Finite dimensional representations of symplectic reflection algebras for wreath products

by

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Abstract

Symplectic reflection algebras are attached to any finite group G of automorphisms of a symplectic vector space V, and are a multi-parameter deformation of the smash product $TV\sharp G$, where TV is the tensor algebra. Their representations have been studied in connection with different subjects, such as symplectic quotient singularities, Hilbert scheme of points in the plane and combinatorics. Let $\Gamma \subset SL(2,\mathbb{C})$ be a finite subgroup, and let S_n be the symmetric group on n letters. We study finite dimensional representations of the wreath product symplectic reflection algebra $H_{1,k,c}(\mathbf{\Gamma}_n)$ of rank n, attached to the wreath product group $\mathbf{\Gamma}_n = S_n \ltimes \Gamma^n$, and to the parameters (k,c), where k is a complex number (occurring only for $n > 1$), and c a class function on the set of nontrivial elements of Γ. In particular, we construct, for the first time, families of irreducible finite dimensional modules when Γ is not cyclic, $n > 1$, and (k,c) vary in some linear subspace of the space of parameters. The method is deformation theoretic and uses properties of the Hochschild cohomology of $H_{1,k,c}(\mathbf{\Gamma}_n)$, and a Morita equivalence, established by Crawley-Boevey and Holland, between the rank one algebra $H_{1,c}(\Gamma)$ and the deformed preprojective algebra $\Pi_\lambda(Q)$, where Q is the extended Dynkin quiver attached to Γ via the McKay correspondence. We carry out a similar construction for continuous wreath product symplectic reflection algebras, a generalization to the case when $\Gamma \subset SL(2,\mathbb{C})$ is infinite reductive. This time the main tool is the definition of a continuous analog of the deformed preprojective algebras for the infinite affine Dynkin quivers corresponding to the infinite reductive subgroups of $SL(2,\mathbb{C})$.

Thesis Supervisor: Pavel Etingof
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Introduction

The study of symplectic reflection algebras was initiated by Etingof and Ginzburg in [EG02], although these algebras (and several of their properties) already appear in the classical work of Drinfeld [Dri86], as a special case of degenerate affine Hecke algebras for a finite group.

Symplectic reflection algebras arise from the action of a finite group of symplectomorphisms $G \subset \mathrm{Sp}(V)$ on a symplectic vector space (V, ω). They form a multi-parameter family of deformations $H_{t,f}(G)$ of the skew group algebra $SV\sharp G$, where SV is the symmetric algebra of V. The parameter t is a complex number, and the parameter f is a conjugation invariant function on the the set $\mathcal{S}$ of symplectic reflections in G, i.e. elements $s \in G$ such that $\mathrm{rk}(\mathrm{Id} - s)|_V = 2$. Symplectic reflections can be considered as analogs of reflections in a symplectic setting, since they fix a codimension two subspace pointwise, and act on the orthogonal complement with nontrivial complex conjugate eigenvalues of norm one. Hence the name.

Explicitly, if we denote by TV the tensor algebra of V, the symplectic reflection algebra $H_{t,f}(G)$ is the quotient of $TV\sharp G$ by the relations

$$x \otimes y - y \otimes x = t\omega(x, y) \cdot 1 + \sum_{s \in \mathcal{S}} f(s) \cdot \omega_s(x, y) \cdot s \quad \forall x, y \in V$$

where ω_s denotes the (possibly degenerate) skew-symmetric form which coincides with ω on $\mathrm{Im}(\mathrm{Id} - s)$, and has $\ker(\mathrm{Id} - s)$ as its radical.

One of the fundamental properties of $H_{t,f}(G)$ is that it satisfies the analog of the Poincaré-Birkhoff-Witt (PBW) theorem for the universal enveloping algebra of a Lie algebra. Namely, if we consider the increasing filtration on $H_{t,f}(G)$ obtained by assigning degree zero to the elements of G and degree one to the elements of V, we get an isomorphism for the associated graded algebra $\mathrm{gr}H_{t,f}(G) \cong SV\sharp G$. The PBW property assures the flatness of the family of deformations $H_{t,f}(G)$.

Rescaling the parameters (t, f) by a non-zero complex number does not change $H_{t,f}(G)$ up to isomorphism. Thus we can distinguish two main cases in the study of symplectic reflection algebras: the *quasi-classical* case when $t = 0$, and the *quantum* case when $t = 1$. Algebras belonging to these two subfamilies present contrasting features that make them interesting for different reasons, both in algebraic geometry and representation theory.

On the geometric side, let us consider the orbit space V/G. This space and the corresponding commutative algebra, the ring of invariants SV^G, might not be the right objects to describe the geometric properties of the G-action, that might instead be connected with some resolutions of V/G. One can then think to approach the study of such properties by replacing SV^G with the non-commutative smash product $SV\sharp G$, and constructing non-commutative deformations of this algebra. In the quasi-classical case, for example, the non-commutative deformation $H_{0,f}(G)$ of $SV\sharp G$ has a big center which is a commutative deformation of SV^G (i.e. corresponding to an actual algebraic variety), and can be used to study symplectic resolutions of some interesting (Poisson) deformations of the orbifold V/G ([EG02]), [GS04]). A class of symplectic reflection algebras of particular interest for algebraic geometry is the one of rational Cherednik algebras. These are symplectic reflection algebras attached to an irreducible finite complex reflection group W in a vector space $\mathfrak{h}$, acting diagonally on $V = \mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h}^*$ denotes the dual of the reflection representation. In this case, the symplectic form is given by the natural pairing. In other words, V is the cotangent bundle of $\mathfrak{h}$ with the standard symplectic structure, and $\mathfrak{h}$, $\mathfrak{h}^*$ are W-stable irreducible Lagrangian subspaces. In the quantum case, the Cherednik algebra $H_{1,f}(\mathfrak{h} \oplus \mathfrak{h}^*, W)$ for W of type A can be regarded in different ways as a non-commutative deformation of the Hilbert scheme of points in the plane $\mathbb{C}^2$ ([GS06],[GS05],[KR]).

On the representation theoretic side, the most challenging case is the quantum one which is the most non-commutative one. Indeed, when $t = 1$ and regarding f as a formal parameter, the family $H_{1,f}(G)$ gives a universal deformation of the smash product $\mathcal{W}\sharp G$, where $\mathcal{W}$ is the Weyl algebra of the symplectic space (V, ω) ([EG02]). The representation theory of such deformations has proved to be very rich and interesting. In particular, for rational Cherednik algebras an analog of category $\mathcal{O}$ for Lie algebras has been defined, as well as a theory of standard modules and formal characters ([BEG03], [GGOR03], [Chm06]). Moreover, in the case of Cherednik algebras, the theory of symplectic reflection algebras is connected to the one of double affine Hecke algebras (of which they are a certain degeneration, cfr [EG02], Introduction), introduced by I. Cherednik and used by the same author to prove some important Macdonald's conjectures ([Che95]). This links the representation theory of symplectic reflection algebras to combinatorics and the study of special functions.

The main topic of this thesis is the representation theory of the symplectic reflection algebras of wreath product type $H_{t,k,c}(\mathbf{\Gamma}_n)$. These are symplectic reflection algebras attached to the semi-direct products $\mathbf{\Gamma}_n := S_n \ltimes \Gamma^n \subset \mathrm{Sp}(2n, \mathbb{C})$. Here S_n is the symmetric group on n letters, Γ is any finite subgroup of $SL(2, \mathbb{C}) = \mathrm{Sp}(2, \mathbb{C})$, and S_n acts on Γ^n by permuting the factors. The deformation parameter f appearing in the definition of symplectic reflection algebra can in this case be written as a pair (k, c), where k is a complex number, and c is a class function on the non-trivial elements of Γ. The integer n is called the rank of the algebra $H_{t,k,c}(\mathbf{\Gamma}_n)$. In particular, when $\Gamma = \mathbb{Z}/m\mathbb{Z}$ the group $S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$ is a complex reflection group (real reflection group of type A if $m = 1$ and of type B if $m = 2$) and one recovers a subfamily of rational Cherednik algebras.

In the quasi-classical case, representations of the wreath product algebras were studied in [EG02] using a geometric approach and the main result of the authors is that, if (k, c) are generic, isomorphism classes of finite dimensional irreducible modules are parametrized by the points of the (smooth) algebraic variety corresponding to the center of $H_{0,k,c}(\mathbf{\Gamma}_n)$.

In the quantum case, in contrast with the case of Cherednik algebras, a uniform approach to the representation theory does not exist yet. Nevertheless, when $n = 1$, there is no parameter k and $H_{1,c}(\Gamma)$ coincides with some non-commutative deformation of the Kleinian singularity $\mathbb{C}^2/\Gamma$ introduced by Crawley-Boevey and Holland, who classified finite dimensional irreducible modules using methods coming from the representation theory of quivers and deformed preprojective algebras, which are some special quotients of path algebras of quivers. In particular, in [CBH98] the authors established a Morita equivalence of $H_{1,c}(\Gamma)$ with the deformed preprojective algebra $\Pi_\lambda(Q)$, where Q is the extended (ADE) Dynkin quiver attached to Γ via the McKay correspondence, and $\lambda \in \mathbb{C}^I$, where I is the set of vertices of Q, is a parameter depending on c. This allowed them to define reflection functors that give equivalences of the categories of modules for different values of the deformation parameters.

The main result of this thesis is the construction of the first (for non-cyclic Γ) families of finite dimensional representations for $H_{1,k,c}(\mathbf{\Gamma}_n)$ when $n > 1$, and (k, c) vary in some linear subspaces of the space of deformation parameters. We use two methods, both arising from simple observations and corresponding natural questions.

1) Irreducible representations of the group $\mathbf{\Gamma}_n$ are well known. They can all be obtained in the following way. Choose any vector with positive integer coordinates $\vec{n} = (n_1, \dots, n_r)$ such that $\sum_{i=1}^r n_i = n$. Let S_{n_i} be the subgroup of S_n moving only the indices j such that $n_1 + \cdots + n_{i-1} < j \leq n_1 + \cdots + n_i$, and consider the subgroup $S_{n_1} \times \cdots \times S_{n_r} \subset S_n$. Take an irreducible representation X of the group $S_{\vec{n}}$: $X := X_1 \otimes \cdots \otimes X_r$, where X_i is an irreducible representation of S_{n_i} for any i. Choose a collection $N_1, \cdots, N_r$ of irreducible pairwise non-isomorphic representations of Γ, and form the irreducible representation $N := N_1^{\otimes n_1} \otimes \cdots \otimes N_r^{\otimes n_r}$ of Γ^n. Then $X \otimes N$ is an irreducible $S_{\vec{n}} \ltimes \Gamma^n$-module (where $S_{\vec{n}}$ acts also on N by permuting the factors) and $M := \mathrm{Ind}_{S_{\vec{n}} \ltimes \Gamma^n}^{\mathbf{\Gamma}_n} X \otimes Y$ is an irreducible $\mathbf{\Gamma}_n$-module. Which such modules can be extended to an irreducible representation of the entire algebra $H_{1,k,c}(\mathbf{\Gamma}_n)$, and for which values of the parameters (k, c)?

2) Since $H_{1,0,c}(\mathbf{\Gamma}_n) = H_{1,c}(\Gamma)^{\otimes n} \sharp S_n$, when k is zero, irreducible finite dimensional representations are known. They can all be obtained with the same procedure used in 1). We get modules $M = \mathrm{Ind}_{H_{1,c}(\Gamma)^{\otimes n} \sharp S_{\vec{n}}}^{H_{1,0,c}(\mathbf{\Gamma}_n)} X \otimes Y$, where this time we take

$Y = Y_1^{\otimes n_1} \otimes \cdots \otimes Y_r^{\otimes n_r}$, for some Y_is irreducible pairwise non-isomorphic $H_{1,c}(\Gamma)$-modules. Since, by the PBW property, $H_{1,k,c+c'}(\mathbf{\Gamma}_n)$ is a flat formal deformation of $H_{1,0,c}(\mathbf{\Gamma}_n)$ a natural question is: can we formally deform a $H_{1,0,c}$-module M to values of the parameters $(k, c+c')$ with $k \neq 0$?

In Theorem 3.2.4, we give an answer to question 1). Using just methods from representation theory of finite groups, we obtain a complete classification of all irreducible $\mathbf{\Gamma}_n$-modules that extend to $H_{1,k,c}(\mathbf{\Gamma}_n)$-modules for $k \neq 0$. Such modules extend if and only if the Young diagram of X_i is a rectangle for any i and $\mathrm{Hom}_\Gamma(N_i, N_j \otimes \mathbb{C}^2) = 0$ for any $i \neq j$, where $\mathbb{C}^2$ is the defining representation of Γ. Such representations have a unique extension obtained by making V act trivially, and the values of the parameters (k, c) for which they extend lie in a codimension r linear subspace, where r is the dimension of the vector $\vec{n}$.

In Theorem 3.3.10, using cohomological methods, we give a partial answer to the second question. We show that sufficient conditions for an irreducible representation of $H_{1,0,c}(\mathbf{\Gamma}_n)$ to formally deform to some values of the parameter with $k \neq 0$ are that X_i has rectangular Young diagram for any i and that $\mathrm{Ext}^1_{H_{1,c}(\Gamma)}(Y_i, Y_j) = 0$ for any $i \neq j$. Such representations actually admit a unique deformation in the formal neighborhood of 0 of a codimension r linear subspace. We also show that in a dense open set of this linear subspace the deformation is not only formal, i.e. $H_{1,k,c+c'}(\mathbf{\Gamma}_n)$ admits an irreducible representation isomorphic to M as a $\mathbf{\Gamma}_n$-module. We want to mention that in [GG05] Gan and Ginzburg introduced a one parameter deformation $\mathcal{A}_{n,\nu,\lambda}(Q)$ (where $\nu \in \mathbb{C}$) of the smash product $\Pi_\lambda(Q)^{\otimes n} \sharp S_n$, Morita equivalent to $H_{1,k,c}(\mathbf{\Gamma}_n)$ for any n, when Q is the McKay quiver of Γ. In ([Gan06]) Gan, using this interpretation of the wreath product symplectic reflection algebras in terms of deformed preprojective algebras, was able to generalize the reflection functors of [CBH98] to the case $n > 1$. This allowed him to prove the necessity of the conditions of Theorem 3.3.10, showing that our result gives an exhaustive classification of representations coming from deformations.

Symplectic reflection algebras have a generalization to reductive algebraic groups called continuous symplectic reflection algebras ([EGG05]). In this case, the role of the group algebra is played by the ring of algebraic distributions $\mathcal{O}(G)^*$, the dual space of the ring of regular functions $\mathcal{O}(G)$. The second topic of this thesis is the study of finite dimensional representations of continuous symplectic reflection algebras of wreath product type, i.e. attached to the groups $S_n \ltimes \Gamma^n$, where $\Gamma \subset SL(2, \mathbb{C})$ is an infinite reductive subgroup. This time the main tool is the definition of some "continuous" analogs of the deformed preprojective algebras for the infinite affine Dynkin quivers corresponding to the reductive subgroups of $SL(2, \mathbb{C})$, and of the corresponding generalization to such quivers of the Gan-Ginzburg algebra $\mathcal{A}_{n,\nu,\lambda}(Q)$. A Morita equivalence between these algebras and the continuous symplectic reflection algebras allows us to easily extend the methods of [CBH98] and [Gan06] to the continuous case. In particular, in Corollary 6.2.2 we give a complete classification of the finite dimensional irreducible representations for $n = 1$. For $n > 1$, in Theorem 6.5.2 and Theorem 6.5.3, we extend the results of Theorem 3.2.4 and Theorem 3.3.10 to the continuous case, giving necessary and sufficient conditions for deforming irreducible finite dimensional representations existing for $k = 0$ to nonzero values of k.

Chapter 1

Basic deformation theory

1.1 Plan of the chapter

In this chapter we first review the basic definitions of the theory of flat formal deformations for associative algebras. We then recall the fundamental role of Hochschild cohomology in this theory and the notion of *universal deformation.* Finally, we briefly discuss deformations of modules.

1.2 Flat formal deformations of associative algebras

Let k be a field, and let A be an associative unital algebra over k. Let U be a finite dimensional k-vector space. Denote by $k[[U]]$ the ring of k-valued formal functions on U, and denote by $\mathfrak{m}$ the unique maximal ideal in $k[[U]]$.

We recall that a $k[[U]]$-module is called *topologically free* if it is isomorphic to $V[[U]]$ for some k-vector space V.

Definition 1.2.1. *A flat formal deformation of A over $k[[U]]$ is an algebra A_U over $k[[U]]$ which is topologically free as a $k[[U]]$-module, together with a fixed isomorphism of algebras $\varphi : A_U/\mathfrak{m}A_U \longrightarrow A$.*

Thus in particular $A_U = A[[U]]$ as a $k[[U]]$-module. If $\dim U = n$, then A_U is said to be an n-parameter flat formal deformation of A.

Two deformations A_U, A'_U are said to be isomorphic if there exists a $k[[U]]$-algebra isomorphism $A_U \cong A'_U$ which is the identity modulo $\mathfrak{m}$. A deformation is said to be trivial if there exists an algebra isomorphism $A_U \cong A[[U]]$ which is the identity modulo $\mathfrak{m}$, where the algebra structure on $A[[U]]$ is given by the usual multiplication of formal power series on U with coefficients in A.

In a similar way, one can define m-th order deformations as deformations over the ring $k[[U]]/\mathfrak{m}^{m+1}$.

If $\hbar_1, \ldots, \hbar_n$ are coordinates on U, then we can identify $\mathbb{C}[[U]]$ with the ring of power series $k[[\hbar_1, ..., \hbar_n]]$, and the ideal $\mathfrak{m}$ with the ideal $(\hbar_1, \ldots, \hbar_n)$. Using the fact that A_U is topologically free we can choose an identification $\tilde{\varphi} : A_U \longrightarrow A[[\hbar_1, \ldots, \hbar_n]]$ as $k[[\hbar_1, ..., \hbar_n]]$-modules, coinciding with the isomorphism φ of Definition 1.2.1 modulo $(\hbar_1, \ldots, \hbar_n)$. Let us denote by $\underline{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n_{\geq 0}$ a multi-index, and let $\mathbf{h}^{\underline{p}}$ be the product $\prod_j \hbar_j^{p_j}$. We can think of A_U as the module $A[[\hbar_1, \ldots, \hbar_n]]$ equipped with a new $k[[\hbar_1, ..., \hbar_n]]$-linear, associative *star-product* determined by a formula

$$a * b = \sum_{\underline{p}} c_{\underline{p}}(a, b)\mathbf{h}^{\underline{p}} \tag{1.1}$$

where $c_{\underline{p}} : A \times A \longrightarrow A$ are k-bilinear maps, and $c_{0,\ldots,0}(a, b) = ab$ for any $a, b \in A$ (the product coincides with the product in A modulo $(\hbar_1, \ldots, \hbar_n)$).

In particular, a one parameter deformation $A_\hbar$ can be thought of as $A[[\hbar]]$ equipped with a $k[[\hbar]]$-linear associative product $*$ such that for any $a, b \in A$

$$a * b = ab + c_1(a, b)\hbar + c_2(a, b)\hbar^2 + \cdots , \tag{1.2}$$

where $c_j : A \times A \longrightarrow A$ are k-bilinear maps, and $c_0(a, b) = ab$ is just the original product in A.

One can think of c_1, and the corresponding first order deformation (deformation over the ring $k[[\hbar]]/\hbar^2$), as the *infinitesimal deformation* or *differential* of the family

$A_\hbar$. This leads to two natural questions. The first is finding a classifying space for infinitesimal deformations. The second is defining a convenient theoretical setting to describe the obstructions to *integrating* infinitesimal deformations, i.e. given a first order deformation c_1, lifting the associativity property of the product $*$ from order one to any order by choosing appropriate c_is for $i > 1$. In his pioneering work ([Ger63],[Ger64]), Gerstenhaber showed how the natural language to approach these problems is the one of homological algebra, specifically the one of Hochschild cohomology that we are going to review in the next section.

1.3 Hochschild cohomology and deformation theory

For an A-bimodule E, consider the following complex (Hochschild complex)

$$0 \longrightarrow C^0(A,E) \xrightarrow{d} \cdots \xrightarrow{d} C^m(A,E) \xrightarrow{d} C^{m+1}(A,E) \xrightarrow{d} \cdots$$

where $C^m(A,E) = \mathrm{Hom}_k(A^{\otimes m}, E)$ is the space of m-linear maps from A^m to E (and $C^0(A,E) := E$), and the differential d is defined as follows:

$$(de)(a): \;=\; ae - ea \qquad \forall\, e \in E,\, a \in A$$

$$\begin{aligned}(df)(a_1,\cdots,a_{m+1}): \;&=\; a_1\, f(a_2,\cdots,a_{m+1})\\ &+\; \sum_{i=1}^{m}(-1)^i f(a_1,\cdots,a_{i-1},a_i\,a_{i+1},a_{i+2},\cdots,a_{m+1})\\ &-\; (-1)^m f(a_1,\cdots,a_m)\,a_{m+1}.\end{aligned}$$

Definition 1.3.1. *The i-th Hochschild cohomology group $H^i(A,E)$ of A with coefficients in the bimodule E is the i-th cohomology group of the Hochschild complex $(C^\bullet, d)$.*

We recall that an A-bimodule structure on a k vector space space E is the same as a left $A \otimes A^o$-module structure, where A^o is the opposite algebra of A, and $A \otimes A^o$ is called the enveloping algebra of A. The following fact will be very useful to us in this thesis.

Proposition 1.3.2. *There exists a natural isomorphism*

$$H^i(A, E) \longrightarrow \mathrm{Ext}^i_{A \otimes A^o}(A, E)$$

Proof. Consider the A-bimodule structure on $A^{\otimes m}$ given by

$$b(a_1 \otimes \cdots \otimes a_m)c = ba_1 \otimes \cdots \otimes a_m c$$

The so called bar resolution of the bimodule A is a projective resolution and is given by

$$\cdots \longrightarrow A^{\otimes 3} \longrightarrow A^{\otimes 2} \longrightarrow A$$

where the differential is

$$\partial(a_1 \otimes \cdots \otimes a_m) = a_1 a_2 \otimes \cdots \otimes a_m - \cdots + (-1)^{m-1} a_1 \otimes \cdots \otimes a_{m-1} a_m.$$

It is now enough to observe that, for any $m \geq 2$, one has a natural isomorphism of vector spaces $\mathrm{Hom}_{A \otimes A^o}(A^{\otimes m}, E) \cong Hom_k(A^{m-2}, E) = C^{m-2}(A, E)$, and that ∂ corresponds to d under this identification.

□

Let us now go back to one parameter deformations and formula (1.2). Imposing the associativity condition

$$(a * b) * c = a * (b * c)$$

one gets a hierarchy of equations

$$dc_m(a,b,c) = \sum_{\substack{i+j=m \\ i,j>0}} c_i(c_j(a,b),c) - c_i(a,c_j(b,c)) \tag{1.3}$$

where $m > 0$ and d is the Hochschild differential. In particular, when $m = 1$ the right hand side is zero, and one gets that associativity in order one holds if and only if c_1 is a Hochschild 2-cocycle with values in the bimodule A. It is easy to see that two first order deformations are isomorphic if and only if the associated 2-cocycles c_1, c_1' are in the same cohomology class in $H^2(A,A)$.

Theorem 1.3.3. *Two first order one parameter deformations of an associative algebra A are isomorphic if and only if the corresponding 2-cocycles c_1, c_1' are in the same Hochschild cohomology class.*

Proof. Suppose c_1, c_1' are in the same cohomology class. Then $c_1 - c_1' = df$, where f is an endomorphism of A as a k-vector space. Then one can check that the assignment

$$a \longrightarrow a + f(a)\hbar$$

for any $a \in A$ defines by $k[[\hbar]]$-linear extension an isomorphism between the corresponding first order deformations. Vice versa suppose that two first order deformations are isomorphic. This means that there exists a $k[[\hbar]]/\hbar^2$-algebra isomorphism between them which is the identity modulo $\hbar$. Such an isomorphism is given by a map

$$a \longrightarrow a + f(a)\hbar$$

where f is a k linear endomorphism of A, and the map is compatible with the star products $*$, $*'$ modulo $\hbar^2$. Imposing this compatibility condition gives exactly $c_1 - c_1' = df$.

□

Thus the cohomology group $H^2(A,A)$ parametrizes infinitesimal deformations up to isomorphism.

For $m = 2$ equation (1.3) gives

$$dc_2(a,b,c) = c_1(c_1(a,b),c) - c_1(a,c_1(b,c))$$

for any $a,b,c \in A$. It can be computed that, if c_1 is a cocycle, then the right hand side defines a 3-cocycle b_1 and its cohomology class depends only on the cohomology class of c_1. The cohomology class of b_1 in $H^3(A,A)$ is the only obstruction to lifting associativity from order one to order two.

In general, it can be shown that if equation (1.3) is satisfied for $m = 1, \ldots, M-1$ then for $m = M$ the right hand side is a cocycle b_M. Thus *all* obstructions lie in $H^3(A,A)$. This time though, the cohomology class of b_M depends not only on the cohomology class $[c_1]$ of c_1 but also on the *entire sequence* $c_2, \ldots, c_{M-1}$. This is why lifting an infinitesimal deformation to a family $A_\hbar$ is a highly non trivial problem. Indeed, if maps $c_1, \ldots, c_{M-1}$ satisfying (1.3) are chosen, then any two maps c_M, c'_M compatible with them must satisfy $d(c_M - c'_M) = 0$, i.e. they must differ by a cocycle. Moreover if two solutions differ by a coboundary they give rise to isomorphic M-th order deformations. In other words, the freedom in choosing the solution at step M lies in $H^2(A,A)$. The problem is that a particular choice of c_M affects the equations in the hierarchy for *all* $m > M$, and can determine an obstruction at *any* of the next steps. If $H^3(A,A)$ is zero though, all obstructions vanish and any first order deformation can be lifted, i.e. it is the differential of a family of flat formal deformations.

From the previous discussion it appears that the space $H^2(A,A)$ is a natural candidate to parametrize a *universal deformation* for the algebra A, in a sense that we are going to clarify.

Let us look at formula (1.1) for a deformation with parameters in an n dimensional vector space U. Imposing the associativity condition and arguing as in the one parameter case, one obtains that $c_{0,\ldots,1_j,\ldots 0}$ must be Hochschild 2-cocycles for each j.

Thus any such deformation defines a natural linear map ϕ from the space of parameters U to $H^2(A,A)$. Such a map is given by the assignment

$$\begin{array}{rccc} \phi: & U & \longrightarrow & H^2(A,A) \\ & (\hbar_1,\cdots,\hbar_n) & \longrightarrow & \sum_j \hbar_j\,[c_{0,\ldots,1_j,\ldots 0}] \end{array}$$

for any $(\hbar_1,\cdots,\hbar_n)\in U$, where $[C]$ stands for the cohomology class of a cocycle C.

Proposition 1.3.4. *If the space $H^2(A,A)$ is finite dimensional and $H^3(A,A)=0$ then there exists a flat formal deformation A^{u} over $k[[U]]$, where $U:=H^2(A,A)$, such that the map ϕ above is the identity. This deformation is unique up to isomorphism (we allow automorphisms of $k[[U]]$).*

Proof. Suppose $\dim H^2(A,A)=n$. For a multi-index $\underline{p}=(p_1,...,p_n)\in\mathbb{Z}^n_{\geq 0}$, let $|\underline{p}|=p_1+\cdots+p_n$ denote its length. For any $j=1,...,n$ let e_j be the multi-index $(0,...,1_j,...,0)$. Fix a basis $[c_{e_1}],\ldots,[c_{e_n}]$ for $H^2(A,A)$ and let $\hbar_1,\ldots,\hbar_n$ be coordinates relative to this basis . We claim that, up to isomorphism and automorphisms of $k[[U]]$, a deformation as in the statement of the theorem must be given by a formula

$$a*b=ab+\sum_{i=1}^{n}c_{e_i}(a,b)\hbar_i+\sum_{|\underline{p}|>1}c_{\underline{p}}(a,b)\mathbf{h}^{\underline{p}} \tag{1.4}$$

for some choice of representatives $c_{e_1},\ldots,c_{e_n}$ of the above basis and for some k-bilinear maps $c_{\underline{p}}:A\times A\to A$. This is true because, first of all, the map ϕ must be the identity. Secondly, arguing as in the proof of Theorem 1.3.3, one can see that different choices of representatives for the basis $[c_{e_1}],\ldots,[c_{e_n}]$ do not affect the representation at order one up to isomorphism. Finally a change of basis in $H^2(A,A)$ changes such a deformation by the corresponding induced automorphism of $k[[U]]\cong k[[\hbar_1,...,\hbar_n]]$.

It is easy to compute that the condition that formula (1.4) defines a deformation over $k[[\hbar_1,...,\hbar_n]]$ gives, for each $\underline{p}$ with $|\underline{p}|>1$, an equation

$$dc_{\underline{p}}=b_{\underline{p}} \tag{1.5}$$

where $b_{\underline{p}}$ is a 3-cocycle whose expression may involve $c_{\underline{q}}$ only for $|\underline{q}| < |\underline{p}|$. Since $H^3(A, A) = 0$ any 3-cocycle is a coboundary, thus we can recursively solve the equations above and find $c_{\underline{p}}$s that give a deformation as desired.

What is left now to show is that such a deformation is unique. We discussed uniqueness at order one. It remains to prove that the deformation does not depend on our choices of $c_{\underline{p}}$s for $|\underline{p}| > 1$ up to isomorphism (possibly involving automorphisms of $k[[\hbar_1, ..., \hbar_n]]$). We will show this by induction on $|\underline{p}|$. Let now $A^{\mathfrak{u}}$ and $A'^{\mathfrak{u}}$ be two deformations given by two sets of maps $\{c_{\underline{p}}\}$, $\{c'_{\underline{p}}\}$ respectively. Let N be the maximal number such that, up to automorphism of $A^{\mathfrak{u}}$ and $A'^{\mathfrak{u}}$, it is possible to set $c_{\underline{p}} = c'_{\underline{p}}$ for any $\underline{p}$ such that $|\underline{p}| < N$. Then for any $\underline{q}$ with $|\underline{q}| = N$ we have $b_{\underline{q}} = b'_{\underline{q}}$, where $b_{\underline{q}}$ and $b'_{\underline{q}}$ are the cocycles on the right hand side of equation (1.5) for $c_{\underline{q}}$ and $c'_{\underline{q}}$ respectively. This means that $d(c_{\underline{q}} - c'_{\underline{q}}) = 0$ i.e. the two maps differ by a cocycle. Thus we can write

$$c_{\underline{q}} = c'_{\underline{q}} + \sum_j \alpha_{\underline{q}j} c_{e_j}$$

with $\alpha_{\underline{q}j} \in \mathbb{C}$. Consider now the automorphism ψ_N of $k[[\hbar_1, ..., \hbar_n]]$ defined by the assignment

$$\hbar_j \longrightarrow \hbar_j + \sum_{|\underline{q}|=N} \alpha_{\underline{q}j} \mathbf{h}^{\underline{q}}.$$

It is easy to see that twisting $A'^{\mathfrak{u}}$ with ψ_N we can set $c_{\underline{q}} = c'_{\underline{q}}$ for any $\underline{q}$ with $|\underline{q}| = N$, without affecting the $c_{\underline{p}}$s with $|\underline{p}| < N$. This contradicts the maximality of N.

□

When it exists, the deformation $A^{\mathfrak{u}} = A^{\mathfrak{u}}_{\hbar_1,\ldots,\hbar_n}$ of Theorem 1.3.4 is called the *universal deformation* of A.

The existence of the universal deformation guarantees that the moduli space of one parameter deformations is a smooth space, given by the formal neighborhood of zero in $H^2(A, A)$. This fact is a consequence of the following proposition stating the universal property of $A^{\mathfrak{u}}$.

Proposition 1.3.5. *For every flat formal one parameter deformation $A_\hbar$ there exists a unique power series $\alpha(\hbar) = (\alpha_1(\hbar), \ldots, \alpha_n(\hbar)) \in \hbar H^2(A, A)[[\hbar]]$ such that there is* *an isomorphism* $A_\hbar \cong A^{\mathrm{u}}_{\alpha_1(\hbar),\dots,\alpha_n(\hbar)}$ *of flat formal deformations. Moreover* $\alpha'(0)$ *is the cohomology class of the differential* c_1 *of the family* $A_\hbar$*.*

Proof. Let us denote by $*_{\mathrm{u}}$ the multiplication in A^{u}. Suppose $\dim H^2(A,A) = n$. For any a, b in A, regard $a *_{\mathrm{u}} b = a *_{\mathrm{u}} b(\hbar_1, \dots, \hbar_n)$ as a formal function from $U := H^2(A,A)$ to A. Consider now a one parameter deformation $A_\hbar = (A[[\hbar]], *)$, and regard $a * b = a * b(\hbar)$ as a formal function from k to A. We have to show that, up to isomorphism, the algebra $A_\hbar$ is given by a formula $a * b(\hbar) = a *_{\mathrm{u}} b(\alpha(\hbar))$ for a unique power series.

Thus we have to find $\alpha_i(\hbar) \in \hbar k[[\hbar]]$ for $i = 1, \dots n$ such that, for any $a, b \in A$, the identity

$$\sum_j c_j(a,b)\hbar^j = \sum_i c_{e_i}(a,b)\alpha_i(\hbar) + \sum_{|p|>1} c_{\underline{p}}(a,b)\alpha_1(\hbar)^{p_1}\cdots\alpha_n(\hbar)^{p_n} \tag{1.6}$$

is satisfied. If $\alpha_i(\hbar) = \sum_{r>1} \alpha_{ir}\hbar^r$ then one can compute that the condition for (1.6) to be satisfied at order one is

$$\sum_i \alpha_{i1}[c_{e_i}] = [c_1].$$

Clearly there exist unique $\alpha_{11}, \dots, \alpha_{1n}$ satisfying this equation. In general one can compute that the condition that (1.6) is satisfied at order n is given by

$$\sum_i \alpha_{in}[c_{e_i}] = [\tilde{c}_n]$$

where $\tilde{c}_n$ is a cocycle depending on α_{ij} with $j < n$. It is clear that this equation has a unique solution.

□

We want to end this section by mentioning that the conditions of Proposition 1.3.5 are not necessary for the existence of the universal deformation. Indeed, all formal obstructions to deformations can vanish even when $H^3(A,A)$ is nonzero, although this fact might be extremely difficult to prove.

1.4 Flat formal deformations of modules

Let now M be a left A-module. Let A_U be a flat formal deformation of A over $k[[U]]$, where U is some finite dimensional vector space, and let $\varphi : A \longrightarrow A_U/\mathfrak{m}A_U$ be the fixed isomorphism of Definition 1.2.1. The following definition formalizes the intuitive notion of a deformation of M to an A_U-module.

Definition 1.4.1. *A flat formal deformation of the module M is a A_U-module M_U which is topologically free as a $k[[U]]$-module together with a fixed isomorphism of A-modules $\gamma : M \longrightarrow M_U/\mathfrak{m}M_U$, where the structure of A module on $M_U/\mathfrak{m}M_U$ is the one induced by the isomorphism φ.*

The fundamental tool for the study of deformations of modules is again Hochschild cohomology.

Indeed, let $\hbar_1, \ldots, \hbar_n$ be coordinates in U and let $\underline{p} = (p_1, \ldots, p_n)$ be a multi-index in $\mathbb{Z}^n_{\geq 0}$. Arguing as in Section 1.2, we can think of M_U as the $k[[\hbar_1, ..., \hbar_n]]$-module $M[[\hbar_1, \ldots, \hbar_n]]$ together with a $k[[\hbar_1, ..., \hbar_n]]$ algebra homomorphism $\widetilde{\rho} : A \longrightarrow \operatorname{End} M[[\hbar_1, \ldots, \hbar_n]]$ given by a formula

$$\widetilde{\rho}(a) = \sum_{\underline{p}} \rho_{\underline{p}}(a) h^{\underline{p}}, \tag{1.7}$$

where $\rho_{\underline{p}} : A \longrightarrow \operatorname{End} M$ are k-linear maps and $\rho_{0,\ldots,0}(a) = \rho(a)$, where ρ is the homomorphism giving the representation M.

Imposing the condition that $\widetilde{\rho}$ is a homomorphism gives a hierarchy of equations

$$d\rho_{\underline{p}}(a, b) = -\sum_{\substack{\underline{q} + \underline{s} = \underline{p} \\ |\underline{q}|, |\underline{s}| > 0}} \rho_{\underline{q}}(a)\rho_{\underline{s}}(b) + \sum_{\substack{\underline{q} + \underline{s} = \underline{p} \\ |\underline{q}|, |\underline{s}| > 0}} \rho_{\underline{q}}(c_{\underline{s}}(a, b)) \tag{1.8}$$

where d is the Hochschild differential and the $c_{\underline{s}}$s are the maps defining the product in A_U as in formula (1.1).

If $\rho_{\underline{q}}$s satisfy the above equation for all $\underline{q}$ with $|\underline{q}| < N$, then for any $\underline{p}$ with $|\underline{p}| = N$

the right hand side is a cocycle. Thus, all the obstruction to the deformation of the module M lie in $H^2(A, \operatorname{End} M)$. At each step, the freedom in choosing a solution for (1.8) lies in $H^1(A, \operatorname{End} M)$.

It is clear from this discussion that the problem of deforming modules presents similar difficulties to the one of deforming algebras, and that in general an A-module M does not admit any deformation to a representation of A_U. Nevertheless, sometimes the (Hochschild-)cohomological properties of A and M are such that it is possible to find deformations, as we will see in the next chapters of this thesis.

Chapter 2

Symplectic reflection algebras

2.1 Plan of the chapter

In this chapter we recall the basics of the theory of symplectic reflection algebras. In Section 2.2 we give the general definition, and we describe the main properties of these algebras such as the PBW property. In Section 2.3 we consider more specifically symplectic reflection algebras associated to wreath product groups, that will be the object of interest of this thesis.

2.2 Definition and properties

Let (V, ω) be a symplectic vector space over $\mathbb{C}$, and let $G \subset \mathrm{Sp}(V)$ be a finite group of symplectomorphisms. Denote by TV the tensor algebra of V and by $\mathbb{C}[G]$ the group algebra. For any two vectors $u, v \in V$ we will write uv for the tensor product $u \otimes v$.

Definition 2.2.1. *The smash product algebra $TV\sharp G$ is the vector space $TV \otimes_{\mathbb{C}} \mathbb{C}[G]$ with the product defined by the formula*

$$(u \otimes g)(v \otimes h) = u(gv) \otimes gh$$

for any $u, v \in V$ and $g, h \in G$.

Note that assigning grade degree zero to the elements of $\mathbb{C}[G]$ and grade degree one to the elements of V the algebra $TV\sharp G$ becomes a graded algebra (with a corresponding filtration).

The main object of our interest will be a family of algebras obtained as quotients of the above smash product.

Definition 2.2.2. *An element $s \in G$ is called a symplectic reflection if* $\mathrm{rk}(\mathrm{Id}-s) = 2$.

Symplectic reflections can be considered as symplectic analogs of complex reflections. Indeed, any symplectic reflection s fixes a (complex) codimension two space pointwise and acts diagonally on a complement with complex conjugate eigenvalues of norm one. In other words there exists a basis such that s is a diagonal matrix

$$s = \begin{pmatrix} \lambda & & & & \\ & \lambda^{-1} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

with $\lambda \neq 1$ and $|\lambda| = 1$. We will denote by $\mathcal{S}$ the set of symplectic reflections. By definition, this set is stable under conjugation. We will denote by $C(\mathcal{S})$ the vector space $\mathbb{C}[\mathcal{S}]^G$ of $\mathbb{C}$-valued class functions on $\mathcal{S}$, and we will write $f_s = f(s)$ for any $f \in C(\mathcal{S})$.

For any $s \in \mathcal{S}$ consider the ω-orthogonal decomposition $\mathrm{Im}(\mathrm{Id}-s) \oplus \mathrm{Ker}(\mathrm{Id}-s)$. Denote by ω_s the skew symmetric form that coincides with ω on $\mathrm{Im}(\mathrm{Id}-s)$ and has $\mathrm{Ker}(\mathrm{Id}-s)$ as its radical.

Definition 2.2.3 ([EG02])**.** *For any $f \in C(\mathcal{S})$ and any constant $t \in \mathbb{C}$ the symplectic reflection algebra $H_{t,f}(G)$ is the quotient of the smash product $TV\sharp G$ by the relations*

$$uv - vu = t\,\omega(u,v) + \sum_{s \in \mathcal{S}} f_s\,\omega_s(u,v)\,s \tag{2.1}$$

for any $u, v \in V$.

Let now $e = \frac{1}{|G|}\sum_{g\in G} g \in \mathbb{C}[G]$ be the averaging idempotent.

Definition 2.2.4. *The spherical subalgebra is the algebra* $eH_{t,f}(G)e \subset H_{t,f}(G)$ *.*

Note that that $eH_{t,f}(G)e$ does not contain the unit element of $H_{t,f}(G)$.

When both t and f are zero, one has $uv - vu = 0$ for any $u, v \in V$. Thus, if we denote by SV the symmetric algebra of V we have:

$$H_{0,0}(G) = SV\sharp G.$$

In particular the algebra $H_{0,0}(G)$ is graded. Moreover, the isomorphism $e(SV\sharp G)e \cong SV^G$, where SV^G is the algebra of invariant polynomials, yields an isomorphism

$$eH_{0,0}(G)e \cong SV^G$$

for the spherical subalgebra.

In general though, the defining relations (2.1) are not homogeneous and the algebra $H_{t,f}(G)$ does not inherit the grading of $TV\sharp G$ but only a filtration $\mathcal{F}_\bullet$. Consider the associated graded algebra

$$\mathrm{gr}(H_{t,f}(G)) = \bigoplus_i \mathcal{F}_i(H_{t,f}(G))/\mathcal{F}_{i-1}(H_{t,f}(G)).$$

Since $uv - vu$ lies in degree two for any $u, v \in V$, while any element of the group algebra lies in degree zero it is clear from the defining relations that u, v commute in $\mathrm{gr}(H_{t,f}(G))$. Thus there exists a surjective homomorphism of graded algebras

$$\phi : H_{0,0}(G) = SV\sharp G \twoheadrightarrow \mathrm{gr}(H_{t,f}(G)).$$

One of the most important properties of the algebra $H_{t,f}(G)$ is that the above homomorphism is also injective, as stated in the next theorem, called *Poincaré-Birkhoff-Witt(PBW)-Theorem* in analogy with the PBW-Theorem for the universal enveloping algebra of a Lie algebra.

Theorem 2.2.5. *([EG02], Theorem 1.3) For any $t \in \mathbb{C}$ and $f \in C(\mathcal{S})$ the above homomorphism is an isomorphism.*

It is clear that the PBW theorem also gives an isomorphism

$$\mathrm{gr}(eH_{t,f}(G)e) \cong SV^G.$$

It is not hard to see that rescaling the parameters t, f by a non-zero complex number does not change $H_{t,f}(G)$ up to isomorphism. Thus, in particular, for any $t \neq 0$ there is an isomorphism $H_{t,f}(G) \cong H_{1,f/t}(G)$. This reduces the study of symplectic reflection algebras to the two cases $t = 0$ (quasi-classical case) and $t = 1$ (quantum case) that present substantial differences. In this thesis we will be concerned with the second case.

Suppose then that $t = 1$. In this case, specializing the parameter f to 0 we get an isomorphism

$$H_{1,0}(G) \cong \mathcal{W}\sharp G$$

where $\mathcal{W}$ is the *Weyl algebra* of the symplectic vector space (V, ω) i.e.

$$\mathcal{W} := \frac{TV}{\langle uv - vu = \omega(u,v)\rangle_{u,v\in V}}.$$

Regard now f as a formal parameter and the family $\{H_{1,f}(G)\}_f$ as a single algebra over $\mathbb{C}[[C(\mathcal{S})]]$. Thanks to the PBW property stated in Theorem 2.2.5, $\{H_{1,f}(G)\}_f$ has no torsion as a $\mathbb{C}[[C(\mathcal{S})]]$-module. Thus the corresponding family is flat over $\mathbb{C}[[C(\mathcal{S})]]$ and can be seen as a flat formal deformation of $\mathcal{W}\sharp G$ (in this case the power series defining the star product is a degree one polynomial in f, thus it converges for any value of f and the deformation is not only formal). The following theorem is due to Etingof and Ginzburg.

Theorem 2.2.6 ([EG02])**.** *The family $\{H_{1,f}(G)\}_f$ is the universal deformation of the algebra $\mathcal{W}\sharp G$ and the family $\{eH_{1,f}(G)e\}_f$ is the universal deformation of the algebra of invariants $\mathcal{W}^G$.*

Proof. In [AFLS00] Alev, Farinati, Lambre and Solotar proved that the dimensions of the Hochschild cohomology groups of $\mathcal{W}\sharp G$ are as follows

$$\dim H^i(\mathcal{W}\sharp G) = \begin{cases} \mathbf{n}(j) & \text{if} \quad i = 2j \\ 0 & \text{if} \quad j \text{ is odd} \end{cases}$$

where

$$\mathbf{n}(j) = \sharp \left\{ \text{conjugacy classes of elements } g \in G \text{ such that } \mathrm{rk}(\mathrm{Id} - g) = 2j \right\}.$$

In particular, $H^3(\mathcal{W}\sharp G, \mathcal{W}\sharp G) = 0$ and $\dim H^2(\mathcal{W}\sharp G, \mathcal{W}\sharp G) = \dim C(\mathcal{S})$. Thus Theorem 1.3.4 guarantees the existence of the universal deformation and the fact that it is parametrized by the space $C(\mathcal{S})$. The only thing left to do is proving that the family $\{H_{1,f}(G)\}_f$ is actually the one satisfying the condition in Theorem 1.3.4. In other words, it must be verified that the cohomology classes corresponding to the family of infinitesimal (order one) deformations $\{H_{1,f}(G)/\mathfrak{m}H_{1,f}(G)\}_f$ (here $\mathfrak{m}$ is the maximal ideal in $\mathbb{C}[[C(\mathcal{S})]]$) span the whole space $C(\mathcal{S})$ and not a smaller one. This is done in [EG02]. The proof for $\{H_{1,f}(G)/\mathfrak{m}H_{1,f}(G)\}_f$ is similar.

□

2.3 The wreath-product construction

In this thesis we will be interested in the study of the symplectic reflection algebras attached to a special family of groups generated by symplectic reflections, provided by the *wreath product* construction.

Let Γ be a finite subgroup of $SL(2,\mathbb{C})$, and let Γ^n be the direct product $\underbrace{\Gamma \times \cdots \times \Gamma}_{n \text{ factors}}$. Let S_n be the symmetric group of rank n.

Definition 2.3.1. *The wreath product* $\mathbf{\Gamma}_n := S_n \ltimes \Gamma^n$ *is the semi-direct product of* S_n *and* Γ^n*, where* Γ^n *is normal, and the action of* S_n *on* Γ^n *by conjugation is the natural one in which* S_n *permutes the direct factors of* Γ^n*.*

Let now L be a 2-dimensional complex vector space with a symplectic form ω_L, and consider the space $V = L^{\oplus n}$, endowed with the induced symplectic form $\omega_V = \omega_L{}^{\oplus n}$.

Choosing a symplectic basis we can identify $Sp(L)$ with $SL(2, \mathbb{C})$. Clearly, the natural (faithful) action of the wreath product group $\mathbf{\Gamma}_n$ on V, where each factor Γ in Γ^n acts on the corresponding summand L in V, and S_n permutes such summands, is symplectic. Thus $\mathbf{\Gamma}_n \subset Sp(V)$.

In the sequel we will write $\gamma_i \in \mathbf{\Gamma}_n$ for any element $\gamma \in \Gamma$ seen as an element in the i-th factor Γ of Γ^n. $\mathbf{\Gamma}_n$ acts by conjugation on the set $\mathcal{S}$ of its symplectic reflections. It is easy to see that there are symplectic reflections of two types in $\mathbf{\Gamma}_n$:

(S) the elements $s_{ij}\gamma_i\gamma_j{}^{-1}$ where $i, j \in [1, n]$, s_{ij} is the transposition $(ij) \in S_n$, and $\gamma \in \Gamma$;

(Γ) the elements γ_i, for $i \in [1, n]$ and $\gamma \in \Gamma \smallsetminus \{1\}$.

Elements of type (S) are all in the same conjugacy class, while elements of type (Γ) form one conjugacy class for any nontrivial conjugacy class in Γ. Thus functions $f \in C(\mathcal{S})$ can be written as pairs (k, c), where k is a number (the value of f on elements of type (S)), and c is a conjugation invariant function on $\Gamma \smallsetminus \{1\}$ (encoding the values of f on the elements of type (Γ)).

Definition 2.3.2. *The wreath product symplectic reflection algebra $H_{t,k,c}(\mathbf{\Gamma}_n)$ is the symplectic reflection algebra attached to the vector space V, the group $\mathbf{\Gamma}_n$ and the parameters $t \in \mathbb{C}$ and $f = (k, c)$.*

We will now give a more explicit presentation of the algebra $H_{t,k,c}(\mathbf{\Gamma}_n)$. For any vector $u \in L$ and any $i \in [1, n]$ we will write $u_i \in V$ for u placed in the i-th summand of V. In particular from now on we will fix a symplectic basis $\{x, y\}$ of L ($\omega_L(x, y) = 1$) and we will denote by $\{x_i, y_i\}$ the corresponding symplectic basis for V. We will also write c_γ for the value of the function c on the element $\gamma \in \Gamma$.

Lemma 2.3.3. *([GG], Lemma 3.1.1) The algebra $H_{1,k,c}(\mathbf{\Gamma}_n)$ is the quotient of $TV\sharp\mathbf{\Gamma}_n$ by the following relations:*

(R1) *For any $i \in [1,n]$:*

$$[x_i, y_i] = t + \frac{k}{2}\sum_{j\neq i}\sum_{\gamma\in\Gamma} s_{ij}\gamma_i\gamma_j^{-1} + \sum_{\gamma\in\Gamma\smallsetminus\{1\}} c_\gamma\gamma_i\,.$$

(R2) *For any $u, v \in L$ and $i \neq j$:*

$$[u_i, v_j] = -\frac{k}{2}\sum_{\gamma\in\Gamma} \omega_L(\gamma u, v) s_{ij}\gamma_i\gamma_j^{-1}\,.$$

□

We will call the integer n the *rank* of the algebra $H_{1,k,c}(\mathbf{\Gamma}_n)$.

We want now to have a closer look at some interesting examples.

Example 2.3.4. *When the rank n is one, there is no parameter k (there are no symplectic reflections of type (S)). Let now $\mathbf{c} = t + \sum_{\gamma\in\Gamma\smallsetminus\{1\}} c_\gamma\gamma$ be the central element of $\mathbb{C}[\Gamma]$ corresponding to the class function coinciding with c on $\Gamma\smallsetminus\{1\}$ and assuming value t on the identity element. If we identify the tensor algebra algebra TL with the ring $\mathbb{C}\langle x, y\rangle$ of noncommutative polynomials in x, y, then the rank one wreath product symplectic reflection algebra is the quotient*

$$H_{t,c}(\Gamma) := \frac{\mathbb{C}\langle x, y\rangle \sharp \Gamma}{\langle [x,y] - \mathbf{c}\rangle}.$$

The algebra $H_{t,c}(\Gamma)$ has interesting connections with the Kleinian singularity $\mathbb{C}^2/\Gamma$ (the spherical subalgebra $eH_{t,c}(\Gamma)e$ is a non-commutative deformation of the ring of invariants $\mathbb{C}[x,y]^\Gamma$) and it was studied by Crawley-Boevey and Holland in [CBH98]. In particular, as we will see in the next chapter, a complete classification of the simple finite dimensional $H_{t,c}(\Gamma)$ module is available for all values of the parameters.

Example 2.3.5. *When $k = 0$ the defining relations (R1), (R2) simplify drastically and there is an isomorphism*

$$H_{t,0,c}(\mathbf{\Gamma}_n) \cong H_{t,c}(\Gamma)^{\otimes n} \sharp S_n. \tag{2.2}$$

Example 2.3.6. *Suppose $\Gamma = \mathbb{Z}/m\mathbb{Z}$ is cyclic of some order m. In this case there is a splitting $V = \mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h}$ is the reflection representation of $S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$ (as a complex reflection group) and $\mathfrak{h}^*$ is its dual. The symplectic form ω_V can be identified with the natural pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$ that become Lagrangian subspaces. Thus the vector space V can be seen as the cotangent bundle of $\mathfrak{h}$ endowed with its natural structure of symplectic manifold and with the diagonal (Hamiltonian) action of $S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$. The algebra $H_{t,k,c}(S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n)$ is a special case of rational Cherednik algebra. As a vector space, this algebra has a decomposition $S\mathfrak{h} \otimes \mathbb{C}[\mathbf{\Gamma}_n] \otimes S\mathfrak{h}^*$, analog to the triangular decomposition for the universal enveloping algebra of a semisimple Lie algebra.*

As already mentioned, in this thesis we will be concerned with the representation theory of the wreath product algebra when $t = 1$. Although our results will be true whenever Γ is nontrivial, we want to recall that when Γ is cyclic, i.e. in the case of the rational Cherednik algebra, there exists a more general (and effective) approach to the study of representations. For rational Cherednik algebras, in fact, an analog of category $\mathcal{O}$ for finite dimensional semisimple Lie algebras has been defined, as well as a theory of standard modules and formal characters ([BEG03], [GGOR03]).

For completeness, we want to end this section with a few words about the case $t = 0$. Suppose $c \neq 0$, the fact that makes the quasi-classical case notably different is that the algebra $H_{0,k,c}(\mathbf{\Gamma}_n)$ has a large center $Z_{0,k,c}$ such that

$$\mathrm{gr}(Z_{0,k,c}) = SV^{\mathbf{\Gamma}_n}$$

and

$$Z_{0,k,c} \cong eH_{0,k,c}(\mathbf{\Gamma}_n)e.$$

The following theorem is due to Etingof and Ginzburg.

Theorem 2.3.7. *([EG02], Corollary 1.14) If the parameters (k, c) are generic, all irreducible $H_{0,k,c}(\mathbf{\Gamma}_n)$-modules are finite dimensional of dimension $|\mathbf{\Gamma}_n| = n!|\Gamma|^n$, and are isomorphic to the regular representation of $\mathbf{\Gamma}_n$ as $\mathbf{\Gamma}_n$-modules. Moreover,*

$\operatorname{Spec} Z_{0,k,c}$ *is a smooth algebraic variety, and irreducible modules are parametrized up to isomorphism by the points of* $\operatorname{Spec} Z_{0,k,c}$ *via the map that assigns to each module its central character.*

Chapter 3

Finite dimensional representations for $H_{1,k,c}(\mathbf{\Gamma}_n)$

3.1 Plan of the chapter

In this chapter we will present two different methods to produce examples of finite dimensional representations for the algebra $H_{1,k,c}(\mathbf{\Gamma}_n)$. Both methods start from simple observations.

In the first place from Definition 2.2.3 we can see that the algebra $H_{1,k,c}(\mathbf{\Gamma}_n)$ contains a copy of the group algebra $\mathbb{C}[\mathbf{\Gamma}_n]$. Thus, the simplest thing to do is trying to classify all irreducible $\mathbf{\Gamma}_n$-modules that extend to representations of the whole algebra $H_{1,k,c}(\mathbf{\Gamma}_n)$. We give a complete answer to this problem in Section 3.2, which is based on the paper [Mon07b].

Secondly, in the rank one case a complete classification of the finite dimensional representations for the wreath product algebra $H_{1,c}(\Gamma)$ is available, thanks to the results of Crawley-Boevey and Holland ([CBH98]). Moreover, as observed in Example 2.3.5

$$H_{1,0,c}(\mathbf{\Gamma}_n) = H_{1,c}(\Gamma)^{\otimes n} \sharp S_n$$

i.e. when the parameter k is zero, the rank n algebra is simply the smash product of the tensor product of n copies of the rank one algebra with S_n (where S_n acts by permuting the factors). As a consequence, in this case, finite dimensional irreducible representations are also known (they can be recovered from a knowledge of the irreducible finite dimensional representations in rank one and from well known results by Macdonald about skew group algebras as explained in Section 3.3.4). We observe now that that the algebra $H_{1,0,c}(\mathbf{\Gamma}_n)$ has a flat formal deformation over the finite dimensional vector space $C(\mathcal{S})$ given by $H_{1,k,c+c'}$. The fact that this deformation is flat follows from the PBW Theorem 2.2.5. Using this observation in Section 3.3 we determine some sufficient conditions for a $H_{1,0,c}(\mathbf{\Gamma}_n)$-module to be deformed to values of the parameters with nonzero k. This last section is based on the papers [EM05] and [Mon07a].

We recall that in the case of cyclic (nontrivial) Γ some finite dimensional representation were constructed by Chmutova and Etingof in [CE03] before our work.

3.2 Extending irreducible $\mathbf{\Gamma}_n$-modules

3.2.1 Irreducible representations of wreath product groups

For the reader's convenience, and in order to introduce some important notation, we recall the classification of irreducible representations for a wreath product group. Everything that follows is true for any finite group Γ and for representations over any algebraically closed field F of characteristic 0. For simplicity we will consider $F = \mathbb{C}$, the field of complex numbers. For complete proofs and details the reader should refer to [JK81], Chapter 4.

A nice property of the wreath product group $\mathbf{\Gamma}_n$ is that the set of its irreducible representations $\mathrm{Irr}(\mathbf{\Gamma}_n)$ can be completely recovered from a knowledge of $\mathrm{Irr}(\Gamma)$, using the representation theory of the symmetric group.

Let $\{N_1, \ldots, N_\nu\}$ denote a complete set of pairwise non-isomorphic representations of Γ over $\mathbb{C}$. Then a complete set of irreducible representations of Γ^n is given by $N = N_{h_1} \otimes \cdots \otimes N_{h_n}$ where $(h_1, \ldots, h_n)$ varies in $[1, \nu]^n$. If n_h denotes the number of indices i s.t. $h_i = h$, i.e. the number of factors of N equal to N_h, then

$$\vec{n} = (n_1, \dots, n_\nu)$$

is called the *type* of N.

We will say that two representations N, N' are *conjugate* if they have the same type. This simply means that $N = N_{h_1} \otimes \cdots \otimes N_{h_n}$ and $N' = N_{h_{\sigma(1)}} \otimes \cdots \otimes N_{h_{\sigma(n)}}$ for some $\sigma \in S_n$, i.e. N' equals the representation N twisted by the outer automorphism of Γ^n that permutes the factors according to σ. It turns out that the role played by conjugate representations of Γ^n in recovering irreducible representations of $\mathbf{\Gamma}_n$ is exactly the same. This is essentially because, as one can easily argue from Definition 2.3.1, the outer automorphism induced by $\sigma \in S_n$ on Γ^n is a restriction of an inner automorphism in $\mathbf{\Gamma}_n$ (conjugation by the element $\sigma \in \mathbf{\Gamma}_n$). So from now on we will consider only the representations of Γ^n that can be written as $N = N_1^{\otimes n_1} \otimes \cdots \otimes N_\nu^{\otimes n_\nu}$. Notice that the representations of this form are a complete set of irreducible, pairwise non-conjugate representations of Γ^n.

For any h, we denote by S_{n_h} the subgroup of S_n consisting of the permutations that move only the indices $\{\sum_{i=1}^{h-1} n_i + 1, \dots, \sum_{i=1}^{h} n_i\}$, corresponding to the factors of N isomorphic to N_h. We agree that $S_{n_h} = \{1\}$ if $n_h = 0$. Thus we can consider the group

$$S_{\vec{n}} = S_{n_1} \times \cdots \times S_{n_\nu} \subset S_n \subset S_n \ltimes \Gamma^n$$

called the *inertia factor* of N. Obviously any irreducible representation X of $S_{\vec{n}}$ is obtained as $X = X_1 \otimes \cdots \otimes X_\nu$, where X_h is an irreducible representation of S_{n_h}.

The *inertia subgroup* of N, instead, is defined to be

$$(\mathbf{\Gamma}_n)_N = S_{\vec{n}} \ltimes \Gamma^n \subset S_n \ltimes \Gamma^n.$$

Let's now consider an irreducible Γ^n-module $N = N_1^{\otimes n_1} \otimes \cdots \otimes N_\nu^{\otimes n_\nu}$. There is a natural action of $(\mathbf{\Gamma}_n)_N$ on N in which Γ^n acts in the obvious way, and $S_{\vec{n}}$ permutes the factors. This representation can be shown to be irreducible. For simplicity we will keep the notation N for this representation.

Another easy way to obtain irreducible representations of $(\mathbf{\Gamma}_n)_N$ is extending an irreducible representation $X = X_1 \otimes \cdots \otimes X_\nu$ of $S_{\vec{n}}$ by making Γ^n act trivially. In this case we will also keep the notation X for this extension.

Let's now consider the tensor product of X and N

$$X \otimes N = (X_1 \otimes \cdots \otimes X_\nu) \otimes \left(N_1^{\otimes n_1} \otimes \cdots \otimes N_\nu^{\otimes n_\nu}\right).$$

Here $S_{\vec{n}}$ acts both on X and on N(permuting the factors), while Γ^n acts only on N. This is also an irreducible representation of $(\mathbf{\Gamma}_n)_N$ ([JK81], page 155).

We can now obtain the induced representation of $\mathbf{\Gamma}_n$:

$$X \otimes N \uparrow := \mathrm{Ind}_{(\mathbf{\Gamma}_n)_N}^{\mathbf{\Gamma}_n} X \otimes N$$

The following theorem holds.

Theorem 3.2.1. *The representation $X \otimes N \uparrow$ is irreducible and runs through a complete system of pairwise non-isomorphic irreducible representations of $\mathbf{\Gamma}_n$ if N runs through a complete system of pairwise non-conjugate irreducible representations of Γ^n and, while N remains fixed, X runs through a complete system of pairwise non-isomorphic irreducible representations of $S_{\vec{n}}$.*

In particular we have that, for a fixed X, the representation $X \otimes N \uparrow$ depends only on the type of N. With abuse of language we will call *type* of $X \otimes N \uparrow$ the type of N as a representation of Γ^n. We remark that the possible types of $X \otimes N \uparrow$ are in bijection with the ν-tuples $(n_1, \ldots, n_\nu)$, $n_h \geq 0$, $\sum_h n_h = n$ and that to any such ν-tuple we can attach a proper partition of n, taking all the non-zero n_hs in $(n_1, \ldots, n_\nu)$ and ordering them in non-increasing order.

Example 3.2.2. *Suppose all the factors of N are the same , i.e. $N = N_h^{\otimes n}$ for some $h \in \{1, \ldots, \nu\}$. The type of N is $(0, \ldots 0, n, 0, \ldots 0)$ with n in the h-th position and is associated to the partition of n of Young diagram a single row of length n, i.e. the partition corresponding to the trivial representation of S_n. For this reason we will* *call these representations of "trivial type". In this case the inertia factor of N is S_n, its inertia subgroup coincides with $S_n \ltimes \Gamma^n$, and we need no induction. For any irreducible representation X of S_n we obtain the irreducible representation $X \otimes N$ of $S_n \ltimes \Gamma^n$.*

3.2.2 McKay correspondence

It is well known (see for example [Cox91], Chapters 6,7) that all the finite subgroups of $SL(2, \mathbb{C})$, or equivalently the finite groups of quaternions, can be distinguished into two infinite series

- the cyclic groups $\mathcal{C}_{m+1}$ for any $m \geq 0$ ($\mathcal{C}_1 = \{1\}$), of order $m + 1$;
- the dicyclic groups $\mathcal{D}_{m-2}$ for $n \geq 4$, of order $4(m - 2)$;

and three exceptional groups that are the double coverings of the groups of rotations preserving regular polyhedra in $\mathbb{R}^3$ via the homomorphism of Lie groups $SU(2) \longrightarrow SO(3, \mathbb{R})$:

- the binary tetrahedral group $\mathfrak{T}$, of order 24;
- the binary octahedral group $\mathfrak{O}$, of order 48;
- the binary icosahedral group $\mathfrak{I}$, of order 120.

The terminology we used refers to the so called *McKay correspondence*, as we are going to explain. In ([McK81]) McKay showed that *"the eigenvectors of the Cartan matrices of affine type $\tilde{A}_m$, $\tilde{D}_m$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ can be taken to be the columns of the character tables of the finite groups of quaternions"*. To this end he attached a graph to any finite subgroup of $SL(2, \mathbb{C})$ in the following way. Consider the set of irreducible non-isomorphic representations of a finite group $\Gamma \subset SL(2, \mathbb{C})$, $I = \{N_1, \dots, N_\nu\}$, and let L be the defining representation of Γ, i.e. the representation of Γ as a subgroup of $SL(2, \mathbb{C})$. Notice that L is a self-dual representation. Now build the graph in which the set of vertices is I, and the number of edges between two vertices N_h and $N_{h'}$ is the multiplicity of the irreducible representation N_h in $N_{h'} \otimes L$ or equivalently, since L is self-dual, the multiplicity of $N_{h'}$ in $N_h \otimes L$. Any such graph turns out to be an extended Dynkin graph with extending vertex corresponding to the trivial representation. If we label each vertex with the dimension of the corresponding representation the result is the following. When $\Gamma = \mathcal{C}_{n+1}$, $n \geq 0$ is cyclic we get the extended Dynkin diagram $\tilde{A}_n$:

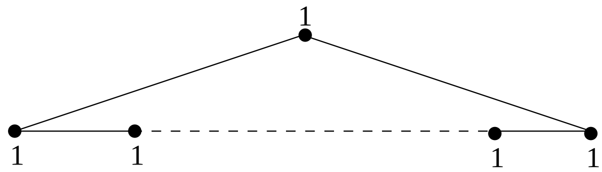

Figure 3-1: Graph $\tilde{A}_n$

When $\Gamma = \mathcal{D}_{n-2}$, $n \geq 4$ is dicyclic we get the diagram $\tilde{D}_n$:

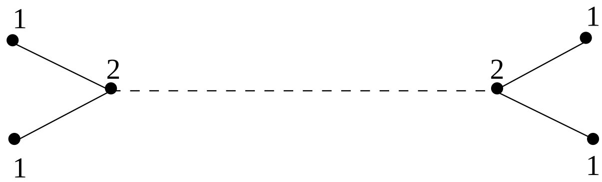

Figure 3-2: Graph $\tilde{D}_n$

When $\Gamma = \mathfrak{T}$ (binary tetrahedral), $\Gamma = \mathfrak{O}$ (binary octahedral) or $\Gamma = \mathfrak{I}$ (binary icosahedral) we get the extended Dynkin diagrams of type $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ respectively:

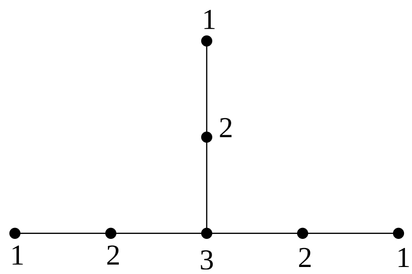

Figure 3-3: Graph $\tilde{E}_6$

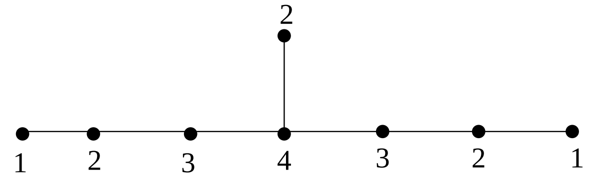

Figure 3-4: Graph $\tilde{E}_7$

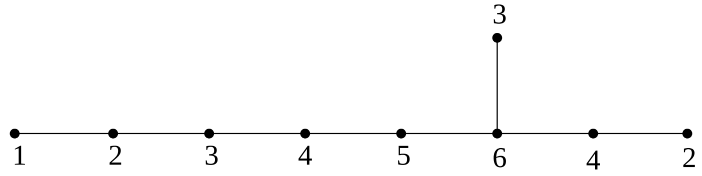

Figure 3-5: Graph $\tilde{E}_8$

In this setting, the adjacent vertices to a fixed vertex N_h correspond to the types of the irreducible components of the representation $N_h \otimes L$, while the number of connecting vertices corresponds to the multiplicities of such types. Note that the decomposition of $N_h \otimes L$ is multiplicity free (i.e. the diagram is simply laced) except when $\Gamma = \mathcal{C}_2$ (i.e. for type $\tilde{A}_1$). Thus, when $\Gamma \neq \mathcal{C}_2$, if for any $i = 1, \ldots, \nu$ we set $d_i = \dim N_i$, and we consider the vector $\delta = \{d_i\} \in \mathbb{Z}^\nu$ (corresponding to the above labeling), we get the "harmonic" property

$$2d_i = \sum_{j \text{adjacent to } i} d_j$$

for the extended Dynkin diagrams above.

3.2.3 Representations of S_n with rectangular Young diagram

In what follows we will use the following standard results from representation theory of the symmetric group. Denote by $\mathfrak{h}$ the reflection representation of S_n. For a Young diagram μ we denote by X_μ the corresponding irreducible representation of S_n and by $C(\mu)$ the content of μ, i.e. the sum of signed distances of the cells from the diagonal.

Lemma 3.2.3. *i)* $\mathrm{Hom}_{S_n}(\mathfrak{h} \otimes X_\mu, X_\mu) = \mathbb{C}^{m-1}$*, where m is the number of corners of the Young diagram μ. In particular* $\mathrm{Hom}_{S_n}(\mathfrak{h} \otimes X_\mu, X_\mu) = 0$ *if and only if μ is a rectangle.*

ii) The element $C = s_{12} + s_{13} + \cdots + s_{1n}$ *acts by a scalar in* X_μ *if and only if* μ *is a rectangle. In this case* $C|_{X_\mu} = \frac{2C(\mu)}{n}$.

iii) If μ *is a rectangular Young diagram of height* a *and width* b*, then* $C(\mu) = \frac{(b-a)\,n}{2}$.

Proof. Let $S_{n-1} \subset S_n$ be the subgroup of permutations fixing the index 1. It is well known that $X_\mu|_{S_{n-1}} = \sum X_{\mu-j}$, where the sum is taken over the corners of μ and $\mu - j$ is the Young diagram obtained from μ by cutting off the corner j. Since $\mathfrak{h} \oplus \mathbb{C} = \mathrm{Ind}_{S_{n-1}}^{S_n}\mathbb{C}$, the assertion *(*i) follows from the Frobenius reciprocity. To prove *(*ii), observe that C commutes with S_{n-1}, so acts by a scalar on each $X_{\mu-j}$. Thus, if μ is a rectangle, C acts as a scalar (as we have only one summand), and the "if" part of the statement is proved. To prove the "only if" part, let Z_n be the sum of all transpositions in S_n. Z_n is a central element in the group algebra, and it is known to act in X_μ by the scalar $\mathbf{c}(\mu)$, where $\mathbf{c}(\mu)$ is the content of μ, i.e. the sum over all cells of the signed distances from these cells to the diagonal. Now, $C = Z_n - Z_{n-1}$, so it acts on $X_{\mu-j}$ by the scalar $\mathbf{c}(j)$, the signed distance from the cell j to the diagonal. The numbers $\mathbf{c}(j)$ are clearly different for all corners j, so if there are 2 or more corners, then C cannot act by a scalar. This finishes the proof of (ii). Part (iii) is a straightforward computation.

□

3.2.4 The main theorem

Our main theorem classifies the irreducible representations of $\mathbf{\Gamma}_n$ that extend to representations of $H_{1,k,c}(\mathbf{\Gamma}_n)$ for values of (k, c) with $k \neq 0$. For $\Gamma = \{1\}$ it is easy to see that the algebra $H_{1,k,c}(S_n)$ has no finite dimensional representations. In fact $H_{1,k,c}(S_n)$ always contains a copy of the Weyl algebra (generated by the elements $x_1 + \cdots + x_n$, $y_1 + \cdots + y_n$) that has no finite dimensional representations. We will thus consider the case $\Gamma \neq \{1\}$. Before stating the theorem we need to introduce some notation:

- ν will denote the number of conjugacy classes $\{C_1, \ldots, C_\nu\}$ of Γ , with $C_1 = \{1\}$,

$|C_s|$ will be the cardinality of the class C_s, and c_s the value of the class function c on C_s;

- for any irreducible representation N_h of Γ, $\chi_{N_h}(C_s)$ will be the value of the character of N_h on the class C_s .

With this notation, the complex number $\frac{|C_s|\,\chi_{N_h}(C_s)}{\dim N_h}$ is the scalar corresponding to the central element $\sum_{\gamma\in C_s}\gamma$ in the irreducible representation N_h.

Theorem 3.2.4. *Let $\Gamma \neq \{1\}$. Then:*

I) If an irreducible $\mathbf{\Gamma}_n$-module M extends to a representation of $H_{1,k,c}(\mathbf{\Gamma}_n)$ then the generators x_i, y_i act by zero on M for any $i = 1,\dots,n$.

II) For $k \neq 0$ an irreducible representation $M = X \otimes N \uparrow$ of $\mathbf{\Gamma}_n$ of type $(n_1,\dots,n_\nu)$ extends to a representation of some associated symplectic reflection algebra $H_{1,k,c}(\mathbf{\Gamma}_n)$ if and only if the following two conditions are satisfied:

i) $X = X_1 \otimes \cdots \otimes X_\nu$, where X_h is an irreducible representation of S_{n_h} with rectangular Young diagram of some size $a_h \times b_h$, for any h s.t. $n_h \neq 0$;

ii) for any $h \neq h'$ s.t. $n_h, n_{h'} \neq 0$, $\mathrm{Hom}_\Gamma(N_h \otimes L, N_{h'}) = 0$, where L is the natural representation of Γ. In other words, any two non-isomorphic representations N_h, $N_{h'}$ of Γ occurring in the type of N must be non-adjacent vertices in the extended Dynkin diagram attached to Γ. We agree that this condition is empty when N is of trivial type (Example 3.2.2).

III) The values of the parameter (k,c) for which $M = X \otimes N \uparrow$ can be extended form a linear subspace of $\mathbb{C}^\nu$, which can be described as the intersection of the hyperplanes

$$\mathcal{H}_h: \qquad \dim N_h + (b_h - a_h)\frac{k}{2}|\Gamma| + \sum_{s=2}^{\nu} c_s\,|C_s|\,\chi_{N_h}(C_s) = 0 \qquad (3.1)$$

for all $h \in \{1,\dots,\nu\}$ s.t. $n_h \neq 0$, i.e. for any representation N_h occurring in the type of N. The space of the solutions of this system of equations has dimension $\nu - r$ where $r = \#\{h$ s.t. $n_h \neq 0\}$.

3.2.5 Proof of Theorem 3.2.4

From now on we will assume $\Gamma \neq \{1\}$. We will divide the proof of Theorem 3.2.4 in several steps.

<u>STEP 1</u> Proof of Theorem 3.2.4 part I)

Without loss of generality consider the elements x_1, $y_1 \in H_{1,k,c}(\mathbf{\Gamma}_n)$. From Section 2.3 we know these elements commute with the elements γ_i for $i \neq 1$, and the action of γ_1 by conjugation on such elements corresponds to the action of γ on the basis vectors x, y respectively in the natural representation L of Γ. Thus we can view x_1, y_1 as a basis for the representation:

$$L \otimes \underbrace{\mathbb{C} \otimes \cdots \otimes \mathbb{C}}_{n-1}$$

of Γ^n, where $\mathbb{C}$ is the trivial one-dimensional representation. So we have that the action of x_1, y_1 on M induces maps of Γ^n-modules:

$$(L \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C}) \otimes M \longrightarrow M.$$

But now from Section 3.2.1 we have that, as a Γ^n-module, M decomposes in irreducibles as

$$\bigoplus_{\sigma} N_{h_{\sigma(1)}} \otimes \cdots \otimes N_{h_{\sigma(n)}}$$

where σ are permutations in S_n and factors may appear with some multiplicity. Thus composing with the Γ^n-module maps given by the injections and projections of the direct factors we have that x_1, y_1 induce Γ^n-module maps:

$$\left(L \otimes N_{h_{\sigma(1)}}\right) \otimes \cdots \otimes N_{h_{\sigma(n)}} \longrightarrow N_{h_{\sigma'(1)}} \otimes \cdots \otimes N_{h_{\sigma'(n)}}$$

for any σ, σ'. Since the N_hs are irreducible Γ-modules, in order for such a map to be non-zero, we must have $N_{h_{\sigma(i)}} \cong N_{h_{\sigma'(i)}}$ for any $i \geq 2$. This implies $N_{h_{\sigma(1)}} \cong N_{h_{\sigma'(1)}}$

and we get a homomorphism:

$$L \otimes N_{h_{\sigma(i)}} \longrightarrow N_{h_{\sigma(i)}}.$$

But such a homomorphism must be zero as explained in Section 3.2.2, as all extended Dynkin diagrams except $\tilde{A}_0$ have no loop-vertices. We deduce x_1, y_1 act trivially on M.

□

Now that we know that the generators x_i, y_i must act trivially on M we can reduce the defining relations *(R1)*, *(R2)* of $H_{1,k,c}(\mathbf{\Gamma}_n)$ in Lemma 2.3.3 to the simpler form:

(R1') For any $i \in [1, n]$:

$$0 = 1 + \frac{k}{2} \sum_{j \neq i} \sum_{\gamma \in \Gamma} s_{ij} \gamma_i \gamma_j^{-1} + \sum_{\gamma \in \Gamma \setminus \{1\}} c_\gamma \gamma_i \, .$$

(R2') For any $u, v \in L$ and $i \neq j$:

$$0 = \frac{k}{2} \sum_{\gamma \in \Gamma} \omega_L(\gamma u, v) \gamma_i \gamma_j^{-1} \, ,$$

where, with abuse of notation, we wrote γ_i, s_{ij} etc... for the images of the corresponding elements of $H_{1,k,c}(\mathbf{\Gamma}_n)$ in the representation M.

This reduction will allow us to prove part II) and III) of Theorem 3.2.4 by using simple classical results from the representation theory of finite groups.

<u>STEP 2</u> The relations (R2')

It turns out that the relations *(R2')* have an easy interpretation in terms of the extended Dynkin diagram attached to the group Γ in the McKay correspondence. Let L be the natural representation of Γ. We have the following proposition.

Lemma 3.2.5. *If $X \otimes Y \uparrow$ is a representation of $\mathbf{\Gamma}_n$ of type $(n_1, \dots, n_\nu)$, then the operators of the corresponding matrix representation satisfy* (R2') *for $k \neq 0$ if and only if for any pair h, h' s.t. $n_h, n_{h'} \neq 0$, $\mathrm{Hom}_\Gamma(L \otimes N_h, N_{h'}) = 0$ i.e. if and only if N_h, $N_{h'}$ are not adjacent vertices of the extended Dynkin diagram associated to Γ in the McKay correspondence.*

Proof. Relations *(R2)* are satisfied for $k \neq 0$ if and only if

$$\sum_{\gamma \in \Gamma} \omega_L(\gamma u, v)\, \gamma_i \gamma_j^{-1} = 0 \qquad \forall\, u, v \in L, \quad \forall\, i \neq j\,.$$

We observe that, for any N, the subgroup Γ^n is contained in the inertia subgroup of N and is normal in $\mathbf{\Gamma}_n$. For this reason the induced representation $X \otimes N \uparrow$ can be written as:

$$\sigma_1 \cdot (X \otimes N) \oplus \dots \oplus \sigma_\ell \cdot (X \otimes N) \tag{3.2}$$

where $\ell = \frac{n!}{n_1! \dots n_\nu!}$, and $\{\sigma_1, \dots, \sigma_\ell\}$ is a set of representatives of the left cosets of the inertia factor $(\mathbf{\Gamma}_n)_N$ in $\mathbf{\Gamma}_n$, that can be chosen to be all in S_n. The action of an element $g \in \mathbf{\Gamma}_n$ on a vector $\sigma_l \cdot v$ is defined as follows:

$$g(\sigma_l \cdot v) = \sigma_r \cdot (g' v) \qquad \text{where } g\sigma_l = \sigma_r g' \qquad g' \in (\mathbf{\Gamma}_n)_N\,.$$

By the normality of Γ^n, all the direct factors of (3.2) are stable under the action of $\sum_{\gamma \in \Gamma} \omega_L(\gamma u, v)\, \gamma_i \gamma_j^{-1}$, thus this operator has a block diagonal form. The l-th block corresponds to the operator $A(s,t) = \sum_{\gamma \in \Gamma} \omega_L(\gamma u, v)\, \gamma_s \gamma_t^{-1}$, with $(s,t) = (\sigma_l^{-1}(i), \sigma_l^{-1}(j))$, in the representation $X \otimes N$ of $(\mathbf{\Gamma}_n)_N$. We are reduced now to show that any such block is zero if and only if the conditions of the proposition are satisfied. Since the action of $A(s,t)$ is trivial on X, we can suppose X to be trivial 1-dimensional, thus $X \otimes N \cong N$. Without loss of generality we can also suppose $s \leq t$. Since the bilinear form ω_L is non degenerate and u, v vary in all L we have:

$$A(s,t) = 0 \Leftrightarrow \sum_{\gamma \in \Gamma} \gamma \otimes \gamma \otimes \gamma^{-1}|_{L \otimes N_{h_s} \otimes N_{h_t}} = 0 \Leftrightarrow \sum_{\gamma \in \Gamma} \gamma \otimes \gamma \otimes \gamma^{-1*}|_{L \otimes N_{h_s} \otimes N_{h_t}} = 0$$

where “ * ” denotes the transposition. Now, if we denote by $N_{h_t}^*$ the dual representation of N_{h_t}, we notice that the last operator corresponds to the operator:

$$\sum_{\gamma\in\Gamma} \gamma\otimes\gamma\otimes\gamma|_{L\otimes N_{h_s}\otimes N_{h_t}^*},$$

that is a multiple of the projector on the invariants of the representation $L\otimes N_{h_s}\otimes N_{h_t}^*$.

Now this is zero if and only if $\mathrm{Hom}_\Gamma(L\otimes N_{h_t}, N_{h_s}) = 0$. From Section 3.2.2 we know this happens exactly when N_{h_s}, N_{h_t} are non adjacent vertices in the Dynkin diagram attached to Γ.

□

Notice that when N is of trivial type $(0,\dots,n,\dots 0)$, i.e. when all the factors of N are the same, Lemma 3.2.5 implies that the conditions *(R2')* are automatically satisfied (since extended Dynkin diagrams corresponding to non-trivial finite subgroups of $SL(2,\mathbb{C})$ have no loop vertices).

<u>STEP 3</u> The relations (R1')

The only thing we are left to do now is analyzing the conditions for relations *(R1')* to be satisfied. We will begin from the easiest case of $M = X\otimes N\uparrow$ with N of type $(0,\dots,n,\dots,0)$. We have the following proposition.

Lemma 3.2.6. *For $k\neq 0$ a representation M of $\mathbf{\Gamma}_n$ of trivial type* $(0,\dots,n,\dots 0)$ *extends to a representation of $H_{1,k,c}(\mathbf{\Gamma}_n)$ if and only if the following conditions are satisfied:*

i) the representation X of S_n corresponds to a rectangular Young diagram;

ii) the parameter (k,c) satisfies the corresponding equation in part III) of Theorem 3.2.4.

Proof. As we observed in the previous subsection, Lemma 3.2.5 implies that in this case relations *(R2')* are satisfied. Thus we only have to consider relations *(R1')*.

We have $M = X \otimes N_h^{\otimes n}$, with X and N_h irreducible modules for S_n and Γ respectively. We will begin with an easy clarifying example. When N_h is one dimensional, it is straightforward to check that Lemma 3.2.6 holds. In this case in fact, the permutation action of S_n on $N_h^{\otimes n}$ is trivial and $X \otimes N_h^{\otimes n} \cong X$ as S_n-modules. Thus the relation *(R1')* for a fixed i looks like:

$$\frac{k}{2}\,|\Gamma| \sum_{j\neq i} s_{ij} = -1 - \sum_{\gamma\in\Gamma\smallsetminus\{1\}} c_\gamma\,\chi_{N_h}(\gamma) = -1 - \sum_{s=2}^{\nu} c_s\,|C_s|\,\chi_{N_h}(C_s)\,,$$

where $\chi_{N_h}(C_s)$ is the value of the character of N_h on the conjugacy class C_s. For $k \neq 0$ we have:

$$\sum_{j\neq i} s_{ij} = \frac{2\,(-1-\sum_{s=2}^{\nu} c_s\,|C_s|\,\chi_{N_h}(C_s))}{k\,|\Gamma|}\,. \tag{3.3}$$

So $\sum_{j\neq i} s_{ij}$ must act as a scalar and Lemma 3.2.3 part *ii*) implies that X must have rectangular Young diagram μ of some size $a \times b$. We remark that Lemma 3.2.3 part *ii*) *iii*) implies that the element $\sum_{j\neq i} s_{ij}$ acts as the scalar $\frac{2\,C(\mu)}{n} = (b-a)$ in this representation. Substituting this value in equation (3.3) we get the result in the 1-dimensional case. Notice that this first consideration solves completely the case when Γ is cyclic.

Let's now suppose dim $X = m$ and dim $N_h = p > 1$. We rewrite relations *(R1')* as follows:

$$-1 - \sum_{\gamma\in\Gamma\setminus\{1\}} c_\gamma \gamma_i = \frac{k}{2}\sum_{j\neq i}\sum_{\gamma\in\Gamma} s_{ij}\gamma_i\gamma_j^{-1}\,. \tag{3.4}$$

We observe that the left hand side of (3.4) is a central element of the group algebra $\mathbb{C}[\Gamma^n]$ (due to the fact that c is a class function), and that, as a Γ^n-module, $X \otimes N$ is isomorphic to a direct sum of $\dim X$ copies of the irreducible representation $N_h^{\otimes n}$. Thus the left hand side acts as a scalar in this representation. More precisely we have:

$$\sum_{\gamma\in\Gamma\smallsetminus\{1\}} c_\gamma\gamma_i = \sum_{s=2}^{\nu} \frac{c_s\,|C_s|\,\chi_{Y_h}(C_s)}{\dim N_h}\,\mathrm{Id}_{X\otimes N}.$$

So we must have that $\frac{k}{2}\sum_{j\neq i}\sum_{\gamma\in\Gamma} s_{ij}\gamma_i\gamma_j^{-1}$ is a scalar. We will show that this operator has a block form that reduces equation (3.4) to equation (3.3).

For this let's take any two bases $\{v_1, \dots, v_p\}$, $\{x_1, \dots, x_m\}$ for N_h and X respectively. The vectors $\{\mathbf{v}_{\underline{i}} = v_{i_1} \otimes \cdots \otimes v_{i_n}\}$, where the multi-index $\underline{i} = (i_1, \dots, i_n)$ varies in $[1, p]^n$, are clearly a basis for $N = N_h^{\otimes n}$. We can give the multi-indices I a total ordering $\underline{i}_1, \dots, \underline{i}_{p^n}$ using the lexicographic order. Consider now the basis of $X \otimes N_h^{\otimes n}$ given by the vectors:

$$Z_1 = x_1 \otimes \mathbf{v}_{\underline{i}_1}, \; \dots \; , Z_m = x_m \otimes \mathbf{v}_{\underline{i}_1}, \; \dots \; , Z_{m(p^n-1)+1} = x_1 \otimes \mathbf{v}_{\underline{i}_{p^n}}, \; \dots \; , Z_{mp^n} = x_m \otimes \mathbf{v}_{\underline{i}_{p^n}} \, .$$

Any transposition $s_{ij} \in S_n$ induces a permutation $\tilde{s}_{ij}$ (of order 2) on the set $\{\underline{i}_1, \dots, \underline{i}_{p^n}\}$ thus on the vectors of the basis $\{Z_1, \dots, Z_{mp^n}\}$. Let's now denote by $A_X(s_{ij})$ the operator (of size $m \times m$) for s_{ij} in the representation X, and by O_m the 0-operator of size $m \times m$. It is easy to see that, using the basis $\{Z_1, \dots, Z_{mp^n}\}$, we can obtain a block form for the operator s_{ij} in the representation $X \otimes N$ from the block diagonal operator

$$\begin{pmatrix} A_X(s_{ij}) & O_m & \cdots & O_m \\ O_m & \ddots & \cdots & O_m \\ \vdots & \vdots & \ddots & \vdots \\ O_m & \cdots & O_m & A_X(s_{ij}) \end{pmatrix}$$

by simply permuting the columns according to $\tilde{s}_{ij}$. Using this, we can compute a block form for $s_{ij} \sum_{\gamma \in \Gamma} \gamma_i \gamma_j^{-1}$. We denote each block, of size $m \times m$, by its position $(\underline{r}, \underline{t})$, where $\underline{r} = (r_1, \dots, r_n)$, $\underline{t} = (t_1, \dots, t_n)$ are multi-indices. We have the following formulas for the blocks :

- for $(\underline{r}, \underline{t})$ with $\underline{r}$ differing from $\underline{t}$ at most for the pair of indices (r_i, r_j):

  $$(\underline{r}, \underline{t}) = A_X(s_{ij}) \sum_{\gamma \in \Gamma} \alpha_{r_j t_i}(\gamma) \, \alpha_{r_i t_j}(\gamma^{-1}) \, ,$$

  where $\alpha_{r_j, t_i}(\gamma)$ are the matrix coefficients in the representation N_h;

- for $(\underline{r},\underline{t})$ with $\underline{r}$ differing from $\underline{t}$ for indices different from r_i, r_j

$$(\underline{r},\underline{t}) = O_m\,.$$

Summing up over $j \neq i$ we can now rewrite relations *(R1')* in block form for each $i \in [1,n]$:

1. for $(\underline{r},\underline{t})$ with $\underline{r}$ differing from $\underline{t}$ at most for the index r_i

$$\frac{k}{2}\sum_{j\neq i}A_X(s_{ij})\sum_{\gamma\in\Gamma}\alpha_{r_jt_i}(\gamma)\alpha_{r_ir_j}(\gamma^{-1})=-\delta_{r_it_i}I_m\left(1+\sum_{s=2}^{\nu}\frac{c_s\,|C_s|\,\chi_{N_h}(C_s)}{\dim N_h}\right) \quad (3.5)$$

where $\delta_{r_it_i} = \begin{cases} 0 & \text{if } r_i \neq t_i \\ 1 & \text{if } r_i = t_i \end{cases}$;

2. for $(\underline{r},\underline{t})$ with $\underline{r}$ differing from $\underline{t}$ at least for an index r_j, $j \neq i$, and at most for the pair of indices (r_i, r_j)

$$\frac{k}{2}\,A_X(s_{ij})\sum_{\gamma\in\Gamma}\alpha_{r_jt_i}(\gamma)\alpha_{r_it_j}(\gamma^{-1}) = 0\,. \quad (3.6)$$

In all the other cases we only obtain trivial relations.

Now we observe that, using the orthogonality property of matrix coefficients of irreducible representations of a finite group, we get:

$$\sum_{\gamma\in\Gamma}\alpha_{r_j,t_i}(\gamma)\,\alpha_{r_i,t_j}(\gamma^{-1}) = \delta_{r_it_i}\,\delta_{r_jt_j}\,\frac{|\Gamma|}{\dim N_h}\,.$$

Substituting these values in equation (3.6) we obtain trivial relations. From equation (3.5), instead, we obtain that $\sum_{j\neq i} A_X(s_{ij})$ must be a scalar operator. Thus Lemma 3.2.3 implies that the Young diagram μ attached to X is a rectangle, of some size $a_h \times b_h$, and that $\sum_{j\neq i} A_X(s_{ij})$ acts on X as the scalar $(b-a)$. Thus from equation

(3.5) we obtain the equation:

$$\dim N_h + (b_h - a_h)\,\frac{k}{2}\,|\Gamma| + \sum_{s=2}^{\nu} c_s\,|C_s|\,\chi_{N_h}(C_s) = 0$$

which is exactly the equation (3.1) for the hyperplane $\mathcal{H}_h$ in Theorem 3.2.4, part *III*). Notice that, in this case, we get a single equation since N_h is the only factor appearing in N.

□

We will now analyze the cases when the inertia factor of N is not the entire S_n and an actual induction is needed to build the representation $X \otimes N \uparrow$. If the type of N is $\vec{n} = (n_1, \ldots, n_\nu)$, then the inertia factor is $S_{\vec{n}} = S_{n_1} \times \cdots \times S_{n_\nu}$ and we have:

$$X \otimes N \uparrow = \sigma_1 \cdot (X \otimes N) \oplus \cdots \oplus \sigma_\ell \cdot (X \otimes N)$$

where $\ell = \frac{n!}{n_1! \cdots n_\nu!}$ and $\{\sigma_1, \ldots, \sigma_\ell\}$ is a set of representatives for the left cosets of $S_{\vec{n}}$ in S_n.

Remark 3.2.7. *Let's denote by $[\sigma]$ the left coset of σ with respect to $S_{\vec{n}}$. An easy computation shows that for any transposition s_{ij} and any permutation σ:*

$$[s_{ij}\sigma] = [\sigma] \Leftrightarrow s_{\sigma^{-1}(i)\sigma^{-1}(j)} \in S_{\vec{n}}\,.$$

Moreover we observe that for any $\sigma \in S_n$ and any $i = 1, \ldots, n$:

$$\gamma_i \sigma = \sigma \gamma_{\sigma^{-1}(i)}\,.$$

We are now ready to prove the following result.

Lemma 3.2.8. *For a representation $X \otimes N \uparrow$ of $\mathbf{\Gamma}_n$ of non-trivial type $(n_1, \ldots, n_\nu)$ relations* (R1) *are satisfied for some non-zero values of k if and only if:*

i) $X = X_1 \otimes \cdots \otimes X_\nu$*, with* X_h *an irreducible representation of* S_{n_h} *with rectangular Young diagram;*

ii) the parameter (k, c) *satisfies the corresponding system of equations in part III) of Theorem 3.2.4.*

Proof. Let N be a representation of Γ^n of type $(n_1, \dots, n_\nu)$. We observe that for any X, if we choose $\{\sigma_1, \dots, \sigma_\ell\} \subset S_n$ representatives of the left cosets of $(\mathbf{\Gamma}_n)_N$ in $\mathbf{\Gamma}_n$:

$$X \otimes N \uparrow = \sigma_1 \cdot (X \otimes N) \oplus \cdots \oplus \sigma_\ell \cdot (X \otimes N)\,, \tag{3.7}$$

is a Γ^n-stable decomposition of $X \otimes N \uparrow$. For any representative σ_l, let's denote by $\sigma_l\, N$ the representation of Γ^n with same underlying vector space as N and with the action on N twisted by the automorphism induced by σ_l on Γ^n (the action of γ_i on $\sigma_l\, N$ is the same as the action of $\gamma_{\sigma^{-1}(i)}$ on N). Since Γ^n acts trivially on X, as a Γ^n-module the subspace $\sigma_l \cdot (X \otimes N)$ is isomorphic to a direct sum of copies of the irreducible representation $\sigma_l\, N$. So, for a fixed i, the Γ^n-central operator $-1 - \sum_{\gamma \in \Gamma \smallsetminus \{1\}} c_\gamma \gamma_i$ preserves the subspaces $\sigma_l \cdot (X \otimes N)$ and acts as a scalar on each of them. For any vector $\sigma_l \cdot v \in \sigma_l(X \otimes N)$ we have:

$$\begin{aligned} \left(-1 - \sum_{\gamma \in \Gamma \smallsetminus \{1\}} c_\gamma \gamma_i\right)(\sigma_l \cdot v) &= \sigma_l \cdot (-v) + \sigma_l \cdot \left(\left(- \sum_{\gamma \in \Gamma \smallsetminus \{1\}} c_\gamma \gamma_{\sigma_l^{-1}(i)}\right) v\right) \\ &= C(\sigma_l \cdot v). \end{aligned} \tag{3.8}$$

with $C \in \mathbb{C}$. The action of $\frac{k}{2} \sum_{j \neq i} \sum_{\gamma \in \Gamma} s_{ij} \gamma_i \gamma_j^{-1}$ on such a vector is instead:

$$\left(\frac{k}{2} \sum_{j \neq i} \sum_{\gamma \in \Gamma} s_{ij} \gamma_i \gamma_j^{-1}\right)(\sigma_l \cdot v) = \sum_{j \neq i} \sigma_{r(ijl)} \cdot \left(\left(\frac{k}{2} \sum_{\gamma \in \Gamma} \tilde{\sigma}_{ijl} \gamma_{\sigma_l^{-1}(i)} \gamma^{-1}_{\sigma_l^{-1}(j)}\right) v\right) \tag{3.9}$$

where $\sigma_{r(ijl)}$ is the representative in the set $\{\sigma_1, \dots, \sigma_\ell\}$ of the coset $[s_{ij}\sigma_l]$ and $\tilde{\sigma}_{ijl} \in S_{\vec{n}}$ is the unique element s.t. $s_{ij}\sigma_l = \sigma_{r(ijl)}\tilde{\sigma}_{ijl}$.

Relations *(R1')* are satisfied if and only if these two actions are the same. In particular $\frac{k}{2} \sum_{j \neq i} \sum_{\gamma \in \Gamma} s_{ij} \gamma_i \gamma_j^{-1}$ must preserve the subspace $\sigma_l \cdot (X \otimes N)$. But let's now look at equation (3.9) and take $j \neq i$ s.t. $[s_{ij}\sigma_l] = [\sigma_r]$, $r \neq l$ i.e. s_{ij} "moves" the subspace $\sigma_l \cdot (X \otimes N)$ sending it to the subspace $\sigma_r \cdot (X \otimes N)$. Then we have: $s_{\sigma_l^{-1}(i)\sigma_l^{-1}(j)} \notin S_{\vec{n}}$. This means that the representations $N_{\sigma_l^{-1}(i)}$, $N_{\sigma_l^{-1}(j)}$ are not isomorphic. As a consequence, arguing as in Lemma 3.2.5, we have that

$$\sum_{\gamma \in \Gamma} \gamma_{\sigma_l^{-1}(i)} \gamma^{-1}_{\sigma_l^{-1}(j)} = 0$$

in the representation $X \otimes N$, hence s_{ij} sends the subspace $\sigma_l \cdot (X \otimes N)$ to 0. This means that $\frac{k}{2} \sum_{j \neq i} \sum_{\gamma \in \Gamma} s_{ij} \gamma_i \gamma_j^{-1}$ indeed preserves the subspace $\sigma_l \cdot (X \otimes N)$ and that relations *(R1)* split up into equations that can be checked on the subspaces $\sigma_l \cdot (X \otimes N)$. So in equation (3.9) it is enough to take the sum over the js s.t. $[s_{ij}\sigma_l] = [\sigma_l]$. Moreover we know that if $[s_{ij}\sigma_l] = [\sigma_l]$ then $s_{ij}\sigma_l = \sigma_l \, s_{\sigma_l^{-1}(i)\sigma_l^{-1}(j)}$ i.e. $\tilde{\sigma}_{ijl} = s_{\sigma_l^{-1}(i)\sigma_l^{-1}(j)}$. Hence, for a fixed i, if $\sigma_l^{-1}(i) = p$ the relations *(R1')* reduce to the following equations:

$$\frac{k}{2} \sum_{\substack{q \neq p \\ s_{pq} \in S_{n_{h_p}}}} s_{pq} \sum_{\gamma \in \Gamma} \gamma_p \gamma_q^{-1} = -1 - \sum_{\gamma \in \Gamma \smallsetminus \{1\}} c_\gamma \gamma_p \tag{3.10}$$

where the identity must be considered in the representation $X \otimes N$ of $S_{\vec{n}} \ltimes \Gamma^n$ and $p \in \{\sigma_1^{-1}(i), \ldots, \sigma_\ell^{-1}(i)\}$. For any p, equation (3.10) is exactly the p-th equation of relations *(R1')* for the extension of the representation of trivial type $X_{h_p} \otimes N_{h_p}^{\otimes n_{h_p}}$ of $S_{n_{h_p}} \ltimes \Gamma^{n_{h_p}}$ to the algebra $H_{1,k,c}(S_{n_{h_p}} \ltimes \Gamma^{n_{h_p}})$. It is easily checked that, letting i and σ_l vary, we obtain all the relations for the extension of the representations $X_h \otimes N_h^{\otimes n_h}$ of $S_{n_h} \otimes \Gamma^{n_h}$ for any $n_h \neq 0$. Using Lemma 3.2.6 we get the result.

□

<u>STEP 4</u> The conditions on the parameter (k, c)

Now that we classified the representations of $\mathbf{\Gamma}_n$ that can potentially be extended to representations of $H_{1,k,c}(\mathbf{\Gamma}_n)$ for $k \neq 0$. We would now like to show that such extensions exist for a non-empty set of values of (k, c). This amounts to proving that the system of equations in Theorem 3.2.4 part III) admits solutions. Fix a representation $M = X \otimes N \uparrow$ of $\mathbf{\Gamma}_n$ of type $(n_1, \dots, n_\nu)$ satisfying conditions i) ii) of Theorem 3.2.4 part II). We have the following lemma.

Lemma 3.2.9. *If $r = \#\{h \text{ s.t. } n_h \neq 0\}$, then the space of the solutions for the system of equations in part II) of Theorem 3.2.4 has dimension $\nu - r$.*

Proof. By condition ii) we have that:

$$r = \#\{h \text{ s.t. } n_h \neq 0\} < \nu = \#\{\text{vertices in the extended Dynkin diagram of } \Gamma\}\,.$$

Without loss of generality we can suppose $n_1, \dots, n_r \neq 0$, $n_h = 0$ for $h > r$. So in matrix form the system has size $r \times \nu$, with $r \leq \nu - 1$:

$$\begin{pmatrix} \frac{(b_1-a_1)\,|\Gamma|}{2} & |C_2|\chi_{N_1}(C_2) & \dots & |C_\nu|\chi_{N_1}(C_\nu) \\ \frac{(b_2-a_2)\,|\Gamma|}{2} & |C_2|\chi_{N_2}(C_2) & \dots & |C_\nu|\chi_{N_2}(C_\nu) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{(b_r-a_r)\,|\Gamma|}{2} & |C_2|\chi_{N_r}(C_2) & \dots & |C_\nu|\chi_{N_r}(C_\nu) \end{pmatrix} \begin{pmatrix} k \\ c_2 \\ \vdots \\ c_\nu \end{pmatrix} = \begin{pmatrix} -dimN_1 \\ -dimN_2 \\ \vdots \\ -dimN_r \end{pmatrix}.$$

But now we have:

$$\text{rk} \begin{pmatrix} |C_2|\chi_{N_1}(C_2) & \dots & |C_\nu|\chi_{N_1}(C_\nu) \\ |C_2|\chi_{N_2}(C_2) & \dots & |C_\nu|\chi_{N_2}(C_\nu) \\ \vdots & \vdots & \vdots \\ |C_2|\chi_{N_r}(C_2) & \dots & |C_\nu|\chi_{N_r}(C_\nu) \end{pmatrix} = \text{rk} \begin{pmatrix} \chi_{N_1}(C_2) & \dots & \chi_{N_1}(C_\nu) \\ \chi_{N_2}(C_2) & \dots & \chi_{N_2}(C_\nu) \\ \vdots & \vdots & \vdots \\ \chi_{N_r}(C_2) & \dots & \chi_{N_r}(C_\nu) \end{pmatrix} = r\,.$$

In fact the rows $R_1, \dots, R_r$ on the RHS are rows of the character table for Γ from which we have erased the entries $\chi_{N_h}(1) = \dim N_h$. If a non-trivial linear combination $\sum_{h=1}^r a_h R_h$ of these rows is zero then the class function $\chi = \sum_{h=1}^r a_h \chi_{N_h}$ satisfies the equation:

$$\chi(\gamma) = 0, \quad \forall\, \gamma \in \Gamma \smallsetminus \{1\}\,.$$

This is possible only if $\chi = m\rho$, where $m \in \mathbb{C}$ and ρ is the character of the regular representation. Now we must have $m \neq 0$ since characters of non-isomorphic irreducible representations are linearly independent. But $m \neq 0$ is also impossible since, by condition ii), $N_1, \dots N_r$ are not a complete set of irreducible representations of Γ while, on the other hand, any irreducible representation of Γ occurs in the regular representation with non-zero multiplicity. So the matrix for the system in part III) of Theorem 3.2.4 has maximal rank and the space of solutions has dimension $\nu - r$.

□

We are finally ready to prove part II) and III) of Theorem 3.2.4.

STEP 5 Proof of Theorem 3.2.4 parts II) and III)

Just combine the results of Lemma 3.2.5, Lemma 3.2.6 and Lemma 3.2.9.

3.3 Deforming irreducible $H_{1,0,c}(\mathbf{\Gamma}_n)$-modules

3.3.1 A proposition in deformation theory

Let A be an associative unital algebra over $\mathbb{C}$ and let A_U be a flat formal deformation over $\mathbb{C}[[U]]$, where U is some finite dimensional vector space. Let M be a left A-module.

There exists a natural map $\eta : U \to H^2(A, \operatorname{End} M)$. The map η is the composition of the map $\phi : U \longrightarrow H^2(A, A)$ of Section 1.3 with the natural map $\psi : H^2(A, A) \longrightarrow H^2(A, \operatorname{End} M)$ induced by functoriality by the homomorphism $\rho : A \longrightarrow \operatorname{End} M$ giving the representation. The next proposition will be our main tool in investigating the possibilty of obtaining $H_{1,k,c+c'}(\mathbf{\Gamma}_n)$-modules for $k \neq 0$ as deformations of $H_{1,0,c}(\mathbf{\Gamma}_n)$-modules.

Proposition 3.3.1. *Assume that η is surjective with kernel K, and moreover that $H^1(A, \operatorname{End} M) = 0$. Then:*

(i) There exists a unique smooth formal subscheme S of the formal neighborhood of the origin in U, with tangent space K at the origin, such that M deforms to a representation of the algebra $A_S := A_U \hat{\otimes}_{\mathbb{C}[[U]]}\mathbb{C}[S]$ (where $\hat{\otimes}$ is the completed tensor product).

(ii) The deformation of M over S is unique.

Proof. Let us realize A_U explicitly as $A[[\hbar_1, ..., \hbar_n]]$ equipped with a product $*$ as in (1.1). We may assume that K is the space of all vectors $(\hbar_1, ..., \hbar_n)$ such that $\hbar_{m+1} = ... = \hbar_n = 0$.

Let D be the formal neighborhood of the origin in K, with coordinates $t_1 = \hbar_1, ..., t_m = \hbar_m$. Let $\theta : D \to U$ be a map given by the formula $\theta(t_1, ..., t_m) = (\hbar_1, ..., \hbar_n)$, where $\hbar_i = t_i$ for $i \leq m$, and

$$\hbar_k = \sum_{k,p_1,...,p_m} \hbar_{k,p_1,...,p_m} t_1^{p_1} ... t_m^{p_m},\ k > m,$$

where $\hbar_{k,p_1,...,p_m} \in \mathbb{C}$. More briefly, we can write $\hbar_k = \sum_{\underline{p}} \hbar_{k\underline{p}} t^{\underline{p}}$, where $\underline{p}$ is a multi-index. For brevity we also let e_j to be the multi-index $(0, ..., 1_j, ..., 0)$.

We claim that there exist unique formal functions $\hbar_k = \hbar_k(t)$, $k > m$, for which we can deform M over D. Indeed, such a deformation would be defined by a series

$$\widetilde{\rho}(a) = \sum_{\underline{p}} \rho_{\underline{p}}(a) t^{\underline{p}},$$

where $\rho_0(a) = \rho(a)$, and ρ is the homomorphism giving the representation M. The condition that $\widetilde{\rho}$ is a representation gives for each $\underline{p}$

$$d\,\rho_{\underline{p}} = \sum_j \hbar_{j\underline{p}} \rho(c_{e_j}) + B_{\underline{p}}, \tag{3.11}$$

where for $j \leq m$, $\hbar_{j\underline{p}} = 1$ if $\underline{p} = e_j$ and zero otherwise, and $B_{\underline{p}}$ is a 2-cocycle whose expression may involve $\rho_{\underline{q}}$ and $\hbar_{k\underline{q}}$ **only** with $|\underline{q}| < |\underline{p}|$. Since the map η is surjective, there are (unique) $\hbar_{m+1,\underline{p}}, ..., \hbar_{n\underline{p}}$ for which the right hand side is a coboundary. For such $\hbar_{m+1,\underline{p}}, ..., \hbar_{n\underline{p}}$ (and only for them), we can solve (3.11) for $\rho_{\underline{p}}$.

This shows the existence of the functions $\hbar_j(t)$, $j > m$, such that the deformation of M over D is possible. To show the uniqueness of these functions, let $\hbar_j$ and $\hbar'_j$ be two sets of functions for which the deformation exists. Let $\rho_{\underline{p}}, \rho'_{\underline{p}}$ be the coeffients of the corresponding representations $\widetilde{\rho}, \widetilde{\rho}'$. Let N be the maximal number such that $\hbar_{j\underline{p}} = \hbar'_{j\underline{p}}$ for $|\underline{p}| < N$. Since $H^1(A, \mathrm{End}\, M) = 0$, the solution $\rho_{\underline{p}}$ of (3.11) is unique up to adding a coboundary. Thus we can use changes of basis in M to modify $\widetilde{\rho}$ so that $\rho_{\underline{p}} = \rho'_{\underline{p}}$ for $|\underline{p}| < N$ (note that this does not affect $\hbar_j$). Then for any $\underline{q}$ with $|\underline{q}| = N$, $B_{\underline{q}}(\widetilde{\rho}) = B_{\underline{q}}(\widetilde{\rho}')$, and hence $\hbar_{j\underline{q}} = \hbar'_{j\underline{q}}$. This contradicts the maximality of N.

Thus, we have shown that the functions $\hbar_j$ exist and are unique; they define a parametrization of the desired subscheme S by D. Our proof also implies that the deformation of M over S is unique, so we are done.

□

We end this section by recalling the following fact from algebraic geometry that will guarantee that the representations we will find in Theorem 3.3.10 are actually irreducible.

Let X be an affine irreducible algebraic variety over $\mathbb{C}$, $R = \mathbb{C}[X]$. Let A be an algebra over R and M an A-module, such that A and M are free as R-modules and M is of finite rank. For $x \in X$, let A_x, M_x be the fibers of A, M at x; so A_x is a $\mathbb{C}$-algebra and M_x a finite dimensional module over A_x.

Proposition 3.3.2. *The set of x for which M_x is irreducible is open in X.*

Proof. Let x be a point of X where M_x is irreducible. We have that the map $f_x : A \to \mathrm{End}\, M_x$ is surjective. This means that there exist elements $a_1, ..., a_{N^2}$ in A, $N = \dim_R M$, such that $f_x(a_i)$ is a basis of $\mathrm{End}\, M_x$. The set U of points z of X such that $f_z(a_i)$ are a basis of $\mathrm{End}\, M_z$ is open and contains x. We found a neighborhood U of x such that, for all z in U, M_z is an irreducible A_z-module, as desired.

□

3.3.2 Deformed preprojective algebras

In order to apply Proposition 3.3.1 to our case, we need to have a closer look at the finite dimensional irreducible representations of $H_{1,0,c}(\mathbf{\Gamma}_n) = H_{1,c}(\Gamma)^{\otimes n}\sharp S_n$. As we already mentioned, in analogy with the case of the wreath product groups, such representations can be recovered from a knowledge of the irreducible representations of the rank one algebra $H_{1,c}(\Gamma)$, using the representation theory of S_n. The complete classification of the finite dimensional irreducible representations of $H_{1,c}(\Gamma)$ was obtained by Crawley-Boevey and Holland in [CBH98] using the representation theory of quivers and deformed preprojective algebras.

In this section we recall the Crawley-Boevey and Holland definition of the deformed preprojective algebra and how it is related with $H_{1,c}(\Gamma)$.

Let Q be a finite quiver (finite oriented graph) and let I be the set of its vertices. If two vertices $i, j \in I$ are connected by an arrow a in such a way that

$$a : i \longrightarrow j$$

we will denote by $h(a) = j$ the head of the arrow a and by $t(a) = i$ its tail. We will denote by $\overline{Q}$ be the double quiver of Q, obtained by adding a reverse arrow

$$a^* : j \longrightarrow i$$

for any arrow $a : i \longrightarrow j$ of Q.

Let $B := \bigoplus_{i\in I} \mathbb{C}e_i$ be the semisimple finite dimensional algebra spanned by orthogonal idempotents e_i corresponding to the vertices. Let let E be the vector space with basis the set of arrows of $\overline{Q}$. We have that E is a B-bimodule and $E = \sum_{i,j\in I} E_{ij}$, where E_{ij} is spanned by all the arrows a with $h(a) = i$, $t(a) = j$. We can form the path algebra of $\overline{Q}$ defined as $\mathbb{C}\overline{Q} := T_B E = \bigoplus_{n\geq 0} T_B^n E$, where $T_B^n E$ is the n-fold tensor product of E over B. Each idempotent e_i corresponds to the trivial path that does not move from the vertex i, and arrows compose as paths on the oriented graph when it is possible (otherwise their composition gives 0). For any $\lambda \in B$ we write

$\lambda = \sum_{i \in I} \lambda_i e_i$. For any $i \in I$ define the element R_i of $\mathbb{C}\overline{Q}$ as follows:

$$R_i := \sum_{\{a \in Q | h(a) = i\}} aa^* - \sum_{\{a \in Q | t(a) = i\}} a^* a \,. \tag{3.12}$$

Definition 3.3.3. *For any $\lambda \in B$, the deformed preprojective algebra $\Pi_\lambda(Q)$ is the quotient*

$$\frac{\mathbb{C}\overline{Q}}{\langle R_i - \lambda_i e_i \rangle_{i \in I}}$$

where $\langle \dots \rangle$ denotes the two-sided ideal generated by the indicated elements.

By [CBH98] Lemma 2.2, the algebra $\Pi_\lambda(Q)$ does not depend on the orientation of Q and it is unchanged up to isomorphism by multiplying λ by a nonzero scalar.

Consider now any quiver Q obtained by assigning any orientation to the extended Dynkin diagram attached to Γ via the McKay correspondence. In this case the set of vertices I is in bijection with the set of isomorphism classes of irreducible Γ-modules $\{N_i\}_{i \in I}$. Consider a parameter $\lambda \in \mathbb{C}^I = \mathbb{C}^\nu$ as above related to the parameter c of the family $H_{1,c}(\Gamma)$ in the following way. If $\mathbf{c} = 1 + \sum_{\gamma \in \Gamma \smallsetminus \{1\}} c_\gamma \gamma$ is the central element of $\mathbb{C}[\Gamma]$ that appears in the definition of $H_{1,c}(\Gamma)$ (see Example 2.3.4) then

$$\lambda_i = \operatorname{tr}_{N_i} \mathbf{c} \,. \tag{3.13}$$

We have the following theorem.

Theorem 3.3.4. *For Q and λ as above the algebra $H_{1,c}(\Gamma)$ is Morita equivalent to the algebra $\Pi_\lambda(Q)$.*

3.3.3 Irreducible representations of $H_{1,c}(\Gamma)$

From the previous section we know that classifying the finite dimensional irreducible representations of $H_{1,c}(\Gamma)$ is equivalent to solving the same problem for the algebra $\Pi_\lambda(Q)$. If we denote by $\cdot$ the standard scalar product in $\mathbb{C}^\nu$, by $\mathcal{R}$ the regular representation of Γ, and by $\delta \in \mathbb{Z}^\nu$ the vector at the end of Section 3.2.2 then

$$\lambda \cdot \delta = \mathrm{tr}_{\mathcal{R}} \mathbf{c}$$

and the condition $1 = \mathrm{tr}_{\mathcal{R}} \mathbf{c}$ corresponds to the condition $\lambda \cdot \delta = 1$. Thus, in particular it is enough to look at parameters λ with $\lambda \cdot \delta \neq 0$.

Any finite dimensional representation Y of $\Pi_\lambda(Q)$ is also a representations of the path algebra of $\overline{Q}$. Thus we can attach to Y a *dimension vector* $\alpha \in \mathbb{Z}^\nu$ such that $\alpha_i = \dim e_i Y$.

We recall now that one can associate an affine root system to the McKay graph Q. The roots of such system can be distinguished into real roots and imaginary roots. The real roots are divided into positive and negative roots, and are the images of the coordinate vectors of $\mathbb{Z}^\nu$ under sequences of some suitably defined reflections, generating the Weyl group W attached to Q. The imaginary roots, instead, are all the non-zero integer multiples of the vector δ. When $\lambda \cdot \delta \neq 0$, let us denote by R_λ the set of real roots α such that $\lambda \cdot \alpha = 0$ (this is a finite set), and by Σ_λ the unique basis of R_λ consisting of positive roots. The following theorem gives a classification of the isomorphism classes of irreducible finite dimensional representations for $H_{1,c}(\Gamma)$.

Theorem 3.3.5 ([CBH98], Theorem 7.4)**.** *If $\lambda \cdot \delta \neq 0$, then $\Pi_\lambda(Q)$ has only finitely many finite-dimensional simple modules up to isomorphisms, and they are in one-to-one correspondence with the set Σ_λ. The correspondence is the one assigning to each module its dimension vector.*

We end this section with three lemmas that will be useful in the sequel.

Lemma 3.3.6. *The characters of the irreducible finite dimensional $H_{1,c}(\Gamma)$-modules are linearly independent.*

Proof. Let $\{Y_h\}_{h=1,\dots,p}$, where $p = |\Sigma_\lambda|$, be a complete collection of finite dimensional, irreducible, pairwise non-isomorphic representations of $H_{1,c}(\Gamma)$. According to Theorem 3.3.5, we write $\Sigma_\lambda \ni \alpha_{(h)} = \left\{\alpha_{(h),j}\right\}_{j=1,\dots,\nu}$ for the dimension vector corresponding to Y_h under the Morita equivalence of Section 3.3.2, where λ corresponds to c as in (3.13). This means we have a decomposition $Y_h = \bigoplus_{j=1}^{\nu} N_j^{\oplus \alpha_{(h),j}}$ as a Γ-module (see [CBH98], § 3). Thus for the character of Y_h as a Γ-module we get $(\chi_{Y_h})|_\Gamma = \sum_{j=1}^{\nu} \alpha_{(h),j}\chi_{N_j}$. By Theorem 3.3.5 the α_h's are a basis of the vector space $\mathrm{Span}_{\mathbb{C}}(R_\lambda) \subset \mathbb{C}^\nu$, thus they are linearly independent vectors in $\mathbb{C}^\nu$. The result now follows from the fact that the χ_{N_j}'s for $j = 1, \ldots, \nu$ are linearly independent functions on Γ.

□

We recall that the *symmetrized Ringel form* attached to the quiver Q is the bilinear form $(-,-)$ on $\mathbb{Z}^I$ defined as follows. For any $\alpha = \{\alpha_i\}$ and $\beta = \{\beta_i\}$ in $\mathbb{Z}^I$ set

$$\langle \alpha, \beta \rangle = \sum_{i \in I} \alpha_i \, \beta_i - \sum_{a \in Q} \alpha_{t(a)} \, \beta_{h(a)}. \tag{3.14}$$

Then

$$(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle. \tag{3.15}$$

This bilinear form is W-invariant, where W is the Weyl goup of Q. Moreover, when the underlying graph of Q is extended Dynkin, this form is positive semidefinite. The radical is generated by the vector δ ([Kac90] Proposition 4.7, Theorem 4.8).

Lemma 3.3.7. *Let Y_h, $Y_{h'}$ be two irreducible, finite dimensional, non isomorphic representations of $H_{1,c}(\Gamma)$ and let $\alpha_{(h)}$, $\alpha_{(h')}$ be the two distinct roots corresponding to them under the Morita equivalence. The following are equivalent*

i) $(\alpha_{(h)}, \alpha_{(h')}) = 0$

ii) $\mathrm{Ext}^1_{H_{1,c}(\Gamma)}(Y_h, Y_{h'}) = 0$

Proof. We can of course prove the result for the corresponding $\Pi_\lambda(Q)$-modules that we will still denote by Y_h, $Y_{h'}$. The result then can be easily deduced from [CBH98], § 7. The two conditions are in fact equivalent when λ is a *dominant* parameter (see [CBH98] § 7 for a definition of dominance) . In this case in fact, by [CBH98] Lemma 7.1, the dimension vectors $\alpha_{(h)}, \alpha_{(h')}$ are simple roots corresponding to two distinct vertices. Thus $\mathrm{Ext}^1_{\Pi_\lambda}(Y_h, Y_{h'}) \neq 0$ if and only if $\alpha_{(h)}, \alpha_{(h')}$ correspond to adjacent vertices i.e. if and only if $(\alpha_{(h)}, \alpha_{(h')}) \neq 0$ (see formulas (3.14), (3.15)). When λ is not dominant but $\lambda \cdot \delta = 1$, there exists an element w of the Weyl group W attached to the quiver Q such that $w\lambda = \lambda^+$ is dominant. In this case the algebra Π_λ is Morita equivalent to the deformed preprojective algebra Π_{λ^+}, and this Morita equivalence acts on the dimension vectors as the element $w \in W$, i.e. in particular preserving the Ringel form (see [CBH98] Corollary 5.2, Lemma 7.2, and Theorem 7.4). Thus we can reduce ourselves to the case when λ is dominant.

□

Lemma 3.3.8. *Let Y be an irreducible $H_{1,c}(\Gamma)$-module. Then* $\mathrm{Ext}^1_{H_{1,c}(\Gamma)}(Y, Y) = 0$.

Proof. We will prove the result for the corresponding $\Pi_\lambda(Q)$-module that we will also denote by Y. But it is known ([CBH98], Corollary 7.6) that $\Pi_\lambda(Q)$ contains only one minimal ideal J among all the nonzero ideals, and $\mathrm{Ext}^1_{\Pi_\lambda(Q)/J}(Y', Y') = 0$ for any irreducible module Y' over the (finite dimensional) quotient algebra $\Pi_\lambda(Q)/J$. Since any finite dimensional $\Pi_\lambda(Q)$-module must factor through $\Pi_\lambda(Q)/J$, we get $\mathrm{Ext}^1_{\Pi_\lambda(Q)}(Y, Y) = 0$, as desired.

□

3.3.4 Irreducible representations of $H_{1,0,c}(\mathbf{\Gamma}_n)$

We recall that $H_{1,0,c}(\mathbf{\Gamma}_n) = H_{1,c}(\Gamma)^{\otimes n} \sharp S_n$. Let $\{Y_h\}_{h=1,\dots,p}$, where $p = |\Sigma_\lambda|$, be a complete set of pairwise non-isomorphic finite dimensional $H_{1,c}(\Gamma)$-modules. In a similar fashion as in Section 3.2.1, let $\mathbf{Y} = Y_1^{\otimes n_1} \otimes \cdots \otimes Y_p^{\otimes n_p}$, with $\sum_i n_i = n$, be a representation of $H_{1,c}(\Gamma)^{\otimes n}$ of type $\vec{n} = (n_1, \cdots, n_p)$. Set $S_{\vec{n}} = S_{n_1} \times \cdots \times S_{n_p} \subset S_n$, and let $X = X_1 \otimes \cdots \otimes X_p$ be an irreducible representation of $S_{\vec{n}}$. As in Section 3.2.1, we will say that a representation $\mathbf{Y'} = Y_{h_1} \otimes \cdots \otimes Y_{h_n}$ is conjugate to $\mathbf{Y}$ (or has the same type as $\mathbf{Y}$) if it has the same factors up to a permutation. The tensor product $X \otimes \mathbf{Y}$ is an irreducible representation of the subalgebra $H_{1,c}(\Gamma)^{\otimes n} \sharp S_{\vec{n}} \subset H_{1,0,c}(\mathbf{\Gamma}_n)$, where $S_{\vec{n}}$ acts both on X and on $\mathbf{Y}$ (permuting the factors of $\mathbf{Y}$). We can form the induced $H_{1,0,c}(\mathbf{\Gamma}_n)$-module:

$$X \otimes \mathbf{Y} \uparrow := \mathrm{Ind}_{H_{1,c}(\Gamma)^{\otimes n} \sharp S_{\vec{n}}}^{H_{1,0,c}(\mathbf{\Gamma}_n)} X \otimes Y.$$

By [Mac80] (paragraph after (A.5)), we have the following theorem.

Theorem 3.3.9. *The representation $X \otimes \mathbf{Y} \uparrow$ is irreducible and runs through a complete system of pairwise non-isomorphic irreducible representations of $H_{1,0,c}(\mathbf{\Gamma}_n)$ if $\mathbf{Y}$ runs through a complete system of pairwise non-conjugate irreducible representations of $H_{1,c}(\Gamma)$ and, while $\mathbf{Y}$ remains fixed, X runs through a complete system of pairwise non-isomorphic irreducible representations of $S_{\vec{n}}$.*

3.3.5 The main theorem

Denote by $M_c = X \otimes \mathbf{Y} \uparrow$ an $H_{1,0,c}(\mathbf{\Gamma}_n)$-module of some type $(n_1, \ldots, n_p)$ for some c. Notice that, by Theorem 3.3.5, such representations exist only for special values of c. Denote by M the underlying vector space of M_c. Let $\mathcal{M} = \mathcal{M}(\mathbf{Y}, X)$ be the moduli space of irreducible representations of $H_{1,k,c}(\mathbf{\Gamma}_n)$ isomorphic to M as $\mathbf{\Gamma}_n$-modules (where (k, c) are allowed to vary). This is a quasi-affine algebraic variety: it is the quotient of the quasi-affine variety $\widetilde{\mathcal{M}}(\mathbf{Y}, X)$ of extensions of the $\mathbf{\Gamma}_n$-module M to an irreducible $H_{1,k,c}(\mathbf{\Gamma}_n)$-module by a free action of the reductive group G of basis changes in M compatible with $\mathbf{\Gamma}_n$ modulo scalars. Let $F : \mathcal{M} \to C(\mathcal{S})$ be the morphism which sends a representation to the corresponding values of (k, c).

Let χ_{Y_h} be the character of the representation Y_h of $H_{1,c_0}(\Gamma)$ and let $\alpha_{(h)}$ be the positive real root attached to Y_h under the Morita equivalence. With the same notation as in Theorem 3.2.4 we have the following result.

Theorem 3.3.10. *Suppose X_h has rectangular Young diagram of some size $a_h \times b_h$ for any h such that $n_h \neq 0$, and that moreover $(\alpha_{(h)}, \alpha_{(h')}) = 0$ for any $h \neq h'$ such that $n_h, n_{h'} \neq 0$ (where we agree that the last condition is empty if $\vec{n} = (0, \ldots, n, \ldots, 0)$ corresponds to the trivial partition). Then:*

(i) For any c_0 the representation M_{c_0} of $H_{1,0,c_0}(\mathbf{\Gamma}_n)$ can be formally deformed to a representation of $H_{1,k,c}(\mathbf{\Gamma}_n)$ along the intersection of the hyperplanes

$$\mathcal{H}_{Y_h,a_h,b_h} : \dim Y_i + \frac{k}{2}|\Gamma|(b_i - a_i) + \sum_{s=2}^{\nu} c_s|C_s|\chi_{Y_i}(C_s) = 0$$

for all $h \in \{1, \ldots, p\}$ such that $n_h \neq 0$, but not in other directions. This deformation is unique. The linear space described by the above equations has dimension $\nu - r$ where $r = \#\{h \text{ s.t. } n_h \neq 0\}$.

(ii) The morphism F maps $\mathcal{M}$ to $\bigcap_{h|n_h\neq 0} \mathcal{H}_{Y_h,a_h,b_h}$ and is étale at M_{c_0} for all c_0. Its restriction to the formal neighborhood of M_{c_0} is the deformation from (i).

(iii) There exists a nonempty Zariski open subset $\mathcal{U}$ of $\bigcap_{h|n_h\neq 0} \mathcal{H}_{Y_h,a_h,b_h}$ such that for $(k, c) \in \mathcal{U}$, the algebra $H_{1,k,c}(\mathbf{\Gamma}_n)$ admits a finite dimensional irreducible representation isomorphic to M as a $\mathbf{\Gamma}_n$-module.

3.3.6 Proof of Theorem 3.3.10

STEP 1 **Homological properties of $H_{1,c}(\Gamma)$**

We recall the following definition (see [vdB98, vdB02, EO06]):

Definition 3.3.11. *An algebra B is defined to be in the class $VB(d)$ if it is of finite Hochschild dimension (i.e. there exists $n \in \mathbb{N}$ s.t. $H^i(B, E) = 0$ for any $i > n$ and any B-bimodule E) and $H^*(B, B \otimes B^o)$ is concentrated in degree d, where it equals B as a B-bimodule.*

The meaning of this definition is clarified by the following result by Van den Bergh ([vdB98, vdB02]).

Theorem 3.3.12. *If $B \in VB(d)$ then for any B-bimodule E, the Hochschild homology $H_i(B, E)$ is naturally isomorphic to the Hochschild cohomology $H^{d-i}(B, E)$.*

□

Proposition 3.3.13. *The algebra $H_{1,c}(\Gamma)$ belongs to the class $VB(2)$.*

Proof. Let us pose $B := H_{1,c}(\Gamma)$. If $\Gamma = \{1\}$, the statement is well known ([vdB98, vdB02]; see also [EO06]). Let us consider the case $\Gamma \neq \{1\}$. We have to show that B has finite Hochschild dimension and that:

$$H^i(B, B \otimes B^o) = 0 \qquad \text{for } i \neq 2$$

$$H^2(B, B \otimes B^o) \cong B \qquad \text{as } B-\text{bimodules.}$$

The algebra $\mathbb{C}\langle x, y\rangle \sharp\Gamma$ has a natural increasing filtration obtained by putting x, y in degree 1 and the elements of Γ in degree 0. This filtration clearly induces a filtration on B: $B = \cup_{n\geq 0} F^n B$, and the associated graded algebra is $B_0 = grB = \mathbb{C}[x, y]\sharp\Gamma$ (by the PBW theorem), which has Hochschild dimension 2. So by a deformation argument we have that B has finite Hochschild dimension (equal to 2) and $H^i(B, B \otimes B^o) = 0$ for $i \neq 2$, as this is true for B_0 (which is easily checked since B_0 is a semidirect product of a finite group with a polynomial algebra).

It remains to show the B-bimodule $E := H^2(B, B \otimes B^o)$ is isomorphic to B. Using again a deformation argument (cf. [vdB98]), we can see that E is invertible and free as a right and left B-module, because this is true for B_0. So $E = B\phi$ where ϕ is an automorphism of B such that $gr\phi = 1$. We will now show that $\phi = 1$, which will conclude the proof.

Define a linear map $\xi : B_0 \to B_0$ as follows: if $z \in B_0$ is a homogeneous element of degree n, and $\widetilde{z}$ is its lifting to B, then $\xi(z)$ is defined to be the projection of the element $\phi(\widetilde{z}) - \widetilde{z}$ (which has filtration degree $n - 1$) to $(grB)_{n-1}$. It is easy to check that ξ is well defined (i.e., independent on the choice of the lifting), and is a derivation of B_0 of degree -1.

Our job is to show that $\xi = 0$. This would imply that $\phi = 1$, since B is generated by $F^1 B$.

It is clear that any homogeneous inner derivation of B_0 has nonnegative degree. Hence, it suffices to show that the degree -1 part of $H^1(B_0, B_0)$ is zero. But it is easy to compute using Koszul complexes that $H^1(B_0, B_0) = \mathrm{Vect}(L)^\Gamma$, the space of Γ-invariant vector fields on L. In particular, vector fields of degree -1 are those with constant coefficients. But such a vector field cannot be Γ-invariant unless it is zero, since the space L has no nonzero vectors fixed by Γ. Thus, $\xi = 0$ and we are done.

□

Corollary 3.3.14. $H^2(H_{1,c}(\Gamma), \mathrm{End}\, Y) = H_0(H_{1,c}(\Gamma), \mathrm{End}\, Y) = \mathbb{C}$.

Proof. Posing again $B := H_{1,c}(\Gamma)$, we can apply Theorem 3.3.12 to obtain the first identity. Furthermore, $H_0(B, \mathrm{End}\, Y) = \mathrm{End}\, Y/[B, \mathrm{End}\, Y] = \mathbb{C}$ as Y is irreducible, so the second identity follows.

□

<u>STEP 2</u> Homological properties of $H_{1,0,c}(\mathbf{\Gamma}_n)$

We would now like to apply Proposition 3.3.1 to the algebra $H_{1,0,c_0}(\mathbf{\Gamma}_n)$, its flat formal deformation $H_{1,k,c_0+c'}(\mathbf{\Gamma}_n)$ over the finite dimensional vector space $U = C(\mathcal{S})$, and any module M satisfying the conditions of Theorem 3.3.10. Our job is to compute the cohomology groups $H^2(H_{1,0,c}(\mathbf{\Gamma}_n), \mathrm{End}\, M)$, $H^1(H_{1,0,c}(\mathbf{\Gamma}_n), \mathrm{End}\, M)$ and to show the surjectivity of the map η.

Proposition 3.3.15. *If M is as in Theorem 3.3.10 and $r \in \mathbb{N}$ is as in part (i) of the same theorem then*

$$H^2(H_{1,0,c}(\mathbf{\Gamma}_n), \mathrm{End}\, M) = \bigoplus_{h|n_h \neq 0} H^2(H_{1,c}(\Gamma), \mathrm{End}\, Y_h) = \mathbb{C}^r.$$

Proof. The second equality follows from Corollary 3.3.14. Let us prove the first equality. For simplicity, let B denote the algebra $H_{1,c}(\Gamma)$ as above, and let A be the algebra $H_{1,0,c}(\mathbf{\Gamma}_n)$. Thus we can write $A = S_n \sharp B^{\otimes n}$. We have:

$$\begin{aligned} H^*(A, \operatorname{End} M) &= \operatorname{Ext}^*_{A\otimes A^o}(A, \operatorname{End} M) \\ &= \operatorname{Ext}^*_{B^{\otimes n}\sharp S_n \otimes B^{o\otimes n}\sharp S_n}(B^{\otimes n}\sharp S_n, \operatorname{End} M) \\ &= \operatorname{Ext}^*_{(B^{\otimes n}\otimes B^{o\otimes n})\sharp S_n\times S_n}(B^{\otimes n}\sharp S_n, \operatorname{End} M) \end{aligned}$$

Now, the $(B^{\otimes n} \otimes B^{o\otimes n})\sharp S_n \times S_n$-module $B^{\otimes n}\sharp S_n$ is induced from the module $B^{\otimes n}$ over the subalgebra $(B^{\otimes n} \otimes B^{o\otimes n})\sharp S_n$, in which S_n acts simultaneously permuting the factors of $B^{\otimes n}$ and $B^{o\otimes n}$ (note that $(B^{\otimes n} \otimes B^{o\otimes n})\sharp S_n$ is indeed a subalgebra of $(B^{\otimes n} \otimes B^{o\otimes n})\sharp S_n \times S_n$ as it can be identified with the subalgebra $(B^{\otimes n} \otimes B^{o\otimes n})\sharp D$ where $D = \{(\sigma, \sigma),\ \sigma \in S_n\} \subset S_n \times S_n$). Applying the Shapiro Lemma we get:

$$\begin{aligned} \operatorname{Ext}^*_{(B^{\otimes n}\otimes B^{o\otimes n})\sharp S_n\times S_n}(B^{\otimes n}\sharp S_n, \operatorname{End} M) &= \operatorname{Ext}^*_{(B^{\otimes n}\otimes B^{o\otimes n})\sharp S_n}(B^{\otimes n}, \operatorname{End} M) \\ &= \left(\operatorname{Ext}^*_{B^{\otimes n}\otimes B^{o\otimes n}}(B^{\otimes n}, \operatorname{End} M)\right)^{S_n}. \end{aligned}$$

We observe now that the subalgebra $B^{\otimes n}$ is stable under the inner automorphisms induced by the elements $\sigma \in S_n \subset A$. Thus setting $M' = X \otimes \mathbf{Y}$ the induced A-module M can be written as:

$$M = \sigma_1 M' \oplus \sigma_2 M' \oplus \cdots \oplus \sigma_\ell M' \tag{3.16}$$

where $\ell = \frac{n!}{n_1! \dots n_r!}$ and $\{\sigma_1, \dots, \sigma_\ell\}$ is a set of representatives for the left cosets of $S_{\vec{n}}$ in S_n. The action of an element $\sigma(b_1 \otimes \cdots \otimes b_n)$ on a vector $\sigma_l \cdot m' \in \sigma_l M'$ is the following:

$$\sigma(b_1 \otimes \cdots \otimes b_n)\,(\sigma_l \cdot m') = \sigma_s \cdot (\sigma'(b_{\sigma_l(1)} \otimes \cdots \otimes b_{\sigma_l(n)})m') \in \sigma_s M' \tag{3.17}$$

where $\sigma' \in S_{\vec{n}}$, $\sigma_s \in \{\sigma_1, \dots, \sigma_\ell\}$ are the only elements satisfying $\sigma\sigma_l = \sigma_h \sigma'$. In particular as a $B^{\otimes n}$-module, each summand $\sigma_l M' = \sigma_l X \otimes \sigma_l Y$ equals:

$$X \otimes Y_{h_{\sigma_l^{-1}(1)}} \otimes \cdots \otimes Y_{h_{\sigma_l^{-1}(n)}}$$

with trivial action of $B^{\otimes n}$ on X.

Thus we have a chain of S_n-equivariant isomorphisms:

$$\begin{aligned}
&\operatorname{Ext}^*_{B^{\otimes n}\otimes B^{o\otimes n}}(B^{\otimes n}, \operatorname{End} M)\\
&= \operatorname{Ext}^*_{B^{\otimes n}\otimes B^{o\otimes n}}\left(B^{\otimes n}, \operatorname{End}\left(\bigoplus_l \sigma_l M'\right)\right)\\
&= \operatorname{Ext}^*_{B^{\otimes n}\otimes B^{o\otimes n}}\left(B^{\otimes n}, \bigoplus_{l,s} \operatorname{Hom}(\sigma_l M', \sigma_s M')\right)\\
&= \bigoplus_{l,s} \operatorname{Ext}^*_{B^{\otimes n}\otimes B^{o\otimes n}}\left(B^{\otimes n}, \operatorname{Hom}(\sigma_l M', \sigma_s M')\right)\\
&= \bigoplus_{l,s} \operatorname{Ext}^*_{B^{\otimes n}\otimes B^{o\otimes n}}\left(B^{\otimes n}, \operatorname{Hom}(\sigma_l X, \sigma_s X)\otimes \operatorname{Hom}(\sigma_l \mathbf{Y}, \sigma_s \mathbf{Y})\right)\\
&= \bigoplus_{l,s} \operatorname{Hom}(\sigma_l X, \sigma_s X)\otimes \operatorname{Ext}^*_{B^{\otimes n}\otimes B^{o\otimes n}}\left(B^{\otimes n}, \operatorname{Hom}(\sigma_l \mathbf{Y}, \sigma_s \mathbf{Y})\right)\\
&= \bigoplus_{l,s} \operatorname{Hom}(\sigma_l X, \sigma_s X)\otimes \operatorname{Ext}^*_{B^{\otimes n}\otimes B^{o\otimes n}}\left(B^{\otimes n}, \bigotimes_{i=1}^{n} \operatorname{Hom}(Y_{h_{\sigma_l^{-1}(i)}}, Y_{h_{\sigma_s^{-1}(i)}})\right) \qquad (3.18)
\end{aligned}$$

where the third identity holds since the action of $B^{\otimes n} \otimes B^{o\otimes n}$ does not permute the direct factors in $\bigoplus_l \sigma_l M'$ and is trivial on X in the module M'. Since all the identities pass to invariants, all is left to do is computing the degree 2 component of the invariance of (3.18).

To this end, we apply the Künneth formula in degree 2. Lemma 3.3.7, Lemma 3.3.8 and our conditions on the Y_h's guarantee $Ext^1_B(Y_h, Y_{h'}) = 0$ for any $h \neq h'$ with $n_h, n'_h \neq 0$. Moreover, for any $h \neq h'$, Y_h, $Y_{h'}$ are non-isomorphic irreducible representations of B, thus $\operatorname{Ext}^0_B(Y_h, Y_{h'}) = \operatorname{Hom}_B(Y_h, Y_{h'}) = 0$. As a consequence we get

$$\left(\bigoplus_{\substack{l,s\\ (\sigma_l^{-1}(i)\sigma_s^{-1}(i))\in S_{\vec{n}}\,\forall i}} \operatorname{Hom}(\sigma_l X, \sigma_s X)\otimes \bigoplus_{i=1}^{n} \operatorname{Ext}^2_{B\otimes B^o}\left(B, \operatorname{Hom}(Y_{h_{\sigma_l^{-1}(i)}}, Y_{h_{\sigma_s^{-1}(i)}})\right)\right)^{S_n}$$

where $(\sigma_l^{-1}(i)\sigma_s^{-1}(i))$ denotes the transposition moving the corresponding indices. Now we have $(\sigma_l^{-1}(i)\sigma_h^{-1}(i)) \in S_{\vec{n}}$, $\forall i$ if and only if $\sigma_l = \sigma_s\sigma$ with $\sigma \in S_{\vec{n}}$. But σ_l, σ_s belong to different left cosets of $S_{\vec{n}}$. Thus we can rewrite the last term as:

$$\begin{aligned}
&\left(\bigoplus_l \operatorname{End}\sigma_l X \otimes \bigoplus_i \operatorname{Ext}^2_{B\otimes B^o}(B, \operatorname{End} Y_{h_{\sigma_l^{-1}(i)}})\right)^{S_n} \\
&= \left(\bigoplus_{h|n_h\neq 0} \operatorname{End} X \otimes \left(\operatorname{Ext}^2_{B\otimes B^o}(B, \operatorname{End} Y_h)\right)^{\oplus n_h}\right)^{S_{\vec{n}}} \\
&= \left(\bigoplus_{h|n_h\neq 0} \left(\bigotimes_{h'|n_{h'}\neq 0} \operatorname{End} X_{h'}\right) \otimes \left(\operatorname{Ext}^2_{B\otimes B^o}(B, \operatorname{End} Y_h)\right)^{\oplus n_h}\right)^{S_{\vec{n}}}.
\end{aligned}$$

Now we have that as an S_{n_h}-module:

$$\left(\operatorname{Ext}^2_{B\otimes B^o}(B, \operatorname{End} Y_h)\right)^{\oplus n_h} = \operatorname{Ext}^2_{B\otimes B^o}(B, \operatorname{End} Y_h) \otimes \mathbb{C}^{n_h}$$

with S_{n_h} acting only on $\mathbb{C}^{n_h}$ by permuting the factors, and $\mathbb{C}^{n_h} = \mathbb{C} \oplus \mathfrak{h}_h$, where $\mathbb{C}$ the trivial representation, and $\mathfrak{h}_h$ is the reflection representation of S_{n_h}. So we have:

$$\begin{aligned}
&\bigoplus_{h|n_h\neq 0} \operatorname{Ext}^2_{B\otimes B^o}(B, \operatorname{End} Y_h) \otimes (\operatorname{End} X_1 \otimes \cdots \otimes (\mathbb{C}^{n_h} \otimes \operatorname{End} X_h) \otimes \cdots \otimes \operatorname{End} X_p)^{S_{\vec{n}}} \\
&= \bigoplus_{h|n_h\neq 0} \operatorname{Ext}^2_{B\otimes B^o}(B, \operatorname{End} Y_h) \otimes \\
&\quad \otimes \left(\operatorname{End}_{S_{n_1}} X_1 \otimes \cdots \otimes (\mathbb{C} \otimes \operatorname{End} X_h \oplus \mathfrak{h}_h \otimes \operatorname{End} X_h)^{S_{n_h}} \otimes \cdots \otimes \operatorname{End}_{S_{n_p}} X_p\right) \\
&= \bigoplus_{h|n_h\neq 0} \operatorname{Ext}^2_{B\otimes B^o}(B, \operatorname{End} Y_h) \otimes \left(\operatorname{End}_{S_{n_h}} X_h \oplus \operatorname{Hom}_{S_{n_h}}(\mathfrak{h}_h \otimes X_h, X_h)\right) \\
&= \bigoplus_{h|n_h\neq 0} \operatorname{Ext}^2_{B\otimes B^o}(B, \operatorname{End} Y_h) = \mathbb{C}^r
\end{aligned}$$

since, by Lemma 3.2.3, the fact that X_h has rectangular Young diagram for any h such that $n_h \neq 0$ guarantees $\operatorname{Hom}_{S_{n_h}}(\mathfrak{h}_h \otimes X_h, X_h) = 0$ for all such hs.

□

Corollary 3.3.16. *If M is as in Proposition 3.3.15, then map*

$$\eta : U \longrightarrow H^2(H_{1,0,c}(\mathbf{\Gamma}_n), \operatorname{End} M)$$

is surjective.

Proof. As in the previous proofs, let $B := H_{1,c}(\Gamma)$ and $A := H_{1,0,c}(\mathbf{\Gamma}_n)$. Let $U_0 \subset U$ be the subspace of vectors $(0, c')$. It is sufficient to show that the restriction of η to U_0 is surjective. But this restriction is a composition of three natural maps:

$$U_0 \to H^2(B, B) \to H^2(A, A) \to H^2(A, \operatorname{End} M).$$

Here the first map $\eta_0 : U_0 \to H^2(B, B)$ is induced by the deformation of B along U_0, the second map $\xi : H^2(B, B) \to H^2(A, A)$ comes from the Künneth formula, and the third map $\psi : H^2(A, A) \to H^2(A, \operatorname{End} M)$ is induced by the homomorphism $A \to \operatorname{End} M$.

Now, by Proposition 3.3.15, the map $\psi \circ \xi$ coincides with the map $\psi_0 : H^2(B, B) \to \bigoplus_{h|n_h \neq 0} H^2(B, \operatorname{End} Y_h)$ induced by the homomorphism $\phi : B \to \bigoplus_{h|n_h \neq 0} \operatorname{End} Y_h$. Let $K_0 = \operatorname{Ker}(\psi_0)$ and $U_0' = \eta_0^{-1}(K_0)$. We have to show that $\operatorname{codim} U_0' \geq r = \sharp \{h \text{ s.t. } n_h \neq 0\}$. By the results of **STEP 1**, Proposition 3.3.1 can be applied to the algebra B and the representation Y_h for any h such that $n_h \neq 0$. Thus, for any such h, the representation Y_h admits a first order deformation along U_0'. So, by the defining relations for the rank one case (see Example 2.3.4), on U_0' we must have $\operatorname{tr}_{Y_h}(\mathbf{c}) = 0$, where $\mathbf{c}$ is as in Section 3.3.2. Since the representations Y_h are non-isomorphic their characters are linearly independent by Lemma 3.3.6, hence so are these linear equations. Thus the $\operatorname{codim}(U_0') \geq r$, and $\eta|_{U_0}$ is surjective, as desired.

□

Proposition 3.3.17. $H^1(H_{1,0,c}(\mathbf{\Gamma}_n), \operatorname{End} M) = 0$.

Proof. Arguing as in the proof of Proposition 3.3.15, and using the same notation, we get that $H^1(A, \operatorname{End} M) = \bigoplus_{h|n_h \neq 0} H^1(B, \operatorname{End} Y_h)$, which is zero by Lemma 3.3.8. This proves the proposition.

□

We have proved the following result.

Proposition 3.3.18. *If the conditions of Theorem 3.3.10 are satisfied then there exists a unique smooth codimension r formal subscheme S of the formal neighborhood of the origin in U such that the representation of* $H_{1,0,c_0}(\mathbf{\Gamma}_n)$ *on the vector space M deforms to a representation of* $H_{1,k,c_0+c'}(\mathbf{\Gamma}_n)$ *along S (i.e., abusing the language, for* $(k,c') \in S$*). Furthermore, such a deformation over S is unique.*

Proof. Corollary 3.3.16 and Proposition 3.3.17 show that our case satisfies all the hypotheses of Proposition 3.3.1. Moreover, from $H^2(A, \operatorname{End} M) = \mathbb{C}^r$ we deduce $\dim \operatorname{Ker} \eta = \dim U - r$, and the Proposition follows.

□

<u>STEP 3</u> The subscheme S and the proof of part (i)

Now we would like to find the subscheme S of Proposition 3.3.18. We will do this computing some appropriate trace conditions for the deformation of the module M. Let $x_i, y_i, \gamma_i, s_{ij}$ be the elements of $H_{1,k,c}(\mathbf{\Gamma}_n)$ that appear in Lemma 2.3.3.

Let again r be as in Theorem 3.3.10 part (ii). For simplicity let us write $\mathbf{Y} = Y_{h_1}^{\otimes n_{h_1}} \otimes \cdots \otimes Y_{h_r}^{\otimes h_r}$ where $h_s \in \{1, \ldots, p\}$ are all distinct, and $X = X_{h_1} \otimes \cdots \otimes X_{h_r}$. Since the Y_{h_s} are finite dimensional, irreducible, non-isomorphic representations of $H_{1,c}(\Gamma)$, Lemma 3.3.6 ensures that for any choice of complex numbers $z_1, \ldots, z_r$ there exists a central element Z of $\mathbb{C}[\Gamma]$ such that $\chi_{Y_{h_s}}(Z) = \operatorname{tr}_{Y_{h_s}}(Z) = z_s$. Fix now $Z(1)$ such that $\operatorname{tr}_{Y_{h_s}}(Z(1)) = 1 - \delta_{1s}$ and consider the element:

$$P_1 = \underbrace{1 \otimes \cdots \otimes 1}_{n_{h_1}} \otimes \underbrace{Z(1) \otimes \cdots \otimes Z(1)}_{n-n_{h_1}} \in H_{1,k,c}(\mathbf{\Gamma}_n).$$

Such element commutes with x_1, y_1. Consider now the relation (R1) in Lemma 2.3.3 for $i = 1$ (and $t = 1$), and multiply it by P_1 on the right. The left hand side becomes

$[x_1, y_1 P_1]$, thus it has trace zero. To compute the trace of the right hand side operator it is convenient to use again the decomposition of the induced module M given in the previous section (cfr. (3.16), (3.17)). The trace of the right hand side reduces to the sum of the three terms:

$$\mathrm{tr}_M P_1 = m\,(\dim Y_{h_1})^{n_{h_1}} \prod_s \dim X_{h_s} \tag{3.19}$$

$$\frac{k}{2}\mathrm{tr}_M\left(\sum_{j=2}^{n_{h_1}}\sum_{\gamma\in\Gamma} s_{1j}\gamma_1\gamma_j^{-1}\right)P_1 = \frac{k}{2}\,m\,\mathrm{tr}_{X_{h_1}\otimes Y_{h_1}^{\otimes n_{h_1}}}\left(\sum_{j=2}^{n_{h_1}}\sum_{\gamma\in\Gamma} s_{1j}\gamma_1\gamma_j^{-1}\right)\prod_{s\neq 1}\dim X_{h_s} \tag{3.20}$$

$$\mathrm{tr}_M\left(\sum_{\gamma\in\Gamma\smallsetminus\{1\}} c_\gamma\gamma_1\right)P_1 = m\left(\sum_{\gamma\in\Gamma\smallsetminus\{1\}} c_\gamma \mathrm{tr}_{Y_{h_1}}(\gamma)\right)(\dim Y_{h_1})^{n_{h_1}-1}\prod_s \dim X_{h_s} \tag{3.21}$$

where $m = \frac{(n-n_{h_1})!}{n_{h_2}!\dots n_{h_r}!}$ is the number of elements σ_l in the set of representatives $\{\sigma_1, \dots, \sigma_\ell\}$ such that $\sigma_l^{-1}(\{1, 2, \dots, n_{h_1}\}) = \{1, 2, \dots, n_{h_1}\}$. Now we claim that:

$$\mathrm{tr}_{X_{h_1}\otimes Y_{h_1}^{\otimes n_{h_1}}}(s_{1j}\,\gamma_1\,\gamma_j{}^{-1}) = \mathrm{tr}_{X_{h_1}}(s_{1j})\dim Y_{h_1}{}^{n_{h_1}-1} \qquad \forall\, j \le n_{h_1}. \tag{3.22}$$

To obtain (3.22), we observe that, for $j \le n_{h_1}$, $s_{1j}\,\gamma_1\,\gamma_j{}^{-1}$ is conjugate in the subgroup $\Gamma_{n_{h_1}} = S_{n_{h_1}} \ltimes \Gamma^{n_{h_1}}$ to s_{1j} and that the character of $S_{n_{h_1}}$ on $X_{h_1} \otimes Y_{h_1}^{\otimes n_{h_1}}$ is simply the product of the characters on X_{h_1} and $Y_{h_1}^{\otimes n_{h_1}}$. An easy computation gives $\mathrm{tr}_{Y_{h_1}^{\otimes n_{h_1}}} s_{ij} = \dim Y_{h_1}{}^{n_{h_1}-1}$, hence the formula.

From the proof of Lemma 3.2.3, we know that the central element $\sum\limits_{i<j\le n_{h_1}} s_{ij}$ in $S_{n_{h_1}}$ acts on X_{h_1} as the scalar $\mathbf{c}(\mu)$, where $\mathbf{c}(\mu)$ is the content of the Young diagram μ attached to X_{h_1}. We can deduce that the trace of each transposition s_{1j} is $\mathrm{tr}_{Y_{h_1}}(s_{1j}) = \frac{\dim\, Y_{h_1}}{n_{h_1}(n_{h_1}-1)/2}\mathbf{c}(\mu)$. Thus if the Young diagram of X_{h_1} is of size $a_{h_1} \times b_{h_1}$ we get:

$$\mathrm{tr}_{X_{h_1}\otimes Y_{h_1}^{\otimes n_{h_1}}}(s_{1j}\gamma_1\,\gamma_j{}^{-1}) = \frac{(b_{h_1} - a_{h_1})\dim X_{h_1}}{n_{h_1} - 1}\dim Y_{h_1}{}^{n_{h_1}-1}. \tag{3.23}$$

where we used Lemma 3.2.3 part (iii) to evaluate $\mathbf{c}(\mu) = (b_{h_1} - a_{h_1})n_{h_1}/2$. Substituting in (3.20), summing up (3.19),(3.20),(3.21) and simplifying we get the equation:

$$\dim Y_{h_1} + \frac{k}{2}|\Gamma|(b_{h_1} - a_{h_1}) + \sum_{\gamma\in\Gamma\smallsetminus\{1\}} c_\gamma \chi_{Y_{h_1}}(\gamma) = 0 \tag{3.24}$$

that (grouping the elements $\gamma \in \Gamma \smallsetminus \{1\}$ according their conjugacy class) is exactly the equation for the hyperplane $\mathcal{H}_{Y_{h_1},a_{h_1},b_{h_1}}$. Analogously we can define $Z(s)$, P_s and obtain the equations of the hyperplane $\mathcal{H}_{Y_{h_s},a_{h_s},b_{h_s}}$ for $s = 2,\dots,r$. We get exactly r independent necessary linear conditions.

This shows that $(0, c_0) + S \subset \bigcap_{s=1}^r \mathcal{H}_{Y_{h_s},a_{h_s},b_{h_s}}$. But since the two subschemes have the same dimension we have that S is the formal neighborhood of zero in $\bigcap_{i=1}^r \mathcal{H}_{Y_{h_s},a_{h_s},b_{h_s}} - (0, c_0)$ and Theorem 3.3.10, (i) is proved.

STEP 4 **Proof of part** (ii) **and** (iii)

Let $\mathcal{M}'$ be the formal neighborhood of M_{c_0} in $\mathcal{M}$. We have shown that the morphism $F : \mathcal{M} \to U$ lands in $\bigcap_{h|n_h\neq 0} \mathcal{H}_{Y_h,a_h,b_h}$, and that $F|_{\mathcal{M}'} : \mathcal{M}' \to (0, c_0) + S$ is an isomorphism. This implies that the map $F : \mathcal{M} \to \bigcap_{h|n_h\neq 0} \mathcal{H}_{Y_h,a_h,b_h}$ is étale at M_{c_0}. This proves part (ii) of Theorem 3.3.10, and, together with Proposition 3.3.2, also implies (iii), since a map which is étale at one point is dominant.

Chapter 4

Continuous symplectic reflection algebras

4.1 Plan of the chapter

Continuous symplectic reflection algebras were introduced by Etingof, Gan, and Ginzburg in [EGG05] as a special case of continuous Hecke algebras. They are a generalization of symplectic reflection algebras to reductive algebraic groups, where the role of the group algebra is played by the ring of algebraic distributions $\mathcal{O}(G)^*$. In this chapter, after recalling some generalities about algebraic distributions in Section 4.2, we define continuous symplectic reflection algebras in Section 4.3 and describe some of their properties. In Section 4.4 we consider more specifically continuous symplectic reflection algebras for wreath product groups. Finally, in Section 4.5 we define infinitesimal Hecke algebras, another special case of continuous Hecke algebras we will be interested in.

4.2 Algebraic distributions

Let G be a reductive algebraic group over $\mathbb{C}$, and denote by $\mathcal{O}(G)$ the algebra of regular functions on G. The algebraic distributions on G are the elements of the dual space $\mathcal{O}(G)^*$.

Example 4.2.1. *For any $g \in G$ there is a unique distribution δ_g such that $\langle \delta_g, f\rangle = f(g)$ for any $f \in \mathcal{O}(G)$. This distribution is called the delta distribution concentrated at the point g.*

The algebra $\mathcal{O}(G)^*$ is naturally equipped with the *weak* or *initial* topology, as we are going to explain. Let $\langle\ ,\ \rangle : \mathcal{O}(G)^* \times \mathcal{O}(G) \longrightarrow \mathbb{C}$ be the natural pairing, and for any $f \in \mathcal{O}(G)$ consider the map

$$\begin{array}{rccc} T_f : & \mathcal{O}(G)^* & \longrightarrow & \mathbb{C} \\ & \mu & \longrightarrow & \langle \mu, f\rangle \end{array}$$

The *weak topology* on $\mathcal{O}(G)^*$ is the coarsest topology making all the functions T_f for $f \in \mathcal{O}(G)$ continuous, where $\mathbb{C}$ is given the discrete topology. To understand how the weak topology looks like, let us realize $\mathcal{O}(G)^*$ as the projective limit

$$\mathcal{O}(G)^* = \varprojlim U^*$$

where $U \subset \mathcal{O}(G)$ ranges over all finite dimensional sub-vector spaces. Then, the weak topology coincides with the *inverse limit* topology on $\mathcal{O}(G)^*$, i.e. the coarsest topology making all the projections $\mathcal{O}(G)^* \twoheadrightarrow U$ continuous (where each U is given the discrete topology). In this topology a system of neighborhoods of $0 \in \mathcal{O}(G)^*$ is given by all the sub-vector spaces of $\mathcal{O}(G)^*$ of finite codimension.

The coalgebra structure on $\mathcal{O}(G)$ induces an algebra structure on $\mathcal{O}(G)^*$ given by the convolution product. To ease notation we will write simply $\mu\mu'$ for the convolution of any two distributions $\mu, \mu' \in \mathcal{O}(G)^*$. If $\Delta : \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \otimes \mathcal{O}(G)$ denotes the coproduct for $\mathcal{O}(G)$, then the convolution product $\mu\mu'$ is the unique distribution on G such that:

$$\langle \mu\mu', f\rangle = \langle \mu \otimes \mu', \Delta(f)\rangle \qquad \forall f \in \mathcal{O}(G)\,. \tag{4.1}$$

Example 4.2.2. *If G is a finite group, the assignment $g \longrightarrow \delta_g$ defines an algebra isomorphism $\mathbb{C}[G] \longrightarrow \mathcal{O}(G)^*$.*

The algebra $\mathcal{O}(G)^*$ is equipped with a natural action of the algebra $\mathcal{O}(G)$. For any $f \in \mathcal{O}(G)$ and $\mu \in \mathcal{O}(G)^*$ define the element $f\mu = \mu f$ as the unique distribution such that:

$$\langle f\mu, h\rangle = \langle \mu, fh\rangle \qquad \forall h \in \mathcal{O}(G). \tag{4.2}$$

Let now Z be a closed subscheme of G and let $I(Z)$ be the defining ideal of Z. An algebraic distribution μ is said to be supported on Z if μ annihilates all functions in $I(Z)$. Using formula (4.2), this is equivalent to say that $I(Z) \subset \mathrm{Ann}_{\mathcal{O}(G)}(\mu)$. It is clear that the space of algebraic distributions on G supported on Z is isomorphic to the space of algebraic distributions on Z.

We recall that there are two left actions of G on $\mathcal{O}(G)$ given by the left and right translation respectively. More specifically, if we denote by $g \longrightarrow \lambda_g \in \mathrm{End}\mathcal{O}(G)$ the homomorphism giving the left translation action, and by $g \longrightarrow \rho_g \in \mathrm{End}\mathcal{O}(G)$ the homomorphism giving the right translation we have

$$\lambda_g(f)(h) = f(g^{-1}h)$$
$$\rho_g(f)(h) = f(hg)$$

for any $f \in \mathcal{O}(G)$ and any $g, h \in G$. Both these actions induce left actions on $\mathcal{O}(G)^*$ in the obvious way. Keeping the same notation for such actions, we have:

$$\langle \rho_g(\mu), f\rangle = \langle \mu, \rho_{g^{-1}}(f)\rangle,\ \forall\, g \in G,\ \mu \in \mathcal{O}(G)^*,\ f \in \mathcal{O}(G)$$

(and similarly for the left translation λ_g). These actions commute and thus make $\mathcal{O}(G)^*$ into a $G \times G$-module. Note that as $G \times G$-modules $\mathcal{O}(G)$ and $\mathcal{O}(G)^*$ have the following decompositions:

$$\mathcal{O}(G) \cong \bigoplus_{i\in I} N_i \otimes N_i^* \cong \bigoplus_{i\in I} \mathrm{End}(N_i), \qquad \mathcal{O}(G)^* \cong \prod_{i\in I} N_i \otimes N_i^* \cong \prod_{i\in I} \mathrm{End} N_i \tag{4.3}$$

where N_i ranges over all (isomorphism classes of) irreducible finite dimensional representations of G, and N_i^* denotes the dual representation.

Example 4.2.3. *For any reductive group G there exists a unique right and left* *translation-invariant algebraic distribution* $\int_G(\cdot)\,dg : \mathcal{O}(G) \longrightarrow \mathbb{C}$ *such that* $\int_G 1\,dg =$ 1*. If we consider the decomposition of* $\mathcal{O}(G)$ *of formula (4.3) and we take* $N_0 \cong \mathbb{C}$ *to be the trivial representation, then* $\int_G(\cdot)\,dg$ *can be described as the projection on the one dimensional subspace* $N_0 \otimes N_0^*$*.*

Similarly the action of G on itself by conjugation induces actions on $\mathcal{O}(G)$ and $\mathcal{O}(G)^*$. These actions can be seen as the restrictions to the diagonal copy of G in $G \times G$ of the $G \times G$ actions on $\mathcal{O}(G)$ and $\mathcal{O}(G)^*$ that we described above. Thus, for any closed $\mathrm{Ad}(G)$-invariant subscheme Z of G, we have an induced action of G on the space $\mathcal{O}(Z)^*$ and a natural identification $(\mathcal{O}(Z)^*)^G = \mathcal{O}(Z/G)^*$. We will denote by $C(Z)$ this last space. We will identify $C(Z)$ with the space of G-invariant distributions supported on Z.

4.3 Continuous symplectic reflection algebras

Let (V, ω) be as in Section 2.2. Let $\{v_i\}$ be any basis of V and let $\{v_i^*\}$ be its dual basis. Let G be a reductive algebraic group with an algebraic representation $\rho : G \longrightarrow \mathrm{Sp}(V)$.

Let TV be the tensor algebra of V.

Definition 4.3.1. *The semidirect product* $TV\sharp\mathcal{O}(G)^*$ *is the algebra generated by* $u \in V$ *and* $\mu \in \mathcal{O}(G)^*$ *with the relations*

$$\mu \cdot u = \sum_i v_i \cdot (v_i^*, gu)\mu$$

where $(v_i^*, gu)\mu$ *denotes the action of the regular function* (v_i^*, gu) *on* μ*.*

Notice that, since $V^{\otimes N}$ is finite dimensional for all $N \in \mathbb{Z}_{\geq 0}$, we have

$$TV\sharp\mathcal{O}(G)^* = \bigoplus_{N\in\mathbb{Z}_{\geq 0}} V^{\otimes N} \otimes \mathcal{O}(G)^* = \bigoplus_{N\in\mathbb{Z}_{\geq 0}} ((V^*)^{\otimes N} \otimes \mathcal{O}(G))^*.$$

We can then give each summand $((V^*)^{\otimes N} \otimes \mathcal{O}(G))^*$ the weak topology and equip

$TV\sharp\mathcal{O}(G)^*$ with the *direct sum topology*, i.e. the finest topology such that all the canonical injections are continuous.

Let now $p : \bigwedge^3 V \longrightarrow V$ be the projection obtained by contracting the first two components using ω, and let Σ be the closed subscheme of G defined by the equation $p \circ \bigwedge^3(1 - g|_V) = 0$. This subscheme is clearly $\mathrm{Ad}(G)$-invariant.

Definition 4.3.2. *For any $t \in (\mathcal{O}(\mathrm{Ker}\,\rho)^*)^G$ and any $\phi \in C(\Sigma)$ the continuous symplectic reflection algebra $\mathcal{H}_{t,\phi}(G)$ is the quotient of the smash product $TV\sharp\mathcal{O}(G)^*$ by the relations*

$$uv - vu = \omega(u, v)t + \omega((1 - g)u, (1 - g)v)\phi \tag{4.4}$$

for any $u, v \in V$.

Moreover, $\mathcal{H}_{0,0}(G) = SV\sharp\mathcal{O}(G)^*$, and assigning grade degree zero to $\mathcal{O}(G)^*$ and grade degree one to V, we obtain a filtration on $\mathcal{H}_{t,\phi}(G)$ and a well defined surjective algebra map

$$\varphi : \mathcal{H}_{0,0}(G) = SV\sharp\mathcal{O}(G)^* \twoheadrightarrow \mathrm{gr}(\mathcal{H}_{t,\phi}(G)).$$

By [EGG05] Theorem 3.1 the map φ is an isomorphism, that is to say $\mathcal{H}_{t,\phi}(G)$ satisfies the PBW property as in the finite case.

Finally, let us consider the following analog of symplectic reflections for the continuous case.

Definition 4.3.3. *The set $\mathcal{S}$ of symplectic reflections is the set of elements $s \in G$ such that* $\mathrm{rk}(1 - s)|_V \leq 2$.

It can be seen that S is contained in the set of closed points $\Sigma(\mathbb{C})$ of Σ and that any semisimple element $g \in \Sigma(\mathbb{C})$ is in S. Using this it is easy to see, looking at the defining relations, that when G is finite and ρ is a faithful representation, the algebra $\mathcal{H}_{t,\phi}(G)$ is the same as the symplectic reflection algebra defined in Section 2.2 (see [EGG05], § 3.1).

4.4 The wreath product case

Let $L = (\mathbb{C}^2, \omega_L)$ and $(V = L^{\oplus n}, \omega_V)$ be as in Section 2.3. Let $\Gamma \subset SL(2,\mathbb{C}) = \mathrm{Sp}(L)$ an infinite reductive subgroup. It is well known that, up to conjugation, there exist only three such groups:

1) $SL(2,\mathbb{C})$;

2) $GL(1,\mathbb{C}) = \mathbb{C}^*$, identified with the maximal torus of diagonal matrices;

3) $\widetilde{O}_2$, identified with the normalizer of the maximal torus.

There is a natural symplectic action of the wreath product $\mathbf{\Gamma}_n = S_n \ltimes \Gamma^n$ on V. Thus we can write $\mathbf{\Gamma}_n \subset Sp(V)$.

Let us now to consider the ring of algebraic distributions over $\mathbf{\Gamma}_n$. This is the ring

$$\mathcal{O}(\mathbf{\Gamma}_n)^* \cong (\mathcal{O}(\Gamma^n)\sharp\mathcal{O}(S_n))^* \cong (\mathcal{O}(\Gamma)^{\otimes n}\sharp\mathcal{O}(S_n))^* = (\mathcal{O}(\Gamma)^{\otimes n})^*\sharp\mathcal{O}(S_n)^*$$

where the last identity holds because $\mathcal{O}(S_n) \cong \mathbb{C}[S_n]$ is finite dimensional. Let $\{N_i\}_{i\in I}$ be a complete collection of pairwise non-isomorphic irreducible finite dimensional Γ-modules, where $I = \mathbb{Z}$ if $\Gamma = GL(1,\mathbb{C})$ and $I = \mathbb{Z}_{\geq 0}$ otherwise. Using decomposition (4.3), we can consider the filtration by ideals of $\mathcal{O}(\Gamma)^*$ given by

$$\mathcal{O}(\Gamma)^* = \mathcal{O}(\Gamma)^*_0 \supset \mathcal{O}(\Gamma)^*_1 \supset \mathcal{O}(\Gamma)^*_2 \supset \cdots$$

where $\mathcal{O}(\Gamma)^*_N := \prod_{|i|\geq N} \mathrm{Mat}(d_i)$ and $d_i = \dim N_i$.

For any $l = 1,\dots,n$ and any $N \in \mathbb{Z}_{\geq 0}$ consider the ideals in $\mathcal{O}(\Gamma)^{*\otimes n}$

$$\mathcal{O}(\Gamma^n)^*_{N,l} = \mathcal{O}(\Gamma)^{*\otimes l-1} \otimes \mathcal{O}(\Gamma)^*_N \otimes \mathcal{O}(\Gamma)^{*\otimes n-l} \text{ and } \mathcal{O}(\Gamma^n)^*_N = \mathcal{O}(\Gamma^n)^*_{N,1} + \cdots + \mathcal{O}(\Gamma^n)^*_{N,n}.$$

We denote by $\mathcal{O}(\Gamma)^{*\hat{\otimes} n}$ the completion of the algebra $\mathcal{O}(\Gamma)^{*\otimes n}$ with respect to the filtration by ideals

$$\mathcal{O}(\Gamma)^{*\otimes n} = \mathcal{O}(\Gamma^n)^*_0 \supset \mathcal{O}(\Gamma^n)^*_1 \supset \mathcal{O}(\Gamma^n)^*_2 \supset \cdots$$

that is to say the projective limit

$$\mathcal{O}(\Gamma)^{*\hat{\otimes} n} := \varprojlim \mathcal{O}(\Gamma)^{*\otimes n} / \mathcal{O}(\Gamma^n)^*_N.$$

We have $\mathcal{O}(\Gamma)^{*\hat{\otimes} n} \cong (\mathcal{O}(\Gamma)^{\otimes n})^*$, and since from Example 4.2.2 we know $\mathcal{O}(S_n)^* = \mathbb{C}[S_n]$, we get:

$$\mathcal{O}(\mathbf{\Gamma}_n)^* \cong \mathcal{O}(\Gamma)^{*\hat{\otimes} n} \sharp S_n.$$

Note also that all the quotients $\mathcal{O}(\Gamma)^{*\otimes n}/\mathcal{O}(\Gamma^n)^*_N$ are finite dimensional vector spaces. Thus, equipping these vector spaces with the discrete topology, the inverse limit topology on $\mathcal{O}(\Gamma)^{*\hat{\otimes} n} \cong \mathcal{O}(\Gamma^n)^*$ coincides also in this case with the weak topology.

We want now to give a description of the subscheme Σ defined in the previous section. Let $\gamma_i, s_{ij} \in \mathbf{\Gamma}_n$ be as in Section 2.3. Let $\mathcal{S}_0 = \mathcal{S} \cap \Gamma^n$ be the set of symplectic reflections in Γ^n and let $\mathrm{Ad}(\mathbf{\Gamma}_n)s_{ij} = \{s_{lm}\gamma_l\gamma_m^{-1} | l \neq m,\ \gamma \in \Gamma\}$ be the conjugacy class of any transposition. Then it is easy to see that $\mathcal{S} = \mathcal{S}_0 \cup \mathrm{Ad}(\mathbf{\Gamma}_n)s_{ij}$. The group $\mathbf{\Gamma}_n$ acts by conjugation on $\mathcal{S}$ preserving this decomposition. Moreover we have $\Sigma = \Sigma_0 \cup \mathrm{Ad}(\mathbf{\Gamma}_n)s_{ij}$, where $\Sigma_0 = \Sigma \cap \Gamma^n$. It can be proved (see [EGG05], Proposition 6.4) that for Γ as in case $1), 2), 3)$ above, the set of orbits $\mathcal{S}/\mathbf{\Gamma}_n$ is a scheme isomorphic to $\Sigma/\mathbf{\Gamma}_n$ (see [EGG05], proof of Proposition 6.4). It follows that $C(\mathcal{S}) = C(\Sigma)$, and $C(\mathcal{S}) = C(\mathcal{S}_0) \oplus \mathbb{C}\Delta$, where Δ is the integration over $\mathrm{Ad}(\mathbf{\Gamma}_n)s_{ij}$, and we have a natural identification $C(\mathcal{S}_0) = C(\Gamma)$. Thus for any $\phi \in C(\mathcal{S})$ we can write $\phi = (k, c)$, $c \in C(\Gamma)$, $k \in \mathbb{C}$.

Since in our case ρ is the defining representation of $\mathbf{\Gamma}_n$ as a subgroup of $Sp(V)$, we have $\mathrm{Ker}\rho = \{1\}$. Thus the parameter t can be identified with a complex number corresponding to a scalar multiple $t\delta_1$ of the delta distribution supported at the identity element. If $\Gamma \subset SL(2,\mathbb{C})$ is infinite reductive, though, the identity element 1 is in the closure of $\mathcal{S} \setminus \{1\}$, and the parameter t can be absorbed in c.[1]

[1]Note that all what we said here is not true for continuous symplectic reflection algebras in general (see for example the case of the continuous Cherednik algebra attached to the group O_n, [EGG05] § 3.3.2) and the extra parameter t becomes essential for a definition including all cases ([EGG05], § 3.1).

We can thus consider the continuous symplectic reflection algebra $\mathcal{H}_{k,c}(\mathbf{\Gamma}_n)$ attached to $\mathbf{\Gamma}_n$ and to the parameters (k, c).

We want now to give an analog of Lemma 2.3.3 for the continuous case. Let $f \in \mathcal{O}(\mathbf{\Gamma}_n) \cong \mathcal{O}(\Gamma)^{\otimes n} \sharp \mathcal{O}(S_n)$ be a "decomposable" function, i.e. $f = \tilde{f}(f_1 \otimes \cdots \otimes f_n)$, with $\tilde{f} \in \mathcal{O}(S_n)$ and $f_i \in \mathcal{O}(\Gamma)$ for any i. Then we can write for the distribution Δ:

$$\begin{aligned} \langle \Delta, f \rangle &= \sum_{i<j} \langle \delta_{s_{ij}}, \tilde{f} \rangle \left(\int_\Gamma f_i(\gamma) f_j(\gamma^{-1})\, d\gamma \prod_{l \neq i,j} \langle \delta_1, f_l \rangle \right) \\ &= \sum_{i<j} \langle \delta_{s_{ij}}, \tilde{f} \rangle \langle \Delta_{i,j}, f_1 \otimes \cdots \otimes f_n \rangle \end{aligned} \tag{4.5}$$

where Δ_{ij} is the distribution on Γ^n acting as shown above. Thus $\Delta = \sum_{i,j | i<j} \delta_{s_{ij}} \Delta_{ij}$. We denote by $\omega_L(\gamma u, v)\Delta_{ij}$ the distribution on Γ^n such that

$$\langle \omega_L(\gamma u, v)\Delta_{ij}, f_1 \otimes \cdots \otimes f_n \rangle = \int_\Gamma \omega_L(\gamma u, v) f_i(\gamma) f_j(\gamma^{-1})\, d\gamma \prod_{l \neq i,j} \langle \delta_1, f_l \rangle \,.$$

Finally for $c \in C(\mathcal{S})$ we will denote by c_i the algebraic distribution on Γ^n given by $\delta_1^{\otimes l-1} \otimes c \otimes \delta_1^{\otimes n-l}$.

Let x_i,y_i, u_i,v_i be as in Lemma 2.3.3.

Lemma 4.4.1. *The algebra $\mathcal{H}_{k,c}(\mathbf{\Gamma}_n)$ is the quotient of $TV \sharp \mathcal{O}(\mathbf{\Gamma}_n)^*$ by the following relations:*

(R1) For any $i \in [1, n]$:

$$[x_i, y_i] = c_i + 2k \sum_{j | j \neq i} \delta_{s_{ij}} \Delta_{ij} \,.$$

(R2) For any $u, v \in L$ and $i \neq j$:

$$[u_i, v_j] = -2k\delta_{s_{ij}} \; (\omega_L(\gamma u, v)\Delta_{ij})$$

Proof. For the sake of clarity let us first look at the rank one example. In this case, according to definition 4.3.2, and absorbing the parameter t in c, the only defining relation should be:

$$[x,y]=\omega((1-\gamma)x,(1-\gamma)y)c \tag{4.6}$$

where x,y is a symplectic basis. From this we get the expression

$$[x,y]=(2-\mathrm{tr}_L(\gamma))c \tag{4.7}$$

where tr_L denotes the trace in the defining representation of Γ on L. Now it is enough to show that the invariant function $2-\mathrm{tr}_L(\gamma)$ is not a zero divisor in $\mathcal{O}(\Gamma)^\Gamma$. Indeed, if this is true then the multiplication by $2-\mathrm{tr}_L(\gamma)$ is an injective linear endomorphism of $\mathcal{O}(\Gamma)^\Gamma$, thus the induced linear endomorphism of $\mathcal{O}(\Gamma)^{*\Gamma}$ is surjective, and any invariant distribution c' can be written as $c'=(2-\mathrm{tr}_L(\gamma))c$, for some $c\in\mathcal{O}(\Gamma)^{*\Gamma}=C(\Gamma)$. But now for $\Gamma=\mathbb{C}^*$ and $\Gamma=SL(2,\mathbb{C})$ the ring $O(\Gamma)^\Gamma$ is clearly a domain. When $\Gamma=\widetilde{O}_2$, the cover of the group O_2, we have $\mathcal{O}(\Gamma)^\Gamma=\mathbb{C}[z,z^{-1}]\oplus\mathbb{C}$, where the two summands come from two connected components (so it has zero divisors), but the function $2-\mathrm{tr}(\gamma)$, which is clearly not identically 0 on the first summand, maps to 2 in the second summand (since $\mathrm{tr}(\gamma)=0$ for γ from the conjugacy class of complex orthogonal reflections), so it is not a zero divisor.

To pass to the higher rank case, we observe first of all that $\omega(u_i,v_j)=0$ if $i\neq j$ and $\omega(x_i,y_j)=\delta_{ij}$. Moreover, since the distribution Δ is supported on the conjugacy class $\mathrm{Ad}(\mathbf{\Gamma}_n)s_{ij}=\{s_{lm}\gamma_l\gamma_m^{-1}|l\neq m,\ \gamma\in\Gamma\}$ and for all $i\neq j$ the orbit of s_{ij} under the action of $\Gamma^n\subset\mathbf{\Gamma}_n$ is $\mathrm{Ad}(\Gamma^n)s_{ij}=\left\{s_{ij}\gamma_i\gamma_j^{-1}|\gamma\in\Gamma\right\}$, we have:

$$\begin{aligned}
&\omega((1-g)u_i,(1-g)v_j)\Delta\\
&=(1-\delta_{ij})\left(-\omega(u_i,(\gamma^{-1}v)_i)-\omega((\gamma u)_j,v_j)\right)\delta_{s_{ij}}\Delta_{ij}\\
&\qquad+\delta_{ij}\sum_{l\neq i}\left(\omega(u_i,v_i)+\omega((\gamma u)_l,(\gamma v)_l)\right)\delta_{s_{il}}\Delta_{il}\\
&=-2(1-\delta_{ij})\delta_{s_{ij}}\left(\omega_L(\gamma u,v)\Delta_{ij}\right)+2\delta_{ij}\omega_L(u,v)\sum_{l\neq j}\delta_{s_{il}}\Delta_{il}
\end{aligned} \tag{4.8}$$

It's now trivial to deduce *(R1)*, *(R2)* from (4.8) and the above observations.

□

As in Section 2.3 we have the following interesting examples.

Example 4.4.2. *If* $n = 1$ *then*

$$\mathcal{H}_c(\Gamma) = \frac{\mathbb{C}\langle x, y\rangle \sharp \mathcal{O}(\Gamma)^*}{\langle [x, y] - c\rangle}$$

is a continuous version of the Crawley-Boevey and Holland algebra of Example 2.3.4.

Example 4.4.3. *When* $k = 0$ *we have*

$$\mathcal{H}_{0,c}(\mathbf{\Gamma}_n) = \mathcal{H}_c(\Gamma)^{\hat{\otimes} n} \sharp S_n.$$

Example 4.4.4. *When* $\Gamma = GL(1, \mathbb{C})$ *is a maximal torus there is a decomposition* $V = \mathfrak{h} \oplus \mathfrak{h}^*$*, where* $\mathfrak{h}$ *is the irreducible representation of* $S_n \ltimes GL(1, \mathbb{C})^n$ *on* $\mathbb{C}^n = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ *where* $(\alpha_1, \ldots, \alpha_n) \in GL(1, \mathbb{C})^n$ *acts as the matrix* $\mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ *and* S_n *permutes the vectors of the standard basis. The symplectic form* ω_V *can be identified with the natural pairing between* $\mathfrak{h}$*,* $\mathfrak{h}^*$ *and this algebra is called continuous Cherednik algebra.*

4.5 Infinitesimal Hecke algebras

The rank 1 algebra $\mathcal{H}_c(\Gamma)$ has an interesting *infinitesimal* counterpart called the *infinitesimal Hecke algebra* (cfr. [EGG05] , Section 4). In this section we recall the definition of such algebra.

For $\Gamma = SL(2, \mathbb{C}), GL(1, \mathbb{C}), \widetilde{O}_2$, let $\mathfrak{g}$ be the Lie algebra of Γ. Then the enveloping algebra $\mathcal{U}\mathfrak{g}$ is naturally isomorphic to the subalgebra of $\mathcal{O}(\Gamma)^*$ of all algebraic distributions set-theoretically supported at the identity element $1 \in \Gamma$ (cf. [DG80], II, § 6). More precisely, if we identify any element $D \in \mathcal{U}\mathfrak{g}$ with the corresponding left invariant differential operator on Γ, then the above mentioned isomorphism sends D to the distribution $\tilde{D}$ such that:

$$\langle \tilde{D}, f\rangle := (Df)(1) \qquad \forall f \in \mathcal{O}(\Gamma) \tag{4.9}$$

where by Df we just mean D applied to f as a differential operator.

In particular, $\mathrm{Ad}(\Gamma)$-invariant distributions supported at the origin can be identified with elements of the center $\mathcal{Z}(\mathcal{U}\mathfrak{g})$ of the enveloping algebra. If the distribution c belongs to the subalgebra $\mathcal{U}\mathfrak{g} \subset \mathcal{O}(\Gamma)^*$ we define the infinitesimal Hecke algebra $\mathcal{H}_c(\mathfrak{g})$ as the quotient:

$$\frac{TV \sharp \mathcal{U}\mathfrak{g}}{\langle [x, y] - c \rangle}.$$

When $\mathfrak{g} = \mathfrak{sl}_2$, representations of the algebra $\mathcal{H}_c(\mathfrak{g})$, called *deformed symplectic oscillator algebra* of rank 1, were studied by Khare in [Kha05]. We will compare his results with our results about finite dimensional representations of the algebra $\mathcal{H}_c(SL(2, \mathbb{C}))$ in Section 6.3.

Chapter 5

Continuous deformed preprojective algebras

5.1 Plan of the chapter

In Section 3.3.2 and Section 3.3.3 we illustrated the fundamental role played by the theory of deformed preprojective algebras in the study of the representations of the rank one algebra $H_{1,c}(\Gamma)$. In the higher rank case, a remarkable development in the representation theory of wreath product symplectic reflection algebras has been the introduction, by Gan and Ginzburg ([GG05]), of the higher rank deformed preprojective algebra $\mathcal{A}_{n,\nu,\lambda}(Q)$ for any quiver Q. The algebra $\mathcal{A}_{n,\nu,\lambda}(Q)$ is a one-parameter deformation of the smash product $\Pi_\lambda(Q)^{\otimes n}\sharp S_n$. In the case when the underlying graph of Q is affine Dynkin, this deformation is Morita equivalent to the higher rank symplectic reflection algebra of wreath product type. Recently, following this interpretation of wreath product symplectic reflection algebras of higher rank in terms of deformed path algebras, Gan defined a version of the reflection functors for the higher rank case ([Gan06]). This allowed him to give a more elegant and transparent formulation and proof of Theorem 3.2.4 and Theorem 3.3.10, as well as a proof of the necessity of the conditions in Theorem 3.3.10.

In the light of these results, and of the extended McKay correspondence between infinite reductive subgroups of $SL(2,\mathbb{C})$ and infinite affine Dynkin diagrams of type

A_∞, $A_{+\infty}$, D_∞, in this chapter we define a continuous version of the deformed preprojective algebra $\mathcal{A}_{n,\nu,\lambda}(Q)$ for any quiver Q with such underlying graph. For appropriate values of the parameters, we establish a Morita equivalence between the continuous symplectic reflection algebra $\mathcal{H}_{k,c}(\mathbf{\Gamma}_n)$ and the algebra $\mathcal{A}_{n,\nu,\lambda}(Q)$, where Q is any quiver with underlying graph corresponding to Γ. This result provides us the link we were looking for between the representation theory of quivers and deformed preprojective algebras and the representation theory of the continuous symplectic reflection algebra. As we will see in Chapter 6 it will allow us to extend Gan's methods to the continuous case.

The structure of the chapter is as follows. In Sections 5.2, and Section 5.3, we recall some generalities about infinite affine Dynkin quivers, the definitions of their root systems and of their Weyl groups. In Section 5.4 we examine some properties of the action of the Weyl group on the space of weights that will be fundamental for the application of reflection functors to the continuous case. All these facts are known, and can be easily deduced from [Kac90], but we explicitly present them here in the form and with the level of detail which is convenient for our purposes. In Section 5.5 we give the formal definition of the continuous deformed preprojective algebra $\mathcal{A}_{n,\nu,\lambda}(Q)$. Finally, in Section 5.6, we establish the above mentioned Morita equivalence.

5.2 Infinite affine quivers and the McKay correspondence for reductive subgroups of $SL(2,\mathbb{C})$

Let $\Gamma \subset SL(2,\mathbb{C})$ be one of the three groups in Section 4.4. With the exact same procedure as in Section 3.2.2 one can associate a graph to Γ. In particular, if $\{N_i\}_{i\in I}$ (where we take $I=\mathbb{Z}$ for $\Gamma = GL(1,\mathbb{C})$ and $I=\mathbb{Z}_{\geq 0}$ for $\Gamma = \widetilde{O}_2$, $SL(2,\mathbb{C})$, and N_0 denotes the trivial representation) is a complete collection of finite dimensional irreducible pairwise non-isomorphic representations of Γ, the set of vertices of the graph attached to Γ is in bijection with I and is thus infinite. It is a classical result that the graphs associated to $GL(1,\mathbb{C})$, $\widetilde{O}_2$, $SL(2,\mathbb{C})$ are the infinite Dynkin diagrams A_∞, D_∞, $A_{+\infty}$, respectively. This can be seen as an extension of the McKay correspondence to the reductive case. We will use the notation Γ for both the group and the corresponding graph.

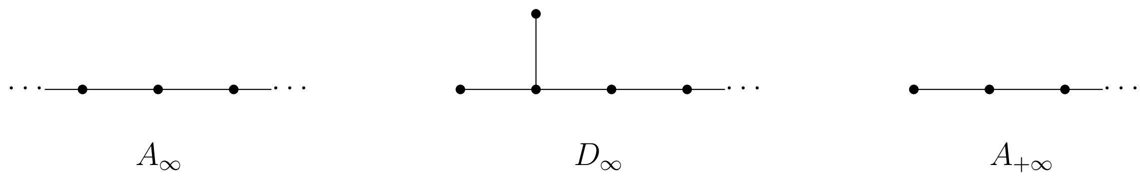

Figure 5-1: Graphs associated to $GL(1,\mathbb{C})$, $\widetilde{O}_2$, and $SL(2,\mathbb{C})$.

We recall that the graphs A_∞, D_∞, $A_{+\infty}$, together with the analog graphs B_∞, C_∞, form the complete list of connected Dynkin diagrams of infinite affine Cartan matrices, i.e. generalized Cartan matrices of infinite order, such that any principal minor of finite order is positive ([Kac90], § 4.10). In particular we get the matrices:

$$A_\infty = \begin{pmatrix} \cdots & \ddots & \vdots & \vdots & \vdots & \cdots \\ \cdots & -1 & 2 & -1 & \vdots & \cdots \\ \cdots & \vdots & -1 & 2 & -1 & \cdots \\ \cdots & \vdots & \vdots & \vdots & \ddots & \cdots \end{pmatrix} \qquad D_\infty = \begin{pmatrix} 2 & 0 & -1 & \cdots & \cdots & \cdots \\ 0 & 2 & -1 & \cdots & \cdots & \cdots \\ -1 & -1 & 2 & -1 & \vdots & \cdots \\ 0 & 0 & -1 & 2 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

$$A_{+\infty} = \begin{pmatrix} 2 & -1 & \cdots & \cdots \\ -1 & 2 & -1 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

In the sequel we will denote by $A = A(\Gamma)$ any such matrix and by $\mathfrak{g}'(A)$ the corresponding Kac-Moody algebra ([Kac90], § 1, 2).

5.3 Infinite rank affine root systems

Here we want to give a description of the root system attached to the graph Γ or, equivalently, to the matrix A.

Consider the space $(\mathbb{Z}^I)_0$ (where $I = \mathbb{Z}$ for $A = A_\infty$ and $I = \mathbb{Z}_+$ for $A = A_{+\infty}$, D_∞) of sequences $\{\alpha_i\}_{i\in I}$ of integer numbers which have only finitely many nonzero entries. This space has a $\mathbb{Z}$-basis $\{\epsilon_i\}_{i\in I}$, where ϵ_i is the sequence with 1 in the i-th position and 0 elsewhere. We will write $\alpha = \sum_{i\in I} \alpha_i \epsilon_i$, where $\alpha_i = 0$ for all but finitely many indices. Denote by $Q = Q(\Gamma)$ the quiver obtained by assigning any orientation to the graph Γ. Since any of the graphs Γ is locally finite (i.e. any vertex has finite valency), formulas (3.14), (3.15) of Section 3.3.3 make sense for any such Q and any two vectors $\alpha, \beta \in (\mathbb{Z}^I)_0$. Thus the symmetrized Ringel form can still be defined for any such Q as a bilinear form on $(\mathbb{Z}^I)_0$. Moreover, the matrix representing the Ringel form in the basis $\{\epsilon_i\}_{i\in I}$ is exactly the Cartan matrix A.

We are now ready to define the root system for A. Our construction works more generally whenever A is the matrix of the symmetrized Ringel form for any locally finite quiver Q, in particular when the quiver is finite it coincides with the usual definition of root system for a quiver ([CBH98], § 6). Moreover our description coincides with the one given in [Kac90] (§ 7.11) for the root system of an infinite rank affine Kac-Moody algebra $\mathfrak{g}'(A)$.

We will say that ϵ_i are the simple roots for A (or for Q or Γ), and we will denote the set of simple roots by $\Pi = \Pi(A)$. Note that, since all the finite order principal minors of A are nondegenerate, the form $(\ ,\ )$ has radical equal to $\{0\}$ on $(\mathbb{Z}^I)_0$.

For any $i \in I$ we will now define the simple reflection $s_i : (\mathbb{Z}^I)_0 \to (\mathbb{Z}^I)_0$ by:

$$s_i(\alpha) = \alpha - (\alpha, \epsilon_i)\epsilon_i.$$

The *Weyl group* W attached to A (equivalently to Γ) is the group of linear automorphisms of $(\mathbb{Z}^I)_0$ generated by the simple reflections s_i, $\forall i \in I$.

The real roots of A, and in general for a locally finite quiver Q, are defined to be the union of the orbits of the simple roots ϵ_i under the action of W, we will denote the set of real roots by $\Delta^{re} = \Delta^{re}(A)$. So we have, by definition, $\Delta^{re} = \bigcup_{w\in W} w\Pi$. It is standard that any such root is positive or negative (i.e. is a sum of simple roots with all non-negative, respectively non-positive, integer coefficients) and that $\Delta^{re}_- = -\Delta^{re}_+$.

The imaginary roots of A, or in general for a locally finite quiver Q, are instead the elements of $(\mathbb{Z}^I)_0$ that are of the form $\pm w\beta$, for $\beta \in F$, where F is the fundamental region:

$$F = \{\beta \in (\mathbb{Z}^I_{\geq 0})_0 | \beta \neq 0, \text{ support of } \beta \text{ connected, and } (\beta, \epsilon^i) \leq 0 \ \ \forall i \in I\}.$$

We denote such vectors by $\Delta^{im} = \Delta^{im}(A)$. The root system for A is the union of real and imaginary roots and we will denote it by $\Delta = \Delta(A) = \Delta^{re} \cup \Delta^{im}$. In our case, when A is an infinite rank affine Cartan matrix, we have $\Delta = \Delta^{re}$, and there are no imaginary roots ([Kac90], § 7.11). This is because any infinite rank affine matrix A (as well as its graph Γ and its root system Δ), can be seen as the limit of a sequence of positive finite rank Cartan matrices $A(n)$, all of the same type, (with their Dynkin diagrams $\Gamma(n)$ and root systems $\Delta(n)$), and for such matrices there are no imaginary roots ([Kac90], § 5.2, Proposition 5.2 c).

All this is in contrast with the theory for finite affine Dynkin quivers, for which the set of imaginary roots is infinite and is given by the nonzero integer multiples of δ. For finite affine Dynkin quivers the vector δ is also the minimal vector with nonnegative integer coordinates among the vectors generating the one dimensional kernel of the corresponding affine Cartan matrix or, equivalently, the radical of the corresponding symmetrized Ringel form, which is positive semi-definite in this case ([Kac90], Theorem 5.6, b). We can recover the analogy with the finite case if we observe that the matrix A makes sense as an endomorphism on the space $\mathbb{Z}^I$ of *all* sequences $\{\alpha_i\}_{i \in I}$ with integer entries. This is because any of its rows has only finitely many non-zero entries. The kernel of this endomorphism of $\mathbb{Z}^I$ is a rank one $\mathbb{Z}$-module generated by the vector $\delta = \{d_i\}$, where $d_i = \dim N_i$. We want to remark that this vector is *not* a root for the Kac-Moody algebra $\mathfrak{g}'(A)$ but it can be seen as a root for a central extension of a completion of $\mathfrak{g}'(A)$ (see [Kac90], § 7.12).

5.4 Action of the Weyl group on weights

Consider the space $(\mathbb{C}^I)_0$ of sequences $\{u_i\}_{i\in I}$ of complex numbers such that $u_i = 0$ for all but finitely many indices. Clearly the space $(\mathbb{Z}^I)_0$ above can be embedded as a lattice in $(\mathbb{C}^I)_0$ and the $\mathbb{Z}$-basis $\{\epsilon_i\}_{i\in I}$ of $(\mathbb{Z}^I)_0$ gives a basis of $(\mathbb{C}^I)_0$ as a complex vector space via this embedding. The dual space of $(\mathbb{C}^I)_0$ is called the weight space and it is isomorphic to the space $\mathbb{C}^I$. We will denote by $\cdot$ the standard pairing between $(\mathbb{C}^I)_0$ and $\mathbb{C}^I$, and by $\{\epsilon_i^*\}$ the "dual basis" (spanning $\mathbb{C}^I$ topologically) of $\{\epsilon_i\}$ with respect to this pairing, that is to say $\epsilon_i^* \cdot \epsilon_j = \delta_{ij}$. For any $\lambda \in \mathbb{C}^I$ we will write $\lambda = \{\lambda_i\}$, where $\lambda = \sum_i \lambda_i \epsilon_i^*$ (where the sum is now possibly infinite).

We will consider $(\mathbb{C}^I)_0$ as embedded in $\mathbb{C}^I$ via the map

$$\begin{array}{cccc} \phi & : & (\mathbb{C}^I)_0 & \longrightarrow & \mathbb{C}^I \\ & & \epsilon_i & \longrightarrow & \sum_j a_{ji}\epsilon_j^* \end{array}$$

where $a_{ji} = (\epsilon_j, \epsilon_i)$. In the basis $\{\epsilon_i\}$, $\{\epsilon_i^*\}$ the map ϕ is given by the Cartan matrix $A(\Gamma)$. Moreover, for any vector $\alpha \in \mathbb{C}^I$ and any $i \in I$, we have:

$$(\alpha, \epsilon_i) = \phi(\alpha) \cdot \epsilon_i = \phi(\alpha)_i \,. \tag{5.1}$$

For any reflection s_i, $i \in I$, we can now consider its dual reflection $r_i : \mathbb{C}^I \to \mathbb{C}^I$ which is uniquely determined by the property

$$r_i\lambda \cdot \alpha = \lambda \cdot s_i\alpha \qquad \forall \lambda \in \mathbb{C}^I, \quad \alpha \in (\mathbb{Z}^I)_0.$$

In other words, we have $(r_i\lambda)_j = \lambda_j - (\epsilon_i, \epsilon_j)\lambda_i$ for any j, which is equivalent to:

$$r_i\lambda = \lambda - \lambda_i\phi(\epsilon_i) \,. \tag{5.2}$$

Thus we can define an action of W on $\mathbb{C}^I$ by the condition

$$\lambda \cdot (w\alpha) = w^{-1}\lambda \cdot \alpha \qquad \forall \lambda \in \mathbb{C}^I, \quad \alpha \in (\mathbb{Z}^I)_0.$$

Let now "$\prec$" be a total ordering on $\mathbb{C}$ satisfying the following properties ([CBH98], § 7):

1. If $a \prec b$, then $a + c \prec b + c$, for any $c \in \mathbb{C}$.

2. On integers $\prec$ coincides with the usual order.

3. For any $a \in \mathbb{C}$ there is $m \in \mathbb{Z}$ with $a \prec m$.

An example of such an order is the lexicographic order with respect to the $\mathbb{R}$-basis $\{1, \sqrt{-1}\}$ of $\mathbb{C}$.

We say that a weight λ is dominant if $\lambda_i \succeq 0$ for all $i \in I$. Let $J \subset I$ be a finite set of indices. For any weight λ let λ_J be the weight such that $(\lambda_J)_i = \lambda_i$, if $i \in J$, $(\lambda_J)_i = 0$ otherwise. Then we say that λ is J-dominant if λ_J is dominant.

Let now J be a finite subset of indices corresponding to some full connected subquiver Q_J. Let $R_{\lambda,J}$ be the set of roots α of Q such that the support of α is contained in J, and $\lambda \cdot \alpha = 0$. Let $\Sigma_{\lambda,J}$ be the set of minimal positive elements of $R_{\lambda,J}$. Denote by W_J the subgroup of W generated by the reflections s_j, for all $j \in J$. For any $\alpha \in R_{\lambda,J}$, let s_α be the automorphism of $(\mathbb{Z}^I)_0$ given by $\beta \to \beta - (\alpha, \beta)\alpha$ and let $W_{\lambda,J}$ be the subgroup of W_J generated by these automorphisms. We have the following lemma.

Lemma 5.4.1. *For any* $\lambda \in \mathbb{C}^I$

1) $R_{\lambda,J}$ is the set of roots of a reduced root system in the (finite dimensional) vector space it generates. The group $W_{\lambda,J}$ acts faithfully on $R_{\lambda,J}$ and identifies with its Weyl group.

2) Any λ is W_J-conjugate to a unique J-dominant weight λ^+, and there exists a unique $w^+ \in W_J$ of minimal length (in W_J) such that $w^+\lambda = \lambda^+$.

3) $\Sigma_{\lambda,J}$ is the unique basis of $R_{\lambda,J}$ consisting of positive roots and $w^+\Sigma_{\lambda,J} = \Sigma_{\lambda^+,J}$.

4) If λ is J-dominant then $\Sigma_{\lambda,J} = \{\epsilon_i | \lambda_i = 0\}$.

Proof. Let us start from part 1). We observe that Q_J is Dynkin. Moreover the roots of Q supported in J are in bijection with the roots of (the finite quiver) Q_J and the bijection is simply the restriction $\alpha \to \alpha|_J$. It is also clear that, using this identification, the group W_J acts on these roots as the Weyl group $W(Q_J)$ of the quiver Q_J. Thus part 1) can be reduced to the analog statement for the quiver Q_J ([CBH98] Lemma 7.2 part (3)).

Let us now prove part 2). Let $J' = J \cup \partial J$, where ∂J is the set of adjacent vertices for the subquiver Q_J (the vertices that are not in Q_J but are joined to Q_J by a path of length one). Let $U_{J'} \subset \mathbb{C}^I$ be the vector space of weights μ satisfying $\mu_i = 0$ for $i \notin J'$ (i.e. the span of ϵ_j^*, $j \in J'$). As above, let $\lambda_{J'}$ be the weight $(\lambda_{J'})_j = \lambda_j$ if $j \in J'$, $(\lambda_{J'})_j = 0$ otherwise. Clearly $\lambda_{J'} \in U_{J'}$. Write $\lambda = \lambda_{J'} + (\lambda - \lambda_{J'})$. Then we have that W_J fixes $(\lambda - \lambda_{J'})$ and preserves $U_{J'}$. Identifying $U_{J'}$ with a finite dimensional vector space of dimension $|J'|$, the weight space for the finite Dynkin quiver $Q_{J'}$, we see that W_J acts on $U_{J'}$ as the parabolic subgroup $W(Q_J) \subset W(Q_{J'})$. The result now follows from the ordinary theory of Dynkin quivers. Indeed, write $\lambda_{J'} = \sum_{j \in J'} b_j \nu(\epsilon_j)$ (this is clearly possible since any principal minor of the Cartan matrix $A(\Gamma)$ is non-degenerate) and define the *height* of λ as $\mathrm{ht}(\lambda) := \sum_{j \in J'} b_j$. Consider now a vector of the form $w\lambda_{J'}$, for $w \in W(Q_J)$, of maximal height with respect to "$\prec$" (this exists since $W(Q_J)$ is finite). If $(w\lambda_{J'})_j \prec 0$ for some $j \in J$ then, from formula (5.2), we get

$$\mathrm{ht}(w\lambda_{J'}) - \mathrm{ht}(r_j w\lambda_{J'}) = \mathrm{ht}(w\lambda_{J'} - r_j w\lambda_{J'}) = \mathrm{ht}((w\lambda_{J'})_j \nu(\epsilon_j)) = (w\lambda_{J'})_j \prec 0$$

Thus $\mathrm{ht}(w\lambda_{J'})$ is not maximal: a contradiction. So we must have $(w\lambda_{J'})_J = (w\lambda)_J$ dominant, and $\lambda^+ := w\lambda$ is J-dominant.

For the uniqueness it is enough to prove that two J-dominant weights cannot be W_J conjugate. Suppose now λ is J-dominant and $w \in W_J$. We have to prove that if $w\lambda$ is J-dominant then $w\lambda = \lambda$. Thanks to the identification with $W(Q_J)$ the group W_J is endowed with a length function. We will prove the result by induction on the length of w. If w is of length one then $w = r_j$ for some $j \in J$. If $r_j\lambda$ is J-dominant then $(r_j\lambda)_i = r_j\lambda \cdot \epsilon_i \succeq 0$ for any $i \in J$. But

$$r_j\lambda \cdot \epsilon_j = \lambda \cdot s_j\epsilon_j = \lambda \cdot (-\epsilon_j) = -(\lambda \cdot \epsilon_j) = -\lambda_j \preceq 0$$

since λ is J-dominant. Thus it must be $\lambda_j = \lambda \cdot \epsilon_j = 0$ and $r_j\lambda = \lambda$ by equation (5.2). Now suppose $w = r_{i_1} \cdots r_{i_s}$ is a reduced expression for w. Since $i_s \in J$ we must have $\lambda \cdot \epsilon_{i_s} \succeq 0$ and therefore $w\lambda \cdot w\epsilon_{i_s} \succeq 0$. But, by Lemma 3.11 of [Kac90], $w\epsilon_{i_s}$ is a negative root (for Q_J) and since $w\lambda$ is J-dominant we must also have $w\lambda \cdot w\epsilon_{i_s} \preceq 0$. Thus $0 = w\lambda \cdot w\epsilon_{i_s} = \lambda \cdot \epsilon_{i_s}$. This implies $r_{i_s}\lambda = \lambda$ and $w\lambda = wr_{i_s}\lambda$, with wr_{i_s} shorter than w and the result follows by induction. It is standard to prove that there exists a unique $w^+ \in W_J$ of minimal length (as an element of $W(Q_J)$) with the property $w\lambda = \lambda^+$.

The proof of part 3) is now straightforward if we use the results of 1) and 2). Part 4) is just a trivial observation.

□

5.5 Definition of the continuous deformed preprojective algebra

5.5.1 The rank one case

We recall that in general ([Eis95], § 7), given an algebra A and a descending filtration by ideals

$$A = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \cdots$$

the completion of A with respect to this filtration is defined as the projective limit

$$\widehat{A} := \varprojlim A/\mathfrak{m}_i.$$

If M is an A-module (A-bimodule) the completion of M is the $\widehat{A}$-module ($\widehat{A}$-bimodule) defined as

$$\widehat{M} := \varprojlim M/\mathfrak{m}_i M \qquad (\widehat{M} := \varprojlim M/\mathfrak{m}_i M + M\mathfrak{m}_i).$$

Let now Q be a quiver with underlying graph an infinite affine Dynkin diagram of type A_∞, D_∞, $A_{+\infty}$ with set of vertices I.

With the same notation as in Section 3.3.2, let $B := \bigoplus_{i\in I} \mathbb{C}e_i \cong (\mathbb{C}^I)_0$ be the algebra spanned by the idempotents e_i for all $i \in I$. Let $\widehat{B} := \prod_{i\in I} \mathbb{C}e_i \cong \mathbb{C}^I$ be the algebra over $\mathbb{C}$ topologically spanned by the same idempotents. Note that $\widehat{B} = B^*$ as vector spaces, and that $\widehat{B}$ is the completion of B with respect to the filtration:

$$B = B_0 \supset B_1 \supset B_2 \supset \cdots$$

where $B_N = \bigoplus_{|i|\geq N} \mathbb{C}e_i$.

Let E be the vector space spanned by the edges of the double quiver $\overline{Q}$, and let $\widehat{E}$ be the vector space with topological basis formed by the same edges. Thus $\widehat{E}$ is a $\widehat{B}$-bimodule, and as such it decomposes as $\widehat{E} = \prod_{i,j\in I} E_{ij}$, where E_{ij} is spanned by all edges $a \in \overline{Q}$ such that $h(a) = i$, $t(a) = j$. Note that the $\widehat{B}$-bimodule $\widehat{E}$ is the completion of the B-bimodule E, i.e. it is the completion of E with respect to the filtration by B-sub-bimodules

$$E = E_0 \supset E_1 \supset E_2 \supset \cdots$$

with $E_N := \bigoplus_{|i|\geq N} E_{ij} + \bigoplus_{|j|\geq N} E_{ij} = B_N E + EB_N$ (thus $E/E_N = \bigoplus_{|i|,|j|<N} E_{ij}$). Moreover, since Q (hence $\overline{Q}$) is locally finite, all the spaces E_{ij} are finite dimensional. Thus $E_{ij}^* \cong E_{ij}$ and we have an identification $\widehat{E} = E^*$ as vector spaces.

For any $k \in \mathbb{Z}_{\geq 0}$, let us now consider the B-bimodule

$$T_B^k E := \underbrace{E \otimes_B \cdots \otimes_B E}_{k \text{ factors}}.$$

As we know, this module is identified with the vector spaced spanned by all the paths of length k on the double quiver $\overline{Q}$, with its natural B-bimodule structure. Thus, the completion $\widehat{T^k_B E}$ of this module can be identified with the vector space topologically spanned by the paths of length k, endowed with its natural $\widehat{B}$-bimodule structure. We will write

$$\widehat{T^k_B E} = \underbrace{\widehat{E}\hat{\otimes}_{\widehat{B}} \cdots \hat{\otimes}_{\widehat{B}}\widehat{E}}_{k \text{ factors}}.$$

To justify this notation we observe that there is an isomorphism of $\widehat{B}$-bimodules

$$\varprojlim T^k_B E/B_N T^k_B E + T^k_B E B_N \cong \varprojlim \widehat{E}\otimes_{\widehat{B}}\cdots\otimes_{\widehat{B}}\widehat{E}/\widehat{B}_N(\widehat{E}\otimes_{\widehat{B}}\cdots\otimes_{\widehat{B}}\widehat{E}) + (\widehat{E}\otimes_{\widehat{B}}\cdots\otimes_{\widehat{B}}\widehat{E})\widehat{B}_N$$

where $\widehat{B}_N = \prod_{|i|\geq N} \mathbb{C}e_i$.

Note that as vector spaces $\widehat{T^k_B E} = (T^k_B E)^*$.

Definition 5.5.1. *The continuous path algebra of $\overline{Q}$ is defined as*

$$\widehat{\mathbb{C}\,\overline{Q}} := \widehat{T_B E} = \bigoplus_{k\geq 0} \widehat{T^k_B E}$$

Observe that, as vector spaces, we have

$$\widehat{T_B E} = \bigoplus_{k\geq 0} \widehat{T^k_B E} = \bigoplus_{k\geq 0} (T^k_B E)^*$$

Thus we can give to each summand the weak topology, and we can equip $\widehat{\mathbb{C}\,\overline{Q}}$ with the direct sum topology (the finest topology such that all the canonical injections are continuous).

We want to remark how this is a reasonable definition of continuous path algebra for the case of infinite locally finite quivers. Indeed, for any locally finite quiver, there are only a finite number of paths of length k (i.e. belonging to the space $\widehat{T^k_B E}$) passing through a fixed vertex. Any element of $\widehat{\mathbb{C}\,\overline{Q}}$ is a, possibly infinite, linear combination of paths such that the length of the summands is bounded, and thus each vertex is contained in at most a finite number of summands. This means that we are avoiding "pathologic cases" of infinite linear combinations of paths passing through the same vertex infinitely many times. Moreover, we want to stress the fact that the algebra of Definition 5.5.1 is unital, with unit $\prod_{i\in I} e_i$, while the usual path algebra $\mathbb{C}\,\overline{Q}$ (where we take $B = \bigoplus_{i\in I} \mathbb{C}e_i$ and $E = \bigoplus E_{ij}$ and usual tensor products) is not unital when the quiver is infinite.

For any $\lambda \in \widehat{B}$ we write $\lambda = \sum_{i\in I} \lambda_i e_i$.

Note that, since the quiver Q is locally finite, for any $i \in I$ the element R_i described by formula (3.12) in Section 3.3.2 is a well defined element of $\widehat{\mathbb{C}\,\overline{Q}}$.

Definition 5.5.2. *The continuous deformed preprojective algebra $\hat{\Pi}_\lambda(Q)$ attached to the infinite affine quiver Q and to the parameter $\lambda \in \widehat{B} = \mathbb{C}^I$ is the quotient:*

$$\hat{\Pi}_\lambda(Q) = \frac{\widehat{\mathbb{C}\,\overline{Q}}}{\langle\langle R_i - \lambda_i e_i \rangle\rangle_{i\in I}}$$

where $\langle\langle \dots \rangle\rangle$ is the closed ideal generated by the indicated elements in the completed path algebra $\widehat{\mathbb{C}\,\overline{Q}}$.

Later in this thesis, we will need to consider a global version of the deformed preprojective algebra. So if R is a commutative unital $\mathbb{C}$-algebra we define $\widehat{R\,\overline{Q}}$ by substituting $\widehat{B}$ with $\widehat{B}_R := \widehat{B}\hat{\otimes}R \cong R^I$ and $\widehat{E}$ with $\widehat{E}_R = \widehat{E}\hat{\otimes}R$. For any $\lambda_i \in R$ for any $i \in I$, we define $\hat{\Pi}_{R,\lambda}(Q)$ to be $\widehat{R\,\overline{Q}}/\langle\langle R_i - \lambda_i e_i \rangle\rangle_{i\in I}$.

5.5.2 The higher rank case

The definition of higher rank continuous deformed preprojective algebra given in this section is just a generalization to the continuous case of the one given by Gan and Ginzburg in [GG05], § 1.2. Namely, let us fix a positive integer $n > 1$ and consider the algebra $\mathcal{B} := B^{\otimes n}$. For any $l = 1, \dots, n$ consider the ideals in $\mathcal{B}$

$$\mathcal{B}_{N,l} = B^{\otimes l-1} \otimes B_N \otimes B^{\otimes n-l} \text{ and } \mathcal{B}_N = \mathcal{B}_{N,1} + \dots + \mathcal{B}_{N,n}.$$

where B_N is as in the previous section.

We denote by $\widehat{\mathcal{B}} = \widehat{B}^{\hat{\otimes} n}$ the completion of $\mathcal{B}$ with respect to the chain of ideals:

$$\mathcal{B} = \mathcal{B}_0 \supset \mathcal{B}_1 \supset \mathcal{B}_2 \supset \cdots .$$

For any $l \in [1, n]$ consider the $\mathcal{B}$-bimodules:

$$\mathcal{E}_l = B^{\otimes(l-1)} \otimes E \otimes B^{\otimes(n-l)} \text{ and } \mathcal{E} = \bigoplus_{1 \le l \le n} \mathcal{E}_l$$

and the completed $\widehat{\mathcal{B}}$-bimodules

$$\widehat{\mathcal{E}_l} = \widehat{B}^{\hat{\otimes}(l-1)} \hat{\otimes} \widehat{E} \hat{\otimes} \widehat{B}^{\hat{\otimes}(n-l)} \text{ and } \widehat{\mathcal{E}} = \bigoplus_{1 \le l \le n} \widehat{\mathcal{E}_l}.$$

Here $\widehat{\mathcal{E}_l}$ is the completion of the $\mathcal{B}$-bimodule $\mathcal{E}_l := B^{\otimes(l-1)} \otimes E \otimes B^{(n-l)}$ with respect to the chain of sub-bimodules

$$\mathcal{E}_l = \mathcal{E}_{l,0} \supset \mathcal{E}_{l,1} \supset \mathcal{E}_{l,2} \supset \cdots$$

with $\mathcal{E}_{l,N} = \mathcal{B}_N \mathcal{E}_l + \mathcal{E}_l \mathcal{B}_N$.

In a similar fashion as in the previous section we can consider the $\mathcal{B}$-bimodule $T^k_{\mathcal{B}}\mathcal{E} := \underbrace{\mathcal{E} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} \mathcal{E}}_{k \text{ factors}}$ and the completed $\widehat{\mathcal{B}}$-bimodule $\widehat{T^k_{\mathcal{B}}\mathcal{E}} := \underbrace{\widehat{\mathcal{E}} \hat{\otimes}_{\widehat{\mathcal{B}}} \cdots \hat{\otimes}_{\widehat{\mathcal{B}}} \widehat{\mathcal{E}}}_{k \text{ factors}}$. We define

$$\widehat{T_{\mathcal{B}}\mathcal{E}} := \bigoplus_{k \ge 0} \widehat{T^k_{\mathcal{B}}\mathcal{E}}.$$

Note that as vector spaces $\widehat{T^k_{\mathcal{B}}\mathcal{E}} = (T^k_{\mathcal{B}}\mathcal{E})^*$. Thus we can equip $\widehat{T^k_{\mathcal{B}}\mathcal{E}}$ with the weak topology and $\widehat{T_{\mathcal{B}}\mathcal{E}}$ with the direct sum topology.

Note that S_n acts naturally on $\widehat{\mathcal{E}}$, thus on $\widehat{T_{\mathcal{B}}\mathcal{E}}$. Now for any $l \in [1, n]$, any path $a \in \widehat{\mathbb{C}\overline{Q}}$ and any $\underline{i} = (i_1, \ldots, i_n) \in I^n$ we consider the elements

$$|_{\underline{i}} := e_{i_1} \otimes \cdots \otimes e_{i_n} \in \widehat{\mathcal{B}}$$

and

$$a_l|_{\underline{i}} := e_{i_1} \otimes \cdots \otimes a e_{i_l} \otimes \cdots \otimes e_{i_n} \in \widehat{T_{\mathcal{B}}\mathcal{E}_l}\,.$$

For an arrow $a \in \overline{Q}$, if $i_l = t(a)$ we define

$$a_l(\underline{i}) := (i'_1, \ldots, i'_n) \in I^n, \text{ where } i'_m = \begin{cases} i_m & \text{if } m \neq l \\ h(a) & \text{if } m = l \end{cases}$$

Definition 5.5.3. *For any $\lambda \in \widehat{B}$ and $\nu \in \mathbb{C}$, define the $\widehat{\mathcal{B}}$-algebra $\mathcal{A}_{n,\nu,\lambda}(Q)$ to be the quotient of $\widehat{T_{\mathcal{B}}\mathcal{E}}\sharp S_n$ by the following relations:*

(I) For any $l \in [1,n]$ and $\underline{i} = (i_1, \ldots, i_n)$:

$$(R_{i_l} - \lambda_{i_l})_l|_{\underline{i}} = \nu \sum_{\{m \neq l \mid i_m = i_l\}} s_{ml}|_{\underline{i}}\,;$$

(II) For any $l, m \in [1,n]$, $l \neq m$, $a, b \in \overline{Q}$ and $\underline{i} = (i_1, \ldots, i_n)$ with $i_l = t(a)$, $i_m = t(b)$:

$$a_l|_{b_m(\underline{i})} b_m|_{\underline{i}} - b_m|_{a_l(\underline{i})} a_l|_{\underline{i}} = \begin{cases} \nu s_{lm}|_{\underline{i}} & \textit{if } b \in Q \textit{ and } a = b^* \\ -\nu s_{lm}|_{\underline{i}} & \textit{if } a \in Q \textit{ and } b = a^* \\ 0 & \textit{else} \end{cases}$$

For $n = 1$ there is no parameter ν and $\mathcal{A}_{1,\lambda}(Q) = \hat{\Pi}_\lambda(Q)$, while for $n > 1$ and $\nu = 0$ we have $\mathcal{A}_{n,\lambda,0}(Q) = \hat{\Pi}_\lambda(Q)^{\hat{\otimes} n}\sharp S_n$.

As in the previous section, if R is a commutative unital $\mathbb{C}$-algebra, we can define $\widehat{\mathcal{B}}_R = \underbrace{\widehat{B}_R \hat{\otimes}_R \cdots \hat{\otimes}_R \widehat{B}_R}_{n \text{ factors}}$ and similarly $\widehat{E}_R$. Taking $\nu \in R$, $\lambda_i \in R$ for any $i \in I$ we can then define $\mathcal{A}_{R,n,\nu,\lambda}$.

5.6 Morita equivalence

Let Q be a McKay quiver for Γ and let χ_{N_i} be the character of the irreducible representation N_i. The following theorem is the analog of Theorem 3.5.2 of [GG05].

Theorem 5.6.1. *The algebra $\mathcal{H}_{k,c}(\mathbf{\Gamma}_n)$ is Morita equivalent to the algebra $\mathcal{A}_{n,\nu,\lambda}(Q)$ for $\nu = 2k$ and $\lambda = \{\lambda_i\}$ where $\lambda_i = \langle c, \chi_{N_i}\rangle$.*

Our proof of this theorem follows very closely the one of [GG05]. We report the proof in detail in Appendix A since proving this Morita equivalence is the key result from which all the results about the representation theory of $\mathcal{H}_{k,c}(\mathbf{\Gamma}_n)$ will follow. Also these computations can help the interested reader to become familiar with the language of algebraic distributions.

Let now $J \subset I$ be a finite subset of indices, and consider the finite dimensional subspace

$$B_J := \bigoplus_{i\in J} \mathbb{C}e_i \subset \prod_{i\in I} \mathbb{C}e_i = \widehat{B}$$

i.e. $B_J = \left\{\lambda \in \widehat{B} | \lambda_i = 0 \,\forall i \notin J\right\}$. Define the finite dimensional vector space $U_J := B_J \times \mathbb{C}$. From the above theorem and the PBW property for the continuous case (see Section 4.3) we can deduce that, for any $\lambda_0 \in \widehat{B}$, the family $\{\mathcal{A}_{n,\nu,\lambda_0+\lambda}\}_{(\nu,\lambda)\in U_J}$ gives a flat formal deformation of $\mathcal{A}_{n,0,\lambda_0}$ in $\dim U_J = |J| + 1$ parameters, where $|J|$ is the cardinality of the set J.

Let $\mathfrak{m}$ be the unique maximal ideal in $\mathbb{C}[[U_J]]$. For any $\mathbb{C}[[U_J]]$-module consider the decreasing filtration $V \supset \mathfrak{m}V \supset \mathfrak{m}^2 V \supset \cdots$ and the associated graded ring

$$\mathrm{gr}_{\mathfrak{m}}(V) := \prod_{h=0}^{\infty} \frac{\mathfrak{m}^h V}{\mathfrak{m}^{h+1} V}.$$

In what follows we will write $\overline{V} := (\mathrm{gr}_{\mathfrak{m}} V)_0 = V/\mathfrak{m}V$.

Let $\hbar'$, $\hbar_i$ for $i \in J$ be coordinate functions on U_J, where $\hbar'$ denotes the projection $B_J \times \mathbb{C} \longrightarrow \mathbb{C}$ and, for each $i \in J$, $\hbar_i$ denotes the projection $B_J \times \mathbb{C} \longrightarrow \mathbb{C}e_i$. Now set $\nu = \hbar'$ and $\lambda_i = \hbar_i$ for any $i \in J$, and regard them as elements in the maximal ideal $\mathfrak{m} \subset \mathbb{C}[[U_J]]$. Consider the algebra $\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda_0+\lambda}$. Then $\mathrm{gr}_{\mathfrak{m}}(\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda_0+\lambda}) \cong \mathcal{A}_{n,0,\lambda_0}[[U_J]]$ as $\mathbb{C}[[U_J]]$-algebras. In particular $\overline{\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda_0+\lambda}} \cong \mathcal{A}_{n,0,\lambda_0}$. This corresponds to the flat formal deformation of $\mathcal{A}_{n,0,\lambda_0}$ over $\mathbb{C}[[U_J]]$ given by the family $\{\mathcal{A}_{n,\nu,\lambda_0+\lambda}\}_{(\nu,\lambda)\in U_J}$.

More generally, let $\nu \in \mathbb{C}[[U_J]]$, and let $\lambda \in B_J \otimes \mathbb{C}[[U_J]]$. Consider the $\mathbb{C}[[U_J]]$-

algebra $\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda}$. The following Lemma is the analog of [Gan06] Lemma 5.13 and thanks to the PBW property of $\mathcal{A}_{n,\nu,\lambda}$ can be proved in exactly the same way.

Lemma 5.6.2. *Assume that $\nu \in \mathbb{C}[U_J]$ and $\lambda \in B_J \otimes \mathbb{C}[U_J] \subset B_J \otimes \mathbb{C}[[U_J]]$. Then $\mathrm{gr}_{\mathfrak{m}}(\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda}) = \overline{\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda}}[[U_J]]$ as $\mathbb{C}[[U_J]]$-algebras.*

Chapter 6

Finite dimensional representations for the continuous case

6.1 Plan of the chapter

In this chapter we use the Morita equivalence established in Section 5.6 to study the representation theory of $\mathcal{A}_{n,\nu,\lambda}$.

In Section 6.2 we classify finite dimensional representations in rank one, and in Section 6.3 we compare our results for the special case $\Gamma = SL(2,\mathbb{C})$ with the results of [Kha05] for the deformed symplectic oscillator algebra of rank one.

Finally, in Section 6.5 we consider the higher rank case and we extend the reflection functors and the results of [Gan06] to the continuous case.

6.2 Representations in the rank one case

The following easy result holds.

Proposition 6.2.1. *Any finite dimensional representation of the continuous deformed preprojective algebra $\hat{\Pi}_\lambda(Q)$ is a finite dimensional representation of some ordinary deformed preprojective algebra $\Pi_{\lambda|_J}(Q_J)$, where $J \subset I$ is a finite subset of vertices, Q_J is the corresponding full subquiver of Q, and $\lambda|_J \in \mathbb{C}^J$ is the restriction of* *the parameter* λ *to set of vertices* J*. Vice versa, any finite dimensional representation of* $\Pi_{\lambda|_J}(Q_J)$ *can be extended to a finite dimensional representation of* $\hat{\Pi}_\lambda(Q)$*.*

Proof. By Definition 5.5.2 we have that a representation Y of $\hat{\Pi}_\lambda(Q)$ is a representation of $\widehat{C\overline{Q}}$ such that, for each $a \in Q$, the corresponding linear maps $a : e_{t(a)}Y \to e_{h(a)}Y$, $a^* : e_{h(a)}Y \to e_{t(a)}Y$ satisfy the relation

$$\sum_{\substack{a \in Q \\ h(a)=i}} aa^* - \sum_{\substack{a \in Q \\ t(a)=i}} a^*a = \lambda_i \mathrm{Id}_{e_iY}$$

for any i. But now, if $\dim Y < \infty$, we must have $\dim e_iY = \alpha_i < \infty$ for all i, and $e_iY = 0$ for all but finitely many i. Thus the representation Y is supported at a finite number of vertices and the result follows.

Conversely, suppose $\Pi_{\lambda|_J}(Q_J)$ admits a finite dimensional representation Y, then we can clearly extend it to a representation of $\hat{\Pi}_\lambda(Q)$ by setting $e_iY = 0$ for $i \notin J$ and $a = a^* = 0$ for $a \notin Q_J$.

□

Proposition 6.2.1 implies the next Corollary.

Corollary 6.2.2. *For any* $\lambda \in \mathbb{C}^I$ *there is a bijection between the set of isomorphism classes of finite dimensional simple* $\hat{\Pi}_\lambda(Q)$*-modules and the set*

$$\widehat{\Sigma}_\lambda := \bigcup_J \Sigma_{\lambda,J}$$

where J *runs over all the finite subsets of indices corresponding to connected subquivers.*

Proof. The result is implied by Proposition 6.2.1 and by the fact, proved by Crawley-Boevey and Holland, that isomorphism classes of simple $\Pi_{\lambda|_J}(Q_J)$-modules are in bijection with $\Sigma_{\lambda,J}$ ([CBH98], § 7).

□

6.3 The $SL(2,\mathbb{C})$ case

We will now compare Khare's result about representation theory of the deformed symplectic oscillator algebra of rank one with the results of Section 6.2 in the case $\Gamma = SL(2,\mathbb{C})$.

We observe that, in this case, the subalgebra of invariant algebraic distributions supported at the identity can be identified with the algebra of polynomials in the quadratic Casimir element $\Delta = \frac{1}{4}(EF + FE + \frac{H^2}{2})$ (where E, F, H are the standard generators of $\mathfrak{sl}_2$), that coincides with the center of the enveloping algebra. Now if we let x, y be a symplectic basis of the standard two dimensional complex symplectic vector space L, and we take $f = f(\Delta)$ to be a polynomial with no constant coefficient, we can see that Khare's deformed symplectic oscillator algebra (cfr [Kha05], § 9)

$$H_f = \frac{TL\sharp\mathcal{U}(\mathfrak{sl}_2)}{\langle [x,y] = 1 + f(\Delta)\rangle}$$

coincides with the infinitesimal Hecke algebra $H_c(SL(2,\mathbb{C}))$ when we take $f = f_c$ to be an appropriate polynomial depending on c.

Let $V_C(i)$, $i \in \mathbb{Z}_{\geq 0}$ be the standard cyclic module of $\mathfrak{sl}_2$ of highest weight i (i.e. the irreducible finite dimensional representation of dimension $i+1$). Denote by b_i be the scalar by which the Casimir Δ acts on $V_C(i)$ ($b_i = i(i+2)/8$).

Khare's classification of finite dimensional representations of H_f can be summarized as follows (see [Kha05] § 15, Theorem 11, and § 9, formula (1)).

I) There exists a unique simple H_f-module of the form

$$(*) \qquad V(r,s) := \bigoplus_{i=s}^{r} V_C(i)$$

for any $s \leq r$ in $\mathbb{Z}_{\geq 0}$ satisfying the two conditions:

i) $$\sum_{i=s}^{r}(i+1)(1+f(b_i)) = 0;$$

ii) $$\sum_{i=k}^{r}(i+1)(1+f(b_i)) \neq 0 \qquad s < k \leq r \ .$$

II) Any finite dimensional irreducible H_f-module is isomorphic to one of the $V(r,s)$.

We observe now that all positive roots for the infinite quiver $A_{+\infty}$ are of the form $\alpha = \alpha_{[s,r]} = \sum_{i=s}^{r} \epsilon_i$ for some $0 \leq s \leq r$, where ϵ_i are coordinate vectors (simple roots) as in Section 5.3 (cfr. [Kac90], § 7.11, where $\epsilon_i = \alpha_i$ in Kac's notation). Thus, according to Corollary 6.2.2, in the case of $SL(2,\mathbb{C})$ all possible simple finite dimensional $\mathcal{H}_c(SL(2,\mathbb{C}))$-modules must have the form $(*)$. Moreover, Corollary 6.2.2 tells us that a simple representation of dimension vector α exists if and only if $\alpha \in \widehat{\Sigma}_\lambda$. This condition is equivalent to the following two conditions on the root α:

a') $\lambda \cdot \alpha = 0$;

b') for any nontrivial decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(n)}$ into positive roots we must have $\lambda \cdot \beta^{(k)} \neq 0$ for some k.

Now, any decomposition as in b') looks like:

$$\alpha_{[s,r]} = \alpha_{[s,s+t_1]} + \alpha_{[s+t_1+1,s+t_1+t_2]} + \cdots + \alpha_{[s+t_1+\cdots+t_n+1,r]}$$

with $s + t_1 + \cdots + t_n + 1 \leq r$. In particular, we can consider the decompositions $\alpha_{[s,r]} = \alpha_{[s,m-1]} + \alpha_{[m,r]}$ for any $s < m \leq r$. Since $0 = \lambda \cdot \alpha_{[s,r]} = \lambda \cdot \alpha_{[s,m-1]} + \lambda \cdot \alpha_{[m,r]}$ our condition implies that in particular $\lambda \cdot \alpha_{[m,r]} \neq 0$. On the other hand, any nontrivial decomposition of $\alpha_{[r,s]}$ contains a root $\alpha_{[m,r]}$ for $s < m \leq r$. Thus, the conditions a'), b') can be rephrased as:

a) $\alpha_{[s,r]} \cdot \lambda = 0$;

b) $\alpha_{[m,r]} \cdot \lambda \neq 0 \qquad s < m \leq r$.

We will now translate the conditions a), b) on the dimension vector α into Khare's conditions $i)$, $ii)$. In order to do this, we have to compare Khare's parameter f with our parameter c. Let's denote by χ_i the irreducible character corresponding to $V_C(i)$. Then, since the χ_i s span the space of invariant functions, we must have

$$\lambda_i = \langle c, \chi_i \rangle = \langle 1 + f(\Delta), \chi_i \rangle . \tag{6.1}$$

For any $i \in \mathbb{Z}_{\geq 0}$. We recall now that constants in $\mathcal{U}(\mathfrak{sl}_2)$ correspond to multiples of the delta distribution δ_1. Moreover for any $D \in \mathcal{U}(\mathfrak{sl}_2)$ one has $D(\chi_i)(1) = \chi_i(D) = \mathrm{tr}_{V_C(i)}(D)$, and, in particular $\mathrm{tr}_{V_C(i)}(\Delta^l) = \dim(V_C(i))b_i^l = (i+1)b_i^l$. It is then easy to compute that

$$\langle 1 + f(\Delta), \chi_i \rangle \;=\; (i+1)(1 + f(b_i))\,.$$

and the equality (6.1) becomes

$$(i+1)(1 + f(b_i)) = \lambda_i\,. \tag{6.2}$$

Thus we can rewrite the conditions $i)$ and $ii)$ as

1) $\sum_{i=s}^{r} \lambda_i = 0$

2) $\sum_{i=k}^{r} \lambda_i \neq 0 \quad s < k \leq r$

which correspond to conditions $a)$, $b)$ above respectively.

6.4 Gan's reflection functors

In [Gan06], Wee Liang Gan constructed reflection functors for higher rank. Gan's reflection functors are defined for any loop-free vertex i of any finite quiver Q and, under some conditions on the parameter λ, ν, they establish an equivalence

$$F_i : \mathcal{A}_{R,n,\nu,\lambda}(Q) - \mathrm{mod} \to \mathcal{A}_{R,n,\nu,r_i\lambda}(Q) - \mathrm{mod}$$

where $r_i\lambda$ denotes the action of the dual simple reflection r_i at the vertex i on the parameter $\lambda \in R^I$ (which we can obtain by extending the action on $\mathbb{C}^I$ by R-linearity). Thanks to this property the functors F_i turned out to be a very powerful tool in the deformation-theoretic approach to the study of finite dimensional representations of higher rank deformed preprojective algebras.

Gan's definition of reflection functors can be pushed ahead, without any change, when Q is an infinite quiver with underlying graph an affine Dynkin diagram of type A_∞, $A_{+\infty}$, D_∞ (and more generally when Q is an infinite locally finite quiver) and we consider modules that are finitely generated over R. Gan's results about the representations of the wreath product symplectic reflection algebra will then naturally extend to the continuous case. The proofs of all the results in this section are exactly as the ones for the analog statements in [Gan06]. Thus we will mostly refer to such proofs and, when necessary, we will explain why and how they can be adapted to the case of infinite affine Dynkin quivers.

From now on, let us suppose Q is a quiver with underlying graph of type A_∞, $A_{+\infty}$, D_∞, and, to ease notation, let us write $\mathcal{A}_{n,\nu,\lambda}$ for $\mathcal{A}_{n,\nu,\lambda}(Q)$, and $\hat{\Pi}_\lambda$ for $\hat{\Pi}_\lambda(Q)$.

We observe that Gan's construction works only for loop free vertices ([GG05], 2.3) but, in our case, any vertex is such. Thus let i be any vertex of Q. Since $\mathcal{A}_{R,n,\nu,\lambda}$ does not depend on the orientation of Q, we can suppose that i is a sink (all arrows at i point toward i). Let V be an $\mathcal{A}_{R,n,\nu,\lambda}$-module which is finitely generated as an R-module. The definition of $F_i(V)$ is the same as in [Gan06] that we are now going to recall.

Let

$$\mathcal{H} := \{a \in Q | h(a) = i\}. \tag{6.3}$$

Remark 6.4.1. *Note that, for any infinite affine quiver, the set $\mathcal{H}$ is finite.*

If $\underline{j} = (j_1, \ldots, j_n) \in I^n$, where I is the set of vertices of Q, let

$$V_{\underline{j}} := |_{\underline{j}} V, \qquad \text{and} \qquad \Delta(\underline{j}) := \{m \in \{1, \ldots, n\} \,|\, j_m = i\}$$

where $|_{\underline{j}}$ is the element $e_{j_1} \otimes \cdots \otimes e_{j_n}$ as in Section 5.5.2.

Remark 6.4.2. *Observe that, even if Q is an infinite quiver, if V is finitely generated as a R-module then $V_{\underline{j}} \neq 0$ only for a finite number of multi-indices $\underline{j}$.*

For any subset $D \subset \Delta(\underline{j})$, consider the finite set

$$\mathcal{X}(D) := \{ \text{ all maps } \xi : D \to \mathcal{H}\}. \tag{6.4}$$

For any $\xi \in \mathcal{X}(D)$ let

$$t(\underline{j},\xi) := (t_1,\dots,t_n) \in I^n, \qquad \text{where} \qquad t_m = \begin{cases} j_m & \text{if } m \notin D \\ t(\xi(m)) & \text{if } m \in D \end{cases}.$$

Set

$$V(\underline{j},D) := \bigoplus_{\xi\in\mathcal{X}(D)} V_{t(\underline{j},\xi)}$$

so that $V(\underline{j},\emptyset) = V_{\underline{j}}$. For any ξ there are a projection and an inclusion map

$$\pi_{\underline{j},\xi} : V(\underline{j},D) \to V_{t(\underline{j},\xi)}, \qquad \mu_{\underline{j},\xi} : V_{t(\underline{j},\xi)} \hookrightarrow V(\underline{j},D).$$

Moreover, for any $p \in D$ there is a restriction map $\rho_p : \mathcal{X}(D) \to \mathcal{X}(D \setminus \{p\})$. Thus for each $\xi \in \mathcal{X}(D)$ we can consider the two compositions

$$V(\underline{j},D) \xrightarrow{\pi_{\underline{j},\xi}} V_{t(\underline{j},\xi)} \xrightarrow{\xi(p)_p|_{t(\underline{j},\xi)}} V_{t(\underline{j},\rho_p(\xi))} \xhookrightarrow{\mu_{\underline{j},\rho_p(\xi)}} V(\underline{j},D\setminus\{p\})$$

$$V(\underline{j},D\setminus\{p\}) \xrightarrow{\pi_{\underline{j},\rho_p(\xi)}} V_{t(\underline{j},\rho_p(\xi))} \xrightarrow{\xi(p)^*_p|_{t(\underline{j},\rho_p(\xi))}} V_{t(\underline{j},\xi)} \xhookrightarrow{\mu_{\underline{j},\xi}} V(\underline{j},D)\ .$$

where we recall that $\xi(p)_p|_{t(\underline{j},\xi)} = e_{t_1} \otimes \cdots \otimes \xi(p)e_{t_p} \otimes \cdots e_{t_n}$ (and similarly for $\xi(p)^*_p|_{t(\underline{j},\rho_p(\xi))}$).

Define now

$$\pi_{\underline{j},p}(D) : V(\underline{j},D) \to V(\underline{j},D\setminus\{p\}), \qquad \pi_{\underline{j},p}(D) := \sum_{\xi\in\mathcal{X}(D)} \mu_{\underline{j},\rho_p(\xi)}\xi(p)_p|_{t(\underline{j},\xi)}\pi_{\underline{j},\xi} \quad (6.5)$$

$$\mu_{\underline{j},p}(D) : V(\underline{j},D\setminus\{p\}) \to V(\underline{j},D), \qquad \pi_{\underline{j},p}(D) := \sum_{\xi\in\mathcal{X}(D)} \mu_{\underline{j},\xi}\xi(p)^*_p|_{t(\underline{j},\rho_p(\xi))}\pi_{\underline{j},\rho_p(\xi)}. \quad (6.6)$$

Let

$$V_{\underline{j}}(D) := \begin{cases} \cap_{p\in D}\mathrm{Ker}(\pi_{\underline{j},p}(D)) & \text{if } D \neq \emptyset \\ V_{\underline{j}} & \text{if } D = \emptyset \end{cases}$$

and let $V'_{\underline{j}} := V_{\underline{j}}(\Delta(\underline{j}))$.

Now, for all $\sigma \in S_n$ we set $\sigma(\underline{j}) = (j_{\sigma^{-1}(1)}, \ldots, j_{\sigma^{-1}(n)})$, so that $\Delta(\sigma(\underline{j})) = \sigma(\Delta(\underline{j}))$. For any $D \subset \Delta(\underline{j})$ and any map $\xi \in \mathcal{X}(D)$, we define $\sigma(\xi)$ as the map $\xi \circ \sigma^{-1} \in \mathcal{X}(\sigma(D))$. We define $\sigma|_{\underline{j}}$ as the map that is given by the formula

$$\sigma|_{\underline{j}} : V(\underline{j}, D) \to V(\sigma(\underline{j}), \sigma(D)), \qquad \sigma|_{\underline{j}} := \sum_{\xi \in \mathcal{X}(D)} \mu_{\sigma(\underline{j}),\sigma(\xi)} \sigma \pi_{\underline{j},\xi}$$

for any $D \subset \Delta(\underline{j})$.

Definition 6.4.3. *We define $F_i(V) := V' = \bigoplus_{\underline{j} \in I^n} V'_{\underline{j}}$ as a $\widehat{\mathcal{B}}_R \sharp S_n$-module (the compatibility of the S_n action can be checked by computation as in [Gan06], Lemma 2.2).*

Note that by Remark 6.4.2 the direct sum in Definition 6.4.3 is really a direct sum over a finite subset of indices in I.

For any $l = 1, \ldots, n$, $a \in \overline{Q}$, $\underline{j} \in I^n$ with $j_l = t(a)$, we have to define a map $a'_l|_{\underline{j}} : V'_{\underline{j}} \to V'_{a_l(\underline{j})}$, where $a_l(\underline{j})$ is as defined in Section 5.5.2. One has three cases.

Case I. If $h(a), t(a) \neq i$ then $l \notin \Delta(\underline{j}) = \Delta(a_l(\underline{j}))$. For any $D \subset \Delta(\underline{j})$ we have a map

$$a_l|_{\underline{j},D} : V(\underline{j}, D) \to V(a_l(\underline{j}), D), \qquad a_l|_{\underline{j},D} := \sum_{\xi \in \mathcal{X}(D)} \mu_{a_l(\underline{j}),\xi} a_l|_{t(\underline{j},\xi)} \pi_{\underline{j},\xi}.$$

We define

$$a'_l|_{\underline{j}} := a_l|_{\underline{j},\Delta(\underline{j})}. \tag{6.7}$$

Case II. If $t(a) = i$, then $l \in \Delta(\underline{j})$ and $\Delta(a_l(\underline{j})) = \Delta(\underline{j}) \setminus \{l\}$ since there are no loop edges at i (hence $h(a) \neq i$). If $l \in D \subset \Delta(\underline{j})$, for each $r \in \mathcal{H}$ there is an injective map

$$\tau_{r,l,D} : \mathcal{X}(D \setminus \{l\}) \hookrightarrow \mathcal{X}(D) : \eta \to \tau_{r,l,D}(\eta)$$

where

$$\tau_{r,l,D}(\eta)(m) := \begin{cases} \eta(m) & \text{if } m \in D \setminus \{l\} \\ r & \text{if } m = l \end{cases}$$

We observe that $t(\underline{j}, \tau_{r,l,D}(\eta)) = t(r_l^*(\underline{j}), \eta)$. Thus there is a projection map

$$\tau^!_{r,l,\underline{j},D} : V(\underline{j}, D) \to V(r_l^*(\underline{j}), D\setminus\{l\}), \qquad \tau^!_{r,l,\underline{j},D} := \sum_{\eta \in \mathcal{X}(D\setminus\{l\})} \mu_{r_l^*(\underline{j}),\eta} \pi_{\underline{j},\tau_{r,l,D}(\eta)},$$

where we used again the fact that there are no loop edges at i and thus, since $t(r^*) = i$, we must have $h(r^*) \neq i$. Similarly there is an inclusion map

$$\tau_{r,l,\underline{j},D_!} : V(r_l^*(\underline{j}), D\setminus\{l\}) \to V(\underline{j}, D), \qquad \tau_{r,l,\underline{j},D_!} := \sum_{\eta \in \mathcal{X}(D\setminus\{l\})} \mu_{\underline{j},\tau_{r,l,D}(\eta)} \pi_{r_l^*(\underline{j}),\eta}.$$

We define

$$a'_l|_{\underline{j}} := \tau^!_{a^*,l,\underline{j},\Delta(\underline{j})}. \tag{6.8}$$

Case III. If $h(a) = i$, then $l \notin \Delta(\underline{j})$ and $\Delta(a_l(\underline{j})) = \Delta(\underline{j}) \cup \{l\}$. For any $D \subset \Delta(\underline{j})$ we have the inclusion map

$$\tau_{a,l,a_l(\underline{j}),D\cup\{l\}_!} : V(\underline{j}, D) \to V(a_l(\underline{j}), D \cup \{l\})$$

as above. We have a map

$$\theta_{a,l,\underline{j},D} : V(\underline{j}, D) \to V(a_l(\underline{j}), D \cup \{l\})$$

defined by

$$\theta_{a,l,\underline{j},D} := \left(-\lambda_i + \mu_{a_l(\underline{j}),l} \pi_{a_l(\underline{j}),l} + \nu \sum_{m \in D} s_{ml}|_{a_l(\underline{j})} \right) \tau_{a,l,a_l(\underline{j}),D\cup\{l\}_!}.$$

We define

$$a'_l|_{\underline{j}} := \theta_{a,l,\underline{j},\Delta(\underline{j})}. \tag{6.9}$$

We have the following proposition

Proposition 6.4.4. *[[GG05], Proposition 2.7] With the above action $F_i(V)$ is a $\mathcal{A}_{R,n,\nu,r_i\lambda}$-module.*

When Q is an infinite affine Dynkin quiver (or in general a locally finite one), the reflection functors satisfy the same properties as in Gan ([Gan06], § 6.2). In particular if i is a loop-free vertex, let Λ_i be the set of all $(\lambda, \nu) \in \widehat{B}_R \times R = R^I \times R$ such that $\lambda_i \pm \nu \sum_{m=2}^{r} s_{1m}$ is invertible in $R[S_r]$ for all $r = 1, \ldots, n$.

Theorem 6.4.5 ([Gan06], Theorem 5.1)**.** *If $(\lambda, \nu) \in \Lambda_i$, then the functor*

$$F_i : \mathcal{A}_{R,n,\nu,\lambda} - mod \to \mathcal{A}_{R,n,\nu,r_i\lambda} - mod$$

is an equivalence of categories with quasi-inverse functor F_i, where we are considering the categories of left modules that are finitely generated as R-modules.

Lemma 6.4.6 ([Gan06], Proposition 5.12)**.** *If $R = \mathbb{C}$ we have:*

$$\Lambda_i = \{(\lambda, \nu) \in \widehat{B} \times \mathbb{C} | \lambda_i \pm p\nu \neq 0 \textit{ for } p = 0, \ldots, n-1\}$$

Let now B_J and U_J be the finite dimensional vectors spaces defined at the end of Section 5.6. Let $R = \mathbb{C}[[U_J]]$. Let $\nu \in \mathbb{C}[[U_J]]$, and $\lambda \in B_J \otimes \mathbb{C}[[U_J]]$. With the same notation as in Section 5.6, we say that a $\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda}$-module V_{U_J} is a flat formal deformation of a $\overline{\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda}}$-module V if $V_{U_J} \cong V[[U_J]]$ as $\mathbb{C}[[U_J]]$-modules and $\overline{V_{U_J}} \cong V$. Proposition 5.14 of [Gan06] extend to the continuous case with analogous proof.

Proposition 6.4.7. *Assume that $\nu \in \mathbb{C}[U_J]$, and $\lambda \in B_J \otimes \mathbb{C}[U_J]$ (as in Lemma* 5.6.2*). Moreover, let $i \in J$ and assume that $(\lambda, \nu) \in \Lambda_i$. If a $\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda}$-module V_{U_J} is a flat formal deformation of a finite dimensional $\overline{\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda}}$-module V, then $F_i(V_{U_J})$ is a flat formal deformation of $F_i(V)$.*

□

6.5 Representations in the higher rank case

For any commutative unital $\mathbb{C}$-algebra R and any $\widehat{\mathcal{B}}_R\sharp S_n$-module V define the set $I_V \subset I$ as follows

$$I_V := \Big\{ i \in I | \exists\, \underline{j} := (j_1, \dots j_n) \in I^n \text{ with } \ i \in \{j_1, \dots, j_n\} \text{ and } |_{\underline{j}} V \neq 0 \Big\}.$$

where we recall that $|_{\underline{j}} = e_{j_1} \otimes \cdots \otimes e_{j_n}$.

It is clear that if V is finitely generated over R then I_V is a finite subset of I (see Remark 6.4.2).

In particular when $n = 1$, $R = \mathbb{C}$ and V is finite dimensional, of dimension vector say α, then $I_V = \{i \in I | e_i V \neq 0\} = \text{supp}(\alpha)$, the support of the dimension vector.

Suppose now I_V is finite, and let $J \subset I$ be any finite subset of indices such that $I_V \subset J$. Consider the finite rank free R-module

$$B_{R,J} := \bigoplus_{i \in J} R e_i \hookrightarrow \prod_{i \in I} R e_i = \widehat{B}_R$$

i.e. $B_{R,J} = \Big\{ \lambda \in \widehat{B}_R | \lambda_i = 0 \ \forall i \notin J \Big\}$. Then the module V factors through the homomorphism $(\widehat{\mathcal{B}}_R\sharp S_n) \twoheadrightarrow B_{R,J}^{\otimes n}\sharp S_n$ defined by

$$\begin{cases} \sigma \longrightarrow \sigma & \text{if } \sigma \in S_n \\ |_{\underline{j}} \longrightarrow |_{\underline{j}} & \text{if } j_s \in J \text{ for all } s \\ |_{\underline{j}} \longrightarrow 0 & \text{otherwise} \end{cases} \tag{6.10}$$

In accordance with the notation of Section 6.2, in all what follows, for any finite set of indices J and any parameter $\lambda \in \widehat{B}_R$, we will denote by $\lambda|_J \in B_{R,J}$ the restriction of the parameter λ to the set J.

Suppose now V is a $\mathcal{A}_{R,n,\nu,\lambda}$-module and $I_V \subset J$. Let Q_J be the full subquiver corresponding to J. Then V factors through the homomorphism $\mathcal{A}_{R,n,\nu,\lambda} \longrightarrow$

$\mathcal{A}_{R,n,\nu,\lambda|_J}(Q_J)$ defined by formulas (6.10) above and by the assignment

$$\begin{cases} a_l|_{\underline{j}} & \longrightarrow & a_l|_{\underline{j}} & \text{if } j_s \in J \text{ for all } s \\ a_l|_{\underline{j}} & \longrightarrow & 0 & \text{otherwise} \end{cases}$$

where we recall that, for any $a \in \overline{Q}$ and $l = 1, \ldots, n$, we denoted by $a_l|_{\underline{j}}$ the element $e_{j_1} \otimes \cdots \otimes ae_{j_l} \otimes \cdots \otimes e_{j_n}$.

Observe that what we said above also implies that the module V factors through an action of the algebra $\mathcal{A}_{R,n,\nu,\lambda_J}$, where λ_J is as in Section 5.4 (i.e. $(\lambda_J)_i = \lambda_i$ if $i \in J$ and $(\lambda_J)_i = 0$ otherwise).

Thus, if we want to study modules over $\mathcal{A}_{R,n,\nu,\lambda}(Q)$ that are finitely generated as R-modules, it is actually enough to consider subfamilies of algebras depending only on finitely many of the parameters λ_i.

Let us now go back to the case $R = \mathbb{C}$. The easiest examples of finite dimensional $\mathcal{A}_{n,\nu,\lambda}$- modules one can look for are the modules which are irreducible as $\widehat{\mathcal{B}}\sharp S_n$-modules. For any $i \in I$, let $\mathcal{N}_i$ be the complex vector space with dimension vector ϵ_i. Consider $\vec{n} = (n_1, \ldots, n_r)$, where $n_i \in \mathbb{Z}_{>0}$, and $\sum_{i=1}^r n_i = n$. Let $\{i_1, \ldots, i_r\}$ be a set of r distinct vertices of Q and let $\mathcal{N} = \mathcal{N}_{i_1}^{\otimes n_1} \otimes \cdots \otimes \mathcal{N}_{i_r}^{\otimes n_r}$. As in Chapter 3, regard S_{n_j} as the group permuting the factors in $\mathcal{N}_{i_j}^{\otimes n_j}$, and consider $S_{\vec{n}} := S_{n_1} \times \cdots \times S_{n_r} \subset S_n$. Let $X = X_1 \otimes \cdots \otimes X_r$ be a simple module for the group $S_{\vec{n}}$. Then $X \otimes \mathcal{N}$ is a simple module for $\widehat{\mathcal{B}}\sharp S_{\vec{n}}$. One can then form the induced $\widehat{\mathcal{B}}\sharp S_n$-module $X \otimes \mathcal{N} \uparrow := \mathrm{Ind}_{\widehat{\mathcal{B}}\sharp S_{\vec{n}}}^{\widehat{\mathcal{B}}\sharp S_n}(X \otimes \mathcal{N})$. Moreover, it is known that any simple finite dimensional $\widehat{\mathcal{B}}\sharp S_n$-module has this form (this is true by [Mac80], paragraph after (A5), when Γ is finite, and remains true for Γ reductive when we consider only finite dimensional representations).

Observe now that we have

$$X \otimes \mathcal{N} \uparrow = \bigoplus_{\sigma} \sigma(X \otimes \mathcal{N}) \tag{6.11}$$

where σ runs over a set of left coset representative of $S_{\vec{n}}$ in S_n (in particular we have $I_{X \otimes \mathcal{N}\uparrow} = \{i_1, \ldots, i_r\}$).

The following lemma is the analog of [Gan06], Lemma 6.1.

Lemma 6.5.1. *Suppose $\nu \in \mathbb{C}[[U_J]]$, and $\lambda \in B_J \otimes \mathbb{C}[[U_J]]$. Let a $\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda}$-module V_{U_J} be a flat formal deformation of the finite dimensional $\overline{\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda}}$-module $\overline{V_{U_J}}$. If $\overline{V_{U_J}}$ is simple as a $\widehat{\mathcal{B}}\sharp S_n$-module, then all elements of $\widehat{\mathcal{E}}_{\mathbb{C}[[U_J]]}$ must act by 0 on V_{U_J}.*

Proof. The proof goes exactly as in [Gan06] when we observe that also in our case $\widehat{\mathcal{B}}\sharp S_n$ is a semisimple algebra. This implies that V_{U_J} must be of the form $(X \otimes \mathcal{N} \uparrow)[[U_J]]$ as a $\widehat{\mathcal{B}}_{\mathbb{C}[[U_J]]}\sharp S_n$-module, with $i_1, \ldots, i_r \in J$ (trivial deformation of an irreducible module $X \otimes \mathcal{N} \uparrow$). Let us fix $l \in \{1, \ldots n\}$. Any $a_l|_{\underline{j}} \in \widehat{\mathcal{B}}_{\mathbb{C}[[U_J]]}\sharp S_n$ induces maps between the different summands in formula (6.11). Now, for any $\underline{j} = (j_1, \ldots, j_n) \in I^n$ and $\sigma \in S_n$, if $j_h = j_{\sigma(h)}$ for any $h \neq l$ then $j_l = j_{\sigma(l)}$. Since Q has no edge loops, this implies that $a_l|_{\underline{j}}$ must act by 0 on $\mathcal{N}$. Since this is true for any a and any l, all elements of $\widehat{\mathcal{E}}_{\mathbb{C}[[U_J]]}$ must act by 0 on V_{U_J}.

The following theorem can be proved as Theorem 6.2 in [Gan06] and is equivalent to Theorem 6.5 in [EGG05].

Theorem 6.5.2. *Assume $\mathbb{C}[[U_J]] \ni \nu \neq 0$ and $\lambda \in B_J \otimes \mathbb{C}[[U_J]]$. The $\widehat{\mathcal{B}}_{\mathbb{C}[[U_J]]}\sharp S_n$-module $(X \otimes \mathcal{N} \uparrow)[[U_J]]$ extends to a $\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda_J}$-module if and only if the following conditions are satisfied:*

(i) For all $l \in \{1, \ldots, r\}$, the simple module X_l of S_{n_l} has rectangular Young diagram of some size $a_l \times b_l$;

(ii) No two vertices in the collection $\{i_1, \ldots, i_r\}$ are adjacent in Q, i.e. $(\epsilon_{i_j}, \epsilon_{i_k}) = 0$ for any $j \neq k \in \{1, \ldots, r\}$;

(iii) For all $l \in \{1, \ldots, r\}$, one has $\lambda_{i_l} = \nu(a_l - b_l)$;

where we agree that condition (ii) is empty if $r = 1$ i.e. if $\vec{n} = (n)$.

Fix now $\lambda_0 \in \widehat{B}$. Let $Y_1, \ldots, Y_r$ be a collection of simple, pairwise non-isomorphic, finite dimensional representations of $\hat{\Pi}_{\lambda_0}$ and denote by $\alpha_{(i)}$ the dimension vector of Y_i. Let $\mathbf{Y} := Y_1^{\otimes n_1} \otimes \cdots \otimes Y_r^{\otimes n_r}$. Then $X \otimes \mathbf{Y}$ is an irreducible representation of $\hat{\Pi}_{\lambda_0}^{\hat{\otimes} n}\sharp S_{\vec{n}}$ and, as in the finite case, we can consider the induced $\mathcal{A}_{n,0,\lambda_0}$-module

$X \otimes \mathbf{Y} \uparrow := \mathrm{Ind}_{\hat{\Pi}_{\lambda_0}^{\hat{\otimes} n} \sharp S_{\vec{n}}}^{\hat{\Pi}_{\lambda_0}^{\hat{\otimes} n} \sharp S_n} X \otimes \mathbf{Y}$. It is known (as before by [Mac80]) that any finite dimensional simple $\mathcal{A}_{n,0,\lambda_0}$-module is of this form. For $i = 1, \ldots, r$, denote by $\alpha_{(i)}$ the dimension vector of Y_i. It is easy to see that:

$$I_{X \otimes \mathbf{Y} \uparrow} = \bigcup_{i=1}^{r} I_{Y_i} = \bigcup_{i=1}^{r} \mathrm{supp}(\alpha_{(i)}).$$

Let J be a finite subset of indices corresponding to some connected subquiver and such that $I_{X \otimes \mathbf{Y} \uparrow} \subset J$. As in Section 5.4, let λ^+ be the unique J-dominant weight W_J-conjugate to λ_0, and w^+ the unique element of minimal length such that $w^+ \lambda_0 = \lambda^+$. Write $w^+ = s_{j_h} \cdots s_{j_1}$ for some simple reflections corresponding to vertices $j_1, \ldots, j_h \in J$. By minimality of the length we know $(r_{j_g} \cdots r_{j_1} \lambda)_{j_{g+1}} \neq 0$ for $g = 0, \ldots, h-1$. Denote by F_{w^+} the composition $F_{j_h} \cdots F_{j_1}$, and by $F_{(w^+)^{-1}}$ the composition $F_{j_1} \cdots F_{j_h}$.

The following theorem is the analog of Theorem 6.3 in [Gan06].

Theorem 6.5.3. *Suppose $\lambda \in B_J \otimes \mathbb{C}[U_J]$ with $\lambda_i \in U_J$ for any $i \in J$, and that $0 \neq \nu \in U_J$. The $\mathcal{A}_{n,0,\lambda_0}$-module $X \otimes \mathbf{Y} \uparrow$ has a flat formal deformation to a $\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda_0+\lambda}$-module if and only if the following conditions are satisfied:*

(i) For all $l \in \{1, \ldots, r\}$, the simple module X_l of S_{n_l} has rectangular Young diagram, of size $a_l \times b_l$;

(ii) $(\alpha_{(l)}, \alpha_{(m)}) = 0$ for all $l \neq m$;

(iii) For all $l \in \{1, \ldots, r\}$, one has $\lambda \cdot \alpha_{(l)} = (a_l - b_l)\nu$;

where we agree that condition (ii) is empty if $r = 1$. When the deformation exists it is unique.

Proof. Take J, λ^+, w^+, F_{w^+}, $F_{(w^+)^{-1}}$ as above. Exactly as in [Gan06], proof of Theorem 6.3, we can prove that $(r_{j_g} \cdots r_{j_1}(\lambda_0 + \lambda), \nu) \in \Lambda_{j_{g+1}}$ for $g = 0, \ldots, h-1$. Indeed, for any $C \in \mathbb{C}[[U_J]][S_n]$ the element $(r_{j_g} \cdots r_{j_1}(\lambda_0 + \lambda))_{j_{g+1}} + \nu C$ has an inverse in $\mathbb{C}[[U_J]][S_n]$. Finding this inverse amounts to solving a system of $n!$ equations in $n!$ variables with matrix $(r_{j_g} \cdots r_{j_1}(\lambda_0 + \lambda))_{j_{g+1}} \mathrm{Id}_{n!} + \nu A$, with A some matrix. Since $\lambda, \nu \in \mathfrak{m}$ and $(r_{j_g} \cdots r_{j_1}\lambda_0)_{j_{g+1}} \neq 0$ (by minimality of the length of w^+), the determinant of this matrix is nonzero modulo $\mathfrak{m}$. Thus the determinant is invertible in $\mathbb{C}[[U_J]]$ and so the matrix is invertible. As a consequence we have that the sequence F_{w^+} establishes an equivalence of categories $\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda_0+\lambda} - \mathrm{mod} \longrightarrow \mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda^+ + w^+(\lambda)} - \mathrm{mod}$.

For any $l = 1, \ldots, r$ we have that $\alpha_{(l)} \in \Sigma_{\lambda_0,J}$. From Lemma 5.4.1 we know that $w^+\Sigma_{\lambda_0,J} = \Sigma_{\lambda^+,J}$. Since λ^+ is J-dominant we know that $\Sigma_{\lambda^+,J} = \{\epsilon_i | i \in J, \lambda_i^+ = 0\}$. For any $l = 1, \ldots r$, define $i_l \in J$ by $w^+(\alpha_{(l)}) = \epsilon_{i_l} \in \Sigma_{\lambda^+,J}$. As we observed at the beginning of this section, since $I_{X\otimes\mathbf{Y}\uparrow} \subset J$ the $\mathcal{A}_{n,0,\lambda_0}$-module $X \otimes \mathbf{Y} \uparrow$ factors through $\mathcal{A}_{n,0,(\lambda_0)|_J}(Q_J)$. Moreover, since $j_1, \cdots, j_h \in J$, we can deduce from the definition of the functors F_{j_l} that the $\mathcal{A}_{n,0,\lambda^+}$-module $F_{w^+}(X \otimes \mathbf{Y} \uparrow)$ factors through $\mathcal{A}_{n,0,(\lambda^+)|_J}(Q_J)$. By [CBH98] Theorem 5.1, we must have $F_{w^+}(X \otimes \mathbf{Y} \uparrow) = X \otimes \mathcal{N} \uparrow$ where $\mathcal{N} = \mathcal{N}_{i_1}^{\otimes n_1} \otimes \cdots \otimes \mathcal{N}_{i_r}^{\otimes n_r}$.

From now the proof goes exactly as in [Gan06] and we report it for the reader's convenience.

We first observe that, for any $l = 1, \ldots, r$, we have $\lambda_0 \cdot \alpha_{(l)} = \lambda^+ \cdot \epsilon_{i_l} = 0$, and that both the products $(\ ,\ )$ and $\cdot$ are W-invariant.

Using this, we have that, if the conditions in the theorem are satisfied, then the $\widehat{\mathcal{B}}_{\mathbb{C}[[U_J]]}\sharp S_n$-module $M := X \otimes \mathcal{N} \uparrow [[U_J]]$ satisfies the conditions of Theorem 6.5.2, and thus extends to a $\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda^+ + w^+(\lambda)}$-module (with $\widehat{\mathcal{E}}_{\mathbb{C}[[U_J]]}$ acting by 0). Thus by Proposition 6.4.7 the $\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda_0+\lambda}$-module $F_{(w^+)^{-1}}(M)$ is a flat formal deformation of $X \otimes \mathbf{Y} \uparrow$.

Vice versa suppose a $\mathcal{A}_{\mathbb{C}[[U_J]],n,\nu,\lambda_0+\lambda}$-module V is a flat formal deformation of $X \otimes \mathbf{Y} \uparrow$. Then, using again Proposition 6.4.7, we have that $F_{w^+}(V)$ is a flat formal deformation of $X \otimes \mathcal{N} \uparrow$. Since $F_{w^+}(V) = (X \otimes \mathcal{N} \uparrow)[[U_J]]$ as a $\widehat{\mathcal{B}}_{\mathbb{C}[[U_J]]}\sharp S_n$-module, then by Theorem 6.5.2 the conditions (i), (ii), (iii) must be satisfied. Moreover, we have that, by Lemma 6.5.1, $\widehat{\mathcal{E}}_{\mathbb{C}[[U_J]]}$ must act by 0. Thus $F_{w^+}(V)$ is the unique flat formal deformation of $X \otimes \mathcal{N} \uparrow$ and V is the unique flat formal deformation of $X \otimes \mathbf{Y} \uparrow$.

$\square$

Assume now that $\lambda_0 \in B_J$ and that the conditions (i), (ii) of Theorem 6.5.3 hold. Let $\nu \in \mathbb{C}[U_J]$ and $\lambda \in B_J \otimes \mathbb{C}[U_J]$ be functions satisfying the condition (iii). Suppose there is a point $o \in U_J$ such that λ specializes to λ_0 and ν specializes to 0 at o. Following Gan's notation let U'_J be the Zariski open set in U_J defined by $(r_{j_g} \cdots r_{j_1}\lambda)_{j_{g+1}} \pm p\nu \neq 0$ for $g = 0, \ldots, h-1$ and $p = 0, \ldots, n-1$. Since $o \in U'_J$ this set is nonempty. Let $\mathbb{C}[U'_J]$ be the ring of regular functions on U'_J and for any $u \in U'_J$ let $\mathfrak{m}_u$ denote the maximal ideal of functions vanishing at u. If V is a $\mathbb{C}[U'_J]$ -module let $V^u := V/\mathfrak{m}_u V$. The following theorem is the analog of [Gan06], Theorem 6.4 and the proof is exactly the same.

Theorem 6.5.4. *There exists a $\mathcal{A}_{\mathbb{C}[U'_J],n,\nu,\lambda}$-module $V_{U'_J}$ such that:*

(i) $V^o_{U'_J} = X \otimes \mathbf{Y}\uparrow$ *as a $\mathcal{A}_{n,0,\lambda_0}$-module, and $V_{U'_J}$ is flat over U'_J;*

(ii) for any point $u \in U'_J$, $V^u_{U'_J}$ is a finite dimensional simple $\mathcal{A}^u_{\mathbb{C}[U'_J],n,\nu,\lambda}$-module, isomorphic to $X \otimes \mathbf{Y}\uparrow$ as a $\widehat{\mathcal{B}}^u_{\mathbb{C}[U'_J]}\sharp S_n$-module.

□

Appendix A

Proof of Theorem 5.6.1

Before getting started we need to introduce some notation. For any $i \in I$, let us choose a basis for the irreducible representation N_i, and let us denote by $E^{N_i}_{pq}(\gamma) \in \mathcal{O}(\Gamma)$ the (p,q)-th matrix coefficient for N_i in such basis, where $1 \leq p, q \leq d_i = \dim N_i$. Since Γ is reductive, we know these functions span the algebra $\mathcal{O}(\Gamma)$. Moreover, if we take matrix coefficients $E^{N_i}_{pq}(\gamma)$, $E^{N_j^*}_{lm}(\gamma)$ with respect to dual bases, the following orthogonality relation holds:

$$\int_\Gamma E^{N_i}_{pq}(\gamma)\, E^{N_j^*}_{lm}(\gamma) d\gamma = \frac{1}{d_i} \delta_{ij} \delta_{pl} \delta_{qm}\,, \tag{A.1}$$

where $\int_\Gamma$ is the linear form described in Section 4.2. Let $\check{E}^{N_i}_{pq}$ be the unique distribution such that $\langle \check{E}^{N_i}_{pq}, E^{N_j}_{lm} \rangle = \delta_{ij}\delta_{pl}\delta_{qm}$. Using (A.1) we can write $\langle \check{E}^{N_i}_{pq}, E^{N_j}_{lm} \rangle = d_i \int_\Gamma E^{N_j}_{lm}\, E^{N_i^*}_{pq}\, d\gamma$. Using the identifications (4.3) of Section 4.2, we can see how these distributions span topologically $\mathcal{O}(\Gamma)^*$.

It is straightforward to compute that, if $\Delta : \mathcal{O}(\Gamma) \longrightarrow \mathcal{O}(\Gamma) \otimes \mathcal{O}(\Gamma)$ denotes the coproduct for $\mathcal{O}(\Gamma)$, then $\Delta(E^{N_i}_{pq}) = \sum_{r=1}^{d_i} E^{N_i}_{pr} \otimes E^{N_i}_{rq}$. In all what follows, when there is no ambiguity, we will just omit the sum sign over repeated indices and write $\Delta(E^{N_i}_{pq}) = E^{N_i}_{pr} \otimes E^{N_i}_{rq}$. Using just the definition of convolution product (formula (4.1), Section 4.2), it is now easy to see that $\check{E}^{N_i}_{pq} \check{E}^{N_j}_{lm} = \delta_{ij}\delta_{ql}\check{E}^{N_i}_{pm}$, so that the identification $\mathcal{O}(\Gamma)^* = \prod_{i\in I} \mathrm{Mat}(d_i)$ is an algebra isomorphism.

We observe now that $V \otimes \mathcal{O}(\Gamma)^{*\hat{\otimes} n}$ is a $\mathcal{O}(\Gamma)^{*\hat{\otimes} n}$-bimodule with right action only on the second factor and left action defined by

$$\mu(w \otimes \mu') = \sum_i \left(x_i \otimes (x_i^*, gw) + y_i \otimes (y_i^*, gw)\right) \mu\mu'$$

for all $\mu, \mu' \in \mathcal{O}(\Gamma)^{*\hat{\otimes} n}$, $w \in V$, where $\{x_i, y_i\}$ is a symplectic basis as in Lemma 4.4.1

and $\{x_i^*, y_i^*\}$ is its dual basis. Let us now denote by

$$T^k_{\mathcal{O}(\Gamma)^{*\hat{\otimes}n}}(V \otimes \mathcal{O}(\Gamma)^{*\hat{\otimes}n}) := (V \otimes \mathcal{O}(\Gamma)^{*\hat{\otimes}n}) \otimes_{\mathcal{O}(\Gamma)^{*\hat{\otimes}n}} \cdots \otimes_{\mathcal{O}(\Gamma)^{*\hat{\otimes}n}} (V \otimes \mathcal{O}(\Gamma)^{*\hat{\otimes}n}).$$

We have

$$TV \sharp \mathcal{O}(\mathbf{\Gamma}_n)^* = \left(\bigoplus_{k \geq 0} T^k_{\mathcal{O}(\Gamma)^{*\hat{\otimes}n}}(V \otimes \mathcal{O}(\Gamma)^{*\hat{\otimes}n}) \right) \sharp S_n = T_{\mathcal{O}(\Gamma)^{*\hat{\otimes}n}}(V \otimes \mathcal{O}(\Gamma)^{*\hat{\otimes}n}) \sharp S_n. \quad \text{(A.2)}$$

Following [CBH98] (§ 3) and [GG05] (§ 3.3) we will now define the idempotents $\varphi_i := \check{E}^{N_i}_{11}$ and $\varphi = \sum_i \varphi_i$ in the algebra $\mathcal{O}(\Gamma)^*$.

For the element element $\varphi^{\otimes n} \in \mathcal{O}(\Gamma)^{*\hat{\otimes}n}$ we have:

$$\varphi^{\otimes n} = \sum_{i_1, \ldots, i_n \in I} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_n}$$

and

$$\begin{aligned} &\sum_{i_1, p_1, \ldots, i_n, p_n} (\check{E}^{N_{i_1}}_{p_1 1} \otimes \cdots \otimes \check{E}^{N_{i_n}}_{p_n 1}) \varphi^{\otimes n} (\check{E}^{N_{i_1}}_{1 p_1} \otimes \cdots \otimes \check{E}^{N_{i_n}}_{1 p_n}) \\ &= \sum_{i_1, p_1, \ldots, i_n, p_n} \check{E}^{N_{i_1}}_{p_1 1} \check{E}^{N_{i_1}}_{1 p_1} \otimes \cdots \otimes \check{E}^{N_{i_n}}_{p_n 1} \check{E}^{N_{i_n}}_{1 p_n} = \delta_1^{\otimes n}. \end{aligned} \quad \text{(A.3)}$$

Since $\delta_1^{\otimes n}$ is the unit element in $\mathcal{O}(\Gamma)^{*\hat{\otimes}n}$, equation (A.3) implies a Morita equivalence $\varphi^{\otimes n} \left(TV \sharp \mathcal{O}(\Gamma)^{*\hat{\otimes}n} \right) \varphi^{\otimes n} \sim TV \sharp \mathcal{O}(\Gamma)^{*\hat{\otimes}n}$.

Now clearly we have an isomorphism

$$\widehat{\mathcal{B}} \xrightarrow{\sim} \varphi^{\otimes n} \mathcal{O}(\Gamma)^{*\hat{\otimes}n} \varphi^{\otimes n} = \prod_{i_1, \ldots, i_n} \mathbb{C} \cdot \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_n} \quad \text{(A.4)}$$

such that

$$e_{i_1} \otimes \cdots \otimes e_{i_n} \longrightarrow \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_n}.$$

Moreover we have bijections

$$\varphi_i \mathcal{O}(\Gamma)^* \varphi_j \leftrightarrow \mathrm{Hom}_\Gamma(N_i, N_j) \qquad \varphi_i \left(L \otimes \mathcal{O}(\Gamma)^* \right) \varphi_j \leftrightarrow \mathrm{Hom}_\Gamma(N_i, L \otimes N_j).$$

Indeed we have that $\mathcal{O}(\Gamma)^* \varphi_j \cong N_j$ and $\left(L \otimes \mathcal{O}(\Gamma)^* \right) \varphi_j \cong L \otimes N_j$. The first is an irreducible finite dimensional representation and, since Γ is reductive, the second is a semisimple representation. Multiplying on the left by the idempotent φ_i corresponds to projecting on the multiplicity space of the component of type N_i of such representations.

Thus we have an isomorphism

$$
\begin{aligned}
&\varphi^{\otimes n}\left(V\otimes\mathcal{O}(\Gamma)^{*\hat{\otimes}n}\right)\varphi^{\otimes n}\\
&=\prod_{i_1,\dots,j_n}(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_n})(L^{\oplus n}\otimes\underbrace{\mathcal{O}(\Gamma)^*\hat{\otimes}\cdots\hat{\otimes}\mathcal{O}(\Gamma)^*}_{n})(\varphi_{j_1}\otimes\cdots\otimes\varphi_{j_n})\\
&=\bigoplus_{l=1}^{n}\prod_{i_1,\dots,j_n}\mathrm{Hom}_\Gamma(N_{i_1},N_{j_1})\otimes\cdots\otimes\mathrm{Hom}_\Gamma(N_{i_l},L\otimes N_{j_l})\otimes\cdots\otimes\mathrm{Hom}_\Gamma(N_{i_n},N_{j_n})\\
&\cong\widehat{\mathcal{E}} \qquad\qquad (A.5)
\end{aligned}
$$

Now from (A.2) and from identities (A.4) and (A.5) it follows that:

$$\varphi^{\otimes n}\left(T_{\mathcal{O}(\Gamma)^{*\hat{\otimes}n}}\left(V\otimes\mathcal{O}(\Gamma)^{*\hat{\otimes}n}\right)\right)\varphi^{\otimes n}\cong\widehat{T_{\mathcal{B}}\mathcal{E}} \qquad (A.6)$$

and

$$
\begin{aligned}
\varphi^{\otimes n}\left(TV\sharp\mathcal{O}(\Gamma)^{*\hat{\otimes}n}\right)\varphi^{\otimes n} &= \varphi^{\otimes n}\left(T_{\mathcal{O}(\Gamma)^{*\hat{\otimes}n}}\left(V\otimes\mathcal{O}(\Gamma)^{*\hat{\otimes}n}\right)\sharp S_n\right)\varphi^{\otimes n}\\
&\cong \widehat{T_{\mathcal{B}}\mathcal{E}}\sharp S_n \qquad (A.7)
\end{aligned}
$$

Now by (A.3) we have that $\mathcal{H}_{k,c}(\mathbf{\Gamma}_n)$ is Morita equivalent to $\varphi^{\otimes n}\mathcal{H}_{k,c}(\mathbf{\Gamma}_n)\varphi^{\otimes n}$. By (A.7) we have that $\varphi^{\otimes n}\mathcal{H}_{k,c}(\mathbf{\Gamma}_n)\varphi^{\otimes n}$ is isomorphic to some quotient of $\widehat{T_{\mathcal{B}}\mathcal{E}}\sharp S_n$. We will show in the next theorem that for an appropriate choice of the parameters this quotient is exactly the one described in Definition 5.5.3.

We will need the following auxiliary lemma, which is the analog for infinite affine quivers of Lemma 3.2 of [CBH98]. Let Q be a quiver attached to Γ (with any orientation). Let ζ be the linear map $\zeta:\mathbb{C}\to L\otimes L$ such that $1\to y\otimes x-x\otimes y$.

Lemma A.0.5. *For any arrow $a:i\to j$ in Q there exist Γ-module homomorphisms*

$$\theta_a:N_i\to L\otimes N_j \quad \textit{and} \quad \phi_a:N_j\to L\otimes N_i$$

such that for any vertex i

$$\sum_{a\in Q,h(a)=i}(\mathrm{Id}_L\otimes\theta_a)\phi_a-\sum_{a\in Q,t(a)=i}(\mathrm{Id}_L\otimes\phi_a)\theta_a=-d_i(\zeta\otimes\mathrm{Id}_{N_i})$$

as maps from N_i to $L\otimes L\otimes N_i$, and such that

$$(\omega_L\otimes\mathrm{Id}_{N_{t(a)}})(\mathrm{Id}_L\otimes\phi_a)\theta_a=-d_{h(a)}\mathrm{Id}_{N_{t(a)}}$$

and

$$(\omega_L\otimes\mathrm{Id}_{N_{h(a)}})(\mathrm{Id}_L\otimes\theta_a)\phi_a=d_{t(a)}\mathrm{Id}_{N_{h(a)}}.$$

Moreover the θ_a, ϕ_a combine to give a basis for each of the spaces $\mathrm{Hom}_\Gamma(N_i,L\otimes N_j)$.

Proof. (of Theorem 5.6.1) In the case Γ is of type A_∞ the same proof as in [CBH98], Lemma 3.2, works without any change. For Γ of type $A_{+\infty}$, D_∞ the proof goes as the one in [CBH98] for type $\tilde{D}_n, \tilde{E}_n$ if we observe that also in our case Q is a (semi-infinite) tree, the $L \otimes N_i$ are all multiplicity free and the vector δ is the unique vector in $\mathbb{C}^I$, up to scalar multiples, such that $(\delta, \epsilon_i) = 0$ for all $i \in I$.

□

Proof. We give a proof for $n \geq 2$ since the proof for $n = 1$ is similar and easier. Our proof rephrases the proof of Theorem 3.5.2 in [GG05] in the language of algebraic distributions on $\mathbf{\Gamma}_n$. Using equations (A.4)-(A.7) and Lemma $A.0.5$ we can define an isomorphism

$$\begin{aligned} \widehat{T_{\mathcal{B}}\mathcal{E}}\sharp S_n &\to \varphi^{\otimes n}\,(TV\sharp\mathcal{O}(\mathbf{\Gamma}_n)^*)\,\varphi^{\otimes n} \\ e_{i_1}\otimes\cdots\otimes e_{i_n}\cdot\sigma &\to \varphi_{i_1}\otimes\cdots\otimes\varphi_{i_n}\cdot\sigma, \\ e_{i_1}\otimes\cdots\otimes a\otimes\cdots\otimes e_{i_n}\cdot\sigma &\to \varphi_{i_1}\otimes\cdots\otimes\phi_a\otimes\cdots\otimes\varphi_{i_n}\cdot\sigma, \\ e_{i_1}\otimes\cdots\otimes a^*\otimes\cdots\otimes e_{i_n}\cdot\sigma &\to \varphi_{i_1}\otimes\cdots\otimes\theta_a\otimes\cdots\otimes\varphi_{i_n}\cdot\sigma, \end{aligned}$$

for all $i_1, \ldots, i_n \in I$, $a \in Q$.

Let us denote by J the subspace of $TV\sharp\mathcal{O}(\mathbf{\Gamma}_n)^*$ topologically spanned by relations *(R1)*, *(R2)* of Definition 4.4.1. Then the algebra $\mathcal{H}_{k,c}(\mathbf{\Gamma}_n)$ is the quotient of $TV\sharp\mathcal{O}(\mathbf{\Gamma}_n)^*$ by the two-sided ideal generated by J and $\varphi^{\otimes n}\mathcal{H}_{k,c}(\mathbf{\Gamma}_n)\varphi^{\otimes n}$ is the quotient of $\varphi^{\otimes n}TV\sharp\mathcal{O}(\mathbf{\Gamma}_n)^*\varphi^{\otimes n}$ by the ideal

$$\begin{aligned} &\varphi^{\otimes n}\,(TV\sharp\mathcal{O}(\mathbf{\Gamma}_n)^*)\,J\,(TV\sharp\mathcal{O}(\mathbf{\Gamma}_n)^*)\,\varphi^{\otimes n} \\ &= \varphi^{\otimes n}\,(TV\sharp\mathcal{O}(\mathbf{\Gamma}_n)^*)\,\varphi^{\otimes n}\mathcal{O}(\Gamma)^{*\hat{\otimes}n}J\mathcal{O}(\Gamma)^{*\hat{\otimes}n}\varphi^{\otimes n}\,(TV\sharp\mathcal{O}(\mathbf{\Gamma}_n)^*)\,\varphi^{\otimes n} \end{aligned}$$

where the identity holds by equation (A.3). Our claim is that the image of the two sided ideal generated by $\varphi^{\otimes n}\mathcal{O}(\Gamma)^{*\hat{\otimes}n}J\mathcal{O}(\Gamma)^{*\hat{\otimes}n}\varphi^{\otimes n}$ under the above isomorphism is exactly the ideal of the defining relations *(I)*, *(II)* for $\mathcal{A}_{n,\nu,\lambda}(Q)$.

Let us first consider the relations *(R1)*. Notice that for any $\varphi \in \mathcal{O}(\Gamma)^*$ and $x, y \in L$ a symplectic basis, we have that in $TL\sharp\mathcal{O}(\Gamma)^*$

$$\varphi(xy - yx) = (xy - yx)\varphi\,. \tag{A.8}$$

In fact

$$\begin{aligned} &\varphi(xy-yx) \\ &= (x(x^*,\gamma x)\varphi + y(y^*,\gamma x)\varphi)\,y - (y(y^*,\gamma y)\varphi + x(x^*,\gamma y)\varphi)\,x \\ &= xx(x^*,\gamma y)(x^*,\gamma x)\varphi + xy(y^*,\gamma y)(x^*,\gamma x)\varphi + yx(x^*,\gamma y)(y^*,\gamma x)\varphi + yy(y^*,\gamma y)(y^*,\gamma x)\varphi + \\ &- yx(x^*,\gamma x)(y^*,\gamma x)\varphi - yy(y^*,\gamma x)(y^*,\gamma y)\varphi - xy(y^*,\gamma x)(x^*,\gamma y)\varphi - xx(x^*,\gamma x)(x^*,\gamma y)\varphi \\ &= (xy-yx)\,((x^*,\gamma x)(y^*,\gamma y) - (x^*,\gamma y)(y^*,\gamma x))\,\varphi \\ &= (xy-yx)(\det\gamma)\,\varphi \\ &= (xy-yx)\varphi \end{aligned}$$

where the last identity holds since $\det\gamma \equiv 1$ as a function on $SL(2,\mathbb{C})$. Also, since c

is a Γ-invariant distribution, for all $\varphi \in \mathcal{O}(\Gamma)^*$ we have

$$\varphi c = c\varphi \in \mathcal{O}(\Gamma)^*. \tag{A.9}$$

Moreover, if for any $\varphi \in \mathcal{O}(\Gamma)^*$ we write $\varphi_i = \delta_1 \otimes \cdots \otimes \varphi \otimes \cdots \otimes \delta_1 \in \mathcal{O}(\Gamma)^{*\hat{\otimes} n}$, where φ is placed in the ith position, we have that

$$\varphi_i \psi_j \left(\delta_{s_{ij}} \Delta_{ij}\right) = \left(\delta_{s_{ij}} \Delta_{ij}\right) \varphi_i \psi_j \tag{A.10}$$

for any $\varphi, \psi \in \mathcal{O}(\Gamma)^*$ and any $i, j \in [1, n]$. To see that (A.10) holds it is enough to test the right and left hand side of the equality on a decomposable function $\tilde{f}(E^{N_{l_1}}_{p_1 q_1} \otimes \cdots \otimes E^{N_{l_n}}_{p_n q_n})$, where $\tilde{f}$ is a function on S_n. Suppose without loss of generality that $i = 1$, $j = 2$. For the right hand side we have

$$\begin{aligned}
&\langle (\delta_{s_{12}} \Delta_{12})\, \varphi_1 \psi_2, \tilde{f}(E^{N_{l_1}}_{p_1 q_1} \otimes \cdots \otimes E^{N_{l_n}}_{p_n q_n}) \rangle \\
&= \langle \delta_{s_{12}}, \tilde{f} \rangle \left(\langle \varphi, E^{N_{l_1}}_{r_1 q_1} \rangle \langle \psi, E^{N_{l_2}}_{r_2 q_2} \rangle \int_\Gamma E^{N_{l_1}}_{p_1 r_1}(\gamma) E^{N_{l_2}}_{p_2 r_2}(\gamma^{-1})\, d\gamma \right) \langle \delta_1, E^{N_{l_3}}_{p_3 q_3} \rangle \cdots \langle \delta_1, E^{N_{l_n}}_{p_n q_n} \rangle \\
&= \tilde{f}(s_{12}) \left(\langle \varphi, E^{N_{l_1}}_{r_1 q_1} \rangle \langle \psi, E^{N_{l_2}}_{r_2 q_2} \rangle \int_\Gamma E^{N_{l_1}}_{p_1 r_1}(\gamma) E^{N^*_{l_2}}_{r_2 p_2}(\gamma)\, d\gamma \right) E^{N_{l_3}}_{p_3 q_3}(1) \cdots E^{N_{l_n}}_{p_n q_n}(1) \\
&= \delta_{l_1 l_2} \tilde{f}(s_{12}) \frac{1}{d_{l_1}} \langle \psi, E^{N_{l_1}}_{p_1 q_2} \rangle \langle \varphi, E^{N_{l_1}}_{p_2 q_1} \rangle E^{N_{l_3}}_{p_3 q_3}(1) \cdots E^{N_{l_n}}_{p_n q_n}(1)
\end{aligned}$$

while for the left hand side we have

$$\begin{aligned}
&\langle \varphi_1 \psi_2 \left(\delta_{s_{12}} \Delta_{12}\right), \tilde{f}(E^{N_{l_1}}_{p_1 q_1} \otimes \cdots \otimes E^{N_{l_n}}_{p_n q_n} \rangle \\
&= \langle \delta_{s_{12}} \varphi_2 \psi_1 \Delta_{12}, E^{N_{l_1}}_{p_1 q_1} \otimes \cdots \otimes E^{N_{l_n}}_{p_n q_n} \rangle \\
&= \langle \delta_{s_{12}}, \tilde{f} \rangle \left(\langle \psi, E^{N_{l_1}}_{p_1 r_1} \rangle \langle \varphi, E^{N_{l_2}}_{p_2 r_2} \rangle \int_\Gamma E^{N_{l_1}}_{r_1 q_1}(\gamma) E^{N_{l_2}}_{r_2 q_2}(\gamma^{-1})\, d\gamma \right) \langle \delta_1, E^{N_{l_3}}_{p_3 q_3} \rangle \cdots \langle \delta_1, E^{N_{l_n}}_{p_n q_n} \rangle \\
&= \tilde{f}(s_{12}) \left(\langle \psi, E^{N_{l_1}}_{p_1 r_1} \rangle \langle \varphi, E^{N_{l_2}}_{p_2 r_2} \rangle \int_\Gamma E^{N_{l_1}}_{r_1 q_1}(\gamma) E^{N^*_{l_2}}_{q_2 r_2}(\gamma)\, d\gamma \right) E^{N_{l_3}}_{p_3 q_3}(1) \cdots E^{N_{l_n}}_{p_n q_n}(1) \\
&= \delta_{l_1 l_2} \tilde{f}(s_{12}) \frac{1}{d_{l_1}} \langle \psi, E^{N_{l_1}}_{p_1 q_2} \rangle \langle \varphi, E^{N_{l_1}}_{p_2 q_1} \rangle E^{N_{l_3}}_{p_3 q_3}(1) \cdots E^{N_{l_n}}_{p_n q_n}(1)
\end{aligned}$$

where in both cases we used the fact that for any finite dimensional representation N_i, if we choose dual bases, we have $E^{N_i}_{pq}(\gamma^{-1}) = E^{N^*_i}_{qp}(\gamma)$ for any $p, q = 1, \ldots, d_i$. Now using (A.8), (A.9), (A.10), if we denote by J_1 the vector space spanned by relations *(R1)* we see that

$$\varphi^{\otimes n} \mathcal{O}(\Gamma)^{*\hat{\otimes} n} J_1 \mathcal{O}(\Gamma)^{*\hat{\otimes} n} \varphi^{\otimes n} = \varphi^{\otimes n} J_1 \varphi^{\otimes n} \mathcal{O}(\Gamma)^{*\hat{\otimes} n} \varphi^{\otimes n}.$$

Then for any choice of $i_1, \dots, i_n \in I$ and $l \in [1, n]$ we have:

$$\begin{aligned}
&\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_n} \cdot [x_l, y_l] \\
&= [x_l, y_l] \cdot \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_n} \\
&= \varphi_{i_1} \otimes \cdots \otimes \frac{1}{d_{i_l}} \left(\sum_{a \in Q,\, h(a)=i_l} \phi_a \theta_a - \sum_{a \in Q,\, t(a)=i_l} \theta_a \phi_a \right) \otimes \cdots \otimes \varphi_{i_n}
\end{aligned} \tag{A.11}$$

and

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_n} c_l = \frac{\lambda_{i_l}}{d_{i_l}} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_n}. \tag{A.12}$$

Indeed we can write $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_n} c_l = \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_l} c \otimes \cdots \otimes \varphi_{i_n}$, and testing on a function $E^{N_j}_{pq} \in \mathcal{O}(\Gamma)$ we have:

$$\begin{aligned}
&\langle \varphi_{i_l} c, E^{N_j}_{pq} \rangle \\
&= \langle \varphi_{i_l}, E^{N_j}_{pr} \rangle \langle c, E^{N_j}_{rq} \rangle \\
&= \delta_{i_l j} \delta_{p1} \langle c, E^{N_j}_{1q} \rangle = \delta_{i_l j} \delta_{p1} \delta_{q1} \langle c, E^{N_{i_l}}_{11} \rangle \\
&= \frac{1}{d_{i_l}} \delta_{i_l j} \delta_{p1} \delta_{q1} \langle c, \chi_{i_l} \rangle.
\end{aligned}$$

The last identities follow from the fact that c is a Γ-invariant distribution, thus a sum of duals of characters. More precisely

$$c = \sum_i \alpha_i \sum_{j=1}^{d_i} \check{E}^{N_i}_{jj} \qquad \alpha_i \in \mathbb{C}\,,$$

and one has

$$\langle c, E^{N_j}_{pq} \rangle = 0 \qquad \text{if } p \neq q\,,$$

so that

$$\langle c, \chi_{i_l} \rangle = \langle \sum_i \alpha_i \sum_{j=1}^{d_i} \check{E}^{N_i}_{jj}, \sum_{j=1}^{d_{i_l}} E^{N_{i_l}}_{jj} \rangle = d_{i_l} \alpha_{i_l} = d_{i_l} \langle c, E^{N_{i_l}}_{11} \rangle.$$

Now we claim that

$$(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_n})\, \delta_{s_{lj}} \Delta_{lj} \, (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_n}) = \frac{\delta_{i_j i_l}}{d_{i_l}} \, (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_n})\, \delta_{s_{lj}} \tag{A.13}$$

Indeed, supposing without loss of generality that $l = 1$ $j = 2$, and testing the left hand side on a decomposable function $\tilde{f}\left(E^{N_{l_1}}_{p_1q_1}\otimes\cdots\otimes E^{N_{l_n}}_{p_nq_n}\right)$ we get:

$$\begin{aligned}
&\langle(\varphi_{i_1}\otimes\varphi_{i_2}\otimes\cdots\otimes\varphi_{i_n})\,\delta_{s_{12}}\Delta_{12},\tilde{f}\left(E^{N_{l_1}}_{p_1q_1}\otimes\cdots\otimes E^{N_{l_n}}_{p_nq_n}\right)\rangle\\
&=\langle\delta_{s_{12}}\,(\varphi_{i_2}\otimes\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_n})\,\Delta_{12},\tilde{f}\left(E^{N_{l_1}}_{p_1q_1}\otimes\cdots\otimes E^{N_{l_n}}_{p_nq_n}\right)\rangle\\
&=\tilde{f}(s_{12})\langle\varphi_{i_2},E^{N_{l_1}}_{p_1r_1}\rangle\langle\varphi_{i_1},E^{N_{l_2}}_{p_2r_2}\rangle\left(\int_\Gamma E^{N_{l_1}}_{r_1q_1}(\gamma)E^{N^*_{l_2}}_{q_2r_2}(\gamma)\,d\gamma\right)\prod_{j\geq 3}\langle\varphi_{i_j},E^{N_{l_j}}_{p_jq_j}\rangle\\
&=\tilde{f}(s_{12})\delta_{i_1i_2}\prod_{j\geq 1}\delta_{i_jl_j}\prod_{j\geq 1}\delta_{p_j1}\prod_{j\geq 3}\delta_{q_j1}\int_\Gamma E^{N_{i_1}}_{1q_1}(\gamma)E^{N^*_{i_1}}_{q_21}(\gamma)\,d\gamma\\
&=\tilde{f}(s_{12})\frac{\delta_{i_1i_2}}{d_{i_1}}\prod_{j\geq 1}\delta_{i_jl_j}\prod_{j\geq 1}\delta_{p_j1}\prod_{j\geq 1}\delta_{q_j1}\\
&=\frac{\delta_{i_1i_2}}{d_{i_1}}\langle(\varphi_{i_1}\otimes\varphi_{i_2}\otimes\cdots\otimes\varphi_{i_n})\,\delta_{s_{12}},\tilde{f}\left(E^{N_{l_1}}_{p_1q_1}\otimes\cdots\otimes E^{N_{l_n}}_{p_nq_n}\right)\rangle.
\end{aligned}$$

By (A.11), (A.12), (A.13) we thus have that relations of type *(R1)* give us exactly the relations *(I)* in Definition 5.5.3.

We will now find the relations that are given by *(R2)*. We will assume without loss of generality that $n = 2$. First of all, for any $u, v \in L$ and any $\varphi, \psi \in \mathcal{O}(\Gamma)^*$, if x, y is any basis for L, we can easily see that

$$\begin{aligned}
&(\varphi\otimes\psi)\cdot[u_1,v_2]\\
&=[x_1,x_2]\,((x^*,hu)\varphi\otimes(x^*,gv)\psi)+[x_1,y_2]\,((x^*,hu))\varphi\otimes(y^*,gv)\psi)\\
&+[y_1,x_2]\,((y^*,hu))\varphi\otimes(x^*,gv)\psi)+[y_1,y_2]\,((y^*,hu))\varphi\otimes(y,gv)\psi)
\end{aligned}\tag{A.14}$$

and similarly

$$\begin{aligned}
&(\varphi\otimes\psi)\,\delta_{s_{12}}\omega_L(\gamma u,v)\Delta_{12}\\
&=\delta_{s_{12}}\omega_L(\gamma x,x)\Delta_{12}\,((x^*,hu)\varphi\otimes(x^*,gv)\psi)+\delta_{s_{12}}\omega_L(\gamma x,y)\Delta_{12}\,((x^*,hu)\varphi\otimes(y^*,gv)\psi)\\
&+\delta_{s_{12}}\omega_L(\gamma y,x)\Delta_{12}\,((y^*,hu)\varphi\otimes(x^*,gv)\psi)+\delta_{s_{12}}\omega_L(\gamma y,y)\Delta_{12}\,((y^*,hu)\varphi\otimes(y^*,gv)\psi)\,.
\end{aligned}\tag{A.15}$$

To prove this last identity we first have to introduce some more notation. For $\varphi, \psi \in \mathcal{O}(\Gamma)^*$ we will write φ_h, ψ_g to indicate the variable with respect to which these distributions are considered (so φ is a linear functional on functions in the variable h etc...). Since the value of the distribution Δ_{12} on any function $f_1 \otimes f_2 \in \mathcal{O}(\Gamma)^{\otimes 2}$ can be written as

$$\langle\Delta_{12},f_1\otimes f_2\rangle=\int_{\substack{\Gamma\times\Gamma\\ g=h}} f_1(g)f_2(h^{-1})\,dg\,dh\,.$$

We will write $\Delta_{12,(g\,h)}$ to keep track of the variables. Finally for $E^{N_{l_1}}_{p_1q_1}\otimes E^{N_{l_2}}_{p_2q_2}\in\mathcal{O}(\Gamma)^{\otimes 2}$

we will write

$$\Delta(E^{N_{l_1}}_{p_1q_1} \otimes E^{N_{l_2}}_{p_2q_2})(g,h,g',h') = E^{N_{l_1}}_{p_1r_1}(g) \otimes E^{N_{l_2}}_{p_2r_2}(h) \otimes E^{N_{l_1}}_{r_1q_1}(g') \otimes E^{N_{l_2}}_{r_2q_2}(h')$$

for the coproduct. Let us now consider the decomposable function $f = \tilde{f}(E^{N_{l_1}}_{p_1q_1} \otimes E^{N_{l_2}}_{p_2q_2})$, where $\tilde{f}$ is a function on S_n. We have

$$\begin{aligned} &\langle(\varphi\otimes\psi)\,\delta_{s_{12}}\omega_L(\gamma u,v)\Delta_{12}\ ,\ f\rangle \\ &= \tilde{f}(s_{12})\langle(\psi_g\otimes\varphi_h)\left(\omega_L(g'u,v)\Delta_{12,(g'\,h')}\right),\ \Delta(E^{N_{l_1}}_{p_1q_1}\otimes E^{N_{l_2}}_{p_2q_2})(g,h,g',h')\rangle \\ &= \tilde{f}(s_{12})\langle\psi_g\varphi_h\,,\ E^{N_{l_1}}_{p_1r_1}(g)E^{N_{l_2}}_{p_2r_2}(h)\rangle\langle\Delta_{12,(g'\,h')}\,,\ \omega_L(g'u,v)E^{N_{l_1}}_{r_1q_1}(g')E^{N_{l_2}}_{r_2q_2}(h')\rangle\,. \end{aligned}$$

Now making the change of variable $(g',h') = (g^{-1}\tilde{g}h, \gamma^{-1}\tilde{h}h)$ and using the fact that the integral is left and right translation invariant we get

$$\tilde{f}(s_{12})\langle\psi_g\varphi_h\,,\ E^{N_{l_1}}_{p_1r_1}(g)E^{N_{l_2}}_{p_2r_2}(h)\rangle\langle\Delta_{12,(g'\,h')}\,,\ \omega_L(g^{-1}\tilde{g}hu,v)E^{N_{l_1}}_{r_1q_1}(g^{-1}\tilde{g}h)E^{N_{l_2}}_{r_2q_2}(h^{-1}\tilde{h}g)\rangle\,. \tag{A.16}$$

We now observe that:

$$\begin{aligned} &\omega_L(g^{-1}\tilde{g}hu,v) = \omega_L(\tilde{g}hu,gv) \\ &= \overbrace{(x^*,hu)(x^*,gv)\omega_L(\tilde{g}x,x)}^{1} + \overbrace{(x^*,hu)(y^*,gv)\omega_L(\tilde{g}x,y)}^{2} \\ &\quad + \overbrace{(y^*,hu)(x^*,gv)\omega_L(\tilde{g}y,x)}^{3} + \overbrace{(y^*,hu)(y^*,gv)\omega_L(\tilde{g}y,y)}^{4} \end{aligned} \tag{A.17}$$

and that

$$E^{N_{l_1}}_{r_1q_1}(g^{-1}\tilde{g}h)E^{N_{l_2}}_{r_2q_2}(h^{-1}\tilde{h}^{-1}g) \tag{A.18}$$

$$= E^{N_{l_1}}_{r_1s_1}(g^{-1})E^{N_{l_1}}_{s_1t_1}(\tilde{g})E^{N_{l_1}}_{t_1q_1}(h)E^{N_{l_2}}_{r_2s_2}(h^{-1})E^{N_{l_2}}_{s_2t_2}(\tilde{h}^{-1})E^{N_{l_2}}_{t_2q_2}(g)\,. \tag{A.19}$$

Using (A.17) we can rewrite (A.16) as a sum of four terms. If we use (A.19) to rewrite the first of these terms, for example, we get

$$\begin{aligned} &\tilde{f}(s_{12})\langle\psi_g\varphi_h\,,\ (x^*,hu)(x^*,gv)E^{N_{l_1}}_{p_1r_1}(g)E^{N_{l_1}}_{r_1s_1}(g^{-1})E^{N_{l_1}}_{t_1q_1}(h)E_{p_2,r_2}(h)E^{N_{l_2}}_{r_2s_2}(h^{-1})E^{N_{l_2}}_{t_2q_2}(g)\rangle\cdot \\ &\qquad\qquad \cdot\langle\omega_L(\tilde{g}x,x)\Delta_{12,(\tilde{g}\,\tilde{h})}\,,\ E^{N_{l_1}}_{s_1t_1}(\tilde{g})E^{N_{l_2}}_{s_2t_2}(\tilde{h})\rangle \\ &= \tilde{f}(s_{12})\langle\psi_g\varphi_h\,,\ (x^*,hu)(x^*,gv)E^{N_{l_1}}_{p_1s_1}(e)E^{N_{l_1}}_{t_1q_1}(h)E_{p_2,s_2}(e)E^{N_{l_2}}_{t_2q_2}(g)\rangle\cdot \\ &\qquad\qquad \cdot\langle\omega_L(\tilde{g}x,x)\Delta_{12,(\tilde{g}\,\tilde{h})}\,,\ E^{N_{l_1}}_{s_1t_1}(\tilde{g})E^{N_{l_2}}_{s_2t_2}(\tilde{h})\rangle \\ &= \tilde{f}(s_{12})\langle\psi_g\varphi_h\,,\ (x^*,hu)(x^*,gv)E^{N_{l_1}}_{t_1q_1}(h)E^{N_{l_2}}_{t_2q_2}(g)\rangle\cdot \\ &\qquad\qquad \cdot\langle\omega_L(\tilde{g}x,x)\Delta_{12,(\tilde{g}\,\tilde{h})}\,,\ E^{N_{l_1}}_{p_1s_1}(e)E^{N_{l_1}}_{s_1t_1}(\tilde{g})E_{p_2,s_2}(e)E^{N_{l_2}}_{s_2t_2}(\tilde{h})\rangle \end{aligned}$$

$$= \tilde f(s_{12})\langle \psi_g\varphi_h\,,\,(x^*,hu)(x^*,gv)E^{N_{l_1}}_{t_1q_1}(h)E^{N_{l_2}}_{t_2q_2}(g)\rangle\cdot\langle \omega_L(\tilde g x,x)\Delta_{12,(\tilde g\,\tilde h)}, E^{N_{l_1}}_{p_1t_1}(\tilde g)E_{p_2,t_2}(\tilde h)\rangle$$
$$= \tilde f(s_{12})\langle\Big(\omega_L(\tilde g u,v)\Delta_{12,(\tilde g\,\tilde h)}\Big)\,((x^*,hu)\varphi_h)\,((x^*,gv)\psi_g)\;,\;\Delta(E^{N_{l_1}}_{p_1q_1}\otimes E^{N_{l_2}}_{p_2q_2})(\tilde g,\tilde h,h,g)\rangle$$
$$= \langle \delta_{s_{12}}\,((x^*,hu)\varphi\otimes(x^*,gu)\psi)\,\omega_L(\gamma x,v)\Delta_{12}\;,\;f\rangle$$

where we just used the properties of the coproduct and counit (evaluation at the identity). It is of course possible to rewrite the remaining three terms in a similar way, so that we get exactly expression (A.15).

Now for any $i,j,k,l\in I$ an easy computation shows that, via the identification $TV\sharp\mathcal{O}(\Gamma)^{*\hat\otimes 2}\cong T_{\mathcal{O}(\Gamma)^{*\hat\otimes 2}}(V\otimes\mathcal{O}(\Gamma)^{*\hat\otimes 2})$, we have

$$\begin{aligned}&(\varphi_i\otimes\varphi_j)\,(\varphi\otimes\delta_1)\,[u_1,v_2]\,(\delta_1\otimes\psi)\,(\varphi_k\otimes\varphi_l)\\ &=(\varphi_i\varphi\otimes\varphi_j)\,(u_1\otimes(\varphi_k\otimes\varphi_j))\bigotimes(\varphi_k\otimes\varphi_j)\,(v_2\otimes(\varphi_k\otimes\psi\varphi_l))\\ &\quad-(\varphi_i\otimes\varphi_j)\,(v_2\otimes(\varphi_i\otimes\psi\varphi_l))\bigotimes(\varphi_i\varphi\otimes\varphi_l)\,(u_1\otimes(\varphi_k\otimes\varphi_l))\end{aligned}\tag{A.20}$$

where $\bigotimes$ denotes the product in $T_{\mathcal{O}(\Gamma)^{*\hat\otimes 2}}(V\otimes\mathcal{O}(\Gamma)^{*\hat\otimes 2})$, and we can see (A.20) as an identity between algebraic distributions on Γ^2 with values in T^2V. On the other hand we trivially have that

$$\begin{aligned}&(\varphi_i\otimes\varphi_j)\,(\varphi\otimes\delta_1)\,(\delta_{s_{12}}\omega_L(\gamma u,v)\Delta_{12})\,(\delta_1\otimes\psi)\,(\varphi_k\otimes\varphi_l)\\ &=\delta_{s_{12}}\,(\varphi_j\otimes\varphi_i\varphi)\,\omega_L(\gamma u,v)\Delta_{12}\,(\varphi_k\otimes\psi\varphi_l)\;.\end{aligned}\tag{A.21}$$

As in [GG05] we observe now that for any arrow $a\in\overline{Q}$ we can find distributions $\varphi_a,\psi_a\in\ \mathcal{O}(\Gamma)^{*\hat\otimes n}$ and vectors $u_a,v_a\in L$ such that

$$\varphi_{t(a)}\varphi_a\left(u_a\otimes\varphi_{h(a)}\right)\neq 0\qquad\text{and}\qquad\varphi_{h(a)}\left(v_a\otimes\psi_a\varphi_{t(a)}\right)\neq 0.$$

Also in our case Q has no loop vertices, thus we have that the spaces $\varphi_i\Big(L\otimes\mathcal{O}(\Gamma)^{*\hat\otimes 2}\Big)\varphi_j$ are at most one dimensional and for any $i,j\in I$ we have an identification:

$$\left(\varphi_i\mathcal{O}(\Gamma)^{*\hat\otimes 2}\otimes L\otimes\mathcal{O}(\Gamma)^{*\hat\otimes 2}\varphi_j\right)^{\Gamma}\to\varphi_i\left(L\otimes\mathcal{O}(\Gamma)^{*\hat\otimes 2}\right)\varphi_j$$

$$\alpha\otimes u\otimes\beta\to\alpha\,(u\otimes\beta)$$

where $\varphi_i\mathcal{O}(\Gamma)^{*\hat\otimes 2}\otimes L\otimes\varphi_j\mathcal{O}(\Gamma)^{*\hat\otimes 2}\cong N_i^*\otimes L\otimes N_j$ as Γ-modules. Moreover again as in [GG05] we have a non degenerate Γ-equivariant pairing

$$\left(\varphi_i\mathcal{O}(\Gamma)^{*\hat\otimes 2}\otimes L\otimes\mathcal{O}(\Gamma)^{*\hat\otimes 2}\varphi_j\right)\bigotimes\left(\varphi_j\mathcal{O}(\Gamma)^{*\hat\otimes 2}\otimes L\otimes\mathcal{O}(\Gamma)^{*\hat\otimes 2}\varphi_i\right)\to\mathbb{C}$$

$$(\alpha\otimes u\otimes\beta)\bigotimes(\alpha'\otimes u\otimes\beta')\to(\alpha\beta')(\alpha'\beta)\omega_L(u,u').$$

As a consequence, we can assume that, for any $a\in\overline{Q}$, we have $\omega_L(u_a,v_a)=1$.

Moreover $\varphi_{t(a)}\varphi\left(v_a \otimes \varphi_{h(a)}\right) = 0$ if $\varphi_{h(a)}\left(v_a \otimes \psi\varphi_{t(a)}\right) \neq 0$, and $\varphi_{h(a)}\left(u_a \otimes \psi\varphi_{t(a)}\right) = 0$ if $\varphi_{t(a)}\varphi\left(u_a \otimes \varphi_{h(a)}\right) \neq 0$.

Note that if $i \neq l$ or $j \neq k$ the expression (A) is zero. To see this, let us evaluate the distribution $(\varphi_j \otimes \varphi_i\varphi)\,\omega_L(\gamma u, v)\Delta_{12}\,(\varphi_k \otimes \psi\varphi_l)$ on a function $E^{N_{l_1}}_{p_1q_1} \otimes E^{N_{l_2}}_{p_2q_2}$. We have

$$\begin{aligned} &\langle(\varphi_j \otimes \varphi_i\varphi)\,\omega_L(\gamma u, v)\Delta_{12}\,(\varphi_k \otimes \psi\varphi_l)\ ,\ E^{N_{l_1}}_{p_1q_1} \otimes E^{N_{l_2}}_{p_2q_2}\rangle \\ =&\ \langle\varphi_j\ ,\ E^{N_{l_1}}_{p_1r_1}\rangle\langle\varphi_i\varphi\ ,\ E^{N_{l_2}}_{p_2r_2}\rangle\langle\omega_L(\gamma u, v)\Delta_{12}\ ,\ E^{N_{l_1}}_{r_1s_1} \otimes E^{N_{l_2}}_{r_2s_2}\rangle\langle\varphi_k\ ,\ E^{N_{l_1}}_{s_1q_1}\rangle\langle\psi\varphi_l\ ,\ E^{N_{l_2}}_{s_2q_2}\rangle\,. \end{aligned}$$

Since last expression is zero if $j, k \neq l_1$ and $i, l \neq l_2$ the above distribution is identically 0 if $j \neq k$ or $i \neq l$.

Thus, if $a, b \in \overline{Q}$ are two arrows such that $b \neq a^*$ or $a \neq b^*$ we get from (A.20), (A) and *(R2)* that

$$(a \otimes h(b))(t(a) \otimes b) - (h(a) \otimes b)(a \otimes t(b)) = 0\,.$$

Suppose now $j = k$ and $i = l$. Consider an edge $a : i \to j$ in $\overline{Q}$ and suppose, for simplicity, $a \in Q$. We have an injection as an irreducible factor $\theta_a : N_i \hookrightarrow L \otimes N_j$. We can choose a basis $\xi := \{\xi_i\}$ of $L \otimes N_j = N_i \oplus \dots$ adapted to this decompositions into irreducibles

$$\xi_1 := \varphi_i = E^{N_i}_{11}, \quad \xi_1 := E^{N_i}_{21}, \quad \xi_3 := E^{N_i}_{31}, \quad \dots, \quad \xi_{d_i} := E^{N_i}_{d_i1}, \quad \dots\,.$$

On the other hand we can choose a basis $\mu := \{\mu_i\}$ for $L \otimes N_j$ adapted to the tensor product

$$\mu_1 := u_a \otimes \varphi_j = u_a \otimes E^{N_j}_{11}, \quad \mu_2 = u_a \otimes E^{N_j}_{21}, \quad \dots, \quad \mu_{2d_j} := v_a \otimes E^{N_j}_{d_j1}.$$

Let's now define the matrix $\tau = (\tau_{pq})$ by $\varphi\mu_q = \sum_p \tau_{pq}\xi_p$ and the matrix $\rho = (\rho_{pq})$ by $\psi\xi_q = \sum_p \rho_{pq}\mu_q$. In other words we have $\tau = {}_\xi\varphi_\mu$, where ${}_\xi\varphi_\mu = ({}_{\xi_p}\varphi_{\mu_q})$ denotes the matrix representing the linear map induced by φ on $L \otimes N_j$ if we choose the basis μ for the domain and ξ for the image. Similarly we have $\rho = {}_\mu\psi_\xi$. Now, recalling that we are using the following identifications

$$N_j \overset{\phi_a}{\hookrightarrow} L \otimes N_j \overset{\vartheta_a}{\hookrightarrow} L \otimes L \otimes N_j \overset{\omega_L \otimes 1}{\to} N_j\,,$$

and that by Lemma $A.0.5$ this composition of morphisms equals $d_j\mathrm{Id}_{N_j}$, we have that

$$\varphi_i\varphi\,(u_a \otimes \varphi_j) = \tau_{11}\varphi_i \qquad \text{and} \qquad \varphi_j\,(v_a \otimes \psi\varphi_i) = -\frac{\rho_{11}}{d_i}\varphi_j\,. \tag{A.22}$$

We now claim that

$$(\varphi_j \otimes \varphi_i\varphi)\,(\omega_L(\gamma u_a, v_a)\Delta_{12})\,(\varphi_j \otimes \psi\varphi_i) = \frac{\tau_{11}\rho_{11}}{d_i}\varphi_j \otimes \varphi_i\,. \tag{A.23}$$

First of all it's easy to see that

$$(\varphi_j \otimes \varphi_i\varphi)\,(\omega_L(\gamma u_a, v_a)\Delta_{12})\,(\varphi_j \otimes \psi\varphi_i) = C\varphi_j \otimes \varphi_i\,,$$

where C is some constant. To compute C we will evaluate the left hand side of (A.23) on the function $E_{11}^{N_j} \otimes E_{11}^{N_i} \in \mathcal{O}(\Gamma)^{\otimes 2}$. We recall that we can see the functions $E_{pq}^{N_i}(\gamma)$ as the matrix coefficients for the action of γ on the direct factor $N_i \subset L \otimes N_j$ in the basis ξ and the functions $E_{rs}^{L}(\gamma)E_{pq}^{N_j}(\gamma)$ as the matrix coefficients for γ on $L \otimes N_j$ in the basis μ. We define the matrix $\alpha = \{\alpha_{pq}\}$ as the matrix of the change of basis $\mu_q = \sum_p \alpha_{pq}\xi_p$ and by $\tilde{\alpha} = (\tilde{\alpha}_{pq})$ its inverse. Accordingly to the previous notation we write ${}_\xi\varphi_\xi = ({}_{\xi_q}\varphi_{\xi_p})$ (respectively ${}_\xi\psi_\xi = ({}_{\xi_q}\psi_{\xi_p})$) for the matrix of the linear map φ (respectively ψ) where we chose the basis ξ both for the domain and the image.

$$\begin{aligned}
&\langle(\varphi_j \otimes \varphi_i\varphi)\,(\omega_L(\gamma u_a, v_a)\Delta_{12})\,(\varphi_j \otimes \psi\varphi_i)\ ,\ E_{11}^{N_j} \otimes E_{11}^{N_i}\rangle \\
&= \sum_{r,p=1}^{d_j}\sum_{r',p'=1}^{d_i} \langle \check{E}_{11}^{N_j}, E_{1r}^{N_j}\rangle\langle\varphi, E_{1r'}^{N_i}\rangle \left(\int_\Gamma \omega_L(\gamma u_a, v_a)E_{rp}^{N_j}(\gamma)E_{r'p'}^{N_i}(\gamma^{-1})\,d\gamma\right)\cdot \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cdot\langle \check{E}_{11}^{N_j}, E_{p1}^{N_j}\rangle\langle\psi, E_{p'1}^{N_i}\rangle \\
&= \sum_{r',p'=1}^{d_i} \langle\varphi, E_{1r'}^{N_i}\rangle \left(\int_\Gamma \omega_L(\gamma u_a, v_a)E_{11}^{N_j}(\gamma)E_{p'r'}^{N_i}(\gamma^{-1})\,d\gamma\right)\langle\psi, E_{p'1}^{N_i}\rangle \\
&= \sum_{r',p'=1}^{d_i} \langle\varphi, E_{1r'}^{N_i}\rangle \left(\int_\Gamma E_{11}^{L}(\gamma)E_{11}^{N_j}(\gamma)E_{p'r'}^{N_i}(\gamma^{-1})\,d\gamma\right)\langle\psi, E_{p'1}^{N_i}\rangle \\
&= \frac{1}{d_i}\sum_{r',p'=1}^{d_i} \langle\varphi, E_{1r'}^{N_i}\rangle\langle\psi, E_{p'1}^{N_i}\rangle\langle\check{E}_{p'r'}^{N_i}, E_{11}^{L}(\gamma)E_{11}^{N_j}(\gamma)\rangle \\
&= \frac{1}{d_i}\sum_{r',p'=1}^{d_i} \langle\varphi, E_{1r'}^{N_i}\rangle\langle\psi, E_{p'1}^{N_i}\rangle\langle\check{E}_{p'r'}^{N_i}, \sum_{s,t=1}^{d_i}\tilde{\alpha}_{1s}\alpha_{t1}E_{st}^{N_i}\rangle \\
&= \frac{1}{d_i}\sum_{r',p'=1}^{d_i} \langle\varphi, E_{1r'}^{N_i}\rangle\langle\psi, E_{p'1}^{N_i}\rangle\tilde{\alpha}_{1p'}\alpha_{r'1} \\
&= \frac{1}{d_i}\left(\sum_{p'=1}^{d_i}\langle\psi, E_{p'1}^{N_i}\rangle\tilde{\alpha}_{1p'}\right)\left(\sum_{r'=1}^{d_i}\langle\varphi, E_{1r'}^{N_i}\rangle\alpha_{r'1}\right) \\
&= \frac{1}{d_i}\left(\sum_{p'=1}^{d_i}{}_{\xi_{p'}}\psi_{\xi_1}\tilde{\alpha}_{1p'}\right)\left(\sum_{r'=1}^{d_i}{}_{\xi_1}\varphi_{\xi_{r'}}\alpha_{r'1}\right) = \frac{\rho_{11}\tau_{11}}{d_i}
\end{aligned}$$

where the last identity holds since $\rho = {}_\mu\psi_\xi = \alpha^{-1}\,{}_\xi\psi_\xi = \tilde{\alpha}\,{}_\xi\psi_\xi$ and $\tau = {}_\xi\varphi_\mu = {}_\xi\psi_\xi\,\alpha$.

Thus we have that $C = \frac{\rho_{11}a\tau_{11}}{d_i}$ and the identity (A.23) holds. So now taking $i = l$, $j = k$ $u = u_a, v = v_a, \varphi = \varphi_a, \psi = \psi_a$ in (A.20) and (A.23), and using (A.22), we have that relation *(R2)* gives us exactly

$$(a^* \otimes h(a))\,(h(a) \otimes a) - (t(a) \otimes a)\,(a^* \otimes t(a)) = 2k\delta_{s_{12}}\,(h(a) \otimes t(a))$$

since $\tau_{11}, \rho_{11} \neq 0$ in this case as observed above. Also taking $u = u_a, v = v_a$ in (A.20) and (A.23) we have that, if $\varphi_i \varphi\,(u_a \otimes \varphi_j) \neq 0$, then $\varphi_j\,(u_a \otimes \psi\varphi_j) = 0$ and so $\rho_{d_j+1} = 0$ (see (A.22)) and both sides of *(R2)* give zero. The same is true if we exchange the roles of u_a and v_a. Thus the relations *(R2)* give exactly the relations *(II)* of Definition 5.5.3.

□

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