Spectral Properties of Kähler Quotients

by

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Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2008

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Abstract

The asymptotic behavior for the spectral measure of a Kähler manifold has been studied by many authors in the context of Kähler quantization. It is well known that the spectral measure has an asymptotic expansion, while the coefficients of this expansion are not known even for very simple examples. In this thesis we study the spectral properties of Kähler manifolds assuming the existence of some symmetry, i.e., a Hamiltonian action.

The main tool we will use is a function which we call the *stability function*. Roughly speaking, it is the function which compares quantum states before reduction with quantum states after reduction. We will study this function in detail, compute the function for many classes of Kähler manifolds, and apply it to study various spectral problems on Kähler quotients.

As for the spectral measure, we will give an explicit way to compute the coefficients in the asymptotic expansion for toric varieties. It turns out that the upstairs spectral measure in this case is described by an interesting integral transform which we will call the *twisted Mellin transform*. We will study both analytic and combinatorial aspects of this transform in the beginning of this thesis.

Thesis Supervisor: Victor W. Guillemin
Title: Professor of Mathematics

Acknowledgments

First and foremost my thanks go out to my thesis adviser, Professor Victor Guillemin, for all the interesting problems he suggested to me, for sharing with me his mathematical ideas, for correcting me numerous of mathematics/grammar mistakes, for helping me to prepare presentations, and for showing me how to teach mathematics. Without his help, this thesis would never be possible.

I would like to thank many Professors in MIT who taught me courses and/or helped me in this way or in that way. They are: Richard Dudley, Pavel Etingof, Sigurdur Helgason, David Jerison, George Lusztig, Arthur Mattuck, Richard Melrose, Richard Stanley, Hartley Rogers and Andras Vasy (Stanford) etc. Thanks also goes to Professors Daniel Burns and Alejandro Uribe in the University of Michigan for many fruitful conservations and suggestions. I would also like to thank the members of "Guillemin group" in MIT: Silvia, Ana Rita, Zhenqi, Liat, and especially Alvaro. Many thanks should also go to Linda and other staff members here in MIT who do me a lot of good to help my life here so enjoyable.

This list should definitely include the big Chinese family in MIT mathematics department, which contains earlier students Damin, Zhou, Xuhua, Fangyun, Huadong and Chuying, and current students, especially those in our MCMSeminar: Fang, Jun, Lu, Qian, Ting and Xiaoguang, for their helps, friendship and pleasant times. Special thanks goes to my best friend Mao Sheng in Mainz for his valuable friendship during these years.

Last, but not least, my thanks go to my parents Hongcai Wang, Rongyuan Shao and my brother Jiasheng Wang, for their unconditional support and love. I am especially thankful to my wife, Dili Ruan, for her love, sharing life with me and giving me strength, and to my son Daniel for his tears and smiles.

Contents

Chapter 1

Introduction

The Bohr correspondence principle in quantum mechanics asserts that the classical system *ought to* be the small $\hbar$ limit of the corresponding quantized system. Roughly speaking, the quantum observables/quantum states are just the eigenvalues/eigenfunctions of some self-adjoint operators acting on the quantum space. So one is led to study the asymptotic behavior of certain quantum operators, in the small $\hbar$, or equivalently, large N, limit. The systematic exploitation of these ideas is now known as *semi-classical analysis*.

Symplectic geometry has grown up as the mathematics framework for classical mechanics. In the case that the phase space is not only symplectic but also also complex, i.e., is a Kähler manifold, Kähler quantization is a very successful quantization scheme. It is well known that for M compact the spectral measure (=trace of the spectrum) in this setting has an asymptotic expansion in inverse power of N as N tends to infinity, while the coefficients in this expansion are not known in general. However, in many situations the phase space can be obtained from a very simple system together with some symmetry. It turns out that the spectral properties of such quotient Kähler manifolds can be derived explicitly from the invariant spectral properties of the upstairs Kähler manifolds, and vice versa.

1.1 Motivation

1.1.1 Kähler quantization

The word "quantization" appears everywhere in mathematics and physics, with many different meanings. Roughly speaking, it is a procedure/correspondence/functor/$\cdots$ from a classical mechanical system to a quantum mechanical system. However, there is no canonical definition of this functor. Different quantization schemes include geometric quantization, deformation quantization, Berezin-Toeplitz quantization, asymptotic quantization, etc. The common nature of all quantization theories is the understanding that classical mechanics and quantum mechanics are just the different descriptions of the same reality. According to Hamilton, classical mechanics can be described via (M, ω, H), where (M, ω) is a symplectic manifold called the pahse space, and H a real-valued function on M called the Hamiltonian, e.g. the energy function of the system. The dynamics of this classical system is described by the equation

$$\frac{df}{dt} = \{f, H\}, \tag{1.1.1}$$

where $f \in C^\infty(M)$ is a classical observable, and $\{\cdot, \cdot\}$ the Poisson bracket induced by the symplectic form on M. On the other hand, a quantum mechanical system is described by a Hilbert space $\mathcal{H}$ together with a self-adjoint operator $\hat{H}$ acting on $\mathcal{H}$, e.g., the Schrödinger operator. The Heisenberg's formalism of the dynamics for a quantum system is given by the equation

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, \hat{H}], \tag{1.1.2}$$

where A is any quantum observable, i.e., an self-adjoint operator acting on $\mathcal{H}$. Thus it is clear that a quantization procedure "should"

1. Assign a Hilbert space $\mathcal{H}$ to the symplectic manifold M.

2. Convert the Poisson bracket structure on $C^\infty(M)$ to the Lie bracket structure

on the associative algebra of self-adjoint operators acting on $\mathcal{H}$.

Let's first briefly review "geometric quantization" as defined by Kostant and Souriau in 1970's. As possibly the most widely used quantization method, geometric quantization can be divided into two (or more) steps. The first step, *pre-quantization*, associate to a symplectic manifold (M, ω) a *pre-quantum* line bundle $\mathbb{L}$, i.e., a Hermitian line bundle with connection whose first Chern class (i.e. the curvature) coincides with the symplectic form,

$$c_1(\mathbb{L}) = [\omega/2\pi].$$

Note that the existence of such line bundle requires (M, ω) to be *pre-quantizable*, i.e. ω should satisfy the integrability condition

$$\left[\frac{\omega}{2\pi}\right] \in \mathrm{Image}(H^2_{\check{C}ech}(M, \mathbb{Z}) \hookrightarrow H^2_{deRham}(M)). \tag{1.1.3}$$

The Hilbert space in pre-quantization is taken to be the space of all square-integrable sections of $\mathbb{L}$, with L^2-norm induced by the Hermitian structure,

$$\langle s_1, s_2 \rangle = \int_M \langle s_1(x), s_2(x) \rangle \frac{\omega^d}{d!}, \tag{1.1.4}$$

and the quantum operator $Q(f)$ associated to $f \in C^\infty(M)$ is

$$f \mapsto Q(f) = \frac{\hbar}{\sqrt{-1}} \nabla_{v_f} + f, \tag{1.1.5}$$

where v_f is the Hamiltonian vector field associated with f. It turns out that the pre-quantum space $\mathcal{H}$ above is too big to give a correct quantization. The second step of geometric quantization, *polarization*, eliminates this problem by taking $\mathcal{H}$ to be those sections which "only depend on half the variables" on M, or in other words, sections that are *constant along half directions*.

In general the choice of polarizations is problematical: they do not always exist, and are not unique if they do exist. However, in the case that M is Kähler, there is a canonical choice of polarization. Recall that a Kähler form ω is a $(1,1)$ form, and the Kähler manifold (M,ω) is pre-quantizable if

$$\left[\frac{\omega}{2\pi}\right] \in \mathrm{Image}(H^2_{\check{C}ech}(M,\mathbb{Z}) \hookrightarrow H^{1,1}_{Dolbeault}(M)), \tag{1.1.6}$$

and the pre-quantum line bundle $\mathbb{L}$ is required to be *holomorphic*. The polarized sections are by definition the square-integrable holomorphic sections of $\mathbb{L}$.

Kähler quantization, also called Berezin-Toeplitz quantization, is a variant of the geometric quantization story above. Suppose (M,ω) is a pre-quantizable Kähler manifold with $\mathbb{L}$ a pre-quantum line bundle. The quantum space is almost the same as above,

$$\mathcal{H}_N = \Gamma_{hol}(\mathbb{L}^N), \tag{1.1.7}$$

where N is a big constant [1]. However, the quantum operator is taken to be a much simpler one, the Toeplitz operator

$$T_N(f) = \pi_N M_f \pi_N, \tag{1.1.8}$$

where $\pi_N : \Gamma(\mathbb{L}^N) \to \Gamma_{hol}(\mathbb{L}^N)$ is the orthogonal projection, and

$$M_f : \Gamma(\mathbb{L}^N) \to \Gamma(\mathbb{L}^N), \quad s \mapsto fs$$

is the operator "multiplication by f".

We end this brief introduction by mentioning the fact that the Toeplitz operators (1.1.8) do satisfy the deformation quantization condition, i.e., in the asymptotic sense the Kähler quantization converts the Poisson structure for classical system to the Lie structure for quantum system. For more details on geometric/Kähler quantization, c.f. [BMS94], [Ber75], [Kos70], [Woo92] etc.

[1] $\frac{1}{N}$ plays the role of the Planck constant $\hbar = 6.62 \times 10^{-34}$.

1.1.2 Semiclassical limits

Quantization transfers information from the classical world to the quantum world. The inverse process, i.e. reading off classical information from quantum information, is the main task of semiclassical analysis. The guideline philosophy is given by the following

Bohr Correspondence Principle. *A classical system should describe the large N (small $\hbar$) limit of the corresponding quantized system.*

Notice that the quantum observables are not the quantum operators themselves, but rather the spectra of these operators. So semiclassical analysis aims at studying the spectral properties of self-adjoint operators on M in a small parameter limit: What can one read off from the spectra of the quantum operators?

As for Kähler quantization, the quantized space is $\mathcal{H}_N = \Gamma_{hol}(\mathbb{L}^N)$, and the quantum operators are $T_N(f) = \pi_N M_f \pi_N$. So the question can be reformulated as

1. Describing the asymptotics of the spectrum of $T_N(f)$.

2. Describing the asymptotics of the eigenstates in $\Gamma_{hol}(\mathbb{L}^N)$.

The first semiclassical result is the following estimate of the dimension of the quantized space, which is a corollary of the Hirzebruch-Riemann-Roch formula,

$$D(N) := \dim \Gamma_{hol}(\mathbb{L}^N) = N^d \mathrm{Vol}(M) + N^{d-1} \int_M c_1(M) \wedge \omega^{d-1} + \cdots \ , \qquad (1.1.9)$$

where d is the dimension of M, and we assume M is compact. As a result, we see that one can read off the dimension and the volume of the classical system from the quantized system.

A more interesting semiclassical problem, which is now a very active area, concerns the ergodicity properties of quantum states. For a Riemannian manifold, the geodesic flow is called *ergodic* if the only measurable subsets which are invariant under the geodesic flow are of measure 0 or of full measure. As we know, the quantum counterpart of the geodesic flow is the Laplacian operator Δ [2]. Suppose

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

are the eigenvalues of Δ, with n^{th} *normalized* eigenfunction

$$\Delta\varphi_n = \lambda_n\varphi_n.$$

Let

$$\nu_n = |\varphi_n(x)|^2 dx.$$

Then the Shnirelman-Zelditch-Colin de Verdiere's quantum ergodicity theorem claims that for a density one [3] subsequence $\{\lambda_{n_k}\}$,

$$\nu_{n_k} \to dx \tag{1.1.10}$$

as $k \to \infty$.

Colin de Verdiere also proved a collective version of quantum ergodicity: Suppose the Riemannian manifold is Zoll, i.e., all of its geodesics are of length 2π. Then the eigenvalues of $\sqrt{\Delta}$, the square root of the Laplacian operator, have the cluster property,

$$\mathrm{Spec}(\sqrt{\Delta}) \subset \bigcup_n (an + b - \frac{c}{n}, an + b + \frac{c}{n})$$

for some constants a, b and c. If we denote the clustered eigenfunctions by $u_{n,i}$,

$$\Delta u_{n,i} = \lambda_{n,i}^2 u_{n,i}$$

with

$$\lambda_{n,i} \in I_n = (an + b - \frac{c}{n}, an + b + \frac{c}{n}).$$

[2] More precisely, the geodesic flow is the projection of the Hamiltonian flow of the function $|\xi|$, whose quantization is the operator $\sqrt{\Delta}$.

[3] A sequence $\{n_k\} \subset \mathbb{N}$ is of density one if $\lim_{n\to\infty} \frac{\#\{k|n_k<n\}}{n} = 1$.

Then

$$\nu_n(f) := \frac{1}{p(n)} \int \sum_i |u_{n,i}(x)|^2 f(x) dx \sim \sum_{k=0}^{\infty} c_k(f) n^{-k}$$

as $n \to \infty$, where $p(n)$ is the number of eigenvalues of $\sqrt{\Delta}$ in the interval I_n. Colin de Verdiere also proved that the multiplicity function $p(n)$ is a polynomial in n of degree $d-1$ for n large. For more details on quantum ergodicity, c.f. the review papers [Col07], [JNT01].

The collective version of quantum ergodicity that we have just described has a natural analogy in the setting of Kähler quantization, which is due to V. Guillemin and L. Boutet de Monvel. Let

$$s_{N,i}, \qquad 1 \leq i \leq D(N)$$

be an *orthonormal* basis of $\Gamma_{hol}(\mathbb{L}^N)$, and let

$$\nu_N = \frac{1}{D(N)} \sum |s_{N,i}|^2 dx,$$

where dx is the Liouville volume measure. Then

Theorem. *For any compact manifold M, $\nu_N(f)$ has an asymptotic expansion*

$$\nu_N(f) \sim \sum_{i=0}^{\infty} c_i(f) N^{-i} \tag{1.1.11}$$

as $N \to \infty$.

Notice that $D(N)\nu_N$ is exactly the trace of the quantum operator,

$$\mu_N(f) := D(N)\nu_N = \mathrm{Tr}(\pi_N M_f \pi_N),$$

so (1.1.11) gives one information on the average value of the spectrum of the quantum operator $T_N(f)$.

Moreover just as in (1.1.10) one can see that the leading term in (1.1.11) is

$$c_0(f) = \int_M f(x)\ dx.$$

However, for higher order terms $c_k(f)$, all one knows is that they are distributional functions of f. One of the main goal of this thesis is to give an explicitly way to compute them for toric varieties, and shed a light on how to compute them in some more general cases.

1.2 Upstairs-vs-Downstairs philosophy

1.2.1 The "stairs"

For general Kähler manifolds the problems we described in the previous section are very hard to analyze. However, in many cases the classical phase space M_{red} is obtained from a much simpler phase space, M, by symplectic reduction. In this case it is natural to ask what we can deduce about the downstairs space M_{red} from this upstairs space M.

More explicitly, suppose G is a connected compact Lie group, $\mathfrak{g}$ its Lie algebra, and τ a holomorphic Hamiltonian action of G on M with a proper moment map Φ. Moreover, assume that there exists a lifting, $\tau^{\#}$, of τ to $\mathbb{L}$, which preserves the Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{L}$. If the G-action on $\Phi^{-1}(0)$ is free, the quotient space

$$M_{red} = \Phi^{-1}(0)/G$$

is a compact Kähler manifold. Moreover, the Hermitian line bundle $(\mathbb{L}, \langle \cdot, \cdot \rangle)$ on M naturally descends to a Hermitian line bundle $(\mathbb{L}_{red}, \langle \cdot, \cdot \rangle_{red})$ on M_{red}, and the curvature form of $\mathbb{L}_{red}$ is the reduced Kähler form $-\omega_{red}$, thus $\mathbb{L}_{red}$ is a pre-quantum line bundle over M_{red} (c.f. section 3.1.3). From these line bundle identifications one gets a natural map

$$\Gamma_{hol}(\mathbb{L}^k)^G \to \Gamma_{hol}(\mathbb{L}^k_{red}) \tag{1.2.1}$$

and one has (at least for M compact)

Theorem (Quantization commutes with reduction for Kähler manifolds). *Suppose that for some $k_0 > 0$ the set $\Gamma_{hol}(\mathbb{L}^{k_0})^G$ contains a nonzero element. Then the map (1.2.1) is bijective for every k.*

The proof of this theorem in [GuS82] implicitly involves the notion of *stability function* which will be our "stairs" connecting the upstairs story to the downstairs story. To define this function let $G_{\mathbb{C}}$ be the complexification of G and let M_{st} be the $G_{\mathbb{C}}$ flow-out of $\Phi^{-1}(0)$. Modulo the assumptions in the theorem above M_{st} is a Zariski open subset of M, and if G acts freely on $\Phi^{-1}(0)$ then $G_{\mathbb{C}}$ acts freely on M_{st} and

$$M_{red} = \Phi^{-1}(0)/G = M_{st}/G_{\mathbb{C}}.$$

Let π be the projection of M_{st} onto M_{red}. The stability function associated to this data is a real-valued C^∞ map

$$\psi : M_{st} \to \mathbb{R}$$

with the defining property

$$\langle \pi^* s, \pi^* s\rangle = e^{\psi} \pi^* \langle s, s\rangle_{red} \tag{1.2.2}$$

for all sections $s \in \Gamma(\mathbb{L}_{red})$. We will show that this function is proper, non-positive, and takes its maximum value 0 precisely on $\Phi^{-1}(0)$. Moreover, for any point $p \in \Phi^{-1}(0)$, p is the only critical point of the restriction of ψ to the "orbit" $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$ (Here $\exp(\sqrt{-1}\mathfrak{g})$ is the "imaginary" part of $G_{\mathbb{C}}$). Let dx be the volume form on this orbit. By applying the method of steepest descent, one gets an asymptotic expansion

$$\int_{\exp(\sqrt{-1}\mathfrak{g})\cdot p} e^{\lambda\psi} dx \sim \left(\frac{\lambda}{\pi}\right)^{-m/2} \left(1 + \sum_{i=1}^{\infty} c_i \lambda^{-i}\right) \tag{1.2.3}$$

for λ large, where m is the dimension of G, and c_i are constants depending on p. (We will always fix the notations $d = \dim_{\mathbb{C}} M, m = \dim_{\mathbb{R}} G$ and $n = d - m = \dim_{\mathbb{C}} M_{red}$.)

1.2.2 Main results

The asymptotic formula (1.2.3) has many applications. First by integrating (1.2.3) over the G-orbit through p, we get

$$\int_{G_{\mathbb{C}}\cdot p} e^{\lambda\psi}\frac{\omega^m}{m!} \sim \left(\frac{\lambda}{\pi}\right)^{-m/2} V(p)\left(1+O(\frac{1}{\lambda})\right) \tag{1.2.4}$$

as $\lambda \to \infty$, where $V(p)$ is the Riemannian volume of the G-orbit through p. Thus for any holomorphic section $s_k \in \Gamma_{hol}(\mathbb{L}^k_{red})$,

$$\left(\frac{k}{\pi}\right)^{m/2} \|\pi^* s_k\|^2 = \|V^{1/2}s_k\|^2_{red} + O(\frac{1}{k}). \tag{1.2.5}$$

This can be viewed as a "$\frac{1}{2}$-form correction" which makes the identification of $\Gamma_{hol}(\mathbb{L}^k_{red})$ with $\Gamma_{hol}(\mathbb{L})^G$ an isometry modulo $O(\frac{1}{k})$. (Compare with [HaK07], [Li07] for similar results on $\frac{1}{2}$-form corrections).

A second application of (1.2.3) concerns the spectral measures associated with holomorphic sections of $\mathbb{L}^k_{red}$: Let μ and μ_{red} be the symplectic volume forms on M and M_{red} respectively. Given a sequence of "quantum states"

$$s_k \in \Gamma_{hol}(\mathbb{L}^k_{red})$$

one can, by (1.2.3), relate the asymptotics of the spectral measures

$$\langle s_k, s_k\rangle\mu_{red} \tag{1.2.6}$$

defined by these quantum states to the asymptotics of the corresponding spectral measures

$$\langle \pi^* s_k, \pi^* s_k\rangle\mu \tag{1.2.7}$$

on M. In the special case where M is $\mathbb{C}^d$ and M_{red} a toric variety the asymptotics of (1.2.7) can be computed explicitly by Mellin transform techniques in chapter 2 and from this computation together with the identity (1.2.3) one gets an alternative proof of the asymptotic properties of (1.2.6) for toric varieties described in [BGU07].

One can also regard the function

$$\langle s_k, s_k \rangle : M_{red} \to \mathbb{R} \tag{1.2.8}$$

as a random variable and study the asymptotic properties of its probability distribution, i.e., the measure

$$\langle s_k, s_k \rangle_* \mu_{red}, \tag{1.2.9}$$

on the real line. These properties, however, can be read off from the asymptotic behavior of the *moments* of this measure, which are, by definition just the integrals

$$m_{red}(l, s_k, \mu_{red}) = \int_{M_{red}} \langle s_k, s_k \rangle^l d\mu_{red}, \qquad l = 1, 2, \cdots \tag{1.2.10}$$

and by (1.2.3) the asymptotics of these integrals can be related to the asymptotics of the corresponding integrals on M viz

$$m(l, \pi^* s_k, \mu) = \int_M \langle \pi^* s_k, \pi^* s_k \rangle^l \mu \ . \tag{1.2.11}$$

In the toric case Shiffman, Tate and Zelditch showed in [STZ04] that if s_k lies in the weight space $\Gamma_{hol}(\mathbb{L}^k)^{\alpha_k}$, where $\alpha_k = k\alpha + O(\frac{1}{k})$, and

$$\nu = (\Phi_P^* \omega_{FS})^n / n!$$

is the pull-back of the Fubini-Study volume form on the projective space via the monomial embedding Φ_P, then, if s_k has L^2 norm 1,

$$\left(\frac{k}{\pi}\right)^{-n(l-1)/2} m_{red}(l, s_k, \nu) \sim \frac{c^l}{l^{n/2}} \tag{1.2.12}$$

as k tends to infinity, c being a positive constant. From this they derived a "universal distribution law" for such measures. We will give below a similar asymptotic result for the moments associated with the volume form, $V\mu_{red}$, which can be derived from (1.2.3) and an analogous, but somewhat simpler version of (1.2.12) for the moments (1.2.11) upstairs on $\mathbb{C}^d$.

Related to these results is another application of (1.2.3): The spectral measure

$$\mu_N(f) = \mathrm{Tr}(\pi_N M_f \pi_N) \tag{1.2.13}$$

can also be written (somewhat less intrinsically) as the sum

$$\mu_N = \sum \langle s_{N,i}, s_{N,i} \rangle \mu, \tag{1.2.14}$$

the $s_{N,i}$'s being an orthonormal basis of $\Gamma_{hol}(\mathbb{L}^N)$ inside $L^2(\mathbb{L}^N, \mu)$. As we have known, $\mu_N(f)$ has an asymptotic expansion,

$$\mu_N(f) \sim \sum_{i=d-1}^{-\infty} a_i(f) N^i \tag{1.2.15}$$

as $N \to \infty$. We will derive a G-invariant version of this result for the upstairs manifold. More precisely, if we let π_N^G be the orthogonal projection

$$\pi_N^G : L^2(\mathbb{L}^N, \mu) \to \Gamma_{hol}(\mathbb{L}^N)^G,$$

then for any G-invariant function f on M we have the asymptotic expansion

$$\mu_N^G(f) = \mathrm{Tr}(\pi_N^G M_f \pi_N^G) \sim \sum_{i=n-1}^{\infty} a_i^G(f) N^i \tag{1.2.16}$$

as $N \to \infty$. Moreover, the identity (1.2.3) enables one to read off this upstairs G-invariant expansion from the downstairs expansion and vise versa. Notice that for this G-invariant expansion, we don't have to require the upstairs manifold to be compact.

For example, for toric varieties, the upstairs space, $\mathbb{C}^d$, is not compact, so the space of holomorphic sections is infinite dimensional, and Colin de Verdiere's result doesn't apply; however the G-invariant version of the upstairs asymptotics can, in this case, be computed directly by Mellin transform techniques in chapter 2 together with an Euler-Maclaurin formula for convex lattice polytopes ([GuS06]) and hence one gets from (1.2.3) an alternative proof of the asymptotic expansion of μ_N for toric varieties obtained in [BGU07].

As a last application we discuss "Bohr-Sommerfeld" issues in the context of GIT theory. Let ∇_{red} be the Kählerian connection on $\mathbb{L}_{red}$ with defining property,

$$curv(\nabla_{red}) = -\omega_{red}.$$

A Lagrangian submanifold $\Lambda_{red} \subset M_{red}$ is said to be *Bohr-Sommerfeld* if the connection $\iota^*_{\Lambda_{red}} \nabla_{red}$ is trivial. In this case there exists a covariant constant non-vanishing section, s_{BS}, of $\iota^*_{\Lambda_{red}} \mathbb{L}_{red}$. Viewing s_{BS} as a "delta section" of $\mathbb{L}_{red}$ and projecting it onto $\Gamma_{hol}(\mathbb{L}_{red})$, one gets a holomorphic section $s_{\Lambda_{red}}$ of $\mathbb{L}_{red}$, and one would like to know

1. Is this section nonzero?
2. What, in fact, is this section?
3. What about the sections $s^{(k)}_{\Lambda_{red}}$ of $\mathbb{L}^k_{red}$? Do they have interesting asymptotic properties as $k \to \infty$? Do they, for instance, "concentrate" asymptotically on Λ_{red}?

These three questions often turn out to be intractable. However, we will show that "downstairs" version of these questions on M_{red} can be translated into upstairs version of these questions on M where they often become more accessible.

1.3 Thesis outline

We will begin with a detailed study of a very elementary integral transform – the twisted Mellin transform, which will be used later in chapter 4 to study the asymptotic behavior of the spectral measures of toric varieties. The definition of the twisted Mellin transform together with many fundamental properties and examples are given in section 2.1. In view of the application alluded to above, we study the asymptotic behavior of the twisted Mellin transform in section 2.2. The application of the twisted Mellin transform to spectral measures on Bargmann space is given in section 2.3.

Chapter 3 is the heart of this thesis. After reviewing the necessary background on pre-quantum line bundles and Kähler quotients in section 3.1, we define the stability function and study analytic properties of this function in section 3.2. In section 3.3 we study the asymptotic behavior of the stability function as well as of various Laplace-type integrals of this function using the steepest decent method. These results are applied to a number of spectral problems on Kähler quotients, including maximum points, spectral measures, moments of spectral probability measure and Bohr-Sommerfeld Lagrangians, in section 3.4.

In chapter 4 we apply this stability theory to toric geometry. We begin by a brief description of toric varieties following Delzant's construction. Then in section 4.2 we compute the stability functions on toric varieties in canonical affine coordinate systems. The application of stability theory together with the twisted Mellin transform techniques to toric varieties is given in section 4.3. Finally in section 4.4 we apply stability theory to study the measure $\langle s_k, s_k\rangle_* d\mu$ for toric varieties, and give an alternate proof of Shiffman-Tate-Zelditch's remarkable universal distribution law for eigenstates on toric varieties.

Chapter 5 is devoted to computing the stability function for some non-toric varieties, with the hope to apply stability theory for such manifolds in the future. Examples includes Grassmannians, coadjoint orbits of the unitary group $U(n)$, and, more generally, quiver varieties. The common feature of these varieties is that they are all quotient space of $\mathbb{C}^d$ by a Hamiltonian unitary group action. It turns out that the stability functions for these varieties are even simpler than in the toric case.

At the end of this thesis we include two appendixes. Appendix A deals with the role of Hamiltonian actions in geometric quantization. We will explain in detail how to lift a Hamiltonian action to a pre-quantum line bundle and how to complexify such actions. (These results are extensive used in chapter 3.) Appendix B is devoted to the theory of generalized Toeplitz operators, in the sense of Boutet de Monvel-Guillemin. We will define these operators and then give a brief proof of the asymptotic formula for spectral measures alluded to above.

Chapter 2

The Twisted Mellin Transform

The standard Mellin transform

$$Mf(s) = \int_0^\infty f(x)x^{s-1}dx$$

is an integral transform that may be regarded as the multiplicative version of the two-sided Laplace transform, and is widely used in analysis, number theory and combinatorics. In this chapter we will study the following "twisted" version of this transform,

$$\mathcal{M}f(s) = \frac{\int_0^\infty f(x)x^s e^{-x}\, dx}{\int_0^\infty x^s e^{-x}\, dx}.$$

The "twisted" transform has a number of remarkable properties, the most remarkable perhaps being that it intertwines the standard differential operator, $\frac{d}{dx}$, and the finite difference analogue of $\frac{d}{dx}$:

$$\nabla f(x) = f(x) - f(x-1).$$

By a theorem of Mullin and Rota ([MuR70]) it is known that there exists an invertible operator intertwining the "umbral" calculi generated by $\frac{d}{dx}$ and ∇; but, as far as we know the explicit expression for this intertwiner is new.

The motivation for us to study this transform is that the d-dimensional version of this transform provides the precise formula for the invariant spectral measure on the Bargmann space $\mathbb{C}^d$. We are mainly interested in the asymptotic behavior of this transform, since, as we have mentioned in chapter 1, the asymptotic behavior of spectral measures on $\mathbb{C}^d$ determines the asymptotics of spectral measures on toric varieties. We will derive such an asymptotic formula at the end of this chapter. For its application to toric varieties, see chapter 4.

2.1 Definition and basic properties

2.1.1 The definition

Recall that a function $f : \mathbb{R}^+ \to \mathbb{R}$ is of polynomial growth if

$$|f(x)| \leq Cx^N \tag{2.1.1}$$

for some N.

Definition 2.1.1. *Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is a function of polynomial growth. We define its* twisted Mellin transform *to be*

$$\mathcal{M}f(s) = \frac{\int_0^\infty f(x)x^s e^{-x}\, dx}{\int_0^\infty x^s e^{-x}\, dx}. \tag{2.1.2}$$

Remark 2.1.2. *Comparing the twisted Mellin transform (2.1.2) to the standard Mellin transform*

$$Mf(s) = \int_0^\infty f(x)x^{s-1}dx \tag{2.1.3}$$

we see that for the twisted Mellin transform:

1. *The numerator is the standard Mellin transform of the function $xe^{-x}f(x)$, so the twisted Mellin transform inherits many properties of the standard Mellin transform, including an inversion formula, a Parseval formula.*

2. *In view of the e^{-x} factor in the integrand, the twisted Mellin transform is defined*

for a much wider class of functions than the standard Mellin transform. For example, the standard Mellin transform is even not well defined for constant functions. As we will see later, we can even define the twisted Mellin transform for tempered distributions.

2.1.2 Some elementary properties

By direct computations one can easily derive the following translation properties of the twisted Mellin transform:

Proposition 2.1.3. *Suppose $a, b \in \mathbb{R}$, $c > 0$, f is a function of polynomial growth, then* (1) *For $g(x) = x^a f(x)$,*

$$\mathcal{M}g(s) = \frac{\Gamma(s+a+1)}{\Gamma(s+1)}\mathcal{M}f(s+a). \tag{2.1.4}$$

(2) *For $g(x) = e^{-cx} f(x)$,*

$$\mathcal{M}g(s) = (c+1)^{-s-1}\mathcal{M}f_c(s), \tag{2.1.5}$$

where $f_c(x)$ is the dilation, $f_c(x) = f(\frac{x}{c+1})$. (3) *For $g(x) = f(x)\ln x$,*

$$\mathcal{M}g(s) = \frac{d}{ds}\mathcal{M}f(s) + \mathcal{M}f(s)\frac{\Gamma'(s+1)}{\Gamma(s+1)}. \tag{2.1.6}$$

Similar to the standard Mellin transform, the twisted Mellin transform also behaves well with respect to differential and integral operations:

Proposition 2.1.4. *Suppose f is a function of polynomial growth, then* (1) *For $g(x) = \frac{df}{dx}(x)$,*

$$\mathcal{M}g(s) = \nabla\mathcal{M}f(s) := \mathcal{M}f(s) - \mathcal{M}f(s-1), \tag{2.1.7}$$

and more generally, for any $n \in \mathbb{N}$ and $g(x) = f^{(n)}(x)$,

$$\mathcal{M}g(s) = \nabla^n(\mathcal{M}f)(s) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}\mathcal{M}f(s-i). \tag{2.1.8}$$

(2) *For* $g(x) = \int_0^x f(t)\ dt$,

$$\mathcal{M}g(s) = \sum_{i=0}^{[s]-1} \mathcal{M}f(s-i) + \mathcal{M}g(s-[s]). \tag{2.1.9}$$

where $[s]$ *is the integer part of* s. *In particular,*

$$\mathcal{M}g(n) = \sum_{i=0}^{n} \mathcal{M}f(i). \tag{2.1.10}$$

Proof. To prove (1), we note that for $g(x) = f'(x)$,

$$\begin{aligned}
\mathcal{M}g(s) &= \frac{\int_0^\infty f'(x)x^s e^{-x}\ dx}{\int_0^\infty x^s e^{-x}\ dx} \\
&= \frac{\int_0^\infty f(x)(x^s - sx^{s-1})e^{-x}\ dx}{\int_0^\infty x^s e^{-x}\ dx} \\
&= \frac{\int_0^\infty f(x)x^s e^{-x}\ dx}{\int_0^\infty x^s e^{-x}\ dx} - \frac{\int_0^\infty f(x)x^{s-1}e^{-x}\ dx}{\int_0^\infty x^{s-1}e^{-x}\ dx} \\
&= \mathcal{M}f(s) - \mathcal{M}f(s-1).
\end{aligned}$$

The property (2.1.8) is easily deduced from (2.1.7) by induction.

To prove (2), we note that by integration by parts,

$$\mathcal{M}g(s) = \mathcal{M}f(s) + \mathcal{M}g(s-1),$$

which implies (2.1.9). As for (2.1.10), this follows from the obvious fact $\mathcal{M}g(0) = \mathcal{M}f(0)$. $\square$

From the definition its easy to see that the twisted Mellin transform is smooth, i.e. it transform a smooth function to a smooth function. Moreover, it transforms a function which is of polynomial growth of degree N to a function which is of polynomial growth of degree N, and Schwartz functions to Schwartz functions:

Proposition 2.1.5. (1) *Suppose* $|f(x)| \leq Cx^N$, *then* $|\mathcal{M}f(s)| \leq C's^N$.

(2) $\mathcal{M}$ *maps Schwartz functions to Schwartz functions.*

Proof. (1) This comes from the definition:

$$|\mathcal{M}f(s)| \leq \frac{\int_0^\infty Cx^N x^s e^{-x}\ dx}{\Gamma(s+1)} = C\frac{\Gamma(s+N+1)}{\Gamma(s+1)} \leq C's^N.$$

(2) Suppose f is a Schwartz function, i.e. for any α, β, there is a constant $C_{\alpha,\beta}$ such that $\sup_x |x^\alpha \partial^\beta f(x)| \leq C_{\alpha,\beta}$. Consider the function $\mathcal{M}f(s)$:

For $\beta = 0$, $|x^\alpha f(x)| \leq C$ implies $|s^\alpha \mathcal{M}f(s)| \leq C'$.

For $\beta = 1$, we apply (2.1.6) and the above result to get $|s^\alpha \frac{d}{ds}\mathcal{M}f(s)| \leq C_\alpha$.

For $\beta \geq 1$, let $\psi(s) = \Gamma'(s)/\Gamma(s)$. Then by repeated applications of (2.1.6) one can see that $\frac{d^n}{ds^n}\mathcal{M}f(s)$ is a linear combination of the functions $\mathcal{M}g_i(s)\psi^{(j)}(s+1)$, where $g_i(x) = f(x)(\ln x)^i$ and

$$\psi^{(m)}(s+1) = \frac{d^m}{ds^m}\psi(s+1)$$

is the polygamma function, which is bounded for each m, as is clear from its integral representation:

$$|\psi^{(m)}(s+1)| = \left|(-1)^{m+1}\int_0^\infty \frac{t^m e^{-(s+1)t}}{1-e^{-t}}dt\right| \leq \int_0^\infty \frac{t^m e^{-t}}{1-e^{-t}}dt = \zeta(m+1)\Gamma(m+1).$$

Thus by induction we easily deduce that $|s^\alpha \partial^\beta \mathcal{M}f(s)| \leq C_{\alpha,\beta}$. □

Remark 2.1.6. *Since the twisted Mellin transform transforms a Schwartz function to a Schwartz function, we can define the twisted Mellin transform on tempered distributions by duality.*

2.1.3 Examples

We will next compute the twisted Mellin transform for some elementary functions such as polynomials, exponentials and trigonometric functions.

(a) For $f(x) = x^a$,

$$\mathcal{M}f(s) = \Gamma(s+a+1)/\Gamma(s+1).$$

In particular, if $f(x) = x^n$, n a positive integer, then

$$\mathcal{M}f(s) = s^{[n]} := (s+1)(s+2)\cdots(s+n).$$

Thus the twisted Mellin transform of a polynomial of degree n is again a polynomial of degree n.

(b) Suppose $a > 1$, then for $f(x) = a^{-x}$,

$$\mathcal{M}f(s) = (\ln a + 1)^{-1-s}.$$

More generally, if $f(x) = x^b a^{-x}$, then

$$\mathcal{M}f(s) = (\ln a + 1)^{-1-b-s}\Gamma(s+b+1)/\Gamma(s+1).$$

(c) For $f(x) = \frac{1}{1-e^{-x}}$,

$$\mathcal{M}f(s) = \zeta(s+1),$$

and as a corollary, for the Todd function $f(x) = \frac{x}{1-e^{-x}}$,

$$\mathcal{M}f(s) = (s+1)\zeta(s+2).$$

(d) For $f(x) = \ln x$, one gets from (2.1.6)

$$\mathcal{M}f(s) = \frac{\Gamma'(s+1)}{\Gamma(s+1)},$$

and in general, for $f(x) = (\ln x)^n$,

$$\mathcal{M}f(s) = \frac{\Gamma^{(n)}(s+1)}{\Gamma(s+1)}.$$

(e) For the trigonometric functions $f(x) = \sin x$ and $g(x) = \cos x$,

$$\mathcal{M}f(s) = \frac{1}{(\sqrt{2})^{s+1}} \sin \frac{(s+1)\pi}{4},$$
$$\mathcal{M}g(s) = \frac{1}{(\sqrt{2})^{s+1}} \cos \frac{(s+1)\pi}{4}.$$

(*Proof.* Let $h(x) = e^{ix}$, then $\mathcal{M}h(s) = \frac{1}{(1-i)^{s+1}}$.)

Similarly for $f(x) = \sin(ax)$ and $g(x) = \cos(ax)$,

$$\mathcal{M}f(s) = (1+a^2)^{-s} \sin(s \arctan a),$$
$$\mathcal{M}g(s) = (1+a^2)^{-s} \cos(s \arctan a).$$

2.1.4 Combinatorial aspects of the twisted Mellin transform

The twisted Mellin transform has applications in the "umbral calculus" of Mullin-Rota. As we have seen, the twisted Mellin transform maps the monomials $\{x^n\}$ to the polynomials

$$s^{[n]} = (s+1)(s+2)\cdots(s+n) = \sum_{k=0}^{n} c(n+1, k+1) s^k, \tag{2.1.11}$$

where $c(n,k)$ is the sign-less Stirling number of first kind. Note that both $\{x^n\}$ and $\{x^{[n]}\}$ are a basis of the polynomial ring $\mathcal{P}$, so $\mathcal{M}$ is in fact an automorphism of $\mathcal{P}$.

To state the umbral calculus applications of $\mathcal{M}$, let's first recall some combinatorial concepts. A sequence of polynomials, $\{p_n(x)\}$, is called a *polynomial sequence of binomial type* if $p_0(x) = 1, \deg p_n = n$ and

$$p_n(x+y) = \sum_k \binom{n}{k} p_k(x) p_{n-k}(y). \tag{2.1.12}$$

For example, both the sequence $\{x^n\}$ and the sequence $\{(x-1)^{[n]}\}$ are polynomial sequences of binomial type. Mullin and Rota ([MuR70]) proved that one can always associate to any binomial type polynomial sequence $\{p_n\}$ a shift-invariant [1] operator called the *delta operator* Q, which by definition satisfies $Qp_n = np_{n-1}$. The sequence $\{p_n\}$ is called the *sequence of basic polynomials* for Q. It is easy to see that the delta operator associated to the sequence $\{x^n\}$ is just the differential operator, $Qp = dp/dx$, while the delta operator associated to the sequence $\{(x-1)^{[n]}\}$ is the backward difference operator, $Qp = \nabla p(x) = p(x) - p(x-1)$.

It is obvious that any two delta operators can be intertwined: one just defines the intertwiner to be the operator that maps one sequence of basic polynomials to the other sequence of basic polynomials. Such operators are called *umbral operators*. By a basic theorem of umbral calculus, any umbral operator would intertwine any delta operator to another delta operator. However, to our knowledge, no explicit formula for any of these operators has been written down in the literature. Our result (2.1.7) shows that the twisted Mellin transform $\mathcal{M}$ *is* the intertwiner between the operators $\frac{d}{dx}$ and ∇:

Proposition 2.1.7. *The twisted Mellin transform $\mathcal{M}$ is the umbral operator intertwine $\frac{d}{dx}$ and ∇.*

2.2 Asymptotic expansion of the twisted Mellin transform

2.2.1 The asymptotic expansion

We can rewrite the twisted Mellin transform as

$$\mathcal{M}f(s) = \frac{\int_0^\infty f(x)e^{s\log x - x}\, dx}{\int_0^\infty e^{s\log x - x}\, dx}. \tag{2.2.1}$$

[1] An operator T is shift-invariant if $TE^a = E^aT$, where E^a is the shift operator $E^af(x) = f(x+a)$.

For the phase function $\varphi(x,s) = s\log x - x$, we have

$$0 = \frac{\partial \varphi}{\partial x} \quad \Longrightarrow \quad x = s,$$

thus the function $\varphi_s(x) = \varphi(x,s)$ has a unique critical point at $x = s$. Moreover, this is a global maximum of $\varphi(x,s)$, since

$$\lim_{x\to+\infty} \varphi(x,s) = -\infty,$$

and

$$\frac{\partial^2\varphi}{\partial x^2} = -\frac{s}{x^2} < 0.$$

Hence if f is a symbol, we can apply the method of steepest descent to both denominator and numerator to get

$$\mathcal{M}f(s) \sim \sum_k g_k(s) f^{(k)}(s). \tag{2.2.2}$$

To compute the functions g_k's we merely take f to be polynomials. Fix any s, applying $\mathcal{M}$ to the Taylor expansion of f at s, we get

$$\mathcal{M}f(s) = \sum_{r=0}^{n} \frac{1}{r!} f^{(r)}(s) g_r(s), \tag{2.2.3}$$

where

$$g_r(s) = \frac{\int_0^\infty (x-s)^r x^s e^{-x}\ dx}{\int_0^\infty x^s e^{-x}\ dx} = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} s^{[i]} s^{r-i}. \tag{2.2.4}$$

2.2.2 The coefficients

From (2.2.4) it seems that g_r is a polynomial of degree r. Let's compute them for $r \leq 5$ small,

$$\begin{aligned} g_0(s) &= g_1(s) = 1, \\ g_2(s) &= 2 + s, \\ g_3(s) &= 6 + 5s, \\ g_4(s) &= 24 + 26s + 3s^2, \\ g_5(s) &= 120 + 154s + 35s^2. \end{aligned}$$

which suggests that $g_r(s)$ is a polynomial of degree $[r/2]$ in s! We will give two separate proofs of this fact, one combinatorial and one analytic.

Proof 1. Putting (2.1.11) into (2.2.4), we get

$$\begin{aligned} g_r(s) &= \sum_{i=0}^{r} \sum_{k=0}^{i} (-1)^{r-i} \binom{r}{i} c(i+1, k+1) s^{r-(i-k)} \\ &= \sum_{j=0}^{r} \left(\sum_{i=j}^{r} (-1)^{r-i} \binom{r}{i} c(i+1, i-j+1) \right) s^{r-j}. \end{aligned}$$

On the other hand, by the definition of the Stirling number,

$$c(n+1, n+1-k) = c(n, n-k) + nc(n, n-(k-1)). \tag{2.2.5}$$

and from this recurrence relation we will show:

Lemma 2.2.1. *There are constants $C_{l,j}$, depending only on l, j, such that*

$$c(i+1, i+1-j) = \sum_{l=j}^{2j} C_{l,j} \ (i)_l, \tag{2.2.6}$$

where $(i)_l = i(i-1)\cdots(i-l+1)$ is the falling factorial.

Proof. This is true for $j = 0$, since $c(n+1, n+1) = 1$. Notice that

$$\sum_{i=l}^{n} (i)_l = l!\left(\binom{l}{l} + \binom{l+1}{l} + \cdots + \binom{n}{l}\right) = l!\binom{n+1}{l+1} = \frac{1}{l+1}(n+1)_{l+1}.$$

Now use induction and the recurrence relation (2.2.5). □

Now suppose $2j \leq r$, then the coefficients of s^{r-j} in $f_r(s)$ is

$$\begin{aligned}\sum_{i=j}^{r}(-1)^{r-i}\binom{r}{i}c(i+1, i-j+1) =& \sum_{i=j}^{r}(-1)^{r-i}\binom{r}{i}\sum_{l=j}^{2j} C_{l,j}\ (i)_l \\ =& \sum_{l=j}^{2j} C_{l,j}(r)_l \sum_{i=l}^{r}(-1)^{r-i}\binom{r-l}{i-l} \\ =& 0,\end{aligned}$$

which proves that g_r is a polynomial of degree $[r/2]$.

Proof 2. First we derive a recurrence relation for $g_r(s)$. Using

$$\frac{d}{dx}(s\log x - x) = -\frac{x-s}{x}$$

and integration by parts we get

$$\begin{aligned}\Gamma(s+1)g_r(s) &= -\int_0^\infty e^{s\ln x - x}x(x-s)^{r-1}\frac{d}{dx}(s\ln x - x)\ dx \\ &= \int_0^\infty e^{s\ln x - x}\frac{d}{dx}(x(x-s)^{r-1})\ dx \\ &= \int_0^\infty e^{s\ln x - x}\frac{d}{dx}((x-s)^r + s(x-s)^{r-1})\ dx \\ &= r\int_0^\infty x^s e^{-x}(x-s)^{r-1}\ dx + (r-1)s\int_0^\infty x^s e^{-x}(x-s)^{r-2}\ dx,\end{aligned}$$

i.e.

$$g_r(s) = rg_{r-1}(s) + (r-1)sg_{r-2}(s). \tag{2.2.7}$$

Moreover, we can compute the initial conditions directly

$$g_1(s) = g_0(s) = 1. \tag{2.2.8}$$

Remark 2.2.2. *The recurrence relation (2.2.7) also follows easily from (2.1.9) and (2.1.4). In fact, if we denote $h_r(x) = (x-s)^r$, then $g_r(s) = \mathcal{M}h_r(s)$, and thus*

$$\begin{aligned} rg_{r-1}(s) &= \mathcal{M}h_r(s) - \mathcal{M}h_r(s-1) \\ &= g_r(s) - (\mathcal{M}(xh_{r-1})(s-1) - s\mathcal{M}h_{r-1}(s-1)) \\ &= g_r(s) - s(\mathcal{M}h_{r-1}(s) - \mathcal{M}h_{r-1}(s-1)) \\ &= g_r(s) - s(r-1)\mathcal{M}h_{r-2}(s). \end{aligned}$$

From (2.2.7), (2.2.8) and induction, it follows again that $g_r(s)$ is a polynomial of degree $[r/2]$. Thus coming back to (2.2.2) we have proved

Theorem 2.2.3. *For any symbolic function f, we have*

$$\mathcal{M}f(s) \sim \sum_r \frac{1}{r!} f^{(r)}(s) g_r(s), \tag{2.2.9}$$

where $g_r(s)$ is the polynomial of integer coefficients of degree $[r/2]$ given by (2.2.4).

2.2.3 Combinatorial aspects of g_r

The polynomials $g_r(s)$ have many interesting combinatorial properties:

(1) Since $g_r(s)$ is a polynomial of degree $[r/2]$, we can write

$$g_r(s) = \sum_{i=0}^{[r/2]} a_{r,i} s^i, \tag{2.2.10}$$

the coefficients satisfying the recurrence relation

$$a_{r,i} = ra_{r-1,i} + (r-1)a_{r-2,i-1} \tag{2.2.11}$$

and initial conditions

$$a_{r,0} = r!, \quad a_{2k,k} = (2k-1)!!,$$

which implies

$$\begin{aligned} a_{r,1} =& r!\left(\frac{1}{r} + \frac{1}{r-1} + \cdots + \frac{1}{2}\right), \\ a_{r,2} =& r!\left(\frac{(r-1)a_{r-2,1}}{r!} + \frac{(r-2)a_{r-3,1}}{(r-1)!} + \cdots + \frac{3a_{2,1}}{4!}\right), \end{aligned}$$

and in general

$$a_{r,k} = r!\left(\frac{(r-1)a_{r-2,k-1}}{r!} + \frac{(r-2)a_{r-3,k-1}}{(r-1)!} + \cdots + \frac{(2k-1)a_{2k-2,k-1}}{(2k)!}\right). \tag{2.2.12}$$

(2) The coefficients, $a_{r,i}$, of $g_r(s)$, are exactly those appeared as coefficients of polynomials used for exponential generating functions for diagonals of unsigned Stirling numbers of the first kind. More precisely, for fixed k, the exponential generating function for the sequence $\{c(n+1, n+1-k)\}_{n\geq 0}$ is given by (c.f. sequence A112486 in "The On-Line Encyclopedia of Integer Sequences" [2])

$$\sum_{n=0}^{\infty} c_{n+1,n+1-k}\frac{x^n}{n!} = e^x \sum_{n=k}^{2k}\left(a_{n,n-k}\frac{x^n}{n!}\right). \tag{2.2.13}$$

(3) The sequence of functions g_r's have a very simple exponential generating function:

$$\begin{aligned} \sum_{r=0}^{\infty} g_r(s)\frac{x^r}{r!} &= \sum_{i=0}^{\infty}\sum_{r=i}^{\infty}(-1)^{r-i}\frac{1}{r!}\binom{r}{i}s^{[i]}s^{r-i}x^r \\ &= \left(\sum_{i=0}^{\infty}\frac{s^{[i]}x^i}{i!}\right)\left(\sum_{r=i}^{\infty}(-1)^{r-i}\frac{s^{r-i}x^{r-i}}{(r-i)!}\right) \\ &= \frac{e^{-sx}}{(1-x)^{1+s}}. \end{aligned}$$

(4) From the generating function above we get a combinatorial interpreting of $g_r(s)$ for integers s: $r!g_r(s)$ is the number of $r \times r$ $\mathbb{N}$-matrices with every row and

[2] Website http://www.research.att.com/ $\sim$ njas/sequences/

column sum equal to $3 + 2s$ and with at most 2 nonzero entries in every row. (c.f. Exercise 5.62 of [Sta97]).

(5) There are also other combinatorial interpreting for small value of s. For example, the sequence $g_r(1)$ count permutations w of $\{1, 2, \cdots, r+1\}$ such that $w(i+1) \neq w(i) + 1$ (c.f. the sequence A000255 of "On-line Encyclopedia of Integer Sequences"). For $s = 2$, we have

$$g_r(2) = \frac{2^{-r^2}}{r!} \sum_{M \in D_r} (\det M)^4,$$

where D_r is the set of all $r \times r$ matrices of ± 1's. (c.f. Exercise 5.64(b) of [Sta97]).

2.3 Applications to spectral measures of Bargmann space

2.3.1 Spectral measures of the Bargmann space

The Bargmann measure on $\mathbb{C}^d$ is

$$\mu = e^{-|z|^2} dz d\bar{z}.$$

Let's assume $d = 1$ first. The spectral measure associated with the quantum eigenstate z^k is

$$\mu_k(f) = \mathrm{Tr}(\pi_k M_f \pi_k),$$

where $f \in C^\infty(\mathbb{C})$ and π_k is the orthogonal projection from $L^2(\mathbb{C}, \mu)$ onto the one dimensional subspace spanned by z^k. By averaging with respect to the $\mathbb{T}^1$-action, we can assume $f \in C^\infty(\mathbb{C})^{\mathbb{T}^1}$, i.e.

$$f(z) = f(r^2),$$

where $r = |z|$ is the modulus of complex number z. Now we can give an explicit expression for the spectral measure,

$$\begin{aligned}\mu_k(f) = \frac{\langle fz^k, z^k\rangle_\mu}{\langle z^k, z^k\rangle_\mu} &= \frac{\int_0^\infty f(r^2) r^{2k+1} e^{-r^2}\, dr}{\int_0^\infty r^{2k+1} e^{-r^2}\, dr} \\ &= \frac{\int_0^\infty f(x) x^k e^{-x}\, dx}{\int_0^\infty x^k e^{-x}\, dx} \\ &= \mathcal{M}f(k).\end{aligned}$$

In other words, the spectral measure of the 1-dimensional Bargmann space is precisely given by the twisted Mellin transform.

If we replace the Bargmann measure, μ, by the generalized Bargmann measure

$$\mu_\alpha = e^{-\alpha|z|^2}\, dz d\bar{z},$$

then we are naturally led, by the same argument above, to studying the "α-twisted Mellin transform"

$$\mathcal{M}_\alpha f(s) = \frac{\int_0^\infty f(x) x^s e^{-\alpha x}\, dx}{\int_0^\infty x^s e^{-\alpha x}\, dx}. \tag{2.3.1}$$

All the properties of the twisted Mellin transform can be easily generalized to $\mathcal{M}_\alpha$. In fact, it is easy to see that

$$\mathcal{M}_\alpha f_c(s) = \mathcal{M}_{\alpha/c} f(s), \tag{2.3.2}$$

where $f_c(x) = f(cx)$, so the α-twisted Mellin transform of a function can easily be computed from the twisted Mellin transform.

Now consider the general case $d > 1$. Suppose G is a torus acting in a Hamiltonian fashion on $\mathbb{C}^d$. For α in the weight lattice of G-action we let

$$\Gamma_\alpha^N = \mathrm{span}\{z_1^{k_1} \ldots z_d^{k_d}\,,\ \sum k_i \alpha_i = N\alpha\} \tag{2.3.3}$$

be the set of invariant functions under G-action, where α_i's are the weights of G-

actions. The spectral measure on $\mathbb{C}^d$ that we are interested in is

$$\nu_N(f) = \text{trace } \pi_N M_f \pi_N \tag{2.3.4}$$

where π_N is the orthogonal projection from $L^2(\mathbb{C}^d\,,\, e^{-N|z|^2}\, dz\, d\bar{z})$ onto Γ_α^N. Since ν_N is $\mathbb{T}^d$-invariant, by averaging under the diagonal $\mathbb{T}^d$-action on $\mathbb{C}^d$ we can assume f is $\mathbb{T}^d$-invariant, i.e.

$$f(z) = f(r_1^2, \cdots, r_d^2), \tag{2.3.5}$$

where $r_i = |z_i|$.

Note that the functions, z^k, $k \in N\Delta \cap \mathbb{Z}^d$, are an orthonormal basis of Γ_α^N and the functions

$$\frac{1}{c_{N,k}} z^k$$

with

$$c_{N,k} = \left(\int_{\mathbb{C}^d} |z^k|^2 e^{-N|z|^2}\, dz\, d\bar{z} \right)^{\frac{1}{2}} \tag{2.3.6}$$

are an orthogonal basis of Γ_α^N. Hence the trace of $\pi_N M_f \pi_N$ is equal to the sum of

$$\frac{\int |z_1|^{2k_1} \dots |z_d|^{2k_d} e^{-N|z|^2} f(z)\, dz\, d\bar{z}}{\int |z_1|^{2k_1} \dots |z_d|^{2k_d} e^{-N|z|^2}\, dz\, d\bar{z}} \tag{2.3.7}$$

over the polytope $k \in N\Delta \cap \mathbb{Z}^d$, where

$$N\Delta = \{(k_1, \cdots, k_d) \in \mathbb{Z}^d \mid \sum k_i \alpha_i = N\alpha\}. \tag{2.3.8}$$

For $\mathbb{T}^d$-invariant function (2.3.5) a simple computation shows that the spectral measure (2.3.4) is given by a d-dimensional twisted Mellin transform,

$$\nu_n(f) = \sum_{k \in N\Delta} \frac{\int_{\mathbb{R}_+^d} x_1^{k_1} \cdots x_d^{k_d} e^{-N \sum x_i} f(x) dx}{\int_{\mathbb{R}_+^d} x_1^{k_1} \cdots x_d^{k_d} e^{-N \sum x_i} dx}. \tag{2.3.9}$$

2.3.2 Asymptotics for summands

We also begins with $d = 1$. Given a symbolic function f, consider the integral

$$A_N(f)(s) = \frac{\int_0^\infty f(x) x^{Ns} e^{-Nx}\, dx}{\int_0^\infty x^{Ns} e^{-Nx}\, dx}, \tag{2.3.10}$$

as $N \to \infty$. By definition, this is just the "N-twisted Mellin transform" $\mathcal{M}_N f(Ns)$, which, according to (2.3.2), equals $\mathcal{M} f_N(Ns)$, where $f_N(x) = f(x/N)$. Thus by Theorem 2.2.3,

$$A_N(f)(s) \sim \sum_k \frac{1}{k!} \left(\frac{1}{N}\right)^k f^{(k)}(s) g_k(Ns). \tag{2.3.11}$$

Note that since $g_k(x)$ is a polynomial of degree $[k/2]$, the above formula does give us an asymptotic expansion. In particular, we have

$$\begin{aligned} A_N(f)(s) = f(s) &+ \frac{1}{N}\left(f'(s) + f''(s)\frac{s}{2}\right) \\ &+ \frac{1}{N^2}\left(f''(s) + f'''(s)\frac{5s}{6} + f^{(4)}(s)\frac{s^2}{8}\right) + O(N^{-3}). \end{aligned}$$

In general for $d > 1$, we would like to consider the d-dimensional "N-twisted" Mellin transform

$$A_N f(x) = \frac{\int_{\mathbb{R}_+^d} e^{N(\sum x_i \log y_i - y_i)} f(y)\, dy}{\int_{\mathbb{R}_+^d} e^{N(\sum x_i \log y_i - y_i)}\, dy}. \tag{2.3.12}$$

Note that the d-dimensional twisted Mellin transform is just the same as applying a sequence of 1-dimensional twisted Mellin transforms, one for each variable. It follows from theorem 2.2.3 that

$$A_N f(x) \sim \sum_\alpha \frac{1}{\alpha!} N^{-|\alpha|} f^{(\alpha)}(x)\; g_\alpha(Nx), \tag{2.3.13}$$

where

$$g_\alpha(x) = g_{\alpha_1}(x_1) \ldots g_{\alpha_d}(x_d)$$

and

$$g_k(s) = \sum_{0 \le \ell \le k} (-1)^\ell \binom{k}{\ell} s^\ell s^{[k-\ell]}. \tag{2.3.14}$$

2.3.3 A generalized Euler-Maclaurin formula

Let $\Delta \subseteq \mathbb{R}^n$ be an n-dimensional convex polytope. By elementary calculus the Riemann integral of a function $f \in \mathcal{C}^\infty(\Delta)$:

$$\int_\Delta f(x)\, dx$$

is approximated by the Riemann sum

$$\frac{1}{N^n} \sum_{k \in N\Delta \cap \mathbb{Z}^n} f\left(\frac{k}{N}\right)$$

up to an error term of order $O(N^{-1})$.

Recently Guillemin and Sternberg ([GuS06]) showed that if Δ is a simple *lattice* polytope, i.e., if its vertices are lattice points, then this $O(N^{-1})$ can be replaced by an asymptotic series in inverse powers of N. In particular for polytopes associated with toric varieties (such as the polytope (2.3.8)) the terms in this series can be explicitly computed by the following method.

Enumerate the facets of Δ, and for the i^{th} facet let $u_i \in \mathbb{Z}^n$ be a primitive lattice vector which is perpendicular to this facet and points "outward" from Δ into $\mathbb{R}^n$. Then Δ can be defined by a set of inequalities

$$\langle u_i, x \rangle \le c_i\,, \quad i = 1, \ldots, r \tag{2.3.15}$$

where r is the number of facets. Let Δ_h be the polytope

$$\langle u_i, x \rangle \le c_i + h_i\,, \quad i = 1, \ldots, r\,. \tag{2.3.16}$$

Theorem 2.3.1 ([GuS06]). *For $f \in \mathcal{C}^\infty(\mathbb{R}^n)$*

$$\frac{1}{N^n} \sum_{k \in \mathbb{Z}^n \cap N\Delta} f\left(\frac{k}{N}\right) \quad \sim \quad \left(\tau\left(\frac{1}{N}\frac{\partial}{\partial h}\right) \int_{\Delta_h} f(x)\, dx\right)(h=0) \qquad (2.3.17)$$

where

$$\tau(w_1, \ldots, w_r) \quad = \quad \prod_{i=1}^{r} \frac{w_i}{1 - e^{-w_i}} \qquad (2.3.18)$$

and $\tau\left(\frac{1}{N}\frac{\partial}{\partial h}\right)$ is the operator obtained from (2.3.18) by making the substitution $w_i \to \frac{1}{N}\frac{\partial}{\partial h_i}$.

Now notice that if we divide (2.3.9) by N^n the right hand side is exactly a Riemann sum of the form above. Hence if we replace $A_N f$ by the series (2.3.11) and apply (2.3.17) to each summand we get an asymptotic expansion of $\nu_N(f)$ in inverse powers of N in which the summands can be read off from the summands on the right hand side of (2.3.17).

Chapter 3

Stability Theory

The "quantization commutes with reduction" theorem establishes a bijection between the set of G-invariant holomorphic sections of a line bundle, $\mathbb{L}$, on a Kähler G-manifold M, with the set of holomorphic sections on the quotient line bundle, $\mathbb{L}_{red}$, on the symplectic quotient, M_{red}, of M. One direction of this map, that an upstairs invariant holomorphic section gives a downstairs holomorphic section, is obvious. As for the other direction, one first pulls back any holomorphic section of the quotient line bundle to an invariant holomorphic section of the upstairs bundle on a dense open subset. Then one compares the norms of the downstairs section and the corresponding upstairs section (under different metrics). The difference of these two norms is measured by the *stability function*. It turns out that this function has a number of remarkable properties, and as a corollary, one can canonically extend the upstairs section on this dense open subset to an invariant holomorphic section over the whole manifold by setting it equal to zero on the complement.

Since the stability function measures the difference of the downstairs section with corresponding upstairs section, it provide a nature bridge between the two stories. In principle, any spectral problem concerning these quantum states on the quotient bundle can be translated to the corresponding problem on the upstairs bundle via the stability function.

3.1 Backgrounds

3.1.1 Kähler reduction

Suppose (M, ω) is a symplectic manifold, G a connected compact Lie group acting in a Hamiltonian fashion on M, and $\Phi : M \to \mathfrak{g}^*$ a moment map, i.e., Φ is equivariant with respect to the given G-action on M and the coadjoint G-action on $\mathfrak{g}^*$, with the defining property

$$d\langle \Phi, v\rangle = \iota_{v_M}\omega, \qquad v \in \mathfrak{g}, \tag{3.1.1}$$

where v_M is the vector field on M generated by the one-parameter subgroup

$$\{\exp(-tv) \mid t \in \mathbb{R}\}$$

of G. Furthermore we assume that Φ is proper, 0 is a regular value and that G acts freely on the zero level set $\Phi^{-1}(0)$. Then by the Marsden-Weinstein theorem, the quotient space

$$M_{red} = \Phi^{-1}(0)/G$$

is a connected compact symplectic manifold with symplectic form ω_{red} satisfying

$$\iota^*\omega = \pi_0^*\omega_{red}, \tag{3.1.2}$$

where $\iota : \Phi^{-1}(0) \hookrightarrow M$ is the inclusion map, and $\pi_0 : \Phi^{-1}(0) \to M_{red}$ the quotient map. Moreover, if ω is integral, so is ω_{red}; and if (M, ω) is Kähler with holomorphic G-action, then M_{red} is a compact Kähler manifold and ω_{red} is a Kähler form.

3.1.2 GIT quotients

The Kähler quotient M_{red} also has the following GIT description:

Let $G_{\mathbb{C}}$ be the complexification of G, i.e., $G_{\mathbb{C}}$ is the unique connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \sqrt{-1}\mathfrak{g}$ which contains G as its maximal compact subgroup. We will assume that the action of G on M extends canonically to a holomorphic action of $G_{\mathbb{C}}$ on M (This will automatically be the case if M is compact. For more details, c.f. appendix A.2.) The infinitesimal action of $G_{\mathbb{C}}$ on M is given by

$$w_M = Jv_M \tag{3.1.3}$$

for $v \in \mathfrak{g}, w = \sqrt{-1}v$, where J is the automorphism of TM defining the complex structure.

The set of stable points [1] , M_{st}, of M (with respect to this $G_{\mathbb{C}}$ action) is defined to be the $G_{\mathbb{C}}$-flow out of $\Phi^{-1}(0)$:

$$M_{st} = G_{\mathbb{C}} \cdot \Phi^{-1}(0). \tag{3.1.4}$$

This is an open subset of M on which $G_{\mathbb{C}}$ acts freely, and each $G_{\mathbb{C}}$-orbit in M_{st} intersects $\Phi^{-1}(0)$ in precisely one G-orbit, c.f. [GuS82]. Moreover, for any G-invariant holomorphic section s_k of $\mathbb{L}^k$, M_{st} contains all p with $s_k(p) \neq 0$. (For a proof, see the arguments at the end of §3.2.2). In addition, if M is compact $M - M_{st}$ is just the common zero sets of these s_k's. Since M_{st} is a principal $G_{\mathbb{C}}$ bundle over M_{red}, the $G_{\mathbb{C}}$ action on M_{st} is proper. The quotient space $M_{st}/G_{\mathbb{C}}$ has the structure of a complex manifold. Moreover, since each $G_{\mathbb{C}}$-orbit in M_{st} intersects $\Phi^{-1}(0)$ in precisely one G-orbit, this GIT quotient space coincides with the symplectic quotient:

$$M_{red} = M_{st}/G_{\mathbb{C}}.$$

In other words, M_{red} is a Kähler manifold with ω_{red} its Kähler form, and the projection map $\pi : M_{st} \to M_{red}$ is holomorphic.

---

[1] In GIT there are several different notations of sbability, called semistable, stable and properly stable respectively. However, under our assumptions all these definitions coincide with our definition here.

3.1.3 Reduction at the quantum level

Suppose $(\mathbb{L}, \langle\cdot,\cdot\rangle)$ is a pre-quantum line bundle over M. There is a unique holomorphic connection ∇ on $\mathbb{L}$, (called the metric connection), which is compatible with the Hermitian inner product on $\mathbb{L}$, i.e., satisfies the compatibility condition for every locally nonvanishing holomorphic section $s : U \to \mathbb{L}$,

$$\frac{\nabla s}{s} = \partial \log \langle s, s\rangle \in \Omega^{1,0}(U). \tag{3.1.5}$$

The pre-quantization condition amounts to requiring that the curvature form of the connection ∇ is $-\omega$, i.e.,

$$curv(\nabla) := -\sqrt{-1}\bar{\partial}\partial \log \langle s, s\rangle = -\omega. \tag{3.1.6}$$

To define reduction on the quantum level, we assume that the G action on M can be lifted to an action $\tau^{\#}$ of G on $\mathbb{L}$ by holomorphic line bundle automorphisms. For details, c.f. appendix A.1. By averaging, we may assume that $\tau^{\#}$ preserves the metric $\langle\cdot,\cdot\rangle$, and thus preserves the connection ∇ and the curvature form ω. The infinitesimal action of $\mathfrak{g}$ on sections of $\mathbb{L}$ is given by Kostant's formula ([Kos70])

$$L_v s = \nabla_{v_M} s - \sqrt{-1}\langle\Phi, v\rangle s \tag{3.1.7}$$

for all smooth sections $s \in \Gamma(\mathbb{L})$ and all $v \in \mathfrak{g}$. Since G acts freely on $\Phi^{-1}(0)$, the lifted action $\tau^{\#}$ is free on $\iota^*\mathbb{L}$. The quotient

$$\mathbb{L}_{red} = \iota^*\mathbb{L}/G$$

is now a holomorphic line bundle over M_{red}.

On the other hand, by [GuS82], the lifted action $\tau^{\#}$ can be extended canonically to an action $\tau^{\#}_{\mathbb{C}}$ of $G_{\mathbb{C}}$ on $\mathbb{L}$. Denote by $\mathbb{L}_{st}$ the restriction of $\mathbb{L}$ to the open set M_{st}, then $G_{\mathbb{C}}$ acts freely on $\mathbb{L}_{st}$, and we get the GIT description of the quotient line bundle,

$$\mathbb{L}_{red} = \mathbb{L}_{st}/G_{\mathbb{C}}.$$

On $\mathbb{L}_{red}$ there is a naturally defined Hermitian structure, $\langle \cdot, \cdot \rangle_{red}$, i.e.,

$$\pi_0^* \langle s, s \rangle_{red} = \iota^* \langle \pi^* s, \pi^* s \rangle \tag{3.1.8}$$

for all $s \in \Gamma(\mathbb{L}_{red})$. Moreover, the induced curvature form of $\mathbb{L}_{red}$ is the reduced Kähler form ω_{red}, c.f. corollary 3.2.8. In other words, the quotient line bundle $(\mathbb{L}_{red}, \langle \cdot, \cdot \rangle_{red})$ is a pre-quantum line bundle over the quotient space (M_{red}, ω_{red}).

3.2 The stability function

3.2.1 Definition of the stability function

Definition 3.2.1. *The stability function $\psi : M_{st} \to \mathbb{R}$ is defined to satisfy*

$$\langle \pi^* s, \pi^* s \rangle = e^{\psi} \pi^* \langle s, s \rangle_{red} \ . \tag{3.2.1}$$

More precisely, suppose U is an open subset in M_{st} and $s : U \to \mathbb{L}_{red}$ a non-vanishing section, then ψ restricted to $\pi^{-1}(U)$ is defined to be

$$\psi = \log \langle \pi^* s, \pi^* s \rangle - \pi^* \log \langle s, s \rangle_{red} \ . \tag{3.2.2}$$

Obviously this definition is independent of the choices of s.

By definition it is easy to see that ψ is a G-invariant function on M_{st} which vanishes on $\Phi^{-1}(0)$, and by (3.1.6),

$$\omega = \pi^* \omega_{red} + \sqrt{-1}\ \bar{\partial} \partial \psi. \tag{3.2.3}$$

Thus ψ can be thought of as a potential function for ω *relative* to ω_{red}.

Remark 3.2.2. *It is easy to see that the stability function associated to the line bundle $\mathbb{L}^N$ is $N\psi$.*

Remark 3.2.3. *(Reduction by stages) Let $G = G_1 \times G_2$ be a product of compact Lie groups G_1 and G_2. Then by reduction in stages M_{red} can be identified with $(M_{red}^{(1)})^{(2)}$, where $M_{red}^{(1)}$ is the reduction of M with respect to G_1 and $(M_{red}^{(1)})^{(2)}$ the reduction of $M_{red}^{(1)}$ with respect to G_2. Let M_{st}^{G} and $M_{st}^{G_1}$ be the set of stable points in M with respect to the G-action and G_1-action respectively, and $(M_{red}^{(1)})_{st}^{G_2}$ the set of stable points in $M_{st}^{(1)}$ with respect to the G_2-action. Denote by π_1 the projection of M_{st} onto $M_{red}^{(1)}$. We claim that $M_{st}^{G} \subset M_{st}^{G_1}$ and $\pi_1^{-1}((M_{red}^{(1)})_{st}^{G_2}) = M_{st}^{G}$. The first of these assertions is obvious and the second assertion follows from the identification*

$$\begin{aligned}
\pi_1^{-1}((M_{red}^{(1)})_{st}^{G_2}) &= \pi_1^{-1}((G_2)_{\mathbb{C}}\bar{\Phi}_2^{-1}(0)) \\
&= (G_1)_{\mathbb{C}}(\pi_1^{-1}((G_2)_{\mathbb{C}}\bar{\Phi}_2^{-1}(0)) \cap \Phi_1^{-1}(0)) \\
&= (G_1)_{\mathbb{C}}(G_2)_{\mathbb{C}}(\pi_1^{-1}(\bar{\Phi}_2^{-1}(0)) \cap \Phi_1^{-1}(0)) \\
&= G_{\mathbb{C}}(\Phi_2^{-1}(0) \cap \Phi_1^{-1}(0)) \\
&= G_{\mathbb{C}}\Phi^{-1}(0).
\end{aligned}$$

Thus $\psi = \psi_1 + \pi_1^* \psi_2^1$, where ψ is the stability function associated with reduction of M by G, ψ_1 the stability function associated with reduction of M by G_1, and ψ_2^1 the stability function associated with the reduction of $M_{red}^{(1)}$ by G_2.

Remark 3.2.4. *(Action on product manifolds) As in the previous remark let $G = G_1 \times G_2$. Let M_i, $i = 1, 2$, be Kählerian G_i manifolds and $\mathbb{L}_i$ pre-quantum line bundles over M_i, satisfying the assumptions in the previous sections. Denote by ψ_i the stability function on M_i associated to $\mathbb{L}_i$. Letting G be the product $G_1 \times G_2$ the stability function on the G-manifold $M_1 \times M_2$ associated with the product line bundle $pr_1^*\mathbb{L}_1 \otimes pr_2^*\mathbb{L}_2$ is $pr_1^*\psi_1 + pr_2^*\psi_2$.*

3.2.2 Two useful lemmas

Recall that by (3.1.3), the vector field w_M for the "imaginary vector" $w = \sqrt{-1}v \in \sqrt{-1}\mathfrak{g}$ is $w_M = Jv_M$.

Lemma 3.2.5 ([GuS82]). *Suppose $w = \sqrt{-1}v \in \sqrt{-1}\mathfrak{g}$, then w_M is the gradient vector field of $\langle \Phi, v\rangle$ with respect to the Kähler metric g.*

Proof.

$$d\langle \Phi, v\rangle = \iota_{v_M}\omega = \omega(-Jw_M, \cdot) = \omega(\cdot, Jw_M) = g(w_M, \cdot).$$

□

Lemma 3.2.6. *Suppose $w = \sqrt{-1}v \in \sqrt{-1}\mathfrak{g}$. Then for any nonvanishing G-invariant holomorphic section $\tilde{s} \in \Gamma_{hol}(\mathbb{L})^G$,*

$$L_{w_M} \log\langle \tilde{s}, \tilde{s}\rangle = -2\langle \Phi, v\rangle. \tag{3.2.4}$$

Proof. Since

$$J(v_M + \sqrt{-1}w_M) = w_M - \sqrt{-1}v_M = -\sqrt{-1}(v_M + \sqrt{-1}w_M),$$

$v_M + \sqrt{-1}w_M$ is a complex vector field of type (0,1). Thus the covariant derivative

$$\nabla_{v_M}\tilde{s} = -\sqrt{-1}\nabla_{w_M}\tilde{s}. \tag{3.2.5}$$

Since $\tilde{s}$ is G-invariant, by Kostant's identity (3.1.7),

$$0 = L_v\tilde{s} = \nabla_{v_M}\tilde{s} - \sqrt{-1}\langle \Phi, v\rangle\tilde{s}. \tag{3.2.6}$$

Thus

$$\nabla_{w_M}\tilde{s} = -\langle \Phi, v\rangle\tilde{s}.$$

By metric compatibility, we have for any G-invariant holomorphic section $\tilde{s}$

$$L_{w_M} \log \langle \tilde{s}, \tilde{s} \rangle = -2\langle \Phi, v \rangle.$$

$\square$

Since $d^{\mathbb{C}} = JdJ^{-1}$, one has

Corollary 3.2.7. $\left(d^{\mathbb{C}} \log\langle \tilde{s}, \tilde{s} \rangle, v_M\right) = -2\langle \Phi, v \rangle.$

Proof. Notice that $J = (\sqrt{-1})^{p-q}$ on $\Lambda^{p,q}$, so we get

$$\begin{aligned}\left(d^{\mathbb{C}} \log\langle \tilde{s}, \tilde{s} \rangle, v_M\right) &= \left(Jd \log\langle \tilde{s}, \tilde{s} \rangle, J^{-1} w_M\right) \\ &= -\left(d \log\langle \tilde{s}, \tilde{s} \rangle, w_M\right) \\ &= -L_{w_M} \log\langle \tilde{s}, \tilde{s} \rangle.\end{aligned}$$

$\square$

This implies that $\mathbb{L}_{red}$ is the prequantum line bundle over M_{red}:

Corollary 3.2.8. $\omega_{red} = \sqrt{-1}\bar{\partial}\partial \log\langle s, s \rangle_{red}$.

Proof. We only need to check

$$\iota^* \omega = \pi_0^*(\sqrt{-1}\bar{\partial}\partial \log\langle s, s \rangle_{red}).$$

Since $dd^{\mathbb{C}} f = 2\sqrt{-1}\bar{\partial}\partial f$, it suffices to show

$$\iota^* d^{\mathbb{C}} \log\langle \pi^* s, \pi^* s \rangle = \pi_0^* d^{\mathbb{C}} \log\langle s, s \rangle_{red}$$

at $p \in \Phi^{-1}(0)$. By definition both sides coincide on $T_{\pi(p)} M_{red}$ viewed as a subspace of $T_p \Phi^{-1}(0)$. Notice that $T_p \Phi^{-1}(0) = \ker(d\pi_0)_p \oplus T_{\pi(p)} M_{red}$, and the right hand side vanishes on $\ker(d\pi_0)_p$, so we only need to show that the left hand side vanishes on

$\ker(d\pi_0)_p$, which follows from corollary 3.2.7 since $\Phi(p) = 0$:

$$(d^{\mathbb{C}} \log\langle \pi^* s, \pi^* s\rangle, v_M)|_p = -2\langle \Phi(p), v\rangle = 0.$$

$\square$

Another corollary of Lemma 3.2.6 is the following: Suppose M is compact and let $\tilde{s}$ be a holomorphic G-invariant section of $\mathbb{L}$ and p a point where $\tilde{s}(p) \neq 0$. The function

$$\langle \tilde{s}, \tilde{s}\rangle : \overline{G_{\mathbb{C}} \cdot p} \to \mathbb{R}$$

takes its maximum at some point q and since $\overline{G_{\mathbb{C}} \cdot p}$ is $G_{\mathbb{C}}$-invariant and

$$\langle \tilde{s}, \tilde{s}\rangle(q) \geq \langle \tilde{s}, \tilde{s}\rangle(p) > 0$$

it follows from (3.2.4) that $\Phi(q) = 0$, i.e. $q \in M_{st}$. But M_{st} is open and $G_{\mathbb{C}}$-invariant. Hence $p \in M_{st}$. Thus we've proved

Proposition 3.2.9. *If $p \in M - M_{st}$, then $s(p) = 0$ for all $s \in \Gamma_{hol}(\mathbb{L})^G$.*

3.2.3 Analytic properties of the stability function

From definition we see that ψ is invariant under the real Lie group G. The heart of stability theory in this chapter is that the stability function also behaves well in the "imaginary" directions:

Proposition 3.2.10. *Suppose $w = \sqrt{-1}v \in \sqrt{-1}\mathfrak{g}$, then $L_{w_M}\psi = -2\langle \Phi, v\rangle$.*

Proof. Suppose s is any holomorphic section of the reduced bundle $\mathbb{L}_{red}$. Since $\pi^* \log\langle s, s\rangle_{red}$ is $G_{\mathbb{C}}$-invariant, we have from (3.2.2),

$$L_{w_M}\psi = L_{w_M} \log\langle \pi^* s, \pi^* s\rangle.$$

Now apply lemma 3.2.6 to the G-invariant section $\pi^* s$. $\square$

The main result of this section is

Theorem 3.2.11. *ψ is a proper function which takes its maximum value 0 on $\Phi^{-1}(0)$. Moreover, for any $p \in \Phi^{-1}(0)$, the restriction of ψ to the orbit $\exp\sqrt{-1}\mathfrak{g}\cdot p$ has only one critical point, namely p itself, and this critical point is a global maximum.*

Proof. As before we take $w = \sqrt{-1}v \in \sqrt{-1}\mathfrak{g}$. Since $G_{\mathbb{C}}$ acts freely on M_{st}, we have a diffeomorphism

$$\kappa : \Phi^{-1}(0) \times \sqrt{-1}\mathfrak{g} \to M_{st},\ (p, w) \mapsto \tau_{\mathbb{C}}(\exp w)p. \tag{3.2.7}$$

We define two functions

$$\psi_0(p, w, t) = (\kappa^*\psi)(p, tw) \tag{3.2.8}$$

and

$$\phi_0(p, w, t) = \langle \kappa^*\Phi(p, tw), v\rangle. \tag{3.2.9}$$

Then proposition 3.2.10 leads to the following differential equation

$$\frac{d}{dt}\psi_0 = -2\phi_0, \tag{3.2.10}$$

with initial conditions

$$\psi_0(p, w, 0) = 0 \tag{3.2.11}$$

and

$$\phi_0(p, w, 0) = 0. \tag{3.2.12}$$

Since w_M is the gradient vector field of $\langle \Phi, v\rangle$, and $t \mapsto \kappa(p, tw)$ is an integral curve of w_M, we see that ϕ_0 is a strictly increasing function of t. Thus ψ_0 is strictly increasing for $t < 0$, strictly decreasing for $t > 0$, and takes its maximal value 0 at $t = 0$. This shows that p is the only critical point in the orbit $\sqrt{-1}\mathfrak{g}\cdot p$.

The fact ψ is proper also follows from the differential equation (3.2.10), since for any $t_0 > 0$ we have

$$\psi_0(p, w, t) \leq C_0 - 2(t - t_0)C_1, \qquad t > t_0$$

where

$$C_0 = \max_{|w|=1} \psi_0(p, w, t_0) < 0$$

and

$$C_1 = \min_{|w|=1} \phi_0(p, w, t_0) > 0.$$

$\square$

Remark 3.2.12. *The proof above also gives us an alternate way to compute the stability function, namely we only need to solve the differential equation (3.2.10) along each orbit* $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$ *with initial condition (3.2.11). Of course a much more complicated step is to write down explicitly the decomposition of* M_{st} *as a product* $\Phi^{-1}(0) \times \sqrt{-1}\mathfrak{g}$.

Corollary 3.2.13. *For any* $s \in \Gamma_{hol}(\mathbb{L}_{red})$*, the norm* $\langle \pi^* s, \pi^* s \rangle(p)$ *is bounded on* M_{st}*, and tends to* 0 *as* p *goes to the boundary of* M_{st}*.*

3.3 Asymptotic behavior of the stability function

3.3.1 The basic asymptotics

From the previous section we have seen that the stability function takes its global maximum 0 exactly at $\Phi^{-1}(0)$. Thus for λ large, $e^{\lambda\psi}$ tends to 0 very fast off $\Phi^{-1}(0)$. So in principle, only a very small neighborhood of $\Phi^{-1}(0)$ will contribute to the asymptotics of the integral

$$\int_{M_{st}} f e^{\lambda\psi} dx$$

for f a bounded function in $C^\infty(M_{st})^G$ and for λ large. In this section we will derive an asymptotic expansion in λ for this integral, beginning with (1.2.3).

The proof of (1.2.3) is based on the following method of steepest descent: Let X be an m-dimensional Riemannian manifold with volume form dx, and $\psi : X \to \mathbb{R}$ a real-valued smooth function which has a unique maximum at a point p. Suppose moreover that p is a nondegenerate critical point of ψ. Then for $f \in C^\infty(X)$ with $fe^{\lambda\psi}$ in $\mathcal{L}^1(X, dx)$,

$$\int_X f(x)e^{\lambda\psi(x)}dx \sim e^{\lambda\psi(p)}\sum_{k=0}^{\infty} c_k\lambda^{-\frac{m}{2}-k}, \qquad \text{as } \lambda \to \infty \tag{3.3.1}$$

where the c_k's are constants. Moreover,

$$c_0 = (2\pi)^{m/2}\tau_p f(p), \tag{3.3.2}$$

where

$$\tau_p^{-1} = \frac{(\det d^2\psi_p(e_i, e_j))^{1/2}}{|dx_p(e_1, \cdots, e_n)|} \tag{3.3.3}$$

for any basis $e_1, \cdots, e_m$ of T_pM.

From this general result we obtain:

Theorem 3.3.1. *Let dx be the Riemannian volume form on $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$ induced by the Kähler-Riemannian metric on M_{st}, and let f be a smooth function on M. Then for any $p \in \Phi^{-1}(0)$ and λ large,*

$$\int_{\exp\sqrt{-1}\mathfrak{g}\cdot p} f(x)e^{\lambda\psi(x)}dx \sim \left(\frac{\lambda}{\pi}\right)^{-m/2}\left(f(p) + \sum_{i=1}^{\infty} c_i\lambda^{-i}\right), \tag{3.3.4}$$

where c_i are constants depending on f, ψ and p.

Proof. We need to compute the Hessian of ψ restricted to $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$ at the point p. By proposition 3.2.10,

$$d(d\psi(w_M)) = d(L_{w_M}\psi) = -2d\langle\Phi, v\rangle = -2\omega(v_M, \cdot),$$

so

$$d^2\psi_p(w_M, w'_M) = -2\omega_p(v_M, w'_M) = -2g_p(v_M, v'_M) = -2g_p(w_M, w'_M).$$

This implies $\tau_p = 2^{-m/2}$. □

3.3.2 Asymptotics on submanifolds of M_{st}

From (3.3.4) we obtain asymptotic formulas similar to (3.3.4) for submanifolds of M_{st} which are foliated by the sets $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$. For example, by the Cartan decomposition

$$G_{\mathbb{C}} = G \times \exp(\sqrt{-1}\mathfrak{g})$$

one gets a splitting

$$G_{\mathbb{C}} \cdot p = G \times \exp(\sqrt{-1}\mathfrak{g}) \cdot p.$$

Moreover, this is an orthogonal splitting on $\Phi^{-1}(0)$. Thus if we write

$$\frac{\omega^m}{m!}(p) = g(x) d\nu \wedge dx,$$

where $d\nu$ is the Riemannian volume form on the G-orbit $G \cdot p$, defined by the Kähler-Riemannian metric, we see that $g(x)$ is G-invariant and $g(p) = 1$ on $\Phi^{-1}(0)$. Thus if we apply theorem 3.3.1 we get

Corollary 3.3.2. *As $\lambda \to \infty$,*

$$\int_{G_{\mathbb{C}} \cdot p} f(x) e^{\lambda\psi} \frac{\omega^m}{m!} \sim V(p) \left(\frac{\lambda}{\pi}\right)^{-m/2} \left(f(p) + \sum_{i=1}^{\infty} c_i(p)\lambda^{-i}\right), \tag{3.3.5}$$

where $V(p) = Vol(G \cdot p)$ is the Riemannian volume of the G orbit through p.

Similarly the diffeomorphism (3.2.7) gives a splitting of M_{st} into the imaginary orbits $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$, and by the same argument one gets

Corollary 3.3.3. *As* $\lambda \to \infty$,

$$\int_{M_{st}} e^{\lambda\psi} \frac{\omega^d}{d!} \sim Vol(\Phi^{-1}(0)) \left(\frac{\lambda}{\pi}\right)^{-m/2} \left(1 + \sum_{i=1}^{\infty} C_i \lambda^{-i}\right) . \tag{3.3.6}$$

3.3.3 The Half form correction

Now we apply corollary 3.3.2 to prove (1.2.5). Since $M_{red} = M_{st}/G_{\mathbb{C}}$, we have a decomposition of the volume form

$$\frac{\omega^d}{d!} = \pi^* \frac{\omega_{red}^n}{n!} \wedge d\mu_\pi, \tag{3.3.7}$$

where $d\mu_\pi$ is the induced volume form on $G_{\mathbb{C}} \cdot p$,

$$d\mu_\pi(x) = h(x) \frac{\omega^m}{m!},$$

with $h(p) = 1$ on $\Phi^{-1}(0)$. Now suppose $s_k \in \Gamma_{hol}(\mathbb{L}_{red}^k)$. Since the stability function of $\mathbb{L}_{red}^k$ is $k\psi$, (1.2.2) becomes

$$\langle \pi^* s_k, \pi^* s_k \rangle = e^{k\psi} \pi^* \langle s_k, s_k \rangle_{red}.$$

By (3.3.5),

$$\begin{aligned} \|\pi^* s_k\|^2 &= \int_{M_{red}} \left(\int_{G_{\mathbb{C}} \cdot p} e^{k\psi} d\mu_\pi \right) \langle s_k, s_k \rangle_{red} \frac{\omega_{red}^n}{n!} \\ &= \left(\frac{k}{\pi}\right)^{-m/2} \left(1 + O(k^{-1})\right) \int_{M_{red}} V(\pi_0^{-1}(q)) \langle s_k, s_k \rangle_{red} \frac{\omega_{red}^n}{n!}. \end{aligned}$$

In other words,

$$\left(\frac{k}{\pi}\right)^{m/2} \|\pi^* s_k\|^2 = \|V^{1/2} s_k\|_{red}^2 + O(\frac{1}{k}), \tag{3.3.8}$$

where V is the volume function $V(q) = V(\pi_0^{-1}(q))$.

The presence of the factor V can be viewed as a "$\frac{1}{2}$-form correction" in the Kostant-Souriau version of geometric quantization. Namely, let $\mathbb{K} = \bigwedge^d (T^{1,0}M)^*$

and $\mathbb{K}_{red} = \bigwedge^n (T^{1,0}M_{red})^*$ be the canonical line bundles on M and M_{red} and let $\ll, \gg$ and $\ll, \gg_{red}$ be the Hermitian inner products on these bundles, then

$$\pi_0^* \mathbb{K}_{red} = \iota^* \mathbb{K}$$

and

$$\pi_0^*(V \ll, \gg_{red}) = \iota^* \ll, \gg .$$

So if $\mathbb{K}^{\frac{1}{2}}$ and $\mathbb{K}_{red}^{\frac{1}{2}}$ are "$\frac{1}{2}$-form" bundles on M and M_{red} (i.e., the square roots of $\mathbb{K}$ and $\mathbb{K}_{red}$), then one has a map

$$\Gamma_{hol}(\mathbb{L}^k \otimes \mathbb{K}^{\frac{1}{2}})^G \to \Gamma_{hol}(\mathbb{L}^k_{red} \otimes \mathbb{K}^{\frac{1}{2}}_{red})$$

which is an isometry modulo an error term of order $O(k^{-1})$. (See [HaK07] and [Li07] for more details on half form correction.)

3.4 Applications to spectral problems

3.4.1 Maximum points of quantum states

Suppose M is a Kähler manifold with quantum line bundle $\mathbb{L}$, and $\tilde{s} \in \Gamma_{hol}(\mathbb{L})$ is a quantum state. The "invariance of polarization" conjecture of Kostant-Souriau is closely connected with the question: where does the function $\langle \tilde{s}, \tilde{s} \rangle$ take its maximum? If C is the set where $\langle \tilde{s}, \tilde{s} \rangle$ takes its maximum, what can one say about C? What is the asymptotic behavior of the function $\langle \tilde{s}, \tilde{s} \rangle^k$ in a neighborhood of C?

To address these questions we begin by recalling the following results:

Proposition 3.4.1. *If C above is a submanifold of M, then*
(a) *C is an isotropic submanifold of M;*
(b) *$\iota_C^* \tilde{s}$ is a non-vanishing covariant constant section of $\iota_C^* \mathbb{L}$;*
(c) *Moreover if M is a Kähler G-manifold and $\tilde{s}$ is in $\Gamma_{hol}(\mathbb{L})^G$ then C is contained in the zero level set of Φ.*

Proof. (a) Let $\alpha = \sqrt{-1}\bar{\partial}\log\langle\tilde{s},\tilde{s}\rangle$. Then $\omega = d\alpha$ and $\alpha_p = 0$ for every $p \in C$, so $\iota_C^*\omega = 0$.

(b) By (3.1.5), $\nabla s = 0$ on C.

(c) By (3.1.7),

$$\nabla_{v_M} s = \sqrt{-1}\langle\phi, v\rangle s = 0$$

along C, therefore since s is non-zero on C, $\langle\Phi, v\rangle = 0$ on C. $\square$

We will call a submanifold C of M for which the line bundle $\iota_C^*\mathbb{L}$ admits a nonzero covariant constant section a *Bohr-Sommerfeld* set. Notice that if s_0 is a section of $\iota_C^*\mathbb{L}$ which is non-vanishing, then

$$\frac{\nabla s_0}{s_0} = \alpha_0 \Longleftrightarrow d\alpha_0 = \iota_C^*\omega,$$

so if s is covariant constant then C has to be isotropic. The most interesting Bohr-Sommerfeld sets are those which are maximally isotropic, i.e., Lagrangian, and the term "Bohr-Sommerfeld" is usually reserved for these Lagrangian submanifolds .

A basic problem in Bohr-Sommerfeld theory is obtaining converse results to the proposition above. Given a Bohr-Sommerfeld set, C, does there exist a holomorphic section, s, of $\mathbb{L}$ taking its maximum on C, i.e., for which the measure

$$\langle s^k, s^k\rangle\mu_{Liouville} \tag{3.4.1}$$

becomes more and more concentrated on C as $k \to \infty$. As we pointed out in the introduction this problem is often intractable, however if we are in the setting of GIT theory with M replaced by M_{red}, then the downstairs version of this question can be translated into the upstairs version of this question which is often easier. In §3.4.2 we will discuss the behavior of measures of type (3.4.1) in general and then in §3.4.5 discuss this Bohr-Sommerfeld problem.

3.4.2 Asymptotics of spectral measures

We will now apply stability theory to the spectral measure (1.2.6) on M_{red}. For f an integrable function on M_{red}, consider the asymptotic behavior of the integral

$$\int_{M_{red}} f\langle s_k, s_k\rangle \mu_{red}, \tag{3.4.2}$$

with $s_k \in \Gamma_{hol}(\mathbb{L}^k_{red})$ and $k \to \infty$. It is natural to compare (3.4.2) with the upstairs integral

$$\int_{M_{st}} \pi^* f\langle \pi^* s_k, \pi^* s_k\rangle \mu. \tag{3.4.3}$$

However, since M_{st} is noncompact, the integral above may not converge in general. To eliminate the possible convergence issues, we multiply the integrand by a cutoff function, i.e., a compactly supported function χ which is identically 1 on a neighborhood of $\Phi^{-1}(0)$. In other words, we consider the integral

$$\int_{M_{st}} \chi\pi^* f\langle \pi^* s_k, \pi^* s_k\rangle \mu. \tag{3.4.4}$$

Obviously different choices of the cutoff function will not affect the asymptotic behavior of (3.4.4).

Using the decomposition (3.3.7) we get

$$\begin{aligned}\int_{M_{st}} \chi\pi^* f\langle \pi^* s_k, \pi^* s_k\rangle \frac{\omega^d}{d!} &= \int_{M_{red}} \left(\int_{G_{\mathbb{C}}\cdot p} e^{k\psi}\chi d\mu_\pi\right) f\langle s_k, s_k\rangle_{red} \frac{\omega^n_{red}}{n!} \\ &\sim \int_{M_{red}} V f\langle s_k, s_k\rangle d\mu_{red},\end{aligned}$$

where $V(q) := V(\pi^{-1}(q))$ is the volume function. We conclude

Proposition 3.4.2. *As $k \to \infty$ we have*

$$\int_{M_{red}} f\langle s_k, s_k\rangle \mu_{red} \sim (\frac{k}{\pi})^{-m/2} \int_M \chi\tilde{f}\tilde{V}^{-1}\langle \pi^* s_k, \pi^* s_k\rangle \mu,$$

where $\tilde{f} = \pi^* f, \tilde{V} = \pi^* V$ and χ is any cutoff function near $\Phi^{-1}(0)$.

Similarly if we apply the same arguments to the spectral measure

$$\mu_N = \sum_i \langle s_{N,i}, s_{N,i} \rangle \mu_{red}, \tag{3.4.5}$$

where $\{s_{N,i}\}$ is an orthonormal basis of $\mathbb{L}^N_{red}$, we get

Proposition 3.4.3. *As $N \to \infty$,*

$$\int_{M_{red}} f \mu_N \sim (\frac{N}{\pi})^{-m/2} \int_{M_{st}} \chi \tilde{f} \tilde{V}^{-1} \mu^G_N, \tag{3.4.6}$$

where

$$\mu^G_N = \sum_i \langle \pi^* s_{N,i}, \pi^* s_{N,i} \rangle \mu$$

is the upstairs G-invariant spectral measure (1.2.7).

3.4.3 Asymptotics of the moments

We next describe the role of "upstairs" versus "downstairs" in describing the asymptotic behavior of the distribution function

$$\sigma_k([t, \infty)) = \mathrm{Vol}\{z \mid \langle s_k, s_k \rangle(z) \geq t\}, \tag{3.4.7}$$

for $s_k \in \Gamma_{hol}(\mathbb{L}^k_{red})$, i.e., of the push-forward measure, $\langle s_k, s_k \rangle_* \mu$, on the real line $\mathbb{R}$. The moments (1.2.10) completely determine this measure, and by theorem 3.3.1 the moments (1.2.10) on M_{red} are closely related to the corresponding moments (1.2.11) on M. In fact, by corollary 3.3.2 and the decomposition (3.3.7),

$$\begin{aligned} \int_{M_{st}} \langle \pi^* s_k, \pi^* s_k \rangle^l \mu &= \int_{M_{st}} (\pi^* \langle s_k, s_k \rangle)^l e^{lk\psi} \pi^* \frac{\omega^n_{red}}{n!} \wedge h(x) \frac{\omega^m}{m!} \\ &\sim \left(\frac{lk}{\pi}\right)^{-m/2} \int_{M_{red}} \langle s_k, s_k \rangle^l V \mu_{red}. \end{aligned}$$

We conclude

Proposition 3.4.4. *For any integer l, the l^{th} moments (1.2.11) satisfy*

$$m(l, \pi^* s_k, \mu) \sim \left(\frac{lk}{\pi}\right)^{-m/2} m_{red}(l, s_k, V\mu_{red}). \tag{3.4.8}$$

as $k \to \infty$.

3.4.4 Asymptotic expansion of the G-invariant spectral measure

For the spectral measure (1.2.14), Colin de Verdiere showed that it admits an asymptotic expansion (1.2.16) in inverse power of N as $N \to \infty$ if the manifold is compact (See appendix B.2 for a proof of this result). By applying stability theory above, we get from the Colin de Verdiere's expansion for the downstairs manifold a similar asymptotic expansion upstairs for the G-invariant spectral measure without assuming M to be compact. Namely, since M_{red} is compact, Colin de Verdiere's theorem gives one an asymptotic expansion

$$\mu_N^{red}(f) = \mathrm{Tr}(\pi_N^{red} M_f \pi_N^{red}) \sim \sum_{i=n-1}^{-\infty} a_i^{red} N^i,$$

and for

$$\pi_N^G : L^2(\mathbb{L}^N, \mu) \to \Gamma_{hol}(\mathbb{L}^N)^G$$

the orthogonal projection onto G-invariant holomorphic sections, we will deduce from this:

Theorem 3.4.5. *For any compactly supported G-invariant function f on M,*

$$\mathrm{Tr}(\pi_N^G M_f \pi_N^G) \sim \sum_{i=n-1}^{-\infty} a_i^G(f) N^i, \tag{3.4.9}$$

as $N \to \infty$, and the coefficients a_i^G can be computed explicitly from a_i^{red}. In particular, the leading coefficient $a_{n-1}^G(f) = a_{n-1}^{red}(f_0 V)$, where $f_0(p) = f(\pi_0^{-1}(p))$.

Proof. Let $\{s_{N,j}\}$ be an orthonormal basis of $\Gamma_{hol}(\mathbb{L}^N_{red})$ with respect to the volume form $V\mu_{red}$, then $\{\pi^* s_{N,j}\}$ is an orthogonal basis of $\Gamma_{hol}(\mathbb{L}^N)^G$, and

$$\mathrm{Tr}(\pi_N^G M_f \pi_N^G) = \int_M \sum_j \frac{\langle \pi^* s_{N,j}, \pi^* s_{N,j}\rangle}{\|\pi^* s_{N,j}\|^2} f\mu,$$

where, by the same argument as in the proof of (3.3.8), we have

$$\|\pi^* s_{N,j}\|^2 \sim \left(\frac{N}{\pi}\right)^{-m/2}\left(1+\sum_i C_i N^{-i}\right),$$

which implies

$$\frac{1}{\|\pi^* s_{N,j}\|^2} \sim \left(\frac{N}{\pi}\right)^{m/2}\left(1+\sum_i \tilde{C}_i N^{-i}\right).$$

Moreover, it is easy to see that

$$\int_{M_{red}} \sum_j \langle s_{N,j}, s_{N,j}\rangle V f_0 \mu_{red} = \mu_N^{red}(f_0 V).$$

Now the theorem follows from straightforward computations

$$\begin{aligned}
\mathrm{Tr}(\pi_N^G M_f \pi_N^G) &\sim \left(\frac{N}{\pi}\right)^{m/2}\left(1+\sum_i \tilde{C}_i N^{-i}\right)\int_{M_{st}} \sum_j \langle \pi^* s_{N,j}, \pi^* s_{N,j}\rangle f\mu \\
&\sim \left(\frac{N}{\pi}\right)^{m/2}\left(1+\sum_i \tilde{C}_i N^{-i}\right)\int_{M_{red}}\left(\int_{G\cdot p}\sum_{i=-m/2}^{-\infty} N^i c_i(f,p)\right)\sum_j \langle s_{N,j}, s_{N,j}\rangle(p) \\
&= \left(\frac{N}{\pi}\right)^{m/2}\left(1+\sum_i \tilde{C}_i N^{-i}\right)\int_{M_{red}}\sum_{i=-m/2}^{-\infty}\left(N^i\int_{G\cdot p} c_i(f,p)d\nu\right)\sum_j \langle s_{N,j}, s_{N,j}\rangle(p) \\
&= \left(\frac{N}{\pi}\right)^{m/2}\left(1+\sum_i \tilde{C}_i N^{-i}\right)\sum_{i=-m/2}^{-\infty} N^i \mu_N^{red}(c_i V) \\
&\sim \sum_{i=n-1}^{-\infty} a_i^G(f) N^i,
\end{aligned}$$

where we used the fact that since f is G-invariant, so is $c_i(f,p)$. This proves (3.4.9).

Moreover, since $c_{-m/2}(f,p) = f(p)/\pi^{m/2}$, we see that

$$a_{n-1}^G(f) = a_{n-1}^{red}(f_0 V),$$

completing the proof. $\square$

3.4.5 Bohr-Sommerfeld Lagrangians

We assume we are in the same setting as before, and denote by ∇_{red} the metric connection on $\mathbb{L}_{red}$. Suppose Λ_{red} is a Bohr-Sommerfeld Lagrangian submanifold of M_{red}, and s_{BS} is a covariant constant section, i.e.,

$$s_{BS} : \Lambda_{red} \to \iota_{\Lambda_{red}}^* \mathbb{L}_{red}, \quad (\iota_{\Lambda_{red}}^* \nabla_{red}) s_{BS} = 0, \tag{3.4.10}$$

where $\iota_{\Lambda_{red}} : \Lambda_{red} \to M_{red}$ is the inclusion map. Let $\Lambda = \pi_0^{-1}(\Lambda_{red})$, then $\Lambda \subset \Phi^{-1}(0)$ is a G-invariant Lagrangian submanifold of M. Since

$$\pi_0^* \nabla_{red} s_{BS} = \iota_\Lambda^* \nabla \pi_0^* s_{BS}, \tag{3.4.11}$$

we see that $\pi_0^* s_{BS}$ is a covariant constant section on Λ. In other words, Λ is a Bohr-Sommerfeld Lagrangian submanifold of M. Conversely, if Λ is a G-invariant Bohr-Sommerfeld Lagrangian submanifold of M, then $\Lambda_{red} = \pi_0(\Lambda)$ is a Bohr-Sommerfeld Lagrangian submanifold of M_{red}.

Fixing a volume form μ_Λ on Λ, the pair (Λ_{red}, s_{BS}) defines a functional l on the space of holomorphic sections by

$$l : \Gamma_{hol}(\mathbb{L}_{red}) \to \mathbb{C}, \quad s \mapsto \int_{\Lambda_{red}} \langle \iota_{\Lambda_{red}}^* s, s_{BS} \rangle \mu_{\Lambda_{red}}.$$

This in turn defines a global holomorphic section $s_{\Lambda_{red}} \in \Gamma_{hol}(\mathbb{L}_{red})$ by duality. In other words, $s_{\Lambda_{red}}$ is the holomorphic section on M_{red} with the defining property

$$\int_{M_{red}} \langle s, s_{\Lambda_{red}} \rangle \mu_{red} = \int_{\Lambda_{red}} \langle \iota^*_{\Lambda_{red}} s, s_{BS} \rangle \mu_{\Lambda_{red}} \tag{3.4.12}$$

for all $s \in \Gamma_{hol}(\mathbb{L}_{red})$. A fundamental problem in Bohr-Sommerfeld theory is to know whether the section $s_{\Lambda_{red}}$ vanishes identically; and if not, to what extent $s_{\Lambda_{red}}$ is "concentrated" on the set Λ_{red}. One can also ask this question for the analogous section of $\mathbb{L}^k_{red}$.

We apply the upstairs-vs-downstairs philosophy to these problems. For the upstairs Bohr-Sommerfeld Lagrangian pair $(\Lambda, \tilde{s}_{BS})$, $\tilde{s}_{BS} = \pi_0^* s_{BS}$, as above one can associate with it a functional $\tilde{l}$ on $\Gamma_{hol}(\mathbb{L})^G$, which by duality defines a global G-invariant section $\tilde{s}_\Lambda \in \Gamma_{hol}(\mathbb{L})^G$. Obviously $l \neq 0$ if and only if $\tilde{l}$ is nonzero on $\Gamma_{hol}(\mathbb{L})^G$. However, since $\tilde{s}_{BS}$ is a G-invariant section,

$$\langle \tilde{s}, \tilde{s}_{BS} \rangle = \langle \tilde{s}^G, \tilde{s}_{BS} \rangle,$$

where $\tilde{s}^G$ is the orthogonal projection of $\tilde{s} \in \Gamma_{hol}(\mathbb{L})$ onto $\Gamma_{hol}(\mathbb{L})^G$. Thus $\tilde{l}$ is nonzero on $\Gamma_{hol}(\mathbb{L})^G$ if and only if it is nonzero on $\Gamma_{hol}(\mathbb{L})$. Thus we proved

Proposition 3.4.6. $s_{\Lambda_{red}} \neq 0$ *if and only if* $\tilde{s}_\Lambda \neq 0$*.*

A natural question to ask is whether $\pi^* s_{\Lambda_{red}}$ coincides with $\tilde{s}_\Lambda$ on M_{st}, or alternatively, whether $\pi_0^* s_{\Lambda_{red}} = \iota^* \tilde{s}_\Lambda$ on $\Phi^{-1}(0)$. In view of the $\frac{1}{2}$-form correction, we will modify the definition of the downstairs section $s_{\Lambda_{red}}$ to be

$$\int_{M_{red}} \langle s, s_{\Lambda_{red}} \rangle V \mu_{red} = \int_{\Lambda_{red}} \langle \iota^*_{\Lambda_{red}} s, s_{BS} \rangle V \mu_{\Lambda_{red}} \ , \tag{3.4.13}$$

for $s, s_{\Lambda_{red}} \in \Gamma_{hol}(\mathbb{L}_{red})$. The upstairs version of this is

$$\int_{M_{st}} \langle \tilde{s}, \tilde{s}_\Lambda \rangle \mu = \int_\Lambda \langle \iota^*_\Lambda \tilde{s}, \pi_0^* s_{BS} \rangle \mu_\Lambda \tag{3.4.14}$$

for $\tilde{s} = \pi^* s$. Since $\Lambda = \pi_0^{-1}(\Lambda_{red})$, the right hand sides of (3.4.13) and (3.4.14) coincide. Thus

$$\int_{M_{st}} \langle \pi^* s, \tilde{s}_\Lambda \rangle \mu = \int_{M_{red}} \langle s, s_{\Lambda_{red}} \rangle V \mu_{red} \tag{3.4.15}$$

for all $s \in \Gamma_{hol}(\mathbb{L}_{red})$.

Now we assume $s_k \in \Gamma_{hol}(\mathbb{L}^k_{red})$, s^k_{BS} being the k^{th} tensor power of s_{BS}, and let $s^{(k)}_{\Lambda_{red}}$ and $\tilde{s}^{(k)}_\Lambda$ be the corresponding holomorphic sections. Then equation (3.4.15) now reads

$$\int_{M_{st}} \langle \pi^* s_k, \tilde{s}^{(k)}_\Lambda \rangle \mu = \int_{M_{red}} \langle s_k, s^{(k)}_{\Lambda_{red}} \rangle V \mu_{red} \tag{3.4.16}$$

for all $s_k \in \Gamma_{hol}(\mathbb{L}^k_{red})$ (However, the sections $\tilde{s}^{(k)}_\Lambda$ and $s^{(k)}_{\Lambda_{red}}$ are no longer the k^{th} tensor powers of $\tilde{s}_\Lambda$ and $s_{\Lambda_{red}}$ above). Notice that we can choose the two sections in (3.2.1) to be different nonvanishing sections and still get the same stability function ψ. Thus applying stability theory, one has

$$\int_{M_{st}} \langle \pi^* s_k, \pi^* s^{(k)}_{\Lambda_{red}} \rangle \mu \sim (\frac{k}{\pi})^{m/2} \int_{M_{red}} \langle s_k, s^{(k)}_{\Lambda_{red}} \rangle V \mu_{red}$$

for all s_k as $k \to \infty$. This together with (3.4.16) implies

Proposition 3.4.7. *Asymptotically we have*

$$\pi^* s^{(k)}_{\Lambda_{red}} \sim (\frac{k}{\pi})^{m/2} \tilde{s}^{(k)}_\Lambda, \; k \to \infty.$$

Chapter 4

Spectral properties of toric varieties

In recent years, toric geometry has become a very active research area in mathematics, in particular as a testing ground for various conjectures in statistical physics. Many general conjectures are easier to understand in the toric case, and are first proved in the toric setting.

There are many different ways to look at toric varieties, and we will take the symplectic point of view: toric varieties are symplectic manifolds with maximally large toric symmetry group. According to Delzant's description, toric varieties gives us the simplest setting possible for applying our upstairs-vs-downstairs methods:

- The upstairs space is the space $\mathbb{C}^d$.
- The group acting on it is an abelian group $G \subset \mathbb{T}^d$.
- The G-action is a linear action.
- The upstairs pre-quantum line bundle is the trivial line bundle $\mathbb{L} = \mathbb{C}^d \times \mathbb{C}$.
- The G-invariant sections of this bundle are (linear combinations of) monomials.
- These monomial sections form an orthogonal basis of the space $\Gamma^G_{hol}(\mathbb{L})$.

All these features make the upstairs theory very computable.

4.1 Delzant's description of toric varieties

4.1.1 The Delzant construction

Let $\mathbb{L} = \mathbb{C}^d \times \mathbb{C}$ be the trivial line bundle over $\mathbb{C}^d$ equipped with the Hermitian inner product

$$\langle 1, 1 \rangle = e^{-|z|^2},$$

where $1 : \mathbb{C}^d \to \mathbb{L}$, $z \mapsto (z, 1)$ is the standard trivialization of $\mathbb{L}$. The line bundle $\mathbb{L}$ is the pre-quantum line bundle for $\mathbb{C}^d$, since

$$curv(\nabla) = -\sqrt{-1}\bar{\partial}\partial \log \langle 1, 1 \rangle = \sqrt{-1} \sum d\bar{z} \wedge dz = -\omega.$$

Let $K = (S^1)^d$ be the d-torus, which acts on $\mathbb{C}^d$ by the diagonal action,

$$\tau(e^{it_1}, \cdots, e^{it_d}) \cdot (z_1, \cdots, z_d) = (e^{it_1} z_1, \cdots, e^{it_d} z_d).$$

This is a Hamiltonian action with moment map

$$\phi(z) = \sum_{i=1}^{d} |z_i|^2 e_i^*, \tag{4.1.1}$$

where $e_1^*, \cdots, e_d^*$ is the standard basis of $\mathfrak{k}^* = \mathbb{R}^d$.

Now suppose $G \subset K$ is an m-dimensional sub-torus of K, $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra, and $\mathbb{Z}_G^* \subset \mathfrak{g}^*$ the weight lattice [1]. Then the restriction of the K-action to G is still Hamiltonian, with moment map

$$\Phi(z) = L \circ \phi(z) = \sum_{i=1}^{d} |z_i|^2 \alpha_i, \tag{4.1.2}$$

where $\alpha_i = L(e_i^*) \in \mathbb{Z}_G^*$, and $L : \mathfrak{k}^* \to \mathfrak{g}^*$ is the transpose of the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{k}$.

We assume that the moment map Φ is proper, or alternatively, that the α_i's are

[1] The weight lattice is by definition the dual of the group lattice $\mathbb{Z}_G = \mathrm{Ker}(\exp : \mathfrak{g} \to G)$.

polarized: there exists $v \in \mathfrak{g}$ such that $\alpha_i(v) > 0$ for all $1 \leq i \leq d$. Let $\alpha \in \mathbb{Z}_G^*$ be fixed, with the property that G acts freely on $\Phi^{-1}(\alpha)$ [2]. Then the symplectic quotient at level α,

$$M_\alpha = \Phi^{-1}(\alpha)/G,$$

is a symplectic toric manifold; and by Delzant's theorem, all toric manifolds arise this way.

The Hamiltonian action of K on $\mathbb{C}^d$ induces a Hamiltonian action of K on M_α, with moment map Φ_α defined by

$$\phi \circ \iota_\alpha = \Phi_\alpha \circ \pi_\alpha, \tag{4.1.3}$$

where $\iota_\alpha : \Phi^{-1}(\alpha) \hookrightarrow \mathbb{C}^d$ is the inclusion map, and $\pi_\alpha : \Phi^{-1}(\alpha) \to M_\alpha$ the projection map. The moment polytope of this Hamiltonian action on M_α is

$$\Delta_\alpha = L^{-1}(\alpha) \cap \mathbb{R}_+^d = \{t \in \mathbb{R}^d \mid t_i \geq 0,\ \sum t_i \alpha_i = \alpha\}. \tag{4.1.4}$$

If we replace $\mathbb{L}$ by $\mathbb{L}^k$, i.e. the trivial line bundle over $\mathbb{C}^d$ with Hermitian inner product $\langle 1, 1 \rangle_k = e^{-k|z|^2}$, then everything proceeds as above, and the moment polytope is changed to $k\Delta_\alpha = \Delta_{k\alpha}$.

4.1.2 Line bundles over toric varieties

As we showed in section 3.1, M_α also admits the following GIT description,

$$M_\alpha = \mathbb{C}^d_{st}(\alpha)/G_{\mathbb{C}},$$

[2] One can show that G acts freely on $\Phi^{-1}(\alpha)$ iff for any vertex $t \in \Delta_\alpha$, the set $\{\alpha_i : i \in I_t\}$ is a lattice basis for $\mathbb{Z}_G^*$, where Δ_α is the moment polytope (4.1.4), and I_t is the index set such that $t_i \neq 0$.

where $G_{\mathbb{C}} \simeq (\mathbb{C}^*)^n$ is the complexification of G, and $\mathbb{C}^d_{st}(\alpha)$ is the $G_{\mathbb{C}}$ flow-out of $\Phi^{-1}(\alpha)$. This flow-out is easily seen to be identical with the set

$$\mathbb{C}^d_{st}(\alpha) = \{z \in \mathbb{C}^d \mid I_z \in I_{\Delta_\alpha}\}, \tag{4.1.5}$$

where

$$I_z = \{i \mid z_i \neq 0\}$$

and

$$I_{\Delta_\alpha} = \{I_t \mid t \in \Delta_\alpha\}.$$

Now let G acts on the line bundle $\mathbb{L}$ by acting on the trivial section, 1, of $\mathbb{L}$, by weight α. (In Kostant's formula (3.1.7) this has the effect of shifting the moment map Φ by α, so that the new moment map becomes $\Phi - \alpha$ and the α level set of Φ becomes the zero level set of $\Phi - \alpha$). This action extends to an action of $G_{\mathbb{C}}$ on $\mathbb{L}$ which acts on the trivial section 1 by the complexification, $\alpha_{\mathbb{C}}$, of the weight α and we can form the quotient line bundle,

$$\mathbb{L}_\alpha = \iota_\alpha^* \mathbb{L}/G = \mathbb{L}_{st}(\alpha)/G_{\mathbb{C}},$$

where $\mathbb{L}_{st}(\alpha)$ is the restriction of $\mathbb{L}$ to $\mathbb{C}^d_{st}(\alpha)$.

The holomorphic sections of $\mathbb{L}^k_\alpha$ are closely related to monomials in $\mathbb{C}^d$. In fact, since $\mathbb{L}$ is the trivial line bundle, the monomials

$$z^m = z_1^{m_1} \cdots z_d^{m_d}$$

are holomorphic sections of $\mathbb{L}$, and by Kostant's formula, z^m is a G-invariant section of $\mathbb{L}$ (with respect to the moment map Φ_α) if and only if

$$\tau^{\#}(\exp v)^* z^m = e^{i\alpha(v)} z^m$$

for all $v \in \mathfrak{g}$; in other words, if and only if m is an integer point in Δ_α. So we obtain

$$\Gamma_{hol}(\mathbb{L})^G = \mathrm{span}\{z^m \mid m \in \Delta_\alpha \cap \mathbb{Z}^d\}. \tag{4.1.6}$$

In view of (4.1.5), $\mathbb{C}^d_{st}(\alpha)$ is Zariski open, so the GIT mapping

$$\gamma : \Gamma_{hol}(\mathbb{L})^G \to \Gamma_{hol}(\mathbb{L}_\alpha)$$

is bijective, although $\mathbb{C}^d$ is noncompact. As a result, the sections

$$s_m = \gamma(z^m), \quad m \in \Delta_\alpha \cap \mathbb{Z}^d. \tag{4.1.7}$$

give a basis of $\Gamma_{hol}(\mathbb{L}_\alpha)$.

To compute the norm of these sections s_m, we introduce the following notation. Let $j : \Delta_\alpha \hookrightarrow \mathbb{R}^+_d$ be the inclusion map, and t_i the standard i^{th} coordinate functions of $\mathbb{R}^d$. Then the *lattice distance* of $x \in \Delta_\alpha$ to the i^{th} facet of Δ_α is $l_i(x) = j^* t_i(x)$. On $\Phi^{-1}(\alpha)$ one has

$$\langle z^m, z^m \rangle = |z_1^{m_1}|^2 \cdots |z_d^{m_d}|^2 e^{-|z|^2},$$

which implies

$$\langle s_m, s_m \rangle_\alpha = (\Phi_\alpha)^* (l_1^{m_1} \cdots l_d^{m_d} e^{-l}) \ , \tag{4.1.8}$$

where $l = l_1 + \cdots + l_d$. As a corollary, we see that the stability function on $\mathbb{C}^d_{st}(\alpha)$ is

$$\psi(z) = -|z|^2 + \log |z^m|^2 - \pi^* \Phi_\alpha^* (\sum m_i \log l_i - l). \tag{4.1.9}$$

Another corollary of (4.1.8) is that the potential function for the Kähler form on is $\Phi_\alpha^*(\sum m_i \log l_i - l)$. Finally by the Duistermaat-Heckman theorem the push-forward of the symplectic measure on M_α by Φ_α is the Lebesgue measure $d\sigma$ on Δ_α, so the L^2 norm of s_m is

$$\langle s_m, s_m \rangle_{L^2} = \int_{\Delta_\alpha} l_1^{m_1} \cdots l_d^{m_d} e^{-l} d\sigma.$$

For toric varieties, the "Bohr-Sommerfeld" issues that we discussed in section 3.4.1 are easily dealt with: Let $\tilde{s}$ be the G-invariant section, $z_1^{m_1} \cdots z_d^{m_d}$, of $\mathbb{L}$, with $(m_1, \cdots, m_d) \in \Delta_\alpha$. Then $\langle \tilde{s}, \tilde{s} \rangle$ take its maximum on the set $\Phi^{-1}(m_1, \cdots, m_d)$, and if $(m_1, \cdots, m_d)$ is in the interior of Δ_α, this set is a Lagrangian torus: an orbit of $\mathbb{T}^d$. Moreover, if s is the section of $\mathbb{L}_\alpha$ corresponding to $\tilde{s}$, $\langle s, s \rangle$ takes its maximum on the projection of this orbit in M_α, which is also a Lagrangian submanifold.

4.1.3 Canonical affines

We end this section by briefly describing a natural coordinate chart on M_α – the canonical affines. (For more details c.f. [DuP07]). Let v be a vertex of Δ. Since Δ is a Delzant's polytope, $\#I_v = n$ and $\{\alpha_i \mid i \in I_v\}$ is a lattice basis of $\mathbb{Z}_G^*$. Denote by

$$\Delta_v = \{t \in \Delta \mid I_t \supset I_v\}, \tag{4.1.10}$$

the open subset in Δ_α obtained by deleting all facets which don't contain v. Let

$$Z_v = \Phi_\alpha^{-1}(\Delta_v).$$

Definition 4.1.1. *The* canonical affines *in M_α are the open subsets*

$$\mathcal{U}_v = Z_v/G. \tag{4.1.11}$$

Since $\{\alpha_i \mid i \in I_v\}$ is a lattice basis, for $j \notin I_v$ we have $\alpha_j = \sum c_{j,i}\alpha_i$, where $c_{j,i}$ are integers. Suppose $\alpha = \sum a_i \alpha_i$, then Z_v is defined by the equations

$$|z_i|^2 = a_i - \sum_{j \notin I_v} c_{j,i}|z_j|^2, \qquad i \in I_v \tag{4.1.12}$$

and the resulting inequalities

$$\sum c_{j,i}|z_j|^2 < a_i. \tag{4.1.13}$$

So $\mathcal{U}_v$ can be identified with the set (4.1.12). The set

$$z_i = \left(a_i - \sum_{j \notin I_v} c_{j,i} |z_j|^2 \right)^{1/2}$$

is a cross-section of the G-action on Z_v, and the restriction to this cross-section of the standard symplectic form on $\mathbb{C}^d$ is $\sqrt{-1}\ \sum_{i \notin I_z} dz_i \wedge d\bar{z}_i$. So the reduced symplectic form is

$$\omega_\alpha = \sqrt{-1}\ \sum_{j \notin I_v} dz_j \wedge d\bar{z}_j, \tag{4.1.14}$$

in other words, the z_j's with $j \notin I_v$ are *Darboux coordinates* on $\mathcal{U}_v$.

4.2 The stability functions on toric varieties

4.2.1 The general formula

In this section we compute the stability functions for the toric varieties M_α above. For $z \in M_{st}$ there is a unique $g \in \exp \sqrt{-1}\mathfrak{g}$ such that $g \cdot z \in \Phi^{-1}(\alpha)$, and by definition, if $s(z) = z^m = \pi^* s_m$,

$$\begin{aligned} \psi(z) &= \log \langle s, s\rangle(z) - \log \langle s, s\rangle(g \cdot z) \\ &= -|z|^2 + \log |z^m|^2 + |g \cdot z|^2 - \log |(g \cdot z)^m|^2. \end{aligned} \tag{4.2.1}$$

Moreover, If the circle group $(e^{i\theta}, \cdots, e^{i\theta})$ is contained in G, or alternatively, if $v = (1, \cdots, 1) \in \mathfrak{g}$, or alternatively if M_α can be obtained by reduction from $\mathbb{CP}^{d-1}$, then

$$|z|^2 = \sum \alpha_i(v) |z_i|^2 = \langle \Phi(z), v\rangle,$$

thus

$$|g \cdot z|^2 = \langle \Phi(g \cdot z), v\rangle = \langle \alpha, v\rangle, \tag{4.2.2}$$

and (4.2.1) simplifies to

$$\psi(z) = -|z|^2 + \log|z^m|^2 + \alpha(v) - \log|(g \cdot z)^m|^2. \tag{4.2.3}$$

Given a weight $\beta \in \mathbb{Z}_G^*$ let $\chi_\beta : G_{\mathbb{C}} \to \mathbb{C}$ be the character of $G_{\mathbb{C}}$ associated to β. Restricted to $\exp(\sqrt{-1}\mathfrak{g})$, χ_β is the map

$$\chi_\beta(\exp i\xi) = e^{-\beta(\xi)}. \tag{4.2.4}$$

Now note that by (4.2.3),

$$\begin{aligned}\psi(z) &= -|z|^2 + \alpha(v) + \log|z^m|^2 - \log(\prod \chi_{\alpha_i}(g)^{2m_i})|z^m|^2 \\ &= -|z|^2 + \alpha(v) - \log \prod \chi_{\alpha_i(g)^{2m_i}}.\end{aligned}$$

But $z^m = \pi^* s_m$ for $s_m \in \Gamma_{hol}(\mathbb{L}_\alpha)$ if and only if m is in Δ_α, i.e. $\sum m_i\alpha_i = \alpha$, so we get finally by (4.2.4), $\prod \chi_{\alpha_i}(g)^{m_i} = \chi_\alpha(g)$ and

$$\psi(z) = -|z|^2 + \alpha(v) - 2\log \chi_\alpha(g). \tag{4.2.5}$$

Recall now that the map

$$\Phi^{-1}(\alpha) \times \exp(\sqrt{-1}\mathfrak{g}) \to \mathbb{C}^d_{st}$$

is bijective, so the inverse of this map followed by projection onto $\exp(\sqrt{-1}\mathfrak{g})$ gives us a map

$$\gamma : \mathbb{C}^d_{st} \to \exp(\sqrt{-1}\mathfrak{g}), \tag{4.2.6}$$

and by the computation above we've proved

Theorem 4.2.1. *The stability function for M_α, viewed as a GIT quotient of $\mathbb{C}^d$, is*

$$\psi(z) = -|z|^2 + \alpha(v) - 2(\log \gamma^*\chi_\alpha)(z). \tag{4.2.7}$$

For example for $\mathbb{CP}^{n-1}$ itself with $\mathbb{C}^n_{st} = \mathbb{C}^n - \{0\}$ and $\alpha = 1$, $\gamma(z) = |z|$ and hence

$$\psi(z) = -|z|^2 + 1 + \log |z|^2. \tag{4.2.8}$$

The formula (4.2.7) is valid modulo the assumption that M_α can be obtained by reduction from $\mathbb{CP}^{d-1}$, i.e. modulo the assumption (4.2.2). Dropping this assumption we have to replace (4.2.7) by the slightly more complicated formula

$$\psi(z) = -|z|^2 + |\gamma(z)^{-1}z|^2 - 2(\log \gamma^* \chi_\alpha)(z). \tag{4.2.9}$$

4.2.2 Stability functions on canonical affines

We can make the formula (4.2.7) more explicit by restricting to the canonical affines, $\mathcal{U}_v$, of §4.1.3. For any vertex v of Δ it is easy to see that

$$\mathcal{U}_v = \mathbb{C}^d_{\Delta_v}/G_{\mathbb{C}},$$

where

$$\mathbb{C}^d_{\Delta_v} = \{z \in \mathbb{C}^d \mid I_z \supset I_v\} \tag{4.2.10}$$

is an open subset of $\mathbb{C}^d_{st}$. By relabelling we may assume $I_v = \{1, 2 \cdots, n\}$. Since the relabelling makes $\alpha_1, \cdots, \alpha_n \in \mathfrak{g}^*$ into a lattice basis of $\mathbb{Z}^*_G$, $\alpha_k = \sum c_{k,i}\alpha_i$ for $k > n$, where $c_{k,i}$ are integers. Let $f_1, \cdots, f_n$ be the dual basis of the group lattice, $\mathbb{Z}_G$, then the map

$$\mathbb{C}^n \to G_{\mathbb{C}}, \quad (w_1, \cdots, w_n) \mapsto w_1 f_1 + \cdots + w_n f_n \mod \mathbb{Z}_G \tag{4.2.11}$$

gives one an isomorphism of $G_{\mathbb{C}}$ with the complex torus $(\mathbb{C}^*)^n$ and in terms of this isomorphism the $G_{\mathbb{C}}$-action on $\mathbb{C}^d_{\Delta_v}$ is given by

$$(w_1, \cdots, w_n) \cdot z = \left(w_1 z_1, \cdots, w_n z_n, (\prod_{i=1}^n w_i^{c_{n+1,i}}) z_{n+1}, \cdots, (\prod_{i=1}^n w_i^{c_{d,i}}) z_d \right).$$

Now suppose $z \in \mathbb{C}^d_{\Delta_v}$. Then the system of equations obtained from (4.1.12) and (4.2.1),

$$r_i^2|z_i|^2 + \sum_{k=n+1}^{d} c_{k,i}(\prod_{j=1}^{n} r_j^{c_{k,i}})^2|z_k|^2 = a_i, \quad 1 \le i \le n,$$

has a unique solution, $g = (r_1(z), \cdots, r_n(z)) \in (\mathbb{R}^+)^n = \exp(\sqrt{-1}\mathfrak{g})$, i.e., the g in (4.2.1) is $(r_1, \cdots, r_n)$. Via the identification (4.2.10) the weight $\alpha \in \mathbb{Z}^*_G$ corresponds by (4.2.11) to the weight $(a_1, \cdots, a_n) \in \mathbb{Z}^n$ and by (4.2.7) and (4.2.9)

$$\psi|_{\mathbb{C}^d_{\Delta_v}} = -|z|^2 + \alpha(v) - 2\sum_i a_i \log r_i(z) \tag{4.2.12}$$

in the projective case and

$$\psi|_{\mathbb{C}^d_{\Delta_v}} = -|z|^2 + \sum_i r_i^2|z_i|^2 + \sum_{k>n}|(\prod_{i=1}^{n} r_i^{c_{k,i}})z_k|^2 - 2\sum_i a_i \log r_i. \tag{4.2.13}$$

in general.

4.2.3 Example: The stability function on the Hirzebruch surfaces

As an example, let's compute the stability function for Hirzebruch surfaces. Recall that the Hirzebruch surface H_n is the toric 4-manifold whose moment polytope is the polygon with vertices $(0,0), (0,1), (1,1), (n+1,0)$. By the Delzant construction, we see that H_n is in fact the toric manifold obtained from the $\mathbb{T}^2$-action on $\mathbb{C}^4$,

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot z = (e^{i\theta_1}z_1, e^{i\theta_2}z_2, e^{i\theta_1 - in\theta_2}z_3, e^{i\theta_2}z_4).$$

By the procedure above, we find the stability function

$$\psi(z) = -|z|^2 - a_1 \log r_1 - a_2 \log r_2 + a_1 + a_2 - n r_1^{2n} r_2^2 |z_3|^2,$$

where r_1, r_2 are the solution to the system of equations

$$r_1^2|z_1|^2 + r_1^{2n} r_2^2 |z_3|^2 = a_1,$$
$$r_2^2|z_2|^2 - n r_1^{2n} r_2^2 |z_3|^2 + r_1^2 |z_4|^2 = a_2.$$

4.3 Semiclassical behavior of the spectral measures of toric varieties

As we have mentioned, the stability theory derived in chapter 3 is particularly useful for toric varieties M_α, since the upstairs space is the complex space, $\mathbb{C}^d$, the Lie group G is abelian, its action on $\mathbb{C}^d$ is linear, and the G-invariant sections of $\mathbb{L}$ are just linear combinations of monomials. As a consequence, the expressions (1.2.7), (1.2.11), (1.2.16) etc. are relatively easy to compute.

For example, consider the spectral measure

$$\mu_N = \sum \langle s_{N,i}, s_{N,i} \rangle \mu_{red},$$

where $\{s_{N,i}\}$ is an orthonormal basis of $\Gamma_{hol}(\mathbb{L}^N)$, then by proposition 3.4.3,

$$\int_{M_\alpha} f \mu_N \sim \left(\frac{N}{\pi}\right)^{-m/2} \int_{\mathbb{C}^d} \frac{\pi^* f}{\pi^* V} \chi \sum \langle \pi^* s_{N,i}, \pi^* s_{N,i} \rangle \mu.$$

The right hand side has a very simple asymptotic expansion using the twisted Mellin transform. Together with the results of section 3.4.4 one gets an alternative proof of theorem 1.1 of [BGU07]:

Theorem 4.3.1. *There exists differential operators $P_i(x, D)$ of order $2i$ such that*

$$\mu_N(f) \sim \sum_i N^{d-m-i} \int P_i(x, D) f(x) dx, \quad N \to \infty.$$

In this way the coefficients of the downstairs spectral measure asymptotics can be computed explicitly by the coefficients of the asymptotic expansion of the invariant upstairs spectral measure asymptotics – the relation of the leading terms is given in theorem 3.4.5, and the other coefficients depend on the asymptotics of the Laplace integral (3.3.1) together with the value of the stability function near $\Phi^{-1}(0)$. Similarly theorem 1.2 of [BGU07] can be derived from the results of section 3.4.2 and upstairs analogues of these results in section 2.3.

We can also apply the same method to study the pointwise asymptotics of $\frac{\langle s_k, s_k\rangle(x)}{\|s_k\|^2_{L^2}}$. For any $c \in \mathrm{Int}\Delta$ let $F(x) = \sum c_i \log l_i - l$.

Lemma 4.3.2. *c is the unique critical point of F in the interior of Δ, and is a nondegenerate maximum.*

Proof. Since $F(x) \to -\infty$ as x tends to the boundary $\partial\Delta$, F has a maximum in interior of Δ. Since $l_i(c) = c_i$, it is obvious that c is the only critical point of F, thus the global maximum. $\square$

Applying the steepest descent method, one get

$$\int_\Delta e^{\lambda F} d\sigma = \tau_c \left(\frac{\lambda}{2\pi}\right)^{-m/2} e^{\lambda F(c)}(1 + O(1/\lambda)). \tag{4.3.1}$$

It follows that

$$\frac{e^{\lambda F}}{\int_\Delta e^{\lambda F} d\sigma} = \tau_c^{-1} \left(\frac{\lambda}{2\pi}\right)^{m/2} e^{\lambda(F - F(c))}(1 + O(1/\lambda)). \tag{4.3.2}$$

Now suppose $k \in \Delta$ is a rational point, $N \in \mathbb{N}$ such that Nk is an integer point. Applying the previous result to $s_{Nk} \in \Gamma_{hol}(\mathbb{L}_\alpha^N)$, we get

$$\frac{\langle s_{Nk}, s_{Nk}\rangle(x)}{\|s_{Nk}, s_{Nk}\|^2_{L^2}} = \tau_x^{-1} \left(\frac{N}{2\pi}\right)^{m/2} e^{-N(F(k) - F(x))}(1 + O(N^{-1})).$$

This is one of the main estimates in [STZ04]. It implies that $\frac{s_{Nk}(x)}{\|s_{Nk}\|}$ tends to a δ-section of $\mathbb{L}_\alpha^N$ concentrated on $\Phi_\alpha^{-1}(k)$ as $N \to \infty$.

4.4 Universal distribution laws on toric varieties

4.4.1 Non-rescaled distribution law for Bargmann space

We turn now to study the asymptotics of the probability distribution (3.4.7) on toric varieties. Let's begin with the upstairs story, where

$$\sigma_{N,k}([t,\infty)) = \mathrm{Vol}\{z \in CC^d \mid \langle s_k, s_k\rangle_N(z) \geq t\}. \tag{4.4.1}$$

Suppose $k = Na$ with $a \in \Delta$. We begin by observing that

$$\|z^k\|_N^2 = \int_{\mathbb{C}^d} \langle z^k, z^k\rangle_N dz d\bar{z} = \left(\frac{\pi}{N}\right)^d \prod_i \frac{(k_i)!}{N^{k_i}},$$

and hence

$$\langle s_k, s_k\rangle_N = \left(\frac{N}{\pi}\right)^d \frac{N^{|k|}}{k!} |z^k|^2 e^{-N|z|^2}. \tag{4.4.2}$$

We first assume that $k = (k_1, \cdots, k_d)$ with $k_i > 0$ for all $1 \leq i \leq d$, and observe that $\sigma_{N,k}([t,\infty))$ is the volume of the region in $\mathbb{C}^d$

$$|z^k|^2 e^{-N|z|^2} > \left(\frac{\pi}{N}\right)^d \frac{k!}{N^{|k|}} t,$$

or, with $a = \frac{k}{N}$, the region

$$|z^a|^2 e^{-|z|^2} > \left(\left(\frac{\pi}{N}\right)^d \frac{k!}{N^{|k|}} t\right)^{1/N}. \tag{4.4.3}$$

By Stirling's formula,

$$k_i! = \sqrt{2\pi k_i} \left(\frac{k_i}{e}\right)^{k_i} \left(1 + O(\frac{1}{N})\right),$$

so the right hand side of (4.4.3) is equal to

$$\lambda_N = \left(\pi^d N^{-d/2} t \prod_i (2\pi a_i)^{1/2}\right)^{1/N} \left(\frac{a}{e}\right)^a \left(1 + O(\frac{1}{N^2})\right).$$

Thus if we set $|z_i|^2 = r_i$ and let $f(r)$ be the function

$$f(r) = \sum_{i=1}^{d} (a_i \log r_i - r_i),$$

the inequality (4.4.3) becomes

$$f(r) \geq \log \lambda_N = \sum_i (a_i \log a_i - a_i) - \frac{d}{2N} \log N + \frac{\log t + \gamma}{N} + O(\frac{1}{N^2}), \qquad (4.4.4)$$

where

$$\gamma = \log \left(\pi^d \prod_i (2\pi a_i)^{1/2}\right). \qquad (4.4.5)$$

We now note that $f(r)$ has a unique maximum at $r = a$ and that in a neighborhood of this maximum,

$$f(r) = \sum_i \left(a_i \log a_i - a_i - \frac{1}{2a_i}(r_i - a_i)^2\right) + \cdots.$$

Hence for N large (ignoring terms in N of order $O(\frac{1}{N})$) (4.4.3) reduces to

$$(1 + O(|r - a|)) \sum_i \frac{1}{2a_i}(r_i - a_i)^2 \leq \frac{d}{2N} \log N + O(\frac{1}{N}),$$

or, since $r_i = |z_i|^2$,

$$(1 + O(|r - a|)) \sum_i \frac{1}{2a_i} \left(|z_i|^2 - a_i\right)^2 \leq \frac{d}{2N} \log N + O(\frac{1}{N}). \qquad (4.4.6)$$

To compute the volume of this set to the leading order, we first note that the volume of the ellipsoid

$$\sum_{i=1}^{d} \frac{1}{2a_i} x_i^2 \leq \varepsilon \tag{4.4.7}$$

in $\mathbb{R}^d$ is

$$\gamma_d \left(\prod 2a_i \right)^{1/2} \varepsilon^{d/2}, \tag{4.4.8}$$

where γ_d is the volume of the unit d-ball. Now consider the map

$$g : \mathbb{R}_+^d \to \mathbb{R}^d, \qquad s_i \mapsto x_i = s_i^2 - a_i.$$

The pre-image of the region (4.4.7) with respect to this map is the set

$$\sum \frac{1}{2a_i} (s_i^2 - a_i)^2 \leq \varepsilon. \tag{4.4.9}$$

If s is a point in this set, then $s_i = \sqrt{a_i} + O(\varepsilon^{1/4})$, so

$$\det(Dg(s)) = \prod_i (2s_i) = 2^d \prod_i \sqrt{a_i}(1 + O(\varepsilon^{1/4})),$$

and thus by (4.4.8) the volume of the region (4.4.9) is equal, modulo $O(\varepsilon^{1/4})$, to

$$\gamma_d \left(\frac{\varepsilon}{2} \right)^{d/2}. \tag{4.4.10}$$

Finally note that the region (4.4.6) is, with $\varepsilon = \frac{d}{2N} \log N$, the pre-image of the region (4.4.9) with respect to the torus fibration, $s_i = |z_i|$. Since each torus fiber has volume $\prod(2\pi s_i)$ and $s_i = \sqrt{a_i} + O(\varepsilon^{1/4})$, the total volume of the region (4.4.6) is equal modulo a factor of $1 + O(\varepsilon^{1/4})$ to

$$(2\pi)^d \gamma_d \left(\prod_i \frac{a_i \varepsilon}{2} \right)^{1/2}, \tag{4.4.11}$$

and hence by substituting $\frac{d\log N}{2N}$ for ε we arrive finally at the asymptotic formula

$$\sigma_{N,k}\left([t,\infty)\right) \sim \pi^d \gamma_d \prod_i \left(a_i \frac{d}{N}\log N\right)^{1/2}. \tag{4.4.12}$$

Remark 4.4.1. *More generally suppose $k = (k_1, \cdots, k_l, 0, \cdots, 0)$ with $k_i > 0$ for $1 \le i \le l$, then $\sigma_{N,k}([t,\infty))$ is, to its leading order, equal to the volume of the region*

$$\sum_{i=1}^{l} \frac{1}{2a_i}(|z_i|^2 - a_i)^2 + \sum_{i=l+1}^{d} |z_i|^2 \le \frac{d}{2N}\log N$$

To compute the volume of this set, we regard it as the pre-image of the l-torus fibration over the $2d-l$ dimensional ellipsoid

$$\sum_{i=1}^{l} \frac{1}{2a_i}(s_i^2 - a_i)^2 + \sum_{i=l+1}^{d} (x_i^2 + y_i^2) \le \frac{d}{2N}\log N,$$

and by the same argument as above, get

$$\sigma_{N,k}\left([t,\infty)\right) \sim 2^{l-d}\pi^l \gamma_{2d-l} \left(\frac{d\log N}{N}\right)^{d-\frac{l}{2}} \prod_i (a_i)^{1/2}. \tag{4.4.13}$$

4.4.2 Rescaled distribution laws for Bargmann space

For simplicity we assume all k_i's are positive. From (4.4.3), (4.4.4) and (4.4.5) we have

$$\sum_{i=1}^{d} \frac{1}{2a_i}\left(|z_i|^2 - a_i\right)^2 \le \varepsilon_N, \tag{4.4.14}$$

where

$$\varepsilon_N = \frac{d}{2N}\log N - \frac{\log t + \gamma}{N} + o\left(\frac{1}{N}\right). \tag{4.4.15}$$

Thus the t term gets absorbed in the $O(\frac{1}{N})$ and doesn't affect the leading asymptotics of $\sigma_N([t,\infty))$. However, we can remedy this problem by rescaling techniques.

The first choice of rescaling is to eliminate the leading term $\frac{d}{2N}\log N$. To do so, we replace t by $N^{d/2}t$. Then

$$\varepsilon_N = (-\log t - \gamma)\frac{1}{N} + o(\frac{1}{N}) \tag{4.4.16}$$

and the computations in the last section show that this rescaled version of $\sigma_{N,k}([t,\infty))$ satisfies (4.4.9) with $\varepsilon = \varepsilon_N$ given by (4.4.16) and hence depends in an interesting way on t. (One proviso, however, is that $\log t$ has to be smaller than $-\gamma$.)

There are also many other interesting choices of rescalings: we may rescale t such that the term containing $\log t$ dominate other terms. For example, we may replace t by $e^{-N^\alpha (\log N)^\beta t}$, where $0 < \alpha < 1$ or $\alpha = 0, \beta > 1$. In this case

$$\varepsilon_N = N^{\alpha-1}(\log N)^\beta t + O(\frac{\log N}{N}). \tag{4.4.17}$$

We may also replace t by N^{-t}, which is the extreme case $\alpha = 0, \beta = 1$ above, then

$$\varepsilon_N = \frac{d+2t}{N}\log N + O(\frac{1}{N}). \tag{4.4.18}$$

4.4.3 Universal rescaled law on toric varieties

In this section we suppose $\beta \in \Delta_\alpha$ is rational, and N is large with $N\beta \in \mathbb{Z}^d$. One of the main results in [STZ04] is the following universal rescaled law for the probability distribution function (4.4.1) on toric varieties,

$$\lim_{N\to\infty} (\frac{N}{\pi})^{n/2}\sigma_{N,N\beta}((\frac{N}{\pi})^{n/2}t) = \frac{(\log c/t)^{n/2}}{c\Gamma(n/2+1)}. \tag{4.4.19}$$

By measure theoretic arguments, they deduce this from moment estimates, (c.f. §4.1 of [STZ04])

$$\int_{M_\alpha} x^l d\nu_N \to \frac{c^{l-1}}{l^{n/2}}, \qquad N \to \infty, \tag{4.4.20}$$

where l is any positive integer, ν_N is the push-forward measure

$$\nu_N = \left(\left|(\frac{N}{\pi})^{-n/4}\phi_{N\beta}\right|^2\right)_* \left((\frac{N}{\pi})^{n/2}\nu\right),$$

with $\phi_{N\beta} = s_{N\beta}/\|s_{N\beta}\|$ and ν the pullback of the Fubini-Study form via a projective embedding. By a simple computation it is easy to see that

$$\int x^l d\nu_N(x) = \left(\frac{N}{\pi}\right)^{-\frac{n(l-1)}{2}} \int_{M_\alpha} |\phi_{N\beta}|^{2l}\nu = \left(\frac{N}{\pi}\right)^{-\frac{n(l-1)}{2}} m_\alpha(l, \phi_{N\beta}, \nu). \qquad (4.4.21)$$

The upstairs analogue of (4.4.20) for toric varieties is rather easy to prove:

Lemma 4.4.2. *For any l, the l^{th} moments*

$$\left(\frac{N}{\pi}\right)^{-d(l-1)/2} m(l, \frac{z^{N\beta}}{\|z^{N\beta}\|}, d\mu) \to \frac{c^{l-1}}{l^{d/2}} \qquad (N \to \infty). \qquad (4.4.22)$$

Proof. Direct computation. □

Thus we can apply proposition 3.4.4 to derive (4.4.20) from (4.4.22). By (3.4.8) and (4.4.8),

$$m_\alpha(l, \frac{s_{N\beta}}{\|s_{N\beta}\|}, \frac{\omega_\alpha^n}{n!}) \sim l^{-m/2}\left(\frac{N}{\pi}\right)^{m(l-1)/2} m(l, \frac{z^{N\beta}}{\|z^{N\beta}\|}, \frac{\omega^d}{d!}).$$

Thus

$$\left(\frac{N}{\pi}\right)^{-n(l-1)/2} m_\alpha(l, \frac{s_{N\beta}}{\|s_{N\beta}\|}, \frac{\omega_\alpha^n}{n!}) \to \frac{c^{l-1}}{l^{n/2}}$$

as $N \to \infty$ for all l. This together with the measure theoretic arguments alluded to above implies the distribution law (4.4.19) for the volume form $V\mu_\alpha$ on M_α.

Remark 4.4.3. *Here we only consider the case when β is an interior point of the Delzant polytope, which corresponds to the case $r = 0$ in [STZ04]. However, one can modify the arguments above slightly to show the same result for general r and $N\beta$ replaced by $N\beta + o(1)$.*

Chapter 5

The stability functions on some non-toric varieties

In this chapter we make a tentative first step toward generalizing the results of chapter 4 to the non-abelian analogues of toric varieties: *spherical varieties.* The simplest examples of spherical varieties are the coadjoint orbits of the unitary group $\mathcal{U}(n)$ viewed as $\mathcal{U}(n-1)$-manifolds. It is well known that the coadjoint orbits of $\mathcal{U}(n)$ can be identified with the sets of isospectral Hermitian matrices $\mathcal{H}(\lambda) \subset \mathcal{H}(n)$, i.e., Hermitian matrices with fixed eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Suppose there is only one strict inequality $\lambda_k > \lambda_{k+1}$ while the others are all equal, then if $k = 1$ or $n-1$ $\mathcal{H}(\lambda)$ $\mathbb{CP}^{n-1}$ (which is toric), and if $1 < k < n-1$ it is $Gr(k, \mathbb{C}^n)$ (which is non-toric). Following Shaun Martin, we will show how these varieties can be obtained by symplectic reduction from a linear action of a compact Lie group on $\mathbb{C}^d$, and compute their stability functions. More generally, we will study the stability functions for several classes of quiver varieties, e.g., polygon spaces.

5.1 The stability function on Grassmannians: an illustrative example

5.1.1 GIT for Grassmannians

Suppose $k < n$. It is well known that the complex Grassmannian $Gr(k, \mathbb{C}^n)$ can be realized as the quotient space of $\mathbb{C}^{kn}$ by symplectic reduction or as a GIT quotient as follows:

Let $M = \mathfrak{M}_{k,n}(\mathbb{C}) \simeq \mathbb{C}^{kn}$ be the space of complex $k \times n$ matrices. We equip $\mathbb{C}^{kn}$ with its standard Kähler metric, the standard trivial line bundle $\mathbb{C} \times \mathbb{C}^{kn} \to \mathbb{C}^{kn}$, and the standard Hermitian inner product on this line bundle,

$$\langle 1, 1 \rangle(Z) = e^{-\operatorname{Tr} ZZ^*}. \tag{5.1.1}$$

Now let $G = \mathcal{U}(k)$ act on $\mathfrak{M}_{k,n}$ by left multiplication. This action preserves the inner product (5.1.1), and thus preserves the Kähler form $\sqrt{-1}\partial\bar{\partial} \operatorname{Tr} ZZ^*$. It is not hard to see that it is a Hamiltonian action with moment map

$$\Phi : \mathfrak{M}_{k,n} \to \mathcal{H}_k, \qquad Z \mapsto ZZ^*, \tag{5.1.2}$$

where $\mathcal{H}_k$ is the space of $k \times k$ Hermitian matrices. Here we identify $\mathcal{H}_k$ with $\sqrt{-1}\mathcal{H}_k = Lie(\mathcal{U}(k))$, and identify $\mathcal{H}_k$ with $Lie(\mathcal{U}(k))^* = \mathcal{H}_k^*$ via the Killing form. Notice that the identity matrix I lies in the annihilator of the commutator ideal,

$$[\mathcal{H}_k, \mathcal{H}_k]^0 = \{a \in \mathcal{H}_k^* \mid \langle [h_1, h_2], a \rangle = 0 \text{ for all } h_1, h_2 \in \mathcal{H}_k\},$$

so $\Phi - I$ is also a moment map, and it's clear that the reduced space

$$M_{red} = \Phi^{-1}(I)/G$$

is the Grassmannian $Gr(k, \mathbb{C}^n)$.

On the other hand, the complexification of $\mathcal{U}(k)$ is $GL(k,\mathbb{C})$, and it's not hard to see that the set of stable points, M_{st}, is exactly the set of $k\times n$ matrices $A \in M$ which have rank k, and that the quotient $M_{st}/GL(k,\mathbb{C})$ is again $Gr(k,\mathbb{C}^n)$. This gives us the GIT description of $Gr(k,\mathbb{C}^n)$.

As for the reduced line bundle, $\mathbb{L}_{red}$, on M_{red}, this is obtained from the trivial line bundle on M_{st} by "shifting" the action of $GL(k,\mathbb{C})$ on the trivial line bundle in conformity with the shifting, "$\Phi \Rightarrow \Phi - I$", of the moment map, i.e. by letting $GL(k,\mathbb{C})$ act on this bundle by the character

$$\gamma : GL(k,\mathbb{C}) \to \mathbb{C}^*, \gamma(A) = \det(A).$$

5.1.2 The stability function on the Grassmannians $Gr(k,\mathbb{C}^n)$

To compute this stability function, we first look for the G-invariant sections of the twisted line bundle. For any index set

$$J = \{j_1, \cdots, j_k\} \subset \{1,2,\cdots,n\}$$

denote by $Z_J = Z_{j_1,\cdots,j_k}$ the $k\times k$ sub-matrix consisting of the $j_1,\cdots,j_k$ columns of Z.

Lemma 5.1.1. *The functions*

$$s_J(Z) = \det(Z_J)$$

are G-invariant sections of the trivial line bundle on $\mathfrak{M}_{k,n}$ for the twisted G-action.

Proof. Let H be any $n\times n$ Hermitian matrix, and v_H the generating vector field of the one-parameter subgroup generated by H. Then by Kostant's identity (3.1.7) one only needs to show

$$\iota_{v_H}\partial \log\langle s_J, s_J\rangle = -\sqrt{-1}\,\mathrm{Tr}\left((ZZ^* - I)H\right).$$

This follows from direct computation:

$$\begin{aligned}
\iota_{v_H}\partial\log\langle s_J, s_J\rangle &= \iota_{v_H}\partial(-\operatorname{Tr} ZZ^* + \log\det(Z_J\bar{Z}_J)) \\
&= -\operatorname{Tr}((\iota_{v_H}dZ)Z^*) + \iota_{v_H}\partial\operatorname{Tr}\log(Z_J Z_J^*) \\
&= -\operatorname{Tr}((\iota_{v_H}dZ)Z^*) + \operatorname{Tr}((\iota_{v_H}dZ_J)Z_J^*(Z_J^*)^{-1}Z_J^{-1}) \\
&= -\sqrt{-1}\operatorname{Tr}(H(ZZ^*-I)),
\end{aligned}$$

completing the proof. □

Now we are ready to compute the stability function for the Grassmannians. Without loss of generality, we suppose

$$\{j_1,\cdots,j_k\} = \{1,\cdots,k\}.$$

For any rank k matrix $Z \in M_{st}$, let $B \in GL(k,\mathbb{C})$ be a nonsingular matrix with $BZ \in \Phi^{-1}(I)$. Thus the stability function at point Z is

$$\begin{aligned}
\psi(Z) &= \log\left(|\det(Z_{1,\cdots,k})|^2 e^{-\operatorname{Tr} ZZ^*}\right) - \log\left(|\det((BZ)_{1,\cdots,k})|^2 e^{-\operatorname{Tr} I}\right) \\
&= k - \operatorname{Tr}(ZZ^*) - \log|\det B|^2
\end{aligned}$$

Since $B^*B = (Z^*)^{-1}Z^{-1}$, we conclude

$$\psi(Z) = k - \operatorname{Tr}(ZZ^*) + \log\det(ZZ^*). \tag{5.1.3}$$

Similarly, if we do reduction at mI instead of I, or alternately, use the moment map $\Phi - mI$, then the invariant sections are given by

$$s_J(Z) = \det(Z_J)^m,$$

and the stability function is

$$\psi(Z) = km - \operatorname{Tr}(ZZ^*) + m^2\log\det(ZZ^*).$$

5.2 The stability functions on coadjoint orbits of $U(n)$

5.2.1 Martin's reduction procedure

For general coadjoint orbit of $\mathcal{U}(n)$, Shaun Martin showed that there is an analogous GIT description. Since he never published this result, we will roughly outline his argument here, focusing for simplicity on the case $\lambda_1 > \cdots > \lambda_n$.

Let

$$M = \mathfrak{M}_{1,2}(\mathbb{C}) \times \mathfrak{M}_{2,3}(\mathbb{C}) \times \cdots \times \mathfrak{M}_{n-1,n}(\mathbb{C}).$$

Then each component of M is a linear symplectic space, and M is just the linear symplectic space $\mathbb{C}^{(n-1)n(n+1)/3}$ with standard Kähler form $\omega = -\sqrt{-1}\partial\bar{\partial}\log\rho$, where ρ is the potential function

$$\rho(Z) = \exp(-\sum_{i=1}^{n-1} \operatorname{Tr} Z_i Z_i^*).$$

Consider the group

$$G = \mathcal{U}(1) \times \mathcal{U}(2) \times \cdots \times \mathcal{U}(n-1)$$

acting on M by the recipe:

$$\tau_{(U_1,\cdots,U_{n-1})}(Z_1,\cdots,Z_{n-1}) = (U_1 Z_1 U_2^*, \cdots, U_{n-2}Z_{n-2}U_{n-1}^*, U_{n-1}Z_{n-1}). \tag{5.2.1}$$

Lemma 5.2.1. *The action above is Hamiltonian with moment map*

$$\Phi(Z_1,\cdots,Z_{n-1}) = (Z_1Z_1^*, Z_2Z_2^* - Z_1^*Z_1, \cdots, Z_{n-1}Z_{n-1}^* - Z_{n-2}^*Z_{n-2}). \tag{5.2.2}$$

Proof. Given any $H = (H_1,\cdots,H_{n-1}) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_{n-1}$, denote by $\mathcal{U}_H(t)$ the one parameter subgroup of G generated by H, i.e.,

$$\mathcal{U}_H(t)Z = \big(\exp(\sqrt{-1}tH_1)Z_1\exp(-\sqrt{-1}tH_2), \cdots, \\ \exp(\sqrt{-1}tH_{n-2})Z_{n-2}\exp(-\sqrt{-1}tH_{n-1}), \exp(\sqrt{-1}tH_{n-1})Z_{n-1}\big).$$

Let v_H be the infinitesimal generator of this group, then

$$\iota_{v_H}(\sqrt{-1}\partial\log\rho) = -\sqrt{-1}\sum \mathrm{Tr}((\iota_{v_H}dZ_i)Z_i^*).$$

Since

$$\iota_{v_H}dZ_i = \frac{d}{dt}(\exp(\sqrt{-1}tH_i)Z_i\exp(-\sqrt{-1}tH_{i+1}))\Big|_{t=0} = \sqrt{-1}(H_iZ_i - Z_iH_{i+1}),$$

we see that

$$\iota_{v_H}(\sqrt{-1}\partial\log\rho) = \sum \mathrm{Tr}(H_iZ_iZ_i^* - H_{i+1}Z_i^*Z_i) = \langle\Phi(Z), H\rangle.$$

This shows that (5.2.2) is a moment map of τ. □

Given $a = (a_1, \cdots, a_n) \in \mathbb{R}_+^n$, let

$$\phi^{-1}(aI) = \Phi^{-1}(a_1I_1, \cdots, a_{n-1}I_{n-1}),$$

and let

$$M_a = \Phi^{-1}(aI)/G$$

be the reduced space at level $(a_1I_1, \cdots, a_{n-1}I_{n-1}) \in [\mathfrak{g}, \mathfrak{g}]^0$. Consider the residual action of $GL(n, \mathbb{C})$ on M,

$$\kappa : GL(n, \mathbb{C}) \times M \to M, \quad \kappa_A Z = (Z_1, \cdots, Z_{n-2}, Z_{n-1}A^{-1}). \tag{5.2.3}$$

Then the actions κ and τ commute, and by the same argument as above we see that

$\kappa|_{\mathcal{U}(n)}$ is a Hamiltonian action with a moment map

$$\Psi : M \to \mathcal{H}_n, \quad \Psi(Z) = Z_{n-1}^* Z_{n-1} + a_n I_n. \tag{5.2.4}$$

We thus get a Hamiltonian action of $\mathcal{U}(n)$ on the reduced space M_a with moment map $\Psi_a : M_a \to \mathcal{H}_n$, which satisfies

$$\Psi \circ i = \Psi_a \circ \pi_0,$$

where, as usual, $i : \Phi^{-1}(aI) \hookrightarrow M$ is the inclusion map and $\pi_0 : \Phi^{-1}(aI) \to M_a$ the projection.

Theorem 5.2.2 ([Mar98]). Ψ_a *is a* $\mathcal{U}(n)$*-equivariant symplectomorphism of* M_a *onto* $\mathcal{H}(\lambda)$*, with* $\lambda_i = \sum_{j=i}^n a_j$.

Proof. First we prove that Ψ_a maps M_a onto the isospectral set $\mathcal{H}(\lambda)$. In view of the relation $\Psi \circ i = \Psi_a \circ \pi_0$, we only need to show Image$(\Psi) = \mathcal{H}(\lambda)$. In fact, if $Z_i Z_i^*$ has eigenvalues $(\mu_1, \cdots, \mu_i)$, then the eigenvalues of $Z_i Z_i^*$ are exactly $(\mu_1, \cdots, \mu_i, 0)$, so it is straightforward to see that $Z_2 Z_2^* = Z_1 Z_1^* + a_2 I_2$ has eigenvalues $a_1 + a_2, a_2$, and in general $Z_i Z_i^*$ has eigenvalues

$$a_1 + \cdots + a_i, a_2 + \cdots + a_i, \cdots, a_i.$$

This proves that Ψ_a maps M_a into $\mathcal{H}(\lambda)$, and since G acts transitively on $\mathcal{H}(\lambda)$, this map is onto.

Next note that by dimension-counting $\dim M_a = \dim \mathcal{H}(\lambda)$, so Ψ_a is a finite-to-one covering. Since the adjoint orbits of $\mathcal{U}(n)$ are simply-connected, we conclude that this map is also injective [1], and thus a diffeomorphism.

[1] This fact can also be proved using tools from elementary matrix theory, e.g., the singular value decomposition.

Since Ψ_a is a moment map, it is a Poisson mapping between M_a and $\mathcal{H}(n)$, i.e.,

$$\{f \circ \Psi_a, g \circ \Psi_a\}_{M_a} = \{f, g\}_{\mathcal{H}(\lambda)} \circ \Psi_a$$

for any $f, g \in C^\infty(\mathcal{H}(\lambda))$. Thus Ψ_a is a symplectomorphism between M_a and $\mathcal{H}(\lambda)$.

Finally the $\mathcal{U}(n)$-equivariance comes from the fact that

$$\Psi(U \cdot Z) = (U^{-1})^* Z_{n-1}^* Z_{n-1} U^{-1} + a_n I_n = U(Z_{n-1}^* Z_{n-1} + a_n I_n)U^{-1} = U \cdot \Psi(Z).$$

This completes the proof. □

The GIT description of this reduction procedure is now clear:

$$Z = (Z_1, \cdots, Z_{n-1}) \in M_{st}$$

if and only if Z_i is of rank i for all i, and

$$M_a = M_{st}/G_{\mathbb{C}}$$

with $G_{\mathbb{C}}$ the product

$$G_{\mathbb{C}} = GL(1, \mathbb{C}) \times \cdots \times GL(n-1, \mathbb{C})$$

whose action is compatible with (5.2.1).

5.2.2 Twisted line bundles over $\mathcal{U}(n)-$coadjoint orbits

As in the toric case, reduction at level zero of the moment map (5.2.4) is not very interesting, since the reduced line bundle is the trivial line bundle. To get the Grassmannian, we shifted the moment map by the identity matrix. Equivalently, we "twisted" the action of $GL(k, \mathbb{C})$ on the trivial line bundle $\mathbb{C} \times \mathbb{C}^{kn}$ by a character of $GL(k, \mathbb{C})$. It is to this shifted moment map/twisted action that we applied the reduction procedure to obtain a reduced line bundle on $Gr(k, \mathbb{C}^n)$.

Similarly, for $\mathcal{U}(n)$-coadjoint orbits we will twist the $G_{\mathbb{C}}$ action on the trivial line bundle over M by characters of $G_{\mathbb{C}}$. Every character of $G_{\mathbb{C}}$ is of the form

$$\gamma = \gamma_1^{m_1} \cdots \gamma_{n-1}^{m_{n-1}}, \tag{5.2.5}$$

where $\gamma_k(A) = \det(A_k)$ for $A = (A_1, \cdots, A_{n-1})$. Let

$$\pi_k : M \to \mathfrak{M}_{k,n}, \quad (Z_1, \cdots, Z_{n-1}) \to Z_k Z_{k+1} \cdots Z_{n-1}.$$

Then π_k intertwines the action of $G_{\mathbb{C}}$ on M with the standard left action of $\mathcal{U}(k)$ on $\mathfrak{M}_{k,n}$, and intertwines the action κ of $\mathcal{U}(n)$ on M with the standard right action of $\mathcal{U}(n)$ on $\mathfrak{M}_{k,n}$. Let $\mathbb{L}_k$ be the holomorphic line bundle on $\mathfrak{M}_{k,n}$ associated with the character

$$\gamma_k : GL(k, \mathbb{C}) \to \mathbb{C}^*, \quad A \mapsto \det(A). \tag{5.2.6}$$

Then the bundle $\pi_k^* \mathbb{L}_k$ is the holomorphic line bundle on M associated with γ_k and

$$\mathbb{L} := \bigotimes_{k=1}^{n-1} (\pi_k^* \mathbb{L}_k)^{m_k} \tag{5.2.7}$$

is the holomorphic line bundle associated with the character γ. In particular if s_k is a $GL(k, \mathbb{C})$-invariant holomorphic section of $\mathbb{L}_k$, then

$$(\pi_1^* s_1)^{m_1} \cdots (\pi_{n-1}^* s_{n-1})^{m_{n-1}} \tag{5.2.8}$$

is a $G_{\mathbb{C}}$-invariant holomorphic section of $\mathbb{L}$, and all $G_{\mathbb{C}}$-invariant holomorphic sections of $\mathbb{L}$ are linear combinations of these sections. Since the representation of $GL(n, \mathbb{C})$ on the space $\Gamma_{hol}(\mathbb{L}_k)$ is its k-th elementary representation we conclude

Theorem 5.2.3. *The representation of $GL(n, \mathbb{C})$ on the space $\Gamma_{hol}(\mathbb{L})$ is the irreducible representation with highest weight $\sum_{i=1}^{n-1} m_i \alpha_i$, where $\alpha_1, \cdots, \alpha_{n-1}$ are the* *simple roots of* $GL(n, \mathbb{C})$.

For the canonical trivializing section of $\mathbb{L}$ its Hermitian inner product with itself is

$$\prod_{i=1}^{n-1} \det(Z_i Z_{i+1} \cdots Z_{n-1} Z_{n-1}^* \cdots Z_i^*)^{-m_i}$$

and hence the potential function for the $\mathbb{L}$-twisted Kähler structure on M is

$$\rho_{\mathbb{L}} = \sum_{i=1}^{n-1} \operatorname{Tr} Z_i Z_i^* - m_i \log \det(Z_i \cdots Z_{n-1} Z_{n-1}^* \cdots Z_i^*) \qquad (5.2.9)$$

and the corresponding $\mathbb{L}$-twisted moment map is

$$\Phi_{\mathbb{L}}(Z_1, \cdots, Z_{n-1}) = (Z_1 Z_1^* - m_1 I_1, \cdots, Z_{n-1} Z_{n-1}^* - m_{n-1} I_{n-1}). \qquad (5.2.10)$$

5.2.3 The stability functions on $\mathcal{U}(n)$-coadjoint orbits

These stability functions are computed in more or less the same way as above. By the same arguments as in the proof of lemma 5.2.1, one can see that

$$s(Z_1, \cdots, Z_{n-1}) = \prod (\det(Z_i)_{1,\cdots,i})^{m_i - m_{i-1}} \qquad (5.2.11)$$

is G-invariant for the moment map $\Phi - (m_1 I_1, \cdots, m_{n-1} I_{n-1})$.

Now suppose $(Z_1, \cdots, Z_{n-1}) \in M_{st}$, then there are $B_i \in GL(i, \mathbb{C})$ such that

$$B_1 Z_1 Z_1^* B_1^* = m_1 I_1 \qquad (5.2.12)$$

and

$$B_i Z_i Z_i^* B_i^* = Z_{i-1}^* B_{i-1}^* B_{i-1} Z_{i-1} + m_i I_i, \qquad 2 \leq i \leq n-1. \qquad (5.2.13)$$

From (5.2.12) we have

$$\det(B_i B_1^*) = m_1 \det(Z_1 Z_1^*)^{-1},$$

and from this and (5.2.13) we conclude

$$\begin{aligned}\det(B_i Z_i Z_i^* B_i^*) &= \det(m_i I_i + B_{i-1} Z_{i-1} Z_{i-1}^* B_{i-1}^*)\\ &= \det((m_i + m_{i-1}) I_{i-1} + B_{i-2} Z_{i-2} Z_{i-2}^* B_{i-2}^*)\\ &= m_1 + \cdots + m_i.\end{aligned}$$

So we get for all i,

$$\det(B_i B_i^*) = (m_1 + \cdots + m_i) \det(Z_i Z_i^*)^{-1}.$$

Now it is easy to compute

$$\begin{aligned}\psi(Z) = \log &\left(e^{-\sum \operatorname{Tr}(Z_i Z_i^*)} \prod |\det(Z_i)_{1,\cdots,i}|^{2m_i - 2m_{i-1}} \right)\\ &- \log \left(e^{-\sum i m_i} \prod |\det(B_i Z_i)_{1,\cdots,i}|^{2m_i - 2m_{i-1}} \right)\\ = &\sum i m_i - \sum \operatorname{Tr}(Z_i Z_i^*) - \sum (m_i - m_{i-1}) \log |\det B_i|^2\\ = &\sum i m_i - \sum \operatorname{Tr}(Z_i Z_i^*) + \sum (m_i - m_{i-1})(m_1 + \cdots + m_i) \log \det(Z_i Z_i^*).\end{aligned}$$

Remark 5.2.4. *Although we only carry out the computations for generic $\mathcal{U}(n)$-coadjoint orbits, i.e., for the isospectral sets with*

$$\lambda_1 < \cdots < \lambda_n,$$

the same argument apply to all $\mathcal{U}(n)$-coadjoint orbits. In fact, for the isospectral set with $\lambda_1 < \cdots < \lambda_r$ whose multiplicities are $i_1, \cdots, i_r$, we can take the upstairs space to be

$$\mathfrak{M}_{i_1 \times (i_1 + i_2)} \times \mathfrak{M}_{(i_1 + i_2) \times (i_1 + i_2 + i_3)} \times \mathfrak{M}_{(n - i_r) \times n}$$

and obtain results for these degenerate coadjoint orbits completely analogous to those above.

5.3 The stability functions on quiver varieties

5.3.1 Quiver Varieties

Let's first recall some notations from quiver algebra theory. A *quiver* Q is an oriented graph (I, E), where $I = \{1, 2, \cdots, n\}$ is the set of vertices, and $E \subset I \times I$ the set of edges. A representation, V, of a quiver assigns a Hermitian vector space V_i to each vertex i of the quiver and a linear map $Z_{ij} \in \mathrm{Hom}(V_i, V_j)$ to each edge $(i, j) \in E$. The dimension vector of the quiver representation V is the vector $l = (l_1, \cdots, l_n)$, where $l_i = \dim V_i$. Thus the space of representations of Q with underlying vector spaces V fixed is the complex space

$$M = \mathrm{Hom}(V) := \bigoplus_{(i,j)\in E} \mathrm{Hom}(V_i, V_j). \tag{5.3.1}$$

We equip M with its standard symplectic form and consider the unitary group

$$U(V) = U(V_1) \times \cdots \times U(V_n)$$

acting on M by

$$(u_1, \cdots, u_n) \cdot (Z_{ij}) = (u_j Z_{ij} u_i^{-1}). \tag{5.3.2}$$

The isomorphism classes of representations of Q of dimension l is in bijection with the $GL(V)$-orbits on $\mathrm{Hom}(V)$. Geometrically this quotient space can have bad singularities, and to avoid this problem, one replaces this quotient by its GIT quotient, or equivalently, the Kähler quotient of $\mathrm{Hom}(V)$ by the $U(V)$-action. These quotients are what one calls *quiver varieties*.

Proposition 5.3.1. *The action (5.3.2) is Hamiltonian with moment map*

$$\mu : Hom(V) \to \mathfrak{g}^*,$$

$$\mu(Z_{ij}) = \left(\sum_{(j,1)\in E} Z_{j1}Z_{j1}^* - \sum_{(1,j)\in E} Z_{1j}^*Z_{1j}, \cdots, \sum_{(j,n)\in E} Z_{jn}Z_{jn}^* - \sum_{(n,j)\in E} Z_{nj}^*Z_{nj} \right). \tag{5.3.3}$$

The proof involves the same computation as in lemma 5.2.1, so we will omit it.

Notice that by (5.3.2) the circle group $\{(e^{i\theta}I_{l_1}, \cdots, e^{i\theta}I_{l_n})\}$ act trivially on M, so we get an induced action of the quotient group $G = U(V)/S^1$. The Lie algebra of G is given by

$$\{(H_1, \cdots, H_n) \mid H_i \text{ Hermitian }, \sum \operatorname{Tr} H_i = 0\}$$

and this G-action also has μ as its moment map. Letting $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ with

$$l_1\lambda_1 + \cdots + l_n\lambda_n = 0,$$

and supposing that the G-action is free on $\mu^{-1}(\lambda I)$, the *quiver variety* associated to λ is by definition the quotient

$$R_\lambda(l) = \mu^{-1}(\lambda I)/G,$$

where $\lambda I = (\lambda_1 I_{l_1}, \cdots, \lambda_n I_{l_n})$.

We can also modify the definition of quiver varieties to get an effective $U(V)$-action. Namely, we attach to Q another collection of Hermitian vector spaces (the "frame"), $\tilde{V} = (\tilde{V}_1, \cdots, \tilde{V}_n)$, with dimension vector $\tilde{l} = (\tilde{l}_1, \cdots, \tilde{l}_n)$, and redefine the space M to be

$$\operatorname{Hom}(V, \tilde{V}) := \bigoplus_{(i,j)\in E} \operatorname{Hom}(V_i, V_j) \oplus \bigoplus_{i\in I} \operatorname{Hom}(V_i, \tilde{V}_i).$$

The group $U(V)$ acts on $\mathrm{Hom}(V, \tilde{V})$ by

$$(u_1, \cdots, u_n) \cdot (Z_{ij}, Y_i) = (u_j Z_{ij} u_i^{-1}, Y_i u_i^{-1}).$$

As above the $U(V)$-action is Hamiltonian, and the k^{th} component of its moment map is

$$(\mu(Z_{ij}, Y_i))_k = \sum_{(j,k)\in E} Z_{jk} Z_{jk}^* - \sum_{(k,j)\in E} Z_{kj}^* Z_{kj} - Y_k^* Y_k.$$

Now the center S^1 acts nontrivially on $\mathrm{Hom}(V, \tilde{V})$ providing that the "frames" $\tilde{V}_i$ are not all zero, and we define the *framed quiver variety* $R_\lambda(l, \tilde{l})$ to be the Kähler quotient of $\mathrm{Hom}(V, \tilde{V})$ by the $U(V)$-action above at the level $\lambda = (\lambda_1 I_{l_1}, \cdots, \lambda_n I_{l_n})$. As examples, the Grassmannian and the coadjoint orbit of $\mathcal{U}(n)$ that we considered in the previous section are just the framed quiver varieties whose underlying quivers are depicted below:

$$\begin{tikzcd} \mathbb{C}^k \arrow[d] \\ \mathbb{C}^n \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}^1 \arrow[r] & \mathbb{C}^2 \arrow[r] & \cdots \arrow[r] & \mathbb{C}^{n-2} \arrow[r] & \mathbb{C}^{n-1} \arrow[d] \\ & & & & \mathbb{C}^n \end{tikzcd}$$

5.3.2 Stability functions

We equip M with the trivial line bundle and, for actions of $\mathcal{U}(V)$ associated with characters $\prod(\det A_i)^{\lambda_i}$, describe the invariant sections.

Proposition 5.3.2. *For fixed $\lambda \in \mathbb{Z}^n$, the sections*

$$s(Z_{ij}) = \prod_{(i,j)\in E} \det((Z_{ij})_J)^{\nu_{ij}} \tag{5.3.4}$$

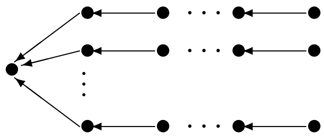
are invariant sections with respect to the moment map $\mu - \lambda I$, where ν_{ij} are integers satisfying

$$\sum_j \nu_{ji} - \sum_j \nu_{ij} = \lambda_i. \tag{5.3.5}$$

The proof is essentially the same proof as that of Lemma 5.2.1.

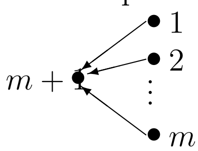
From now on we will require that the quiver, Q, be noncyclic, otherwise there will be infinitely many G-invariant sections. (Moreover, in the cyclic case the quiver variety is not compact.) For a general quiver variety whose underlying quiver is noncyclic, we can, in principle, compute the stability function, using the G-invariant sections above, as we did for toric varieties in section 4.2; but in practice the computation can be quite complicated.

However, in the special case that the quiver is a star quiver, i.e., is of the following shape:



one can write down the stability functions fairly explicitly: on each "arm", we just apply the same technique we used for the coadjoint orbits of $\mathcal{U}(n)$.

As an example, we'll compute the stability function for *polygon space*. This is by definition a quiver variety whose underlying quiver is the oriented graph



and for which the V_i's satisfy $\dim V_i = 1$ for $1 \le i \le m$ and $\dim V_{m+1} = 2$. Thus

$$\mathrm{Hom}(V) = \bigoplus \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) = (\mathbb{C}^2)^m \tag{5.3.6}$$

and

$$G = (S^1)^m \times U(2)/S^1 \simeq (S^1)^m \times SO(3). \tag{5.3.7}$$

The moment map for this data is

$$(Z_1, \cdots, Z_m) \mapsto (-|Z_1|^2, \cdots, -|Z_m|^2, Z_1 Z_1^* + \cdots + Z_m Z_m^*), \tag{5.3.8}$$

where $Z_i = (x_i, y_i) \in \mathbb{C}^2$.

Now consider the quiver variety $\mu^{-1}(\lambda I)/G$, with $\lambda = (\lambda_1, \cdots, \lambda_m, \lambda_{m+1})$ satisfying

$$\lambda_1 + \cdots + \lambda_m + 2\lambda_{m+1} = 0$$

and $\lambda_i < 0$ for $1 \leq i \leq m$. Let's explain why this variety is called "polygon space". The $(S^1)^m$-action on $(\mathbb{C}^2)^m$ is the standard action, so reducing at level $(\lambda_1, \cdots, \lambda_m)$ gives us a product of spheres $S^2_{-\lambda_1} \times \cdots \times S^2_{-\lambda_m}$ of radii $-\lambda_1, \cdots, -\lambda_m$. So we can think of an element of $S^2_{-\lambda_1} \times \cdots \times S^2_{-\lambda_m}$ as a polygon path in $\mathbb{R}^3$ whose i^{th} edge is a vector of length $-\lambda_i$ in $S^2_{-\lambda_i}$. The $SO(3)$-action on this product of spheres is the standard diagonal action, and the moment map sums up the points, i.e. takes as its value the endpoint of the polygon path. However, under the identification (5.3.7), the Lie algebra of $SO(3)$ gets identified with $\mathcal{H}(2)/\{aI_2\}$. Thus the fact that the last entry of the moment map (5.3.8) equals $\lambda_{m+1}I_2$ implies that this endpoint is the origin in the Lie algebra of $SO(3)$. In other words, our polygon path is a polygon. So the quiver variety $R_\lambda(1, \cdots, 1, 2)$ is just the space of all polygons in $\mathbb{R}^3$ whose sides are of length $-\lambda_1, \cdots, -\lambda_m$, up to rotation.

Using the invariant section $s(Z) = \prod_{i=1}^m x_i^{-\lambda_i}$ to compute the stability function for this space we have

$$\begin{aligned}\psi(Z) &= -\sum(|x_i|^2 + |y_i|^2) + \sum(-\lambda_i)\log|x_i|^2 + \sum(-\lambda_i) - \sum(-\lambda_i)\log\frac{-\lambda_i|x_i|^2}{|x_i|^2+|y_i|^2} \\ &= 2\lambda_{m+1} - |Z|^2 + \sum \lambda_i \log\frac{-\lambda_i}{|Z_i|^2}.\end{aligned}$$

Finally we point out that everything we said above applies to framed quiver varieties, in which case the $U(V)$-action is free on $\Phi^{-1}(\lambda I)$. The coadjoint orbits of $\mathcal{U}(n)$ are just special cases of quiver varieties of this type.

Appendix A

Hamiltonian actions in geometric quantization

A.1 Quantizing Hamiltonian actions

We will start by recalling some concepts in symplectic Hamiltonian geometry. Let (M, ω) be a symplectic manifold. For any smooth function $f \in C^\infty(M)$, the *Hamiltonian vector field* associated to f is the vector field v_f on M satisfying

$$\omega(v_f, Y) = df(Y) \tag{A.1.1}$$

for all vector fields Y on M. The existence of such a vector field is guaranteed by the non-degeneracy of ω. Given any two smooth functions $f, g \in C^\infty(M)$, their *Poisson bracket* is defined to be

$$\{f, g\} = \omega(v_f, v_g). \tag{A.1.2}$$

In a local Darboux coordinates $\{x_1, \cdots, x_n, \xi_1, \cdots, \xi_n\}$ the Poisson bracket above has the explicit expression

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i} - \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i}\right).$$

It is well known that the set of Hamiltonian vector fields form a Lie sub-algebra in the Lie algebra of all vector fields on M,

$$v_{\{f,g\}} = -[v_f, v_g]. \tag{A.1.3}$$

Now suppose G is a connected compact Lie group, and

$$\tau : G \times M \to M$$

a smooth action of G on M. Moreover, we suppose that the G-action preserves the symplectic structure, in other words, for every $g \in G$,

$$\tau_g^* \omega = \omega. \tag{A.1.4}$$

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ its dual Lie algebra. Every $v \in \mathfrak{g}$ generates a one-parameter subgroup of G,

$$\{\exp(-tv) \mid t \in \mathbb{R}\}.$$

Denote by $v^{\#}$ the vector field generated by this subgroup. The G-action τ is called a *Hamiltonian action* if there exists a map $\Phi : M \to \mathfrak{g}^*$, called the *moment map*, such that

1. $v^{\#}$ is the Hamiltonian vector field associated to the function

$$\Phi^v(\cdot) := \langle \Phi(\cdot), v \rangle. \tag{A.1.5}$$

2. Φ is equivariant with respect to the G-action on M and the coadjoint action of G on $\mathfrak{g}^*$.

Note that in the case G is a torus, the equivariant condition above is reduced to the condition that Φ is G-invariant.

Now suppose (M, ω) is pre-quantizable, i.e. the cohomology class $[\omega]$ is integral. Let $\mathbb{L}$ be a pre-quantum line bundle over (M, ω). Recall that in the geometric quantization procedure of Kostant and Souriau, any $f \in C^\infty(M)$ is quantized to a quantum operator $Q(f)$ on $\Gamma(\mathbb{L})$,

$$Q(f) = -\sqrt{-1}\hbar\nabla_{v_f} + f. \tag{A.1.6}$$

A natural question arise: how to quantize the Hamiltonian action τ? In other words, how to lift the G-action on M to a "suitable" G-action on $\mathbb{L}$?

In general it is not possible to find a lifting of G-action. However, at the infinitesimal level we do have a canonical lifting, i.e. a canonical representation of $\mathfrak{g}$ on the space of smooth sections of $\mathbb{L}$. In fact, given any $v \in \mathfrak{g}$ the function Φ^v defined by (A.1.5) has $v^\#$ as its Hamiltonian vector field. Now in view of (A.1.6), one can define the $\mathfrak{g}$-action on $\Gamma(\mathbb{L})$ by

$$L_v s = \nabla_{v^\#} s - \sqrt{-1}\langle \Phi, v\rangle s. \tag{A.1.7}$$

In the following we will make the assumption

Assumption I. *The $\mathfrak{g}$-action (A.1.7) can be integrated to a global G-action on $\mathbb{L}$.*

For example, when G is connected and simply connected, we can always lift the G-action on M to a linear G-action on $\mathbb{L}$ whose infinitesimal action is given by (A.1.7).

In general the obstruction for such a lifting lies in a G-equivariant cohomology class,

Theorem A.1.1 ([Rie01]). *The G-action on M lifts to a linear action on $\mathbb{L}$ iff*

$$c_1(\mathbb{L}) \in \iota^* H^2_G(M; \mathbb{Z}),$$

where $H_G(M)$ is the equivariant cohomology ring of M and ι^* the "forgetfulness" map $H^*_G \to H^*$.

We can always assume that the lifted action preserves the Hermitian metric (via averaging) and the connection ∇, and thus preserves the curvature. These assumptions can be viewed as the "quantum version" of the fact that the G-action on M preserves the symplectic structure.

Finally let's compute an example. Let $M = \mathbb{C}^d$ be the complex space, $\mathbb{L} = \mathbb{C}^d \times \mathbb{C}$ the trivial line bundle with the Hermitian metric

$$\langle 1, 1 \rangle = e^{-|z|^2}.$$

Let $G = S^1$ acting on M by the diagonal action,

$$e^{i\theta} \cdot (z_1, \cdots, z_d) = (e^{i\theta} z_1, \cdots, e^{i\theta} z_d). \tag{A.1.8}$$

The moment maps of this action are

$$\Phi^{(\alpha)}(z) = \alpha - \sum |z_k|^2,$$

where $\alpha \in \mathbb{Z}_G^* \subset \mathfrak{g}^* \simeq \mathbb{R}$ is a real number. Given any $v \in \mathfrak{g} \simeq \mathbb{R}$, we have

$$v^\# = -\sqrt{-1} v \sum (z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i}).$$

Thus for the section $s = (z, 1)$,

$$\nabla_{v^\#} s + \sqrt{-1} \langle \Phi^{(\alpha)}, v \rangle s = \sqrt{-1} \alpha v s,$$

It follows that the S^1-action (A.1.8) is lifted to

$$e^{i\theta}(z, w) = (e^{i\theta} z, e^{i\alpha\theta} w) \tag{A.1.9}$$

on $\mathbb{L}$. (Note that the lifted action depends on the choice of the moment map.)

A.2 Quantizing GIT actions

Now we explain how the results in the previous section extend to GIT actions. We suppose (M, ω) is not just symplectic, but rather Kähler. Moreover, assume that the G-action τ on M is not only Hamiltonian, but also holomorphic. One way to give a complex structure to the symplectic quotient M_{red} is to realize M_{red} as the GIT quotient of some complex manifold with respect to a holomorphic complex group action, The complex group being the complexification of G, which we will denote by $G_{\mathbb{C}}$. By definition $G_{\mathbb{C}}$ is the unique complex Lie group satisfying

- $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \sqrt{-1}\mathfrak{g}$.
- $G_{\mathbb{C}}$ has G as its maximal compact subgroup.

For example, the complexification of S^1 is $\mathbb{C}^*$, and the complexification of $U(n)$ is $GL(n, \mathbb{C})$.

We would like to extend the Hamiltonian holomorphic G-action on M to a holomorphic $G_{\mathbb{C}}$-action on M. As in the lifting case, this is not always possible. However, at the infinitesimal level there is a canonical way to extend the $\mathfrak{g}$-action to $\mathfrak{g}_{\mathbb{C}}$-action. In fact, for $w = \sqrt{-1}v \in \sqrt{-1}\mathfrak{g}$, we just define

$$w^{\#} := Jv^{\#}, \tag{A.2.1}$$

where J is the almost complex structure on M. We will assume

Assumption II. *The $\mathfrak{g}_{\mathbb{C}}$-action defined by (A.2.1) can be integrated to a global holomorphic $G_{\mathbb{C}}$-action on M.*

It was proved by V. Guillemin and S. Sternberg in [GuS82] that if M is a compact Kähler manifold and G is a compact connected Lie group acting on M which preserves the polarization [1], then the G-action can be canonically extended to a $G_{\mathbb{C}}$-action preserving the polarization. In particular, if we take the polarization to be the

[1] A polarization is an involutive Lagrangian sub-bundle of the complexified tangent bundle $TM^{\mathbb{C}} = TM \otimes \mathbb{C}$.

Kähler polarization, we see that a holomorphic action τ on M can be extended to a holomorphic $G_{\mathbb{C}}$-action $\tau_{\mathbb{C}}$ on M. (However, the $G_{\mathbb{C}}$ action is no longer Hamiltonian.)

Now suppose (M, ω) is a pre-quantizable Kähler manifold, and $\mathbb{L}$ a pre-quantum line bundle over M. Suppose the G-action τ on M lifts to a G-action $\tau^{\#}$ on $\mathbb{L}$. To quantize the $G_{\mathbb{C}}$-action $\tau_{\mathbb{C}}$, it is enough to complexifying the G-action $\tau^{\#}$ on $\mathbb{L}$ to a $G_{\mathbb{C}}$-action $\tau_{\mathbb{C}}^{\#}$ on $\mathbb{L}$. Under the previous assumptions one can show that this is always possible. Moreover, it is obvious that at the infinitesimal level the action $\tau_{\mathbb{C}}^{\#}$ should be given by

$$L_w s = \sqrt{-1} L_v s = \nabla_{w^{\#}} s + \langle \Phi, v \rangle s, \quad \forall s \in \Gamma(\mathbb{L}) \tag{A.2.2}$$

for $w = \sqrt{-1} v \in \sqrt{-1}\mathfrak{g}$. For more details, c.f. [GuS82].

As an example, it is easy to see that the complexification of the action (A.1.8) is

$$\tilde{z} \cdot (z_1, \cdots, z_d) = (\tilde{z} z_1, \cdots, \tilde{z} z_d)$$

for $\tilde{z} \in \mathbb{C}^*$, which can be lifted to the holomorphic action on $\mathbb{C}^d \times \mathbb{C}$,

$$\tilde{z} \cdot (z, w) = (\tilde{z} z, \tilde{z}^{\alpha} \omega).$$

Finally we remark that the $G_{\mathbb{C}}$-action on M and on $\mathbb{L}$ are totally determined by the corresponding G-action. There are two corollaries of this fact that are very important for us:

1. If G acts freely on $\Phi^{-1}(0)$, then $G_{\mathbb{C}}$ acts freely on the set $M_{st} = G_{\mathbb{C}} \circ \Phi^{-1}(0)$.

2. A (holomorphic) section of $\mathbb{L}$ is G-invariant if and only if it is $G_{\mathbb{C}}$-invariant.

A.3 Quantization commutes with reduction

The results concerning quantization and reduction, Hamiltonian action and GIT action, which we described in section 3.1, appendix A.1 and Appendix A.2, can be depicted graphically by the diagram:

$$
\begin{tikzcd}
\underline{\text{quantum level:}} & \mathbb{L} \arrow[dd] & \supset & \iota^*\mathbb{L} \arrow[dd] \arrow[dr] & & \mathbb{L}_{st} \arrow[dl] \arrow[dd] & \subset & \mathbb{L} \arrow[dd] \\
& & & & \mathbb{L}_{red} \arrow[dd] & & & \\
\underline{\text{classical level:}} & M & \overset{\iota}{\supset} & \Phi^{-1}(0) \arrow[dr, "\pi_0"] & & M_{st} \arrow[dl, "\pi"'] & \subset & M \\
& & & & M_{red} & & &
\end{tikzcd}
$$

Hamiltonian G-action   *GIT $G_{\mathbb{C}}$-action*

By definition, $\mathbb{L}_{red} = \mathbb{L}_{st}/G_{\mathbb{C}}$, thus we have a bijective map

$$\Gamma_{hol}(\mathbb{L}_{st})^{G_{\mathbb{C}}} \xrightarrow{\sim} \Gamma_{hol}(\mathbb{L}_{red}). \tag{A.3.1}$$

As we pointed out at the end of appendix A.2, a section is G-invariant if and only if it is $G_{\mathbb{C}}$-invariant, i.e.,

$$\Gamma_{hol}(\mathbb{L}_{st})^{G} \xrightarrow{\sim} \Gamma_{hol}(\mathbb{L}_{st})^{G_{\mathbb{C}}}. \tag{A.3.2}$$

Composing these two maps with the restriction map

$$\Gamma_{hol}(\mathbb{L})^{G} \longrightarrow \Gamma_{hol}(\mathbb{L}_{st})^{G},$$

we get the GIT map

$$\gamma : \Gamma_{hol}(\mathbb{L})^{G} \longrightarrow \Gamma_{hol}(\mathbb{L}_{red}). \tag{A.3.3}$$

Similarly one can define for every $k \in \mathbb{N}$ the GIT map

$$\gamma_k : \Gamma_{hol}(\mathbb{L}^k)^{G} \longrightarrow \Gamma_{hol}(\mathbb{L}^k_{red}). \tag{A.3.4}$$

The quantization commutes with reduction theorem claims that these maps are bijection for all k. Let's briefly describe how the stability function emerged in the proof of this theorem in [GuS82].

For simplicity we assume that M is compact. In this case there exists some $k_0 > 0$ such that the set $\Gamma_{hol}(\mathbb{L}^{k_0})^G$ is nonempty. (This is a highly nontrivial result. For a proof, c.f. the appendix of [GuS82]). Let $\tilde{s}$ be an element in this set. Then the argument at the end of section 3.2.2 tells us that the set M_{st} is Zariski open: its complement $M - M_{st}$ is contained in a codimension one subvariety of M. (If M is noncompact, we need to assume that there exists some $\tilde{s} \in \Gamma_{hol}(\mathbb{L}^{k_0})^G$ such that the function $\langle \tilde{s}, \tilde{s} \rangle$ attains its maximum on each closure of orbit, $\overline{G_{\mathbb{C}} \circ p}$, for all p satisfying $\tilde{s}(p) \neq 0$. Assuming this, the same argument applies.)

To prove that γ_k is bijective, we only need to find a unique extension of the holomorphic section $\pi^* s_k$ of $\mathbb{L}^k_{st}$ to a G-invariant holomorphic section of $\mathbb{L}^k$. This can be done as follows: Combining the stability theory, corollary 3.2.9, with the argument above, we see that the only way to extend $\pi^* s_k$ from $\mathbb{L}^k_{st}$ to $\mathbb{L}^k$ is to define $\pi^* s_k = 0$ on $M - M_{st}$, and that if we define s_k this way then by Liouville's theorem s_k is holomorphic on all of M. This proves that γ_k is bijective, for all k.

Appendix B

(Generalized) Toeplitz Operators

B.1 The algebra of Toeplitz operators

Let Ω be an open, strictly pseudoconvex domain with compact closure and smooth boundary in a complex analytic manifold of complex dimension m. Denote by $X = \partial\Omega$ its boundary. We will equip with X the measure ν intrinsically defined as follows: let ρ be a defining function of Ω, i.e.,

- $\rho > 0$ in the interior of Ω,
- $\rho = 0$ on the boundary X,
- $d\rho \neq 0$ near X.

Then the one-form

$$\alpha = \frac{1}{2\sqrt{-1}}(\partial\rho - \bar{\partial}\rho)\Big|_X \tag{B.1.1}$$

is a contact form on X. In other words,

$$\nu = \alpha \wedge (d\alpha)^{n-1} \tag{B.1.2}$$

is a volume form on X.

Let H be the *Hardy space* on X, i.e. the closure in L^2 of the space of C^∞ functions on X which can be extended to holomorphic functions in Ω. It is a closed Hilbert subspace in $L^2(X)$. The orthogonal projection

$$\pi : L^2(X) \longrightarrow H$$

is called the (generalized) *Szegö projector.*

Definition B.1.1. *An operator* $T : C^\infty(X) \to C^\infty(X)$ *is called a* Toeplitz operator of order k *if it can be written in the form*

$$T = \pi P \pi, \tag{B.1.3}$$

where $P : L^2(X) \to L^2(X)$ *is a classical pseudodifferential operator of order* k *on* X*.*

It can be shown that, like the algebra of classical pseudodifferential operators, the Toeplitz operators form a (noncommutative) ring under composition, filtered by their degrees. For Toeplitz operators we also have the concept of principle symbol, which plays a similar role as for pseudodifferential operators. We will describe it now. Let

$$\Sigma^+ = \{(x, \xi) \mid \xi = t\alpha_x,\, t > 0\} \tag{B.1.4}$$

be the principal $\mathbb{R}^+$-bundle on X generated by α in T^*X. It is the symplectic cone corresponding to the contact manifold X. The principle symbol of T, denoted by $\sigma(T)$, is defined to be the restriction of $\sigma(P)$ to Σ^+, i.e.

$$\sigma(T) = \sigma(P)|_{\Sigma^+}, \tag{B.1.5}$$

where P is a pseudodifferential operator on X such that $T = \pi P \pi$. It can be shown that the principle symbol (B.1.5) is well-defined, and has properties like the principal symbol of a classical pseudodifferential operator:

1. $\sigma(T_1)\sigma(T_2) = \sigma(T_1 T_2)$,

2. $\sigma([T_1, T_2]) = \{\sigma(T_1), \sigma(T_2)\}$,

3. If T is of order k with $\sigma(T) = 0$, then T is of order $k-1$.

A Toeplitz operator T is called *elliptic* if its principal symbol $\sigma(T) \neq 0$ everywhere. From the symbolic properties above one can construct a parametrix T' for any elliptic Toeplitz operator T, i.e. T' is a Toeplitz operator of order $-k$ such that both $I - TT'$ and $I - T'T$ are smoothing operators. As in the pseudodifferential case, any Toeplitz operator of order $k < 0$ is a compact operator. It follows that the spectrum of T is discrete and has no finite point of accumulation.

As been pointed out in [Bou79] and [BoG81], the Toeplitz operators form an algebra of pseudo-local operators, and a pseudodifferential operator can be viewed as a special kind of Toeplitz operator. Many classical results for classical pseudodifferential operators can be extended to Toeplitz operators. For example, we have the following trace formula,

Proposition B.1.2 ([BoG81]). *Let f be a compactly supported smooth function on the real line $\mathbb{R}$, and let T and G be Toeplitz operators as above. Moreover, assume Q is Zoll* [1]*. Then*

$$trace(e^{\sqrt{-1}tQ} f(T)) \sim \sum_k a_k(f)\chi_k(t), \tag{B.1.6}$$

where $\chi_k(t)$ is the distribution

$$\chi_k(t) = \sum_{n>0} n^k e^{ink}. \tag{B.1.7}$$

Moreover, the leading coefficient is

$$a_0(f) = \gamma \int_{\sigma(Q)=1} f(\sigma(T))(z)dz. \tag{B.1.8}$$

[1] i.e., assume Q is self-adjoint and elliptic and that $\mathrm{spec}(Q) = \mathbb{Z}_+$.

B.2 The asymptotics of spectral measures

Let (M, ω) be a pre-quantizable Kähler manifold, $(\mathbb{L}, h)$ a pre-quantum line bundle over M. Denote by $\mathbb{L}^*$ the dual line bundle of $\mathbb{L}$. Let

$$D(\mathbb{L}^*) = \{(x, \xi) \in \mathbb{L}^* \mid x \in M, \langle \xi, \xi \rangle \leq 1\} \tag{B.2.1}$$

be the disc bundle in the dual bundle. As observed by Grauert, $D(\mathbb{L}^*)$ is a strictly pseudoconvex domain in $\mathbb{L}^*$. The manifold we are interested in is its boundary,

$$X = \partial D = \{(x, \xi) \in \mathbb{L}^* \mid \|\xi\| = 1\}, \tag{B.2.2}$$

the unit circle bundle in the dual bundle.

There is a natural S^1-action on this unit circle bundle. Let Q be its infinitesimal generator,

$$Q = \pi \frac{1}{\sqrt{-1}} \frac{\partial}{\partial \theta} \pi, \tag{B.2.3}$$

where π is the Szegö projector.

Lemma B.2.1 ([Gui79]). *Q is a first order self-adjoint elliptic Toeplitz operator. Moreover, Q is a Zoll operator whose eigenspace corresponding to the eigenvalue k can be identified with the space of holomorphic sections of $\mathbb{L}^k$, $\Gamma_{hol}(\mathbb{L}^k)$.*

For any smooth function $f \in C^\infty(X)$, let M_f be the operator "multiplication by f". We may view $\Gamma_{hol}(\mathbb{L}^n)$ as a subspace of H^2, and denote by

$$\pi_n : L^2(\mathbb{L}^n) \to \Gamma_{hol}(\mathbb{L}^n)$$

the orthogonal projection. Now we can prove

Theorem B.2.2. *There is an asymptotic expansion*

$$trace(\pi_n M_f \pi_n) \sim \sum_{k=d-1}^{-\infty} a_k(f) n^k, \quad n \to \infty, \tag{B.2.4}$$

where $d = \dim M$.

Proof. By the trace formula (B.1.6),

$$\operatorname{trace}(e^{itQ}M_f) \sim \sum a_k \chi_k(t)$$

where

$$\chi_k(t) = \sum_{n>0} n^k e^{int}.$$

On the other hand,

$$\operatorname{trace}(e^{itQ}M_f) = \sum e^{int}\operatorname{Tr}\pi_n M_f \pi_n.$$

By comparing the coefficient of e^{int}, we get the theorem. $\square$

Finally we point out that the coefficients a_k in the asymptotic expansion above are given by the noncommutative residue trace on the algebra of Toeplitz operators, [Gui93]. Recall that a trace functional for an associative algebra $\mathcal{A}$ is by definition a linear map $\tau : \mathcal{A} \longrightarrow \mathbb{C}$ with the property

$$\tau(AB) = \tau(BA)$$

for all $A, B \in \mathcal{A}$. In [Gui93], V. Guillemin showed that up to a multiplicative constant, there is a unique trace on the algebra of Toeplitz operators associated with a strictly pseudoconvex domain, defined as following: Fixing a first order self-adjoint elliptic Toeplitz operator Q, consider the zeta function

$$\zeta_T(z) = \operatorname{trace}(Q^z T). \tag{B.2.5}$$

This is well-defined and holomorphic for $\Re(z) \ll 0$, and can be extended to a meromorphic function to $\mathbb{C}$ with simple isolated poles at $z = -m-1, -m, -m+1, \cdots$. The residue trace of T with respect to Q is then defined to be the residue of this meromorphic function at $z = 0$:

$$\operatorname{res}_Q(T) = \operatorname{res}_{z=0}(\zeta_T(z)). \tag{B.2.6}$$

Now for any $\Re(z) \gg 0$, theorem B.2.2 gives

$$\operatorname{trace}(Q^{-z}\pi_n M_f \pi_n) \sim \sum_{k=d-1}^{-\infty} a_k n^{k-z}.$$

Summing over n,

$$\operatorname{trace}(Q^{-z} M_f) \sim \sum_k a_k \zeta(z-k),$$

where ζ is the classical zeta function. This implies

$$a_{k-1} = \operatorname{res}_{z=k}(Q^{-z} M_f),$$

which is exactly the noncommutative residue.

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