Equivariant Resolution of Singularities and Semi-stable Reduction in Characteristic 0

by

Jianhua Wang

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Author ....
Department of Mathematics
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Certified by ....
Steven L. Kleiman
Professor of Mathematics
Thesis Supervisor

Accepted by ....
Richard B. Melrose, Chairman
Departmental Graduate Committee
Department of Mathematics

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Abstract

In this thesis, I shall prove the equivariant resolution of singularities theorem first, then using this theorem and the barycentric subdivision technique, I shall prove the equivariant semi-stable reduction theorem. Both results are over algebraically closed fields of characteristic 0 and their proofs are purely algebraic in nature. In the statement of the equivariant semi-stable theorem, besides giving the equivariant version of classic theorem, I shall describe more precisely what the base curve could be. I shall also discuss a stronger form of the theorem when the dimension of the fiber is less than or equal to 2.

Thesis Supervisor: Steven L. Kleiman
Title: Professor of Mathematics

to my parents and grandparents

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Contents

Introduction

In this thesis, k is always assumed to be an algebraically closed field of characteristic 0. By a variety we shall mean an integral scheme of finite type over k without further remark. Our main goal is to prove the following theorem:

Theorem 0.1. *Let C be a nonsingular algebraic curve, let $O \in C$, and let $f : X \longrightarrow C$ be a proper morphism of a variety X onto C such that the morphism*

$$res\ f : X - f^{-1}(O) \longrightarrow C - O$$

is smooth. If G is a finite group acting on X and C, f is G-equivariant, and O is invariant under the action of G, then there exist a nonsingular curve C' and a finite morphism

$$\pi : C' \longrightarrow C$$

with $\pi^{-1}(O) = O'$, and there exist an variety X', a finite group G' determined by C', and a morphism $p : X' \longrightarrow X \times_C C'$ so that we have the commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{p} & X \times_C C' & \longrightarrow & X \\ \Big\downarrow f' \circ p & & \Big\downarrow f' & & \Big\downarrow f \\ C' & = & C' & \xrightarrow{\pi} & C \end{array}$$

with the following properties:

a) G is a quotient group of G'. There are natural actions of G' on all the varieties in the above diagram, in particular, G' acts on X and C in the obvious sense. Moreover, every morphism in the diagram is G'-equivariant,

b) p is an isomorphism over $(f')^{-1}(C' - O')$,

c) p is projective. In fact, p is obtained by blowing up a sheaf of ideas Γ with $\Gamma|_{(f')^{-1}(C'-O')} = O_{X \times_C C'}|_{(f')^{-1}(C'-O')}$,

d) X' is nonsingular, and the fiber $(f' \circ p)^{-1}(O')$ is reduced with nonsingular components crossing normally.

Remark. We only state the existence of $C', \pi,$ and G' right now. We will give a more precise form of the theorem and state what exactly $C', \pi,$ and G' are in the fourth chapter.

This theorem is the equivariant version of the semi-stable reduction theorem in [KKMS]. They treated the problem by associating a conical polyhedral complex and a compact polyhedral complex to a toroidal embedding, hence reducing the semi-stable reduction problems to purely combinatorial problems on combinatorial objects, namely, polyhedral complexes. We use a similar approach here, constructing subdivisions of polyhedral complexes more carefully to solve the equivariant case. First, we shall prove the following equivariant version of Hironaka's famous resolution of singularities theorem, which is an indispensable ingredient of the proof of theorem 0.1.

Theorem 0.2. *Let X be a variety, let $Z \subset X$ be a proper closed subset, and let $G \subset Aut_k(Z \subset X)$ be a finite group. Then there is a projective G-equivariant modification $r : X_1 \to X$ such that X_1 is a nonsingular variety and $r^{-1}(Z)$ is a G-strict divisor of normal crossings. Moreover, G acts on $X_1 \setminus r^{-1}(Z) \longrightarrow X_1$ toroidally.*

This theorem was announced by Hironaka, but a complete proof was not easily accessible for a long time. The situation was remedied by E. Bierstone and P. Milman [B-M2], who gave a construction of completely canonical resolution of singularities. Their construction builds on a thorough understanding of the effect of blowing up. They carefully constructed an invariant pointing to the next blowup.

The proof we give here comes from joint work of Professor Dan Abramovich and the author [ℵ-W]. We assume the existence of resolution of singularities without group actions. We first reduce the problem to the toroidal embedding case using resolution of singularities, and then we further reduce the question to a combinatorial problem on the conical polyhedral complex associated to this toroidal embedding. Finally we solve the problem by using barycentric subdivision, which is a very powerful tool in solving equivariant problems.

CHAPTER I

Preliminaries

In this chapter we give a brief introduction on toric varieties and toroidal embeddings. Most of the material comes from [F2] and [KKMS]. We omit all proofs, the interested reader is referred to the references.

1. Toric Embedding

The idea of toric variety comes from the study of compactification problems [NA]. The compactification description gives a simple way of saying what a toric variety is: It is a normal variety X containing a torus T as a dense open subset, together with an action

$$T \times X \longrightarrow X$$

of T on X which extends the natural action of T on itself. The simplest compact example is the projective space ${P_k}^n$, regarded as the compactification of k^n as usual:

$$(k^*)^n \hookrightarrow k^n \hookrightarrow {P_k}^n$$

Any product of affine and projective spaces can also be realized as toric varieties.

Let N be a lattice isomorphic to $\mathbb{Z}^n$ for some n. A **polyhedral cone** in $N_{\mathbb{R}}$ is a cone which has its apex at the origin and is generated by a finite number of vectors. We call a polyhedral cone **rational** if it is generated by vectors in the lattice, and call it **strongly convex** if it does not contain any lines passing through the origin. We abuse terminology and simply say a cone in N when we refer to a strongly convex rational polyhedral cone.

Denote the dual lattice $Hom(N, \mathbb{Z})$ by M and denote the dual pairing of N by $<,>$. If σ is a cone in N, define

$$\sigma^{\vee} = \{u \in M_{\mathbb{R}} :< u, v >\geq 0 \text{ for all } v \in \sigma\}.$$

This dual cone determines a commutative semigroup

$$S_{\sigma} = \sigma^{\vee} \cap M = \{u \in M :< u, v >\geq 0 \text{ for all } v \in \sigma\}.$$

The following lemma is the starting point of toric variety theory; it gives us the foundation to construct toric varieties from cones.

Lemma 1.1 (Gordon's Lemma). *S_{σ} is a finitely generated semigroup.*

So the group algebra $k[S_{\sigma}]$ is a finitely generated commutative k-algebra, and we have an affine variety $U_{\sigma} = Spec\ (k[S_{\sigma}])$.

If τ is a face of σ, then S_{σ} is contained in S_{τ}, so $k[S_{\sigma}]$ is a subalgebra of $k[S_{\tau}]$. The homomorphism $k[S_{\sigma}] \longrightarrow k[S_{\tau}]$ induces a morphism $U_{\tau} \longrightarrow U_{\sigma}$.

Lemma 1.2. *If τ is a face of σ, then the morphism $U_{\tau} \longrightarrow U_{\sigma}$ embeds U_{τ} as a* *principal open subset of* U_σ.

In particular, the torus $T_N = U_0$ can be embedded into the affine toric varieties U_σ for all cones σ in N.

We can define an action of the torus T_N on U_σ as follows. A point $t \in T_N$ can be identified with a map of groups $M \longrightarrow k^*$, and a point $x \in U_\sigma$ can be identified with a map of semigroups $S_\sigma \longrightarrow k$; the product $t \times x$ is the map of semigroups $S_\sigma \longrightarrow k$ given by

$$u \longmapsto t(u)x(u)$$

These maps are compatible with inclusions of the open subsets of U_σ corresponding to faces of σ. In particular, they extend the action of T_N on itself.

A **finite rational partial polyhedral decomposition**(abbreviated to f.r.p.p. decomposition) of $N_\mathbb{R}$ is a finite set Δ of strongly convex rational cones in $N_\mathbb{R}$ such that:

(1) if σ is an element in Δ, then all faces of σ are elements in Δ,

(2) for any $\sigma, \tau \in \Delta$, $\sigma \cap \tau$ is a face of σ and τ.

From an f.r.p.p. decomposition we can construct a toric variety $X(\Delta)$ in the following way. We take the disjoint union of all affine toric varieties U_σ for $\sigma \in \Delta$ and glue as follows: for cones σ and τ, the intersection $\sigma \cap \tau$ is a face of both σ and τ, so $U_{\sigma\cap\tau}$ is identified as a principal open subvariety of U_σ and U_τ; we glue U_σ and

U_τ by this identification on open subvarieties.

Theorem 1.3. *$X(\Delta)$ is a well-defined separated variety.*

The actions of T_N on the varieties U_σ described previously are compatible with the patching isomorphisms. It gives an action of T_N on $X(\Delta)$ which extends the product in T_N:

$$\begin{array}{ccccc} T_N & \times & X(\Delta) & \longrightarrow & X(\Delta) \\ \uparrow & & \uparrow & & \uparrow \\ T_N & \times & T_N & \longrightarrow & T_N. \end{array}$$

The converse is also true: any separated, normal variety X containing a torus T_N as a dense open subvariety, with compatible action as above, can be realized as a toric variety $X(\Delta)$ for a unique f.r.p.p. decomposition of $N_{\mathbb{R}}$.

Suppose $\phi : N' \longrightarrow N$ is a homomorphism of lattices, Δ an f.r.p.p. decomposition of N, and Δ' an f.r.p.p. decomposition of N' such that for each cone $\sigma' \in \Delta'$, there is some cone $\sigma \in \Delta$ containing $\phi(\sigma')$. Then there exist morphisms $U_{\sigma'} \longrightarrow U_\sigma \subset X(\Delta)$ for all $\sigma' \in \Delta$. These morphism are easily seen to be independent of the choice of σ's, and they patch together to give a morphism

$$\phi_* : X(\Delta') \longrightarrow X(\Delta)$$

Theorem 1.4. *A toric variety $X(\Delta)$ is compact if and only if its support $|\Delta|$ is the whole space $N_{\mathbb{R}}$.*

Theorem 1.5. *The map* $\phi_* : X(\Delta') \longrightarrow X(\Delta)$ *is proper if and only if* $\phi^{-1}(|\Delta|) = |\Delta'|$.

Theorem 1.6. *An affine toric variety* U_σ *is nonsingular if and only if* σ *is generated by part of a basis for the lattice* N*, i.e., if and only if the cone has index one in* N*, in which case*

$$U_\sigma \cong k^i \otimes (k^*)^{n-i}, \ i = dim(\sigma).$$

We henceforth call a cone σ nonsingular if U_σ is nonsingular, i.e., if it is of index one. We say that an f.r.p.p. decomposition is nonsingular if all its cones are nonsingular, or equivalently, if the corresponding toric variety is nonsingular.

Theorem 1.7. *The rings* $A_\sigma = k[S_\sigma]$ *are integrally closed.*

2. Toroidal Embedding

A large portion of the terminology in this section is borrowed from [ℵ-dJ] and [KKMS]. Let Z be a variety, Z_i the irreducible components of Z, G a group acting on Z. We say that Z **is** G**-strict** if the union of translates $\cup_{g \in G} g(Z_i)$ of each component Z_i is a normal variety, or equivalently, if each Z_i is normal and whenever there is an element of G mapping Z_i to Z_j, $i \neq j$, then $Z_i \cap Z_j = \phi$. We say that Z is **strict** if it is G-strict with respect to the trivial group, i.e., if every Z_i is normal.

A **modification** is a proper birational morphism of irreducible varieties.

A divisor $D \subset X$ is called a **divisor of normal crossings** if étale locally at every point it is the zero set of $u_1 \cdots u_k$ for some $u_1, \ldots, u_k$ belonging to a regular system of parameters. Thus, in a strict divisor of normal crossings D, all components of D are nonsingular.

An open embedding $U \hookrightarrow X$ is called a **toroidal embedding** if locally in the étale topology (or in classical topology when $k = \mathbb{C}$) it is isomorphic to a toric variety $T \hookrightarrow V$ (see [KKMS], II§1). Let $E_i, i \in I$ be the irreducible components of $X \backslash U$. A finite group action $G \subset Aut(U \hookrightarrow X)$ is said to be **toroidal** if the stabilizer of every point can be identified on some neighborhood with a subgroup of the torus T. We say that a toroidal action is **G-strict** if $X \backslash U$ is G-strict. In particular, the toroidal embedding itself is said to be strict if $X \backslash U$ is strict. This is the same as the notion of **toroidal embedding without self-intersections** in [KKMS]. For any subset J of I, the components of the sets $\cap_{i \in J} E_i - \cup_{i \notin J} E_i$ define a stratification of X. Each component is called a **stratum**.

A **conical(resp. compact) polyhedral complex** Δ is a topological space $|\Delta|$ with a finite family of closed subsets $\{\sigma_\alpha\}$ (called its cells) and finite-dimensional real vector spaces V_α of real-valued continuous functions on σ_α such that

1. via a basis $f_1, ..., f_{n_\alpha}$ of V_α, we get a homeomorphism

$$\phi_\alpha : \sigma_\alpha \longrightarrow \sigma'_\alpha \subset \mathbb{R}^{n_\alpha},$$

where $\sigma_\alpha{}'$ is a conical convex polyhedron in $\mathbb{R}^{n_\alpha}$ not contained in a hyperplane(resp. V_α contains $\mathbb{R}$, the constant functions, and via a basis $1, f_1, ..., f_{n_\alpha}$ of V_α, we get a homeomorphism

$$\phi_\alpha : \sigma_\alpha \longrightarrow \sigma'_\alpha \subset P^{n_\alpha}$$

where σ'_α is a compact convex polyhedron in P^{n_α} not contained in a hyperplane),

2. ϕ_α^{-1}(faces of σ'_α) is the union of some σ_β's for $\sigma_\beta \in \Delta$. We call the σ_β's faces of σ_α and ϕ_α^{-1}(interior of σ'_α) the interior of σ_α,

3. $|\Delta|$ is the disjoint union of the interiors of σ_α's,

4. if σ_β is a face of σ_α, then V_β consists of restrictions of elements of V_α on σ_β.

An **integral structure** on a conical(resp. compact) polyhedral complex is a family of finitely generated abelian groups $L_\alpha \subset V_\alpha$ such that:

1(compact case only). L_α contains the set of all constant functions with values in $n\mathbb{Z}$ for some integer n,

2. $L_\alpha \times \mathbb{R} \cong V_\alpha$,

3. If σ_β is a face of σ_α, then $res_{\sigma_\beta} L_\alpha = L_\beta$.

The motivation to introduce conical polyhedral complex with integral structure comes from the following theorem; it gives us a perfect analogue between toric varieties and strictly toroidal embeddings.

Theorem 1.8. *To every strictly toroidal embedding $U \subset X$, we can associate a conical polyhedral complex with an integral structure $\Delta = (|\Delta|, \sigma^Y, M^Y)$ whose cells are in 1-1 correspondence with the strata of X.*

From now on, when we refer to a conical polyhedral complex, it is understood that the complex is endowed with an integral structure.

We introduce the following construction, which we will use in proving the semi-stable reduction theorems. Suppose that for a strictly toroidal embedding $U \subset X$, we are given a positive Cartier divisor D with support $X \setminus U$. We associate to the triple (X, U, D) a compact polyhedral complex Δ^* with an integral structure, where

$$|\Delta^*| = \{x \in |\Delta| | < D, x >= 1\},$$

$$\delta^{*Y} = |\Delta^*| \cap \delta^Y, \text{ for all cells } \delta^Y \subset \Delta,$$

and the integral structure is given by $res_{\delta^{*Y}}(M^Y)$. In [KKMS, p.86 (Definition 2)] one defines **a finite rational partial polyhedral decomposition** Δ' of a conical polyhedral complex Δ. As in the previous section, we abbreviate it to f.r.p.p decomposition. We restrict our attention to the case where $|\Delta'| = |\Delta|$, and we simply call this a **polyhedral decomposition** or a **subdivision**.

The utility of polyhedral decompositions is given in the following theorem; it establishes a correspondence between allowable modifications of a given strictly toroidal embedding (which in our terminology are proper) and polyhedral decompositions of the conical polyhedral complex.

Theorem 1.9 [KKMS]. *The correspondence $\Delta' \longmapsto Z_{\Delta'}$ defines a bijection between the f.r.p.p. decompositions of Δ and the isomorphism classes of modifications of*

X.

In order to guarantee that a modification is projective, one needs a bit more. Following [KKMS, p.91], a function $ord : \Delta \to \mathbb{R}$ defined on a conical polyhedral complex with integral structure is called an **order function** if:

(1) $ord(\lambda x) = \lambda \cdot ord(x), \forall \lambda \in \mathbb{R}_+$,

(2) ord is continuous and piecewise-linear,

(3) $ord(N^Y \cap \sigma^Y) \subset \mathbb{Z}$ for all strata Y,

(4) ord is convex on each cone $\sigma \subset \Delta$.

For an order function on the conical polyhedral complex corresponding to X, we can define canonically a complete coherent sheaf of fractional ideals on X, and vice versa (see [KKMS, I§2]). An order function is positive if and only if its corresponding sheaf is a genuine ideal sheaf. We have the following important theorem [KKMS]:

Theorem 1.10. *Let Γ be a coherent sheaf of ideals corresponding to a positive order function ord, and let $B_\Gamma(X)$ be the normalized blowup of X along Γ. Then $B_\Gamma(X) \to X$ is an allowable modification of X, described by the decomposition of $|\Delta|$ obtained from subdividing the cones into the largest subcones on which ord is linear.*

A polyhedral decomposition is said to be **projective** if it is obtained from a positive order function. It is clear from theorem 1.10 that a modification obtained from a projective decomposition is a projective morphism.

Lemma 1.11. *Let Δ be a polyhedral complex. If Δ' is a projective subdivision of Δ and Δ'' is a projective subdivision of Δ', then Δ'' is a projective subdivision of Δ.*

Given a cone σ and a rational ray $\tau \subset \sigma$, it is natural to define a subdivision of σ centered at τ, whose cones are of the form $\sigma' + \tau$, where σ' runs over faces of σ disjoint from τ. Given a polyhedral complex Δ and a rational ray τ, we take the subdivision centered at τ of all cones containing τ, and again call the resulting subdivision of Δ the subdivision centered at τ.

A very important subdivision is the barycentric subdivision. Let σ be a cone with integral structure, and let $e_1, \ldots, e_k$ be integral generators of its edges. The **barycenter** of σ is the ray $b(\sigma) = \mathbb{R}_{\geq 0} \sum e_i$. The **barycentric subdivision** of a polyhedral complex Δ of dimension m is the minimal subdivision $B(\Delta)$ in which the barycenters of all cones in Δ appear as cones in $B(\Delta)$. It may be obtained by first taking the subdivisions centered at the barycenters of m-dimensional cones, then taking the subdivisions of the resulting complex centered at the barycenters of the cones of dimension $m-1$ of the *original* complex Δ, and so on.

One can also obtain the barycentric subdivision inductively in a different way: The barycentric subdivision of an m-dimensional cone δ is formed by first taking the barycentric subdivisions of all its faces and each of the resulting cones σ, including the cone $\sigma + b(\delta)$. Hence, it is clear that $B(\Delta)$ is a simplicial subdivision.

Lemma 1.12. *let Δ be a polyhedral complex and τ a rational ray, then the sub-*

division centered at τ *is projective. In particular, the barycentric subdivision of a polyhedral complex is projective.*

CHAPTER II

Equivariant Toroidal Modification

In this chapter we are going to prove the equivariant resolution of singularities for toroidal embeddings.

Lemma 2.1. *Let X be a variety, $f : \overline{X} \longrightarrow X$ the normalization of X, and G a subgroup of $Aut(X)$. We can define a canonical action of G on $\overline{X}$ such that f is a $G-$equivariant map.*

Proof. Let g be any element of G, we abuse notation and use g to denote the automorphism of X induced by g. From the universal property of the normalization map, we see that $g \circ f$ factors through f, i.e., there exists a morphism $\overline{g} : \overline{X} \longrightarrow \overline{X}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \overline{X} & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \overline{g}} & & \downarrow{\scriptstyle g} \\ \overline{X} & \xrightarrow{f} & X \end{array}$$

It is clear that if g is the identity map of X, then $\overline{g}$ is the identity map of $\overline{X}$.

Take any two elements g_1, g_2, we have the following commutative diagram:

$$\begin{array}{ccc} \overline{X} & \xrightarrow{f} & X \\ \big\downarrow \overline{g_1} & & \big\downarrow g_1 \\ \overline{X} & \xrightarrow{f} & X \\ \big\downarrow \overline{g_2} & & \big\downarrow g_2 \\ \overline{X} & \xrightarrow{f} & X \end{array}$$

From the diagram we see clearly that $\overline{g_1} \circ \overline{g_2} = \overline{g_1 \circ g_2}$. Take $g_1 = g$ and $g_2 = g^{-1}$, we get $\overline{g} \circ \overline{g^{-1}} = \overline{g \circ g^{-1}} = \overline{id} = id$. This shows that for any g in G, $\overline{g}$ is an automorphism of X. We hence define the action of g on $\overline{X}$ by $\overline{g}$, from the construction of $\overline{g}$ we know that f is G-equivariant.

Lemma 2.2. *Let $U \subset X$ be a strictly toroidal embedding, and let G be a finite subgroup of $Aut(U \subset X)$. Then:*

(1) *The group G acts linearly on $\Delta(X)$.*

(2) *If the action of G is strictly toroidal, $g \in G$, and $\delta \subset \Delta(X)$ is a cone such that $g(\delta) = \delta$, then $g_{|\delta} = id$.*

Proof.

(1) Clearly, G acts on the stratification of $U \subset X$. Note that from Definition 3 of [KKMS, P. 59], $\Delta(X)$ is built up from the groups M^Y of Cartier divisors on $Star(Y)$ supported on $Star(Y) \setminus U$, as Y runs through the strata. Since

$g \in G$ canonically transforms M^Y to $M^{g^{-1}Y}$ in a linear manner, our claim follows.

(2) Assume $g(\delta) = \delta$ and $g_{|\delta} \neq id$. Then there exists an edge $e_1 \in \delta$ such that $g(e_1) \neq e_1$. Denote $g(e_1)$ by e_2. Let E_1 and E_2 be the divisors corresponding to e_1 and e_2. Since $g(e_1) = e_2$ we have $g(E_1) = E_2$. As e_1, e_2 are both edges of δ, $E_1 \cap E_2 \neq \phi$. So $\cup_{g \in G} g(E_1)$ can not be normal since it has two intersecting components. This is a contradiction to the fact that G acts strictly on X.

Lemma 2.3. *Let G be a finite subgroup of $Aut(U \subset X)$ acting toroidally on X. Let Δ_1 be a G-equivariant subdivision of Δ, with corresponding modification $f : X_1 \to X$. Then G acts toroidally on X_1 and f is G-equivariant. Moreover, if G acts strictly on X, it also acts strictly on X_1.*

Proof. The fact that there is a natural G-action on X_1 such that f is G-equivariant follows from the canonical manner in which X_1 is constructed from the decomposition Δ_1, see Theorems 6* and 7* of [KKMS, §2.2.].

Now for any $a \in X_1$ and any $g \in Stab_a$, we have $g \circ f(a) = f \circ g(a) = f(a)$, hence $g \in Stab_{f(a)}$. So $Stab_a$ is a subgroup of $Stab_{f(a)}$, which is identified with a subgroup of the torus in a neighborhood of $f(a)$. This shows that $Stab_a$ can be identified with a subgroup of the torus in a neighborhood of a.

It remains to show that if G acts strictly on X, then it also acts strictly on X_1. Assume this is not the case. Then there exist two edges τ_1, τ_2 in Δ_1 which are both edges of a cone δ' and $g(\tau_1) = \tau_2$ for some $g \in G$. Choose a cone δ' of minimal dimension among all cones in Δ containing τ_1 and τ_2. Since G acts strictly on X, τ_1 and τ_2 cannot both be edges in Δ. Without loss of generality, assume τ_2 is not an edge in Δ, then τ_2 must be in the interior of a cone δ in Δ containing δ'. Now since $\tau_2 \subset \delta' \cap g(\delta')$ and τ_2 is contained in the interior of δ, we conclude that the intersection of the interior of δ and the interior of $g(\delta)$ is nonempty, from which it follows that $g(\delta) = \delta$. By the previous lemma, $g_{|\delta} = id$, so $g_{|\delta'} = id$ as well, a contradiction. □

Proposition 2.4.

(1) *There is a one-to-one correspondence between the edges in the barycentric subdivision $B(\Delta)$ and the positive dimensional cones in Δ. We denote this correspondence by $\tau \longleftrightarrow \delta_\tau$.*

(2) *Let $\tau_i \neq \tau_j$ be edges of a cone $\hat{\delta} \in B(\Delta)$. Then δ_{τ_i} and δ_{τ_j} are of different dimensions.*

(3) *If G is a finite group acting toroidally on a strictly toroidal embedding $U \subset X$, then the action of G on $X_{B(\Delta)}$ is strict.*

Remark. Using this proposition, the argument at the end of [ℵ-dJ] can be significantly simplified: there is no need to show the G-strictness of the toroidal embedding obtained there, since the barycentric subdivision automatically gives a G-strict modification.

Proof.

1. Define b : {positive dimensional cones in Δ} $\longrightarrow$ { edges in $B(\Delta)$} by

$$b(\delta) = \text{ the barycenter of } \delta$$

and define δ : { edges in $B(\Delta)$} $\longrightarrow$ { positive dimensional cones in Δ} by

$$\delta(\tau) = \text{ the unique cone whose interior contains } \tau.$$

It is easy to see that b and δ are inverses of each other.

2. We proceed by induction on $\dim \Delta$. The case $\dim \Delta = 1$ is trivial. Since τ_i and τ_j are two edges of $\hat{\delta} \in B(\Delta)$, the cone $\mathbb{R}_+\tau_i + \mathbb{R}_+\tau_j$ must lie inside some cone of Δ, say δ^*, which we can choose to be of minimal dimension. We recall the second construction of the barycentric subdivision described in the Preliminaries. If $\dim \delta^* \leq m - 1$, δ is in the barycentric subdivision of the $(m-1)$-skeleton of Δ, and the statement follows by the induction hypothesis. If $\dim \delta^* = m$, exactly one of τ_1 and τ_2 must be the barycenter of δ^*(otherwise a proper face of δ^* which contains τ_i and τ_j is a cone of Δ with smaller dimension than δ^*), hence one of δ_{τ_i} has dimension m and the other has dimension strictly less than m.

3. Since the decomposition $B(\Delta)$ of Δ is equivariant, by lemma 2.3 we know that G acts toroidally on $X_B(\Delta)$. Let $E_1, E_2 \subset X_B(\Delta) \setminus U$ be irreducible components and e_1, e_2 edges in $B(\Delta)$ correspond to E_1, E_2. If $E_1 \cap E_2 \neq \phi$, there is a cone in $B(\Delta)$ containing e_1, e_2 as edges. From part (2), $\dim \delta_{e_1} \neq \dim \delta_{e_2}$, so $g(e_1) \neq e_2$ for

any $g \in G$. This proves that G acts strictly on $X_{B(\Delta)}$. $\square$

Proposition 2.5. *There is a positive G-equivariant order function on $B(\Delta)$ such that the associated ideal Γ induces a blowing up $B_\Gamma X_{B(\Delta)}$, which is a nonsingular G-strict toroidal embedding on which G acts toroidally.*

Proof. By the previous proposition, we know that G acts toroidally and strictly on $X_{B(\Delta)}$. It follows from lemma 2.2 that the quotient $B(\Delta)/G$ is a conical polyhedral complex, since no cone has two edges in $B(\Delta)$ which are identified in the quotient. We can use the argument in [KKMS, I§2, lemmas 1-3] to get a positive order function $ord : B(\Delta)/G \to \mathbb{R}$ which induces a simplicial subdivision of $B(\Delta)/G$ such that all its cells are of index 1. Let $q : B(\Delta) \to B(\Delta)/G$ be the quotient map. Then $ord \circ q$ is a positive order function which induces a G-equivariant subdivision of $B(\Delta)$ into simplicial cones of index 1. Let Γ be the corresponding ideal sheaf. By theorem 1.6, the blow up $X_{B(\Delta)}$ along Γ is a nonsingular strictly toroidal embedding $U \subset B_\Gamma X_{B(\Delta)}$. By lemma 2.4, G acts strictly and toroidally on $X_{B(\Delta)}$. Finally by lemma 2.3, G acts strictly and toroidally on $B_\Gamma X_{B(\Delta)}$. $\square$

Theorem 2.6. *Let $U \subset X$ be a strictly toroidal embedding, and let $G \subset \mathrm{Aut}(U \subset X)$ be a finite group whose action is toroidal. Then there is a G-equivariant toroidal ideal sheaf Γ such that the normalized blowup of X along Γ is a nonsingular G-strict toroidal embedding.*

Proof. In the previous two propositions, we performed barycentric subdivision of Δ and found G-equivariant subdivision of $B(\Delta)$ to get a subdivision $\{\delta_I\}$ whose cells are all of index 1. We consider $\{\delta_I\}$ as a subdivision of Δ. Let $Y \to X$ be the modification associated to this subdivision. Clearly Y is nonsingular and G acts strictly and toroidally on Y. Since we know the composition of two projective subdivisions is projective from lemma 1.11, $\{\delta_I\}$ is projective, so it is obtained from a positive order function. Let Γ be the coherent sheaf of ideals corresponding to this order function, then Y is the normalized blowup of X along Γ. $\square$

Remark. With a little more work we can obtain a **canonical** choice of toroidal equivariant resolution of singularities. We observe that the cones in the barycentric subdivision have canonically ordered coordinates agreeing on intersecting cones: for a cone δ, choose the unit coordinate vectors e_i to be primitive lattice vectors generating the edges τ, where $i = \dim \delta_\tau$, the dimension of the cone of which τ is a barycenter. Recall that to resolve singularities, one successively takes subdivisions centered at lattice points w_j which are not integrally generated by the vectors e_i. These w_j are partially ordered with respect to the lexicographic ordering of their canonical coordinates, in such a way that if $w_j \neq w_k$ have the same coordinates (e.g. if $g(w_1) = w_2$), they do not lie in the same cone. Therefore we can take the centered subdivisions simultaneously.

We conclude this chapter with a simple proposition which is implicitly used in [ℵ-dJ] and will be used in the next chapter.

Proposition 2.7. *Let $U \subset X$ be a strictly toroidal embedding, and let $G \subset Aut(U \subset X)$ be a finite group acting strictly and toroidally. Then $(X/G, U/G)$ is a strictly toroidal embedding.*

Proof: Since the quotient of a toric variety by a finite subgroup of the torus is toric, X/G is still a toroidal embedding, by definition of a toroidal embedding. We need to show that it is strict. Let $q : X \to X/G$ be the quotient map. Let $Z \subset X \setminus U$ be a divisor. Then $q(Z) = q(\cup_g g(Z))$. Since the action is strict, we have $q(\cup_g g(Z)) \simeq Z/Stab(Z)$, which is normal. $\square$

CHAPTER III

Equivariant Resolution of Singularities

In this chapter, we are going to prove the equivariant resolution of singularities theorem for the general case. We have already proven the theorem for the case of toroidal embeddings in the second chapter. Hence, it suffices to reduce the problem to the toroidal case. To achieve this goal, we first need the following theorem:

Theorem 3.1. *Let S be a smooth variety, let S' be a normal variety, and let $f : S' \longrightarrow S$ be a finite morphism. Suppose Z is a divisor of S of normal crossings and $T = S - Z$. If $f^{-1}(T)$ is étale over T, then $f^{-1}(T) \longrightarrow S'$ is a strictly toroidal embedding. Moreover, $Gal(S'/S)$ acts on S' toroidally.*

To prove this theorem we need the following lemma[G]:

Lemma 3.2 (Abhyankar's lemma). *Let X be a regular local scheme, let $D = \sum_{1\leq i\leq r} div\ f_i$ be a divisor of normal crossings, i.e., $f_1, f_2, ..., f_r$ belong to a regular system of parameters. Let $Y =$ Supp D and $U = X - Y$. Given a finite étale* *covering V of U, there exist $n_1, ..., n_r \in \mathbb{Z}^+$ such that for*

$$X' = X[x_1, ...x_r]/({x_1}^{n_1} - f_1, ..., {x_r}^{n_r} - f_r),$$

$U' = U \times_X X'$, and $V' = V \times_X X'$, the étale covering $V' \longrightarrow U'$ can be extended uniquely to an étale covering $X'' \longrightarrow X'$, where X'' is a variety containing V' as an open subvariety.

Proof of theorem 3.1. Since the question is local, we can assume that S is a regular local scheme and $Z = \sum_{1 \leq i \leq r} \text{div } f_i$ is a divisor of normal crossings. Applying Abhyankar's lemma we get $n_1, ..., n_r \in \mathbb{Z}_+$ and

$$S'' = S[x_1, ...x_r]/({x_1}^{n_1} - f_1, ..., {x_r}^{n_r} - f_r)$$

such that $S' \times_S S''$ is étale over S''. We pass everything to its completion and still use the same notation. Since $S' \times_S S''$ is étale over S'', after taking completion they are isomorphic. Hence we have a map $p : S'' \longrightarrow S'$, which is a quotient map under a subgroup $G \subset Gal(S''/S)$. Since S'' is strictly toroidal, $Gal(S''/S)$ is a subgroup of the torus $S'' - Z''$. It follows that G is also a subgroup of the torus $S'' - Z''$. Hence $S' = S''/G$ is strictly toroidal(proposition 2.7.) and $Gal(S'/S) = Gal(S''/S)/G \subset (S'' - Z'')/G = S' - Z'$, i.e., $Gal(S'/S)$ acts toroidally on S'. □

Proof of theorem 0.2.

Suppose Z, X, and G are as given in the statement of the theorem. Let $Y = X/G$, Z/G be the quotients, and B the branch locus. Define $W = (Z/G) \cup B$. Let

$(Y', W') \to (Y, W)$ be a resolution of singularities of Y where W' a strict divisor of normal crossings. Let X' be the normalization of Y' in $K(X)$, let Z' be the inverse image of W', and let $U = X' \setminus Z'$. From theorem 3.1 we know that $U \subset X'$ is a strictly toroidal embedding on which $G = Gal(X/Y)$ acts toroidally. Applying proposition 2.5, we obtain a nonsingular strict toroidal embedding $U \subset X_1 \to X'$ as required. □

The following conjecture is suggested by Professor Dan Abramovich, it is a generalization of theorem 3.1.

Conjecture. *Let $T \hookrightarrow S$ be a strictly toroidal embedding, let $Z = S - T$, and let $f : S' \longrightarrow S$ be a finite morphism where S' is normal. If $f^{-1}(T)$ is étale over T, then $f^{-1}(T) \longrightarrow S'$ is a strictly toroidal embedding. Moreover, $Gal(S'/S)$ acts on S' toroidally.*

The next theorem will not be used later in this thesis. It is the analogue of theorem 3.1 in the toric variety case. It is interesting that neither of them seems to imply the other.

Theorem 3.3. *Let $T \longrightarrow S$ be an affine toric variety, and let $f : S' \longrightarrow S$ be a finite morphism where S' is normal. If $f^{-1}(T)$ is étale over T, then $f^{-1}(T) \longrightarrow S'$ is also an affine toric variety. Moreover, $Gal(S'/S)$ can be identified with a subgroup of $f^{-1}(T)$, i.e., $Gal(S'/S)$ acts toroidally on S'.*

Proof: Let $n = \dim S$. Then T is a torus isomorphic to k^{*n}. Since $f^{-1}(T)$ is finite étale over T, from classical results in complex tori theory and the Leftschetz principle, we know $f^{-1}(T)$ is also isomorphic to k^{*n}. Now if we identify $f^{-1}(T)$ with *Spec* $k[y_1, y_2, ..., y_n, {y_1}^{-1}, {y_2}^{-1}, ..., {y_n}^{-1}]$ and T with *Spec* $k[x_1, x_2, ..., x_n, {x_1}^{-1}, {x_2}^{-1}, ..., {x_n}^{-1}]$, then the morphism f is induced from a homomorphism of their coordinate rings, which we also denote by f. Using the following lemma we can find suitable coordinates of $f^{-1}(T)$ and T such that the map is monomial with respect to the new coordinates.

Lemma 3.4. *Let f be a homomorphism of rings from $k[x_1, x_2, ..., x_n, {x_1}^{-1}, {x_2}^{-1}, ..., {x_n}^{-1}]$ to $k[y_1, y_2, ..., y_n, {y_1}^{-1}, {y_2}^{-1}, ..., {y_n}^{-1}]$. Then f is étale $\Leftrightarrow f = g \circ h \circ e$, where e is an automorphism of $k[x_1, x_2, ..., x_n, {x_1}^{-1}, {x_2}^{-1}, ..., {x_n}^{-1}]$, g is an automorphism of $k[y_1, y_2, ..., y_n, {y_1}^{-1}, {y_2}^{-1}, ..., {y_n}^{-1}]$, and $h(y_i) = (y_i)^{m_i}$, $m_i \in \mathbb{Z}_+$, for $i = 1, 2, ..., n$.*

Without loss of generality, we assume these suitable coordinates are $\{x_i\}$ and $\{y_i\}$, and $f(x_i) = {y_i}^{m_i}, m_i \in \mathbb{Z}_+$, for $i = 1, 2, ..., n$.

Since S is an affine toric variety, we can assume it to be *Spec* $k[\chi^{\lambda_1}, \chi^{\lambda_2}, ..., \chi^{\lambda_s}]$, where $\lambda_1, \lambda_2, ..., \lambda_s$ are generators of a cone δ in $N = \mathbb{Z}^n$. Let $e_1, e_2, ..., e_n$ be the generators of the lattice N, then $\frac{e_1}{m_1}, \frac{e_2}{m_2}, ..., \frac{e_n}{m_n}$ generate a lattice N' such that $N \subset N'$. We use δ' to denote the cone in N' generated by λ_i, $i = 1, 2, ..., n$, and use $S_{\delta'}$ to denote the affine toric variety corresponding to this cone. We immediately have the following fiber product diagram:

$$\begin{array}{ccc} T & \longrightarrow & S_{\delta'} \\ \Big\downarrow & & \Big\downarrow f' \\ T & \longrightarrow & S \end{array}$$

It is easy to see that $f' : T \longrightarrow T$ is identical to $f : f^{-1}(T) \longrightarrow T$. Since $S_{\delta'}$ and S' share the same open set $f^{-1}(T)$, they have the same function fields. Moreover, f and f' are both affine morphisms, $S_{\delta'}$ and S' are both normal varieties, so f and f' are both normalization maps and hence are identical.

We thus conclude that $f^{-1}(T) \longrightarrow S'$ is a toric variety.

To see that $Gal(S'/S)$ can be identified with a subgroup of $f^{-1}(T)$, note that $Gal(S'/S)$ acts on S' and S is the quotient of the action. The restriction of $Gal(S'/S)$ on $f^{-1}(T)$ obviously agrees with the action of a subgroup G' of $f^{-1}(T)$ on $f^{-1}(T)$. Since $f^{-1}(T)$ is an open subscheme of S', we conclude that the action of $Gal(S'/S)$ on S' agrees with the action of G' on S'. Hence we can identify $Gal(S'/S)$ with G'. $\square$

Proof of lemma 3.4.

$\Longleftarrow$: Obvious.

$\Longrightarrow$: we assume $f(x_i) = C_i {y_1}^{a_{i1}} {y_2}^{a_{i2}} ... {y_n}^{a_{in}}, a_{ij} \in \mathbb{Z}$. A better way to see what this map looks like is to take the logarithm of the coordinates x_i's and y_i's. We have

$$f : (\log \frac{x_1}{C_1}, \log \frac{x_2}{C_2}, ... \log \frac{x_n}{C_n})^T = A \times (\log y_1, \log y_2, ... \log y_n)^T$$

where A has a_{ij} as its (i, j)th entry. In other words, the homomorphism between the coordinate rings of the torus can be realized as a linear function between the logarithms of their coordinates. It is easy to see that the morphism f is an isomorphism if and if only the matrix A has determinant 1 or -1.

Since all entries of A are integers , we can performs column and row transformations to diagonalize A, in other words, there exist matrices S and R with integer entries such that $A = SA'R$, S and R have determinants 1 or -1, and $A' = diag(a_1, a_2, ..., a_n)$ where all a_i are integers. Since f is étale, A is nonsingular and a_i are nonzero. We can further assume that a_i are all positive, by choosing appropriate S and R.

Now let's set

$$(\log x_1', \log x_2', ... \log x_n')^T = S^{-1}(\log \frac{x_1}{C_1}, \log \frac{x_2}{C_2}, ... \log \frac{x_n}{C_n})^T$$

and

$$(\log y_1', \log y_2', ... \log y_n')^T = R(\log y_1, \log y_2, ... \log y_n)^T$$

Then we define e, g, h by

$$e : x_i \longrightarrow x_i'$$

$$h : x_i' \longrightarrow {y_i'}^{a_i}$$

$$g : {y_i}' \longrightarrow y_i.$$

e, g are isomorphisms from our previous discussion. It is clear that $f = g \circ h \circ e$. □

CHAPTER IV

Equivariant Semi-stable Reduction

Lemma 4.1.

Let C be a smooth curve and G a finite subgroup of $Aut(C)$. Denote the function field of C by K. Let field K' be a Galois extension of K which is also a Galois extension of K^G. Let $\pi : C' \longrightarrow C$ be the normalization of C corresponding to the field extension $K \subset K'$, and let $G' = Gal(K'/K^G)$. Then G' is a finite group with a well-defined action on C', and if we define an action of G' on C in the obvious sense, then π is G'-equivariant.

Proof. Denote the completion of C and C' by $\overline{C}$ and $\overline{C'}$, respectively, π can be extended to a morphism $\overline{\pi} : \overline{C'} \longrightarrow \overline{C}$.

Since $G' = Gal(K'/K^G)$, there is a canonical surjection $p : G' \longrightarrow G$. To simplify notation we will use $\overline{g}$ to denote $p(g)$ for $g \in G'$.

We can extend the action of G on C to $\overline{C}$. Since every element g' of G' induces an automorphism of $\overline{C'}$, we can define an action of G' on $\overline{C'}$. The surjection $p : G' \longrightarrow G$

induces an action of G' on $\overline{C}$ which is compatible with the action of G on $\overline{C}$.

Since $\overline{g}$ is the restriction of g on K, the following diagram is commutative

$$\begin{array}{ccc} K' & \xleftarrow{i} & K \\ \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle \overline{g}} \\ K' & \xleftarrow{i} & K \end{array}$$

.

It follows that the diagram

$$\begin{array}{ccc} \overline{C'} & \xrightarrow{\overline{\pi}} & \overline{C} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \overline{g}} \\ \overline{C'} & \xrightarrow{\overline{\pi}} & \overline{C} \end{array}$$

is commutative. Hence $\overline{\pi}$ is G'-equivariant. Let $g \in G'$ and $c \in C'$. Then $\overline{\pi} \circ g(c) = \overline{g} \circ \overline{\pi}(c)$. But $\overline{\pi}(c) \in C$ and hence $\overline{g} \circ \overline{\pi}(c) \in C$, so $g(c) \in \overline{\pi}^{-1}(C) = C'$. This proves that g maps points of C' to C'. Similarly we can prove that g maps points of $\overline{C'} - C'$ to $\overline{C'} - C'$. It follows that the restriction of g to C' induces an isomorphism of C' to itself, so G' acts on C'. Finally, since $\overline{\pi}$ is G'-equivariant and π is the restriction of $\overline{\pi}$ on C', π is G'-equivariant. □

Lemma 4.2. *Let X, X', Y be varieties and let G be any group acting on X, X', Y. If $f : Y \longrightarrow X$, $p : X' \longrightarrow X$ are G-equivariant morphisms, then there is a natural action of G on $X' \times_X Y$ such that the following diagram is G-equivariant*

$$\begin{array}{ccc} X' \times_X Y & \xrightarrow{p'} & Y \\ \Big\downarrow f' & & \Big\downarrow f \\ X' & \xrightarrow{p} & X \end{array}$$

Proof. Let $g \in G$. We have the following commutative diagram:

$$\begin{array}{ccc} X' \times_X Y & \xrightarrow{g \circ p'} & Y \\ \Big\downarrow g \circ f' & & \Big\downarrow f \\ X' & \xrightarrow{p} & X \end{array}$$

By the universal property of fiber products there exists a morphism $g' : X' \times_X Y \longrightarrow X' \times_X Y$ such that the following diagram is commutative,

$$\begin{array}{ccccc} X' \times_X Y & = & X' \times_X Y & \xrightarrow{p'} & Y \\ \| & & \Big\downarrow g' & & \Big\downarrow g \\ X' \times_X Y & \xrightarrow{g'} & X' \times_X Y & \xrightarrow{p'} & Y \\ \Big\downarrow f' & & \Big\downarrow f' & & \Big\downarrow f \\ X' & \xrightarrow{g} & X' & \xrightarrow{p} & X \end{array}$$

So $p' \circ g' = g \circ p'$ and $f' \circ g' = g \circ f'$. Clearly, if g is the trivial element of G, it corresponds to the identity morphisms on X, X', and Y, and g' is the identity map on $X' \times_X Y$. Moreover, ${g_1}' \circ {g_2}' = (g_1 \circ g_2)'$ for any $g_1, g_2 \in G$. Hence $g' \circ g^{-1'} = (g \circ g^{-1})' = id$, proving that g' is an automorphism. We define the action of g on $X' \times_X Y$ by the automorphism g'. From our construction this G-action clearly makes the diagram G-equivariant. □

Before we state the equivariant semi-stable reduction theorem more precisely, we need a few more machineries.

Let C be a smooth curve, let G be a finite group acting on C, and let O be a point on C which is invariant under the action of G. Denote the local ring of O by $A_{O,C}$ and the maximal ideal of $A_{O,C}$ by $m_{O,C}$.

Lemma 4.3. *Under the above conditions, the image of G in $Aut(C)$, which we denote by H, is a cyclic group and we can find a local parameter t of $A_{O,C}$ such that $K = K^H(t)$ and the minimal polynomial of t is $x^{|H|} - a$ for some $a \in K^H$.*

Proof. We abuse notation and don't distinguish between automorphisms of the curve C and automorphisms of its function field $K(C)$. Since H acts on C and O is invariant under the action of H, $H \subset Aut(A_{O,C})$. Passing to the completion of $A_{O,C}$, we consider H as a subgroup of $Aut(\hat{A}_{O,C})$.

Since C is nonsingular, $A_{O,C}$ is a regular ring and $\hat{A}_{O,C} \cong k[[s]]$ for any regular element $s \in m_{O,C}$. For any $g \in H$, $s^{-1}g(s)$ invertible, so $\prod_{g\in H} s^{-1}g(s)$ is invertible in $k[[s]]$. Let $n = |H|$. Since k is algebraically closed, any invertible element of $k[[s]]$ has nth roots. Hence there exists $r \in k[[s]]$ such that $\prod_{g\in H} s^{-1}g(s) = r^n$, i.e., $\prod_{g\in H} g(s) = (sr)^n$.

Clearly, $\prod_{g\in H} g(s)$ is invariant under H, so $(sr)^n$ is invariant under H. Since $g((sr)^n) = (sr)^n$, we have $g(sr) = \xi_g sr$, where ξ_g is an nth root of unity. For different g, ξ_g must be different, so $\{\xi_g | g \in H\}$ is exactly the set of all nth roots of unity. It follows immediately that H is cyclic.

Since H is cyclic, the action of any element of H on $\frac{m_{O,C}}{{m_{O,C}}^2} \cong k$ is a multiplication by an nth root of unity. Moreover, different elements of H correspond to different nth roots of unity. Let t' be a generator of $m_{O,C}$, and denote the image of t' in $\frac{m_{O,C}}{{m_{O,C}}^2}$ by $\overline{t'}$. Let g be the element of H which acts on $\frac{m_{O,C}}{{m_{O,C}}^2}$ by $g(\overline{t'}) = \xi\overline{t'}$. Here ξ is a primitive nth root of unity. Consider the element $t = \frac{1}{n} \sum_{0 \le i \le n-1} \xi^{-i} g^i(t')$. It is easy to check that $g(t) = \xi t$ and $t - t' \in {m_{O,C}}^2$. Hence t is a local parameter of $A_{O,C}$ and the minimal polynomial of t is $x^n - t^n$. This proves that $K = K^H(t)$. □

We fix this generator t of $m_{O,C}$. For all $d \ge 1$, let C_d be the normalization of C corresponding to the field extension generated by $t^{\frac{1}{d}}$, let $\pi_d : C_d \longrightarrow C$ be the canonical morphism, and let $O_d = {\pi_d}^{-1}(O)$. Let $K(C)$ and $K(C_d)$ be the function fields of C and C_d, respectively. Clearly, $K(C_d)$ is a Galois extension of both $K(C)$ and $K(C)^H$. We use H_d to denote the group $Gal(K(C_d)/K(C)^H)$. By lemma 4.1, H_d acts on C_d, Hence $G_d := H_d \times_H G$ acts on C_d. Given a smooth variety X and a morphism $f : X \longrightarrow C$, suppose $f^{-1}(O)$ is a divisor of normal crossings. We write $f^{-1}(O) = \sum_{1 \le i \le N} n(i) E_i$. For $d \ge 1$, let X_d be the normalization of $X \times_C C_d$, let $f_d : X_d \longrightarrow C_d$ be the projection, and let $U_d = f_d^{-1}(C_d - O_d)$. From Lemma 2.1, 4.1

and 4.2 we have canonical G_d-actions on X_d and C_d so that the following diagram

$$\begin{array}{ccc} X_d & \xrightarrow{\pi_d f} & X \\ \Big\downarrow f_d & & \Big\downarrow f \\ C_d & \xrightarrow{\pi_d} & C \end{array}$$

is G_d-equivariant. Henceforth, whenever we do a base change $C_d \longrightarrow C$, we will extend the group action of G to a group action of G_d as above.

Lemma 4.4 [KKMS, P. 102]. *$U_d \subset X_d$ is a strict toroidal embedding.*

Remark. This lemma is also an easy corollary of theorem 3.3.

Lemma 4.5. [KKMS, P. 103]. *Let $v = l.c.m.(n(1), n(2), ...n(N))$. If $v|d$, then $X_d \longrightarrow X_v \times_{C_v} C_d$ is an isomorphism, hence the closed fibers $f_d^{-1}(O_d)$ are independent of d, and the projection $X_v \longrightarrow X_d$ induces a bijection between the strata of $X_v - U_v$ and the strata of $X_d - U_d$. Moreover, $f_d^{-1}(O_d)$ is a reduced subscheme of X_d.*

In the following discussion and lemma 4.6, we always assume $v|d$.

Let Δ_d be the polyhedral complex associated to $U_d \subset X_d$, then there is a canonical polyhedral isomorphism between Δ_d and Δ_v.

However, when we replace Δ_v by Δ_d, the integral structure changes. The integral structures on the corresponding polyhedrals δ^{Y_d} and δ^{Y_v} are given by the functions defined by M^{Y_d} and M^{Y_v}, respectively. There is the following lemma:

Lemma 4.6 [KKMS, P. 105]. *Every Cartier divisor D on Star Y_d supported by $f_d{}^{-1}(O_d)$ is of the form $p^*D_1 + a(t^{\frac{1}{d}})$ for some $a \in \mathbb{Z}$, where p is the morphism from X_d to X_v.*

For $U_d \subset X_d$, we have a positive Cartier divisor D_d with support $X_d - U_d$, namely $f_d^{-1}(O_d)$. D_d defines a function $l_d : \Delta_d \longrightarrow R_+$. Note that via the canonical isomorphism $\Delta_d \cong \Delta_v, l_d = \dfrac{v}{d} l_v$, we can define a compact polyhedral complex

$$\Delta_d{}^* = \{x \in \Delta_d | l_d(x) = 1\}$$

By restriction, we get an integral structure $M_d{}^*$ on $\Delta_d{}^*$. Moreover, by central projection and the canonical isomorphism between Δ_d and Δ_v, we get a canonical isomorphism of $\Delta_d{}^*$ and $\Delta_v{}^*$. By lemma 4.6, $M_d{}^* = \dfrac{d}{v} M_v{}^* + \mathbb{Z}$. Hence, we obtain isomorphism between the integral lattice $(\Delta_d{}^*)_{\mathbb{Z}}$ in $\Delta_d{}^*$ and the lattice of $(\Delta_v{}^*)_{\frac{v}{d}\mathbb{Z}}$ of points in $\Delta_v{}^*$ with coordinates in $\dfrac{v}{d}\mathbb{Z}$. We recall the following theorem [KKMS]. It reveals the connection between subdivisions of the compact polyhedral complex and the corresponding modifications of X.

Theorem 4.7. *Given $U \subset X$ and a divisor D whose support is in $X - U$, let Δ' be a subdivision of Δ, let $(\Delta^*)'$ be the associated subdivision of Δ^*, and let $f : Z_{\Delta'} \longrightarrow X$ be the corresponding modification. Then*

a) The vertices of $(\Delta^*)'$ are in $(\Delta^*)_{\mathbb{Z}}$ iff $f^{-1}(D)$ vanishes to order one on each component of $Z_{\Delta'} - U$.

b) If (a) holds, the volume of every polyhedron τ_0 *in* Δ_0' *is* $\frac{1}{(dim\tau_0)!}$ *iff* $Z_{\Delta'}$ *is nonsingular.*

We call a subdivision of a compact polyhedral complex **reduced** if the subdivision satisfies the condition (a), and we call it **nonsingular** if it satisfies condition (b). We also call a compact polyhedral complex **reduced**(resp. **nonsingular**) if its trivial subdivision is reduced(resp. nonsingular).

Theorem 4.8 (Equivariant Semi-stable Reduction Theorem I).

Let C *be a nonsingular algebraic curve,* $O \in C$*,* X *a nonsingular variety, and* $f : X \longrightarrow C$ *a proper morphism of a variety* X *onto* C *such that*

$$resf : X - f^{-1}(O) \longrightarrow C - O$$

is smooth and $f^{-1}(O)$ *is a divisor with normal crossings. Suppose* $f^{-1}(O) = \sum_{1 \le i \le N} n(i)E_i$ *and let* v *be the least common multiple of* $n(i), 1 \le i \le N$*. If* G *is a finite subgroup acting on* X *and* C*,* f *is* G*-equivariant,* O *is invariant under the action of* G*, and* $f^{-1}(O)$ *is* G*-strict, then*

1. There exists an $e \in \mathbb{Z}_+$ *such that for any* $d \in \mathbb{Z}_+$*, we have a variety* X'*, a morphism* $p : X' \longrightarrow X \times_C C'$*, and the* G_{ved}*-equivariant commutative diagram*

$$\begin{array}{ccccc}
X' & \xrightarrow{p} & C_{ved} \times_C X & \xrightarrow{\pi_{ved}{}^f} & X \\
\downarrow f_{ved} \circ p & & \downarrow f_{ved} & & \downarrow f \\
C_{ved} & = & C_{ved} & \xrightarrow{\pi_{ved}} & C
\end{array}$$

with the following properties:

a) p is an isomorphism over ${f_{ved}}^{-1}(C_{ved} - O_{ved})$,

b) p is projective. In fact, p is obtained by blowing up a sheaf of ideals Γ *with* $\Gamma|_{{f_{ved}}^{-1}(C_{ved}-O_{ved})} = O_{C_{ved}\times_C X}|_{{f_{ved}}^{-1}(C_{ved}-O_{ved})}$,

c) X' *is nonsingular, and the fiber* $(f_{ved} \circ p)^{-1}(O_{ved})$ *is reduced with non-singular components crossing normally.*

2. Suppose $\dim X \leq 3$, *then* $e = 1$ *suffices.*

Proof. Since X is nonsingular and $f^{-1}(O)$ is a divisor of normal crossings of X, $X - f^{-1}(O) \to X$ is a toroidal embedding. We use Δ to denote its conical polyhedral complex and Δ^* to denote the compact polyhedral complex corresponding to $f^{-1}(O)$ as usual. We first construct the following commutative diagram:

$$\begin{array}{ccc} X_v & \xrightarrow{\pi_v f} & X \\ \downarrow f_v & & \downarrow f \\ C_v & \xrightarrow{\pi_v} & C \end{array}$$

After this base change, we know from lemma 4.5 that ${f_v}^{-1}(O_v)$ is already a reduced subscheme of X_v. From our previous discussion we know that G_v acts naturally on X_v and the above diagram is G_v-equivariant. Moreover, since $f^{-1}(O)$ is G-strict, $f_v^{-1}(O_v)$ is G_v-strict.

We consider the conical polyhedral complex Δ_v and the compact polyhedral complex ${\Delta_v}^*$ corresponding to X_v and $f_v^{-1}(O_v)$, respectively. Since $f_v^{-1}(O_v)$ is G_v-strict, the G_v-quotient of Δ_v is a well-defined conical polyhedral complex by lemma 2.2. Moreover, since $f_v^{-1}(O_v)$ is invariant under the action of G_v, ${\Delta_v}^*$ is invariant under the action of G_v, so its G_v-quotient is also a well-defined compact polyhedral complex on Δ_v/G_v.

We have shown that ${\Delta_v}^*$ has all its vertices in $({\Delta_v}^*)_{\mathbb{Z}}$, i.e., ${\Delta_v}^*$ is reduced. So ${\Delta_v}^*/G_v$ is also reduced. If we can find a nonsingular subdivision of ${\Delta_v}^*/G_v$, this will induce a G_v-equivariant nonsingular subdivision of ${\Delta_v}^*$, and we will be done. However, a general reduced compact polyhedral complex may not have a nonsingular subdivision, as shown in the following counterexample.

Example. let A be the tetrahedron in $\mathbb{Z}^3$ with the four vertices (0,0,0), (1,1,0), (1,0,1), (0,1,1). Clearly, there is no lattice point besides these four vertices in A, so A is the only reduced subdivision of itself. However, A has volume $\frac{2}{3}$, so it is not nonsingular.

Fortunately, the following theorem assures us that we can get a nonsingular subdivision of any compact polyhedral complex if we are allowed to refine the integral structure(by a refinement of an integral structure we mean a larger integral structure containing the original one).

Theorem 4.9. *Given a polyhedron $\sigma \in \mathbb{R}^n$ with integral vertices, there exist an integer e and a subdivision of σ into simplices τ_α such that for all α:*

1) vertices of $\tau_\alpha \in \frac{1}{e}\mathbb{Z}^n$*,*

2) volume$(\tau_\alpha) = \frac{1}{e^n n!}$.

This theorem was proved in [KKMS]. They introduced very complicated subdivisions and proved theorem 4.9 with those subdivisions. Moreover, they also proved that these subdivisions are all projective, hence the subdivision in the above theorem is projective. For any integer e, they introduced a subdivision for a simplex called the e-regular subdivision. Intuitively, an e-regular subdivision of a simplex of dimension n is just a subdivision of the original simplex using $n+1$ families of hyperplanes, each family containing $e+1$ parallel hyperplanes. An e-regular subdivision subdivides an n-dimensional simplex into e^n identical simplices(see [KKMS] for detail). Any regular subdivision is also projective, we will use this fact later.

The above theorem, combined with the discussion after lemma 4.6, gives us a way of using a base change to get a nonsingular subdivision.

Returning to our proof, we already know that $\Delta_v{}^*/G_v$ is reduced, applying the above theorem, we get an integer e and a subdivision τ_α of $\Delta_v{}^*/G_v$ such that for all α:

1) vertices of $\tau_\alpha \in \frac{1}{e}\mathbb{Z}^n$, and

2) $volume(\tau_\alpha) = \frac{1}{e^n n!}$.

Now for any integer d, we further perform a d-regular subdivision of τ_α and obtain a subdivision σ_β such that for all β:

1) vertices of $\sigma_\beta \in \frac{1}{ed}\mathbb{Z}^n$,

2) $volume(\sigma_\beta) = \frac{1}{(ed)^n n!}$.

This subdivision leads to a G_v-equivariant subdivision of $\Delta_v{}^*$ which we also call σ_β.

Hence, if we interpret σ_β as a subdivision of $\Delta_{ved}{}^*$, then this subdivision is G_{ved}-equivariant. This subdivision induces a G_{ved}-equivariant modification $p : X' \longrightarrow X_{ved}$ which satisfies all the conditions.

In the cases X is a variety with dimension less or equal than 3, $e = 1$ suffices. Indeed, the following Pick's theorem and proposition 4.11 provide us a nonsingular subdivision of $\Delta^*{}_v$. A proof of Pick's theorem can be found in [EGH]. □

Lemma 4.10 (Pick's Theorem). *Let $\mathbb{Z} \times \mathbb{Z}$ be a lattice and P a convex polygon whose vertices are all lattice points. If A is the number of lattice points inside P and L is the number of lattice points on the boundary of P, then the area of P is $A + \frac{L}{2} - 1$. In particular, for a triangle with no lattice points on it other than the vertices, the area is $\frac{1}{2}$.*

Proposition 4.11. *Let P be a compact polyhedral complex of dimension at most 2. There exists a nonsingular subdivision of P.*

Proof. If P is of dimension 1, the result is obvious.

If $dimP = 2$, we consider any maximal reduced subdivision P' of P. Then P' must also be nonsingular. Indeed, there is no lattice point in any triangle of the subdivision (since otherwise, we can subdivide this triangle to get a finer subdivision), so the area of any triangle of P' is $\frac{1}{2}$ by Pick's theorem. □

Theorem 4.12 (Equivariant Semi-stable Reduction Theorem II). *Let C be a non-singular algebraic curve, $O \in C$, X a variety, and $f : X \longrightarrow C$ a proper morphism of a variety X onto C such that*

$$resf : X - f^{-1}(O) \longrightarrow C - O$$

is smooth. If G is a finite subgroup acting on X and C, f is G-equivariant, and O is invariant under the action of G, then there exists an $e \in \mathbb{Z}_+$ such that for any $d \in \mathbb{Z}_+$, we have a variety X', a morphism $p : X' \longrightarrow X \times_C C'$, and the G_{ed}-equivariant commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{p} & C_{ed} \times_C X & \xrightarrow{\pi_{ed}f} & X \\ \Big\downarrow f_{ed} \circ p & & \Big\downarrow f_{ed} & & \Big\downarrow f \\ C_{ed} & = & C_{ed} & \xrightarrow{\pi_{ed}} & C \end{array}$$

with the following properties:

a) p is an isomorphism over ${f_{ed}}^{-1}(C_{ed} - O_{ed})$,

b) p is projective. In fact, p is obtained by blowing up a sheaf of ideals Γ with $\Gamma|_{{f_{ed}}^{-1}(C_{ed}-O_{ed})} = O_{C_{ed}\times_C X}|_{{f_{ed}}^{-1}(C_{ed}-O_{ed})}$, *c) X' is nonsingular, and the fiber $(f_{ed} \circ p)^{-1}(O_{ed})$ is reduced with non-singular components crossing normally.*

Proof. By theorem 0.2, we can find a G-equivariant resolution

$$g : Y \longrightarrow X$$

such that Y is nonsingular and $g^{-1}(f^{-1}(O))_{red}$ is a union of non-singular components crossing transversely. We perform the barycentric subdivision on $\Delta(Y)$ and let the corresponding modification be $h : B(Y) \longrightarrow Y$. Let v be the l.c.m of all the coefficients of $h^{-1} \circ g^{-1} \circ f^{-1}(O)$. From proposition 2.4 we know that G acts naturally on $B(Y)$, h is G-equivariant, and $h^{-1} \circ g^{-1} \circ f^{-1}(O)$ is G-strict. By theorem 5.3, we can find an $e' \in \mathbb{Z}_+$ such that for any $d \in \mathbb{Z}_+$, there exist a variety X' and a morphism $p : X' \longrightarrow C_{ve'd} \times_C B(Y)$ satisfying all the conditions stated in theorem 5.3. In particular, the following diagram is commutative and G_{ved}-equivariant

$$\begin{array}{ccccc} X' & \xrightarrow{p} & C_{ve'd} \times_C B(Y) & \xrightarrow{\pi_{ve'd}^{f \circ g \circ h}} & B(Y) \\ & & \Big\downarrow f' \circ g' \circ h' & & \Big\downarrow f \circ g \circ h \\ & & C_{ve'd} & \xrightarrow{\pi_{ve'd}} & C \end{array}$$

By lemma 5.2, it follows that in the following diagram, $G_{ve'd}$ acts naturally on $C_{ve'd} \times_C X$ and the whole diagram is $G_{ve'd}$-equivariant

$$
\begin{array}{ccccccc}
X' & \xrightarrow{p} & (C_{ve'd} \times_C X) \times_X B(Y) & \xrightarrow{\pi_{ved}{}^{f \circ g \circ h}} & B(Y) \\
\downarrow{\scriptstyle g'' \circ h} & & \downarrow{\scriptstyle g' \circ h} & & \downarrow{\scriptstyle g \circ h} \\
C_{ve'd} \times_C X & = & C_{ve'd} \times_C X & \xrightarrow{\pi_{ved}{}^{f}} & X \\
\downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
C_{ve'd} & = & C_{ve'd} & \xrightarrow{\pi_{ve'd}} & C
\end{array}
$$

Thus we can let $e = ve'$. For any d, $(X', g' \circ p)$ satisfies all the conditions we need. □

Remark. In theorem 4.8, for the case $\dim X \leq 3$ we assumed that X is a smooth variety and $f^{-1}(O)$ is G-strict divisor of normal crossings, so that we could have a good control of e. In fact, if we drop the G-strict condition, we can perform the barycentric subdivision as in the proof of theorem 4.12 and show that e can be chosen to be 2.

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