Regularity and Boundary Variations for the Neumann Problem

by

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Abstract

The formula for the first variation of the Neumann eigenvalues of the Laplace operator under domain perturbation of a smooth submanifold of a Riemannian manifold is calculated. It is shown that this first variation formula remains finite when the smooth submanifold is replaced by a Lipschitz domain in $\boldsymbol{R}^n$. Existence and uniqueness is established for the solution to the inhomogeneous Neumann problem in Lipschitz domains with data in L^p Sobolev spaces.

Thesis Supervisor: David S. Jerison
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Chapter 1

Introduction

1.1 Preview of the Thesis

Given a Riemannian manifold M of dimension n, let $u(\epsilon, \cdot) \in C^\infty(M)$, $\epsilon \in \boldsymbol{R}$, be a smooth one-parameter family of Neumann eigenfunctions of the Laplacian satisfying

$$\begin{aligned} \Delta u(\epsilon, \cdot) + \lambda(\epsilon) u(\epsilon, \cdot) &= 0 \text{ on } \Omega_\epsilon && (1.1) \\ \frac{\partial u(\epsilon, \cdot)}{\partial \nu(\epsilon, \cdot)} &= 0 \text{ on } \partial\Omega_\epsilon, && (1.2) \end{aligned}$$

with $\overline{\Omega}_\epsilon$ a corresponding collection of smoothly-varying n-dimensional submanifolds with boundary and $\nu(\epsilon, \cdot)$ the normal vector field to $\partial\Omega_\epsilon$ so that the boundary condition in (1.2) is the normal-derivative-0 condition. The function $\epsilon \mapsto \lambda(\epsilon)$ is differentiable, and our primary enterprise in Chapter 2 of this dissertation is to calculate its derivative, that is, to calculate the first variation of the Neumann eigenvalues.

Denote by $\nabla_{\partial\Omega}$ the Riemannian gradient on $\partial\Omega$ and by $\mathrm{v}(p)$ the normal variation or speed in the normal direction at $p \in \partial\Omega$ of the perturbation (see equation (2.15)). Writing $\lambda = \lambda(0)$ and $u = u(0, \cdot)$, we prove the following theorem.

Theorem A.(also Theorem 2.3) *The first variation of the Neumann eigenvalues of the Laplacian is given by*

$$\lambda'(0) = \int_{\partial\Omega} (|\nabla_{\partial\Omega} u|^2 - \lambda u^2) \mathrm{v} \, dA.$$

One of the key ingredients of the proof (see sec. 2.2) of the formula in Theorem A is the existence of a smooth curve $\gamma_p(\epsilon)$ for each fixed $p \in \partial\Omega$ whose derivative in the normal direction to Ω at p is the normal variation there and which satisfies $\gamma_p(\epsilon) \in \partial\Omega$ for all ϵ near 0 (Proposition 2.2). This is used to differentiate the perturbed Neumann conditions (1.2) with respect to ϵ in order to obtain Neumann boundary conditions on $\partial\Omega$ for the derivative $\partial_\epsilon u(0,\cdot)$ of the eigenfunction (equation (2.31)). The calculation of these Neumann conditions is arguably the cornerstone of the argument. Another highlight of the proof, however, is the utilization of a simple relationship between the Laplace operator on M and that on $\partial\Omega$ when Neumann conditions are satisfied (see (2.38)). Section 2.2 also includes a numerical verification of our eigenvalue variation formula for the example of the unit disk in the plane perturbed at constant speed (Example 2.4).

We conclude Chapter 2 with some remarks pertaining to the possibility of extending the first variation formula to bounded Lipschitz domains Ω in $\boldsymbol{R}^n$. We use a regularity theorem of Jerison-Kenig [13] for the homogeneous Neumann problem to show that the trace of ∇u_k lies in $L^2(\partial\Omega)$. The conclusion is that the formula is well-defined for this nonsmooth case.

A regularity theory for the inhomogeneous Laplace equation with Neumann boundary boundary conditions

$$\Delta w = F \text{ on } \Omega \tag{1.3}$$
$$\frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega \tag{1.4}$$

on bounded (connected) Lipschitz domains Ω is the principal objective of Chapter 3. One relevant physical model – there are many – for this boundary value problem is that of the steady-state temperature distribution within a perfectly insulated domain in space in the presence of both generation and dissipation of heat.

Recall that Green's identities permit us to restate the classical problem (1.3)-(1.4) in a much weaker fashion. Given a bounded linear functional $F \in (L_1^2(\Omega))^*_{1\perp}$ acting on $L_1^2(\Omega)$, with

$$(L_1^2(\Omega))^*_{1\perp} = \{F \in (L_1^2(\Omega))^* \,|\, F(1) = 0\},$$

we say that $w \in L_1^2(\Omega)$ is a weak solution of the inhomogeneous Neumann problem with data F if

$$F(v) = -\int_\Omega < \nabla w, \nabla v > dV \tag{1.5}$$

for all $v \in L^2_1(\Omega)$. By the well-known Hilbert space theoretic methods of Lax-Milgram, a solution to this problem always exists, with resulting regularity estimate

$$||w||_{L^2_1(\Omega)} \leq C||F||_{(L^2_1(\Omega))^*}. \tag{1.6}$$

Moreover, it is necessarily unique, modulo constants.

For $1 < p < \infty$ and $-\infty < \alpha < \infty$ let L^p_α denote the Sobolev (potential) spaces on $\boldsymbol{R}^n$, and, for $\alpha \geq 0$, denote by $L^p_\alpha(\Omega)$ the Sobolev spaces on Ω. Also, write χ_Ω for the truncation operator (corresponding to Ω) defined initially on functions on $\boldsymbol{R}^n$ as multiplication by the indicator function of Ω, and, for $\alpha \geq 0$, let $E_\Omega : L^p_\alpha(\Omega) \to L^p_\alpha$ denote Stein's extension operator (see section 3.1 within the main body of the thesis for more precise definitions of these operators and their boundedness properties). Finally, for $\alpha \geq 0$, set

$$(L^p_\alpha(\Omega))^*_{1^\perp} \equiv \{F \in (L^p_\alpha(\Omega))^* \,|\, F(1) = 0\}.$$

The central theorem of Chapter 3 is a very broad extension of the estimate (1.6), and it may be stated in the following way.

Theorem B.(also Theorem 3.6) *There is $\epsilon, 0 < \epsilon \leq 1$, depending only on the Lipschitz constant of Ω, such that, given $1 < p < \infty$ and $F \in (L^q_{2-\alpha}(\Omega))^*_{1^\perp}$, $1/p + 1/q = 1$, there exists a unique (modulo constants) $w \in L^p_\alpha(\Omega)$ satisfying*

$$F(v) = - < \chi_\Omega \nabla(E_\Omega(w)), \nabla(E_\Omega(v)) > \tag{1.7}$$

for all $v \in L^q_{2-\alpha}(\Omega)$ as well as the estimate

$$||w||_{L^p_\alpha(\Omega)} \leq C||F||_{(L^q_{2-\alpha}(\Omega))^*}, \tag{1.8}$$

provided one of the following holds

$$(a) \quad p_0 < p < p_0' \text{ and } \frac{1}{p} < \alpha < 1 + \frac{1}{p}$$

$$(b) \quad 1 < p \leq p_0 \text{ and } 3/p - 1 - \epsilon < \alpha < 1 + \frac{1}{p}$$

$$(c) \quad p_0' \leq p < \infty \text{ and } \frac{1}{p} < \alpha < 3/p + \epsilon$$

wherein $1/p_0 = 1/2 + \epsilon/2$ *and* $1/p_0' = 1/2 - \epsilon/2$.

When $F \in L^p(\Omega)$, $w \in L_1^p(\Omega)$, and $v \in L_1^q(\Omega)$, the seemingly inscrutable Neumann conditions in (1.7) become simply

$$\int_\Omega Fv\,dV = -\int_\Omega < \nabla w, \nabla v > dV.$$

The range for α and p for which this theorem holds is an open hexagon in $(\alpha, 1/p)$-space with vertices $(0,0), (1/p_0, 1/p_0), (2-\epsilon, 1), (2,1), (2-1/p_0, 1/p_0')$, and $(\epsilon, 0)$. Moreover, it is identical to the range for which estimates are true for the inhomogeneous Dirichlet problem (Jerison-Kenig [12]). In their paper, Jerison and Kenig prove that this range is in fact the best possible for the inhomogeneous Dirichlet problem. The same optimality conclusion is conjectured for the Neumann problem though this will not be proved in this thesis.

Let us now outline the results in Chapter 3 and how they will be assembled to prove Theorem B. In section 3.1, we restate Theorem B, providing the necessary background detail.

Our primary mission in section 3.2 is to appropriately define the boundary function spaces we will require in the following two sections. Of paramount importance here is that for the positive Sobolev and Besov boundary spaces Lipschitz surface density is not present as a weight factor when the defining norm for these spaces is written in terms of the norm for the corresponding space on $\boldsymbol{R}^{n-1}$, whereas for the negative range of spaces it is. Another way of saying this is that if $\partial\Omega$ is the domain above a Lipschitz graph ψ on $\boldsymbol{R}^{n-1}$ then the boundary Sobolev space $L_s^p(\partial\Omega), s \geq 0$, is identical to the space $L_s^p(\boldsymbol{R}^{n-1})$, but, for the negative range, if $g \in L_{-s}^p(\partial\Omega), s \geq 0$, then we have instead $g\omega \in L_{-s}^p(\ \boldsymbol{R}^{n-1})$, where $\omega = \sqrt{1 + |\nabla\psi|^2}$ denotes Lipschitz hypersurface density. This difference makes these function spaces amenable to interpolation. Also included in section 3.2 is a statement of Jerison and Kenig's estimates for the inhomogeneous Dirichlet problem.

In sec 3.3 we establish estimates for the operator sending Neumann boundary values to the Dirichlet boundary values of the harmonic function exhibiting those Neumann boundary conditions, loosely speaking the inverse of the operator often known as the Calderón operator and which we denote by Υ. These estimates, given in Theorem 3.22, are certainly the key step in the proof of Theorem 3.6. We begin the proof of Theorem 3.22 with estimates for the homogeneous Neumann problem proved by Dahlberg and Kenig [5](see Theorem 3.9 of this thesis). Using these estimates it is possible to establish an $L_1^p(\partial\Omega)$-estimate on Υ in the range $1 < p < 2+\delta$ for some $\delta > 0$ (Theorem 3.12 and Corollary 3.13). The idea is then to take the adjoint of the operator Υ to obtain estimates on the dual spaces. We also appeal to a theorem of Brown [1](Theorem 3.17) giving estimates for the Neumann problem in Hardy spaces of order less than 1 on the boundary to obtain end-point results for the $L^1(\partial\Omega)$-end of the range. Since the duals of the Hardy classes of order less than 1 are the Hölder classes we get L^∞ estimates as well.

The proof of Theorem 3.22 is completed using interpolation. Since we need to interpolate between Sobolev and Hardy spaces on the boundary we seek to apply the interpolation theory for the inhomogeneous Triebel-Lizorkin scale of spaces $F_p^{\alpha,q}$ on $\boldsymbol{R}^n$, which includes both the Sobolev and local Hardy classes based on $\boldsymbol{R}^n$. This application is possible because the fashion in which Brown's boundary Hardy spaces are defined is consistent with the way in which we define boundary function spaces in section 3.2. Once again this point is most easily understood when $\partial\Omega$ is the boundary of the domain above a Lipschitz graph. In this case if f is a member of Brown's space $H_1^{p'}(\partial\Omega)$, with p' a real number less than but near 1, then $\nabla f \in H^{p'}(\boldsymbol{R}^{n-1}, dx)$, with dx denoting Lebesgue measure on $\boldsymbol{R}^{n-1}$, and this is just the usual Hardy space of order p'. On the other hand, $H^{p'}(\partial\Omega)$ is in fact the space $H^{p'}(\boldsymbol{R}^{n-1}, d\sigma)$, where $d\sigma$ denotes Lipschitz surface measure. Thus we can reduce both interpolation formulas between $H_1^{p'}(\partial\Omega)$ and the positive boundary spaces of section 3.2 and interpolation formulas between $H^{p'}(\partial\Omega)$ and the negative boundary spaces to formulas for the corresponding function spaces on euclidean space via the usual retraction arguments (Theorem 3.20).

The actual proof proper of the existence assertion in Theorem B is confronted in section 3.4, and the proof draws upon a classical strategy. Let us very briefly describe this strategy in a formal way here. Given $F \in C^\infty(\overline{\Omega})$ with $\int_\Omega F\, dV = 0$ extend F by

0 to all of $\boldsymbol{R}^n$. Then write the proposed solution w as

$$w = (F * N)|\Omega - u, \tag{1.9}$$

where N denotes the Newtonian potential on $\boldsymbol{R}^n$ so that $\Delta(F * N) = F$, and u satisfies the homogeneous Neumann problem

$$\begin{aligned} \Delta u &= 0 \qquad\qquad \text{on}\, \Omega & (1.10) \\ \frac{\partial u}{\partial n} &= \frac{\partial(F * N)}{\partial n} \,\text{on}\, \partial\Omega. & (1.11) \end{aligned}$$

Thus from (1.10)-(1.11) it is clear that in order to obtain estimates for the inhomogeneous Neumann problem it is necessary to combine the estimates for the inhomogeneous Dirichlet problem and those for the inverse Calderón operator in an appropriate way, and this is done in detail in sec. 3.4.

Finally, we establish uniqueness of the solution w to the inhomogeneous Neumann problem in section 3.5. When w lies in $L^2(\Omega)$ uniqueness is quite straightforward and is a consequence of the L^2 expansion of w as an infinite series of eigenfunctions. When w is not a member of L^2, we call upon L^∞ estimates for the Neumann eigenfunctions of the Laplacian on Ω which follow from our own existence results to recast the problem as one involving solutions to the heat equation. Invoking a theorem of Beurling and Deny on positivity of the heat kernel, we are able to prove the convergence of the heat semigroup as $t \to 0$ and deduce uniqueness in $L^p, 1 < p < 2$.

1.2 Preliminary Definition and Notations

Definition of a Bounded Lipschitz Domain. We say that a bounded, open, connected subset $\Omega \subset \boldsymbol{R}^n$ is a bounded Lipschitz domain if to each $P \in \partial\Omega$ may be associated a Lipschitz function $\psi : \boldsymbol{R}^{n-1} \to \boldsymbol{R}$ with Lipschtiz constant less than some fixed $m = m(\Omega) > 0$ (called the **Lipschitz constant** of Ω), an isometry $\phi : \boldsymbol{R}^n \to \boldsymbol{R}^n$, and a radius $r > 0$ for which

$$\begin{aligned} \phi(B(P,r) \cap \Omega) &= \{(x', x_n) \in \boldsymbol{R}^n | x_n > \psi(x'), |(x', x_n)| \leq r\} \\ \phi(B(P,r) \cap \partial\Omega) &= \{(x', x_n) \in \boldsymbol{R}^n | x_n = \psi(x'), |(x', x_n)| \leq r\}, \end{aligned}$$

where $B(P,r)$ is the closed ball in $\boldsymbol{R}^n$ of radius r about P.

In Chapter 3, Lipschitz surface measure on $\partial\Omega$ will be denoted by $d\sigma$ and the usual n-dimensional Lebesgue (volume) measure in $\boldsymbol{R}^n$by dV.

Also, we write

$$C^\infty(\partial\Omega) = \{f|\partial\Omega \mid f \in C^\infty(\boldsymbol{R}^n)\}.$$

Chapter 2

The Formula for the First Variation of the Neumann Eigenvalues

2.1 Riemannian Geometric Preliminaries

In this section we introduce certain notations and concepts from Riemannian geometry which will be required in the next section. Useful treatments of the background behind much of this material may be found in Gallot, Hulin, Lafontaine [9] or Chavel [4]. We conclude the present section with some remarks pertaining to domain perturbations in Riemannian manifolds.

Let M be an oriented,n-dimensional, connected C^∞ Riemannian manifold with Riemannian metric $<,>_{T_pM}$ on each tangent space M_p to M at p. If $(U, x_1, ..., x_n)$ is a local coordinate system at p and $\partial_1, ..., \partial_n$ denote the corresponding coordinate vector fields, then we will write $g_{jk}(q) =< \partial_j|_q, \partial_k|_q >_{T_qM}$, $q \in U$, for the entries of the Riemannian metric in this coordinate system. The inverse of the matrix $(g_{jk}(q))$ will be denoted by $(g^{jk}(q))$.

Write $X(M)$ for the set of smooth vector fields on M. Then $D_Z Y \in X(M)$ will signify the **Levi-Civita connection** on M applied to the vector fields $Z, Y \in X(M)$.

We have, locally,

$$D_Z Y = \sum_k (Z(y_k) + \sum_{i,j} z_i y_j \Gamma^k_{ij})\partial_k \tag{2.1}$$

for vector fields $Z = \Sigma z_i \partial_i$ and $Y = \Sigma y_i \partial_i$, wherein the $\Gamma^k_{ij} \in C^\infty(U)$ are the **Christoffel symbols** defined by setting $D_{\partial_j}\partial_i = \Sigma \Gamma^k_{ij}\partial_k$. We have

$$\Gamma^k_{ij} = \frac{1}{2}\sum_m (\partial_i g_{jm} + \partial_j g_{mi} - \partial_m g_{ij})g^{mk}. \tag{2.2}$$

Since it is clear from (3.1) that $D_Z Y(p)$ only depends on the value $Z(p)$ of Z at p we will sometimes replace the vector field Z by vectors $w \in T_pM$.

We denote by $\frac{DV}{dt}$ the **covariant derivative** (with respect to the Levi-Civita connection) of the vector field V along the differentiable curve $\gamma : [a, b] \to M$. Its local expression is given by

$$\frac{DV}{dt} = \sum_k \left[\frac{dv^k}{dt} + \sum_{i,j} v^j \frac{dx_i}{dt}\Gamma^k_{ij}\right]\partial_k, \tag{2.3}$$

wherein $V(t) = V(\gamma(t)) = \Sigma_k v^k(t)\partial_k|_{\gamma(t)}$, $t \in [a, b]$.

Given a function $f \in C^\infty(M)$, we define its **gradient** to be the unique vector field ∇f on M for which

$$\langle \nabla f(p), \xi\rangle = \xi(f) \tag{2.4}$$

for all $\xi \in T_pM$. In local coordinates this becomes

$$\nabla f = \Sigma_{k,l}(g^{kl}\partial_l f)\partial_k. \tag{2.5}$$

The **divergence** of a vector field Y, denoted div Y, is the real-valued function on M given by

$$(\operatorname{div} Y)(p) \;=\; \operatorname{trace}(Z \mapsto D_Z Y). \tag{2.6}$$

The **Laplacian** of a function $f \in C^\infty(M)$ is the smooth function

$$\Delta f = \operatorname{div}\nabla f. \tag{2.7}$$

It is a simple matter to show that

$$\frac{1}{2}\Delta(u^2) = |\nabla u|^2 + u\Delta u, \tag{2.8}$$

wherein, on M, $|\nabla u|^2 \equiv <\nabla u, \nabla u>$.

Finally, we define the **Hessian**, $\nabla^2 f(p)$, at $p \in M$ of a function $f \in C^\infty(M)$ to be the symmetric bilinear form on T_pM given by

$$\nabla^2 f(p)(Y(p), \mathrm{w}) = < D_Y(\nabla f)(p), \mathrm{w} >_{T_pM},$$

for all $Y \in \mathrm{X}(M)$ and $\mathrm{w} \in T_pM$. It is not difficult to show – and in any case is well-known – that

$$\Delta f = \operatorname{trace} \nabla^2 f. \tag{2.9}$$

Now let $\overline{\Omega}$ be an oriented, compact, connected, smooth submanifold of M of dimension n with $(n-1)$-dimensional boundary $\partial\Omega$ and open interior Ω. The Riemannian metrics associated to these submanifolds will be those induced from that on M. To avoid confusion, we will in general sub- or superscript the operators defined above with our notation for the manifold on which we wish them to be considered, as in "$D_Z^{\partial\Omega}Y$" for the Levi-Civita connection with respect to $\partial\Omega$ applied to the vector fields Z and Y. However, absence of such a subscript will always indicate that the subscript "M" is implied.

We write $\nu(p)$ for the **outward unit normal vector field** to $\partial\Omega$ at $p \in \partial\Omega$.

Proposition 2.1. *Let $p \in \partial\Omega$ and $w \in T_p(\partial\Omega)$ and assume that $\overline{Y} \in X(M)$ is a vector field whose restriction Y to $\partial\Omega$ is a member of $X(\partial\Omega)$. Then*

$$(D_w^M\overline{Y})(p) = (D_w^{\partial\Omega}Y)(p) + < (D_w^M\overline{Y})(p), \nu(p) >_{T_pM} \nu(p). \tag{2.10}$$

Rather similarly, for $f \in C^\infty(\overline{\Omega})$,

$$\nabla_M f(p) = \nabla_{\partial\Omega}(f|\partial\Omega)(p) + < \nabla_M f(p), \nu(p) > \nu(p). \tag{2.11}$$

Proof. To prove (2.10) one selects a coordinate system $(U, x_1, ..., x_n)$ about p considered as a member of M but adapted to $\partial\Omega$ so that

$$U \cap \partial\Omega = \{q \in U | x_n(q) = 0\},$$

and $(U, x_1, ..., x_{n-1})$ is a coordinate system about p considered as a member of $\partial\Omega$. By a linear-change-of-coordinates if necessary, this coordinate system also satisfies $\partial_n|_p = \nu(p)$.

Now write $w = \sum_{i=1}^{n-1} w_i \partial_i|_p$, $\overline{Y} = \sum_{i=1}^{n} \overline{y}_i \partial_i$, and $Y = \sum_{i=1}^{n-1} y_i \partial_i$ (Of course we have $\overline{y}_i|(U \cap \partial\Omega) = y_i$.) and consider that (2.1) implies that

$$
\begin{aligned}
(D_w^M \overline{Y})(p) &= \sum_{k=1}^{n} [w(\overline{y}_k) + \sum_{i,j=1}^{n} w_i \overline{y}_j(p) \Gamma_{M\,ij}^k(p)] \partial_k|_p \\
&= \sum_{k=1}^{n-1} [w(y_k) + \sum_{i,j=1}^{n-1} w_i y_j(p) \Gamma_{M\,ij}^k(p)] \partial_k|_p \\
&+ [w(\overline{y}_n) + \sum_{i,j=1}^{n} w_i \overline{y}_j(p) \Gamma_{M\,ij}^n(p)] \partial_n|_p \\
&= \sum_{k=1}^{n-1} [w(y_k) + (\sum_{i,j=1}^{n-1} w_i y_j \Gamma_{\partial\Omega\,ij}^k(p))] \partial_k|_p \\
&+ < D_w^M \overline{Y}(p), \nu(p) >_{T_pM} \nu(p) \\
&= (D_w^{\partial\Omega} Y)(p) + < (D_w^M \overline{Y})(p), \nu(p) >_{T_pM} \nu(p),
\end{aligned}
$$

wherein the second equality follows because $w_n = \overline{y}_n(p) = 0$ and the third follows from (2.2) since $g_{nk}(p) =< \nu(p), \partial_k|_p >_{T_pM}= 0$ for all $k = 1, ..., n-1$ so that $g^{nk}(p) = 0$ for $k = 1, ..., n-1$ as well.

Equation (2.11) is proved in analogous fashion using (2.5). □

The **normal derivative** of a function $f \in C^\infty(\overline{\Omega})$ is the smooth function

$$\frac{\partial f}{\partial \nu} =< \nabla_M f, \nu >_{TM}$$

on $\partial\Omega$. For future reference we state the Riemannian Green's formulas

$$\int_\Omega [h\Delta f + < \nabla h, \nabla f >] dV = \int_{\partial\Omega} h \frac{\partial f}{\partial \nu} dA \tag{2.12}$$

and

$$\int_\Omega [h\Delta f - f\Delta h] dV = \int_{\partial\Omega} [h \frac{\partial f}{\partial \nu} - f \frac{\partial h}{\partial \nu}] dA, \tag{2.13}$$

for $h, f \in C^\infty(\overline{\Omega})$, where of course dV denotes n-dimensional Riemannian volume measure in M and dA denotes $(n-1)$-dimensional Riemannian volume ("area") measure on $\partial\Omega$ (inherited from M).

A **perturbation** of $\overline{\Omega} = \overline{\Omega}_0$ is a family $\{\overline{\Omega}_\epsilon\}$ of oriented, compact, connected, n-dimensional, smooth, submanifolds (of M) with boundary, parametrized by $\epsilon \in (-\epsilon_0, \epsilon_0)$ with $\epsilon_0 > 0$, to which is associated a smooth real-valued function Ψ defined on $(-\epsilon_0, \epsilon_0) \times M$ such that

(i) for each $\epsilon \in (-\epsilon_0, \epsilon_0)$, $\partial\Omega_\epsilon$ is the zero set of the function $\Psi(\epsilon, \cdot)$

(ii) $|\nabla_M \Psi(\epsilon, p)| > 0$ for all $p \in \partial\Omega_\epsilon$, $\epsilon \in (-\epsilon_0, \epsilon_0)$.

The function $\Psi(\cdot, \cdot)$ we call the **defining function** of the perturbation.

The outward unit normal vector field to $\partial\Omega_\epsilon$ at p will be denoted $\nu(\epsilon, p)$ with $\nu(p) \equiv \nu(0, p)$, and naturally we have

$$\nu(\epsilon, p) = \frac{\nabla_M \Psi(\epsilon, p)}{|\nabla_M \Psi(\epsilon, p)|}. \tag{2.14}$$

Next we define the **normal variation** of the perturbation $(\{\overline{\Omega}_\epsilon\}, \Psi)$ to be the smooth real-valued function v defined on a neighborhood of $\partial\Omega$ given by

$$\mathrm{v}(p) = -\frac{\partial_\epsilon \Psi(0, p)}{|\nabla_M \Psi(0, p)|}. \tag{2.15}$$

The following proposition provides the reason behind this terminology.

Proposition 2.2. *For each $p \in \partial\Omega$ there is an $\epsilon_0 > 0$ to which is associated a smooth curve $\gamma_p : (-\epsilon_0, \epsilon_0) \to M$ for which $\gamma_p(0) = p$, $\gamma_p'(0) = \mathrm{v}(p)\nu(p)$, and $\Psi(\epsilon, \gamma_p(\epsilon)) = 0$ for all $\epsilon \in (-\epsilon_0, \epsilon_0)$.*

Proof. Let $p \in \partial\Omega$ and recall the adapted coordinate system $(U, x_1, ..., x_n)$ about p satisfying $\partial_n|_p = \nu(p)$ from the proof of Proposition 2.1. Write

$$\tilde{\Psi}(\epsilon, x_1, ..., x_n) = \Psi(\epsilon, x^{-1}(x_1, ..., x_n)),$$

wherein x^{-1} denotes the inverse of the coordinate chart x. Since

$$\begin{aligned} \partial_{x_n} \tilde{\Psi}(0, x_1(p), ..., x_{n-1}(p), 0) &= < \nabla_M \Psi(0, p), \nu(p) > \\ &= |\nabla_M \Psi(0, p)| > 0, \end{aligned}$$

we may apply the Implicit Function Theorem to the equation

$$\tilde{\Psi}(\epsilon, x_1, ..., x_n) = 0. \tag{2.16}$$

This gives a mapping

$$(\epsilon, q) \mapsto t(\epsilon, q)$$

defined on $(-\epsilon_0, \epsilon_0) \times (U \cap \partial\Omega)$, with $\epsilon_0 > 0$, satisfying

$$\Psi(\epsilon, x^{-1}(x_1(p), ..., x_{n-1}(p), t(\epsilon, p))) = 0 \tag{2.17}$$

for $\epsilon \in (-\epsilon_0, \epsilon_0)$.

So simply define

$$\gamma_p(\epsilon) = x^{-1}(x_1(p), ..., x_{n-1}(p), t(\epsilon, p)). \tag{2.18}$$

It is of course immediate that $\Psi(\epsilon, \gamma_p(\epsilon)) = 0$ for all $\epsilon \in (-\epsilon_0, \epsilon_0)$. Moreover,

$$\begin{aligned} \gamma_p'(0) &= \sum_{i=1}^{n} \frac{d(x_i \circ \gamma_p)}{d\epsilon}|_{\epsilon=0} \partial_i|_p \\ &= \frac{d(x_n \circ \gamma_p)}{d\epsilon}|_{\epsilon=0} \partial_n|_p \\ &= c_p \nu(p), \end{aligned}$$

wherein $c_p = \frac{d(x_n \circ \gamma_p)}{d\epsilon}|_{\epsilon=0}$.

Finally we simply differentiate (2.17)

$$\begin{aligned} 0 &= \partial_\epsilon(\Psi(\epsilon, \gamma_p(\epsilon)))|_{\epsilon=0} \\ &= \partial_\epsilon \Psi(0, p) + c_p < \nabla_M \Psi(0, p), \nu(p) > . \end{aligned}$$

and solve for c_p to conclude that $c_p =$v(p). □

2.2 Statement and Proof of the Formula

Let $(\{\overline{\Omega}_\epsilon\}, \Psi)$ be a perturbation of the smooth submanifold $\overline{\Omega} \subseteq M$. Consider the parametrized family of Neumann eigenvalue problems

$$\Delta u(\epsilon, \cdot) + \lambda(\epsilon) u(\epsilon, \cdot) = 0 \,\text{on}\, \Omega_\epsilon \tag{2.19}$$

$$\frac{\partial u(\epsilon, \cdot)}{\partial \nu(\epsilon, \cdot)} = 0 \,\text{on}\, \partial\Omega_\epsilon \tag{2.20}$$

It is a classical result that there corresponds to the problem (2.19)-(2.20) a discrete set of eigenvalues

$$0 = \lambda_0(\epsilon) < \lambda_1(\epsilon) < \lambda_2(\epsilon)...$$

and that the corresponding eigenfunctions $u(\epsilon,\cdot)$ lie in $C^\infty(\overline{\Omega}_\epsilon)$. The eigenfunctions $u(\epsilon,\cdot)$ will always be assumed to be normalized so that $\int_{\Omega_\epsilon} |u(\epsilon,\cdot)|^2\,dV = 1$.

Now let $\{u(\epsilon,\cdot)\}$ be a smooth one-parameter family of normalized eigenfunctions corresponding to the problem (2.19)-(2.20) with associated eigenvalues $\lambda(\epsilon)$. In other words, we assume that we have $u(\cdot,\cdot) \in C^\infty(\boldsymbol{R}\times M)$ and that each $u(\epsilon,\cdot)$ satisfies (2.19)-(2.20) on the domain $\overline{\Omega}_\epsilon$. It follows from equation (2.19) that the ordinary derivative of the real-valued function $\epsilon \mapsto \lambda(\epsilon)$ must exist and so may be calculated. Thus, setting $u = u(0,\cdot)$ and $\lambda = \lambda(0)$ and recalling that by $\mathrm{v}(p)$ we denote the normal variation at $p \in \partial\Omega$ of our given perturbation (see (2.15)), our goal is to prove the following theorem.

Theorem 2.3. *The formula for the first variation of the Neumann eigenvalues of the Laplacian is given by*

$$\lambda'(0) = \int_{\partial\Omega} (|\nabla_{\partial\Omega} u|^2 - \lambda u^2)\mathrm{v}\,dA. \tag{2.21}$$

Example 2.4. We remark here that we can actually verify this formula numerically when, for example, Ω is the unit disk $D = D(1)$ in the plane, perturbed in the normal direction to S^1 with constant magnitude ϵ so that $\Omega_\epsilon = D(1+\epsilon)$ is the disk of radius $1+\epsilon$ centered at the origin. In this case we have $\mathrm{v} \equiv 1$ all the way around the unit circle. Letting the first zero of the derivative of the first Bessel function J_1 be designated by the letter k, we calculate that

$$\begin{aligned}
\int_0^{2\pi}\int_0^{1+\epsilon} J_1^2\left(\frac{k}{1+\epsilon}r\right)(\cos^2\theta) r\,dr\,d\theta &= \int_0^{2\pi}\cos^2\theta\,d\theta \int_0^{1+\epsilon} J_1^2\left(\frac{k}{1+\epsilon}r\right) r\,dr \\
&= \frac{\pi(1+\epsilon)^2}{k^2}\int_0^k J_1^2(t)t\,dt \\
&= \frac{1}{2k^2}(\pi(1+\epsilon)^2(k^2-1)J_1^2(k)),
\end{aligned}$$

employing the formula $\frac{d}{dt}\frac{1}{2}[t^2(J_1')^2(t) + (t^2-1)J_1^2(t)] = tJ_1^2(t)$ which follows from Bessel's differential equation. Thus a normalized eigenfunction of the Laplacian on $D(1+\epsilon)$ is given in polar coordinates by

$$u(\epsilon,r,\theta) = \frac{\sqrt{2}kJ_1(\frac{k}{1+\epsilon}r)\cos\theta}{J_1(k)(1+\epsilon)\sqrt{\pi(k^2-1)}}, \tag{2.22}$$

and these eigenfunctions clearly form a smooth one-parameter family. Now for each ϵ near 0, the eigenfunction $u(\epsilon, r, \theta)$ corresponds to the Neumann eigenvalue $\lambda(\epsilon) = \frac{k^2}{(1+\epsilon)^2}$ on $D(1+\epsilon)$. Hence

$$\lambda'(0) = -2k^2. \tag{2.23}$$

Meanwhile,

$$\begin{aligned}
\int_{S^1} (|\nabla_{S^1} u|^2 - \lambda u^2)\, dA &= \frac{2k^2}{\pi J_1^2(k)(k^2-1)} \int_0^{2\pi} (J_1^2(k)\cos^2\theta - k^2 J_1^2(k)\cos^2\theta) \mathrm{d}\theta \\
&= \frac{-2k^2}{\pi} \int_0^{2\pi} \cos^2\theta \, d\theta \\
&= -2k^2,
\end{aligned} \tag{2.24}$$

giving (2.21) for this special case.

The Proof of Theorem 2.3. We begin by differentiating (2.19) with respect to ϵ, obtaining

$$\Delta \partial_\epsilon u(0,\cdot) + \lambda \partial_\epsilon u(0,\cdot) = -\lambda'(0) u(0,\cdot), \tag{2.25}$$

valid on Ω. To find boundary conditions corresponding to (2.25), we fix $p \in \partial\Omega$ and differentiate the Neumann conditions (2.20) with respect to ϵ:

$$\begin{aligned}
0 &= \partial_\epsilon \left(\frac{\partial u(\epsilon, \gamma_p(\epsilon))}{\partial \nu(\epsilon, \gamma_p(\epsilon))} \right)|_{\epsilon=0} \\
&= \partial_\epsilon (< \nabla_M u(\epsilon, \gamma_p(\epsilon)), \nu(\epsilon, \gamma_p(\epsilon)) >_{T_{\gamma_p(\epsilon)}M})|_{\epsilon=0} \\
&= < \frac{D}{d\epsilon}(\nabla_M u(\epsilon, \gamma_p(\epsilon)))|_{\epsilon=0}, \nu(0,p) >_{T_pM} \\
&+ < \nabla_M u(0,p), \frac{D}{d\epsilon}\nu(\epsilon, \gamma_p(\epsilon))|_{\epsilon=0} >_{T_pM},
\end{aligned} \tag{2.26}$$

wherein γ_p is the curve introduced in Proposition 2.2.

Let's first consider the leading term on the right. To simplify our calculations, we fix a Riemannian normal coordinate system $(U, x_1, ..., x_n)$ about p on a neighborhood U with respect to the ambient manifold M. Recall that, with respect to such a coordinate system, the Riemannian metric (g_{ij}) (as well as its inverse (g^{ij})) is the identity to first order at p so that all Christoffel symbols vanish at p. Using the local expression (2.3), we find that

$$\frac{D}{d\epsilon}((\nabla_M u(\epsilon,\cdot))(\gamma_p(\epsilon)))|_{\epsilon=0} = \sum_k \frac{d}{d\epsilon}(((\nabla_M u(\epsilon,\cdot))_k(\gamma_p(\epsilon)))|_{\epsilon=0}\partial_k|_p. \tag{2.27}$$

wherein $\nabla_M u(\epsilon,q)=\sum_{k=1}^{n}(\nabla_M u(\epsilon,q))_k\partial_k|_q$. But, for all $k=1,...,n$,

$$
\begin{aligned}
\frac{d}{d\epsilon}((\nabla_M u(\epsilon,\cdot))_k(\gamma_p(\epsilon)))|_{\epsilon=0} &= (\nabla_M\partial_\epsilon u(0,p))_k \\
&+ <\nabla_M((\nabla_M u(0,\cdot))_k)(p),\gamma_p'(0)>_{T_pM} \\
&= \partial_k\partial_\epsilon u(0,p)+\gamma_p'(0)((\nabla_M u(0,\cdot))_k) \\
&= (\nabla_M\partial_\epsilon u)_k(0,p) \\
&+ (D_{(\mathrm{v}(p)\nu(p))}(\nabla_M u(0,\cdot)))_k(p), \qquad (2.28)
\end{aligned}
$$

citing Proposition 2.2 and the local formula (2.1). Thus

$$
\begin{aligned}
<\frac{D}{d\epsilon}(\nabla_M u(\epsilon,\gamma_p(\epsilon)))|_{\epsilon=0},\nu(0,p)>_{T_pM} &= <\nabla_M\partial_\epsilon u(0,p),\nu(p)>_{T_pM} \\
&+ \mathrm{v}(p)\nabla_M^2 u(\nu(p),\nu(p)). \qquad (2.29)
\end{aligned}
$$

Still working within our normal coordinate system, we analyze the second term in (2.26) by writing

$$
\begin{aligned}
<\nabla_M u(0,p),\frac{D}{d\epsilon}\nu(\epsilon,\gamma_p(\epsilon))|_{\epsilon=0}>_{T_pM} &= <\nabla u(0,p),\frac{D}{d\epsilon}(\frac{\nabla_M\Psi(\cdot,\gamma_p(\cdot))}{|\nabla_M\Psi(\cdot,\gamma_p(\cdot))|})|_{\epsilon=0}>_{T_pM} \\
&= <\nabla_M u(0,p),\sum_{k=1}^{n}\frac{d}{d\epsilon}(\frac{\partial_k\Psi(\cdot,\gamma_p(\cdot))}{|\nabla_M\Psi(\cdot,\gamma_p(\cdot))|})|_{\epsilon=0}\partial_k|_p>_{T_pM} \\
&= <\nabla_M u(0,p),\frac{1}{|\nabla_M\Psi(0,p)|}\cdot \\
&\quad \sum_k\frac{d}{d\epsilon}(\partial_k\Psi(\cdot,\gamma_p(\cdot)))|_{\epsilon=0}\partial_k|_p>_{T_pM} \\
&\quad (\text{ since Neumann conditions are satisfied }) \\
&= <\nabla_M u(0,p),\frac{1}{|\nabla_M\Psi(0,p)|}\sum_k\partial_k\partial_\epsilon\Psi(0,p)\partial_k|_p \\
&+ \frac{1}{|\nabla_M\Psi(0,p)|}\sum_{j,k}\mathrm{v}(p)\nu_j(p)\partial_j\partial_k\Psi(0,p)\partial_k|_p>_{T_pM} \\
&\quad (\text{ wherein } \nu(p)=\sum_j\nu_j(p)\partial_j|_p\ ) \\
&= <\nabla_M u(0,p),\frac{1}{|\nabla\Psi(0,p)|}((\nabla_M\partial_\epsilon\Psi)(0,p)) \\
&+ \partial_\epsilon\Psi(0,p)\nabla_M(\frac{1}{|\nabla_M\Psi(0,\cdot)|})(p)>_{T_pM} \\
&= -<\nabla_M u(p),\nabla_M\mathrm{v}(p)>_{T_pM} \\
&= -<\nabla_{\partial\Omega}u(p),\nabla_{\partial\Omega}\mathrm{v}(p)>_{T_p(\partial\Omega)}, \qquad (2.30)
\end{aligned}
$$

wherein the last equality is verified using (2.11) and the Neumann boundary condition.

Combining the sets of equations (2.26),(2.29), and (2.30) we find that the boundary condition on $\partial\Omega$ at p corresponding to (2.25) is

$$\frac{\partial(\partial_\epsilon u(0,\cdot))}{\partial\nu(0,\cdot)}|_p =< \nabla_{\partial\Omega}u(p), \nabla_{\partial\Omega}\mathrm{v}(p) >_{T_p(\partial\Omega)} -\mathrm{v}(p)\nabla^2_M u(p)(\nu(p),\nu(p)) \tag{2.31}$$

Next we multiply (2.25) by the eigenfunction u itself, obtaining

$$u\Delta\partial_\epsilon u(0,\cdot) + \lambda u\partial_\epsilon u(0,\cdot) = -\lambda'(0)u^2. \tag{2.32}$$

Similarly, we multiply

$$\Delta u + \lambda u = 0$$

by $\partial_\epsilon u(0,\cdot)$ to get

$$\partial_\epsilon u(0,\cdot)\Delta u + \lambda\partial_\epsilon u(0,\cdot)u = 0. \tag{2.33}$$

Now we subtract (2.32) from (2.33), integrate over Ω, and apply the Green's formula (2.13):

$$\begin{aligned} \lambda'(0) &= \lambda'(0)\int_\Omega u^2 dV \\ &= \int_\Omega (\partial_\epsilon u(0,\cdot)\Delta u - u\Delta(\partial_\epsilon u(0,\cdot)))dV \\ &= \int_{\partial\Omega}\left(\partial_\epsilon u(0,\cdot)\frac{\partial u}{\partial\nu} - u\frac{\partial(\partial_\epsilon u(0,\cdot))}{\partial\nu(0,\cdot)}\right)dA \\ &= -\int_{\partial\Omega} u\frac{\partial(\partial_\epsilon u(0,\cdot))}{\partial\nu(0,\cdot)}\,dA \end{aligned} \tag{2.34}$$

Upon inserting our perturbed boundary condition (2.31) and then applying (2.12), with the boundaryless submanifold $\partial\Omega$ in place of Ω, followed by (2.8) this becomes

$$\begin{aligned} \lambda'(0) &= \int_{\partial\Omega}(u\mathrm{v}\nabla^2_M u(\nu,\nu) - u < \nabla_{\partial\Omega}u, \nabla_{\partial\Omega}\mathrm{v} >)dA \\ &= \int_{\partial\Omega}(u\nabla^2_M u(\nu,\nu) + \frac{1}{2}(\Delta_{\partial\Omega}(u^2)))\mathrm{v}\,dA \\ &= \int_{\partial\Omega}(u\nabla^2_M u(\nu,\nu) + |\nabla_{\partial\Omega}u|^2 + u\Delta_{\partial\Omega}u)\mathrm{v}dA. \end{aligned} \tag{2.35}$$

Letting $t_1,...,t_{n-1}$ denote tangent vectors comprising an orthonormal basis for $T_p(\partial\Omega)$ for any fixed $p \in \partial\Omega$, we obtain, invoking (2.9), that

$$\Delta_{\partial\Omega}u(p) \;=\; \mathrm{trace}(\nabla^2_{\partial\Omega}u(p))$$

$$
\begin{aligned}
&= \sum_{k=1}^{n-1} \nabla^2_{\partial\Omega} u(p)(t_k, t_k) \\
&= \sum_{k=1}^{n-1} < D^{\partial\Omega}_{t_k}(\nabla_{\partial\Omega} u)(p), t_k >_{T_p(\partial\Omega)} \qquad (2.36)
\end{aligned}
$$

But, notice that (2.11) and our Neumann boundary conditions imply that $\nabla_{\partial\Omega} u$ is the restriction to $\partial\Omega$ of the vector field $\nabla_M u$ defined on M. Thus appealing to (2.10) we find that

$$
\begin{aligned}
\Delta_{\partial\Omega} u(p) &= \sum_{k=1}^{n-1} < D^M_{t_k}(\nabla_M u)(p) - < D^M_{t_k}(\nabla_M u)(p), \nu(p) >_{T_pM} \nu(p), t_k >_{T_pM} \\
&= \sum_{k=1}^{n-1} < D^M_{t_k}(\nabla_M u)(p), t_k >_{T_pM} \\
&= \sum_{k=1}^{n-1} \nabla^2_M u(p)(t_k, t_k). \qquad (2.37)
\end{aligned}
$$

Therefore, for all $p \in \partial\Omega$,

$$
\begin{aligned}
\Delta_{\partial\Omega} u(p) + \nabla^2_M u(p)(\nu(p), \nu(p)) &= \left(\sum_{k=1}^{n-1} \nabla^2_M u(p)(t_k, t_k) \right) + \nabla^2_M u(p)(\nu(p), \nu(p)) \\
&= \Delta_M u(p). \qquad (2.38)
\end{aligned}
$$

Consequently, inserting (2.38) into (2.35), we find that

$$
\begin{aligned}
\lambda'(0) &= \int_{\partial\Omega} (|\nabla_{\partial\Omega} u|^2 + u\Delta_M u) \mathrm{v} \, dA \\
&= \int_{\partial\Omega} (|\nabla_{\partial\Omega} u|^2 - \lambda u^2) \mathrm{v} \, dA,
\end{aligned}
$$

thereby confirming the validity of our formula. □

2.3 Some Remarks Regarding the Nonsmooth Case

Let $\Omega \subset \boldsymbol{R}^n$ be a bounded Lipschitz domain. Anticipating some methods and terminology from Chapter 3, consider the weak Neumann eigenvalue problem for the Laplacian:

$$
\lambda_k \int_\Omega u_k v \, dV = \int_\Omega < \nabla u_k, \nabla v > dV \qquad (2.39)
$$

for all $v \in L^2_1(\Omega)$. Lax-Milgram methods (and the Elliptic Regularity Theorem) imply there exists a unique solution $u_k \in L^2_1(\Omega) \cap C^\infty(\Omega)$ for all $k = 0, 1, 2, \ldots$. Since we can also view u_k as the solution to the weak inhomogeneous Neumann problem with data $-\lambda_k u_k$, we have

$$u_k = u + (\tilde{u} * N)|\Omega, \tag{2.40}$$

which is exactly the decomposition we use in Chapter 3 to solve the generalized inhomogeneous Neumann problem. Here, $\tilde{u} \in L^2(\boldsymbol{R}^n)$ denotes the function whose restriction to Ω is $-\lambda_k u_k$ but which is identically 0 outside Ω, and $(\tilde{u} * N|\Omega) \in L^2_2(\Omega)$ denotes the restiction to Ω of the convolution of $\tilde{u}$ with the Newtonian potential N on $\boldsymbol{R}^n$. The function u is the solution to the homogeneous problem

$$\begin{aligned} \Delta u &= 0 \qquad\qquad \text{on}\,\Omega && (2.41) \\ \frac{\partial u}{\partial n} &= -\frac{\partial(\tilde{u} * N)}{\partial n}\,\text{on}\,\partial\Omega, && (2.42) \end{aligned}$$

wherein by $\frac{\partial(\tilde{u}*N)}{\partial n}$ we mean, appealing to the Trace Theorem (Theorem 3.7), the function $<\mathrm{Tr}\nabla(\tilde{u} * N|\Omega), n> \in L^2(\partial\Omega)$. Also, we may assume that boundary values in (2.41)-(2.42) are attained nontangentially (see section 3.3).

Availing ourselves of a theorem of Jerison-Kenig [13], we know immediately that $\mathrm{Tr}\nabla u \in L^2(\partial\Omega)$, which means that $\mathrm{Tr}\nabla u_k = \mathrm{Tr}\nabla u + \mathrm{Tr}\nabla((\tilde{u} * N)|\Omega) \in L^2(\partial\Omega)$ as well, so that the analogue for bounded Lipschitz domains of the eigenvalue variation in (2.21) is well-defined. (For this case, the counterpart to the normal variation v will be an $L^\infty(\partial\Omega)$ function.) Therefore, we conclude that we may expect a similar variational formula even in this nonsmooth setting.

Chapter 3

Estimates for the Inhomogeneous Neumann Problem in Lipschitz Domains

3.1 Statement of the Estimates

Our objective in this chapter will be to establish existence and uniqueness of the solution to the inhomogeneous Neumann problem

$$\Delta w = F \text{ on } \Omega \tag{3.1}$$

$$\frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega \tag{3.2}$$

on a bounded (connected) Lipschitz domain Ω in $\boldsymbol{R}^n$ with data in Sobolev spaces.

As discussed in the Introduction, the most general statement of our theorem will require us to interpret this boundary value problem in some suitably weak form, involving a rather flexible integration-by-parts formula. To obtain this formulation, we will require two background theorems, one due to Stein and the other to Strichartz, that will enable us to pass from function spaces defined on the domain Ω to function spaces on $\boldsymbol{R}^n$ and back again. But first, we present some essential definitions.

Let $1 < p < \infty$ and $-\infty < \alpha < \infty$. We define the **Sobolev (potential) space**

$L^p_\alpha = L^p_\alpha(\boldsymbol{R}^n)$ to be the collection of all tempered distributions f on $\boldsymbol{R}^n$ for which

$$((1+|\xi|^2)^{\alpha/2})\hat{f} = \hat{g} \tag{3.3}$$

for some $g \in L^p = L^p(\boldsymbol{R}^n)$, where, of course, the circumflex operator above denotes Fourier transformation of tempered distributions. We impose a norm on this space by setting

$$\| f \|_{L^p_\alpha} = \| g \|_{L^p} . \tag{3.4}$$

We remark here that it is well-known that the dual space of L^p_α is the space $L^q_{-\alpha}$, where 1/p + 1/q = 1. Now let $R_\Omega f$ denote the restriction of a function f on $\boldsymbol{R}^n$ to Ω.

For $1 < p < \infty$ and $\alpha \geq 0$ we define the **Sobolev space on $\boldsymbol{\Omega}$** with indices p and α as $L^p_\alpha(\Omega) = R_\Omega L^p_\alpha$ with the standard quotient norm

$$\|f\|_{L^p_\alpha(\Omega)} = \inf\{\|\mathrm{g}\| \mid \mathrm{R}_\Omega \mathrm{g} = \mathrm{f}\} \tag{3.5}$$

We may now introduce the familiar extension operator of Stein [17].

Theorem 3.1. *For any bounded Lipschitz domain Ω, there is a bounded linear extension operator E_Ω mapping $L^p_k(\Omega)$ into $L^p_k(\boldsymbol{R}^n)$* simultaneously *for all nonnegative integers k and every p, $1 < p < \infty$.*

By complex interpolation we have (See [12])

Proposition 3.2. *The map E_Ω extends to a bounded linear operator mapping $L^p_\alpha(\Omega)$ into L^p_α simultaneously for all $\alpha \geq 0$ and all p with $1 < p < \infty$.*

Remark 3.3. Notice that $R_\Omega : L^p_\alpha \to L^p_\alpha(\Omega)$ induces a bounded linear map $R^*_\Omega : (L^p_\alpha(\Omega))^* \to L^q_{-\alpha}$ between corresponding dual spaces via

$$(R^*_\Omega F)(g) = F(R_\Omega(g))$$

for $g \in L^p_\alpha$. Similarly, $E_\Omega : L^p_\alpha(\Omega) \to L^p_\alpha$ induces a bounded linear map $E^*_\Omega : L^q_{-\alpha} \to (L^p_\alpha(\Omega))^*$. Since $R_\Omega \circ E_\Omega = \mathrm{Id}$, we have $E^*_\Omega \circ R^*_\Omega = \mathrm{Id}$ as well.

Using a well-known characterization of L^p_α due to Strichartz (see Theorem 3.4 of Jerison and Kenig [12]), it is possible to establish the veracity of a certain boundedness property of the truncation operator χ_Ω which operates on functions f defined on $\boldsymbol{R}^n$by setting $(\chi_\Omega f)(x) = f(x)$ for $x \in \Omega$ and $(\chi_\Omega f)(x) = 0$ for $x \in \boldsymbol{R}^n \backslash \Omega$.

Proposition 3.4. *Let Ω be a bounded Lipschitz domain. Suppose that $1 < p < \infty$ and $0 \leq \alpha < 1/p$. Then*

$$\|\chi_\Omega f\|_{L^p_\alpha} \leq C\|f\|_{L^p_\alpha}. \tag{3.6}$$

For the proof, see [12], Proposition 3.5.

Of course the dualized statement also holds and will be very useful.

Corollary 3.5. *Assume that $1 < p < \infty$ and that $-1/q < \alpha \leq 0$. Then, for $f \in L^p_\alpha$,*

$$\|\chi_\Omega f\|_{L^p_\alpha} \leq C\|f\|_{L^p_\alpha}. \tag{3.7}$$

It is proven first for elements of $C_0^\infty(\boldsymbol{R}^n)$ and then extended to all of L^p_α via density.

We are now ready to describe precisely the sense in which the Neumann problem will be solved. Suppose that $1 < p < \infty, \frac{1}{p} < \alpha < 1 + \frac{1}{p}$, and $F \in (L^q_{2-\alpha}(\Omega))^*_{1^\perp}$, wherein by definition

$$(L^q_{2-\alpha}(\Omega))^*_{1^\perp} = \{F \in (L^q_{2-\alpha}(\Omega))^* \,|\, F(1) = 0\}.$$

Then a function $w \in L^p_\alpha(\Omega)$ satisfies the **generalized inhomogeneous Neumann problem** $\mathrm{NP}(p, \alpha, F)$ with data F if

$$F(v) = - < \chi_\Omega \nabla(E_\Omega(w)), \nabla(E_\Omega(v)) > \tag{3.8}$$

for all $v \in L^q_{2-\alpha}(\Omega)$.

The standard definitions of elementary distribution theory show that whenever we have $w \in L^p_1(\Omega)\, 1 < p < \infty$, and $v \in L^q_1(\Omega)$ the right-hand side of (3.8) may be written simply as $\int_\Omega < \nabla w, \nabla v > dV$. This implies, by density and continuity, that the bilinear form on the right-hand side in (3.8) is independent of the particular extension operators used as long as they share with Stein's the boundedness properties expressed in Prop. 3.2.

We present, at last, the principal result of this chapter.

Theorem 3.6. *Let Ω be a bounded Lipschitz domain in $\boldsymbol{R}^n, n \geq 3$, and let $1 < p, q < \infty$ satisfy $1/p + 1/q = 1$. There is $\epsilon, 0 < \epsilon \leq 1$, depending only on the*

Lipschitz constant of Ω*, such that, for every* $F \in (L^q_{2-\alpha}(\Omega))^*_{1\perp}$*, there exists a solution* $w \in L^p_\alpha(\Omega)$ *to the generalized inhomogeneous Neumann problem* NP(p, α, F)*, provided one of the following holds:*

$$\begin{aligned} &(a) \quad p_0 < p < p_0' \text{ and } \frac{1}{p} < \alpha < 1 + \frac{1}{p} \\ &(b) \quad 1 < p \leq p_0 \text{ and } 3/p - 1 - \epsilon < \alpha < 1 + \frac{1}{p} \\ &(c) \quad p_0' \leq p < \infty \text{ and } \frac{1}{p} < \alpha < 3/p + \epsilon \end{aligned}$$

wherein $1/p_0 = 1/2 + \epsilon/2$ *and* $1/p_0' = 1/2 - \epsilon/2$*. Moreover, for all* $F \in (L^q_{2-\alpha}(\Omega))^*_{1\perp}$*, we have the estimate*

$$\|w\|_{L^p_\alpha(\Omega)} \leq C\|F\|_{(L^q_{2-\alpha}(\Omega))^*}. \tag{3.9}$$

Finally, modulo constants, this solution w *is unique.*

As remarked in the Introduction, the range for α and p for which this theorem holds is best understood as the open hexagon in $(\alpha, 1/p)$ space with vertices $(0,0), (1/p_0, 1/p_0), (2-\epsilon, 1), (2,1), (2-1/p_0, 1/p_0')$, and $(\epsilon, 0)$.

3.2 Boundary Function Spaces and Estimates for the Dirichlet Problem

Recall the general strategy for proving this theorem discussed in the Introduction. The key (classical) idea introduced there was to obtain the solution to our inhomogeneous Neumann problem by reduction to a corresponding homogeneous Dirichlet problem through inversion of the Calderón operator. Consequently, in this section we introduce the boundary function spaces we will require, review some known results pertaining to the homogeneous Dirichlet problem, and recall the statement of the related Trace Theorem; treatment of the inverse Calderón Operator, however, is deferred until section 3.3.

Let $1 \leq p < \infty$ and suppose that $0 < \alpha < 1$. We define the **Besov space** $B^p_\alpha = B^p_\alpha(\boldsymbol{R}^n)$ to be the space of all functions f in L^p such that the norm

$$\|f\|_{L^p} + \left(\int_{R^n} \frac{\|f(x+t)-f(x)\|_{L^p}^p}{|t|^{n+p\alpha}} dt\right)^{1/p}$$

is finite. For $p = \infty$ we decree that B_α^∞ will denote the familiar Hölder spaces with exponent α; thus $f \in B_\alpha^\infty$ if

$$|f(x) - f(y)| \leq C|x-y|^\alpha,$$

for all $x, y \in \boldsymbol{R}^n$.

Consider a region Ω above the graph of a Lipschitz function ψ. For $0 < s < 1, 1 < p \leq \infty$ define the **boundary Besov space** $B_s^p(\partial\Omega)$ as the space of functions $f(x, \psi(x)) = g(x)$, where $g \in B_s^p(\boldsymbol{R}^{n-1})$. To extend this to boundaries $\partial\Omega$ of all bounded Lipschitz domains, let $\{B_j = B(P_j, r) | j = 1, 2, ..., M\}$ be a covering of $\partial\Omega$ by balls as in the definition of a Lipschitz domain and let $\eta_j \in C_0^\infty(\boldsymbol{R}^n)$ be such that $\operatorname{supp}\eta_j \subset B_j$, $0 \leq \eta_j \leq 1$, and $\sum_j \eta_j = 1$ on $\partial\Omega$. Let ϕ_j be an isometry of $\boldsymbol{R}^n$ such that

$$\phi_j(B_j \cap \Omega) = \{(x', x_n) | x_n > \psi_j(x'), |(x', x_n)| \leq r\}. \tag{3.10}$$

Denote by $C^\infty(\partial\Omega)$ the set of functions on $\partial\Omega$ that are restrictions to $\partial\Omega$ of functions in $C^\infty(\boldsymbol{R}^n)$. Define $B_s^p(\partial\Omega)$ as the completion of $C^\infty(\partial\Omega)$ with respect to the norm

$$||g||_{B_s^p(\partial\Omega)} = \sum_{j=1}^M ||(\eta_j g)(\phi_j^{-1}(\cdot, \psi_j(\cdot))||_{B_s^p},\ g \in C^\infty(\partial\Omega). \tag{3.11}$$

The negative space $B_{-s}^p(\partial\Omega)$,$1 < p < \infty, 0 < s < 1$, is defined as the dual space

$$B_{-s}^p(\partial\Omega) = (B_s^q(\partial\Omega))^*$$

under the pairing

$$< g, f >= \int_\Omega gf\, d\sigma,\ f \in B_s^q(\partial\Omega),\ g \in L^p(\partial\Omega).$$

In other words the norm is

$$||g||_{B_{-s}^p(\partial\Omega)} = \sup\{|\int_{\partial\Omega} gf\, d\sigma|\ |\ f \in B_s^q(\partial\Omega), ||f||_{B_s^q(\partial\Omega)} \leq 1\}$$

for $g \in L^p(\partial\Omega)$. An equivalent norm is

$$||g||_{B^p_{-s}(\partial\Omega)} = \sum_{j=1}^{M} ||(\eta_j g)(\phi_j^{-1}(\cdot,\psi_j(\cdot))\sqrt{1+|\nabla\psi_j(\cdot)|^2}||_{B^p_{-s}}, \tag{3.12}$$

(**Note:** The observation that Lipschitz hypersurface density is included as a weight factor in the equivalent definition (3.12) of the norm for the negative Besov spaces but is absent from the definition of the norm (3.11) for the positive Besov spaces will be of critical importance within the proofs of Theorems 3.20 and 3.22. Another way of saying the same thing is that if $\partial\Omega$ is the boundary of a domain above a Lipschitz graph ψ on $\boldsymbol{R}^{n-1}$ then the analogous definitions in this case for the boundary Besov spaces imply that if $g \in B^p_s(\partial\Omega), s > 0$, then $g \in B^p_s$ as well, whereas if $g \in B^p_{-s}(\partial\Omega), s > 0$, then we have instead $g\omega \in B^p_{-s}$, where $\omega = \sqrt{1+|\nabla\psi|^2}$ is Lipschitz hypersurface density. Similar observations hold for the boundary Sobolev spaces which we are about to define, and they will also be of crucial importance.)

For $1 < p < \infty, 0 \le s \le 1$, the **boundary Sobolev space** $L^p_s(\partial\Omega)$ is defined as the completion of $C^\infty(\partial\Omega)$ with respect to the norm

$$||g||_{L^p_s(\partial\Omega)} = \sum_{j=1}^{M} ||(\eta_j g)(\phi_j^{-1}(\cdot,\psi_j(\cdot))||_{L^p_s},\ g \in C^\infty(\partial\Omega).$$

The dual space $L^p_{-s}(\partial\Omega)$,$1 < p < \infty, 0 \le s \le 1$, is defined via

$$L^p_{-s}(\partial\Omega) = (L^q_s(\partial\Omega))^*$$

under the pairing

$$< g, f >= \int_\Omega gf\, d\sigma,\ f \in L^q_s(\partial\Omega),\ g \in L^p(\partial\Omega).$$

In other words the norm is

$$||g||_{L^p_{-s}(\partial\Omega)} = \sup\{|\int_{\partial\Omega} gf\, d\sigma|\mid f \in L^q_s(\partial\Omega), ||f||_{L^q_s(\partial\Omega)} \le 1\}$$

for $g \in L^p(\partial\Omega)$. Once again an equivalent norm is

$$||g||_{L^p_{-s}(\partial\Omega)} = \sum_{j=1}^{M} ||(\eta_j g)(\phi_j^{-1}(\cdot,\psi_j(\cdot))\sqrt{1+|\nabla\psi_j(\cdot)|^2}||_{L^p_{-s}},$$

It will in addition be relevant later on that Hölder's inequality implies that $L^p(\partial\Omega)$ acts on $B^q_{-s}(\partial\Omega), -1 < s < 0$, via integration:

$$g \mapsto \int_{\partial\Omega} fg d\sigma, g \in B^q_s(\partial\Omega),$$

and thus $L^p(\partial\Omega)$ is naturally embedded in $B^p_s(\partial\Omega)$ for this range of s.

Since we are interested in the solution to the Neumann problem we will certainly also want to consider mean-value-0 versions of these boundary function spaces. Recall that any $f \in B^p_s(\partial\Omega)$,$1 < p \leq \infty$, $-1 < s < 1, s \neq 0$, acts on the real line either via integration over $\partial\Omega$ (i.e., $< f, r >= \int_{\partial\Omega} rf\, d\sigma$ for all real numbers r) if $s \geq 0$ or by virtue of the identification of $B^p_s(\partial\Omega)$ as a space of linear functionals on an appropriate function space if $s < 0$. Thus we set

$$B^p_s(\partial\Omega)_{1^\perp} = \{f \in B^p_s(\partial\Omega) |\, f(1) = 0\}, \tag{3.13}$$

and similarly for the boundary Sobolev spaces for the ranges for which they have been defined.

We remark that if $\mathcal{F} \subseteq L^1(\partial\Omega)$ is a dense subset of some space $B^p_s(\partial\Omega)$ (resp. $L^p_s(\partial\Omega)$) then $\mathcal{F}_{1^\perp} \equiv \{f - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} f\, d\sigma | f \in \mathcal{F}\}$ is a dense subset of $B^p_s(\partial\Omega)_{1^\perp}$ (resp. $L^p_s(\partial\Omega)_{1^\perp}$). One easily shows this by using the fact that $L^p(\partial\Omega)$-convergence implies $L^1(\partial\Omega)$-convergence (in the case for which $s \geq 0$) or that norm convergence implies weak-* convergence (in the case for which $s < 0$).

Theorem 3.7(Trace Theorem for Lipschitz Domains). *Let* $1 < p < \infty$ *and suppose that* $1/p < \alpha < 1 + 1/p$ *and* $s = \alpha - 1/p$, *then the mapping* Tr, *initially defined on* $C^\infty(\overline{\Omega})$ *as the restriction to* $\partial\Omega$, *extends to a bounded linear operator from* $L^p_\alpha(\Omega)$ *to* $B^p_s(\partial\Omega)$.

Theorem 3.7 follows easily from a special case of a theorem due to Jonsson and Wallin [14], Theorem 1, p.182 (also see [12]). Now let's review the estimates for the homogeneous Dirichlet problem.

Theorem 3.8. *Consider* ϵ *such that* $0 < \epsilon \leq 1$. *Define* p_0 *and* p_0' *by* $1/p_0 = (1+\epsilon)/2$ *and* $1/p_0' = (1-\epsilon)/2$. *Let* s *and* p *be numbers satisfying one of the following:*

(a) $\quad p_0 < p < p_0'$ *and* $0 < s < 1$

(b) $1 < p \leq p_0$ *and* $2/p - 1 - \epsilon < s < 1$

(c) $p_0' \leq p < \infty$ *and* $0 < s < 2/p + \epsilon$.

Let Ω *be a bounded Lipschitz domain in* $\boldsymbol{R}^n$ *for some* $n \geq 3$. *There exists* ϵ *depending only on the Lipschitz constant of* Ω *such that for every* $g \in B_s^p(\partial\Omega)$ *there exists a unique harmonic function* u *such that* Tr $u{=}g$ *and* $u \in L^p_{s+1/p}(\Omega)$. *Moreover,*

$$\|u\|_{L^p_{s+1/p}(\Omega)} \leq C\|g\|_{B_s^p(\partial\Omega)}. \tag{3.14}$$

For the proof we refer the reader to Jerison-Kenig[12].

3.3 Estimates for the Inverse Calderón Operator

Let Ω be a bounded Lipschitz domain and suppose that $Q \in \partial\Omega$. We define the (interior)**nontangential cone** $\Gamma_\alpha(Q)$ for $\alpha > 0$ via

$$\Gamma_\alpha(Q) = \{X \in \Omega | |X - Q| < (1 + \alpha)\text{dist}(X, \partial\Omega)\}.$$

If u is a function on $\partial\Omega$ we may define its **nontangential maximal function** $M(u)$ by setting

$$M(u)(Q) = \sup\{|u(P)| | P \in \Gamma_1(Q)\}.$$

We say that u has a **nontangential limit** at $Q \in \partial\Omega$ if there is a finite, well-defined limit (which we will call) $u(Q)$ as $P \to Q$ from within $\Gamma_\alpha(Q)$ for all $\alpha > 0$.

Next recall the classical method of layer potentials to solve Laplace's equation with Neumann boundary conditions. Given $g \in L^1(\partial\Omega)$, its **single layer potential** (SLP) is the function defined via

$$Sg(X) = \frac{-1}{\omega_n(n-2)} \int_{\partial\Omega} \frac{g(Q)}{|X - Q|^{n-2}}\, d\sigma(Q), \tag{3.15}$$

where ω_n is the surface area of the unit sphere in $\boldsymbol{R}^n$. We will occasionally write $N(X, Q)$ for the kernel $|X - Q|^{2-n}$ – which is the Newtonian potential. The integral in (4.15) is in general pointwise well-defined on all of $\boldsymbol{R}^n \backslash \partial\Omega$ and well-defined a.e. $d\sigma$ on $\partial\Omega$ in the sense that it converges absolutely a.e. $d\sigma$(since the singularity present is locally integrable). Moreover, differentiating twice under the integral ensures that Sg is harmonic on $\boldsymbol{R}^n \backslash \partial\Omega$. In addition, recalling the definition of a bounded Lipschitz domain, let $\{B_j = B(P_j, r) | j = 1, 2, ..., M\}$ be a covering of $\partial\Omega$ by balls and let $\eta_j \in C_0^\infty(\boldsymbol{R}^n)$ be such that $\operatorname{supp}\eta_j \subset B_j$, $0 \leq \eta_j \leq 1$, and $\sum_j \eta_j = 1$ on $\partial\Omega$. It will be helpful to know later on that, letting g be a given function on $\partial\Omega$ to which is associated some fixed $j \in \{1, ..., M\}$ with $\eta_j \equiv 1$ on supp$g$,

$$Sg(X) = c_n \int_{\pi_n(\phi_j(\text{supp}g))} \frac{\sqrt{1 + |\nabla\psi_j(x')|^2}}{[|x' - z'|^2 + |\psi_j(x') - y|^2]^{\frac{n-2}{2}}} g(x')dx' \tag{3.16}$$

where $\phi_j(X) = (z', y)$, $g(x') = g(\phi_j^{-1}(x', \psi_j(x')))$, dx' denotes Lebesgue measure on $\boldsymbol{R}^{n-1}$, and c_n represents the constant factor to the right of the equality sign in (3.15).

Of course, since we are interested in the inverse of the Calderon operator, we will be interested in both the Neumann and Dirichlet boundary values of these single layer potentials.

To bring the Neumann boundary values into consideration, define operators

$$K_\epsilon^* f(P) = \frac{1}{\omega_n} \int_{\{Q \in \partial\Omega | |P-Q| \geq \epsilon\}} \frac{< P - Q, n(P) >}{|P - Q|^n} f(Q)d\sigma(Q) \tag{3.17}$$

for $f \in L^1(\partial\Omega)$,$\epsilon > 0$, and set

$$\begin{aligned} K^* f(P) &= \text{p.v.}\frac{1}{\omega_n} \int_{\partial\Omega} \frac{< P - Q, n(P) >}{|P - Q|^n} f(Q)d\sigma(Q) \\ &\equiv \lim_{\epsilon \to 0} K_\epsilon^* f(P), \end{aligned} \tag{3.18}$$

(wherein $n(P)$ denotes the outward unit normal vector to $\partial\Omega$ defined a.e. $d\sigma$ on any Lipschitz domain) whenever $f \in L^1(\partial\Omega)$ is such that the limit in (3.18) converges for a.e. $d\sigma\, P \in \partial\Omega$. Also write

$$T = \frac{1}{2}I - K^*. \tag{3.19}$$

See [5] or [16] for more information on the operators (3.17)-(3.19).

In 1987 Dahlberg and Kenig (See [5],Theorems 4.17 and 4.18 as well as equations (1.1) and (1.2)) extended an L^2 result of Verchota's [20] by establishing the following theorem.

Theorem 3.9. *Let $\Omega \subseteq \boldsymbol{R}^n$ be a bounded Lipschitz domain whose complement is connected. Then there is $\epsilon = \epsilon(\Omega) > 0$ such that, whenever $1 < p < 2 + \delta$, T is* *an invertible mapping from* $L^p(\partial\Omega)_{1^\perp}$ *onto* $L^p(\partial\Omega)_{1^\perp}$, *and* S *is an invertible mapping from* $L^p(\partial\Omega)$ *onto* $L^p_1(\partial\Omega)$.

Moreover given $f \in L^p(\partial\Omega)_{1^\perp}$ *with* $1 < p < 2 + \delta$ *and writing* $u = ST^{-1}f$, *i.e.*,

$$u(X) = \frac{1}{\omega_n(n-2)} \int_{\partial\Omega} |X - Q|^{2-n} (\frac{1}{2}I - K^*)^{-1}(f)(Q) d\sigma(Q), \tag{3.20}$$

it follows that u *is the unique (modulo constants) harmonic function on* Ω *such that the nontangential maximal function* $M(\nabla u)$ *is bounded in* $L^p(\partial\Omega)$ *and* $\frac{\partial u}{\partial n} = f$ *nontangentially a.e. on* $\partial\Omega$ *in the sense that* $< \nabla u(X), n(Q) > \to f$ *as* $X \to Q$ *nontangentially for a.e.* $Q \in \partial\Omega$. *Finally, we have*

$$||M(\nabla u)||_{L^p(\partial\Omega)} \leq C||f||_{L^p(\partial\Omega)} \tag{3.21}$$

for all $f \in L^p(\partial\Omega)_{1^\perp}$.

Remark 3.10. From (3.21) and the Fundamental Theorem of Calculus we have that the maximal function $M(u)$ on $\partial\Omega$ is also bounded in L^p.

To determine the Dirichlet boundary values of our single layer potentials (see Proposition 3.12) we first prove a requisite lemma.

Lemma 3.11. *Let* $g \in L^p(\partial\Omega), 1 \leq p < \infty$. *Then*

$$Sg(X) \to Sg(Q) \text{ as } X \to Q$$

nontangentially, for a.e. $Q \in \partial\Omega$.

Proof. Given $Q, Q' \in \partial\Omega$, we have

$$||X - Q'|^{2-n} - |Q - Q'|^{2-n}| \leq C|X - Q|^{1/2}|Q - Q'|^{(3/2)-n}$$

for all $X \in \Gamma_\alpha(Q)$, some fixed nontangential cone at Q. Hence

$$|Sg(X) - Sg(Q)| \leq C|X - Q|^{1/2} \int_{\partial\Omega} |Q - Q'|^{(3/2)-n} |g(Q')| d\sigma(Q').$$

Write

$$M(Q) = \int_{\partial\Omega} |Q - Q'|^{3/2-n} |g(Q')| d\sigma(Q').$$

Then the generalized Young's inequality implies that, since $g \in L^p(\partial\Omega)$, $M(Q) \in L^p(\partial\Omega)$ as well. Thus $M(Q) < \infty$ for a.e. $d\sigma$ $Q \in \partial\Omega$. We conclude that

$$|Sg(X) - Sg(Q)| \leq C|X - Q|^{1/2} M(Q) \to 0$$

at every point at which $M(Q) < \infty$, and this readily implies the Lemma. $\square$

Proposition 3.12. *Let $1 < p < 2 + \delta$. Then, given $f \in L^p(\partial\Omega)_{1^\perp}$, we have that $u(X) = ST^{-1}f(X) \to ST^{-1}f(Q)$ as $X \to Q$ nontangentially from within Ω for a.e. $Q \in \partial\Omega$.*

Proof. Take $g = T^{-1}f \in L^p(\partial\Omega)$ and apply Lemma 3.11. $\square$

Now for $1 < p < 2 + \delta$ define the **inverse Calderón** or **Neumann to Dirichlet operator** $\Upsilon : L^p(\partial\Omega)_{1^\perp} \to L_1^p(\partial\Omega)$ by setting

$$\Upsilon(f) = (ST^{-1}(f))|\partial\Omega \tag{3.22}$$

for $f \in L^p(\partial\Omega)_{1^\perp}$. Then Theorem 3.9 and Proposition 3.12 imply a corollary

Corollary 3.13. *Given $f \in L^p(\partial\Omega)_{1^\perp}$, $1 < p < 2 + \delta$, we have the bound*

$$||\Upsilon f||_{L_1^p(\partial\Omega)} \leq C||f||_{L^p(\partial\Omega)} \tag{3.23}$$

In addition, the harmonic function $u = ST^{-1}f$ has nontangential Neumann boundary data f and nontangential Dirichlet boundary data $\Upsilon(f) = (ST^{-1}f)|\partial\Omega$.

Theorem 3.9 tells us that the function u has certain nontangential Neumann boundary values but we are, of course, also interested in weaker solution properties as well. To obtain these we use a version of Green's formulas on Lipschitz domains.(In the next section we will employ a very related formula directly within the proof of Theorem 3.6.) For a more general statement of the following lemma and direction to its proof, see Grisvard [11], Theorem 1.5.3.1, p.52.

Lemma 3.14.*Let Ω be a bounded Lipschitz domain, and assume that $\phi_0, \phi_1 \in C^\infty(\overline{\Omega})$. Then*

$$\int_\Omega \phi_0 \Delta\phi_1 \, dV = -\int_\Omega < \nabla\phi_0, \nabla\phi_1 > dV + \int_{\partial\Omega} \phi_0 < \nabla\phi_1, n > d\sigma,$$

where N is the outer unit normal vector field defined a.e. on $\partial\Omega$.

Theorem 3.15. *Suppose that $1 < p < 2 + \delta$. Assume that $f \in L^p(\partial\Omega)_{1^\perp}$ and let $u = ST^{-1}f$ be its single layer potential considered as a function on Ω. Then $u \in L_1^p(\Omega)$ and*

$$\int_{\partial\Omega} f \, \mathrm{Tr} v \, d\sigma = \int_\Omega < \nabla u, \nabla v > dV, \tag{3.24}$$

for all $v \in L^q_1(\Omega)$.

Proof. That $u|\tilde{\Omega} \in L^p(\tilde{\Omega})$ for any subdomain $\tilde{\Omega}$ with $\overline{\tilde{\Omega}} \subset$interior$(\Omega)$ is of course immediate since u is harmonic. That $u|(\Omega\backslash\tilde{\Omega}) \in L^p(\Omega\backslash\tilde{\Omega})$ follows from Remark 3.10 employing a method quite similar to the one we are about to use to prove (3.24) (but significantly simpler). Thus $u \in L^p(\Omega)$. Similarly, using (3.21), we have that $\nabla u \in L^p(\Omega)$ so that $u \in L^p_1(\Omega)$.

By density and continuity we need only establish (3.24) for $v \in C^\infty(\overline{\Omega})$. The idea is to find a sequence of subdomains $\Omega_K, K = 1, 2, ...$ of Ω for which $\overline{\Omega}_K \subset$interior$(\Omega_{K+1}) \subset$ interior(Ω). Since u is harmonic we will have $u \in C^\infty(\overline{\Omega}_K)$ for all K. This will enable us to apply Lemma 3.14 on each of these smaller domains. Then we only need to prove that

$$|\int_{\Omega_K} < \nabla u, \nabla v > dV - \int_\Omega < \nabla u, \nabla v > dV| \leq 1/K \tag{3.25}$$

and

$$|\int_{\partial\Omega_K} v < \nabla u, n_K > d\sigma_K - \int_{\partial\Omega} v f d\sigma| \leq 1/K \tag{3.26}$$

for all K, wherein n_K is the outer unit normal vector field defined a.e. on $\partial\Omega_K$ and $d\sigma_K$ denotes surface measure on $\partial\Omega_K$. That (3.25) holds will follow readily from dominated convergence as long as measure$(\Omega\backslash\Omega_K) \to 0$ fast enough since $\nabla u \in L^p(\Omega)$ and $\nabla v \in L^q(\Omega)$ implies that $< \nabla u, \nabla v >\in L^1(\Omega)$. In any case, to define the subdomains Ω_K, we invoke the covering of $\partial\Omega$ by balls $\{B_j = B(P_j, r)|j = 1, 2, ..., M\}$ as in the definition of a Lipschitz domain and recall our isometries ϕ_j of $\boldsymbol{R}^n$ which satisfy

$$\phi_j(B_j \cap \Omega) = \{(x', x_n)|x_n > \psi_j(x'), |(x', x_n)| \leq r\}.$$

Now for $\epsilon > 0$ consider Lipschitz hypersurfaces $\Sigma_{j\epsilon}$ contained in $B_j \cap \Omega$ of the form

$$\Sigma_{j\epsilon} = \{\phi_j^{-1}(x', \psi_j(x') + \epsilon)|x' \in \mathcal{O}_j\}, \tag{3.27}$$

where $\mathcal{O}_j$ is some open subset of $\pi_n \circ \phi_j(B_j \cap \partial\Omega) \subseteq \boldsymbol{R}^{n-1}$, and π_n defined on $\boldsymbol{R}^n$ is projection onto the first $n-1$ coordinates. Denoting by $n_{j\epsilon}$ the outward unit normal field on $\Sigma_{j\epsilon}$ and by $d\sigma_{j\epsilon}$ surface measure on $\Sigma_{j\epsilon}$ and letting $I_j : \boldsymbol{R}^{n-1} \to \boldsymbol{R}^n$ denote the inclusion $x' \mapsto (x', \psi_j(x'))$, note that

$$
\begin{aligned}
&|\int_{\Sigma_{j\epsilon}} v < \nabla u, n_{j\epsilon} > d\sigma_{j\epsilon} - \int_{\phi_j^{-1} o I_j(\mathcal{O}_j)} v f d\sigma| \\
\leq & \int_{\pi_n o \phi_j(B_j \cap \partial\Omega)} |v(\phi_j^{-1}(x', \psi_j(x') + \epsilon)) < \nabla u(\phi_j^{-1}(x', \psi_j(x') + \epsilon)), \frac{(d\phi_j)^{-1}((\nabla\psi_j(x'), -1))}{|(\nabla\psi_j(x'), -1)|} > \\
& -v(\phi_j^{-1}(x', \psi_j(x')))f(\phi_j^{-1}(x', \psi_j(x')))|\sqrt{1 + |\nabla\psi_j(x')|^2} dx' \qquad (3.28)
\end{aligned}
$$

(where $d\phi_j$ denotes the orthogonal matrix corresponding to the derivative of the isometry ϕ_j). But, the right-hand side of (3.28) tends to 0 as $\epsilon \to 0$ by dominated convergence since the integrand on the right in (3.28) is dominated by $C(M(\nabla u) + |f|) \in L^1(\partial\Omega)$ and $< \nabla u, n > \to f$ nontangentially a.e. $d\sigma$ (see Theorem 3.9).

Given K it is now a simple matter to piece together a finite number of subsets of the form $\Sigma_{j\epsilon}$ to form complete boundaries $\partial\Omega_K$ of (Lipschitz) subdomains Ω_K of Ω satisfying (3.26) (and (3.25)), though we note that the values for ϵ used may depend on both j and K. $\square$

Remark 3.16. Suppose that $1 < p < \infty$, and assume that the function $u \in L_1^p(\Omega)$ whose maximal function $M(u)$ is bounded in $L^p(\partial\Omega)$ has nontangential Dirichlet boundary values g in $L^p(\partial\Omega)$. Then $g = \mathrm{Tr}\, u$ a.e. $d\sigma$. The proof of this fact is rather similar in spirit to that of Theorem 3.15, and so we leave the intrepid reader to pursue the details on his own.

As its title suggests, the principal goal of this section it to state and prove a theorem yielding estimates for the inverse Calderón operator for a range sufficiently large to prove Thm. 3.6. Using interpolation, the estimates (3.23) of Cor. 3.13 are nearly all we need. However, to state the complete theorem, we will require an additional estimate, which is in the nature of an "ϵ - improvement" over what would have been possible with the estimates (3.23) alone. This estimate involves the Neumann problem for data in Hardy spaces on Lipschitz hypersurfaces and will require some preliminary definitions.

Let $\Delta(Q_0, r) = \{P \in \partial\Omega | |P - Q_0| < r\}$, and assume that r is less than $\mathrm{diam}(\partial\Omega)$. Also let $d = n - 1$ denote the dimension of $\partial\Omega$. Following Brown [1], we say that the function a is an **atom** for $H^{p'}(\partial\Omega)$, with $1 - \delta_m < p' < 1$ (where the subscript m

represents dependence on the Lipschitz constant m of $\partial\Omega$), if for some Q_0 and r we have

$$\begin{array}{ll} (i) & \operatorname{supp} a \subset \Delta(Q_0, r) \\ (ii) & \int_{\Delta(Q_0,r)} a(Q)d\sigma(Q) = 0 \\ (iii) & \|a\|_{L^2(\Delta(Q_0,r))} \leq Cr^{-d(1/p'-1/2)}. \end{array}$$

When $1 \geq p' > \frac{d}{d+1}$, we denote by $H^{p'}(\partial\Omega)$ the **Hardy space on $\partial\Omega$** (with index p') and define it as the collection

$$\{f | f = \Sigma\lambda_j a_j \text{ with } \Sigma|\lambda_j|^{p'} < \infty\}$$

for some sequence of atoms a_j and complex numbers λ_j. The quasi-norm for $H^{p'}(\partial\Omega)$ is given by

$$\|f\|^{p'}_{H^{p'}(\partial\Omega)} = \inf\{\Sigma|\lambda_j|^{p'} \,|f = \Sigma\lambda_j a_j\}.$$

We also need the related space $\tilde{H}^{p'}(\partial\Omega)$ which is defined in the same way as $H^{p'}(\partial\Omega)$ except that we also include as an atom the characteristic function $\chi_{\partial\Omega}$ on $\partial\Omega$.

The **space of distributions on $\partial\Omega$ with one Hardy space derivative,** is denoted $H_1^{p'}(\partial\Omega)$. Once again we give a precise definition by first defining atoms. Thus we say that A is an **atom** for $H_1^{p'}(\partial\Omega)$ if for some $Q_0 \in \partial\Omega$ and $r > 0$, we have

$$\begin{array}{ll} (i) & \operatorname{supp} A \subset \Delta(Q_0, r) \cap \partial\Omega \\ (ii) & \|\nabla_{tan} A\|_{L^2(\partial\Omega)} \leq r^{-d(1/p'-1/2)}, \end{array}$$

wherein the tangential derivative $\nabla_{tan} f$ is defined for smooth f in a neighborhood $B(P_j, r) \cap \partial\Omega$ of $P_j = P \in \partial\Omega$ as the vector

$$\nabla_{tan} f(P) = (\frac{\partial f}{\partial T_1}(P), ..., \frac{\partial f}{\partial T_{n-1}}(P))$$

where

$$\frac{\partial f}{\partial T_i}(P) = \frac{\partial}{\partial x_i}(f \circ \phi_j^{-1}(\cdot, \psi(\cdot)))|_{x=x'} \text{ for } i = 1, ..., n-1, \; P = \phi_j^{-1}(x', \psi_j(x')),$$

Just as in the case above for $H^{p'}(\partial\Omega)$ we now define $H_1^{p'}(\partial\Omega)$ to be the $l^{p'}$-span of these atoms.

(**Note:** The key point of the definition for the space $H_1^{p'}(\partial\Omega)$ is that the way in which its atoms have been defined makes them essentially equivalent to atoms on $\boldsymbol{R}^{n-1}$. This observation is most easily understood when $\partial\Omega$ is the domain above a Lipschitz graph defined on $\boldsymbol{R}^{n-1}$. Then, defining the spaces $H_1^{p'}(\partial\Omega)$ and $H^{p'}(\partial\Omega)$ just as we have defined them above for boundaries of bounded Lipschitz domains, the derivatives of the resulting atoms for the space $H_1^{p'}(\partial\Omega)$ will in fact be members of the standard euclidean Hardy space $H^{p'}(\boldsymbol{R}^{n-1}, dx)$, where dx is Lebesgue measure. In contrast $H^{p'}(\partial\Omega)$ is identical to the space $H^{p'}(\boldsymbol{R}^{n-1}, d\sigma)$.)

Having set forth these definitions, we may now state an existence theorem for the Neumann problem in Hardy spaces.

Theorem 3.17[1]*Let* $1 > p' > 1 - \delta_m$ *and suppose that* $f \in H^{p'}(\partial\Omega)$. *Then the interior Neumann problem with data* f *has a solution* u *which satisfies*

$$||u|\partial\Omega||_{H_1^{p'}(\partial\Omega)} + ||M(\nabla u)||_{L^{p'}(\partial\Omega)} \leq C||f||_{H^{p'}(\partial\Omega)}. \tag{3.29}$$

when u *is normalized by* $u(0) = 0$.

We shall also need R. Brown's corresponding uniqueness result.

Theorem 3.18[1]*If* $1 - \delta_m < p' < 1$, *and* u *satisfies*

$$\begin{cases} \Delta u = 0 \\ M(\nabla u) \in L^{p'}(\partial\Omega) \end{cases} \tag{3.30}$$

with $\frac{\partial u}{\partial n}$ *vanishing in the* $H^{p'}$*-sense, then* u *is a constant.*

We will not need to know precisely what it means to "vanish in the $H^{p'}$-sense" (though of course the interested reader may find a definition for this in Brown's paper). Suffice it to say that the condition that $\frac{\partial u}{\partial n} = 0$ nontangentially a.e. on $\partial\Omega$, which we *will have*, will be strong enough to imply it.

Before venturing any further, let's define the **positive boundary Hölder space** as the dual space

$$B_\alpha^\infty(\partial\Omega) = (H^{p'(\alpha)}(\partial\Omega))^*,$$

where α is positive but near 0 and $p'(\alpha) \equiv \frac{n-1}{\alpha+n-1}$, under the pairing

$$< g, f >= \int_{\Omega} gf \, d\sigma$$

with $f \in C^{\infty}(\partial\Omega)$, $\int_{\partial\Omega} f \, d\sigma = 0$, and $g \in C^{\infty}(\partial\Omega)$ Thus the norm is

$$||g||_{B_{\alpha}^{\infty}(\partial\Omega)} = \sup\{| \int_{\partial\Omega} gf \, d\sigma| \mid f \in C^{\infty}(\partial\Omega), \int_{\partial\Omega} f \, d\sigma = 0, ||f||_{H^{p'(\alpha)}(\partial\Omega)} \leq 1\}.$$

An equivalent norm is

$$||g||_{B_{\alpha}^{\infty}(\partial\Omega)} = \sum_{j=1}^{M} ||(\eta_j g)(\phi_j^{-1}(\cdot, \psi_j(\cdot)))||_{B_{\alpha}^{\infty}},$$

The **negative boundary Hölder space** $B_{\alpha-1}^{\infty}(\partial\Omega)$, once again with α positive but near 0, is defined as the dual space

$$B_{\alpha-1}^{\infty}(\partial\Omega) = (H_1^{p'(\alpha)}(\partial\Omega))^*$$

under the pairing

$$< g, f >= \int_{\partial\Omega} gf \, d\sigma$$

with $f, g \in C^{\infty}(\partial\Omega)$. So the corresponding norm is

$$||g||_{B_{\alpha-1}^{\infty}(\partial\Omega)} = \sup\{| \int_{\partial\Omega} gf \, d\sigma| \mid f \in C^{\infty}(\partial\Omega), ||f||_{H_1^{p'(\alpha)}(\partial\Omega)} \leq 1\}$$

for $g \in C^{\infty}(\partial\Omega)$.An equivalent norm is

$$||g||_{B_{\alpha-1}^{\infty}(\partial\Omega)} = \sum_{j=1}^{M} ||(\eta_j g)(\phi_j^{-1}(\cdot, \psi_j(\cdot))\sqrt{1 + |\nabla\psi_j(\cdot)|^2}||_{B_{\alpha-1}^{\infty}},$$

We note that we have not yet defined the space $B_{-s}^{\infty}(\boldsymbol{R}^{n-1}), s > 0$, and do not intend to do so, due to the technicalities involved. Instead, we refer the reader to Definition 2, p.45 of Triebel [18] for a precise definition. (We in fact define $B_{-s}^{\infty}(\boldsymbol{R}^{n-1})$ as Triebel's $B_{\infty,\infty}^{-s}(\boldsymbol{R}^{n-1})$ space.)

(These two boundary Hölder space definitions are made possible by the corresponding duality statement on $\boldsymbol{R}^{n-1}$:

$$(F_{p'}^{s,2}(\boldsymbol{R}^{n-1}))^* = B_{-s+\alpha(p')}^{\infty}(\boldsymbol{R}^{n-1}),$$

where $-\infty < s < \infty$, $0 < p' < 1$, $\alpha(p') = (n-1)(1-p')/p'$ (see Triebel [18], Theorem 2.11.3, p. 180) and $F^{s,2}_{p'}(\boldsymbol{R}^{n-1})$ is a Triebel-Lizorkin space on $\boldsymbol{R}^{n-1}$(see the proof of Theorem 3.20 for more information on these spaces and direction to references for them), as well as the redefinition (3.41) of the quasi-norm for $H^{p'}_1(\partial\Omega)$ and the corresponding redefinition of the quasi-norm on $H^{p'}(\partial\Omega)$ in terms of that for the euclidean local Hardy spaces $h^{p'} = h^{p'}(\boldsymbol{R}^{n-1})$ (see [10]):

$$||f||_{H^{p'}(\partial\Omega)} = \sum_{j=1}^{M} ||(\eta_j f)(\phi_j^{-1}(\cdot,\psi_j(\cdot)))\sqrt{1+|\nabla\psi_j(\cdot)|^2}||_{h^{p'}}.$$

for $f \in C^\infty(\partial\Omega)$, $\int_{\partial\Omega} f\, d\sigma = 0$.)

The final ingredients required prior to stating our estimates for the inverse Calderon operator are the necessary interpolation results. These we collect within Theorem 3.20, after introducing the following lemma, whose proof is routine.

Lemma 3.19(Retraction Lemma). *Suppose A_i, B_i, $i = 0, 1$, are quasi-Banach spaces (i.e., complete quasi-normed spaces in the sense of Triebel [18]) and assume there exist linear operators $I : A_0 + A_1 \to B_0 + B_1$ and $P : B_0 + B_1 \to A_0 + A_1$ such that we have bounded restrictions $I : A_0 \to B_0$, $I : A_1 \to B_1$, $P : B_0 \to A_0$, and $P : B_1 \to A_1$. In addition assume that for some quasi-Banach space $C \subseteq A_0 + A_1$ and some $\theta \in [0,1]$ we have $I(C) \subseteq [B_0, B_1]_\theta$ as well as $P([B_0, B_1]_\theta) \subseteq C$ and that $P \circ I$ is the identity on $A_0 + A_1$. Then*

$$C = [A_0, A_1]_\theta. \tag{3.31}$$

Theorem 3.20. *Let Ω be a bounded Lipschitz domain.*

$$[L^{p_0}_{s_0}(\partial\Omega), L^{p_1}_{s_1}(\partial\Omega)]_\theta = L^p_s(\partial\Omega), \tag{3.32}$$

where $1 < p_0, p_1 < \infty, -1 \leq s_0, s_1 \leq 0$ or $0 \leq s_0, s_1 \leq 1, s = (1-\theta)s_0 + \theta s_1$, and $1/p = (1-\theta)/p_0 + \theta/p_1$.

$$[L^p_{s_0}(\partial\Omega), L^p_{s_1}(\partial\Omega)]_{\theta,p} = B^p_s(\partial\Omega), \tag{3.33}$$

wherein $1 < p < \infty$,$-1 \leq s_0 \neq s_1 \leq 0$ or $0 \leq s_0 \neq s_1 \leq 1$, $s = (1-\theta)s_0 + \theta s_1$, and $1/p = (1-\theta)/p_0 + \theta/p_1$.

$$[B^{p_0}_{s_0}(\partial\Omega), B^{p_1}_{s_1}(\partial\Omega)]_\theta = B^p_s(\partial\Omega), \tag{3.34}$$

whenever $1 < p_0, p_1 \leq \infty$, $-1 < s_0 \neq s_1 < 0$ *or* $0 < s_0 \neq s_1 < 1$, $s = (1-\theta)s_0 + \theta s_1$, *and* $1/p = (1-\theta)/p_0 + \theta/p_1$.

$$[L^{p_0}(\partial\Omega), H_1^{p'}(\partial\Omega]_\theta = L_s^p(\partial\Omega), \tag{3.35}$$

with $0 < p' < 1$, $1 < p_0 \leq 2$, $s = \theta$, $1/p = (1-\theta)/p_0 + \theta/p'$,*and* $0 \leq \theta < \frac{1}{1/p' - 1/p_0}(1 - 1/p_0)$ *(This remarkable upper bound for* θ *has been chosen merely so that* p *remains greater than* 1.*). Lastly,*

$$[L_{-1}^{p_0}(\partial\Omega), \tilde{H}^{p'}(\partial\Omega)]_\theta = L_s^p(\partial\Omega), \tag{3.36}$$

within the ranges $0 < p' < 1$,$1 < p_0 \leq 2$, $s = \theta - 1$, $1/p = (1-\theta)/p_0 + \theta/p'$, *and* $0 \leq \theta < \frac{1}{1/p' - 1/p_0}(1 - 1/p_0)$.

The second of these interpolation results follows from the real method of interpolation and the others are contained within the theory of complex interpolation.

Proof. All of these interpolation formulas follow from corresponding known formulas for function spaces on $\boldsymbol{R}^{n-1}$. Let us illustrate this reduction to the euclidean case by establishing (3.36). For this formula the function spaces required are contained within the inhomogeneous Triebel-Lizorkin scale $F_p^{\alpha,q} = F_p^{\alpha,q}(\boldsymbol{R}^{n-1}), -\infty < \alpha < \infty, 0 < p < \infty, 0 < q \leq \infty$, of quasi-Banach spaces. (Definitions for these spaces may be found in Triebel [18] or [19], in Frazier-Jawerth [7], or in Frazier-Jawerth-Weiss [8].). For these spaces we have

$$[F_{p_0}^{\alpha_0,q_0}(\boldsymbol{R}^{n-1}), F_{p_1}^{\alpha_1,q_1}(\boldsymbol{R}^{n-1})]_\theta = F_p^{\alpha,q}(\boldsymbol{R}^{n-1}) \tag{3.37}$$

wherein $-\infty \leq \alpha_0, \alpha_1 \leq \infty, 0 < p_0, p_1 < \infty, 0 < q_0, q_1 \leq \infty, \alpha = (1-\theta)\alpha_0 + \theta\alpha_1, 1/p = (1-\theta)/p_0 + \theta/p_1$, and $1/q = (1-\theta)/q_0 + \theta/q_1$, see [18]. It will certainly be crucial to the present argument that, as is rather well-known, the Sobolev spaces $L_\alpha^p(\boldsymbol{R}^{n-1}) = F_p^{\alpha,2}(\boldsymbol{R}^{n-1})$ for $-\infty < \alpha < \infty$ and $1 < p < \infty$ and that the local Hardy spaces of Goldberg [10] (see also Stein [16]) $h^{p'}(\boldsymbol{R}^{n-1}) = F_{p'}^{0,2}(\boldsymbol{R}^{n-1})$ for $0 < p' < 1$.

To prove (3.36) let $\{B(P_j, r) | j = 1, 2, ..., M\}$ be a covering of $\partial\Omega$ by balls as in the definition of a Lipschitz domain and let $\eta_j \in C_0^\infty(\boldsymbol{R}^n)$ such that $\operatorname{supp}\eta_j \subset B(P_j, r)$, $0 \leq \eta_j \leq 1$, and $\sum_j \eta_j = 1$ on $\partial\Omega$. Let ϕ_j be an isometry of $\boldsymbol{R}^n$such that

$$\phi_j(B(P_j, r) \cap \Omega) = \{(x', x_n) | x_n > \psi_j(x'), |(x', x_n)| \leq r\}. \tag{3.38}$$

Now, considering our retraction lemma, we select $A_1 = \tilde{H}^{p'}(\partial\Omega)$ and $A_0 = L^{p_0}_{-1}(\partial\Omega)$ and decree that B_1 is the space defined as the product of N copies of the space $F^{0,2}_{p'}(\boldsymbol{R}^{n-1}) = h^{p'}(\boldsymbol{R}^{n-1})$ and B_0 is the space defined as the product of N copies of the space $F^{-1,2}_{p_0}(\boldsymbol{R}^{n-1}) = L^{p_0}_{-1}(\boldsymbol{R}^{n-1})$. If g is a measurable function on $\partial\Omega$ we define Ig by requiring that its jth component be given by

$$Ig(x')_j = (\eta_j g)(\phi_j^{-1}(x', \psi_j(x')))\omega_j(x'), \tag{3.39}$$

$x' \in \boldsymbol{R}^{n-1}$, wherein $\omega_j(x') = \sqrt{1 + |\nabla\psi_j(x')|^2}$ for $j = 1, ..., M$ is surface density measure corresponding to the Lipschitz mapping ψ_j.

To define the operator P we let $\overline{\eta}_j \in C_0^\infty(B(P_j, 2r)), j = 1, ..., M$, be functions satisfying $\overline{\eta_j} = 1$ on $B(P_j, r)$, and we let $\pi_n(x', x_n) = x'$ be projection on the first $n-1$ coordinates. Now define

$$P((f_j))_{1\le j\le N})(Q) = \sum_{j=1}^{M} \overline{\eta}_j(Q) f_j(\pi_n \circ \phi_j(Q))(\omega_j(\pi_n \circ \phi_j(Q)))^{-1} \tag{3.40}$$

for $Q \in \partial\Omega$ and functions f_j on $\boldsymbol{R}^{n-1}$. With the operators I and P defined in this way (3.36) follows at once from (3.37) and our retraction lemma 3.19.

Formulas (3.32), (3.33), and (3.34) follow in a similar way from known results for Sobolev and Besov spaces on $\boldsymbol{R}^{n-1}$. For all three of these formulas, when $-1 \le s_0, s_1 \le 0$, the operators I and P are defined just as they were above for the proof of (3.36). However, for the positive range $0 \le s_0, s_1 \le 1$, the weight factor in (3.39) and its reciprocal in (3.40) are removed.

The derivation of (3.35) is also analogous to that for (3.36); to obtain it we again invoke the spaces $F^{\alpha,q}_p(\boldsymbol{R}^{n-1})$. Note that the quasi-Banach space $H^{p'}_1(\partial\Omega)$ may be redefined as the completion of $C^\infty(\partial\Omega)$ with respect to the (quasi-)norm

$$\sum_{j=1}^{M} ||(\eta_j g)(\phi_j^{-1}(\cdot, \psi_j(\cdot))||_{F^{1,2}_{p'}}. \tag{3.41}$$

(This is a consequence of the definition of the atoms for $H^{p'}_1(\partial\Omega)$ and the atomic characterization of $F^{1,2}_{p'}$ given in Triebel [19]; see the theorem in sec. 3.2.3 as well as proposition (ii) in sec. 3.2.4.) Thus to adapt our argument for (3.36) to prove (3.35) we use $F^{1,2}_{p'}(\boldsymbol{R}^{n-1})$ in place of $F^{0,2}_{p'}(\boldsymbol{R}^{n-1})$ and $F^{0,2}_{p_0}(\boldsymbol{R}^{n-1}) = L^{p_0}(\boldsymbol{R}^{n-1})$ in place of

$F^{-1,2}_{p_0}(\boldsymbol{R}^{n-1})$ (and of course eliminate the weight factors and their reciprocals in the definitions of the operators I and P respectively). $\square$

Remark 3.21. Any one of the interpolation results (3.32),(3.33),(3.34), or (3.36) in Theorem 3.20 remains true when each of the usual function spaces occurring within its statement is replaced by its mean-value-0 counterpart. (Of course, by the mean-value-0 counterpart of $\tilde{H}^{p'}(\partial\Omega)$ we simply mean $H^{p'}(\partial\Omega)$ itself.) Once again, this is a simple consequence of retraction.

Our Besov space estimates for the inverse Calderón operator are contained in the following theorem.

Theorem 3.22. *There exists ϵ with $0 < \epsilon \leq 1$ so that the inverse Calderón operator Υ introduced in (3.22) satisfies*

$$\|\Upsilon g\|_{B^p_s(\partial\Omega)} \leq C\|g\|_{B^p_{s-1}(\partial\Omega)}, \tag{3.42}$$

provided p and s lie within the ranges

(a) $p_0 < p < p_0'$ *and* $0 < s < 1$

(b) $1 < p \leq p_0$ *and* $2/p - 1 - \epsilon < s < 1$

(c) $p_0' < p < \infty$ *and* $0 < s < 2/p + \epsilon$,

wherein $1/p_0 = \frac{1+\epsilon}{2}, 1/p_0' = \frac{1-\epsilon}{2}$.

The range for p and s for which estimates hold may be described as the open hexagon in $(s, 1/p)$-space with vertices $(0,0), (0, 1/p_0), (1-\epsilon, 1), (1,1), (1, 1/p_0')$, and $(\epsilon, 0)$.

The Proof of Theorem 3.22. Recall from Corollary 3.13 that, for $1 < p < 2+\delta$, we have the operator $\Upsilon : L^p(\partial\Omega)_{1^\perp} \to L^p_1(\partial\Omega)$. Considering the Banach space adjoints of these operators we get mappings $\Upsilon^*_q : L^q_{-1}(\partial\Omega) \to L^q(\partial\Omega)_{1^\perp}$ and hence mappings $\Upsilon^*_q : L^q_{-1}(\partial\Omega)_{1^\perp} \to L^q(\partial\Omega)$ for $2 - \delta < q < \infty$. We will obtain our theorem by interpolating between a rather wide selection of pairs of these operators. Thus we wish to prove that, wherever their domains of definition intersect, any two of these operators coincide.

Our strategy will be to show that all of these maps agree with $\Upsilon : L^2(\partial\Omega)_{1^\perp} \to L^2_1(\partial\Omega)$, whenever their domains intersect with $L^2(\partial\Omega)_{1^\perp}$, and this will suffice because

$C^\infty(\overline{\Omega})|\partial\Omega$ is a common dense subset of all of the spaces $L^p(\partial\Omega), L^p_{-1}(\partial\Omega), 1 < p < \infty$. Of course all of the maps $\Upsilon : L^p(\partial\Omega)_{1^\perp} \to L^p_1(\partial\Omega)$, $1 < p < 2+\delta$, coincide on their intersections, since the definitions of the single layer potential S and the operator $T = \frac{1}{2}I - K^*$ do not depend on the particular value of p.

Now let's show that $\Upsilon_2^* : L^2_{-1}(\partial\Omega)_{1^\perp} \to L^2(\partial\Omega)$ restricted to $L^2(\partial\Omega)_{1^\perp}$ is compatible with $\Upsilon : L^2(\Omega)_{1^\perp} \to L^2_1(\partial\Omega)$. Suppose that $f, g \in L^2(\partial\Omega)_{1^\perp}$, and let u_f and u_g be the corresponding unique harmonic functions of Cor. 3.13 with Neumann data f and g respectively. Then from Theorem 3.15 and Remark 3.16 it follows that

$$\Upsilon_2^* f(g) = \int_{\partial\Omega} f\Upsilon(g) d\sigma = \int_\Omega < \nabla u_f, \nabla u_g > dV = \int_{\partial\Omega} \Upsilon(f) g d\sigma = (\Upsilon f)(g), \quad (3.43)$$

which is as we wished.

To show that $\Upsilon : L^2(\partial\Omega)_{1^\perp} \to L^2_1(\partial\Omega)$ is compatible with $\Upsilon_q^* : L^q_{-1}(\partial\Omega)_{1^\perp} \to L^q(\partial\Omega)$ for *all* values of $q \in (2-\epsilon, \infty)$, we can use what we just proved in the previous paragraph, since, for $f, g \in C^\infty(\overline{\Omega})|\partial\Omega$,

$$(\Upsilon_q^* f)(g) = \int_{\partial\Omega} f\Upsilon(g) d\sigma = (\Upsilon_2^* f)(g) = (\Upsilon f)(g). \quad (3.44)$$

Furthermore, we note that, from the uniqueness result (3.30) of Thm. 3.18 it follows that, whenever $1 < p < 2+\delta$ and $1-\delta_m < p' < 1$, we have that $\Upsilon|(L^p(\partial\Omega)_{1^\perp}) \cap H^{p'}(\partial\Omega) = \Upsilon_{p'}|(L^p(\partial\Omega)_{1^\perp}) \cap H^{p'}(\partial\Omega)$, provided $\Upsilon_{p'}$ refers to the mapping

$$\Upsilon_{p'} : H^{p'}(\partial\Omega) \to H_1^{p'}(\partial\Omega) \quad (3.45)$$

defined by setting $\Upsilon_{p'} f = u|\partial\Omega$, wherein $u|\partial\Omega$ is as in Theorem 3.17.

Finally, we dualize (3.45) to obtain an operator

$$\Upsilon_{p'}^* : (H_1^{p'}(\partial\Omega))^* \to (H^{p'}(\partial\Omega))^*. \quad (3.46)$$

Recall our definition (located between Theorem 3.18 and Lemma 3.19)

$$(H^{p'}(\partial\Omega))^* = B^\infty_{\alpha(p')}(\partial\Omega), \quad (3.47)$$

wherein $\alpha(p') = (n-1)(1-p')/p'$. Also recall that the duality statement

$$(F^{1,2}_{p'}(\boldsymbol{R}^{n-1}))^* = B^\infty_{-1+\alpha(p')}(\boldsymbol{R}^{n-1}) \quad (3.48)$$

on $\boldsymbol{R}^{n-1}$ implies that

$$(H_1^{p'}(\partial\Omega))^* = B^{\infty}_{-1+\alpha(p')}(\partial\Omega). \tag{3.49}$$

This means that we may rewrite (3.46) as

$$\Upsilon^*_{p'} : B^{\infty}_{-1+\alpha(p')}(\partial\Omega) \to B^{\infty}_{\alpha(p')}(\partial\Omega), \tag{3.50}$$

and thus restriction yields a mapping

$$\Upsilon^*_{p'} : B^{\infty}_{-1+\alpha(p')}(\partial\Omega)_{1^\perp} \to B^{\infty}_{\alpha(p')}(\partial\Omega), \tag{3.51}$$

That this mapping is also consistent with $\Upsilon : L^2(\partial\Omega)_{1^\perp} \to L^2_1(\partial\Omega)$ for the purposes of interpolation one readily sees by noting that $L^2(\partial\Omega)_{1^\perp} \cap B^{\infty}_{-1+\alpha(p')}(\partial\Omega)$ is dense in $B^{\infty}_{-1+\alpha(p')}(\partial\Omega)_{1^\perp}$ and considering that

$$\Upsilon^*_{p'}(f)(a) = f(\Upsilon_{p'}(a)) = f(\Upsilon(a)) = \Upsilon(f)(a), \tag{3.52}$$

for all $f \in L^2(\partial\Omega)_{1^\perp}$ and all $a \in L^2(\partial\Omega)_{1^\perp}$ an atom for $H^{p'}(\partial\Omega)$, using our conclusions from the preceding paragraph.

The fact that any two of these myriad operators coincide on the intersection of their domains now justifies our eliminating all subscripts as well as the superscript * wherever they occur and simply referring to all of these mappings as Υ.

Having decisively settled all relevant compatibility questions, it is at last time to interpolate. Recalling the identifications and interpolation results discussed prior to the statement of the present theorem, we use (3.35) and (3.36) to interpolate between $\Upsilon : L^{p_0}_{-1}(\partial\Omega)_{1^\perp} \to L^{p_0}(\partial\Omega)$ and $\Upsilon : H^{p'}(\partial\Omega) \to H^{p'}_1(\partial\Omega)$, with $1 - \delta_m < p' < 1$ to obtain an operator

$$\Upsilon : L^p_{2/p-2-\epsilon}(\partial\Omega)_{1^\perp} \to L^p_{2/p-1-\epsilon}(\partial\Omega) \tag{3.53}$$

for $1 < p \leq p_0$ and $\epsilon > 0$ defined via $\epsilon \equiv \frac{2}{p'} - 2$.

Now to obtain (3.42) in case (a) we simply fix $p \in (p_0, p_0')$ and use (3.33) to interpolate between $\Upsilon : L^p(\partial\Omega)_{1^\perp} \to L^p_1(\partial\Omega)$ and $\Upsilon : L^p_{-1}(\partial\Omega)_{1^\perp} \to L^p(\partial\Omega)$. For case (b), we fix p with $1 < p \leq p_0$ and use (3.33) to interpolate between (3.53) and the operator $\Upsilon : L^p(\partial\Omega)_{1^\perp} \to L^p_1(\partial\Omega)$.

To obtain estimates for the range $p_0' \leq p < \infty$ of case (c), we first observe that Υ satisfies

$$\Upsilon : B^p_{s-1}(\partial\Omega)_{1^\perp} \to B^p_s(\partial\Omega) \tag{3.54}$$

for $0 < s < \frac{2+\epsilon p_0'}{p}$ and $p_0' \leq p < \infty$. This is because, fixing p and s within this range, we can choose p_1 so large that the straight line l joining $(0, 1/p_1)$ and $(1, 1/p_0')$ (in $(s, 1/p)$-space) intersects the horizontal line passing through $(0, 1/p)$ at a point $(s_0, 1/p)$ to the right of $(s, 1/p)$. Then of course $s < s_0$ and we can use (3.32) to interpolate between $\Upsilon : L^{p_0'}(\partial\Omega)_{1^\perp} \to L^{p_0'}_1(\partial\Omega)$ and $\Upsilon : L^{p_1}_{-1}(\partial\Omega)_{1^\perp} \to L^{p_1}(\partial\Omega)$ to obtain an operator

$$\Upsilon : L^p_{s_0-1}(\partial\Omega)_{1^\perp} \to L^p_{s_0}(\partial\Omega). \tag{3.55}$$

To realize (3.54) we can then interpolate between (3.55) and the map $\Upsilon : L^p_{-1}(\partial\Omega)_{1^\perp} \to L^p(\partial\Omega)$ using (3.33).

Using Besov space interpolation (3.34) and our estimate $\Upsilon : B^\infty_{-1+\alpha(p')}(\partial\Omega)_{1^\perp} \to B^\infty_{\alpha(p')}(\partial\Omega)$ for $1 - \delta_m < p' < 1$ while noting that $\epsilon \equiv 2(1/p' - 1) \leq \alpha(p') \equiv (n - 1)(1/p' - 1)$, it is easy to see how to extend the bound (3.54) to the full range in case (c) of the statement of our theorem. □

3.4 Proof of Existence

Let $F \in (L^q_{2-\alpha}(\Omega))^*$, with $1/p + 1/q = 1$ and p and α falling within the appropriate range for the theorem. Of course $R^*_\Omega F \in L^p_{\alpha-2}$ with supp $R^*_\Omega F \subseteq \overline{\Omega}$. Setting $h = N * (R^*_\Omega F)$, where N is the Newtonian potential on $\boldsymbol{R}^n$, it follows that $\Delta h = R^*_\Omega F$ on $\boldsymbol{R}^n$ and, via the Elliptic Regularity Theorem, that $h = h|\Omega \in L^p_\alpha$; in fact we have $||h||_{L^p_\alpha(\Omega)} \leq C||R^*_\Omega F||_{L^p_{\alpha-2}}$. Moreover, since the relevant definitions imply $||R^*_\Omega F||_{L^p_{\alpha-2}} \leq C||F||_{(L^q_{2-\alpha}(\Omega))^*}$, we obtain

$$||h||_{L^p_\alpha(\Omega)} \leq C||F||_{(L^q_{2-\alpha}(\Omega))^*}. \tag{3.56}$$

Now define the **normal derivative** $L_h \in B^p_{\alpha-1-1/p}(\partial\Omega)$ of the function h by setting

$$< L_h, g >=< F, v_g > + < \chi_\Omega \nabla(E_\Omega(h)), \nabla(E_\Omega(v_g)) > \tag{3.57}$$

for all $g \in B^q_{1+1/p-\alpha}(\partial\Omega)$, where v_g is the unique harmonic function in $L^q_{2-\alpha}(\Omega)$ with $\mathrm{Tr}\, v_g = g$ given by the estimate for the homogeneous Dirichlet problem, Thm. 3.8. The normal derivative L_h is a bounded linear functional on $B^q_{1+1/p-\alpha}(\partial\Omega)$. In fact we even have

Proposition 3.23. *With f, h and L_h as above,*

$$||L_h||_{B^p_{\alpha-1-1/p}(\partial\Omega)} \leq C||F||_{(L^q_{2-\alpha}(\Omega))^*} \tag{3.58}$$

Proof. Consider first the leading term in (3.57). The estimate for the homogeneous Dirichlet problem gives

$$\begin{aligned} |< F, v_g >| &\leq ||F||_{(L^q_{2-\alpha}(\Omega))^*}||v_g||_{L^q_{2-\alpha}(\Omega)} \\ &\leq C||F||_{(L^q_{2-\alpha}(\Omega))^*}||g||_{B^q_{1+1/p-\alpha}(\partial\Omega)}. \end{aligned} \tag{3.59}$$

Meanwhile, taking stock of the second term, we find that

$$\begin{aligned} |< \chi_\Omega\nabla(E_\Omega(h)), \nabla(E_\Omega(v_g)) >| &\leq ||\chi_\Omega\nabla(E_\Omega(h))||_{L^p_{\alpha-1}}||\nabla(E_\Omega(v_g))||_{L^q_{1-\alpha}} \\ &\leq C||\nabla(E_\Omega(h))||_{L^p_{\alpha-1}}||\nabla(E_\Omega(v_g))||_{L^q_{1-\alpha}} \\ &\leq C||E_\Omega(h)||_{L^p_\alpha}||E_\Omega v_g||_{L^q_{2-\alpha}} \\ &\leq C||h||_{L^p_\alpha(\Omega)}||v_g||_{L^q_{2-\alpha}(\Omega)} \\ &\leq C||F||_{(L^p_{2-\alpha}(\Omega))^*}||g||_{B^q_{1+1/p-\alpha}(\partial\Omega)}, \end{aligned} \tag{3.60}$$

upon employing, respectively, the boundedness of the truncation operator (Prop. 3.4 and Cor. 3.5), Stein's Extension Thm. (See Prop. 3.2), and the estimate for the homogeneous Dirichlet problem (Theorem 3.8). □

Recall that we are attempting to find a solution to the generalized inhomogeneous Neumann problem NP(p, α, F). Considering (3.8), it seems clear that we must show that we can replace v_g in (3.57) with general $v \in L^q_{2-\alpha}(\Omega)$,i.e.,not just those functions in $L^q_{2-\alpha}(\Omega)$ which happen to be harmonic. To accomplish this we need a density lemma.

Lemma 3.24. *The space $L^p(\Omega)$ is dense in $(L^q_s(\Omega))^*$ for $s \geq 0$.*

Proof. Let $F \in (L^q_s(\Omega))^*$. Then $R^*_\Omega F \in L^p_{-s}$. Since it is well-known that $C^\infty_0(\boldsymbol{R}^n)$ is dense in $L^p_{-s}(\boldsymbol{R}^n)$, we can choose a sequence $< \phi_i >$ in $C^\infty_0(\boldsymbol{R}^n)$ such that

$\phi_i \to R^*_\Omega F$ in L^p_{-s}. Hence, applying E^*_Ω to both sides of this limiting process gives $E^*_\Omega \phi_i \to F$ in $(L^q_s(\Omega))^*$ by Remark 4.3, and it also follows from this remark that, for all i, $E^*_\Omega \phi_i \in L^p(\Omega)$. □

Using this lemma it's a simple matter to show that $C^\infty(\overline{\Omega})$ and even $C^\infty_0(\Omega)$ are likewise dense in $(L^q_s(\Omega))^*$, $s \geq 0$.

We shall also require another integration-by-parts formula on Lipschitz domains. Once again for direction to its proof consult Grisvard [11], Thm. 1.5.3.1.

Lemma 3.25. *For every $u \in L^p_2(\Omega)$ and $v \in L^q_1(\Omega)$ we have*

$$\int_\Omega v \Delta u \, dV = -\int_\Omega < \nabla u, \nabla v > dV + \int_{\partial\Omega} \mathrm{Tr\,v} < \mathrm{Tr}(\nabla u), n > d\sigma, \tag{3.61}$$

Now we show, as promised, that

$$< L_h, \mathrm{Tr}\, v > = < F, v > + < \chi_\Omega \nabla(E_\Omega(h)), \nabla(E_\Omega(v)) > \tag{3.62}$$

for all $v \in L^q_{2-\alpha}(\Omega)$. Let's first prove this for $F \in L^p(\Omega) = (L^q(\Omega))^*$ and $v \in C^\infty(\overline{\Omega})$ The same arguments used in the first paragraph of this section show that $||h||_{L^p_2(\Omega)} \leq C||F||_{L^p(\Omega)}$. Thus we can apply Lemma 3.25, which implies that

$$\begin{aligned} \int_{\partial\Omega} < \mathrm{Tr}(\nabla h), n > (v|\partial\Omega) d\sigma &= \int_\Omega F v dV + \int_\Omega < \nabla h, \nabla v > dV \\ &= < F, v > \\ &+ < \nabla(E_\Omega(v)), \chi_\Omega \nabla(E_\Omega(h)) > \end{aligned} \tag{3.63}$$

On the other hand, setting $g = v|\partial\Omega, \in C^\infty(\overline{\Omega})|\partial\Omega \subset L^q_{1-1/q}(\partial\Omega)$, we recall that there exists a unique harmonic function v_g in $L^q_1(\Omega)$ with $v_g|\partial\Omega = g$. Consequently, employment of Lemma 3.25 once again gives

$$\begin{aligned} \int_{\partial\Omega} < \mathrm{Tr}(\nabla h), n > (v|\partial\Omega) d\sigma &= \int_{\partial\Omega} < \mathrm{Tr}(\nabla h), n > (\mathrm{Tr}\, v_g) d\sigma \\ &= \int_\Omega F v_g dV + \int_\Omega < \nabla v_g, \nabla h > dV \\ &= < F, v_g > + < \chi_\Omega \nabla(E_\Omega(h)), \nabla(E_\Omega(v_g)) > \\ &= < L_h, g >, \end{aligned} \tag{3.64}$$

so that comparison of (3.63) and (3.64) yields the result, as long as $F \in L^p(\Omega)$ and $v \in C^\infty(\overline{\Omega})$. Since Lemma 3.24 ensures us that $L^p(\Omega)$ is dense in $(L^q_s(\Omega)^*$ for $s \geq 0$ and it is well-known that $C^\infty(\overline{\Omega})$ is dense in $L^q_s(\Omega)$ for $s \geq 0$, a simple density-continuity argument now establishes (3.62).

To complete our proof of the existence assertion of Theorem 3.6 we seek to apply our estimates for the inverse Calderon operator (Theorem 3.22) and the estimates for the homogeneous Dirichlet problem (Theorem 3.8). First consider the case $1 < p < p_0'$ and select α from within the range designated by Theorem 4.6. Since $L^p(\partial\Omega)$ is dense in $B^p_{\alpha-1-1/p}(\partial\Omega)$, it follows that $L^p(\partial\Omega)_{1\perp}$ is dense in $B^p_{\alpha-1-1/p}(\partial\Omega)_{1\perp}$.

Of course, given $f \in L^p(\partial\Omega)_{1\perp}$, Cor. 3.13 gives us a unique harmonic function $u \in L^p_1(\Omega)$ with nontangential Neumann boundary data f and nontangential Dirichlet data $\Upsilon(f) \in L^p_1(\partial\Omega)$. By Theorem 3.15 we have

$$\int_{\partial\Omega} f(\mathrm{Tr}\, v) d\sigma = \int_\Omega < \nabla u, \nabla v > dV \tag{3.65}$$

for all $v \in C^\infty(\overline{\Omega})$, which may be rewritten

$$\int_{\partial\Omega} f(\mathrm{Tr}\, v) =< \chi_\Omega \nabla(E_\Omega(u)), \nabla(E_\Omega(v)) > . \tag{3.66}$$

By Theorems 3.8 and 3.22 we have

$$\|u\|_{L^p_\alpha(\Omega)} \leq C\|\Upsilon(f)\|_{B^p_{\alpha-1/p}(\partial\Omega)} \leq \|f\|_{B^p_{\alpha-1-1/p}(\partial\Omega)} \tag{3.67}$$

Thus by density and continuity we can extend our mapping $f \mapsto u$ to a linear operator mapping all of $B^p_{\alpha-1-1/p}(\partial\Omega)$ into $L^p_\alpha(\Omega)$ for which

$$\|u_L\|_{L^p_\alpha(\partial\Omega)} \leq C\|L\|_{B^p_{\alpha-1-1/p}(\partial\Omega)} \tag{3.68}$$

for all $L \in B^p_{\alpha-1-1/p}(\partial\Omega)$ and

$$L(\mathrm{Tr}\, v) =< \chi_\Omega \nabla(E_\Omega(u_L)), \nabla(E_\Omega(v) > \tag{3.69}$$

for all $v \in L^p_{2-\alpha}(\Omega)$.

To extend the results just obtained to the range $p_0' < p < \infty$, fix such a p and once more select α from within the range of Theorem 3.6. Note that $L^\infty(\partial\Omega)_{1\perp}$ is a dense subset of $B^p_{\alpha-1-1/p}(\partial\Omega)_{1\perp}$ that is also a subset of all of the spaces $L^{\tilde{p}}(\partial\Omega)_{1\perp}$, $2 < \tilde{p} < p_0'$. But, as before, for each $f \in L^\infty(\partial\Omega)_{1\perp}$, there exists a unique harmonic function $u_{\tilde{p}} \in L_1^{\tilde{p}}(\partial\Omega)$ with nontangential Neumann boundary data f and Dirichlet data $\Upsilon(f) \in L_{1-1/\tilde{p}}^{\tilde{p}}(\partial\Omega)$ in the sense of the Trace Theorem such that (4.69) holds with $u \equiv u_{\tilde{p}}$ $L \equiv f$ and $v \in C^\infty(\overline{\Omega})$.

On the other hand, by Theorems 3.8 and 3.22, we have

$$\|u_p\|_{L_\alpha^p(\partial\Omega)} \leq C\|\Upsilon(f)\|_{B_{\alpha-1/p}^p(\partial\Omega)} \leq C\|f\|_{B_{\alpha-1-1/p}^p(\partial\Omega)}. \tag{3.70}$$

Here u_p is the unique harmonic function in $L_\alpha^p(\Omega)$ for which Tr $u_p = \Upsilon(f)$, considered as a member of $B_{\alpha-1/p}^p(\partial\Omega)$. But upon examination of the proof of Theorem 3.8 in Jerison-Kenig [](Theorem 5.1 in their paper) we find that the estimate for the homogeneous Dirichlet problem in the range $[p_0', \infty)$ is actually obtained via interpolation from that for the range $[2, p_0')$. What this means is that we must have $u_{\tilde{p}} = u_p$. In this way we obtain (3.68) and (3.69) for the entire range permitted for p and α described in Theorem 3.6.

The final step in our proof of the existence assertion of Theorem 3.6 is merely to combine all of the preceding work in the appropriate fashion. Given $F \in (L_{2-\alpha}^q(\Omega))_{1\perp}^*$ with $1/p+1/q = 1$ and p and α falling within the stated ranges, let $h = (N*(R_\Omega^* F))|\Omega$ be as in the opening paragraph of the present section and define L_h by (3.57). Using the Sobolev Embedding Theorem to show that $\nabla(E_\Omega(1))|\Omega = 0$, it follows from the definition that $< L_h, 1 >= 0$. Therefore, combining (3.58) and (3.68) (with $L \equiv L_h$) and equating (3.62) with (3.69) (once more with $L \equiv L_h$), our entire existence assertion readily follows upon defining $w \equiv h - u_{L_h}$. $\square$

3.5 Proof of Uniqueness

Recall that the well-known methods of Lax-Milgram ensure that that there exists a sequence of eigenfunctions $u_k \in C^\infty(\Omega) \cap L_1^2(\Omega), k = 0, 1, 2, \ldots$, with corresponding eigenvalues λ_k satisfying the (weak) Neumann eigenvalue problem for the Laplacian

$$\lambda_k \int_\Omega u_k v \, dV = \int_\Omega < \nabla u_k, \nabla v > dV, \tag{3.71}$$

for all $v \in L_1^2(\Omega)$.

To establish uniqueness for the inhomogeneous Neumann problem, we first observe that for all $k = 0, 1, 2, \ldots$ we have $u_k \in L^p_\alpha(\Omega) \cap L^q_{2-\alpha}(\Omega)$ with $1 < p < \infty$, α chosen from within the admissible range for Theorem 3.6, and $1/p + 1/q = 1$, and that, moreover, we have the $L^\infty(\Omega)$-bound

$$||u_k||_{L^\infty(\Omega)} \leq C(1 + \lambda_k)^{[n/2]+1}, \tag{3.72}$$

where [ ] denotes the greatest integer function. These assertions are verified by first noting that (3.71) implies that

$$||u_k||_{L^2_1(\Omega)} \leq (1 + \lambda_k). \tag{3.73}$$

Note as well that u_k may be viewed as the unique Lax-Milgram solution in $L^2_1(\Omega)$ to the weak inhomogeneous Neumann problem (3.71) with data $\lambda_k u_k$ and that this is precisely the solution $w \in L^2_1(\Omega)$ to the generalized inhomogeneous Neumann problem with data $F \equiv \lambda_k u_k$ that we have constructed in our own existence proof.

Now one uses the Sobolev inclusions

$$\begin{aligned} (i)\ L^p_s &\subseteq L^r_t \text{ for } 1 < p < r < \infty \text{ and } 1/p - 1/r = (1/n)(s - t) \\ (ii)\ L^p_s &\subseteq L^\infty \text{ for } p > n/s \end{aligned}$$

in conjunction with the estimates in our own existence theorem to bootstap up to the estimate (3.72). To illustrate, we use the estimate (3.73) as our starting point and we obtain an $L^{p_1}(\Omega)$-estimate on u_k, where p_1 satisfies $1/p_1 = 1/2 - 1/n$, in the following way (writing p_1' for the conjugate exponent to p_1 and $\tilde{u}_k$ for the extension of u_k by 0 to a function on $\boldsymbol{R}^n$):

$$\begin{aligned} ||u_k||_{L^{p_1}(\Omega)} &\leq C(1 + \lambda_k)||u_k||_{(L^{p_1'}_1(\Omega))^*} \\ &\quad \text{(using (3.9) in Thm. 3.6)} \\ &= C(1 + \lambda_k)||E^*_\Omega(\tilde{u}_k)||_{(L^{p_1'}_1(\Omega))^*} \\ &\leq C(1 + \lambda_k)||\tilde{u}_k||_{L^{p_1}_{-1}} \\ &\leq C(1 + \lambda_k)||\tilde{u}_k||_{L^2} \end{aligned}$$

$$\begin{aligned} &\quad \text{(using } (i) \text{ above)} \\ &= C(1+\lambda_k)||u_k||_{L^2(\Omega)} \\ &\leq C(1+\lambda_k)^2 \end{aligned}$$

We iterate this procedure at most $[n/2]+1$ times with $p_0 = 2$ and $1/p_{i+1} = 1/p_i - 1/n$ to obtain

$$||u_k||_{L^{\tilde{p}}(\Omega)} \leq C(1+\lambda_k)^{[n/2]+1}, \tag{3.74}$$

with $\tilde{p} \geq n$. One final application of the Sobolev inclusion (i) above then gives (3.74) for all $p \geq \tilde{p}$, and hence yet another application of the estimates in the existence theorem gives

$$||u_k||_{L^p_\epsilon(\Omega)} \leq C(1+\lambda_k)^{[n/2]+1}$$

with ϵ as in the statement of the existence theorem and p as large as one may desire. In particular, if $p > n/\epsilon$ then the Sobolev inclusion (ii) above implies (3.72). Also, since $u_k \in L^p(\Omega) \subseteq (L^q_{2-\alpha}(\Omega))^*$, applying (3.9) once more shows that $u_k \in L^p_\alpha(\Omega)$ for all values for p and α in the ranges $(a), (b)$, and (c) of Theorem 3.6. Finally, notice that $2-\alpha$ falls within the corresponding range for the conjugate exponent q. Thus u_k is a member of all of the spaces $L^q_{2-\alpha}(\Omega)$ as well.

Now let $w \in L^p_\alpha(\Omega)$, and assume that

$$< \chi_\Omega \nabla(E_\Omega(w)), \nabla(E_\Omega(v)) >= 0 \tag{3.75}$$

for all $v \in L^q_{2-\alpha}(\Omega)$. Since, as just noted, $u_k \in L^q_{2-\alpha}(\Omega)$, it follows from (3.71) that

$$\lambda_k \int_\Omega w u_k \, dV = 0 \tag{3.76}$$

for all k. In case $w \in L^2(\Omega)$, this means that for $k > 0$ the coefficient of u_k in the $L^2(\Omega)$-eigenfunction expansion of w is 0. Consequently, in this case w is a multiple of the constant eigenfunction and so must be constant.

For uniqueness within the range $1 < p < 2$, let $f \in L^p(\Omega), 1 < p < 2$. Write

$$c_k = \int_\Omega f u_k \, dV,$$

and set

$$T_t f(x) = \sum_{k=1}^{\infty} c_k e^{-t\lambda_k} u_k(x) \tag{3.77}$$

for $t > 0$. The series in (3.77) converges absolutely for all $x \in \Omega$, and this follows very easily from the L^∞ estimates (3.72) for the eigenfunctions and Weyl's asymptotic formula (on Lipschitz domains) for the distribution of eigenvalues, which implies that

$$0 < C'(\Omega) \leq \lambda_k / k^{2/n} \leq C(\Omega) < \infty$$

as $k \to \infty$. When $f \in L^2(\Omega)$ the function $T_t f$ is in fact the solution to the heat equation on Ω with initial temperature distribution $T_0 f(x) = f(x)$.

Consider that, by the dominated convergence theorem, we can rewrite $T_t f(x)$ in the form

$$T_t f(x) = \int_\Omega h(t, x, y) f(y) dy, \tag{3.78}$$

wherein the kernel

$$h(t, x, y) = \sum_{k=0}^{\infty} e^{-t\lambda_k} u_k(x) u_k(y) \tag{3.79}$$

converges absolutely for all $(t, x, y) \in (0, \infty) \times \Omega \times \Omega$. (Of course this is once again a consequence of (3.72) and Weyl's formula.) By a theorem of Beurling and Deny the kernel h is positive for all $t > 0$. Since a simple computation shows that

$$\begin{aligned} \sup_{x \in \Omega} \int_\Omega h(t, x, y) dy &= 1 \\ \sup_{y \in \Omega} \int_\Omega h(t, x, y) dx &= 1, \end{aligned}$$

we may apply the generalized Young's inequality to find that

$$||T_t f||_{L^p(\Omega)} \leq ||f||_{L^p(\Omega)}. \tag{3.80}$$

If in fact $f \in L^2(\Omega)$ then, since $\sum_k c_k^2 < \infty$ in this case, we know that

$$\lim_{t \to 0^+} T_t f = f \tag{3.81}$$

in $L^2(\Omega)$-norm. Because $L^2(\Omega)$ is dense in $L^p(\Omega)$, using (3.80), we have convergence in (3.81) in $L^p(\Omega)$-norm for all $f \in L^p(\Omega), 1 < p < 2$.

Finally let $f \equiv w \in L^p_\alpha$ with $1 < p < 2$ and α lying within the admissible range for the existence theorem, and assume that (3.75) is satisfied. Equation (3.76) then implies that $c_k = 0$ for all $k > 0$. Hence $T_t w \equiv c_0 u_0$ for all $t > 0$, where u_0 is the constant (normalized) eigenfunction. Therefore w itself is constant. $\square$

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