($'$) denotes the derivative with respect to normalized time.

The number of parameters of interest in this problem has been reduced to three — two damping ratios and the frequency ratio, λ.

Recasting the equation of motion into state space form and normalizing velocities by frequency yields the evaluation model which will be used in this problem:

$$\frac{d}{d\tau} \underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_1' \\ q_2'/\lambda \end{bmatrix}}_{x_e} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda \\ -1 & 0 & -2\zeta_1 & 0 \\ 0 & -\lambda & 0 & -2\zeta_2\lambda \end{bmatrix}}_{A_e} \begin{bmatrix} q_1 \\ q_2 \\ q_1' \\ q_2'/\lambda \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1/\lambda \end{bmatrix}}_{B_e} \mu$$

$$y = \underbrace{\begin{bmatrix} 1 & -1 & & \\ & & 1 & -\lambda \end{bmatrix}}_{C_e} \begin{bmatrix} q_1 \\ q_2 \\ q_1' \\ q_2'/\lambda \end{bmatrix} \tag{3.64}$$

The objective is to minimize a weighted sum of the quadratic response of the measured output, y, and the control effort, μ.

$$J = E\left[\int_0^\infty \left(x^T Q_m x + \rho^2 \mu^2\right) d\tau\right]$$

$$\text{or} \qquad J = \lim_{T\to\infty} E\left[x^T Q x + \rho^2\mu^2\right]$$

$$Q_m = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (3.65)$$

The control is to be designed with knowledge of only the first mode. An appropriate design model is:

$$x_m' = A_m x_m + B_m \mu \qquad y = C_m x_m$$

$$x_m = \begin{bmatrix} q_1 \\ q_1' \end{bmatrix} \qquad A_m = \begin{bmatrix} 0 & 1 \\ -1 & 2\zeta_1 \end{bmatrix} \qquad B_m = \begin{bmatrix} 1 \\ 1 - \frac{1}{\lambda^2} \end{bmatrix} \qquad C_m = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \qquad (3.66)$$

The appearance of the term $(1 - \frac{1}{\lambda^2})$ in the control matrix for the design model, B_d, is a static correction used to give the design model the same static gain as the evaluation model. Figure 3.16 shows the frequency response of both the evaluation model and the design model. It is clear that the design model captures the low frequency behavior of the evaluation model perfectly.

The control is designed to optimize the performance of the design model and is therefore the same as the LQR controller for the single mass typical section.

$$\mu = -B_c x_m \qquad B_c = \frac{1}{\rho} B_m^T P \qquad P A_m + A_m^T P + Q_m - \frac{1}{\rho} P B_m B_m^T P = 0 \qquad (3.67)$$

The matrix Q_m is selected to penalize displacement of the single mode.

Normally, in a perfectly modelled system, the control penalty, ρ is chosen based upon how much power is available for control, or the the maximum force or stroke the actuators can exert. When there are unmodelled dynamics however, this parameter must be made large enough to keep the control from destablising the system. This implies that if one is not overly concerned with control expense, then a level of control penalty exists which yields an optimal tradeoff between performance in the modelled modes and performance degradation or instability in the unmodelled modes. To explore this effect, ρ is left as a

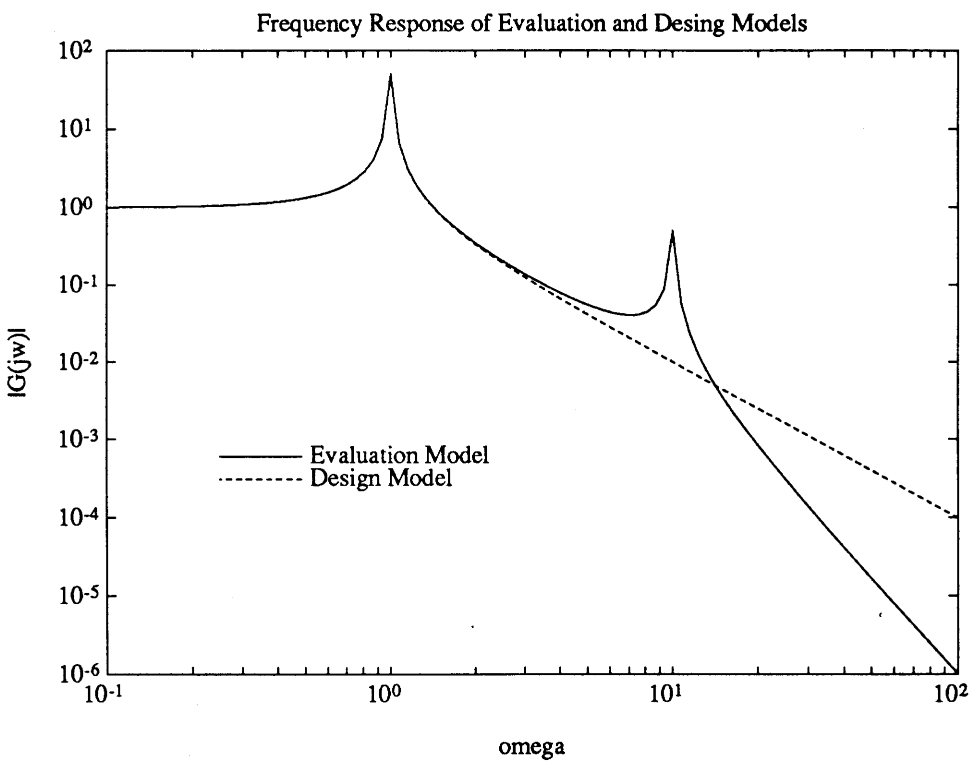

Figure 3.16: Frequency response from control input to sensor output for evaluation and design models of two mass typical section: $\lambda = 10$, $\zeta_2 = .01$.

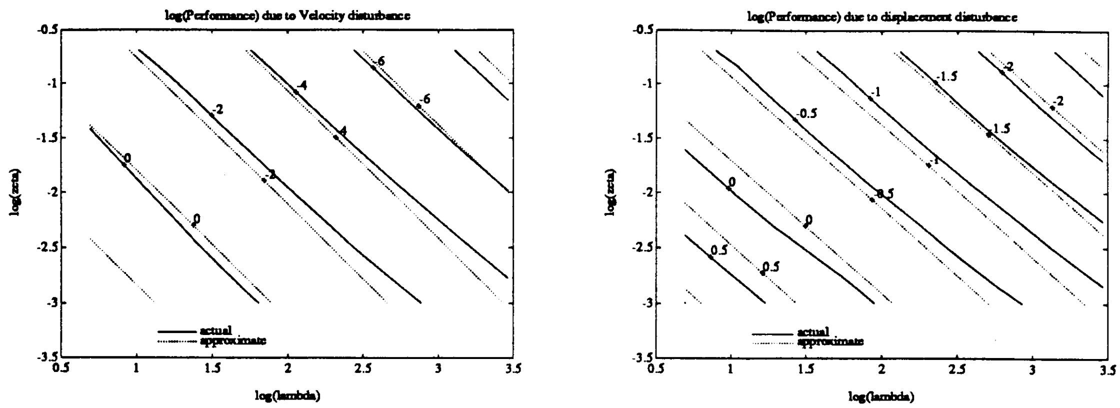

Figure 3.17: Contour plots of log performance of spillover typical section problem.

free parameter and it is adjusted to optimize the performance of the evaluation model with the design model controller in place.

It is desired to study the effect of ζ_2 and λ on this system. From the single mass typical section, the influence of ζ_1 is already known, therefore, setting this parameter to zero will not hurt the analysis. To complete the formulation of the problem, two disturbances will be considered — a velocity disturbance and a displacement disturbance. Furthermore, it will be assumed that the disturbance only has significant energy in the modelled mode. Hence the disturbance matrices are:

$$S_d = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \qquad S_v = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \tag{3.68}$$

This problem is too intractable to solve easily in closed form. It is necessary to resort to numerical methods to determine how the frequency ratio and damping influence the cost when the control penalty is optimized. The equations for algebraically computing the cost and its gradient were given in the previous chapter (Equations 2.28, refeqbb, refeqbc, and 2.58). These equations and the above definitions can be given to a computer program which can pick the optimal ρ given λ and ζ_2. This was done for several values of λ and ζ_2 for both disturbance types. The results are shown in Figure 3.17. The solid lines in both graphs are contour plots of the log of the performance cost plotted over the log of the frequency and damping ratios. The numbers on the contours depict the value of $\log(J)$ for that contour. The straightness and approximately equal spacing of the contours suggest that the performance cost has the following form:

$$\log(J) = a_0 + a_1 \log(\lambda) + a_2 \log(\zeta_2) \tag{3.69}$$

Using the method of least squares, it is possible to estimate the constants a_0, a_1, and a_2. Plugging these into the above equation and exponentiating both sides yields approximate expressions for the displacement and velocity disturbance costs.

$$J_d \approx \frac{.7245}{\zeta^{.651}\lambda^{.786}} \qquad J_v \approx \frac{.4196}{\zeta^{1.93}\lambda^{2.59}} \tag{3.70}$$

These two expressions can be interpreted to give the following design rules:

Design rule 14 *The importance of damping in the unmodelled modes increases as one moves from the open loop to the closed loop problems for a velocity disturbance and decreases (albeit not to zero) for a displacement disturbance.*

Design rule 15 *In going from open to closed loop, the sensitivity of the performance cost to the frequency of an unmodelled mode increases for a velocity disturbance and decreases for a displacement disturbance.*

Again, there are conflicting results for the two disturbance types. The velocity disturbance cost has higher sensitivity to both frequency ratio and the damping ratio than does the displacement disturbance cost. Figure 3.18 helps explain this. It shows the response of the evaluation model to both disturance types for different values of λ and ζ_2. In each case, the control penalty was adjusted to optimize the response to the particular disturbance. Again, the distinguishing feature is the ability of the control to reduce the peak response for the velocity disturbance and not for the displacement disturbance. This favors stronger control in the velocity disturbance and hence increased robustness in the unmodelled mode to allow it.

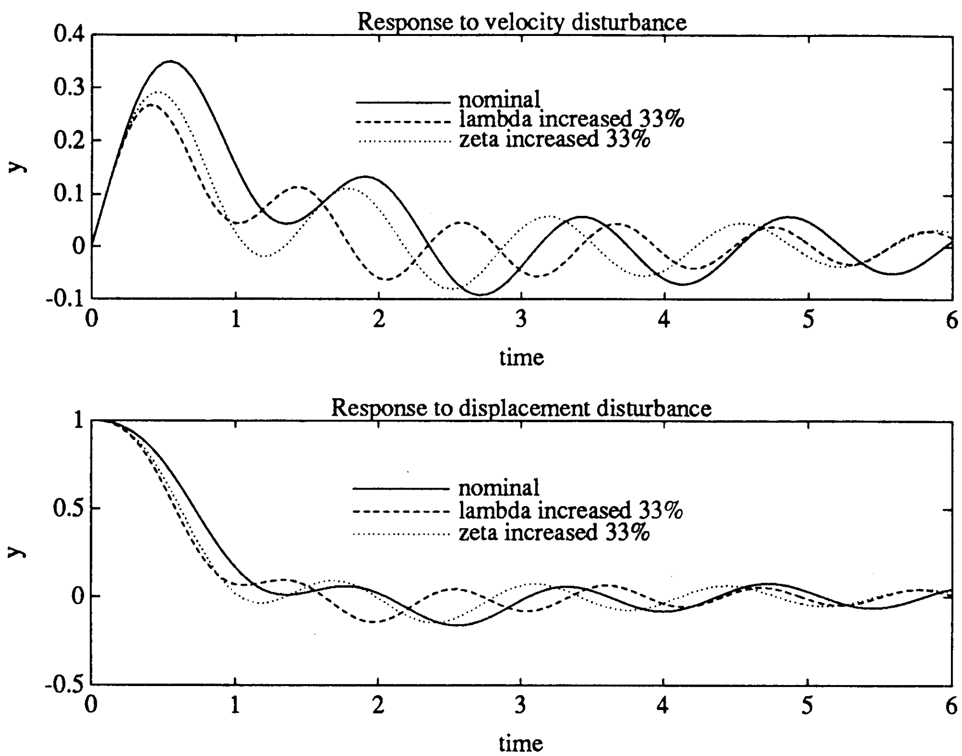

Figure 3.18: Time response for two mass typical section with spillover. Nominal system: $\lambda = 4$, $\zeta_2 = .3$.

Table 3.4: Performance costs for the controlled structure typical sections

Control Type	Disturbance Type	
	Velocity	*Displacement*
Open Loop	$J_v = \frac{\mathcal{G}^2}{4\zeta\omega}(\mathcal{N}_d^2 + \mathcal{N}_v^2)$	$J_d = \frac{\mathcal{G}^2}{4\zeta\omega}(\mathcal{N}_d^2(1+4\zeta^2) + \mathcal{N}_v^2)$
Expensive Control	$J_v = \frac{\mathcal{G}^2\rho}{\mathcal{F}}\sqrt{\mathcal{N}_d^2 + \mathcal{N}_v^2}$	$J_d = \frac{\mathcal{G}^2\rho}{\mathcal{F}}\sqrt{\mathcal{N}_d^2 + \mathcal{N}_v^2}$
Cheap Control Small β_v	$J_v = \frac{\mathcal{G}^2\sqrt{\mathcal{N}_d\rho^3\omega}}{\sqrt{\mathcal{F}^3}}$	$J_d = \frac{\mathcal{G}^2\sqrt{\mathcal{N}_d^3\rho}}{\sqrt{\mathcal{F}\omega}}$
Cheap Control Large β_v	$J_v = \frac{\mathcal{G}^2\mathcal{N}_v\rho}{\mathcal{F}}$	$J_d = \frac{\mathcal{G}^2\mathcal{N}_d\mathcal{N}_v}{\omega}$
Spillover Control	$J_v = \frac{.420}{\zeta^{1.93}\lambda^{2.59}}$	$J_d = \frac{.725}{\zeta^{.651}\lambda^{.786}}$

3.4 Conclusions and Summary

In this chapter, two very simple typical sections were defined. Quantities corresponding to the concepts of controllability ($\mathcal{F}$), disturbability ($\mathcal{G}$), and observability ($\mathcal{N}_d$ and $\mathcal{N}_v$) were developed. These quantities were used to compute the quadratic performance costs in closed form for the cases of open loop control and optimal LQR control for a single mode typical section, and numerically for a two mode typical section which used LQR control based on a reduced design model. The expressions obtained for the different problems are summarized in Table 3.4.

Based on the equations in this table, a series of design rules for controlled structures were stated.

3.4.1 Design Rules for Typical Sections

Open loop design rules

Design rule 1 *Disturbance and output isolation reap similar benefits for all of the problem types.*

Design rule 2 *When a displacement penalty is used in the presence of a displacment disuturbance, there exists a finite amount of damping which will give the optimal performance. Otherwise, increased damping always give the same benefit for all problem types.*

Design rule 3 *Increasing the natural frequency for a constant damping ratio decreases the time constant in the open loop problem. The performance cost is inversely proportional to frequency due to this effect for all of the defined problem types.*

Design rule 4 *In open loop problems which penalize the potential and kinetic energy of the system, increasing the frequency increases the observability in the system. For this effect, the cost is proportional to the square of the frequency.*

Design rule 5 *Increasing the frequency of a mode decreases its disturbability in all of the problem types. In problems which use a displacement disturbance, the cost is inversely proportional to the fourth power of the frequency, and in problems which use a velocity disturbance, the cost is inversely proportional to the square of the frequency due to this effect.*

Closed loop, modelled modes

Design rule 6 *Decreasing the expense of using active control (increasing β) decreases the importance of passive damping to dynamic performance.*

Design rule 7 *For a heavily damped ($\zeta \gg \zeta_{cr}$), lightly controlled ($\beta_d \ll 1$) system, the contribution of the control to improving performance is insignificant when compared to the benefits of the passive damping.*

Design rule 8 *In a lightly damped system ($\zeta \ll \zeta_{cr}$), the benefits of the passive damping and the natural frequency are insignificant when measured relative to the benefits derived from the active control.*

Design rule 9 *For a lightly damped, lightly controlled system, the sensitivities of the performance cost to observability, disturbability, frequency, and controllability are identical for a velocity and a displacement disturbance and a velocity and a displacement state penalty.*

Design rule 10 *The sensitivity to observability of a lightly controlled, lightly damped structure is less than that of the same structure open loop.*

Design rule 11 *For cheap control, the sensitivity of the cost to frequency is postive for a velocity disturbance and negative for a displacement disturbance. In both cases, the magnitude of the sensitivity is less than it would be in the open loop case.*

Design rule 12 *For cheap control, the sensitivity of the performance cost to observability for a displacement disturbance is greater than it was in the expensive control case and less than it was in the expensive control case for a velocity disturbance.*

Design rule 13 *The sensitivity of the performance cost to controllability increases for a velocity disturbance and decreases by for a displacement disturbance in moving from the expensive control case to the cheap control case.*

Closed loop, unmodelled modes

Design rule 14 *The importance of damping in the unmodelled modes increases as one moves from the open loop to the closed loop problems for a velocity disturbance and decreases (albeit not to zero) for a displacement disturbance.*

Design rule 15 *In going from open to closed loop, the sensitivity of the performance cost to the frequency of an unmodelled mode increases for a velocity disturbance and decreases for a displacement disturbance.*

Table 3.5: Ordering of sensitivities in typical section problems: Velocity disturbance and no velocity penalty

Open Loop	Expensive Control	Cheap Control	Unmodelled Mode
2. Disturbance	2. Disturbance	2. Disturbance	1. Frequency
2. Observability	4. Observability	3. Control	2. Disturbance
4. Frequency	4. Control	5. Observability	2. Damping
4. Damping		5. Frequency	

Table 3.6: Ordering of sensitivities in typical section problems: Displacement disturbance and no velocity penalty

Open Loop	Expensive Control	Cheap Control	Unmodelled Mode
2. Disturbance	2. Disturbance	2. Disturbance	2. Disturbance
2. Observability	4. Observability	3. Observability	4. Frequency
4. Frequency	4. Control	5. Control	5. Damping
4. Damping‡		5. Frequency	

‡Finite values of optimal damping

The above design rules are rather cumbersome to use in practice. However, their content can be transformed into a more concise format. The goal of the typical sections was to determine which of the mechanisms for improving the controlled performance of a structure was most likely to drive the design. In the absence of any other information, Table 3.4 and the design rules suggest that a loose ordering can be applied to the modal parameters in the problem, based on the magnitude of the exponent that appears with that parameter in the cost. Tables 3.5 and 3.6 show what this ordering would look like for a displacement penalty with a velocity disturbance and a displacement disturbance, respectively. Each column of these tables corresponds to a different form of control. Within each column, the disturabability, observability, controllability, open loop frequency, and damping ratio are assigned numerical ranks, with lower ranks given to quantities to which the cost will be more sensitive. The rankings remain consistent across columns and between the two tables. (e.g. A quantity with a rank of two in any column of either table will tend to drive the cost as strongly as any other quantity in the tables that also has a rank of two.) This faciliates understanding of how the roles of these quantities change as the control or disturbance type is changed.

These tables must be used carefully, for they lack a great deal of information about any particular controlled structure problem. If it was possible to vary any of the quantities in the above tables with equal ease, and these quantities were all approximately equal initially, then one could pick the quantity which drives the design directly off the tables. For modelled modes, it would be disturbability, and for unmodelled modes, it would be disturbability or frequency, depending on the problem type. Unfortunately, the sensitivity of the quantities in Tables 3.5 and 3.6 to incremental changes in the design are highly dependent on the particular problem. This information must be added to the knowledge contained in the tables to correctly surmise the important parameters in the problem.

A suitable design strategy based on this might go as follows. The designer is given a set of design criterion that the controlled structure must meet. The first step in this problem should be to determine a model for the disturbances. The type of these disturbances, either velocity, or displacment, will determine which of the two tables should be used in the preliminary design. Once this is done, the open-loop column of the appropriate table and the designers knowledge of how changes in the structure influence distubability, observabilility, frequency and damping should be employed in an attempt to meet the design requirements passively. The reason for this is two-fold. First, if the design requirements can be met passively, and the mass of the structure can be kept to within allowable limits, then no control is needed and the preliminary design is complete. If, on the other hand, this cannot be done, the need for the control is justified. The second reason this passive analysis is useful is that if it turns out that the control needs to produce a large amount of performance, then the effects of damping in the modelled modes can be dismissed. This statement is based on Design Rule 8 which noted that the damping dissappeared from the cost once the control level became large enough to produce significant gains in the performance.

The next step of the design process centers around a nominal structure. This is the starting point for the design of the controlled structure. The control necessary for the modelled modes of this system to meet the design requirements should then be computed. The magnitude of the control effort (as reflected in the parameter β_d), determines whether the control for the modelled modes lies in the expensive or cheap control columns of the tables.

At this point, the designer now has a good understanding of how the disturbability, controllability, observability, and frequency in the modelled modes, and disturbability, frequency and damping in the unmodelled modes influences the cost. If similar insight can be obtained for how incremental changes in the structure affect these quantities, then it is possible to formulate a good preliminary design of the controlled structure.

The catch in this problem lies in understanding the relationship between structural parameters and modal quantities. This is not a trivial task. It will be a considerable challenge to generate techniques for this part of the problem that can be applied to a general controlled structure. In the following chapter, a detailed analysis is done for a cantilevered beam. The analysis begins to address how modal quantities can be changed in this particular structure. Some of the examples from Chaper One will be reexamined with the knowledge gained from the typical sections in order to understand further how the sensitivities of modal parameters to design changes can influence the design process.

Chapter 4

Optimization and Analysis of a Beam Model

While the conclusions of Chapter Three are accurate for very simple systems, there are some facets of the controlled structure problem which the typical sections do not capture. Among these are the effect of coupling between two or more modes. Ideally, one would like to find some transformation of the state vector such that the state feedback matrix, state penalty matrix, and disturbance matrix are all diagonalized simultaneously. Also, there should be a transformation of the control vector such that each element of the transformed control vector influences only one mode of the system, and the corresponding control penalty matrix is also diagonal. In that case, the controlled structure problem can be treated as a set of uncoupled modes working in parallel, and the performance cost of the system is just the sum of the performance costs of the modes (Figure 4.1) where the cost for each mode can be obtained directly from the equations developed in Chapter Three. Hence all of the conclusions reached there would be valid.

Unfortunately, it is usually impossible to decouple the system completely. Diagonalizing the state feedback matrix and control penalty matrix will still leave modes coupled through the disturbance, control, and performance output (Figure 4.2). Coupling here means that there are off diagonal in these matrices. The equations for the single mode

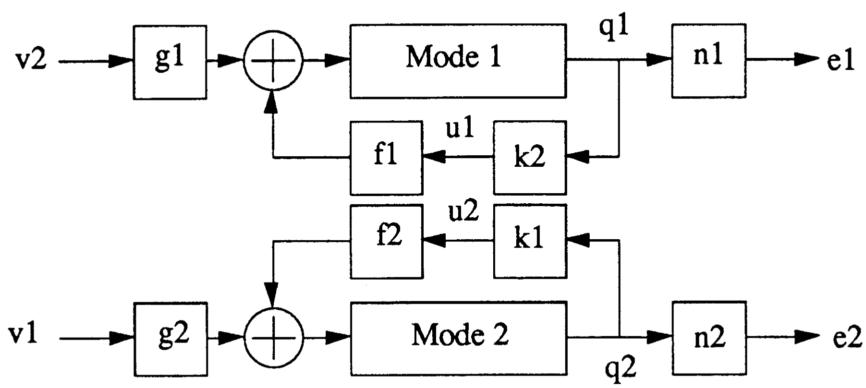

Figure 4.1: System with uncoupled modes

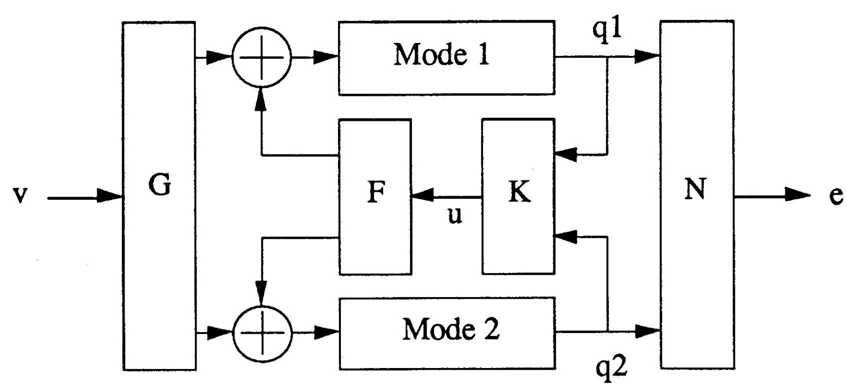

Figure 4.2: System with coupled modes

typical section no longer provide exact solutions to this problem. The hope here is that the typical sections emulate the dominant behavior of a controlled structure, and the coupling effects are less significant. This would justify extrapolating the design rules based on the typical sections to larger systems.

A second part of the controlled structure problem not well handled by the typical sections are the relative costs and the means by which modal parameters can be varied. The typical sections give a good idea of what types of changes should be made to improve the controlled performance, but they do not indicate how these changes should be made in terms of modifying physical structural parameters.

This chapter uses a Bernoulli-Euler beam model to help gain insight into these issues. The first section gives a description of the model and discusses some of the actual issues in the design of flexible space structures which this model confronts. The next section concentrates on the problems surrounding the ideally modeled, undamped system. Vectors which are analogous to the typical section sensitivities to frequency, controllability, disturbability, and observability are developed for this model. It will be shown that the magnitudes of these vectors behave exactly like the typical section sensitivities when only one mode is used in the beam model, but cases differently, in some cases, when more modes are used. The sources of these differences are discussed. Finally, the beam model is optimized for both a displacement and a velocity disturbance, and different values of control penalty. The results of this analysis and the typical section analysis are then used to explain some of the results obtained in the optimization examples presented in Chapter One.

4.1 Description of beam model

Figure 4.3 shows the clamped free beam to be analyzed in the succeeding sections. The use of this model is motivated by the set of problems one might encounter in attempting space-based, optical interferometry (ref [1]). One of the drivers in the design of these

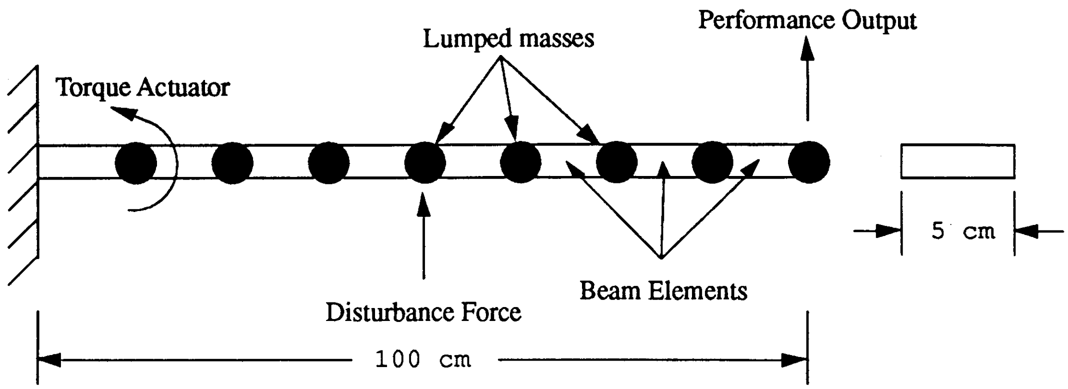

Figure 4.3: Cantilevered aluminim beam model. $E = 73\text{GPa}$, $\rho = 2700 kg/m^3$.

types of structures is the isolation of several siderostats or telescopes from disturbances on the structure. Even tens of nanometers of motion of the structure at these points can seriously degrade the imaging quality of the instrument. A further complication is that the active controller for suppressing disturbances may not be collocated with the mounting point for the siderostats or telescopes. The beam model was selected to emulate both of these effects. The design goal is to suppress motion out at the tip of the beam much as one might want to suppress the motion of the mounting point for a telescope or siderostat out at the end of a space structure. The actuator is a torque actuator located at the free node nearest the clamped end of the beam which emulates the effects of using an "active bay" in a truss structure. Such types of actuators need to be located at the points of highest strain in the structure for them to be effective hence the need to locate the actuator near the root. (Ref. []). The disturbance in this problem is a transverse force introduced at the midpoint of the beam. This position was selected primarily to avoid collocation of the disturbance with the actuator or the performance output. This will make the results of the analysis less ambiguous. Two types of distubances are considered in this problem. The first is the initial displacement of the beam which results from a unit load applied at the midspan (displacement disturbance). The second is an impulsive or stochastic load applied at the same point (velocity disturbance). In the terminology of Chapter Three, the beam with the former disturbance constitutes a Type IV problem, while the beam with the latter disturbance constitutes a Type II problem.

The main structure of the beam is made out of aluminum, and is modeled as eight cubic finite elements. All of the elements have a width of 1 cm, and their thicknesses are included as design variables. Lumped masses are placed at each free node of the beam. The magnitudes of these masses are also included as design parameters. This type of model is very similar to others used in the literature [3].

Before commencing with the analysis, some scaling and constraint issues should be addressed. As mentioned before, the static metric in these problems will always be the total mass of the system. The mass of the system is arbitrarily constrained to be less than or equal to that of a uniform beam with a thickness of 1 cm and no lumped masses . This type of constraint is consistent with the constraints which might be put on an actual spacecraft where mass is limited by lauch capabilities. In all of the cases considered, this constraint was always active at the end of optimization, hence what are really being examined are constant mass designs. The arbitrary selection of the maximum allowable mass is not a cause for concern. Increasing or decreasing this constant is equivalent to changing only the time scale in the problem. (e.g. doubling the size of a uniform beam changes its eigenvalues, but does not change its eigenvectors.)

The temptation with the design parameters is to leave them as the physical thicknesses of the beam elements and the magnitude of the lumped masses. However, in this chapter, a great deal of attention will be paid to the gradient of the cost with respect to these design variables. It is desireable that the magnitude of gradients in this problem somehow reflect the benefit of making changes to the structure. This requires that changes in the structure be quantified. Because the materials budget in this example is on mass, a natural way to quantify a change in the design is by how much mass has to be moved from one point on the structure to another. Therefore the design variables should be rescaled so that an equal change in any design variable implies an equal change in mass.

$$\text{Beam element thickness} \qquad t_i^\star = \frac{t_i w_b h \rho_{al}}{(.01\text{cm}) w_b h \rho_{al}} = \frac{t_i}{.01\text{cm}} \tag{4.1}$$

$$\text{Lumped mass} \qquad m_i^\star = \frac{m_i}{(.01\text{cm}) w_b h \rho_{al}} \tag{4.2}$$

As a simplification, these design variables have also been normalized by the mass of a beam element 1 cm thick.

For all problem types with the single mass typical section, it was found that passive damping was unimportant in problems where the active control played a major role in reducing the cost and robustness was not an issue. Because the analyses here do not consider the effect of unmodelled modes in the cost, not including damping in the model is justified.

This completes the definition of the physical model. In the next section, this model is used to extend the treatment given to the LQR controlled, single mass typical section to more general problems.

4.2 Analysis of LQR controlled, undamped system

The objective of this section is to explore how the controller influences the structural design and draw parallels between results obtained here with those obtained for the typical section. The analysis of the beam will naturally be more complicated than the analysis of the typical sections. The following subsection defines several vectors which are useful in understanding the problem and also the relative importance of disturbability, controllability, observability, and open loop frequency in the optimization solution.

4.2.1 Definition of Subgradients and Subsensitivies

In Chapter Three, quantities were defined for a single mode system which corresponded directly to the concepts of open loop dynamics, disturbability, controllability, and observability. For systems with more modes, it is again possible to define variables which capture these ideas. As in the typical section case, one begins by converting the system to modal coordinates and normalizing the velocity of each mode by its natural frequency.

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix} r \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}\begin{bmatrix} r \\ \dot{r} \end{bmatrix} + \begin{bmatrix} 0 \\ F \end{bmatrix} u \\
\begin{bmatrix} r \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} T & 0 \\ 0 & T\Lambda^{1/2} \end{bmatrix}\begin{bmatrix} q \\ \Lambda^{-1/2}\dot{q} \end{bmatrix} \\
\frac{d}{dt}\begin{bmatrix} q \\ \Lambda^{-1/2}\dot{q} \end{bmatrix} &= \begin{bmatrix} 0 & \Lambda^{1/2} \\ -\Lambda^{1/2} & 0 \end{bmatrix}\begin{bmatrix} q \\ \Lambda^{-1/2}\dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ \Lambda^{-1/2}T^T F \end{bmatrix} u \\
\dot{x}_q &= A_q x_q + B_q u
\end{aligned} \tag{4.3}
$$

The state penalty and disturbance matrices must be transformed to reflect the new coordinates.

$$
\begin{aligned}
Q_q &= \begin{bmatrix} T^T N_d^T N_d T & 0 \\ 0 & \Lambda^{1/2} T^T N_v^T N_v T \Lambda^{1/2} \end{bmatrix} \\
S_q &= \begin{bmatrix} \Lambda^{-1} T^T G_d G_d^T T \Lambda^{-1} & 0 \\ 0 & \Lambda^{-1/2} T^T G_v G_v^T T \Lambda^{-1/2} \end{bmatrix}
\end{aligned} \tag{4.4}
$$

where G_d corresponds to displacement disturbances, and G_v corresponds to velocity disturbances. The matrices A_q, B_q, Q_q, and S_q reflect the effects of open loop dynamics, controllability, observability, and disturbability in the problem. A further feature of this diagonalization is that it makes truncation of the model very easy. Most of the analyses of the beam will use models which use less than eight of the sixteen available modes. The simplifies numeric computations which must be performed.

In Chapter Two, equations for computing the performance cost and gradient for the case of LQR control with a perfectly modeled plant were presented. The gradient in particular is of special interest here. Recall that if P and H are the solutions of the Ricatti and Lyapunov equations:

$$
\begin{aligned}
PA_q + A_q^T P + Q_q - PB_q R^{-1} B_q^T P &= 0 \\
H\left(A_q - B_q R^{-1} B_q^T P\right)^T + \left(A_q - B_q R^{-1} B_q^T P\right) H + S_q &= 0
\end{aligned} \tag{4.5}
$$

then the gradient can be expressed by the sequence of expressions:

$$\frac{\partial J}{\partial \alpha_i} = \text{tr}\left\{ P\frac{\partial S_q}{\partial \alpha_i} + H\left(P\frac{\partial A_q}{\partial \alpha_i} + \frac{\partial A_q^T}{\partial \alpha_i}P + \frac{\partial Q_q}{\partial \alpha_i} - P\frac{\partial}{\partial \alpha_i}\left(B_q R^{-1} B_q^T\right) P \right)\right\} \tag{4.6}$$

where α_i is the i^{th} element of the design vector, α.

This gradient can be divided into four subgradients, each of which represents changes in the cost due solely to one of the four basic mechanisms for improving performance discussed here.

- Frequency Subgradient

$$(\delta J_{fre})_i = \text{tr}\left\{ H\left(P\frac{\partial A_q}{\partial \alpha_i} + \frac{\partial A_q^T}{\partial \alpha_i}P \right)\right\} \tag{4.7}$$

- Disturbability Subgradient

$$(\delta J_{dis})_i = \text{tr}\left\{ P\frac{\partial S_q}{\partial \alpha_i} \right\} \tag{4.8}$$

- Observability Subgradient

$$(\delta J_{obs})_i = \text{tr}\left\{ H\frac{\partial Q_q}{\partial \alpha_i} \right\} \tag{4.9}$$

- Controllability Subgradient

$$(\delta J_{con})_i = -\text{tr}\left\{ HP\frac{\partial}{\partial \alpha_i}\left(B_q R^{-1} B_q^T\right) P \right\} \tag{4.10}$$

The goal of optimization is to minimize the cost function, hence it is the negatives of these subgradients which will be of interest because they indicate directions which lead to reduced cost. The negative of a subgradient represents the best direction (in the design space) to move from a given design if the only changes induced in the system are due to the mechanism with which the subgradient corresponds. For example, if the controllability, observability, and open loop dynamics could somehow be held constant, then the disturbability subgradient would be the best direction to move in reducing the cost.

Often one is interested in a design which lies against one or more design constraints (In fact, this will always be the case here. In the rest of this chapter, all of the designs to be considered will lie against the maximum mass constraint). Unfortunately, the negative of the gradient, or one of its subgradients can cross the constraint and design changes of this nature are not of interest as they lie outside of the design space. For example, it would not be surprising to find that the negative of the gradient almost always suggests design changes which increase the mass of the structure. If the mass of the structure is already at the maximum allowed value, then the design changes that are really of interest are those that conserve mass.

The solution to this problem is to project the gradient and subgradients onto any offending constraints. There are standard techniques for accomplishing this [36] and therefore they will not be discussed here. For the rest of this work, it can be assumed that gradients and subgradients have been projected wherever necessary.

In Chapter Three, a great deal of use was made of the normalized sensitivity of the cost. This is simply the derivative of the cost with respect to some modal parameter normalized by the cost. This normalization will be used again here. The subsensitivity vectors are defined here to be the subgradients of the performance cost normalized by the performance cost.

It is important to note that the subsensitivity vectors contain more information than the sensitivities did in the typical section problems. For example if the system had a single mode then the disturbance subsensitivity vector would be:

$$\frac{\delta J_{dis}}{J} = \frac{\frac{\partial J}{\partial \mathcal{G}}}{J} \frac{\partial \mathcal{G}}{\partial \alpha} \tag{4.11}$$

This vector is made up of two factors. The first captures changes in the cost due to changes in the disturbability, while the second captures changes in the disturbability due to changes in the design vector. Hence, in addition to the effects of coupling, the means by which physical design parameters influence modal parameters will have a significant effect in the analysis. The subsensitivity vectors can now be employed in analyzing the undamped beam model for the case of LQR control based on a perfect model.

4.2.2 Analysis of the Beam Model

Any structural design needs a starting point. In this case, the initial design of the beam is arbitrarily defined to be a uniform beam with all of the element thicknesses set to 1 cm and all of the lumped mass magnitudes set to zero. Notice, that this design has the maxium allowable mass. As stated before, the performance output in this problem is the displacement of the tip, hence N_d is set to penalize the displacement of the tip of the beam, and N_v is set to zero. Because there is only one actuator in this problem, the control penalty matrix is a scalar just as it was in the typical section case.

- Control penalty $\quad R = \rho^2$

Figures 4.4 and 4.5 are plots of the magnitudes of the subsensitivity vectors for the uniform beam when only one mode is used in the model. They show how the magnitudes of the four subsensitivity vectors vary with control penalty for both a displacement and a velocity disturbance. Note that the left sides of the plots correspond to cheap control while the right sides show the expensive control case. The vertical lines in the plots represent the point at which the quantity:

$$\beta_d = \left| \frac{(T_i^T F)(N T_i)}{\rho \omega_i^2} \right| \tag{4.12}$$

for the first mode is equal to unity. In the typical section, this point was defined to separate the cheap control case from the expensive control case. For this model, it is clear that $beta_d = 1$ again marks the point at which the character of this controlled structure begins to change.

Because the system has only one mode and hence is identical to the single mass typical section, it is not surprising that the sensitivities behave exactly as the typical section predicted (Table 3.4). The magnitude of the open loop frequency subsensitivity vector

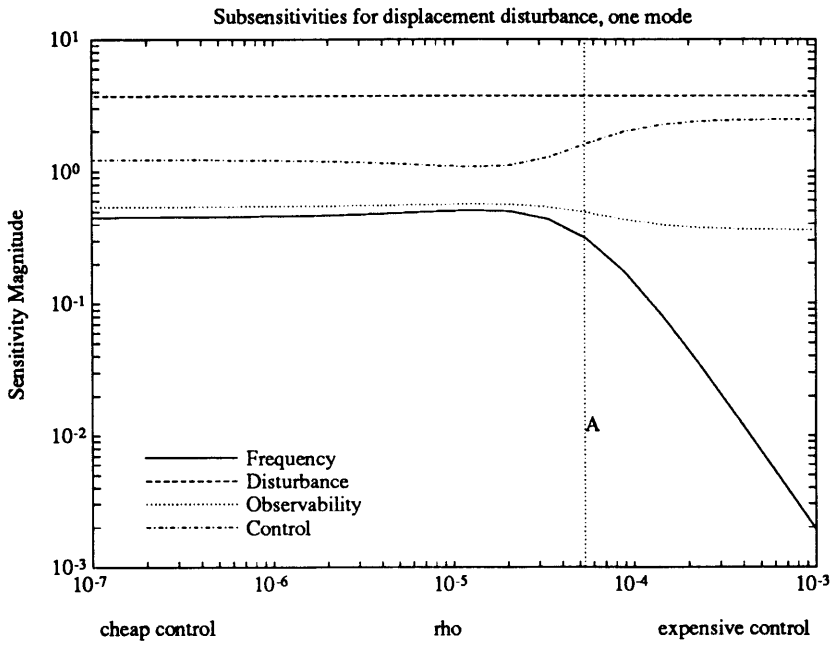

Figure 4.4: Magnitudes of subsensitivity vectors for uniform beam with one mode and displacement disturbance versus control weighting in performance cost: A: β_d for mode 1 = 1

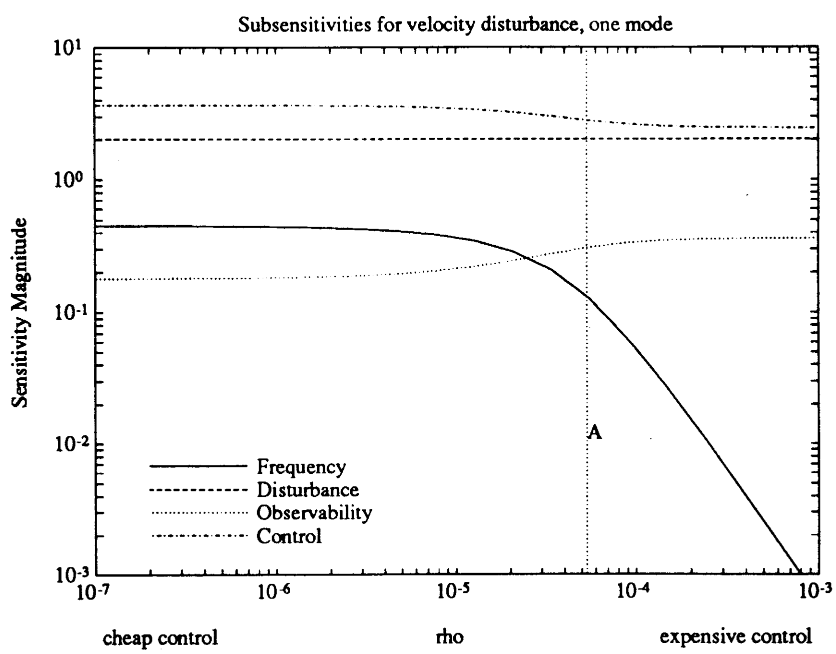

Figure 4.5: Magnitudes of subsensitivity vectors for uniform beam with one mode and velocity disturbance versus control weighting in performance cost: A: β_d for mode 1 = 1

goes to zero for expensive control and the magnitude of the disturbance subsensitivity vector remains independent of the control effort for both disturbance types. For the displacement disturbance, cheap control sees an increase in the observability sensitivity by a factor of two and a decrease in the control sensitivity by a factor of two over expensive control. For the velocity disturbance, the trend is exactly the opposite. All of this was predicted by the design rules.

Things change however when the number of modes in the model is increased. Figures 4.6 and 4.7 show the sensitivities when the number of modes included in the model is increased to four. In addition to the point where β_d crosses unity for the first mode (A), the points at which β_d crosses unity for the next three modes are also shown (B, C, and D, respectively).

The four mode and single mode cases have similar observability and control subsensitivity vectors for expensive control indicating that the addition of modes does not significantly alter the solution for the open-loop or lightly controlled cases. However, there are two major differences which are apparent for the cheap control case. For the displacement disturbance, the magnitude of the control subsensitivity vector for cheap control is lower than expected in the four mode case, but it still decreases from expensive to cheap control is the same. For the velocity disturbance, the magnitude of the observability subsensitivity vector is markedly larger for the four mode case than it was in the single mode case for cheap control. Now, instead of decreasing for the cheap control case, as it did for the single mode case, it increases. A detailed investigation of the source of these differences is beyond the scope of this thesis and will be left for future work. However, it is interesting to note that the largest differences are associated with the controllability and observability and manifest themselves at higher levels of control. This would suggest that the culprit here is coupling of the modes through the controller and not through the disturbance or output observability which are open loop matrices.

Although the magnitudes of the sensitivities leave some questions, their shapes are predictable. Figures 4.8 and 4.9 show graphical depictions of the negatives of the

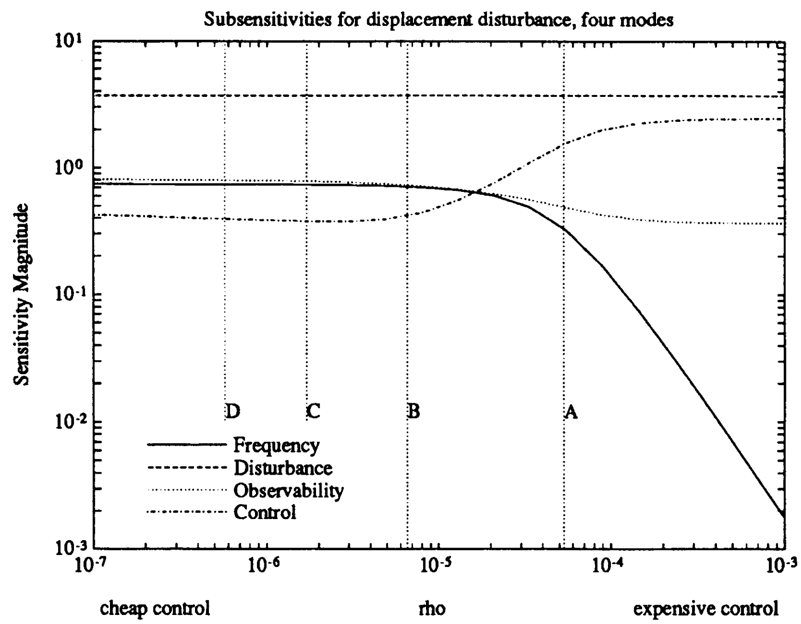

Figure 4.6: Magnitudes of subsensitivity vectors for uniform beam with four modes and displacement disturbance versus control weighting in performance cost: A: β_d for mode 1 = 1, B: β_d for mode 2 = 1, C: β_d for mode 3 = 1, D: β_d for mode 4 = 1

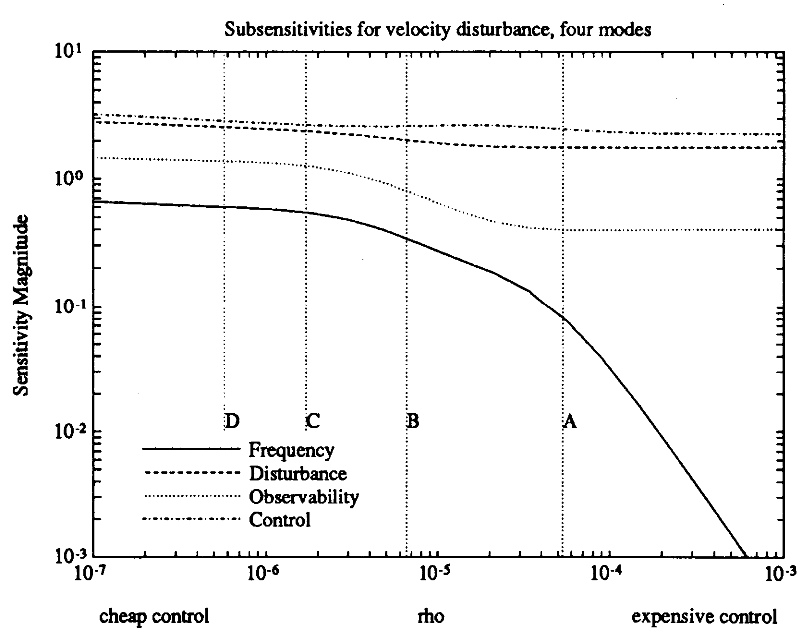

Figure 4.7: Magnitudes of subsensitivity vectors for uniform beam with four modes and velocity disturbance versus control weighting in performance cost: A: β_d for mode 1 = 1, B: β_d for mode 2 = 1, C: β_d for mode 3 = 1, D: β_d for mode 4 = 1

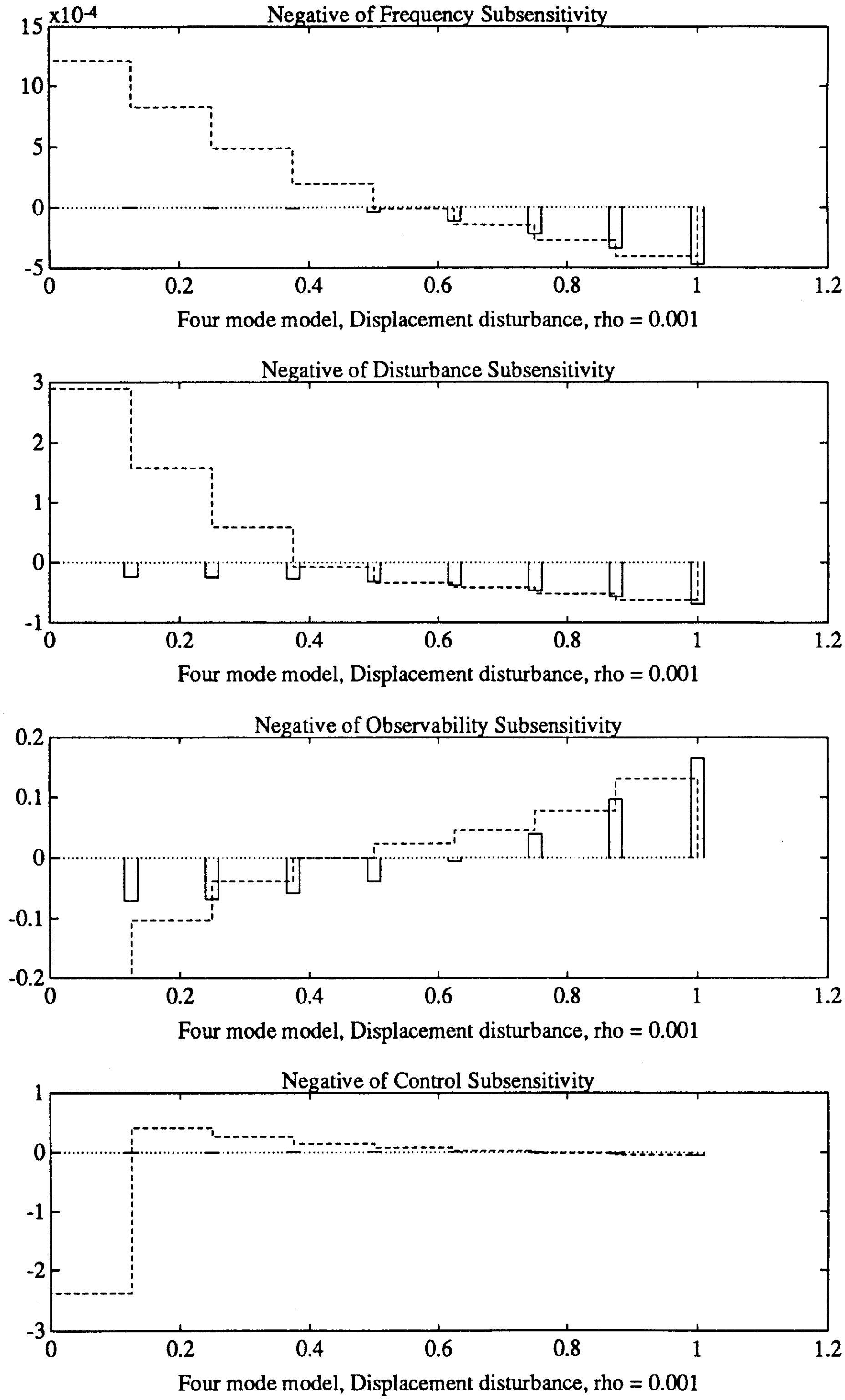

Figure 4.8: Negatives of subsensitivity vectors for model using first four modes of uniform beam and displacement disturbance. Control penalty, $\rho = .001$. All subsensitivity vectors are projected onto constant mass constraint

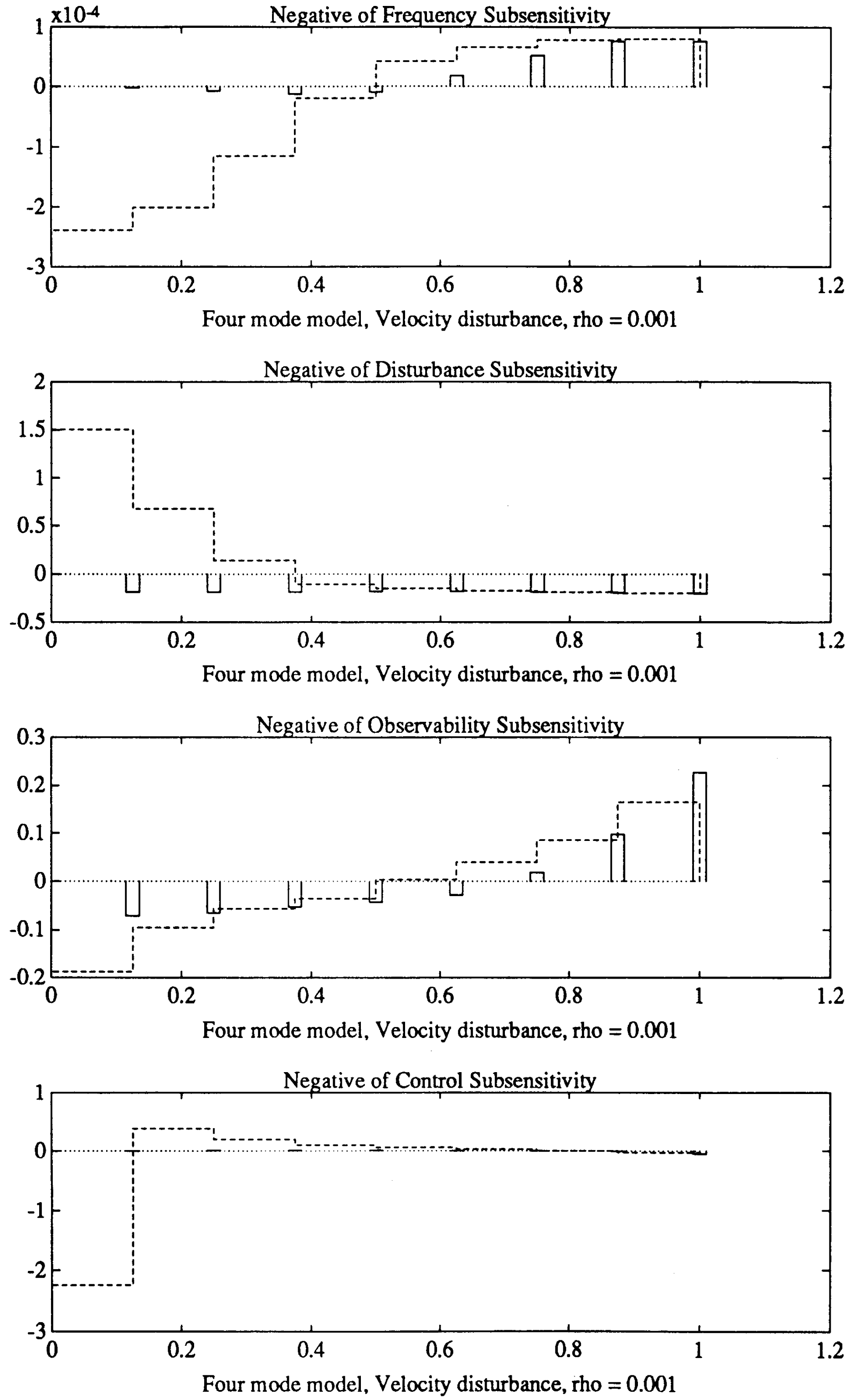

Figure 4.9: Negatives of subsensitivity vectors for model using first four modes of uniform beam and velocity disturbance. Control penalty, $\rho = .001$. All subsensitivity vectors are projected onto constant mass constraint

subsensitivity vectors for both disturbance types at a control penalty level of $\rho = 0.001$. These figures represent directions for which the portions of the cost due to controllability, observability, disturbability, and open loop frequency are decreasing locally. They do not represent optimal designs. Instead they are the best directions on one should move from the uniform beam to reach the optimal design. The dashed lines in the figures represent the changes in the thicknesses of the beam elements, and the narrow bars represent changes in the lumped mass magnitudes. The subsensitivity vectors for lesser values of control penalty were not found to have differing magnitudes as indicated in Figures 4.4, 4.5, 4.6, and 4.7, but similar shapes.

The negative of the frequency subsensitivity vector has two distinctly different characters for the two disturbance types. The changes favored for the displacement disturbance move material from the low strain areas at the tip of the beam to the higher strain areas at the root. Clearly, these changes are aimed at increasing frequency. The changes favored for the velocity disturbance are exactly the opposite, suggesting an attempt to lower the natural frequencies of the beam. Both of these results are in agreement with what was seen in Table 3.4. There, higher frequencies were desired with the displacement disturbance to reduce the time for which the initial error existed, and lower frequencies were desired with the velocity disturbance to give the controller more time to reduce the peak response.

The disturbance subsensitivity vectors are very similar for both of the disturbances. It is interesting to note that the key to reducing the disturbability in this problem lies in stiffening the structure in such a way that it opposes the motion of the point where the disturbance enters the structure at $x = .5$.

The observability subsensitivity vector indicates that mass should be moved out toward the tip of the beam. Notice that this subsensitivity vector makes the greatest use of the lumped masses by placing a large one at the tip of the beam. Concentrations of mass at any point in a system will tend to draw the nodes of the system's eigenvectors toward that point. Thus, this will tend to reduce the motion at that point. This is an interesting result when contrasted with how disturbability is reduced in this system. Basically, there are two funadamental methods on can use to reduce the eigenvectors of a system—by adding mass and by adding stiffness. The chief difference between these two is that adding stiffness to reduce the eigenvectors of a system will also tend to increase its eigenvalues, while adding mass for the same purpose will decrease the eigenvalues. Observability, as defined in Chapter Three, has no dependence on frequency, therefore, the most effiecient means of reducing the eigenvector at the output should be exploited. In this problem, that means adding mass. On the other hand, disturbability is increased by decreasing natural frequency, hence this favors adding stiffness to reduce motion at the disturbance input.

Finally, it is clear that the control subsensitivity vector indicates a substantial reduction of material at the root of the beam would improve controllability. This produces a sort of "hinge" at the root. Physically, the reason for this hinge is to remove stiffness in the system which would directly oppose actuation.

Figures 4.10 and 4.11 show the results of optimizing this system for both disturbance types and for an expensive control case ($\rho = 10^{-3}$) and cheap control case ($\rho = 10^{-7}$). The optimal designs for the displacement disturbance represent a compromise between controllability and disturbability. As predicted by the subsensitivity vector magnitude plots (Figures 4.4 and 4.6), the "hinge" which aids controllability and is present in the expensive control design, disappears in the cheap control case as the magnitude of the control subsensitivity vector decreases relative to the magnitude of the disturbance subsensitivity vector. For the velocity disturbance, this compromise between control and disturbance is also apparent. However, decreasing the control penalty has the opposite effect. Instead of disappearing the hinge becomes much more pronounced. This type of behavior is exactly in accordance with the predictions from the typical sections. For cheap control, the system with the displacement disturbance should be made less disturbable in order to reduce the initial error in the output. On the other hand, the system with the velocity disturbance should be made more controllable to reduce the

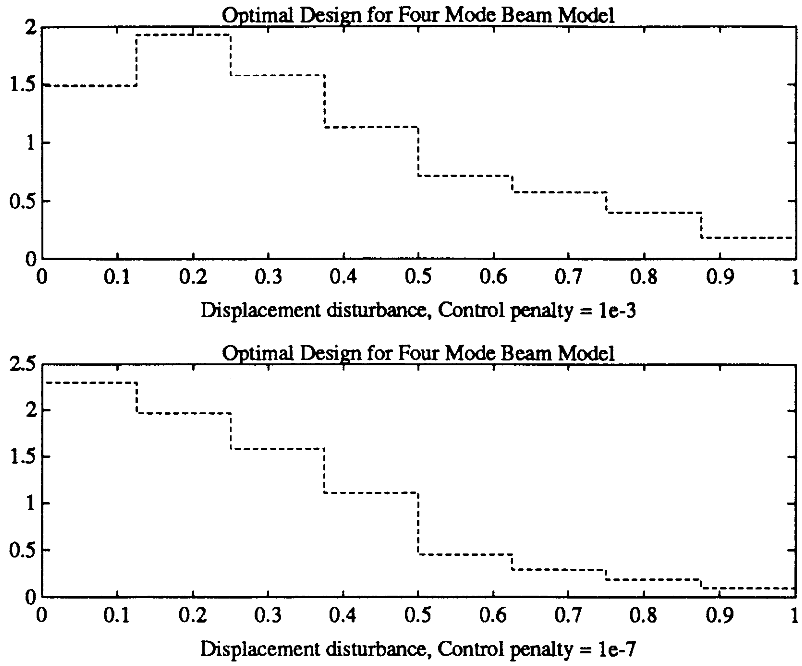

Figure 4.10: Optimal designs for the four mode beam model with a displacement disturbance. Expensive control case: $\rho = .001$. Cheap control case, $\rho = 10^{-7}$.

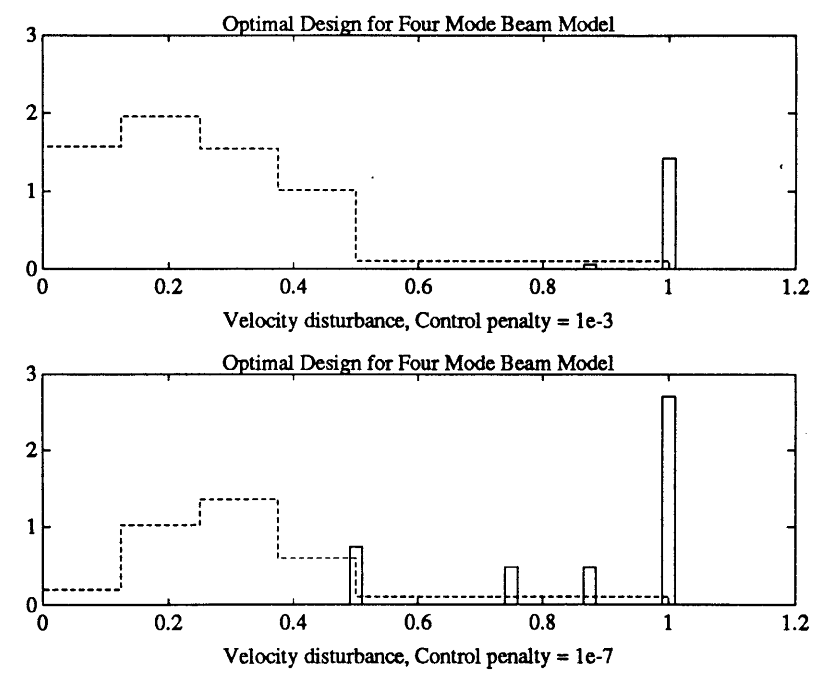

Figure 4.11: Optimal designs for the four mode beam model with a velocity disturbance. Expensive control case: $\rho = .001$. Cheap control case, $\rho = 10^{-7}$.

peak response.

Another striking feature of the velocity disturbance designs is the movement of material from the beam elements outboard of the disturbance to the lumped masses outboard of the disturbance. This is something that one would not expect from looking at the magnitudes of the subsensitivity vectors. In Figure 4.7 the magnitude of the observability subsensitivity vector is substantially lower than the magnitudes of the control and disturbance subsensitivity vectors. The reason that observability plays a role in the optimal design can be accounted for in the shapes of the subsensitivity vectors. The observability sensitivity has a great deal of its magnitude at the tip of the beam, whereas the bulk of the control and disturbance subsensitivity vectors are located further inboard. The reason that this effect appears for the velocity disturbance and not the displacement disturbance is that the disturbance subsensitivity vector (which favors less lumped mass at the tip) has a lesser magnitude for the velocity disturbance.

Finally, it is important to note that the open loop frequencies of the system do not seem to play a role in any of the optimal designs. This does not mean that open loop frequency is unimportant. Instead, it means that the only important effects of open loop frequency in this problem are in how it influences controllabiltiy and observabiliyt. This was predicted by the typical sections for the expensive control case where the open loop dynamics dropped out of the cost entirely. However, what the typical sections could not predict was the small size of the frequency subgradient relative to the overall gradient for the cheap control case. In future work, it might be useful to determine if this is a general feature of the controlled structure problem.

The conclusions from the analysis of the beam model can be summarized as follows:

- A good design of the beam model had to balance three things, controllability, disturbability, and observability. The open loop frequencies were found to have a small influence on the cost for cheap control and to have no influence at all for expensive control.

- All designs showed a compromise between controllability and disturbability. To make the system more controllable, it was found that one should make the root of the beam as flexible as possible, while to make the beam less disturbable, one should make the root as stiff as possible. Clearly, these two design approaches were diametrically opposed.

- The optimal design for the displacement disturbance favored an decrease in disturbability over an increase in controllability as the control penalty was decreased in order to reduce the initial error. On the other hand, the optimal design for the velocity disturbance favored an increase in controllability over a decrease in disturbability as the control penalty was decreased in order to permit the controller to reduce the peak response. Both of these trends were predicted by the typical sections.

- In the velocity disturbance case, the observability subgradient was sufficiently large to drive the design toward reducing the motion of the tip of the beam. This was accomplished simply by placing mass at that location. It was determined that a similar technique could not be used to reduce disturbability due to the sensitivity of disturbability to frequency in this example.

- The effects of coupling in the controlled structure problem are most pronounced at high levels of control and seem to have the greatest effect on the importances ofobservability for the velocity disturbance case and controllability for the displacement disturbance case. In particular, cheap control for the velocity disturbance case favored an increase in the importance of reducing observability which was not predicted by the typical sections.

This beam example represents a single point in the space of optimally controlled structures. In the next section, the results obtained from the typical sections and this example are applied to the examples of Chapter One. This should broaden the understanding of the controlled structure problem.

4.2.3 Application of typical section and beam results to examples of Chapter One

In Chapter One several examples of controlled structure optimization were presented along with an ad hoc interpretation of the results. It is now desirable to examine some of these examples and note how the results of the typical section and beam analyses change those interpretations.

Example 2: Truss example of Miller and Shim

In the truss examples of Miller and Shim, it was suggested that the optimal designs were aimed at reducing the disturbability of the system. One can add a great deal of conviction to this through a process of elimination. First of all, the performance outputs in this problem were potential and kinetic energy. It was shown in Chapter Three (Equation 3.10) that the same modal transformation which diagonalized the open loop plant also diagonalizes the disturbance matrix in this problem with the observability of each mode proportional to its natural frequency. Hence, changes in the observability of this plant will focus on only changing the eigenvalues of the system and not the eigenvectors. It therefore makes sense in this case to lump the benefits of changing observability in this case with those of changing the frequency.

Similarly, there is one actuator at each node of this system. In order to significantly alter the eigenvector portion of the controllability one would have to find design changes which reduce the magnitude of the eigenvectors at all four of these points. This is so difficult to accomplish that it is all but certain that changing the eigenvectors for this purpose will be insignificant. Again, one is left with the frequency portion mechanism, and hence it can be lumped with decreasing frequency to reduce observability and increasing frequency to improve the open loop dynamics.

The net effect of the appearance of frequency in all of these terms is to reduce its importance as a driver to the design. In some cases, higher frequencies would help, (e.g.

for disurbability, and open loop dynamics), but in others, the higher frequencies are a hindrance (e.g. controllability and observability). These benefits tend to cancel each other. On the other hand, changes in the eigenvectors of this system influence only the disturbability. Hence this system is designed to minimize disturbability.

Example 3: Beam Example of Onoda and Haftka

The beam example of Onada and Haftka used a Gaussian White Noise disturbance and a state penalty matrix which penalized displacement of any point on the beam, hence this was a Type II problem. In this example, one had an opportunity to see the designs for both expensive and cheap control. In Chapter One, it was suggested that the expensive control design (Figure 1.8) was an attempt to raise the natural frequencies of the flexible modes in the system, however, it was not clear to what purpose these frequencies were being increased.

The disturbance used in this example excites primarily the rigid body mode in the beam, leaving the flexible modes undisturbed. This is possible because the disturbance is a force distributed over the length of the beam. Unfortunately, the actuator which must be used to control the rigid body mode is not distributed. Any actuation forces to correct rigid body errors will introduce disturbances into the flexible modes. Therefore, a useful way to view this problem is as two separate systems—the rigid body mode of the beam being driven by the Gaussian White Noise, and the flexible modes of the beam being driven by the rigid body control forces (Figure 4.12).

The rigid body system is fairly simple to analyze. The only thing that can be done to change its characteristics is to change its rotary inertia. Increasing this would make the rigid body mode less disturbable and observable, but also less controllable. Looking at the cheap control column of Table 3.5 (β_d is always infinite for a rigid body mode because the frequency is zero), it can be seen that the advantage of decreasing the disturbability and observability together will outweigh the effects of the decreasing controllability. Therefore, a design directed at improving the rigid body characteristics

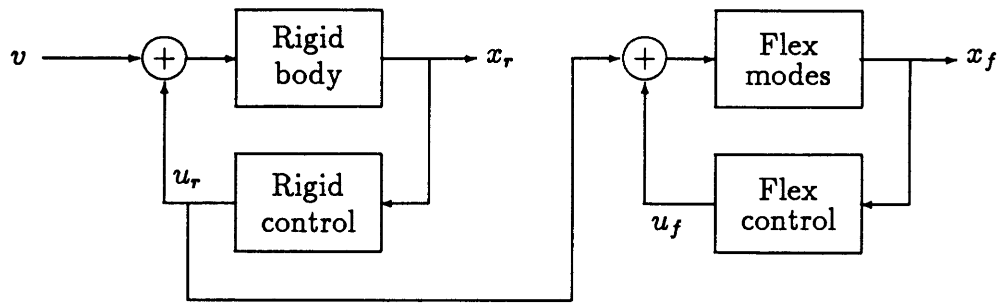

Figure 4.12: Visualization of beam example of Haftka and Onoda as two coupled systems

should increase rotary inertia. This does not appear to be the case in the expensive control solution, and one can conclude that it is the flexible characteristics of the beam which are driving the design.

Observability of the flexible modes is not significant because displacement is penalized everywhere on the beam. Furthermore, open loop frequency (outside of its effects on the disturbability and controllability) does not enter the cost in the expensive control case. Therefore, it is the controllability and observability of the flexible modes which are important in this part of the design.

One can take advantage of the fact that the disturbance and the control for the flexible modes in the system originate from the same actuator. This means that the controllability and disturbability of these modes always have equal values. In Table 3.4 it is apparent that in all cases, the disturbance sensitivity outweighs the control sensitivity. Hence, for expensive control, the performance of the flexible modes is improved by decreasing their controllability/observability. This is clearly the goal of the optimal design for the expensive control case in Figure 1.8.

For cheap control, however, the optimal design was radically different. Looking at Table 3.4 it can be seen that the edge that disturbability has over controllability is diminished by cheap control. The advantage of making the flexible modes less sensitive to the actuator is reduced. The net result is that making the rigid body mode less sensitive

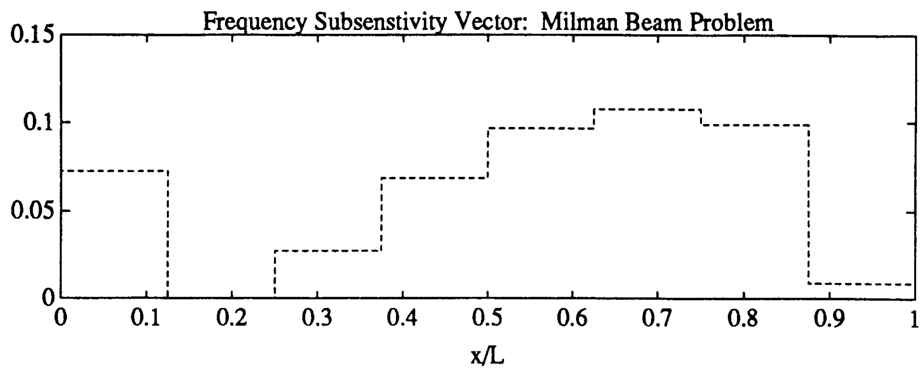

Figure 4.13: Frequency subsenstivity vector for beam problem of Milman *et.al.*: constant mass design.

to the disturbance becomes a viable method for improving the performance. Therefore, the optimal design places the bulk of the mass at the tip of the beam, maximizing its rotary interia.

Example 5: Cantilevered Beam Example of Milman et.al.

In the cantilevered example used by Milman *et. al.*, a rather complicated optimal shape was found for the beam. The source of this shape is very easy to explain in terms of subsensitivity vectors. Figure 4.13 shows the negative of the frequency subsensitivity vector for a uniform cantilevered beam (projected onto constant mass). The similarity between this shape and the optimal shape for the beam strongly suggests that the frequency behavior is the driver in this design. The typical sections tell one that this shape should be an attempt to lower the frequencies of critical modes in the system because this problem uses a velocity disturbance.

This is a surprising result. The performance costs are always least sensitive to changes in open loop frequency. In this example, it must be the case that the senstivities of the frequencies to changes in the structure are very large when compared to similar sensitivites for the controllability, observability, and observability, however, the reasons for this are not clear, and a thorough investigation into the nature of this problem will have to be left for future work.

4.3 Summary

In this chapter, a beam model was presented and analyzed. For the most part, this model verified the design rules for the typical sections with the exception of two important differences noted in the plots of the subsensitivity vector magnitudes. These differences are included in one final design rule:

Design rule 16 *Coupling effects can cause the magnitudes of the sensitivities of the performance cost to observability and controllability to deviate from the exact values predicted by the typical sections.*

The beam examples of Onoda and Haftka and Milman *et. al.* and the truss example of Miller and Shim were examined in light of the results from the typical sections and the beam analysis. In both of these cases, it was found that the behavior of these systems was explainable in terms of the design rules presented here.

Chapter 5

Conclusions

5.1 Summary and Description of Prelimary Design Process

In the design of a controlled structure, there are five basic qualities of the structure which one must consider. These are observability, controllability, disturbability, open loop frequency, and damping ratio. The sensitivities of the quadratic performance costs to these quantities were computed for two typical sections. One of them was an optimally controlled system consisting of a single spring, mass and dashpot. The other was a two mass-spring-dashpot system, where the control was computed using only the first mode (the spillover problem). One can apply a rank ordering to these sensitivities (with the first representing the highest sensitivity). The ordering of these sensitivites for two disturbance types and several types of control are shown in Tables 5.1 and 5.2. Similar results were obtained for the expensive and cheap control cases of an undamped beam model. With the exceptions of those items in the table marked by daggers, the results were identical. Modal coupling in the cheap control case was probably responsible for the discrepancies.

These results suggest a design strategy. To illustrate this strategy, two examples are

Table 5.1: Ordering of sensitivities in typical section problems: Velocity disturbance and no velocity penalty

Open Loop	Expensive Control	Cheap Control	Unmodelled Mode
2. Disturbance	2. Disturbance	2. Disturbance	1. Frequency
2. Observability	4. Observability	3. Control†	2. Disturbance
4. Frequency	4. Control	5. Observability†	2. Damping
4. Damping		5. Frequency	

†Results different in beam model

Table 5.2: Ordering of sensitivities in typical section problems: Displacement disturbance and no velocity penalty

Open Loop	Expensive Control	Cheap Control	Unmodelled Mode
2. Disturbance	2. Disturbance	2. Disturbance	2. Disturbance
2. Observability	4. Observability	3. Observability	4. Frequency
4. Frequency	4. Control	5. Control†	5. Damping
4. Damping‡		5. Frequency	

†Results different in beam model ‡Finite values of optimal damping

used—one is an example of a system which is subjected to velocity disturbances (either stochastic or impulsive forces), the other is subjected to displacement disturbances (initial displacements).

The velocity disturbance example is a space-based, optical interferometer. One of the most demanding missions for future spacecraft is space based imaging interferometry. In a nut shell, interferometry works by combining the starlight coming from several widely spaced telescopes or mirrors located on the spacecraft. Key information is then extracted from the resulting interference pattern. By combining the information from interference patterns taken with the telescopes or mirrors in many different orientations, it is possible to build up an extremely high resolution image. The major problem from the view of the controlled structure designer is that in order for this to work, the light paths from the telescopes or primary mirrors to the combining optics and the positions of the telescopes or primary mirrors in inertial space must be controlled down to a fraction of the wavelength of the light of the incoming image. This ensures interference of the same wave front at the combining optics. For the optical range of light, this means these relative distances must be controlled down to tens of nanometers. In the terminology of this thesis, this is a system with a stochastic force (velocity) disturbance and a displacment penalty. In Chapter Three, these qualities defined a problem type which was designated as Type II.

The displacement disturbance example will be the slew of a robotic arm. The task is to start with the arm initially at rest, and then move the tip of the arm (sometimes called the end effector) to some specified point in space. If the desired final position of the arm is defined to be the zero state, then the objective of the control is to reject an initial displacement. In this problem, only the position of the tip of the arm should be penalized, hence the penalty type is also displacement. In the terminology of Chapter Three, this would be a Type IV problem.

A controlled structure must be designed for two regimes—modelled modes and unmodelled modes. The modelled modes are those modes which are used in the computation of the control law and are generally the lower frequency modes in the structure. The unmodelled modes are all of the other modes which must not be destabilized by the control.

One of the results of the typical section analysis was that the usefulness of damping and active control in the modelled modes of the structure were mutually exclusive. In other words, when the control was large enough to reduce a significant portion of the cost, the damping terms dropped out. The conclusion that can be reached from this is that if control is needed, the presence or absence of damping in the modelled modes is insignificant.

In the unmodelled modes, however, the damping is needed to keep the contributions of the unmodelled modes to the performance cost down. This brings up the first important difference between the displacement and velocity disturbance problems. Although both will require damping in the unmodelled modes, the interferometer will require substantially more damping and higher frequencies in the unmodelled modes than the robotic arm. The reason for this is that the control is emphasized more by a velocity disturbance. This is due to the fact that the control can reduce the peak response of a system due to a velocity disturbance, whereas, the same cannot be said for a displacement disturbance.

The sensitivities of the the robotic arm and the interferometer should behave similarly for the open loop design case and identically for the closed loop design case. The only difference is with the damping. While damping in the interferometer always helps, there is only a finite amount of damping which can be present in the joints of the robotic arm before the damping hinders the control efforts.

In the controlled modes, both systems are most sensitive to disturbances. In the instance of the interferometer, this means that persuing active or passive isolation of disturbance sources might be worthwhile. The same would be true for the robotic arm as well, however because the initial and final states are specified regardless of the structural design, nothing can be done to reduce the disturbance, with the exception of perhaps input command shaping. Hence this will not drive the design of the structure at all.

The observability and controllability share equal roles in both examples at low control levels. However, at high control levels, the sensitivity to control increases and the sensitivity to observability decreases in the interferometer, and the opposite happens in the robotic arm. This implies that at high control levels, effort is better spent designing better actuators in the interferometer than it is in isolating telescope mounting points. For the robotic arm, it would be more important to reduce observability, however, this cannot be done for the same reasons that disturbability could not be reduced. Hence, the controllability of the robotic arm is the first priority. This would favor as light a design as possible to reduce the inertia of the arm, but as internally stiff as possible to keep actuation from disturbing the flexible modes.

Finally, both of the designs are least sensitive to changes in frequency in the modelled modes. This means that frequency will drive the design only when it is impossible or extremely difficult to change controllability, observability, or disturbability. This will be very likely for the robotic arm where it proved impossible to change its disturbability or its observability, and less likely in the interferometer which gave a better selection of design options.

To create preliminary designs for these two examples it will be necessary to understand how specific changes in the structure will influence the modal parameters. One way to gain this insight would be to compute the subsensitivity vectors for the structure for one or more nominal designs.

5.2 Future Work

This thesis has begun to take a detailed look at the design of controlled structures, however, much more needs to be done. The typical sections are very good at capturing the temporal nature of the controlled structure problem but they do not capture the spatial nature of the problem. In other words, the typical sections show which features (controllability, disturbability, observability and open loop frequency) are important in the problem, but they do not suggest what types of physical changes should be made to the structure to influence them. This is one area that future work must concentrate on. It is imperative that one understand the relationship between the structure and these variables in order to execute a good preliminary design. The work done here was basically a confirmation that the results obtained by optimzation were indeed the correct ones. However, more needs to be done to make this approach reliable when the optimal design is not known *a priori*.

A second area to be expanded is the role of damping in the problem. Damping in the modelled modes of a structure was not found to be of significance in this work. On the other hand, it was found through a rather ad hoc treatment of a two mass typical section that damping in the unmodelled modes is extremely important. While the roles of disturbability, controllability, observability and open loop frequency were verified through analysis of a beam model, this was not done for damping. Clearly this should be done, and a more detailed analysis of the use of damping in general should also be done.

Finally, this work considered a very particular class of controller, the Optimal Linear Quadratic Regulator (LQR). In practice this type of control is never implemented because it is impossible to measure the state of a system directly, and its closest cousin, LQG, is is not very robust. The effects of using more realistic types of controllers in the problem needs to be investigated.

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