THE LARGE ORDER BEHAVIOR OF PERTURBATION THEORY

by

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ABSTRACT

We develop methods for computing the large order behavior of the Rayleigh Schroedinger perturbation series for the energy eigenvalues of a quantum mechanical system. In particular, we study systems of coupled anharmonic oscillators. A dispersion relation in the coupling constant is derived which converts the calculation into a tunneling problem which is then solved by semi-classical methods. Entirely new multidimensional WKB techniques are introduced and used to study systems of coupled anharmonic oscillators. If the unperturbed oscillator system is isotropic, then the exact large-order behavior of the perturbation series may be computed analytically. A perturbation scheme is developed to deal with small anisotropies in the unperturbed system. Finally, we discuss the coupled oscillator systems which arise from cutoff quantum field theories. Here the unperturbed system has large anisotropies. Extensive computer calculations have been performed to verify our theoretical predictions, and excellent agreement is found.

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I would like to express my thanks to my adviser, Professor Carl Bender, for his constant aid and encouragement during the course of this work. That I learned an enormous amount of mathematics from him goes without saying, but more importantly, I learned something about perseverance and the place of plain hard work in theoretical research. Not a few of the calculations in this thesis would have been abandoned in despair if it had not been for Carl's insistence that we would soon see light at the end of the tunnel (to coin a phrase).

I have also benefitted from the sage advice of Professors Hung Cheng and T. T. Wu. It should be pointed out, however, that I have had little direct contact with Wu. Our interaction was mediated primarily by the exchange of a Bender.

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CHAPTER I

INTRODUCTION

In an earlier and happier time in the history of physics, a proposal to investigate the convergence of a perturbation series would probably have been met with polite laughter. Many important physical problems were, in that fortunate era, exactly soluble and even when approximation proved necessary, the correct answer could usually be obtained from the first few terms of a simple and intuitive perturbation scheme. Why then pursue a question of convergence up some mathematical back alley while so much good physics was passing by in the main street?

Unfortunately, the modern theoretical physicist can no longer afford the luxury of these comfortable metaphors. Interesting physics now lurks in the back alleys beyond the limits of the Born approximation. It behooves us then to make some attempt to tame these uncivilized regions, and to discover the inner simplicity behind their rank jungles of indices. We will see in this thesis that at least in the problem of the large order behavior of perturbation theory results can be obtained which compare in beauty and simplicity with any calculation based on the Born approximation.

The area of physics in which knowledge of the convergence properties of the perturbation series would be most useful is

the theory of elementary particles.

Nowhere is the insufficiency of the Born approximation more evident; and here the problem is compounded with two others of equal magnitude. There exist no convincing non-perturbative approximation methods for most of the problems that arise in elementary particle physics, and there are no nontrivial model theories which satisfy all of the basic physical principles (Lorentz Invariance, Unitarity, Crossing etc.) which one believes to be operative in this field. In fact, the consistency of these principles has been challenged more than once in the past forty years.[1]

It is not suprising then that several important problems in elementary particle physics lead one to investigate the behavior of perturbation theory in more detail than has previously been deemed necessary.

I do not have the time here to discuss all of these questions but one of them is of sufficient importance to merit some explanation. This is the failure of field theoretic perturbation theory to reproduce the scaling behavior of deep inelastic scattering which has been observed at SLAC.[2] I believe that this is the first time in the history of field theory that such a point of qualitative disagreement with experiment has been reached. [Of course one can always argue that the observed scaling is a transitory, pre-asymptotic

phenomenon which does not contradict the predictions of perturbation theory[3]]. All realistic field theories when evaluated to any finite order of perturbation theory beyond the Born approximation produce conflict with (the extropolation to infinite energy of) the results of the SLAC experiments.

There appear to be two ways out of this dilemma. One must either produce a mechanism which assures the dominance of the Born approximation at high energy despite the strength of the coupling[4], or one must show that the theory's large energy behavior when summed to all orders is very different from that in any finite order. It has long been known that this latter alternative can occur if the bare coupling constant satisfies the so-called Gell-Mann-Low eigenvalue condition.[5] This hypothesis is clearly nonperturbative in nature.

Recently, Adler[6] has shown that if the Gell-Mann - Low eigenvalue exists, then it is an essential singularity of all of the Green's functions of the field theory. Since an essential singularity cannot be detected in any finite order of perturbation theory, a study of the large order behavior of the series seems indicated.

Previous work on this question has mostly been devoted to obtaining bounds on the n^{th} term in the perturbation series for superrenormalizable field theries. These bounds showed

that many of these series diverged. (For detailed references, see the papers of Jaffee[7] and Simon[8]). However, the bounds are not sufficiently strong to determine the nature of the singularity which causes the divergence. In this thesis I will present a method for computing the exact large order behavior of a quantum mechanical perturbation series. This enables one to determine the exact radius of convergence of the series, and suggests qualitative nature of any singularity which causes the series to diverge. The method is a generalization of one invented by Bender and Wu in their study of the one-dimensional anharmonic oscillator.[9]

I have initially set my sights on the ϕ^4 field theory (and other theories whose interactions are polynomials in Bose fields). If spatial and ultraviolet cutoffs are imposed on this model it becomes equivalent[10] to a finite set of coupled anharmonic oscillators. One would hope to compute the large order behavior of perturbation theory for this oscillator system as a function of the cutoffs and then study the limit as the cutoffs go to infinity. Although I have not yet achieved this goal I have made substantial progress toward it. The methods introduced in this thesis enables one to compute the large order behavior of the perturbation series for the energy eigenvalues of essentially any Hamiltonian of the form

$$H = \sum_{i=1}^{N} Pi^2 + m^2 x_i^2 + \lambda P(x) \qquad (1.1)$$

where P(x) is an arbitrary even polynomial. (The restriction to even polynomials is merely a convenience, we can deal with arbitrary polynomials as long as the leading coefficient is even). Eq. [1.1] differs from the Hamiltonians encountered in cutoff field theories in one important respect: the unperturbed oscillator system is completely isotropic (or as we shall say, it is an equal mass oscillator). In Chpater V we will show how to deal with "small" anisotropies in the unperturbed system but the anisotropy in a cutoff field theory cannot be considered small. The problem of anisotropy (unequal mass oscillators) is thus the major obstacle which stands in our path. It should also be noted however, that we are computing eigenvalues while the real objects of interest in quantum field theory are scattering amplitudes. I will make some remarks about this more difficult problem in Chapter VII.

The remainder of this thesis is organized as follows: In Chapter II we discuss analytic properties of the eigenvalues as functions of the coupling constant. We derive an important dispersion relation which enables us to convert the problem of computing the large order behavior of perturbation theory into a barrier penetration problem. The dispersion relation

also provides us with a remarkable quantitative restatement of Dyson's famous argument about the divergence of perturbation theory.

In Chapter III we compute the large order behavior of the perturbation series for a one-dimensional anharmonic oscillator with polynomial self interactions. This simple system serves as an introduction to WKB methods and helps us to motivate certain computational shortcuts which are very useful in multidimensional problems. It also gives us an understanding of the manner in which "mass renormalization" and "Wick ordering" may affect the large-order behavior of perturbation theory.

Chapter IV begins the main body of the thesis. We give a general discussion of barrier penetration problems in N dimensions, and argue that in the limit of a very high and very wide barrier, the escaping probability current is concentrated along a particular trajectory. This trajectory, which is a solution of the classical equations of motion, is called the "most probable escape path" (MPEP). We then study systems of coupled equal mass anharmonic oscillators and show that their MPEP's are straight lines. This enables us to compute the exact large order behavior of the perturbation series for these systems.

In Chapter V we attack the problem of unequal mass oscillators. We first derive a formula which completely determines the large order behavior of perturbation theory once the MPEP is known. Thus we are left with the completely classical problem of finding the MPEP. We then show how to solve this problem perturbatively when the MPEP is almost a straight line. Finally we apply these techniques to a simple two-dimensional oscillator system.

The arguments presented throughout this thesis are highly nonrigorous. For this reason I have done extensive computer calculations to compare with my predictions of the large order behavior of perturbation theory. These calculations are described in Chapter VI and the results are tabulated. Excellent agreement between computer and theoretical calculations is found.

The seventh and final chapter is devoted to conclusions and speculations. I discuss the possible extension of the present work to Green's functions, real field theories and theories with Fermions.

The appendices contain several special calculations not covered inthe text, and some general theorems about the MPEP's for equal mass oscillators.

CHAPTER II

ANALYTIC PROPERTIES OF EIGENVALUES AND THE DISPERSION REPRESENTATION FOR THE RAYLEIGH SCHROEDINGER COEFFICIENTS

A. Introduction and Analytic Continuation of $E(\lambda)$

The Rayleigh Schroedinger perturbation series for a general eigenvalue problem is generated by a recursion relation which determines the nth order contribution to the energy in terms of its predecessors and of integrals over all the lower order contributions to the wave function. For the systems which we will be studying in this thesis, (N dimensional harmonic oscillators with polynomial perturbations) it is possible to reduce the general recursion relation to a single nonlinear second order partial difference equation in N+1 dimensions. In one dimension the problem of finding the large order behavior of perturbation theory can be solved by a direct attack on this difference equation, but in many dimensions the relevant approximation methods become unwieldy.

We will therefore proceed by using a technique discovered by Bender and Wu,[1]: we convert the computation of the large order behavior of perturbation theory to a barrier penetration problem. Such a reformulation of the problem is very important

because it allows us to apply physical intuition to what originally appeared to be a purely mathematical problem. The crucial mathematical tool which enables us to accomplish this remarkable transmogrification is a dispersion relation for the eigenvalue $E(\lambda)$.

The proof of our dispersion relation like that of most of its relatives depends on the analytic and asymptotic properties of the function $E(\lambda)$. However before we can discuss these we will first have to <u>define</u> $E(\lambda)$ for complex λ.

The Hamiltonians which we will be studying all have the form

$$H = Ho + \lambda V = -\nabla^2 + \sum_{i=1}^{N} \frac{m_i{}^2 x_i{}^2}{4} + \lambda V(\vec{x}) \qquad (2.1)$$

where V is a polynomial of order 2M (for simplicity we will always assume that V is even). We will assume that for $\lambda>0$ H is bounded below in the sense that $<\psi|H|\psi> > c$ for all $|\psi>$ in the domain of H. This condition quarantees that for $\lambda>0$ there exist $E_n(\lambda)$ such that the solutions of

$$H(\lambda)\psi = E(\lambda)\ \psi \qquad (2.2)$$

are normalizable

$$\int_{-\infty}^{\infty} \psi^*_{(x)}\ \psi_{(x)} d^N x \ < \infty \qquad (2.3)$$

It is easy to see that this condition cannot be used to define $E(\lambda)$ for all complex λ. For example, when $\lambda < 0$ $H(\lambda)$ is not bounded below for real x, and we do not expect it to have normalizable eigenfunctions.

The definition of $E(\lambda)$ for complex λ is most easily arrived at by the use of a scaling transformation first introduced by Szymanzik.[2] We write

$$V(\vec{x}) = \sum_{j=1}^{M} V^{(j)}(\vec{x}) \tag{2.4}$$

where $V^{(j)}$ is homogeneous of order 2j, and consider the differential equation

$$-\nabla^2 + \alpha \sum_{i=1}^{N} \frac{m_i^2 x_i^2}{4} + \sum_{j=1}^{M} \lambda_j V^{(j)}(\vec{x}) - E(\alpha, \lambda_j)\ \psi = 0 \tag{2.5}$$

For $\alpha > 0$ $\lambda_j > 0$ this operator has eigenfunctions which obey

$$\int \psi^* \psi < \infty$$

Now define the operator $U(\beta)$ for $\beta > 0$ by

$$[\, U(\beta)\psi \,]\ (x) = \beta^{1/4} \psi(\beta^{1/2}\, x) \tag{2.6}$$

It is easy to see that U is unitary and that

$$U^{-1}(\beta)\left(-\nabla^2 + \alpha\sum\frac{m_i^2x_i^2}{4} + \sum_j \lambda_j V^{(j)}\right) U(\beta) \tag{2.7}$$

$$= \left(-\beta\nabla^2 + \alpha\beta^{-1}\sum\frac{m_i^2x_i^2}{4} + \sum_j \lambda_j\beta^{-j}V^{(j)}\right)$$

Since unitarily equivalent Hamiltonians have identical eigenvalues we see that

$$E(\alpha,\lambda_j) = \beta\, E(\alpha\beta^{-2},\lambda_j\beta^{-j-1}) \tag{2.8}$$

In particular

$$E(1,\lambda_j=\lambda) = \beta\, E(\beta^{-2},\lambda\beta^{-j-1}) \tag{2.9}$$

and if $\beta = \lambda^{\frac{1}{M+1}}$

$$E(\lambda) = E(1,\lambda_j=\lambda) = \lambda^{\frac{1}{M+1}}\, E(\lambda^{\frac{-2}{M+1}},\ \lambda_j = \lambda^{1-(\frac{j+1}{M+1})}) \tag{2.10}$$

Kato's work on regular perturbations of operators in Hilbert space[3] shows that for all complex λ, the right hand side of [2.7] (with $\lambda_j = \lambda$; $\beta = \lambda^{1/M+1}$) may be defined as an operator in Hilbert space with normalizable eigenfunctions. Thus, the right hand side of [2.10] is defined for all complex λ. Since any

analytic continuation of $E(\lambda)$ must satisfy [2.10], this equation defines $E(\lambda)$ for all λ.

This definition of $E(\lambda)$ allows us to derive some useful boundary conditions for the eigenfunctions of the unscaled Hamiltonian. In one dimension these boundary conditions give us an alternative definition of $E(\lambda)$ in terms of a boundary value problem in the complex plane.

Let us consider a coordinate direction x_i such that

$$V(x) \underset{|x_i|\to\infty}{} a x_i^{2M} \quad \text{if } |x_k| = 0(1) \quad i \neq k$$

In this region the eigenfunctions of the right hand side of [2.7] satisfy (with $\beta = \lambda^{\frac{1}{M+1}}$)

$$(-\nabla^2 + a x_i^{2M})\ \psi(x) = 0$$

The asymptotic solution of this equation which is normalizable along the real axis is

$$\psi(x) = e^{-\sqrt{a}\,\frac{x_i^{M+1}}{M+1}} f(x_k) \quad i \neq k$$

where f is any twice differentiable function of the small coordinates. Applying the transformation [2.6] we find that the wave function of the unscaled Hamiltonian is

$$\tilde{\psi}(x) = (\text{constant})\ e^{-\sqrt{a}\lambda\ \frac{x_i^{M+1}}{M+1}}\ f(\lambda^{\frac{1}{2(N+1)}} x_k)$$

This wave function satisfies the boundary condition

$$\lim_{|x_i|\to\infty} \tilde{\psi}(x) = 0$$

$$\left|\arg(\pm x_i) + \frac{1}{2(M+1)} \arg \lambda\right| < \frac{\pi}{2(M+1)} \quad , \quad |x_k| = 0(1)$$

which is the result we wished to obtain. It is easy to see that in one dimension the imposition of the above boundary condition on the unscaled differential equation [2.2] defines an eigenvalue problem for all complex λ, and that this definition of the eigenvalue is equivalent to [2.10].

If V is a homogeneous polynomial we can use eq. [2.10] to determine the behavior of $E(\lambda)$ as $|\lambda|\to\infty$. [2.1] says that $E(\lambda)$ is determined by the boundary value problem

$$-\nabla^2 + \lambda^{\frac{-2}{M+1}} \sum \frac{m_i{}^2 x_i{}^2}{4} + V(x) - \lambda^{-1/M+1} E(\lambda) \quad \psi = 0$$

$$\lim_{r\to\infty} \psi(\vec{x}) = 0 \qquad (2.11)$$

For large $|\lambda|$ we can drop the quadratic term since it is a regular perturbation of V[4]:

$$-\nabla^2 + V(x) - \lambda^{-1/M+1} E(\lambda) \quad \psi = 0 \tag{2.12}$$

λ does not appear anywhere in the differential operator so $\lambda^{-\frac{1}{M+1}}E(\lambda)$ must be a constant. Thus,

$$E(\lambda) \underset{|\lambda|\to\infty}{\sim} \lambda^{-1/M+1} \tag{2.13}$$

B. Analyticity Properties and the Dispersion Relation

We will now proceed to derive the dispersion relation for $E(\lambda)$. The proof depends on a theorem due to Simon:[5]

> For any Hamiltonian of the form [2.1] with homogeneous V, $E(\lambda)$ is analytic in the region shown in Figure 1. (The complement of a cut annulus) and the Rayleigh Schrodinger series is asymptotic to E for sufficiently small λ in this region.

Using this result and Cauchy's theorem we can write

$$F(\lambda) = \frac{1}{2\pi i}\int_C \frac{F(x)}{x-\lambda} \tag{2.14}$$

where $F(x) = x^{-1}[E(x)-E(o)]$ and c is the dashed contour

in Fig. 1. Since $F(\lambda) \underset{\lambda\to\infty}{\frac{E(\lambda)}{\lambda}\to\lambda^{\frac{-M}{M+1}}}$ and $F(\lambda) \underset{\lambda\to 0}{=} o(\lambda)$

(Since the perturbation series is asymptotic) we can neglect the contribution from the circles at zero and infinity and write

$$F(\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{-R_2}\frac{D(x)}{x-\lambda} + \int_{-R_3}^{0}\frac{D(x)}{x-\lambda} + \int_{B}\frac{F(x)}{x-\lambda} \tag{2.15}$$

where

$$D(x) = \lim_{\varepsilon\to 0}\left[F(x+i\varepsilon) - F(x-i\varepsilon)\right] \tag{2.16}$$

$R_3(R_2)$ is the outer (inner) radius of the annulus and B is its boundary.

From equation [2.15] we can derive an exact expression for the Rayleigh Schrodinger coefficients

$$E(\lambda) = \sum_{0}^{\infty}\lambda^n A_n \tag{2.17}$$

by expanding $\frac{1}{x-\lambda}$.

$$E(\lambda) = E(o) + \frac{\lambda}{2\pi i}\sum_{n=0}^{\infty}\lambda^n\left\{\int_{-\infty}^{-R_2}D(x)x^{-n-1} + \int_{-R_3}^{0}D(x)x^{-n-1} + \int_{B}F(x)x^{-n-1}\right\} \tag{2.18}$$

$$A_n = \frac{1}{2\pi i}\left[\int_{-\infty}^{-R_2} D(x)x^{-n} + \int_{-R_3}^{0} D(x)x^{-n} + \int_{B} F(x)x^{-n}\right] \qquad (2.19)$$

We are interested in the behavior of A_n as $n\to\infty$ and we will now argue that only the second integral in brackets contributes in this limit, Let D be the maximum value that $|D(x)|$ attains on the interval $(-\infty,-R_2)$ and F the maximum that $|F|$ attains on the contour B. Then if L is the length of the contour B we have

$$\int_{-\infty}^{-R_2} D(x)x^{-n} < D\,\frac{(-R_2)^{1-n}}{1-n}$$

$$\int_{B} F(x)x^{-n} < F\,(-R_3)^{-n} \qquad (2.20)$$

On the other hand, if $|D(x)|$ is non zero on any interval $(-c,-d)$ $b>c>d>0$ then

$$\int_{-R_3}^{0} D(x)x^{-n} > K\,\frac{(-d)^{1-n} - (-c)^{1-n}}{1-n} \qquad (2.21)$$

where K is a constant. Thus, unless $D(x)=0$ on the entire interval $(-R_3,0)$, the second integral dominates as $n\to\infty$, and

$$A_n \underset{n\to\infty}{} \frac{1}{2\pi i} \int_{-R_3}^{0} D(x)x^{-n} \tag{2.22}$$

$$\to \frac{1}{2\pi i} \int_{-\infty}^{0} D(x)x^{-n}$$

This can be written in a more suggestive form if we notice that $D(x) = \frac{2i\,\mathrm{Im}E(x)}{x}$

$$A_n \underset{n\to\infty}{} \frac{1}{\pi} \int_{-\infty}^{0} \mathrm{Im}E(x)x^{-n-1} \tag{2.23}$$

Eq.[2.23] should be written in orange dayglo ink. It is the basis not only of our entire computational program but also of our intuitive understanding of singular perturbation theory. The only assumption that we have made in arriving at [2.23] is that $D(x)$ does not vanish identically on the interval $(-R_3,0)$. This will be verified by explicit calculation.

Eq. [2.23] is extremely attractive because it relates the large n behavior of A_n to the behavior of the imaginary part of E for small negative coupling constant. States with complex energy are unstable and ImE is a measure of the width or lifetime of the state. Thus the large n behavior of A_n depends on whether or not the bound states of the unperturbed Hamiltonian remain stable when we turn on the perturbation with small negative coupling. We can be even more precise than this. If the perturbation series is to converge for any λ then A_n can grow no faster than K^n for some K. Eq. [2.23] tells us that this can only happen if ImE vanishes for $-\frac{1}{K}<x<0$. We see that the radius of convergence of the perturbation series is the largest value of -x for which ImE = 0.

Figure [2] illustrates the behavior of the potential V which corresponds to convergence or divergence of the perturbation series. It is clear from this picture that we are dealing with a tunneling problem. Thus the value of ImE will be related to a barrier penetration factor, and we can use an elementary argument to express ImE in terms of the probability of tunneling through the potential barrier. We write the Schrodinger Equation as

$$(-\nabla^2 + U - E)\psi = 0$$

The conjugate equation is

$$(-\nabla^2 + U - E^*)\ \psi^* = 0$$

We multiply the first equation by ψ^*, the second by ψ subtract and integrate over a volume V:

$$\int_V (E-E^*)\ \psi^*\psi \;=\; -\int_V (\psi^*\nabla^2\psi - \psi\nabla^2\psi^*)$$

Now we use Gauss' theorem to obtain

$$\mathrm{Im}E = \frac{\int_S \bar{J} \cdot \overline{ds}}{\int_V \psi^*\psi} \tag{2.24}$$

where S is the surface surrounding V and

$$\vec{J} = \frac{i}{2}\,[\psi^*\nabla\psi - \psi\nabla\psi^*] \tag{2.25}$$

$\bar{J}$ is the familiar probability current. In Chaps. III-IV, we will show how to compute $\bar{J}$ for small negative λ.

Before we begin our computation I would like to point out to the reader the similarity between our discussion and that given by Dyson[6] in his ancient paper on the divergence of perturbation theory in quantum electrodynamics. Dyson's argument runs as follows: if QED has a convergent perturbation series then we can analytically continue the states and amplitudes of the theory to a world where α, the fine structure constant, is negative so that oppositely charged particles

repel. Now consider a state containing N electron positron pairs with electrons and positrons in separate regions of space. Dyson argues that for N large enough the negative energy from the Coulomb repulsion of the pairs will be larger than their kinetic energy, and therefore this state will have lower energy than the vacuum. Although it is separated from the vacuum by a large potential barrier (of height > $2NMe \tilde{=}$ N MeV) quantum mechanical tunneling will cause the vacuum to decay into this negative energy state. This instability means that QED with negative α cannot be described by analytic functions and the perturbation series must diverge.

It is clear that Eq. [2.23] is nothing but a quantification of Dyson's intuitive discussion. Of course [2.23] has not been proven for QED.

It should be emphasized that we have only proven this important equation for homogeneous V. The scaling argument that was used to prove that $\lim_{\lambda\to\infty} \frac{E(\lambda)}{\lambda} = 0$ does not work for non-homogeneous perturbations. Since I will occasionally want to discuss such perturbations I will assume that [2.23] continues to hold in the non-homogeneous case. In Chap. III, we use [2.23] to calculate the large order behavior of the perturbation series for a one dimensional oscillator with arbitrary polynomial perturbation. The results are in good agreement with computer calculations and this provides numer-

ical "proof" of the correctness of [2.23] in the non-homogeneous case.

CHAPTER III

ANHARMONIC OSCILLATOR WITH POLYNOMIAL SELF INTERACTIONS

A. Introduction

The anharmonic oscillator with polynomial self interactions[1] is defined by the differential equation:

$$0=[-\frac{d^2}{dx^2} + \frac{x^2}{4} + \lambda(a(\frac{x^2}{2})^N + b(\frac{x^2}{2})^{N-1} + c(\frac{x^2}{2})^{N-2} + \ldots\ldots) - E_K(\lambda,a,b,c\ldots)\ ]\ \psi \qquad (3.1)$$

with the boundary condition

$$\lim_{|x| \to \infty} \psi(x) = 0 \quad \text{for} \qquad (3.2)$$

$$|\arg(\pm x) + (2N+2)^{-1}\arg\lambda| < \pi(2N+2)^{-1}$$

The perturbation series for the Kth eigenvalue is

$$E_K(\lambda,a,b,c,\ldots) = \sum_{n=0}^{\infty} \lambda^n A_n^K\ (a,b,c,\ldots)$$

We shall solve equation [3.1] for small negative λ ($\lambda = -\varepsilon$ $0 < \varepsilon << 1$) and use equations [2.23] and [2.25] to determine the large n behavior of A_n^K (a,b,c....). Throughout that calculation it is assumed that N is small compared with $\varepsilon^{-\alpha}$ ($\alpha>0$).

The approximate solution of [3.1] will be obtained by the method of asymptotic matching. We divide the real axis into seven overlapping regions and approximate the differential equation in each region. The general solution of these seven approximate equations contains fourteen arbitrary constants. Two of these constants are determined by the boundary conditions at $\pm\infty$ and a third corresponds to overall normalization. The remaining constants are determined by requiring that the solutions in two adjacent regions agree on the overlap between the two regions. We choose our regions so that as $\varepsilon\to 0$ the size of the overlap becomes infinite, and adjacent wave functions must agree on a larger and larger domain. Thus the approximation to the wave function becomes better and better as $\varepsilon\to 0$.

It should be noted that the number of matching conditions exceeds the number of free constants by one. If we start at $-\infty$ and we proceed to the right by asymptotic matching we will find that there will be no free constants left to satisfy

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the matching condition at $+\infty$. As we shall see, this last condition determines the energy eigenvalue.

Let us now describe the seven regions:

$$\text{I:} \quad 0 \leq |x| < \varepsilon^{-1/4(N-1)}$$

$$\text{II:} \quad 1 \ll x \ll \varepsilon^{-1/2(N-1)}$$

$$\text{III:} \quad x \sim \varepsilon^{-1/2(N-1)}$$

$$\text{IV:} \quad x \gg \varepsilon^{-1/2(N-1)}$$

Regions V, VI, and VII are the mirror images of II, III, and IV respectively. Note that the regions overlap and that the size of the overlap becomes infinite as $\varepsilon \to 0$. (I should point out that by $x \sim \varepsilon^{-\frac{1}{2(N-1)}}$ I mean that $x = \varepsilon^{-\frac{1}{2(N-1)}}(1 \pm a)$ with $a \ll 1$ as $\varepsilon \to 0$. a could be as large as $\varepsilon^{\frac{3}{4(N-1)}}$ for example, in which case the overlap between III and II or IV certainly becomes infinite as $\varepsilon \to 0$.)

B. Parity and Reality of ψ_K

The differential equation [3.1] is invariant under $x \to -x$, and consequently the eigenfunctions may be chosen to have definite parity. This reduces our labor by half, for if we choose a definite parity solution in region I, we need only

perform asymptotic matches on the positive real axis.

It will be convenient to separate ψ_K and E into their real and imaginary parts

$$\psi_K = \phi_K + i\ \chi_K$$

$$E_K = E_K^{(1)} + i\ E_K^{(2)}$$

Referring to Eq.[2.23] we see that $E_K^{(2)}(\varepsilon)$ must go to zero faster than any power of ε as $\varepsilon \to 0$, for otherwise A_n would be infinite. On the other hand, $E_K^{(1)}(\varepsilon) \to K+\frac{1}{2}$ as $\varepsilon \to 0$. We will see that in the region $0 \ll x \ll \varepsilon^{-\frac{1}{2(N-1)}}$, ψ_K will consist of an increasing and a decreasing part. Without loss of generality we can choose the phase of ψ_K so that the decreasing part is purely real. Therefore the behavior of ϕ_K and χ_K will be as shown in Fig. [3].

The upshot of this discussion is that for small ε we can write the real and imaginary parts of Eq.[3.1] as

$$[-\frac{d^2}{dx^2} + \frac{x^2}{4} - \varepsilon(a(\frac{x}{2})^N + ...) - E_K^{(1)}]\phi_K = 0 \qquad (3.3)$$

$$[-\frac{d^2}{dx^2} + \frac{x^2}{4} - \varepsilon(a(\frac{x^2}{2})^N + ...) - E_K^{(1)}]\chi_K = E_K^{(2)}]\phi_K \qquad (3.4)$$

Eq. [3.4] is exact and in [3.3] we have dropped $E_K^{(2)}\chi_K$ compared to $E_K^{(1)}\phi_K$.

We can now outline our complete procedure for solving [3.1]. In regions I and II we will solve Eq. [3.3] approximately, choosing the solution which decreases for increasing x in region II and which has definite parity in region I. (Such a solution only exists for discrete values of $E_K^{(1)}$ so this will fix our eigenvalue.) In regions III and IV we will solve Eq. [3.1] approximately since there is no simple criterion for identifying the real and imaginary parts of ψ_K in these regions.

The boundary condition [3.2] enables us to determine ψ_K in region III up to a multiplicative constant. The value of this constant is found by matching the part of ψ_{KIII} which decreases for increasing x to ϕ_K in region II.

C. ϕ_K In Region I

In region I we can approximate [3.1] by

$$(-\frac{d^2}{dx^2} + \frac{x^2}{4} - E_K^{(1)})\phi_{KI} = 0 \qquad (3.5)$$

The even (odd) solution of this equation is

$$\phi_{KI} = \frac{1}{2}\{D_{E_K^{(1)}-1/2}(x) \pm D_{E_K^{(1)}-1/2}(-x)\} \qquad (3.6)$$

where D_ν is the parabolic cylinder function.[2] We have freely chosen the normalization of ϕ_{KI} .

If we now inquire whether either of these solutions decreases for increasing x we run into trouble. The asymptotic expansions of D (x) and D (-x) for $x \to \infty$ are [3]

$$D_\nu(x) \longrightarrow e^{-1/4 x^2} x^\nu \tag{3.7}$$

$$D_\nu(-x) \longrightarrow e^{-1/4 x^2} x^\nu e^{\nu\pi i} + \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{1/4 x^2} \tag{3.8}$$

Thus, unless $\Gamma(\frac{1}{2} - E_K^{(1)}) = \infty$, ie $E_K^{(1)} = K + \frac{1}{2}$, both definite parity solutions have components which increase as x increases. We see that to the order we are working $E_K^{(1)} = K + \frac{1}{2}$. Furthermore since[4]

$$D_K(Z) = e^{K\pi i} D_K(-Z) \tag{3.9}$$

we have an even parity solution for K even and an odd solution for K odd.

D. ϕ_K In Region II

In region II we try a WKB solution

$$\phi_{KII} = C\left[\frac{x^2}{4} - \varepsilon\left(a\left(\frac{x^2}{2}\right)^N + \ldots.\right) - (K + 1/2)\right]^{-1/4}$$

$$\exp\left[-\int_{x_o}^{x} \sqrt{\frac{x^2}{4} - \varepsilon\left(a\left(\frac{x^2}{2}\right)^N + \ldots\right) - (K+1/2)}\right] \qquad (3.10)$$

where $x_o = \sqrt{4E_K}$ is (approximately) the zero of $\frac{x^2}{4} - \varepsilon\left[a\left(\frac{x^2}{2}\right)^N + \ldots\right] - E_K$ which lies near the origin. In the future we will refer to x_o as the nearby turning point. As usual the validity of this approximation follows from the fact that for $\varepsilon \to 0$ $\frac{1}{V}\frac{dV}{dx}$ is small in region II. In the overlap between regions I and II x is much greater than one but much less than $\varepsilon^{-\frac{1}{2(N-1)}}$. For example we could take the overlap to be the region $\varepsilon^{-\frac{1}{6(N-1)}} < x < \varepsilon^{-\frac{1}{4(N-1)}}$. For such values of x we can approximate [3.10] by

$$\phi_{KII} = C\left(\frac{2}{x}\right)^{1/2} e^{-\frac{\left[t\sqrt{t^2-4(K+1/2)} - 4(K+1/2)\ln\left(t+\sqrt{t^2-4(K+1/2)}\right)\right]^x}{4(K+1/2)}}$$

$$\approx C\, e^{-\frac{x^2}{4}}\, x^K\, e^{1/2}\, \{(K+1/2)(1-\ln(K+1/2)+\ln 2\} \qquad (3.11)$$

Comparing this with the asymptotic expansion of [3.6] as given by [3.7] and [3.8] we find that

$$C = \exp -\frac{1}{2}\{(K+1/2)\ (1-\ln(K+1/2)\ ) + \ln 2\} \qquad (3.12)$$

Notice that the constants a,b, etc. do not appear anywhere in C.

E. ψ_K In Region III

Near the upper turning point of V (which we shall denote by x_1) the WKB approximation breaks down. This is signalled by the singularity of [3.10] at x_1, for x_1 is a regular point of Eq.[3.1] and the exact solution must be analytic there.

Alternatively we could remark that $\frac{1}{V^2}\frac{dV}{dx}$ is no longer small.

To find an appropriate approximation for ψ_K in region III we notice that the distant turning point x_1 is given by

$$x_1 \sim \left(\frac{2^{N-2}}{\varepsilon a}\right)^{\frac{1}{2(N-1)}} - \left(\frac{\varepsilon a}{2^{N-2}}\right)^{\frac{1}{2(N-1)}} \frac{[\frac{b}{a} + 2E_K]}{N-1} + 0\left(\varepsilon^{3/2(N-1)}\right) \tag{3.13}$$

and that near this point we can approximate V by

$$V \sim \frac{N-1}{2} x_1(x_1-x) \tag{3.14}$$

Thus if we introduce the new variable $r \equiv \left(\frac{N-1}{2}x_1\right)^{1/3}(x_1-x)$, we can rewrite the differential equation [3.1] as

$$(-\frac{d^2}{dr^2} + r)\psi_{KIII} = 0 \tag{3.15}$$

This is the familiar Airy equation[5] whose linearly independent solutions are denoted by Ai(r) and Bi(r). It is very important to notice that this equation has no dependence on the constants c,d,... . The effect of these terms is to shift the position of x_1 by $o(\varepsilon^{\frac{3}{2(N-1)}})$ and for $\varepsilon<<1$ this shift is negligible compared with $\varepsilon^{-\frac{1}{2(N-1)}}$

F. ψ_K In Region IV

In region IV we can again use the WKB approximation for ψ_K

$$\psi_{KIV} = V^{-1/4} e^{\pm \int^x \sqrt{V}} \tag{3.16}$$

Since V is negative throughout this region, the phase of the +(-) solution increases decreases throughout the region. Moreover for $x\to\infty$

$$\psi_{KIV}(x) \longrightarrow \frac{e^{\pm i \frac{\sqrt{\varepsilon a}}{N+1} \frac{x^{N+1}}{2^{N/2}}}}{x^{N/2}} \tag{3.17}$$

According to the boundary condition [3.2]

$$\psi_{K}IV\ (x-i\beta) \xrightarrow[x\to\infty]{} 0 \qquad (x>0,\quad 0<\beta<<1)$$

so that the correct solution is the one whose phase decreases as x increases.

We now find the linear combination of Airy functions which matches to the solution in Region IV. This must be the linear combination with decreasing phase as $x\to\infty$. The asymptotic behavior of the Airy functions for large negative r is[6]

$$Ai(r) \xrightarrow[r\to-\infty]{} \pi^{-1/2}(-r)^{-1/4}\ \sin(\tfrac{2}{3}(-r)^{3/2}+\tfrac{\pi}{4})-\cos(\tfrac{2}{3}(-r)^{3/2}+\tfrac{\pi}{4}) \qquad (3.18)$$

$$Bi(r) \xrightarrow[r\to-\infty]{} \pi^{-1/2}(-r)^{-1/4}\ \sin(\tfrac{2}{3}(-r)^{3/2}+\tfrac{\pi}{4})+\cos(\tfrac{2}{3}(-r)^{3/2}+\tfrac{\pi}{4}) \qquad (3.19)$$

and the linear combination with decreasing phase is

$$\psi_{K}III \;=\; D\ [Bi(r)-iA_i(r)] \qquad (3.20)$$

where D is a constant.

G. Matching Regions II and III

When x is the overlap between regions II and III r is large and positive and we can use the asymptotic expansion of the Airy functions[7]:

$$\mathrm{Ai}(r)_{r\to\infty} \tfrac{1}{2}\pi^{-1/2} r^{-1/4} e^{-2/3 r^{3/2}} \tag{3.21}$$

$$\mathrm{Bi}(r)_{r\to\infty} \pi^{-1/2} r^{-1/4} e^{-2/3 r^{3/2}} \tag{3.22}$$

According to our discussion in Sect. B the part of ψ_{KIII} which decreases for increasing x in the overlap region should match to ϕ_{KII}. Thus we should have

$$\phi_{KII}(x) \simeq \frac{D}{\pi^{1/2}} r^{-1/4} e^{2/3 r^{3/2}} \tag{3.23}$$

where

$$r = \left[\frac{(N-1-x_i}{2}\right]^{1/3} (x_1-x) \qquad (3.24)$$

To evaluate D we must find the behavior of ϕ_{KII} for $x \to x_1$. In the first factor in [3.10] we make the approximation

$$V(x) = \frac{x^2}{4} - \varepsilon\left[a\left(\frac{x^2}{2}\right)^N + \ldots\right] - (n+\tfrac{1}{2}) \approx \frac{N-1}{2} x_1 (x_1-x)$$

$$= \left(\frac{N-1}{2} x_1\right)^{2/3} r \qquad (3.25)$$

We cannot however make this approximation in the integral $\int_{x_0}^{x} \sqrt{V(x)}$ because the integration region is so large. Instead we write

$$\int_{x_0}^{x} \sqrt{V(x)}\, dx = \int_{x_0}^{x_1} \sqrt{V(x)}\, dx - \int_{x}^{x_1} \sqrt{V(x)}\, dx$$

$$\equiv A - \int_{x}^{x_1} \sqrt{V(x)}\, dx \qquad (3.26)$$

A is a real constant which we will evaluate in a moment. The x dependent part of the integral can now be approximated

$$\int_{x}^{x_1} dx\sqrt{V(x)} \approx \int_{o}^{r} \sqrt{r}\ dr = \tfrac{2}{3} r^{3/2} \qquad (3.27)$$

Therefore

$$\phi_{KII} \quad C\left(\frac{N-1}{2}x_1\right)^{-1/6} \quad e^{-A} \quad r^{-1/4} \quad e^{2/3\, r^{3/2}} \qquad (3.28)$$

and comparing with Eq. [3.23] we find that

$$D = \pi^{1/2} \; C\left(\frac{N-1}{2}x_1\right)^{-1/6} e^{-A} \qquad (3.29)$$

Note that the only dependence of D on b, c, etc. is through the constants A and x_1 .

The evaluation of A is most simply performed by splitting the integration region into two parts

$$A = \int_{x_0}^{\bar{x}} \sqrt{V}\ dx + \int_{\bar{x}}^{x_1} \sqrt{V}\ dx \equiv A_1 + A_2 \qquad (3.30)$$

where $x_0 << \bar{x} << x_1$.

In the first integral we approximate V by

$$V \approx \frac{x^2}{4} - (K+{}^1\!/_2) \tag{3.31}$$

so that

$$A_1 \simeq \int_{x_0}^{\bar{x}} \sqrt{\frac{x^2}{4} - (K+{}^1\!/_2)} \approx \tfrac{1}{4}\bar{x}^2 \tfrac{1}{2} - (K+\tfrac{1}{2}) - (K+\tfrac{1}{2}) \ln\frac{2\bar{x}}{x_o} \tag{3.32}$$

In the integral for A_2 we make a change of variables $u = \frac{x}{x_1}$

$$A_2 = \tfrac{1}{2}x_1{}^2 \int_{\frac{\bar{x}}{x}}^{1} \{ u\,du \quad 1-4\varepsilon[\tfrac{a}{2}x_1{}^{2N-2}(\tfrac{u^2}{2})^{N-1} + \tfrac{b}{2}x_1{}^{2N-4}(\tfrac{u^2}{2})^{N-2}$$

$$+ \tfrac{c}{2}x_1{}^{2N-6}(\tfrac{u^2}{2})^{N-3} + \ldots.] - \frac{4(K+{}^1\!/_2)}{u^2 x_1{}^2}\}^{1/2} \tag{3.33}$$

Since the integral is now taken over a range which stays finite as $\varepsilon \to 0$ the magnitude of a term may simply be read off from its coefficients. Using Eq. [3.13] we see that the first two terms in the square root are of order one (with a correction of $0(\varepsilon^{\frac{1}{(N-1)}})$, that the terms proportional to b and

$(K+\frac{1}{2})$ are $o(\varepsilon^{\frac{1}{N-1}})$, that the term proportional to c is $o(\varepsilon^{\frac{2}{N-1}})$ etc. Thus we can use the binomial expansion and Eq. [3.13] to write an approximate expression for A_2:

$$A_2 \approx \tfrac{1}{2}\left[\left(\frac{2^{N-2}}{\varepsilon a}\right)^{\frac{1}{N-1}} - \frac{2\left[\frac{b}{a}+2E_K\right]}{N-1} \right] \int_{\bar{x}/x}^{1} udu \sqrt{1-u^{2N-2}}$$

$$-\tfrac{1}{2}\int_{\bar{x}/x}^{1} \frac{udu}{\sqrt{1-u^{2N-2}}} \left[-2\left[\frac{b}{a}+2E_K\right]u^{2N-2} + \frac{2b}{a}u^{2N-4} + \frac{4(K+\frac{1}{2})}{u^2}\right] \tag{3.34}$$

$$- \frac{1}{N-1}\left(\frac{2N-2}{\varepsilon a}\right) + o(\varepsilon^{N-1}) .$$

To calculate the exact leading behavior of perturbation theory we must keep all terms in [3.34] which do not vanish as $\varepsilon \to 0$. There are five such terms, and we evaluate them as follows:

$$\frac{1}{2}\left(\frac{2^{N-2}}{\varepsilon a}\right)^{\frac{1}{N-1}} \int_{\bar{x}/x_1}^{1} u\,du \quad \sqrt{1-u^{2N-2}}$$

$$\cong \frac{1}{2}\left(\frac{2^{N-2}}{\varepsilon a}\right)^{\frac{1}{N-1}} \int_{0}^{1} u\,du\,\sqrt{1-u^{2N-2}} - \frac{1}{2}\left(\frac{2^{N-2}}{\varepsilon a}\right)^{\frac{1}{N-1}} \frac{1}{2}\left(\frac{\bar{x}}{x}\right)^2$$

$$= \frac{1}{4}\left(\frac{2^{N-2}}{\varepsilon a}\right)^{\frac{1}{N-1}} \int_{0}^{1} \frac{dt}{N-1}\, t^{-\frac{N-2}{N-1}} (1-t)^{1/2} - \frac{1}{4}\bar{x}^2 + o(\varepsilon^{1/N-1}) \tag{3.35}$$

$$\cong \frac{1}{4}\left(\frac{2}{\varepsilon a}\right)^{\frac{N-2}{N-1}} \frac{\Gamma\left(\frac{1}{N-1}\right)\Gamma\left(\frac{3}{2}\right)}{(N-1)\Gamma\left(\frac{1}{N-1}+\frac{3}{2}\right)} - \frac{1}{4}\bar{x}^2$$

In [3.35] we have used the substitution

$$t = u^{2N-2} \tag{3.36}$$

The next two terms may be combined to give

$$[\frac{b}{a} + 2E_K] \int_{\bar{x}/x_1}^{1} udu \left[\frac{1}{2} \frac{u^{2N-2}}{\sqrt{1-u^{2N-2}}} - \frac{1}{N-1} \sqrt{1-u^{2N-2}} \right]$$

$$\tilde{=} [\frac{b}{a} + 2E_K] \int_{0}^{1} udu \left[\frac{u^{2N-2}}{2\sqrt{1-u^{2N-2}}} - \frac{\sqrt{1-u^{2N-2}}}{N-1} \right] + o(\varepsilon^{\frac{1}{N-1}}) \tag{3.37}$$

Both integrals in (3.37) give Beta functions when we make the substitution (3.36) and it is easy to show that they cancel exactly!

The fourth term is

$$-\frac{b}{2a} \int_{\bar{x}/x_1}^{1} \frac{u^{2N-3}du}{\sqrt{1-u^{2N-2}}} = -\frac{b}{4(N-1)a} \left[-2\sqrt{1-t} \right]_{(\bar{x}/x_1)^{2N-2}}^{1} \tag{3.38}$$

$$\tilde{=} -\frac{b}{2(N-1)a} + o(\varepsilon)$$

and the fifth is:

$$-(K+{}^1/_2)\int_{\bar{x}/x_1}^{1}\frac{du}{u\sqrt{1-u^{2N-2}}}$$

$$=-\frac{(K+{}^1/_2)}{2N-2}\int_{\left(\frac{\bar{x}}{x_1}\right)^{2N-2}}^{1}\frac{dt}{t\sqrt{1-t}} \qquad (3.39)$$

$$\simeq \frac{(K+{}^1/_2)}{2N-2}\ln\tfrac{1}{2}+(K+\tfrac{1}{2})\ln\bar{x}-\frac{(K+\frac{1}{2})}{2N-2}\ln\left(\frac{2^{N-2}}{\varepsilon a}\right)+o(\varepsilon^{1/N-1})$$

Combining Eqs. (3.34-3.39) we find that

$$A_2=\tfrac{1}{4}\left(\frac{2^{N-2}}{\varepsilon a}\right)^{\frac{1}{N-1}}\frac{\Gamma(\frac{1}{N-1})\ \Gamma(\frac{3}{2})}{(N-1)\Gamma(\frac{1}{N-1}+\frac{3}{2})}+\frac{K+{}^1/_2}{(2N-2)}\ln\left(\frac{2^{N-2}}{\varepsilon a}\right)$$

$$-\frac{(K+{}^1/_2)}{2N-2}\ln 2\ -\frac{b}{2(N-1)a}-\frac{1}{4}\bar{x}^2+(K+{}^1/_2)\ln\bar{x}+o(\varepsilon^{1/N-1})$$

(3.40)

Thus

$$A = A_1 + A_2 = \frac{1}{2}\left(\frac{1}{2\varepsilon a}\right)^{\frac{1}{N-1}} \frac{\Gamma\left(\frac{1}{N-1}\right)\Gamma\left(\frac{3}{2}\right)}{(N-1)\Gamma\left(\frac{1}{N-1}+\frac{3}{2}\right)} + \frac{K+\frac{1}{2}}{2N-2}\ln\varepsilon a$$

$$+ \frac{1}{2}\left(K+\tfrac{1}{2}\right)\left[\ln\left[\tfrac{1}{2}k + \tfrac{1}{4}\right] - 1\right] - \frac{b}{2(N-1)a} + o\left(\varepsilon^{1/N-1}\right) \qquad (3.41)$$

Referring to Eq. (3.29) and using (3.13) we find that the multiplicative factor in front of the Airy function is

$$D = \pi^{1/2} \quad C\left[(N-1)\left(\frac{2^{-N}}{\varepsilon a}\right)^{\frac{1}{2N-2}}\right]^{-1/6} e^{-A} + o\left(\varepsilon^{1/N-1}\right) \qquad (3.42)$$

with C given by (3.12) and A by (3.41).

H. The Large Order Behavior of the Perturbation Series

Our matching program complete, we are now ready to compute the imaginary part of the energy eigenvalue. Since we are in one dimension, the surface integral in [2.24] means simply an evaluation of the current at two points on the positive and negative x axis.

$$\int \vec{J}\cdot d\vec{s} \quad \rightarrow [J(x)-J(-x)] \tag{3.43}$$

The wave function has definite parity so that this is just

$$\int \vec{J}\cdot d\vec{s} = 2J(x) \quad x > o \tag{3.44}$$

The only region in which we know both the real and imaginary parts of the wave function is region III. To simplify our calculation we evaluate the current at the extreme upper end of this region where

$$\psi_k \sim D\pi^{-1/2}(-r)^{-1/4}[1+i]\ e^{-i[\frac{2}{3}(-r)^{3/2}+\frac{\pi}{4}]}$$

$$= D(\tfrac{\pi}{2})^{-1/2}(-r)^{-1/4}\ e^{-\frac{2i}{3}(-r)^{3/2}} \tag{3.45}$$

Remembering that D is a real constant and that

$$r = \left(\frac{N-1}{2}\, x_1\right)^{1/3} (x_1 - x) \qquad \text{we have}$$

$$\int \vec{J}\cdot d\vec{s} = 2J(x) = \left(\frac{N-1}{2} x_1\right)^{1/3} \frac{4D^2}{\pi} \tag{3.46}$$

Thus

$$\int \vec{J}\cdot d\vec{s} = 2C^2 e^{-2A} \tag{3.47}$$

To evaluate $\int \psi_k^* \psi_k$ it is sufficient to approximate ψ_K by D_k since the dominant contribution comes from region I.

$$\int \psi_k^* \psi_k \sim \sqrt{2\pi}\, k\, ! \tag{3.48}$$

Combining [3.48], [3.47], [3.41] and [3.12], and using the Legendre duplication formula[8] for the Gamma function we obtain our final result for A_n

$$A_n^K(a,b,c,\ldots) = -\frac{N-1}{\pi^{3/2}} \frac{2^K}{K!} e^{b/(N-1)a} \left(\frac{-a}{2}\right)^n \Gamma\left((N-1)n+(K+\tfrac{1}{2})\right) \times \left[\frac{\Gamma\left(\frac{2N}{N-1}\right)}{\Gamma^2\left(\frac{N}{N-1}\right)}\right]^{\left[(N-1)n+(K+\frac{1}{2})\right]} \tag{3.49}$$

I. Discussion

Eq. [3.49] has many interesting features, not the least of which is its simple dependence on the constants b,c, etc. We see that adding a new term to the interaction Hamiltonian of an x^{2N} oscillator has no effect at all on the leading large order behavior of the perturbation series if the term grows no faster at infinity than x^{2N-4}. The effect of a term bx^{2N-2} is simply to scale the large order perturbation coefficients for all the energy levels by the same constant factor $e^{b/a(N-1)}$.

In particular if we add a "mass renormalization" term $\delta m x^2$ where $\delta m = o(\lambda)$ as $\lambda \to 0$, we will not affect the large order growth of the series for an $x^{2N}(N>2)$ oscillator and will merely scale the large order coefficients of an x^4 oscillator. Thus "mass renormalization" does not seem to effect the convergence of perturbation theory. Similar remarks may be made about Wick ordering[9].

Another important property of [3.49] emerges if we ask where the different contributions to the large order behavior of perturbation theory come from. A quick review of the last few sections convinces us that the entire factorial and constant to the n behavior (that is the behavior that controls the convergence of the series) comes from WKB wave function

in region II. In particular, if we had simply said that

$$\text{ImE} = (\text{const.})\ e^{-2\int_{x_0}^{x_1} \sqrt{V}} \tag{3.50}$$

we would have found the correct factorial and constant to the n behavior of A_n. This is a very general result and will recur throughout our work. The exponential factor in [3.50] is familiar from most work on the WKB approximation. It is called the barrier penetration factor.[10]

If we were only interested in finite oscillator systems, Eq. [3.50] would shorten our work immensely for it enables us to decide whether perturbation theory converges without ever solving a differential equation or performing an asymptotic match. However, in field theory, we are interested in the constant (independent of n) terms in A_n since it is here that we expect that notorious field theoretic divergences to arise. [See Sec. IV.F]. Thus the constant term out front of A_n must be found if we wish to renormalize our results for the large order behavior of perturbation theory.

It will be useful then to find a computational method intermediate between Eq. [3.50] and the long trek through sections C-H: a method which shortens the work but is sufficiently accurate to compute the exact leading behavior of A_n. Such a shortcut can indeed be found. It eliminates the

asymptotic match at the upper turning point.

Let us return to the beginning of section E where we noted that the region II solution becomes invalid as we move up the real axis towards the upper turning point x_1. Suppose now that instead of continuing on this collision course with the turning point we make a detour into the complex plane as shown in Fig. [4]. If we give the turning point a wide enough berth then the WKB wave function [3.10] is a valid approximation all along the contour Γ in Fig. [4]. In particular we can evaluate the wave function at a point at the upper end of region III

$$\psi(x) = C\left(\frac{x^2}{4} - \varepsilon \left(a\left(\frac{x^2}{2}\right)^N + \ldots\ldots\right) - E_k\right)^{-1/4}$$

$$X \quad \exp - \int_{x_0}^{x} \sqrt{\frac{t^2}{\Delta} - \varepsilon \left(a\left(\frac{t^2}{2}\right)^N + \ldots\right) - E_k}\, dt \qquad (3.51)$$

The integral in [3.51] is to be taken along the contour Γ. However, since the only singularity of the integrand in the region of interest is a branch point at the distant turning point, we can deform the contour until it lies parallel to and infinitesimally above the real axis so that

$$\psi(x) = C \left(\frac{x^2}{4} - \varepsilon\left(a\left(\frac{x^2}{2}\right)^N + \ldots.\right) - E_k \right)^{-1/4}$$

$$\times \exp - \int_{x_0}^{x} \sqrt{\frac{t^2}{4} - \varepsilon\left(a\left(\frac{t^2}{2}\right)^N + \ldots.\right) - E_k}\, dt \qquad (3.52)$$

where the integral is now taken along the real axis. As in section we split the integral into two parts

$$\int_{x_0}^{x} \sqrt{V} = \int_{x_0}^{x_1} \sqrt{V} + \int_{x_1}^{x} \sqrt{V} \qquad (3.53)$$

$$= A + \int_{x_1}^{x} \sqrt{V}$$

with A given by [3.41]. The second integral in [3.53] is purely imaginary, while the first is purely real. Using this fact we plug [3.52] into the definition of the current and find that

$$\int J \cdot ds = 2J(x) = 2C^2 e^{-2A}$$

in complete agreement with [3.47].

Despite the agreement the cautious reader may have grave misgivings about the piece of legerdermain that we have just pulled off. We seem to have used the WKB approximation outside its domain of validity, and to have solved a boundary value problem without using the boundary condition at infinity. Furthermore [3.10] is a formula for the real part of the wave function in region II and we have used it to obtain an expression for the whole wave function at the upper end of region III.

The first and third objections may be dealt with summarily. We have not used the WKB approximation in a region where it is not valid. Eq. [3.52] does not purport to represent the wavefunction near the turning point x_1, but only at points far into the upper end of region III where the WKB approximation is again valid. This explanation may also allay the reader's bourgeois fear that we have "gotten something for nothing". We have lost information by avoiding turning point analysis: namely we no longer know the behavior of ψ near x_1.

To the question about the entire wave function coming out of its real part the only answer I can supply is an affirmation that this is indeed true. The functional form which represents the real part of ψ in region II develops an imag-

inary part as we move past the turning point, and this newly complex function describes the dominant behavior of both the real and imaginary part of ψ at the upper end of region III. Remember that in section G we determined the constant D by matching the decreasing part of the region III wave function to the real part of ψ in region II. We could then have used this result to find the imaginary part of ψ_{II}. By going into the complex plane we could have continued $Im\psi_{II}$ into the upper end of region III and we would then have found that it gave an exponentially small contribution to J.

We can now understand how we have used the boundary condition at infinity in our discussion. This condition determines the correct linear combination of increasing and decreasing WKB solutions that is to be used in region II. Without it we would be unable to affirm that the contribution of $Im\psi_{II}$ was exponentially small at the upper end of region III. Actually, we have treated this question rather cavalierly even in our "careful" analysis. A more detailed treatment (for the case of an x^4 oscillator) which constructs both $Re\psi$ and $Im\psi$ in all regions may be found in a paper by Bender and Wu.[11]

Once we have done the detailed analysis that justifies our "trick" we can give a simple prescription for obtaining the current which flows out to infinity in an arbitrary re-

flection invariant barrier penetration problem: First solve the Schrödinger equation near the origin, choosing a solution with definite parity. Then match this solution to a decreasing WKB solution in the tunneling region. This match determines the lowest order value for the real part of the energy. The escaping current is now computed by evaluating the tunneling region WKB solution at a point far beyond the distant turning point. This is the procedure that we will follow when we turn to multidimensional barrier penetration problems in the following Chapters.

CHAPTER IV

COUPLED ANHARMONIC OSCILLATORS

Introduction

We will now extend the techniques developed in Chapter III to multidimensional systems of oscillators. To begin with, we study a simple two-dimensional system whose Hamiltonian is

$$-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{x^2+y^2}{4} + \frac{\lambda}{4}(ax^4+by^4+2cx^2y^2) \ . \qquad (4.1)$$

It will be clear that much of our discussion applies to systems more general than [4.1] and in later sections we will discuss more complicated oscillators.

The basic strategy for obtaining the large order behavior of perturbation theory for [4.1] is to solve the differential equation by asymptotic matching for small negative λ. We will use the shortcut outlined in Chapter III, so that only one asymptotic match needs to be performed. Thus, we only need to solve the differential equation near the origin and in the large region between the two turning points. (In two dimensions the turning points are really turning lines, and in N dimensions they are "N-1 dimensional turning hypersurfaces".

I find such phrases awkward, and I will continue to use "turning point" even in multidimensional situations).

It is easy to solve [4.1] near the origin (say for $x^2+y^2<\varepsilon^{-1/2}$) for there it reduces to a simple harmonic oscillator equation. In the tunneling region (between the two turning points), however, things are more difficult. It is to this problem that we now turn.

A. Semiclassical Approach to Multidimensional Tunneling Problems

Discussion of Method of Solution

Our method for dealing with tunneling through N dimensional potential barriers is based on a simple physical picture. A particle in an unstable state centered at the origin will ultimately penetrate the barrier and escape to infinity. The total amplitude for escape is the sum of the amplitudes over all possible paths of escape. We will show that there exist most probable escape paths (MPEP's) and that the relative amplitude to escape along other paths is exponentially small. The dominant contribution to the escape amplitude comes from regions, which we call tubes, surrounding the MPEP's. The probability current is negligible outside of these tubes

during tunneling and flows outward in narrow beams. We will show that for a system without a rotational symmetry the number of tubes is finite and that they are well-separated. We will then use semi-classical (WKB related) methods to approximate the solutions to the Schroedinger equation within these tubes.

We have introduced the notion of a tube in order to reduce our nonspherical multidimensional problem to one which is approximately one-dimensional. It is natural to try to solve a tunneling problem using WKB techniques, but standard WKB analysis has proved computationally useful only for systems with one degree of freedom or those which can be reduced to one dimension by symmetry considerations[1]. The zeroth-order WKB equation for the phase S of a wave function with energy E is[2]

$$(\vec{\nabla}S)^2 = V - E \ . \tag{4.2}$$

This is just the Hamilton-Jacobi equation for a classical system with Hamiltonian $\vec{p}^2+V$. In one dimension it reduces to $(dS/dx)^2=V-E$, whose solution is $S=\pm\int(V-E)^{1/2}$. For the general multidimensional case it is a nonlinear partial differential equation. Of course, if the Hamiltonian has a

continuous symmetry, Eq.[4.2] will be separable. However, Eq.[4.2] is nontrivial in general. The new multidimensional techniques which we have discovered simplify the problem of solving Eq.[4.2] because now we need to solve it only in a small, approximately one-dimensional region. Our technique is expressly designed to deal with problems which do not have continuous symmetries, and is thus complementary to the separation of variables idea.

We briefly review the path integral formalism. The amplitude for a particle of energy E to take a particular path P in a potential V is

$$\frac{1}{N} e^{-\int_P (V-E)^{1/2}} \tag{4.3}$$

where $\int_P (E-V)^{1/2}$ is the classical action and the normalization factor N is slowly varying and depends only on the endpoints of the path. The total amplitude is just the sum of Eq.[4.3] over all paths P.

In the tunneling region, (V-E) is positive, and the amplitude is exponentially damped. Therefore, the dominant contribution to the amplitude comes from regions near the paths which minimize the action integral and thus satisfy

$$\delta \int (V-E)^{1/2} = 0 \tag{4.4}$$

The Euler-Lagrange equations following from Eq.[4.2] are[3]

$$2(V-E)\frac{d^2x_i}{ds^2} + \frac{dx_i}{ds}\left[\sum_j \frac{dx_j}{ds}\frac{dV}{dx_j}\right] = \frac{\partial V}{\partial x_i} \quad , \tag{4.5}$$

where s is the path length. All solutions of Eq.[4.5] are local stationary points of the action. However, we are interested in the global minima. This will eliminate all except a discrete set of paths which are just the MPEP's.[4]

Once we have found a set of MPEP's, we must find approximate solutions to the Schroedinger equation along these trajectories. As in any semiclassical or ray description of a wave phenomenon, we must distinguish two levels of approximation. At the first level, called geometrical optics or the eikonal approximation, the phase of the wave function is approximated by a line integral along the trajectory, while its amplitude is assumed to be constant. This is just zeroth-order WKB. The second level, called physical optics or first-order WKB, takes into account the variation of the amplitude and the spread of the wave function into the region around the trajectory. Thus, physical optics is characterized

by a set of tubes through which most of the probability current flows.

These two levels of approximation are clearly distinguished in our results for the large-order behavior of perturbation theory. We find that in general for large n

$$A_n \sim KL^n \Gamma(Mn+J)\left(1+0\left(\frac{1}{n}\right)\right) \quad . \tag{4.6}$$

The constants L, M, and J are determined by geometrical optics alone. Physical optics is needed to find the value of K. This is entirely in accord with the results of Chap. III.

B. Determination of Most Probable Escape Paths

The program we outlined in part A of this section for finding the MPEP's is, of course, very difficult. It involves actually finding closed-form solutions to Eq.[4.5] and explicitly selecting those solutions which minimize $\int (V-E)^{1/2}$. Fortunately, in many cases, a heuristic argument enables us to guess the most probable paths without solving Eq.[4.5], and these turn out to be straight lines. In fact, it is generally true that the MPEP's for the equal mass oscillators

defined in Eq.[1.1] are straight lines. The more difficult problem of unequal mass oscillators, which have curved MPEP's, will be discussed in the next chapter. It is easy to show that the straight MPEP's satisfy Eq.[4.5], but we have no way of proving that they are global minima of the action. The only convincing evidence we have for this is the excellent agreement of our results with our computer calculations.

In this section we use geometrical optics to treat the special case of Eq.[4.1] for which a=b=1. This simplifies the notation without obscuring any of the important features of the problem. In the next section we use physical optics to treat this same case. Equation [4.1] is solved in general in Sec. E.

We expect a straight MPEP to satisfy certain reasonable criteria. It should be a "path of least resistance" to tunneling and thus should pass through a saddle point of the potential $V=x^2/4+y^2/4-\varepsilon(x^4+y^4+2cx^2y^2)/4$. The saddle point should be oriented along the path (which is a radial line). A saddle point of V satisfies the equations

$$\frac{\partial V}{\partial x} = \frac{x}{2} - \varepsilon(x^3+cxy^2) = 0 \ ,$$

$$\frac{\partial V}{\partial y} = \frac{y}{2} - \varepsilon(y^3+cx^2y) = 0 \ . \qquad (4.7)$$

Equations [4.7] have nine solutions, namely

$$(x,y) = (0,0) \ ,$$

$$[\pm(2\varepsilon)^{-1/2},0] \ , \ [0, \pm(2\varepsilon)^{-1/2}] \ ,$$

$$+ \text{ and } - \{[2(c+1)\varepsilon]^{-1/2} \ , \ \pm [2(c+1)\varepsilon]^{-1/2}\} \ . \qquad (4.8)$$

To identify those critical points in Eq.[4.8] which are radially oriented saddle points, we compute the Hessian matrix H (matrix of second partial derivatives). We demand that H have one positive and one negative eigenvalue at the critical point and further require that the eigenvector having negative eigenvalue must lie along the radial line connecting the critical point with the origin. We find that

$$H = \begin{bmatrix} \frac{1}{2} - \varepsilon(3x^2+cy^2) & -2c\varepsilon xy \\ -2c\varepsilon xy & \frac{1}{2} - \varepsilon(3y^2+cx^2) \end{bmatrix} . \qquad (4.9)$$

It follows that (0,0) is not a saddle point, that $(\pm(2\varepsilon)^{-1/2},0)$ and $(0,\pm(2\varepsilon)^{-1/2})$ are acceptable saddle points when $-1<c<1$, and + and - $\{[2(c+1)\varepsilon]^{-1/2}, \pm[2(c+1)\varepsilon]^{-1/2}\}$ are acceptable saddle points when $c>1$. When $c=1$, there are no saddle points. This is the spherically symmetric case where all straight-line paths contribute equally to the amplitude, and it is treated separately in Appendix B.

It is now easy to show that radial lines through the saddle points are solutions of Eq.[4.5][5]. Equation [4.5] reduces to

$$\frac{dx}{ds}\left\{\frac{dx}{ds}\frac{\partial V}{\partial x}+\frac{dy}{ds}\frac{\partial V}{\partial y}\right\}=\frac{\partial V}{\partial x}\,,$$

$$\frac{dy}{ds}\left\{\frac{dx}{ds}\frac{\partial V}{\partial x}+\frac{dy}{ds}\frac{\partial V}{\partial y}\right\}=\frac{\partial V}{\partial y}\,, \qquad (4.10)$$

because our straight line paths have the property that

$$\frac{d^2x}{ds^2}=\frac{d^2y}{ds^2}=0.$$

Equation [4.10] is satisfied by $(\frac{dx}{ds}, \frac{dy}{ds}) = (0,\pm1)$, $(\pm1,0)$, and $(\pm\sqrt{2}/2, \pm\sqrt{2}/2)$ because of three properties of V:

$$\frac{\partial V}{\partial y}\Big|_{y=0} = 0,$$

$$\frac{\partial V}{\partial x}\Big|_{x=0} = 0,$$

$$\frac{\partial V}{\partial x} = \pm \frac{\partial V}{\partial y} \quad \text{when } x = \pm y. \tag{4.11}$$

C. Geometrical Optics

We outline here a brief and heuristic treatment. A careful and mathematically detailed approach is given in the next section. We follow the procedure described at the end of Chap. III. Up to multiplicative constants the wave function $\psi(x,y)$ in the tunneling region on the MPEP is given by

$$e^{-\int_{s_o}^{(x,y)} \sqrt{V-E}\, ds}.$$

We are ignoring all paths except the 4 MPEP's. E is the unperturbed value of the energy, namely 1. The integral is taken along the MPEP from the inner turning point $s_o = 0(1)$ (solution of V-E=0) to the argument of $\psi(x,y)$.

We are interested in computing the current flowing out to infinity. We compute the current at a point just beyond the distant turning point $s_1=0(\varepsilon^{-1/2})$. The total current J is the sum of the currents along each MPEP. J is proportional to

$$\exp[-2\int_{s_0}^{s_1}(V-E)^{1/2}ds]$$

for each MPEP. This reduces to

$$\exp\{-2\int_{s_0}^{s_1}[\frac{s^2}{4}-\frac{\varepsilon}{4}s^4\alpha-1]^{1/2}ds\},$$

where $\alpha=\cos^4\theta+\sin^4\theta+2c\ \sin^2\theta\ \cos^2\theta$,

and θ is the angle between the MPEP and the x-axis. Computing the above integral approximately gives

$$J\propto\varepsilon^{-1/2}e^{-1/(3\varepsilon\alpha)}$$

for each MPEP. From Eq. [2.23] and [2.25], we thus obtain

$$A_n\propto\int_{-\infty}^{0}d\varepsilon(-\varepsilon)^{-n-3/2}e^{1/(3\varepsilon\alpha)}$$

$$\propto\Gamma(n+\tfrac{1}{2})\ (-3\alpha)^n\ . \qquad (4.12)$$

When $-1 \leq c < 1$, the MPEP's are along the axes and $\alpha = 1$. When $c > 1$, the MPEP's are at 45° to the axes and $\alpha = \frac{c+1}{2}$. Thus, for large n,

$$A_n \propto (-3)^n \Gamma(n+\tfrac{1}{2}), \quad -1 \leq c < 1,$$

and

$$A_n \propto [-3(c+1)/2]^n \Gamma(n+\tfrac{1}{2}), \quad c > 1. \tag{4.13}$$

Equation [4.13] is continuous in c at c=1.

Equation [4.13] clearly illustrates the phenomenon of <u>decoupling</u> that takes place in the large-order behavior of perturbation theory. When the coupling of the oscillators is strong enough (c>1), A_n depends on the coupling term. But when $-1 \leq c < 1$, the system seems to behave as if the oscillators were completely uncoupled. Actually, when c<1 the multiplicative constant K, which we will determine in the next section, still depends on c. Nevertheless, the decoupling of the <u>dominant</u> behavior of A_n for large n is quite remarkable and is typical of the simplification that we observe in large order.

D. Physical Optics

In this section we use physical optics to approximate the imaginary part of the ground-state energy for the system

$$\{-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+\frac{x^2}{4}+\frac{y^2}{4}-\frac{\varepsilon}{4}(x^4+y^4+2cx^2y^2)-E\}\psi=0. \tag{4.14}$$

We will solve the problem explicitly only for $-1\leq c<1$. When $c>1$ we use the following symmetry transformation to reduce the problem back to the $c<1$ case:[6]

$$x\rightarrow\bar{x}=\frac{x+y}{\sqrt{2}}\ ,$$

$$y\rightarrow\bar{y}=\frac{x-y}{\sqrt{2}}\ . \tag{4.15}$$

Equation [4.15] converts Eq.[4.14] into

$$\left\{-\frac{\partial^2}{\partial\bar{x}^2}-\frac{\partial^2}{\partial\bar{y}^2}+\frac{\bar{x}^2}{4}+\frac{\bar{y}^2}{4}-\frac{\varepsilon}{4}\left(\frac{1+c}{2}\right)(\bar{x}^4+\bar{y}^4+2\,\frac{3-c}{1+c}\,\bar{x}^2\bar{y}^2)-E\right\}\psi=0. \tag{4.16}$$

We then make the additional transformations

$$c\rightarrow\bar{c}=(3-c)/(1+c),$$

$$\varepsilon\rightarrow\bar{\varepsilon}=\varepsilon(1+c)/2\ , \tag{4.17}$$

and observe that $c>1$ implies that $|\bar{c}|<1$.

1. The Physical Optics Approximation

We will solve Eq.[4.14] in a tube of thickness 0(1) surrounding the positive x-axis. (The x-axis is a MPEP when $|c|<1$.) To do so, we break the tube into two regions: Region I, where y=0(1) and $x<\varepsilon^{-1/4}$, and Region II, where y=0(1) and $\varepsilon^{-1/6}<x<<\varepsilon^{1/2}$. Notice that the regions overlap.

In region I we approximate Eq.[4.14] by

$$[-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+\frac{x^2}{4}+\frac{y^2}{4}-1]\psi = 0 \ , \tag{4.18}$$

whose solution is the unperturbed ground-state wave function

$$\psi_I = e^{-(x^2+y^2)/4} \tag{4.19}$$

We have freely chosen the normalization of ψ_I.

In region II, we approximate Eq.[4.14] by

$$[-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+\frac{x^2}{4}+\frac{y^2}{4}-\frac{\varepsilon}{4}(x^4+2cx^2y^2)-1]\psi = 0, \tag{4.20}$$

where we have neglected εy^4 compared with y^2. We seek a solution which is exponentially decaying with increasing x in the tunneling region. Thus, we factor off a decreasing

WKB-type function of the x variable:

$$\psi=\chi\left(\frac{x^2}{4}-\frac{\varepsilon x^4}{4}-\frac{1}{2}\right)^{-1/4}\exp\left[-\int_{\sqrt{2}}^{x}\left(\frac{t^2}{4}-\frac{\varepsilon t^4}{4}-\frac{1}{2}\right)^{1/2}dt\right]. \tag{4.21}$$

We have chosen the lower endpoint of the integral arbitrarily. The new equation satisfied by χ is

$$(x^2-\varepsilon x^4)^{1/2}\chi_x-\chi_{yy}+\left(\frac{y^2}{4}-\frac{1}{2}-\frac{\varepsilon}{2}cx^2y^2\right)\chi=0. \tag{4.22}$$

The change of scale

$$\varepsilon x^2=z^2 \tag{4.23}$$

eliminates all reference to ε from Eq.[4.9] and gives

$$z(1-z^2)^{1/2}\chi_z-\chi_{yy}+\left(\frac{y^2}{4}-\frac{1}{2}-\frac{c}{2}y^2z^2\right)\chi=0. \tag{4.24}$$

In Eq.[4.21] we factored off the rapidly changing geometrical optics behavior. Equation [4.24] contains the next order correction to this behavior which we have referred to as physical optics. That is, Eq.[4.24], when solved, will provide the multiplicative constants that were missing in Eq.[4.13]. However, there are no further approximations to

be made because all quantities in Eq.[4.24] are of order 1. Equation [4.24] must now be solved exactly!

The change of variables

$$w = (1-z^2)^{1/2} \tag{4.25}$$

is useful because the resulting equation,

$$(w-1)\chi_w - \chi_{yy} + [\frac{y^2}{4} - \frac{1}{2} - \frac{c}{2} y^2(1-w^2)]\chi = 0 \tag{4.26}$$

no longer contains a square-root term.

One strategy for solving Eq.[4.26] is to transform the dependent variable so that a Fourier transform in the y variable gives a (hopefully soluble) first-order partial differential equation. Of course, an immediate Fourier transform of Eq.[4.26] is useless because of the y^2 term. We are thus led to the substitution.

$$\chi = e^{-y^2 f(w)/4} A. \tag{4.27}$$

The undetermined function f(w) will be chosen to eliminate the y^2 term from the differential equation for A. It will then be profitable to Fourier transform that equation because the highest power of y will be one. The above constraint on f(w) takes the form of a Riccati equation:

$$-(w^2-1)f'(w) - f^2(w) + 1 - 2c + 2cw^2 = 0. \tag{4.28}$$

When Eq.[4.15] is satisfied, the equation for A simplifies to

$$(w^2 - 1)A_w - A_{yy} + yf(w)\ A_y + \frac{1}{2}[f(w) - 1]A = 0. \tag{4.29}$$

First we solve Eq.[4.28]. A standard substitution which linearizes the Riccati equation is

$$f(w) = (w^2 - 1)u'(w)/u(w). \tag{4.30}$$

We obtain

$$(1 - w^2)u'' - 2wu' + u(2c - \frac{1}{1-w^2}) = 0. \tag{4.31}$$

We gratefully recognize that Eq.[4.31] is the associated Legendre equation[7]. Solutions to this equation are

$$u(w) = P_\nu^\mu(w),\ Q_\nu^\mu(w), \tag{4.32}$$

where

$$\nu(\nu + 1) = 2c,\ \mu^2 = 1. \tag{4.33}$$

For definiteness we choose[8]

$$u(w) = P^{-1}_\nu(w)\ . \tag{4.34}$$

Next, we return to Eq.[4.29] and complete its solution. It is clear that the strategy of the substitution in Eq.[4.27] has succeeded. That is, if we Fourier transform in the y variable, the resulting equation will be first order and should yield to the method of characteristics. However, we are fortunate that there is an even simpler approach. We change to new independent variables

$$(w,y) \rightarrow (w, s = y/u(w)) . \tag{4.35}$$

In terms of these variables, Eq.[4.29] becomes

$$(w^2 - 1)\, u^2(w) A_w + \frac{1}{2} u^2(w)[f(w) - 1]A = A_{ss} , \tag{4.36}$$

which is separable.

We now argue that the separation constant for Eq.[4.36] is 0. To justify this contention explicitly we separate

$$A(w,s) = B(w)C(s) . \tag{4.37}$$

For separation constant α^2, the equation for C(s) is $C''(s) = \alpha C(s)$, whose solution is $C(s) \propto \cosh(\alpha s) = \cosh[\alpha y/u(w)]$. Here we have kept only the even solution in y because only an even solution can be matched to ψ_I in Eq.[4.19]. However, u(w) in Eq.[4.34] vanishes (see Eq.[4.42]) at w=1 and w≲1 is in the

overlap of regions I and II (see Eqs.[4.23] and [4.25]). Therefore, there is no asymptotic match of ψ across regions I and II unless, of course, $\alpha^2=0$.

Having shown that $\alpha^2=0$ and thus that C(s) is constant, it is straightforward to solve for B(w). We obtain (omitting a multiplicative constant)

$$B(w) = u^{-1/2}[(1-w)/(1+w)]^{1/4} . \tag{4.38}$$

Note that as $w\to 1$, $B(w)\to$ a finite constant (see Eq.[4.44]). Thus, it is possible to match ψ_I and ψ_{II} asymptotically in the overlap region.

We have now solved Eq.[4.14] in region II up to an overall multiplicative constant β. Our final result is

$$\psi_{II}=\beta\left(\frac{x^2}{4} - \frac{\varepsilon x^4}{4} - \frac{1}{2}\right)^{-1/4} \exp\left[-\int_{\sqrt{2}}^{x}\left(\frac{t^2}{4} - \frac{\varepsilon t^4}{4} - \frac{1}{2}\right)^{1/2}dt\right]$$
$$\times \exp[-y^2 f(w)/4][u(w)]^{-1/2}[(1-w)/(1+w)]^{1/4} . \tag{4.39}$$

It is easy to identify the physical meaning of the three types of terms in Eq.[4.39]. There is a rapidly varying term from geometrical optics and several slowly varying terms that do not depend on y. These describe the amplitude along the MPEP.

Finally, the term $\exp[-y^2 f(w)/4]$ describes the falloff of probability current in the tube surrounding the x-axis. A quick calculation shows that as x approaches the turning line [the line along which $V(x,y)-E=0$] at the end of the tunnel near $x=\varepsilon^{-1/2}$, w approaches 0, and, for positive c, the tube gradually widens. At the turning line the tube flares out like the bell of a trumpet. When $-1 \leq c < 0$, the tube narrows as $w \to 0$. When $c=0$, the thickness of the tube is constant along its length.

It might appear that our solution Eq.[4.39] is the result of an amazing sequence of lucky substitutions whose application is rather limited. Actually, these techniques immediately generalize to <u>all</u> straight-line path problems [see IV E-F and Appendix C]. Moreover, when we study an arbitrary curved path problem in the next chapter, we show that factoring off the geometrical optics behavior always leads to a Riccati equation whose solution is related to the thickness of the tube. So, on the contrary, the substitutions we have made are both natural and general.

2. Asymptotic Matching of Regions I and II

We now determine β by requiring that ψ_I in Eq.[4.19] and ψ_{II} in Eq.[4.39] become asymptotically equal in the overlap of regions I and II. In this overlap region x is large compared with 1 but small compared with $\varepsilon^{-1/2}$. Thus, we approximate

$$\left(\frac{x^2}{4} - \frac{\varepsilon x^4}{4} - \frac{1}{2}\right)^{-1/4} \sim \left(\frac{x}{2}\right)^{-1/2}, \tag{4.40}$$

$$- \int_{\sqrt{2}}^{x} \left(\frac{t^2}{4} - \frac{\varepsilon t^4}{4} - \frac{1}{2}\right)^{1/2} dt \sim - \frac{1}{2} \int_{\sqrt{2}}^{x} (t^2-2)^{1/2} dt$$

$$= - \frac{x^2}{4} + \frac{1}{2} \log x + \frac{1}{4} + 0\left(\frac{1}{x^2}\right). \tag{4.41}$$

Also, in the overlap of regions I and II, $w \sim 1$. Thus, we use[9]

$$u(w) = P_{\nu}^{-1}(w) \sim 2^{-1/2} (1-w)^{1/2}, \text{ when } w \sim 1. \tag{4.42}$$

Equation [4.42] implies that

$$f(w) = \frac{(w^2-1)u'(w)}{u(w)} \sim 1, \tag{4.43}$$

$$u^{-1/2}\left(\frac{1-w}{1+w}\right)^{1/4} \sim 1. \tag{4.44}$$

Combining Eqs.[4.39]-[4.44] gives

$$\psi_{II} \sim \beta e^{-(x^2+y^2)/r_e 1/4\sqrt{2}}\ . \tag{4.45}$$

Thus, comparing Eq.[4.45] and Eq.[4.19] gives

$$\beta = 2^{-1/2} e^{-1/4}. \tag{4.46}$$

Now ψ in region II is completely determined relative to the normalization of ψ in region I.

3. Determination of the Probability Current J(x)

At the end of Chap. III we developed a trick for evaluating the probability current for values of x further from the origin than the turning line without ever doing turning point analysis. Without further explanation we use this technique to obtain the magnitude of the probability current emerging from the end of the tube along the positive x-axis

$$J(y) = \beta^2 e^{-y^2 f(0)/2} u^{-1}(0) \exp\left[-\int_{\sqrt{2}}^{x_1} (t^2-\varepsilon t^4-2)^{1/2} dt\right]. \tag{4.47}$$

x_1 is the distant zero of the integrand. Note that J is a function of y only.

The evaluation of the integral is given in Chap. III as

$$\exp\left[-\int_{\sqrt{2}}^{x_1} (t^2-\varepsilon t^4-2)^{1/2} dt\right] \sim e^{1/3 \varepsilon 2(e/\varepsilon)^{1/2}}. \tag{4.48}$$

To compute u(0) and f(0) we use the formulas[10]

$$u(0) = P_\nu^{-1}(0) = \frac{1}{2}\pi^{-1/2} \sin\frac{\pi\nu}{2}\, \Gamma(\tfrac{\nu}{2})/\Gamma(\tfrac{3}{2}+\tfrac{\nu}{2}),$$

$$u'(0) = \frac{d}{dw} P_\nu^{-1}(w)\Big|_{w=0} = -\pi^{-1/2} \cos\frac{\pi\nu}{2}\, \Gamma(\tfrac{1}{2}+\tfrac{\nu}{2})/\Gamma(1+\tfrac{\nu}{2}). \tag{4.49}$$

From Eqs.[4.30] and [4.49] we have

$$\frac{f(0)}{2} = \frac{\cos\frac{\pi\nu}{2}\, \Gamma(\frac{1}{2}+\frac{\nu}{2})\Gamma(\frac{3}{2}+\frac{\nu}{2})}{\sin\frac{\pi\nu}{2}\, \Gamma(\frac{\nu}{2})\Gamma(1+\frac{\nu}{2})}. \tag{4.50}$$

Finally, we combine Eqs.[4.46]-[4.48] and obtain

$$J(y) = e^{-\frac{1}{3\epsilon}} \epsilon^{-1/2} \exp[-y^2 f(0)/2]/u(0), \qquad (4.51)$$

with $u(0)$ and $f(0)/2$ given above.

4. Computation of ImE

To calculate ImE, we use Eq.[2.24] and proceed to evaluate the integrals in the denominator and numerator in turn.

The integral in the denominator is done by replacing ψ with ψ_I in Eq.[4.11] and allowing V, the region of integration, to be unbounded. This is a good approximation because the dominant contribution comes from region I. We obtain

$$\int_V \psi^* \psi \, dv \sim 2\pi \ . \qquad (4.52)$$

The integral in the numerator is a surface integral which reduces to an integral over y. We allow the endpoints of the integral to be unbounded, use Eq.[4.51], and find that

$$\int_{S_V} \vec{J}\cdot\vec{ds} \sim \frac{e^{-\frac{1}{3\varepsilon}}\ \varepsilon^{-1/2}}{u(0)} \int_{-\infty}^{\infty} dy\ e^{-y^2 f(0)/2}$$

$$\sim e^{-\frac{1}{3\varepsilon}} \left[\frac{2\pi}{\varepsilon f(0) u^2(0)}\right]^{1/2} . \qquad (4.53)$$

After using Eqs.[4.49] and [4.50], Eq.[4.53] simplifies drastically. From this result and Eqs.[2.24] and [4.52], we have finally

$$\mathrm{ImE} = 2\sqrt{2}\ e^{-1/3\varepsilon}\ [\nu(\nu+1)/\varepsilon\ \sin(\pi\nu)]^{1/2} . \qquad (4.54)$$

We have multiplied by an extra factor of 4 to obtain Eq.[4.54] because the contributions to the integral for ImE come from four equal tubes. As much current flows out along the positive x-axis as along the negative x-axis and symmetrically along the y-axis in both directions.

5. Perturbation Theory in Large Order

From Eqs.[2.23] and [4.54], we have for large n

$$A_n = \left[\frac{\nu(\nu+1)\pi}{\sin(\pi\nu)}\right]^{1/2} \frac{2\sqrt{2}}{\pi^{3/2}} \int_0^\infty d\varepsilon e^{-\frac{1}{3\varepsilon}} \varepsilon^{-n-\frac{3}{2}} (-1)^{n+1}$$

$$= 2\left[\frac{2c\pi}{\sin(\pi\nu)}\right]^{1/2} \frac{\sqrt{6}}{\pi^{3/2}} (-1)^{n+1} 3^n \Gamma(n+\tfrac{1}{2}) \ . \qquad (4.55a)$$

Equation [4.42a] is valid for $c<1$, $\nu(\nu+1)=2c$.

When $c>1$, we use Eqs.[2.23]and[4.17]to obtain

$$A_n \sim 2\left[\frac{2\bar{c}\pi}{\sin(\pi\bar{\nu})}\right]^{1/2} \frac{\sqrt{6}}{\pi^{3/2}} (-1)^{n+1} \left(\frac{3c+6}{2}\right)^n \Gamma(n+\tfrac{1}{2}) \ , \qquad (4.55b)$$

where $\bar{\nu}(\bar{\nu}+1) = 2\bar{c} = (3-c)/(1+c)$.

For $c=1$, we cite the result in Eq.[B.18], to wit

$$A_n \sim (-1)^{n+1} \frac{6}{\pi} 3^n \Gamma(n+1). \qquad (4.55c)$$

6. Discussion

We can immediately verify Eq.[4.55a] for the case $c=0 \Rightarrow \nu=0$. As $\nu \to 0$, the quantity in square brackets approaches 1, and we obtain exactly twice the result given in Ref. [1] for the one-dimensional oscillator. This is because now we have two uncoupled anharmonic oscillators.

The other, and more interesting, limit to investigate is $c \to 1 \Rightarrow \nu \to 1$. Now the term in square brackets blows up. This singularity corresponds with the onset of spherical symmetry. Recall that our analytical procedures necessarily break down for that case because there are no isolated MPEP's - all radial paths are equally probable. The sudden increase from 4 to an infinite number of MPEP's allows the probability current to escape to infinity faster, reducing the lifetime of the unstable state. We thus observe a constructive interference phenomenon which causes an enhancement in the rate of divergence of perturbation theory - A_n in Eq.[4.55c] is larger than A_n in Eqs.[4.55a] and [4.55b] by a factor of $n^{1/2}$.

One encounters many similar phenomena in optics. Consider, for example, a light beam parallel to the z-axis

and incident upon a flat elliptical plate centered about and perpendicular to the z-axis. We use ray tracing to determine the amplitude of the scattered wave at a point on the z-axis behind the ellipse. Only two rays scattering off the edge of the ellipse, namely those at the ends of the minor axes, contribute appreciably to the amplitude. However, when the lengths of the major and minor axes become equal, the scattering amplitude suffers a discontinuous jump because of constructive interference. All rays scattering off the edge of the now circular disc contribute equally to the scattered wave.

We describe the numerical verification of Eq.[4.55] in Chap. VI.

E. The Case $a,b \neq 1$

The generalization of the discussion of Secs. C and D to the case where $a,b \neq 1$ in Eq.[4.1] is entirely straightforward. As before, we find the saddle points of V by solving

$$\frac{\partial V}{\partial x} = \frac{x}{2} - \varepsilon(ax^3+cxy^2) = 0 ,$$

$$\frac{\partial V}{\partial y} = \frac{y}{2} - \varepsilon(by^3+cx^2y) = 0 ,$$

and requiring that the Hessian matrix

$$H = \begin{bmatrix} \frac{1}{2} - \varepsilon(3ax^2+cy^2) & -2cxy\varepsilon \\ -2cxy\varepsilon & \frac{1}{2} - \varepsilon(3by^2+cx^2) \end{bmatrix}$$

have negative determinant.

The critical points are $(0,0)$, $[0,\pm(2b\varepsilon)^{-1/2}]$, $[\pm(2a\varepsilon)^{-1/2},0]$, and + and - $[2\varepsilon(c^2-ab)]^{-1/2}[(c-b)^{1/2},\pm(c-a)^{1/2}]$. $[0,\pm(2b\varepsilon)^{-1/2}]$ are saddle points if $c/b<1$ and $[\pm(2a\varepsilon)^{-1/2},0]$ are saddle points if $c/a<1$. If $c>\max(a,b)$ then the off-axis critical points are saddle points. (Recall that for the Hamiltonian to be bounded below, we must have $a\geq 0$, $b\geq 0$, and $c\geq -\sqrt{ab}$.) All of the saddle points are radially directed.

As in Sec. D, the off-axis case can be reduced to the case $c/a<1$ by a rotation. To simplify the algebra we introduce the following notation:

$$D = ab-c^2 = \text{Det}\begin{vmatrix} a & c \\ c & b \end{vmatrix} , \tag{4.56a}$$

$$D_1 = b-c = \text{Det}\begin{vmatrix} 1 & c \\ 1 & b \end{vmatrix} , \tag{4.56b}$$

$$D_2 = a-c = \text{Det}\begin{vmatrix} a & 1 \\ c & 1 \end{vmatrix} , \tag{4.56c}$$

$$S = D_1 + D_2 = a + b - 2c . \tag{4.56d}$$

Then a suitable rotation is

$$x = (D_1/S)^{1/2}\ \bar{x} \pm (D_2/S)^{1/2}\ \bar{y} ,$$

$$y = \mp(D_2/S)^{1/2}\ \bar{x} + (D_1/S)^{1/2}\ \bar{y} . \tag{4.57}$$

In terms of the new variables the potential is

$$V = \frac{1}{4}(\bar{x}^2+\bar{y}^2) - \frac{1}{4}\ (\bar{a}\bar{x}^4+\bar{b}\bar{y}^4+2\bar{c}\bar{x}^2\bar{y}^2), \tag{4.58}$$

where

$$\bar{a} = D/S ,$$

$$\bar{b} = [(D_1-D_2)^2+D]/S ,$$

$$\bar{c} = 3D/S-2c . \tag{4.59}$$

Observe that

$$\bar{c} - \bar{a} = 2D/S - 2c = \frac{2D_1 D_2}{S} < 0$$

when $a>0, b>0$, and $c>\max(a,b)$.

Without loss of generality, then, we assume $c<a$. We will solve the Schroedinger equation in a narrow tube surrounding the x-axis. In region I where $0 \leq x < \varepsilon^{-1/4}$, we have

$$\psi_I \sim e^{-(x^2+y^2)/4} . \tag{4.60}$$

In region II, $\varepsilon^{-1/8} < x < \varepsilon^{-1/2}$ and we can approximate the differential equation by

$$\left[-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{x^2}{4} - \frac{\varepsilon}{4}(ax^4+2cx^2y^2) - E\right]\psi_{II} = 0 . \tag{4.61}$$

The substitutions

$$a\varepsilon \to \varepsilon ,$$
$$c/a \to c , \tag{4.62}$$

reduce Eq.[4.61] to Eq.[4.20], which has already been solved.

We, therefore, immediately deduce that the large-order behavior of perturbation theory is

$$A_n = -\frac{\sqrt{6}}{\pi^{3/2}}\ \alpha(-3\beta)^n\ \Gamma(n+\tfrac{1}{2})\ , \tag{4.63}$$

where

$$\alpha = \left\{\frac{-2\pi c}{a\ \cos\ [\frac{\pi}{2}(1+8c/a)^{1/2}]}\right\}^{1/2}\ ,$$

$$\beta = a, \tag{4.64}$$

for the case $a>b>0$, $a>c\geq -\sqrt{ab}$.

There is a factor of 2 missing from the expression for α in Eq.[4.64] relative to Eq.[5.55] because the contribution from the tube along the y-axis is negligible when $a>b$. Also, we have used [4.33] to write ν in terms of c. When $b>a$, we have similar results:

$$\alpha = \left\{\frac{-2\pi c}{b\ \cos[\frac{\pi}{2}(1+8c/b)^{1/2}]}\right\}^{1/2}\ ,$$

$$\beta = b, \tag{4.65}$$

where $b>a>0$, $b>c\geq -\sqrt{ab}$.

For the off-axis case we use Eq.[4.59] to deduce that

$$\alpha = \left\{ \frac{8\pi(2ac+2bc-3ab-c^2)}{(ab-c^2)\cos\left[\frac{\pi}{2}\left(25 - \frac{16c(a+b-2c)}{ab-c^2}\right)^{1/2}\right]} \right\}^{1/2}$$
$$\beta = \frac{ab-c^2}{a+b-2c}\ , \tag{4.66}$$

where $a>0$, $b>0$, $c>\max(a,b)$.

The results in Eqs.[4.64]-[4.66] agree to 6 places with the numbers in Table I of Chap. VI.

F. Generalization to N Dimensions

It is natural to try to extend the techniques we have developed for two mode oscillators to more complicated problems. In this section we will show that such an extension is possible for a large class of N-mode equal-mass oscillator systems. Our aim will be to present a brief overview of what can be accomplished in N dimensions, and we will not dwell on algebraic or numerical details.

We begin by studying systems having potentials of the form

$$V = \sum_{i=1}^{N} \frac{x_i^2}{4} + \frac{\lambda}{4} \sum_{i,j=1}^{N} a_{ij} x_i^2 x_j^2, \tag{4.67}$$

where a_{ij} is a real symmetric matrix. The Hamiltonian in [4.67] must be bounded below, and, therefore, a must satisfy

$$\sum_{i,j} a_{ij}y_iy_j \geq 0 \text{ for } y_i \geq 0 \ . \tag{4.68}$$

When $\lambda=-\varepsilon(\varepsilon>0)$ the critical points of V are the solutions of

$$0 = \frac{\partial V}{\partial x_i} = \frac{x_i}{2} - \varepsilon x_i \sum_{j=1}^{N} a_{ij}x_j^{\,2} \ . \tag{4.69}$$

Thus

$$x_i = 0, \text{ or } \frac{1}{2\varepsilon} = \sum_{j=1}^{N} a_{ij}x_j^{\,2} \tag{4.70}$$

We will first consider the case where all $x_i \neq 0$. Then Eq.[4.70] has a solution if a is nonsingular. We define D_i to be the determinant of the matrix obtained from a by replacing each element of the i^{th} column by 1. Then $\Sigma a_{ij}D_j=\det(a)$ for all i, and

$$x_j^{\,2} = \frac{1}{2\varepsilon}\frac{D_j}{\det(a)} \ . \tag{4.71}$$

If this is to correspond to a point in real space, we must have

$$\frac{D_i}{\det(a)} > 0, \quad 1 \leq i \leq N \ . \tag{4.72}$$

Of course, there are 2^{N-1} vectors which satisfy Eq.[4.71] because we can choose the sign of each component of x_i independently.

The Hessian matrix at the critical point is

$$H_{ij} = \frac{\delta_{ij}}{2} - \varepsilon\delta_{ij} \sum_k a_{ik}x_k^2 - 2\varepsilon x_i x_j a_{ij} \ . \tag{4.73}$$

Using Eq.[4.71], we can rewrite this as

$$H_{ij} = -2\varepsilon x_i x_j a_{ij} = -\frac{\sigma_i\sigma_j}{\det(a)} \sqrt{D_i D_j}\, a_{ij} \ , \tag{4.74}$$

where σ_i is the sign of x_i. Note that D_iD_j is always positive (see Eq.[7.72]). Equations [4.69] and [4.74] imply that

$$\sum_j H_{ij}\sigma_j \sqrt{D_j} = -\sigma_i \sqrt{D_i} \ , \tag{4.75}$$

which means that the radial line through each critical point is one of the critical point's principal axes. Furthermore, the minus sign in Eq.[4.75] implies that along this radial line, V has a maximum at the critical point. Thus, if all the other eigenvalues of H at the critical point are positive,

we have a radially directed saddle point.

A necessary condition for H to have one negative and N-1 positive eigenvalues is

$$\det H < 0 \ . \tag{4.76}$$

But,

$$\det H = [-\det(a)]^{-N}\det[\sigma_i\sigma_j\sqrt{D_iD_j}\ a_{ij}]$$

$$= - [-\det(a)]^{-N-1}\prod_{i=1}^{N} D_i \ .$$

Using Eq.[4.72], we see that

$$\det H < 0 <=> (-1)^N\det(a) < 0 \ . \tag{4.77}$$

In two dimensions, Eq.[4.77] is indeed satisfied by the off-axis saddle points that we discussed in Sec. VI. In fact, this condition and Eq.[4.72] imply that c>max(a,b) which we have shown to be a necessary and sufficient condition for an off-axis saddle point in two dimensions. In higher dimensions, however, we can have det H<0 without having one negative and N-1 positive eigenvalues. It is difficult to give a simple necessary and sufficient condition for saddle points in the general N-dimensional case.

If several of the x_i are zero the procedure for finding a saddle point is slightly more complicated. We choose to label the axes so that the first M x_i are zero. Then, the condition for a critical point becomes

$$x_i = 0, \; i = 1,\ldots M,$$

$$\frac{1}{2\varepsilon} = \sum_{j=M+1}^{N} a_{ij} x_j{}^2, \; i = M+1,\ldots N \; . \tag{4.78}$$

The discussion proceeds as before in the subspace of nonzero components. We find that the Hessian matrix is given by

$$\left[\begin{array}{c|c} \begin{array}{c} \frac{1}{2} - \varepsilon \sum_{k=M+1}^{N} a_{1k} x_k^2 \\ \\ \frac{1}{2} - \varepsilon \sum_{k=M+1}^{N} a_{Mk} x_k{}^2 \end{array} & \\ \hline & -2\varepsilon x_i x_j a_{ij} \end{array}\right]$$

where the x_i's are the solutions of Eq.[7.78]. Thus, in addition to the usual conditions on the N-M dimensional matrix $x_i x_j a_{ij}$, we must have

$$\frac{1}{2} - \varepsilon \sum_{M+1}^{N} a_{ik} x_k^2 > 0 \ , \ i=1,\ldots M \ .$$

The special case of an on-axis critical point, where all but one of the x_i vanish, is important because any other configuration can be reduced to this one by a rotation. Here the condition for a saddle point becomes

$$\frac{1}{2} - \varepsilon a_{iN} x_N^2 > 0, \ i=1,\ldots N-1,$$

where

$$x_N^2 = \frac{1}{2\varepsilon a_{NN}} \ . \tag{4.79}$$

It is now easy to generalize the arguments of Secs. IV and VI to compute the contribution to the large-order behavior of perturbation theory from this saddle point. We observe that in the tube where $\varepsilon^{-1/6} < x_N < \varepsilon^{-1/2}$, $x_i = 0(1)$, $i \neq N$, we can make the approximation

$$\frac{\varepsilon}{4} \sum_{i,j} a_{ij} x_i^2 x_j^2 \sim \frac{\varepsilon}{4} x_N^2 \sum_{j=1}^{N-1} a_{Nj} x_j^2 + \frac{\varepsilon}{4} a_{NN} x_N^4 \ .$$

Following the procedure of Sec. D, we factor a rapidly varying WKB function of the x_N variable out of the wave

function and neglect all terms in the resulting equation which vanish as $\varepsilon \to 0$. After a simple change of variables (see Eq.[4.25]), we obtain a partial differential equation which must be solved exactly:

$$(w^2-1)\chi_w + \sum_{i=1}^{N-1} [- \frac{\partial^2}{\partial x_i^2} + \frac{x_i^2}{4} - \frac{1}{2} - \frac{a_{iN}}{2a_{NN}} x_i^2(1-w^2)]\chi = 0.$$

The ansatz

$$\chi = A\exp[- \frac{1}{4} \sum_{i=1}^{N-1} f_i(w) x_i^2]$$

generates N-1 Riccati equations whose solutions govern the thickness of the tube of probability current in the directions perpendicular to the MPEP. Then the change of variables

$$w = w$$

$$s_i = x_i/u_i(w) \quad , \quad 1 \leq i \leq N-1 \ ,$$

with

$$f_i = (w^2-1)u_i'(w)/u_i(w) \ ,$$

reduces the equation for A to one that is separable:

$$(w^2-1)A_w + \frac{1}{2}\sum_{i=1}^{N-1}[f_i(w)-1]A = \sum_{i=1}^{N-1}\frac{1}{u_i^2(w)}\frac{\partial^2}{\partial S_i^{\,2}}A \ .$$

Finally, we require that the wave function in the tunneling region match to a harmonic oscillator wave function (the solution of the Schroedinger equation near the origin). As in Sec. IV, this implies that A is a function only of w. It is then easy to determine the probability current and evaluate the dispersion integral. The resulting contribution to the large-order behavior of perturbation theory is

$$A_n = -\prod_{i=1}^{N-1}\left[\frac{8\pi a_{iN}}{a_{NN}\sin(\pi\nu_i)}\right]^{1/2}\sqrt{6}\,\pi^{-3/2}(-3a_{NN})^n\Gamma(n+\tfrac{1}{2}) \ , \tag{4.80}$$

where $\nu_i(\nu_i+1) = 2a_{iN}/a_{NN}$.

This expression will be equal to the true large-order behavior if the x_N axis is the MPEP. However, as we mentioned above, we can use Eq.[7.14] even if the dominant saddle point does not lie on this axis. As an example, let us consider the case of a dominant critical point whose coordinates are nonvanishing. Then the following rotation will align the radial line through the saddle point with the $\bar{x}_N$ axis:

$$x_i = \sum_{j=1}^{N} R_{ij}\bar{x}_j \ , \tag{4.81}$$

$$R = \begin{bmatrix} a_{1\,1} & a_{N-1\,1} & (D_1/\sum_{i=1}^{N} D_i)^{1/2} \\ a_{1\,N} & a_{N-1\,N} & (D_N/\sum_{i=1}^{N} D_i)^{1/2} \end{bmatrix}$$

We choose the first N-1 column vectors of R to lie along the other principal axes of the saddle point. In the new coordinate system V will no longer have the simple form in Eq.[4.67]. It will contain terms like $\bar{x}_i^3\,\bar{x}_j$. However, it is easy to see that there are no terms of the form $\bar{x}_N^3\,\bar{x}_j$ $(j\neq N)$ or $\bar{x}_N^2\bar{x}_i\bar{x}_j$ $(i,j\neq N)$. Such terms would give a nonvanishing contribution to $\partial^2 V/\partial\bar{x}_i\partial\bar{x}_j$ at the critical point. (The coordinates of the critical point are $\bar{x}_j = 0$, $1\leq j\leq N-1$, $\bar{x}_N\neq 0$.) Thus V has the form

$$V = \sum_{i=1}^{N} \frac{\bar{x}_i^{\,2}}{4} - \frac{\varepsilon}{4}\left(\bar{a}_{NN}\,\bar{x}_N^4 + \sum_{i=1}^{N-1} \bar{a}_{iN}x_i^{\,2}x_N^{\,2} + b\right),$$

where b depends at most linearly on $\bar{x}_N$. In the tube where $\bar{x}_i = 0(1)$, $1\leq i\leq N-1$, and $\varepsilon^{-1/6}\leq x_N\leq\varepsilon^{-1/2}$ we can clearly approximate V by neglecting b entirely.

We then use Eq.[4.80] to compute the large-order behavior of perturbation theory as before. The computation of $\bar{a}_{iN}$ for all i is a tedious algebraic problem. Still it is easy to find the value of $\bar{a}_{NN}$, the coefficient of $\bar{x}_N^4$ in $\sum_{i,j} a_{ij} x_i^2 x_j^2$. Using Eq.[4.81], it is

$$\bar{a}_{NN} = \sum_{i,j} a_{ij} \frac{D_i D_j}{(\Sigma D_j)^2} = \frac{1}{(\Sigma D_j)^2} \sum_j D_j \det(a) = \frac{\det(a)}{\Sigma D_j} .$$

Thus, the large-order behavior of perturbation theory is

$$A_n \sim -\sqrt{6}\, \pi^{-3/2}\, 2^{N-1} K_N \left[\frac{-3 \det(a)}{\Sigma D_j}\right]^n \Gamma(n+\tfrac{1}{2}) . \qquad (4.82)$$

The constant K_N may be determined from Eq.[4.80] once we have computed $\bar{a}_{iN}$ for $i \neq N$. The factor 2^{N-1} reflects the possible choices of the sign of x_j in Eq.[4.71]. Similar formulas exist for the case where some of the x_i vanish at the dominant saddle point.

Equation [4.80] may also be used to find the large-order behavior of the perturbation series for systems having an infinite number of degrees of freedom. As an example, consider

the sequence of potentials

$$V^{(N)} = \frac{1}{4} \sum_{i=1}^{N} x_i^2 - \frac{\varepsilon}{4} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} x_i^2 x_j^2 . \tag{4.83}$$

The critical points of $V^{(N)}$ are given by

$$0 = \frac{\partial V^{(N)}}{\partial x_i} = \frac{x_i}{2} - \frac{\varepsilon}{4} x_i \sum_{j \neq 1} x_j^2 , \tag{4.84}$$

whose solutions are

$$x_i = 0 \quad \text{or} \quad x_i^2 = [2\varepsilon(M-1)]^{-1} . \tag{4.85}$$

M is the number of nonzero x_i. Note that M can never be one, so Eq. [4.85] always makes sense. The Hessian matrix is

$$H_{ij}^{(N)} = - \frac{\delta_{ij}}{2(M-1)} - 2\varepsilon x_i x_j + 3\varepsilon \delta_{ij} (x_i^2) . \tag{4.86}$$

Observe from Eq. [4.86] that we cannot have a saddle point if any of the x_i vanish because Eq. [4.79] is violated. Hence, all $x_i \neq 0$ and

$$H_{ij}^{(N)} = \frac{\delta_{ij}}{2(N-1)} - \frac{\sigma_i \sigma_j}{(N-1)} ,$$

where σ_i is the sign of x_i. The eigenvalues of this matrix are

$\frac{1}{2} - \frac{N}{N-1}$ which is negative and $\frac{2}{N-1}$, the latter having multiplicity N-1. Therefore, we have a saddle point. There is a different saddle point for each of the 2^{N-1} choices of sign for x_i.

To determine the large-order behavior of perturbation theory for this system, we must compute $D_i^{(N)}$ and $\det[a^{(N)}]$ (see Eq.[4.82]). The matrix $a^{(N)}$ is given by

$$\begin{bmatrix} 0 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 0 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 0 & & & \\ & & & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \end{bmatrix}$$

Thus,

$$\det[a^{(N)}] = (N-1)(-1)^{N-1}$$

and

$$D_i^{(N)} = (-1)^{N-1}, \quad 1 \leq i \leq N .$$

We can also compute $\bar{a}_{iN}^{(N)}$ because the potential is so symmetric. We find that

$$\bar{a}_{iN}^{(N)} = \frac{N-3}{N} \ , \ 1 \leq i \leq N-1 \ ,$$

$$\bar{a}_{NN}^{(N)} = \frac{N-1}{N} \ . \tag{4.87}$$

Therefore, from Eqs.[4.80] and [4.87], the large-order behavior of perturbation theory is

$$A_n^{(N)} = \sqrt{6}\ \pi^{-3/2} \left[\frac{-8\pi \frac{N-3}{N-1}}{\cos\frac{3\pi}{2}\sqrt{1-\frac{16}{9(N-1)}}} \right]^{\frac{1}{2}(N-1)} (-3\tfrac{N-1}{N})^n \Gamma(n+\tfrac{1}{2}) \ . \tag{4.88}$$

Now consider the limit as $N \to \infty$. This limit defines an infinite-mode oscillator system which strictly speaking, is some nonlocal field theory. The leading contribution to Eq.[4.88] which comes from geometrical optics remains finite in this limit:

$$(-3\tfrac{N-1}{N})^n \Gamma(n+\tfrac{1}{2}) \to (-3)^n \Gamma(n+\tfrac{1}{2}) \ .$$

However, the constant from physical optics blows up. The divergence of this constant derives from two sources. The factor $(\sqrt{8\pi})^{N-1}$ occurs in any N-mode problem in which no axis passes through the dominant saddle point. The vanishing of the cosine term as $N \to \infty$ is a more singular divergence of the form $N^{N/2}$. It reflects the disappearance of the saddle

point. The extreme symmetry of the potential makes the saddle become flat as N becomes large. This kind of symmetry is not present in potentials arising from $(\phi^N)_2$ quantum field theories.

It is amusing that we can eliminate this divergence by a mass renormalization. We will argue that by adding a lower-order mass term to the potential we can insure that the $N\to\infty$ limit of $A_n^{(N)}$ exists. Consider the effect of adding a term of the form $\varepsilon R^{(N)} \sum_{i=1}^{N} \frac{1}{4} x_i^2$. Because this term is at most $0(1)$ in the tunneling region, it cannot affect the determination of the MPEP. This term is merely a correction of order $0(\varepsilon)$ to the mass, and thus can actually be viewed as a mass renormalization. Following the scaling arguments of Sec. VII, Ref.II.1 we find that this term contributes an overall multiplicative constant (independent of n) to the large-order growth of perturbation theory, namely

$$\exp\left[\frac{-NR^{(N)}}{2(N-1)}\right] .$$

On the other hand, we easily determine from Eq.[4.88] that the two large-N divergences which we discussed give precisely

$$[6(N-3)]^{\frac{1}{2}(N-1)} .$$

Hence, if we choose

$$R^{(N)} = \frac{(N-1)^2}{N} \ln(6N-18) , \tag{4.89}$$

then the limit $N\to\infty$ may be taken. We obtain the remarkable result that the large-order behavior of the Rayleigh-Schroedinger coefficients A_n for the ground-state energy perturbation series of the infinite-mode system described by the potential

$$V = \lim_{N\to\infty} \{\tfrac{1}{4}[1-\lambda R^{(N)}] \sum_{i=1}^{N} x_i^2 + \tfrac{1}{4}\lambda \sum_{\substack{i,j=1 \\ i\neq j}}^{\infty} x_i^2 x_j^2\} , \tag{4.90}$$

is

$$A_n \sim -\sqrt{6}\, \pi^{-3/2} (-3)^n \Gamma(n+\tfrac{1}{2}) , \tag{4.91}$$

where $N\to\infty$ in such a way that λN is small. This is precisely the result in Eq.[3.49] for the one-mode anharmonic oscillator. Of course, it can be argued that the choice of the potential in Eq.[4.90] is somewhat contrived. Nonetheless, we feel that Eq.[4.91] is just one more example of the extraordinary simplification that takes place in the large-order limit of perturbation theory.

CHAPTER V

SYSTEMS WITH CURVED MPEPS

A. Introduction

The reader should be convinced by now that the techniques we have introduced are practical means of computing the large order behavior of perturbation theory for equal mass oscillators. The essential feature which makes these systems so tractable is the simplicity of their MPEP's - they are straight lines. In this chapter we begin a study of systems with curved MPEP's. The results obtained are far from complete but they do show that our methods work for curved MPEP's.

Our interest in curved path problems is not merely academic. We will see in Chapter VII that cut off scalar field theories are equivalent to systems of anharmonically coupled oscillators with unequal masses (the "masses" here are just the energies of the field modes). These systems have curved MPEP's so it seems likely that any extension of our work to real quantum field theories will have to deal with the problem of curved paths.

The work which we present is organized as follows.

Section B gives a formal solution of the problem assuming that the MPEP is known. Specifically, we show how to com-

pute the physical optics (first-order WKB) approximation to the wave function of a tunneling particle in terms of the MPEP. The expressions we derive are surprisingly simple and bear a remarkable resemblance to the expressions we found in the straightline MPEP case treated in Chpater IV. As in IV we obtain a Riccati equation which determines the thickness of the tube surrounding the (curved) MPEP. In Section C we use a perturbative approach to formally attack the problem of finding the MPEP. We study potentials of the form $V=U_0+\eta U_1$ where U_0 has straight-line MPEP's and η is small. We obtain a perturbative expression for the MPEP and derive a compact form for the geometrical optics approximation to the wave function valid to second order in η.

In Sec. D we apply the general techniques of the previous two sections to the specific potential

$$V = \frac{1}{4}(x^2+y^2) + \frac{\lambda}{4}(x^2+y^2+2c^2y^2) + \frac{\eta}{4}y^2 \quad . \qquad (5.1)$$

We calculate the large-order behavior of the ground-state energy perturbation series (a power series in λ) to second order in η. Chapter VI gives a comparison between the theoretical predictions in Section D and extensive computer calculations. The agreement is excellent.

B. Tunneling Along Curved Paths

In this section we present a formal semiclassical treatment of tunneling along a curved path. For simplicity we consider only two-dimensional problems defined by

$$(-\nabla^2+V-E)\psi=0 \tag{5.2}$$

As explained in Chap. IV, the wave function is concentrated in a narrow region surrounding the most probable escape path (MPEP), which is the trajectory that minimizes the action integral $\int(V-E)^{1/2}$. We emphasize that although the MPEP may be curved, it is assumed to be known. Finding the MPEP is a difficult but classical problem. Thus, in this section we show how to reduce the quantum mechanical problem of tunneling to a purely classical one.

We suppose the MPEP to be given parametrically by

$$\begin{aligned} x&=\phi_1(s) \quad , \\ y&=\phi_2(s) \quad , \end{aligned} \tag{5.3}$$

where s is the path length. Since we are interested in the region surrounding the MPEP, it is convenient to introduce a suitable curvilinear coordinate system. We take one coordinate to be the path length s and the other to be the perpendicular

distance n from the curve (see Fig. 5). Of course, this is only a local definition valid for small n. It will not be necessary to describe the global nature of the coordinate system.

The tangent vector to the curve is

$$[\phi_1'(s),\ \phi_2'(s)] \quad .$$

This vector is a <u>unit</u> vector because

$$\left(\frac{d\phi_1}{ds}\right)^2 + \left(\frac{d\phi_2}{ds}\right)^2 = \frac{(dx)^2+(dy)^2}{(ds)^2} = 1 \quad .$$

A unit vector normal to the curve (that is, to the tangent vector) is

$$[-\phi_2'(s),\phi_1'(s)] \quad .$$

Hence the relation between the (s,n) and (x,y) coordinate systems is

$$\begin{aligned} x &= \phi_1(s) - n\phi_2'(s) \quad , \\ y &= \phi_2(s) + n\phi_1'(s) \quad . \end{aligned} \tag{5.4}$$

Again we emphasize that these relations are local and must be altered for sufficiently large n.[1]

To solve Eq.(5.2), we will need an expression for the scalar product of two vectors in the (s,n) coordinate system. This is most easily obtained in terms of the metric tensor $g_{\mu\nu}$, which is given in the (s,n) coordinate system by[2]

$$\begin{bmatrix} g_{ss} & g_{sn} \\ g_{sn} & g_{nn} \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 & \frac{\partial x}{\partial n}\frac{\partial x}{\partial s} + \frac{\partial y}{\partial n}\frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial n}\frac{\partial x}{\partial s} + \frac{\partial y}{\partial n}\frac{\partial y}{\partial s} & \left(\frac{\partial x}{\partial n}\right)^2 + \left(\frac{\partial y}{\partial n}\right)^2 \end{bmatrix} \tag{5.5}$$

$g_{\mu\nu}$ in Eq.(5.5) may be simplified using the relations

$$\begin{aligned} \frac{\partial x}{\partial s} &= \phi'_1 - n\phi'' \quad , \\ \frac{\partial y}{\partial s} &= \phi'_2 + n\phi''_1 \quad , \\ \frac{\partial x}{\partial n} &= -\phi'_2 \quad , \\ \frac{\partial y}{\partial n} &= \phi'_1 \quad . \end{aligned}$$

We obtain

$$\begin{aligned} g_{ss} &= 1+2n(\phi'_2\phi''_1-\phi'_1\phi''_2)+n^2[(\phi''_1)^2+(\phi''_2)^2]=(1+n\rho)^2 \quad , \\ g_{nn} &= (\phi'_1)^2+(\phi'_2)^2 = 1 \quad , \\ g_{sn} &= 0, \end{aligned} \tag{5.6}$$

where ρ, the curvature of the path, is given by $\phi_2'\phi_1''-\phi_1'\phi_2''$. The off-diagonal elements of the metric tensor vanish because Eq.(5.4) describes an orthogonal coordinate system. Finally, we observe that the reciprocal of the metric tensor is given by

$$g^{ss}=(1+n\rho)^{-2}$$

$$g^{nn}=1 \quad ,$$

$$g^{ns}=0 \quad . \tag{5.7}$$

Having established Eq.(5.7), we proceed to find the WKB approximation to ψ in Eq.(5.2). Substituting

$$\psi=Ae^{-S} \tag{5.8}$$

into Eq.(5.2) gives

$$-\nabla^2 A+2\vec{\nabla}A\cdot\vec{\nabla}S+A\nabla^2 S-A(\vec{\nabla}S)^2+A(V-E)=0. \tag{5.9}$$

The WKB approximation follows from the assumption that S^2 and V-E are large and of the same order of magnitude. Equating powers of S gives the eikonal equation

$$(\vec{\nabla}S)^2=V-E \tag{5.10}$$

and the transport equation

$$2\vec{\nabla}A\cdot\vec{\nabla}S+A\nabla^2 S=0. \tag{5.11}$$

We have disregarded the term $\nabla^2 A$. We simplify Eq. (5.11) by

multiplying by A:

$$\vec{\nabla}\cdot(A^2\vec{\nabla}S)=0. \tag{5.12}$$

The n-dimensional eikonal equation is very hard to solve in general. However, we need only solve it in the neighborhood of the MPEP, which is an approximately one-dimensional region. We expand V-E, S, and A as power series in n:

$$V-E=V_0(s)+nV_1(s)+n^2V_2(s)+...., \tag{5.13}$$

$$S=S_0(s)+nS_1(s)+n^2S_2(s)+n^3S_3(s)+...., \tag{5.14}$$

$$A=A_0(s)+.... \quad . \tag{5.15}$$

We will assume that the linear term S_1 in the expansion of S vanishes. This is the first place where we use the assumption that $\vec{\phi}(s)$ describes a MPEP. We expect the amplitude to reach a maximum on the MPEP and to fall of exponentially on either side. This is not true if $S_1\neq 0$.

Using the expansion in Eq. (5.14) and Eq.(5.7), we express the scalar product $(\vec{\nabla}S)$ as

$$\begin{aligned} g^{\mu\nu}\nabla_\mu S\nabla_\nu S &= (1+n\rho)^{-2}S_s^2+S_n^2 \\ &= S_0'^2 - 2\rho n S_0'^2 \\ &\quad +n^2(3\rho^2 S_0'^2+2S_0'S_2'+4S_2^2) \\ &\quad +0(n^3). \end{aligned} \tag{5.16}$$

Matching powers of n reduces the eikonal equation [Eq.(5.10)] to three equations:

$$S_0'^2 = V_0,$$
$$-2\rho S_0'^2 = V_1 ,$$
$$3\rho^2 S_0'^2 + 2S_0'S_2' + 4S_2^2 = V_2 . \qquad (5.17)$$

The first of these equations has the familiar solution $S_0'=\pm\sqrt{V_0}$, whence

$$S_0 = + \int^s \sqrt{V_0}\ ds \quad . \qquad (5.18)$$

We choose the plus sign because we are describing tunneling. The real part of the wave function ψ should decrease with increasing path length.

The second equation may then be rewritten as

$$\rho = - \frac{V_1}{2V_0} \quad . \qquad (5.19)$$

This equation relates the path directly to the potential and makes no reference to ψ. We therefore view Eq.(5.19) as a consistency condition for our approximation scheme. This condition arises because we have assumed that $S_1=0$. In Appendix A we show that Eq.(5.19) is a consequence of the classical equations of motion (the Euler-Lagrange equations

of $\delta\int\sqrt{V-E}=0$). Thus, our approximation is valid along classical paths.

The third equation is a Riccati equation:

$$S_2'+2V_0^{-1/2}S_1^2=\frac{1}{2}V_0^{-1/2}(V_2-3\rho^2V_0) \quad . \tag{5.20}$$

To convert Eq.(5.20) into a linear second-order differential equation, we substitute

$$S_1=\frac{1}{2}V_0^{1/2}u'/u \quad . \tag{5.21}$$

Then,

$$u''+\frac{1}{2}(V_0'/V_0)u'=u(V_2-3\rho^2V_0)/V_0 \quad , \tag{5.22}$$

or in self-adjoint form

$$(u'V_0^{1/2})'+u(3\rho^2V_0^{1/2}-V_2V_0^{-1/2})=0 \quad . \tag{5.23}$$

This equation can be further simplified to Schroedinger form by introducing the new independent variable

$$t(s)=\int^{s} dsV_0^{-1/2} \quad . \tag{5.24}$$

We obtain

$$-\frac{d^2}{dt^2}u+u(V_2-3\rho^2V_0)=0 \quad . \tag{5.25}$$

We have thus reduced the computation of S_2, which describes the thickness of the beam of probability current, to the solution of a second-order ordinary differential equation. In Chap. IV, where we considered only straight-line ($\rho=0$) MPEP's, Eq.(5.23) was an associated Legendre equation.

This completes our study of the eikonal equation. The transport equation [Eq.(5.12)] need only be solved to zeroth order in n. In this order it is

$$\frac{d}{ds}[A_0^2 S_0'] + 2S_2 A_0^2 = 0 \ ,$$

or

$$\frac{dB}{dS} + 2S_1 V_0^{-1/2} B = 0 \quad , \tag{5.26}$$

where $B = A_0^2 V_0^{1/2}$. (5.27)

Hence,

$$B = (\text{const.})\exp[-2\int^s S_1 V_0^{-1/2} ds] \ ,$$

and $A_0 = (\text{const.}) V_0^{-1/4} u^{-1/2}$, (5.28)

where u is defined in Eq.(5.21).

Combining Eqs. (5.8), (5.14), (5.18), (5.21), and (5.28), our final WKB expression for the wave function is

$$\psi = (\text{const.})V_0^{-1/4}u^{-1/2}e^{-\int^s V_0^{1/2}} e^{-\frac{1}{2}n^2 V_0^{1/2}u'/u} \tag{5.29}$$

where u satisfies Eq.(5.22). To define u (and thus ψ) uniquely, boundary conditions must be imposed on Eq.(5.22). These are determined by matching ψ asymptotically to the solutions of Eq.(5.2) which are valid outside the tunneling region. The matching procedure is exactly analogous to the one for straight-line MPEP's described in Chap. IV.

Equation (5.29) is remarkably similar to the wave function we found in IV [eq.(4.39)]. As in that expression we can identify three physically distinct terms in the wave function ψ. Along the MPEP, ψ is given by a rapidly varying exponential term from geometrical optics and a slowly varying term independent of n. The term $\exp(-\frac{1}{2}n^2 V_0^{1/2}u'/u)$ describes the spread of probability current into the area surrounding the MPEP.

There are, of course, two difficulties present in Eq.(5.29) which were not encountered in the straight-line case. Equation (5.22) is a second-order differential equation and cannot always be solved. Moreover, the functions

V_0, V_2, and ρ appearing in Eqs.(5.22) and (5.29) are undetermined until we have found the MPEP.

The first difficulty is not as bad as it might seem. Because Eq.(5.22) contains a small parameter (the anharmonic coupling constant) for the class of problems in which we are interested, it may be possible to treat it perturbatively. Furthermore, even if we cannot find u, Eq.(5.9) still enables us to compute the dominant large-order behavior of perturbation theory. We demonstrated in the proceding chapters that the leading behavior (the factorial and power growth) could be computed directly from the geometrical optics approximation to the wave function. This approximation is given be

$$\psi \sim e^{-\int^s \sqrt{V_0}\, ds}$$

and requires no knowledge of u.

The second difficulty, that of actually determining the MPEP, is more serious and we have not found a general way to avoid it. It might at first appear that the classical equations may be treated perturbatively for small λ. However, as we show in the next section, this is not true. A simple scaling of the dependent and independent variables completely eliminates the anharmonic coupling constant from

the classical equations, leaving a complicated system of coupled nonlinear equations which must be solved exactly. In the next section we show how to solve these equations when the MPEP is very nearly a straight line.

C. Slightly Curved MPEP's

In this section we show how to obtain the MPEP perturbatively when it differs only slightly from a straight line.

A MPEP is a solution of the classical equations which makes the action $\int\sqrt{V-E}\ ds$ a global minimum. The classical equations are the Euler-Lagrange equations obtained from

$$\delta\int\sqrt{V-E}\ ds = 0 \quad , \tag{5.30}$$

where s is the path length. Equation (5.30) is a constrained variational problem because the path always satisfies

$$[\frac{d}{ds}\ \vec{\phi}(s)]^2 = 1 \quad . \tag{5.31}$$

We can proceed by introducing either a Lagrange multiplier or a dummy parameter t. Following the second method, we derive the Euler-Lagrange equations of the new variation problem

$$\delta\int\sqrt{V-E}\ [(\frac{d}{dt}\vec{\phi})\ ]^{1/2}\ dt = 0 \qquad (5.32)$$

and then set t=s. Either way we obtain

$$2(V-E)\frac{d^2\vec{\phi}}{ds^2} + \frac{d\vec{\phi}}{ds}\ (\frac{d\vec{\phi}}{ds}.\vec{\nabla}\ V)=\vec{\nabla}V \quad , \qquad (5.33)$$

where $(\frac{d\vec{\phi}}{ds})^2=1$ and $\vec{\nabla}V$ means $\partial V/\partial\vec{\phi}$.

For the specific potential V in Eq.(5.1), Eq.(5.33) is a pair of equations, the first of which is

$$[\frac{1}{2}(x^2+y^2+\eta y^2)-\frac{\varepsilon}{2}\ (x^4+y^4+2cx^2y^2)-2E]x''(s)$$

$$+[s'(s)]^2\ (\frac{1}{2}x-\varepsilon x^3-\varepsilon cxy^2)+x'(s)y'(s)(\frac{1}{2}y-\varepsilon y^3-{}^3cyx^2)$$

$$= \frac{1}{2}x-\varepsilon x^3-\varepsilon cxy^2 \ , \qquad (5.34)$$

where we must have $\lambda=-\varepsilon<0$ for tunneling to occur. The second equation of the pair is similar. Equation (5.34) is very difficult to solve in general and the following scaling argument shows that it cannot even be solved perturbatively for small ε. Simply letting $x\to x\varepsilon^{-1/2}$, $y\to y\varepsilon^{-1/2}$, $s\to s\varepsilon^{-1/2}$, and neglecting E compared with x^2 in the tunneling region gives a new equation, almost identical to Eq.(5.34), in which all reference to ε has vanished. Fortunately, we do not have to find all solutions to Eq. (5.34). The MPEP we seek is a special

solution and has in the past been relatively easy to find. In BBWI we found straight-line MPEP's for the equal-mass case (η=0) even though the general solution remains unknown. Here we show how to solve equations like Eq.(5.34) for the MPEP as a perturbation series in η.

We proceed formally by assuming a potential of the form

$$V-E = U_0 + \eta U_1 \ , \tag{5.35}$$

where U_0 has a straight-line MPEP and η is small. U_0 and U_1 are functions of $\vec{\phi}(s)$ and thus implicitly functions of s.

The straight-line solution of the unperturbed problem is just

$$\vec{\phi}_0(s) = s\vec{\phi}_0' \ . \tag{5.36}$$

$\vec{\phi}_0'$ is a constant vector pointing along the path. Plugging this result into Eq.(5.33) gives

$$\vec{\phi}_0' \times \{\vec{\phi}_0' \cdot \vec{\nabla}U_0[\vec{\phi}_0(s)]\} = \vec{\nabla}U_0[\vec{\phi}_0(s)], \tag{5.37}$$

in which $(\vec{\phi}_0')^2$=1. Eq.(5.37) may be written more simply as

$$\vec{\nabla}U_0[\vec{\phi}_0(s)] = \frac{d}{ds}U_0[\vec{\phi}_0(s)]\vec{\phi}_0' \ . \tag{5.38}$$

From here on we simplify our notation by suppressing the argument of any function which is evaluated at $\vec{\phi}_0(s)$. This will shorten many of the formulas which are to follow, but we

caution that it can lead to some confusion. $\vec{\nabla}W$ means compute $\partial W/\partial\vec{\phi}$ and evaluate the result at $\vec{\phi}=\phi_0$. Thus, $a(s)\cdot\vec{\nabla}W=0$ does not imply that $(a(s)\cdot\vec{\nabla})^2W=0$!

We now expand $\vec{\phi}(s)$ in a power series in η:

$$\vec{\phi} = \vec{\phi}_0+\eta\vec{\phi}_1+\eta^2\vec{\phi}_2+\ldots \quad . \tag{5.39}$$

The constraint [Eq.(5.31)] gives

$$1=(\vec{\phi}_0')^2+\sum_{n=1}^{\infty}\eta^n\sum_{m=0}^{n}\vec{\phi}_m'\cdot\vec{\phi}_{n-m}' \quad ,$$

whence

$$\sum_{m=0}^{n}\vec{\phi}_m'\cdot\vec{\phi}_{n-m}' = 0 \quad . \tag{5.40}$$

Also, we have

$$U_{0,1}(\vec{\phi}) = \sum_{n=0}^{\infty}\frac{1}{n!}\{[(\sum_{m=1}^{\infty}\eta^m\vec{\phi}_m)\cdot\vec{\nabla}]^nU_{0,1}\}(\vec{\phi}_0) \quad . \tag{5.41}$$

Plugging these expressions into Eq.(5.33) and keeping terms to first order in η gives

$$\vec{\phi}_1'\cdot\vec{\phi}_0' = 0 \quad , \tag{5.42}$$

and

$$2U_0\vec{\phi}_1'' + \vec{\phi}_1' \frac{d}{ds} U_0(s) + \vec{\phi}'[\vec{\phi}_1' \cdot \vec{\nabla}U_0 + (\vec{\phi}_0' \cdot \vec{\nabla})(\vec{\phi}_1 \cdot \vec{\nabla}U_0) +$$

$$\vec{\phi}_0' \cdot \vec{\nabla}U_1] = \vec{\nabla}(\vec{\phi}_1 \cdot \vec{\nabla}U_0 + U_1) \quad . \tag{5.43}$$

Equation (5.42) implies that $\vec{\phi}_1 \cdot \vec{\phi}_0'$ is a constant and we choose the constant to be 0 without loss of generality. Using Eq.(5.38) we then have

$$\vec{\phi}_1 \cdot \vec{\nabla}U_0 = 0 \quad . \tag{5.44}$$

Furthermore, since the component of $\vec{\phi}_1$ parallel to $\vec{\phi}_0'$ vanishes, we need only consider the components of Eq.(5.43) which are perpendicular to $\vec{\phi}_0'$. The equation for these components simplifies to

$$2U_0(\vec{\delta} \cdot \vec{\phi}_1)'' + \frac{d}{ds}U_0(s)(\vec{\delta} \cdot \vec{\phi}_1)' - \vec{\delta} \cdot \vec{\nabla}U_1 - (\vec{\delta} \cdot \vec{\nabla})(\vec{\phi}_1 \cdot \vec{\nabla})U_0 = 0, \tag{5.45}$$

where $\vec{\delta}$ is a unit vector perpendicular to $\vec{\phi}_0'$.

We have not yet specified the boundary conditions that $\vec{\phi}_1$ must satisfy. We will see later that the requirements that $\vec{\phi}_1$ be finite and have a finite derivative are sufficient to completely define the solution of Eq.(5.45).

1. Calculation to First Order in η

It is surprising that we can calculate the wave function in the geometrical optics approximation [see Eq.(5.18)],

$$\psi = e^{-\int \sqrt{V_0}} ,$$

to first order in η without even solving Eq. (5.44). The solution of Eq.(5.45) is needed to compute ψ to second order in η.

To first order in η we have

$$V_0 = U_0 + \eta U_1 + \eta \vec{\phi}_1 \cdot \vec{\nabla} U_0 , \tag{5.46}$$

but Eq.(5.44) eliminates the last term from this equation. Hence

$$\psi = \exp -\int [(U_0+\eta U_1)(\vec{\phi}_0(s))] \} \tag{5.47}$$

This expression is indeed independent of $\vec{\phi}_1$.

2. Calculation to Second Order in η

In Chap. III we showed that the large-order behavior of perturbation theory was determined up to a multiplicative constant by the geometrical optics approximation to the wave function. In particular, the barrier penetration factor P,

given by

$$P = \exp[-2 \int_{s_0}^{s_1} V_0^{1/2}(s)ds] \ , \tag{5.48}$$

is required. [s_0 and s_1, the nearby and distant turning points, are zeroes of $V_o(s)$.]

We now calculate ℓnP to second order in η. We have

$$V_0 = U_0 + \eta U_1 + \eta^2[\vec{\phi}_2\cdot\vec{\nabla}U_0 + \tfrac{1}{2}(\vec{\phi}_1\cdot\vec{\nabla})^2 U_0 + \vec{\phi}_1\cdot\vec{\nabla}U_1] \ . \tag{5.49}$$

Equation (5.38) gives

$$\vec{\phi}_2\cdot\vec{\nabla}U_0 = \vec{\phi}_2 \cdot \vec{\phi}_0' \frac{dU_0}{ds} \tag{5.50}$$

When n=2, Eq. (5.40) gives

$$\vec{\phi}_2'\cdot \vec{\phi}_0' = -\tfrac{1}{2}\vec{\phi}_1'^{\,2} \ , \tag{5.51}$$

from which we have

$$\vec{\phi}_2 \cdot \vec{\phi}_0' = -\frac{1}{2} \int_{\bar{s}_0}^{s} [\vec{\phi}_1'(s)]^2 ds \ . \tag{5.52}$$

$\bar{s}_0$ is the nearby zero of $U_0[\vec{\phi}_o(s)]$ and it differs from s_o, the nearby zero of V_0, by terms of order η.

We now approximate the expression for ℓnP as follows:

$$\ell nP=-2\int_{s_0}^{s_1}\{U_0+\eta U_1+\eta^2[\vec{\phi}_2\cdot\vec{\nabla}U_0+\tfrac{1}{2}(\vec{\phi}_1\cdot\vec{\nabla})^2U_0+\vec{\phi}_1\cdot\vec{\nabla}U_1]\}^{1/2}ds$$

$$\sim -2\int_{s_0}^{s_1}(U_0+\eta U_1)^{1/2}\quad -\eta^2\int_{s_0}^{s_1}ds\frac{-\frac{1}{2}\frac{dU_0}{ds}\int_{\bar{s}_0}^{s}\vec{\phi}_1'^2+\frac{1}{2}(\vec{\phi}_1\cdot\vec{\nabla})^2U_0+(\vec{\phi}_1\cdot\vec{\nabla})U_1}{(U_0+\eta U_1)^{1/2}}$$

(5.53)

Here we have used Eqs.(5.49)-(5.52). At this point in the computation all reference to ϕ_2 has been eliminated. Because the second integral in Eq.(5.53) is multiplied by η^2, we can change its present limits, s_0 and s_1, to $\bar{s}_0$ and $\bar{s}_1$, the zeroes of U_0. Also we can neglect the U_1 term in the denominator. Hence,

$$\ell nP=-]\int_{s_0}^{s_1}(U_0+\eta U_1)^{1/2}-\eta^2\int_{\bar{s}_0}^{\bar{s}}dsU_0^{1/2}\,,\{-\frac{1}{2}\frac{dU_0}{ds}\int_{\bar{s}_0}^{s}\vec{\phi}_1'^2$$

$$+\tfrac{1}{2}(\phi_1\cdot\vec{\nabla})^2U_0+(\phi_1\cdot\vec{\nabla})U_1\}+0(\eta^3)\quad .\qquad (5.54)$$

Some manipulation of integrals must be done to simplify Eq. (5.54). We define

$$J = -\int_{\bar{s}_0}^{\bar{s}} \frac{1}{2} ds U_0^{-1/2} \{ \frac{dU_0}{ds} \int_{\bar{s}_0}^{s} (\vec{\phi}_1')^2 \} \quad . \tag{5.55}$$

Integrating by parts gives

$$J = \left[-U_0^{1/2} \int_{\bar{s}_0}^{s} (\phi_1')^2 ds \right] \Bigg|_{\bar{s}_0}^{\bar{s}_1} + \int_{\bar{s}_0}^{\bar{s}_1} ds U_0^{1/2} (\phi_1')^2 \quad .$$

Integrating by parts again gives

$$J = \left[-U_0^{1/2} \int_{\bar{s}_0}^{s} (\vec{\phi}_1')^2 ds \right] \Bigg|_{\bar{s}_0}^{\bar{s}_1} + U_0^{1/2} \vec{\phi}_1 \cdot \vec{\phi}_1' \Bigg|_{\bar{s}_0}^{\bar{s}_1}$$

$$- \int_{\bar{s}_0}^{\bar{s}_1} \frac{1}{2} ds \, U_0^{-1/2} \frac{dU_0}{ds} \vec{\phi}_1' \cdot \vec{\phi}_1 - \int_{\bar{s}_0}^{\bar{s}_1} ds U_0^{1/2} \vec{\phi}_1 \cdot \vec{\phi}_1 \quad . \tag{5.56}$$

Next, we use the differential equation for $\vec{\phi}_1$ [Eq.(5.43) to evaluate the last integral:

$$-\int_{\bar{s}_0}^{\bar{s}_1} ds U_0^{1/2}\,\vec{\phi}_1''\cdot\vec{\phi}_1 = -\int_{\bar{s}_0}^{\bar{s}_1} \frac{1}{2} ds U_0^{-\frac{1}{2}}\left\{\vec{\phi}_1\cdot\vec{\nabla}U_1+(\vec{\phi}_1\cdot\vec{\nabla})^2 U_0-\vec{\phi}_1\cdot\vec{\phi}_1'\frac{dU_0}{ds}\right\} \tag{5.57}$$

We have used the result that $\vec{\phi}_1\cdot\vec{\phi}_0'=0$ to simplify Eq. (5.57).

Plugging Eq.(5.57) into Eq. (5.56) gives

$$J=\left[-U_0^{1/2}\int_{\bar{s}_0}^{s}(\vec{\phi}_1')^2 ds\right]\Bigg|_{\bar{s}_0}^{\bar{s}_1} + U_0\vec{\phi}_1'\cdot\vec{\phi}_1\Bigg|_{\bar{s}_0}^{\bar{s}_1}$$

$$-\int_{\bar{s}_0}^{\bar{s}_1}\frac{1}{2} ds U_0^{-1/2}\,\vec{\phi}_1\cdot\vec{\nabla}U_1+(\vec{\phi}_1\cdot\vec{\nabla})^2 U_0 \quad . \tag{5.58}$$

Finally, we substitute Eq. (5.58) into Eq. (5.54):

$$\ell nP=-2\int_{s_0}^{s_1}(U_0+\eta U_1)^{1/2} ds-\eta^2\left\{\int_{\bar{s}_0}^{\bar{s}_1}\frac{1}{2} ds U_0^{-1/2}(\vec{\phi}_1\cdot\vec{\nabla})U_1\right.$$

$$\left.-U_0^{1/2}\int_{\bar{s}_0}^{s}(\phi_1')^2 ds\Bigg|_{\bar{s}_0}^{\bar{s}_1}+U_0^{1/2}\,\phi_1'\cdot\phi_1\Bigg|_{\bar{s}_0}^{\bar{s}_1}\right\}+0(\eta^3). \tag{5.59}$$

The linear differential equation for $\vec{\phi}_1$ [Eq.(5.43) has singular points at the zeroes of U_0, namely $\bar{s}_0$ and $\bar{s}_1$. Therefore, the boundary terms in Eq.(5.59) do not obviously vanish even though $U_0^{1/2}$ vanishes at both $\bar{s}_0$ and $\bar{s}_1$. Fortunately, we are not interested in the general solution of Eq. (5.43) because $\vec{\phi}_1$ must be interpreted as a real path. Consequently, it is required that $\vec{\phi}_1$ be everywhere finite. Furthermore, since $(\vec{\phi}_1')^2=1$ all along the path, $\vec{\phi}_1'$ must also be finite. Thus, we disregard surface terms and obtain

$$\ell nP=-2\int_{s_0}^{s_1} ds(U_0+\eta U_1)^{1/2}-\eta^2\int_{\bar{s}_0}^{\bar{s}_1}\frac{1}{2}dsU_0^{-1/2}(\vec{\phi}_1\cdot\vec{\nabla})U_1+0(\eta^3). \tag{5.60}$$

This is our final expression for ℓnP, which we will evaluate for the specific potential in Eq. (5.1) in the next section. We will, at that time, pursue the question of the boundary conditions for ϕ_1 in greater detail.

3. Comments on the n^{th}-Order Calculation

Unfortunately, the nonlinearity of Eq.(5.33) makes the perturbation calculation that we have outlined quite complex as the order n increases. However, we will show that there are some features of the calculation that are true in all

orders.

First, we emphasize that Eq.(5.60) is correct to second order in η. The first-order term is independent of the perturbed path, while the second-order term depends on the first-order correction to the path. In general, the constraint in Eq.(5.40) allows us to express the n^{th}-order contribution to P in terms of $\vec{\phi}_0$, $\vec{\phi}_1$,...., $\vec{\phi}_{n-1}$, but not $\vec{\phi}_n$.

Second, in every order, the equation comparable to Eq.(5.33) is a linear inhomogeneous differential equation. The differential operator is the <u>same</u> in all orders; only the inhomogeneous term changes from order to order. To verify these assertions we introduce the following notation:

$$U_0[\vec{\phi}(s)] = \sum_{n=0}^{\infty} \eta^n U_0^{(n)}(\vec{\phi}_0, s) \quad ,$$

$$U_1[\vec{\phi}(s)] = \sum_{n=0}^{\infty} \eta^n U_1^{(n)}(\vec{\phi}_0, s) \quad ,$$

where $U_i^{(n)}(\vec{\phi}_0, s)$ is the coefficient of η^n in

$$\sum_{n=0}^{\infty} \frac{1}{n!} \{ [\sum_{m=1}^{\infty} \eta^m \vec{\phi}_m(s)] \cdot \vec{\nabla} \}^n U_i \Big|_{\vec{\phi}_0(s)} \quad .$$

$U_i^{(n)}$ can of course be written in terms of multinomial co-

efficients but the resulting expression is not very illuminating. We now express the n^{th} order equation as

$$2\sum_{j=1}^{n}[U_0^{(n-j)}+U_1^{(n-j-1)}]\vec{\phi}_j''(s) + \sum_{j=0}^{n}[\frac{d}{ds}U_0^{(n-j)}+\frac{d}{ds}U_1^{(n-j-1)}]\vec{\phi}_j'(s)$$

$$=\vec{\nabla}[U_0^{(n)}+U_1^{(n-1)}] \quad , \tag{5.61}$$

in which the relation

$$\vec{\phi}(s)\cdot\vec{\nabla}U[\vec{\phi}(s)] = \frac{d}{ds}U[\vec{\phi}(s)]$$

and the convention

$$U_i^{(n)} = 0 \ , \qquad n<0$$

have been used.

Next we observe that in Eq.(5.61) the only term in $U_i^{(n)}$ that contains $\vec{\phi}_n$ is $\vec{\phi}_n\cdot\vec{\nabla}U_i$. Also, $\vec{\phi}_n$ does not appear in $U_i^{(m)}$ for $m<n+i$. Thus, we rewrite Eq.(5.61) as

$$2U_0^{(0)}\vec{\phi}_n'' + \frac{d}{ds}U_0^{(0)}\vec{\phi}_n' + \vec{\phi}_0'\frac{d}{ds}[\vec{\phi}_n\cdot\vec{\nabla}U_0^{(0)}]-\vec{\nabla}[\vec{\phi}_n\cdot\vec{\nabla}U_0^{(0)}]=\vec{h}_n \quad , \tag{5.62}$$

where $\vec{h}_n$ does not depend on $\vec{\phi}_n$. Hence, only the inhomo-

geneous term of Eq.(5.62) changes from order to order. Since Eq.(5.40) determines $\vec{\phi}_n \cdot \vec{\phi}_0'$ in terms of $\vec{\phi}_1, \ldots, \vec{\phi}_{n-1}$, we need only solve it for directions perpendicular to $\vec{\phi}_0'$. Hence, if $\vec{\delta} \cdot \vec{\phi}_0' = 0$ we have

$$2U_0^{(0)} \vec{\delta} \cdot \vec{\phi}_n'' + \frac{dU_0^{(0)}}{ds} \vec{\phi}_n' \cdot \vec{\delta} - \vec{\delta} \cdot \vec{\nabla} (\vec{\phi}_n \cdot \vec{\nabla}) U_0 = \vec{\delta} \cdot \vec{h}_n \ . \tag{5.63}$$

Observe the similarity between Eqs. (5.45) and (5.63). The computation of $\vec{\phi}_n$ to all orders is possible in principle if we can solve the homogeneous equation corresponding to Eq. (5.63) exactly. If we cannot, further approximations must be made.

D. Illustrative Calculation

Our presentation to this point has been extremely formal and general. In this section we illustrate the techniques we have developed in the previous two sections by specializing to a particular oscillator. We consider the unequal mass version of the two-mode oscillator that we studied in Secs. C and D of Chap. IV:

$$V - E = \frac{1}{4}(x^2 + y^2) - \frac{\varepsilon}{4}(x^4 + 2cx^2y^2 + y^4) + \frac{\eta}{4}y^2 - E \equiv U_0 + \eta U_1 . \tag{5.64}$$

The specific calculation that we perform here is a geo-

metrical optics determination of the large-order behavior of perturbation theory to second order in η for the potential in Eq. (5.64). We will do this by evaluating the expression for ℓnP given in Eq.(5.60). As is the case in IV [see Eq.(4.6)] geometrical optics gives the factorial and power growths of the Rayleigh-Schroedinger coefficients. At the end of this section, we show that our results compare favorably with a computer calculation.

1. Discussion of Saddle Points

Because we work to leading order in ε, we may neglect E in Eq.(5.64). Recall that when $\eta=0$, U_0 has radially directed saddle points at $x=\pm y$, $y=\pm[2\varepsilon(c+1)]^{-1/2}$ when $c>1$. The analysis of Chap. IV. tells us that the MPEP's are straight lines through these saddle points.

When $\eta\neq 0$, we have saddle points at

$$x=\pm(2\varepsilon)^{-1/2},\ y=0,\ 1+\eta>c, \tag{5.65a}$$

$$y=\pm(2\varepsilon)^{-1/2},\ x=0,\ 1+\eta<\frac{1}{c}, \tag{5.65b}$$

$$x=\pm\left(\frac{c-1+c\eta}{2\varepsilon c^2-2\varepsilon}\right)^{1/2},\ y=\pm\left(\frac{c-1-c\eta}{2\varepsilon c^2-2\varepsilon}\right)^{1/2},\ \frac{1}{c}<1+\eta<c. \tag{5.65c}$$

The first two sets of saddle points face the origin but the off-axis saddle points do not; we have straight-line MPEP's if $1+\eta<c$ or $1+\eta>\frac{1}{c}$, but otherwise we must solve a curved path

problem. To solve this more difficult problem we assume that for small η the MPEP's are perturbations of the four straight lines through $x= y$, $y= [2\ (c+1)]^{-1/2}$. By symmetry we need only perturb about one of these lines.

2. Perturbative Determination of the Path to First Order in

In the notation of Sec. C we have $\vec{\phi}_0=\frac{1}{2}\sqrt{2}\ s(1,1)$. This is the expression for the MPEP to lowest order in η and is all that is necessary to evaluate the first integral in Eq.(5.60) for ℓnP. However, we defer this calculation until part C of this section. We proceed to compute the first-order correction to the MPEP because this result is needed to evaluate the second integral in ℓnP.

The equation for $\vec{\phi}_0$ allows us to express the following quantities as functions of s:

$$U_0(\vec{\phi}_0)=\frac{1}{4}\ s^2[1-\varepsilon\alpha s^2] \quad ,$$

$$U_1(\vec{\phi}_0)=\frac{1}{8}\ s^2 \quad ,$$

$$\vec{\nabla}U_1(\vec{\phi}_0)=\frac{1}{4}\sqrt{2}\ \ s(0,1) \quad , \tag{5.66}$$

where

$$\alpha\equiv\frac{1}{2}(c+1) \quad .$$

Equation (5.34) implies that $\vec{\phi}_1$ has the form

$$\vec{\phi}_1=(\phi_1,-\phi_1) \ . \tag{5.67}$$

Hence, Eqs. (5.64), (5.65), and (5.66) give

$$(\phi_1\cdot\nabla)\nabla U_0=\phi_1\cdot\nabla[\tfrac{1}{2}x-\varepsilon(x^3+cxy^2),\ \tfrac{1}{2}y-\varepsilon(y^3+cyx^2)]\Big|_{x=y=\frac{1}{2}\sqrt{2}\ s}$$

$$=\phi_1[\tfrac{1}{2}-\tfrac{1}{2}\varepsilon(3-c)s^2,\ -\tfrac{1}{2}+\varepsilon(3-c)s^2] \ .$$

Thus, $(\vec{\phi}_0'\cdot\vec{\nabla})(\vec{\phi}_1\cdot\vec{\nabla})U_0=0$ and

$$(\vec{\phi}_0'\cdot\vec{\nabla})U_1=\tfrac{1}{4}\ s \ . \tag{5.68}$$

Plugging the above expressions into the formal result in Eq. (5.45), we finally obtain the differential equation for $\phi_1(s)$:

$$s^2[1-\varepsilon\alpha s^2]\frac{d^2\phi_1(s)}{ds^2}+s[1-2\varepsilon\alpha s^2]\frac{d\phi_1(s)}{ds}+(\varepsilon\omega s^2-1)\phi_1(s)$$

$$=-\tfrac{1}{4}\sqrt{2}\ s \ , \tag{5.69}$$

where

$$\omega\equiv 3-c \ .$$

We convert Eq. (5.69) to a more familiar form by introducing the variables

$$z=(1-\varepsilon\alpha s^2)^{1/2} \tag{5.70}$$

and

$$f(z)=\varepsilon^{-1/2}\phi_1(s) \ . \tag{5.71}$$

In terms of these variables Eq. (5.69) becomes

$$(1-z^2)f''(z)-2zf'(z)+[\frac{\omega}{\alpha} - (1-z^2)^{-1}]f(z)=-\frac{1}{2}[2\alpha(1-z^2)]^{-1/2} \tag{5.72}$$

We recognize this equation, from which all ε dependence has disappeared, as an inhomogeneous Legendre equation.[3] Homogeneous solutions are associated Legendre functions of the form

$$P_\nu^{\pm 1} \ , \quad Q_\nu^{\pm 1} \ ,$$

where $\nu(\nu+1)=\omega/\alpha$.

A particular solution of Eq.(5.72) is found by noticing that

$$[(1-z^2)\frac{d^2}{dz^2}-2z\frac{d}{dz} - \frac{1}{1-z^2}] \ (1-z^2)^{-1/2} = 0, \tag{5.73}$$

so that

$$[(1-z^2)\frac{d^2}{dz^2} -2z\frac{d^2}{dz^2} - \frac{1}{1-z^2} + \omega/\alpha]\sigma(1-z^2)^{-1/2}=\frac{\sigma\omega}{\alpha}(1-z^2)^{-1/2} \quad .$$

Therefore, if $\sigma=-\frac{\sqrt{2\alpha}}{4\omega}$, then

$$\sigma(1-z^2)^{-1/2} \tag{5.74}$$

is a particular solution of Eq. (5.72)

The general solution is then

$$f(z)=\beta P^1_\nu(z)+\gamma Q^1_\nu(z) -\frac{\sqrt{2\alpha}}{4\omega}(1-z^2)^{-1/2} \quad . \tag{5.75}$$

β and γ are still arbitrary but, as we argued in Sec. C, f(s) and f'(s) must be finite along the path. In particular, f and its derivative must be finite at the endpoints of the path, which in the notation of Sec. C, are the points s_1 and s_0. s_1 and s_0 are the distant and nearby turning points of $V_0(s)$ [see Eq. (5.49)]. Actually, we will require finiteness at $\bar{s}_1$ and $\bar{s}_0$, the turning points of U_0, because it is simplest to work at these points. [Choosing to use $\bar{s}_0$ and $\bar{s}_1$ instead of s_0 and s_1 can only affect the evaluation of ℓnP to third order in η because $s_0-\bar{s}_0=0(\eta)$ and $s_1-\bar{s}_1=0(\eta)$.]

As $s\to\bar{s}_0$, $z\to 1$ and [4]

$$\lim_{z\to 1} f(z)=\lim_{z\to 1}\left[-\frac{\gamma}{\sqrt{2}}(1-z)^{-1/2} - \frac{\sqrt{\alpha}}{4\omega}(1-z)^{-1/2}\right] \quad .$$

Thus, we must choose

$$\gamma = -\sqrt{2}\alpha \,/\, (4\omega) . \tag{5.70}$$

The distant zero of U_0 is $\bar{s}_1=(\varepsilon\alpha)^{-1/2}$, which corresponds to z=0. f(z) is finite at z=0. However, Eq.(5.70) gives

$$\frac{df}{ds} = - \frac{\varepsilon\alpha s}{1-\varepsilon\alpha s^2} f'(z) \quad ,$$

and thus the derivative of the path at $\bar{s}_1$ is finite only if f'(0)=0. But

$$f'(0) = \beta \frac{d}{dz} P_\nu^1(z) \Big|_{z=0} + \gamma \frac{d}{dz} Q_\nu^1(z) \Big|_{z=0} \quad .$$

This expression reduces to[5]

$$f'(0)=\beta \left[\frac{4\Gamma(\frac{\nu}{2}+\frac{3}{2}) \sin\frac{\pi(\nu+1)}{2}}{\sqrt{\pi}\,\Gamma(\frac{\nu}{2})}\right] + \gamma \left[\frac{2\sqrt{\pi}\,\Gamma(\frac{\nu}{2}+\frac{3}{2}) \cos\frac{\pi(\nu+1)}{2}}{\Gamma(\frac{\nu}{2})}\right] \quad .$$

The vanishing of f'(0) thus requires that

$$\beta = \frac{1}{2}\gamma\pi\tan(\tfrac{1}{2}\pi\nu) \quad . \tag{5.77}$$

Combining Eqs. (5.75-5.77) completely determines f(z):

$$f(z)= -\frac{\sqrt{2}\alpha}{4\omega} \left[\tfrac{1}{2}\pi\tan(\tfrac{1}{2}\pi\nu) P_\nu^1(z) + Q_\nu^1(z) + (1-z^2)^{-1/2}\right] . \tag{5.78}$$

This is our final result for the MPEP to first-order in η.

To review, the first-order correction to the straight-line path is given by $\vec{\phi}_1$ through Eqs.(5.70) and (5.71). The finiteness conditions of f(s) assure us that the boundary terms in Eq. (5.59) vanish.

3. Evaluation of ℓnP. First Integral in Eq. (5.60)

The evaluation of the first term of ℓnP in Eq. (5.60) is relatively easy. The integral is formally given by

$$-2\int_{s_0}^{s_1} ds\,(U_0+\eta U_1)^{1/2} \quad .$$

For the case of the potential in Eq. (5.64) this integral is just

$$-\int_{s_0}^{s_1} s\sqrt{1+\tfrac{1}{2}\eta-\varepsilon\alpha s^2}\; ds$$

$$= -\frac{1}{3\varepsilon\alpha}\,(1+\tfrac{\eta}{2}-\varepsilon\alpha s^2)^{3/2}\Big|_{s_0}^{s_1} \quad . \qquad (5.79)$$

s_1 is the larger zero of $V_0(s)=U_0+\eta U_1+0(\eta^2)$ [see Eq.(5.49).

Therefore, $(1+\frac{\eta}{2} - \varepsilon\alpha s_1^2)^{3/2} = [0(\eta^2)]^{3/2} = 0(\eta^2)$. Furthermore, s_0 is $0(1)$. Thus, $\varepsilon\alpha s_0^2$ is negligible relative to $1+\frac{1}{2}\eta$. This implies that we can approximate Eq. (5.79) by

$$\frac{1}{3\varepsilon\alpha}\,(1+\tfrac{1}{2}\eta)^{3/2} = \frac{1}{3\varepsilon\alpha}\,(1+\tfrac{3}{4}\eta + \tfrac{3}{32}\eta^2)+0(\eta^3) \quad . \tag{5.80}$$

Equation (5.80) is the desired expression for the first contribution to ℓnP. Observe that it was not necessary to know $f(z)$ to obtain Eq. (5.80).

4. Evaluation of ℓnP. Second Integral in Eq. (5.60)

The second integral contributing to ℓnP is formally

$$-\int_{\bar{s}_0}^{\bar{s}_1} \frac{ds}{2\sqrt{U_0}}\,(\vec{\phi}_1\cdot\vec{\nabla})U_1 \quad .$$

But, according to Eq. (5.66)

$$\frac{(\vec{\phi}_1\cdot\vec{\nabla})U_1}{2\sqrt{U_0}} = -\frac{1}{2\varepsilon}\,\frac{f(z)}{z} \quad .$$

Furthermore, from Eq. (5.70)

$$ds = -\frac{1}{\sqrt{\varepsilon\alpha}}\,\frac{z}{1-z^2}\,dz \quad .$$

Hence, the integral in Eq. (5.79) reduces to

$$\frac{1}{2\varepsilon\sqrt{2\alpha}} \int_0^1 \frac{f(z)dz}{1-z^2} \quad . \tag{5.81}$$

A glance at Eq. (5.78) shows that each of the three contributions to Eq. (5.81) is separately divergent at z=1. To extract a finite answer we must integrate up to x<1 and then let x tend to 1 at the end of the calculation. The third term gives

$$\int_0^x \frac{dz}{1-z^2} = \frac{1}{2}\ell n\ \frac{1+x}{1-x} \sim \frac{1}{2}\ell n2 - \frac{1}{2}\ell n(1-x) \quad , \tag{5.82}$$

for $x \sim 1$.

The first and second terms are slightly more complicated. Using the differential equations satisfied by P_ν^1, Q_ν^1, and $(1-z^2)^{-1/2}$ [Eqs. (5.72) and (5.73)] it is easy to show that[6]

$$\int_0^x \frac{W(z)dz}{\sqrt{1-z^2}} = \frac{1}{\nu(\nu+1)} (1-z^2)^{1/2} \left[\frac{zW}{1-z^2} - W'\right] \Bigg|_{z=0}^{z=x} \quad ,$$

where $W=P_\nu^1$ or $Q^1{}_\nu$. A substantial amount of algebra now gives

$$\int_0^x \frac{dz}{\sqrt{1-z^2}} \left[\frac{1}{2}\pi\tan\left(\frac{1}{2}\pi\nu\right)P_\nu^1(z)+Q_\nu^1(z)\right]$$

$$= -\frac{1}{\nu(\nu+1)}\frac{d}{dx}\left\{\frac{1}{2}\pi\tan\left(\frac{1}{2}\pi\nu\right)P_\nu^1(x)\sqrt{1-x^2}+Q_\nu^1(x)\sqrt{1-x^2}\right\} \tag{5.83}$$

We must evaluate this expression for x near 1. $P_\nu^1(x)$ is finite near x=1, so it is sufficient to consider its leading behavior there:[7] $P_\nu^1(x)\sim-\nu(\nu+1)\left(\frac{1-x}{2}\right)^{1/2}$. Thus,

$$\frac{d}{dx}\left[P_\nu^1(x)\sqrt{1-x^2}\right]\sim\nu(\nu+1). \tag{5.84}$$

We must be more careful with the second term in Eq. (5.83). We use an expression for $Q_\nu^1(x)$ in terms of hypergeometric functions[8]

$$Q_\nu^1(x) = \frac{2\pi^{3/2}}{\sqrt{1-x^2}}\left\{\frac{x\tan\left(\frac{1}{2}\pi\nu\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(-\frac{1}{2}-\frac{\nu}{2}\right)}F\left(-\frac{\nu}{2},\frac{1}{2}+\frac{\nu}{2};\frac{3}{2};x^2\right)\right.$$

$$\left.+\frac{\cot\left(\frac{1}{2}\pi\nu\right)}{2\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)}F\left(-\frac{1}{2}-\frac{\nu}{2},\frac{\nu}{2};\frac{1}{2};x^2\right)\right\}. \tag{5.85}$$

and an asymptotic expansion of F near x=1[9]

$$F(a,b;a+b+1;x^2) \sim \frac{\Gamma(a+b+1)}{\Gamma(a)\Gamma(b)} \{\frac{1}{ab} + (1-x^2) [\ell n(1-x^2) - h_{ab}]\}$$

$$+ 0[(1-x^2)^2 \ell n(1-x^2)] . \qquad (5.86)$$

In Eq. (5.86)

$$h_{ab} = \psi(1) + \psi(2) - \psi(a+1) - \psi(b+1) , \qquad (5.87)$$

where ψ is the logarithmic derivative of Γ.

We combine Eqs. (5.85)-(5.87) with the second term of Eq. (5.83) and simplify. The leading divergence drops out leaving one that is only logarithmic:

$$- \frac{d}{dx}[\frac{\sqrt{1-x^2}\ Q_\nu^1(x)}{\nu(\nu+1)}] = \frac{\sin^2(\frac{1}{2}\pi\nu)}{\nu(\nu+1)} + \frac{1}{2} + \frac{1}{2}\ell n2 + \frac{1}{2}\ell n(1-x)$$

$$- \frac{1}{2}[h_1 \sin^2(\tfrac{1}{2}\pi\nu) + h_2 \cos^2(\tfrac{1}{2}\pi\nu)]$$

$$+ 0[(1-x)\ell n(1-x)] , \qquad (5.88)$$

where

$$h_1 = \psi(1)+\psi(2)-\psi(1-\tfrac{\nu}{2})-\psi(\tfrac{3}{2}+\tfrac{\nu}{2}) \ ,$$

$$h_2 = \psi(1)+\psi(2)-\psi(\tfrac{1}{2}-\tfrac{\nu}{2})-\psi(1+\tfrac{\nu}{2}) \quad . \tag{5.89}$$

Observe that the divergent terms in Eqs. (5.82) and (5.88) just cancel.

Next, we combine all the expressions in Eqs. (5.78) and (5.81) - (5.88) and obtain a finite result:

$$-\frac{1}{8\varepsilon\omega}\left[\ell n2+\frac{1}{2}+\frac{\sin^2(\frac{1}{2}\pi\nu)}{\nu(\nu+1)}-\frac{1}{2}\pi\tan(\tfrac{1}{2}\pi\nu)\right.$$

$$\left.-\tfrac{1}{2}h_1\sin^2(\tfrac{1}{2}\pi\nu)-\tfrac{1}{2}h\cos^2(\tfrac{1}{2}\pi\nu)\right] \quad . \tag{5.90}$$

This expression can be simplified using Eq. (5.89) and some well-known identities for the ψ function.[10] Our final result for the second contribution to ℓnP is

$$-\frac{1}{8\varepsilon\omega}\left[\gamma+\psi(\nu+1)-\tfrac{1}{2}\pi\tan(\tfrac{1}{2}\pi\nu)\right] \ , \tag{5.91}$$

where γ is Euler's constant (=.5772156....).

5. The Rayleigh-Schroedinger Coefficients

Adding together the results in Eqs. (5.80) and (5.91) completes the evaluation of ℓnP:

$$\ell nP = -\frac{1}{3\varepsilon\alpha}\left\{1+\frac{3}{4}\eta+\frac{3\alpha}{8\omega}\left[\frac{\omega}{4\alpha}+\gamma+\psi(\nu+1)-\frac{1}{2}\pi\tan\left(\frac{1}{2}\pi\nu\right)\right]\eta^2+0(\eta^3)\right\}, \tag{5.92}$$

where $\omega=3-c$, $\alpha=\frac{1}{2}(c+1)$, and $\nu(\nu+1)=\omega/\alpha$.

It is more convenient to invert the expression in curly brackets, to wit

$$\ell nP=\left\{-3\varepsilon\alpha\left[1-\frac{3}{4}\eta+I\eta^2+0(\eta^3)\right]\right\}^{-1} , \tag{5.93}$$

where we have introduced

$$I = \frac{3}{8}\left[\frac{5}{4}-\frac{\alpha}{\omega}\left(\gamma+\psi(\nu+1)-\frac{\pi}{2}\tan\frac{\pi\nu}{2}\right)\right] . \tag{5.94}$$

The barrier penetration factor P is just the geometrical optics contribution to the imaginary part of the energy. Thus, inserting P into the dispersion representation for A_n [Eq. (2.23) of Chap.II] we find that the large-n behavior of the perturbation series for the potential in Eq. (5.64) is

$$A_n \propto -[-3\alpha(1-\frac{3\eta}{4}+\eta^2 I)\ ]^n \Gamma(n+\frac{1}{2}) \ . \qquad (5.95)$$

This result is correct up to a multiplicative factor independent of n.

CHAPTER VI

COMPARISON WITH NUMERICAL DATA

In this chapter I will give a brief account of the computer calculations that I have done to verify the theoretical predictions of Equations [3.49], [4.55], [4.64-66], and [5.95]. The excellent agreement between theoretical and numerical results attests to the essential correctness of our intuitive but nonrigorous assumptions [in particular our choice of straight line MPEP's for equal mass oscillators].

In the case of the one dimensional anharmonic oscillator with polynomial self-interactions, the computer work was more than just a check, it served to motivate the entire study. Let me describe how this came about. In earlier work[1] on the x^4 oscillator, it was realized that for this system Wick ordering had an essentially trivial effect on the large order behavior of perturbation theory: it merely shifted the mass of the oscillator in a λ dependent way. A simple scaling argument could then be used to find an exact relation between the Wick ordered and unordered perturbation series. Asymptotic analysis of this relation yielded

$$\lim_{n\to\infty} \frac{A_n}{A_n(\text{wick})} = e^3 \tag{6.1}$$

The question then arose whether a similar result held for the Wick Ordered x^{2N} oscillator. For such a system Wick ordering does more than merely shift the mass and the ordered and unordered systems appear to be quite different.

To study this question I performed a computer evaluation of the first terms in the perturbation series for an x^6 oscillator and found that

$$\lim_{n\to\infty} \frac{A_n^{(6)}}{A_n^{(6)}(\text{wick})} = 42.521082 \tag{6.2}$$

The number on the RHS of [6.2] is (to the accuracy available) $e^{1\frac{5}{4}}$. This remarkable result motivated the WKB calculations that appear in Chapter III.[2]

To get down then to the details of the computer calculations. As stated in Chapter II, the Rayleigh Schrodinger recursion relations for a harmonic oscillator with polynomial perturbations may be reduced to a single nonlinear partial difference equation. For the Hamiltonian in Eq.[4.1] the difference equation is:

$$(2i+2j)C_{n,i,j}=(i+1)(2i+1)C_{n,i+1,j}+(j+1)(2j+1)C_{n,i,j+1}$$
$$+aC_{n-1,i-2,j}+bC_{n-1,i,j-2}+2cC_{n-1,i-1,j-1}$$
$$-\sum_{k=1}^{n-1} D_{n-k}C_{k,i,j}, \tag{6.3}$$

where

$$C_{n,1,0}+C_{n,0,1}=D_n=(-1)^{n+1}A_n \tag{6.4}$$

and

$$C_{0,0,0}=1,$$

$$C_{n,0,0}=0 \quad \text{for } n>0,$$

$$C_{n,i,j}=0 \quad \text{for } i+j>2n$$

$$C_{n,i,j}=0 \quad \text{for } i<0 \text{ or } j<0.$$

Equation [6.3] is derived by substituting

$$E(\lambda)=1-\sum_{n=1}^{\infty} D_n(-\lambda)^n, \tag{6.5}$$

$$\psi(x)=e^{-(x^2+y^2)/4}\{1+\sum_{n=1}^{\infty} B_n(x,y)(-\lambda)^n\} \tag{6.6}$$

$$B_n(x,y) = \sum_{i,j=0}^{2n} (x^2/2)^i (y^2/2)^j C_{n,i,j} \ , \qquad (6.7)$$

into Eq. [4.1] and collecting powers of $(x^2/2)^i, (y^2/2)^j$ and λ^n.

The computer program for solving Eq. 6.3 is straightforward. However, the number of entries in the matrix C_{nij} grows as n^3 and the convolution term in [6.3] requires us to store C_{nij} for all n less than the order of perturbation theory being computed. The limitations of core memory prevent a calculation of A_n for n>20, and this is not sufficiently far into the asymptotic region to pull out the precise leading behavior. We therefore proceeded as follows: The calculation of A_{20} was repeated using Eq.[6.3] with all but the first and last terms of the convolution omitted. The difference between the exact and approximate calculation occurred only in the sixth decimal place. Therefore, we used the approximate difference equation to compute A_n for $n \leq 65$. Further justification of this approximation may be found in Ref.[1]of Chap. II.

The computer program then fits (to six significant figures) the raw Rayleigh-Schroedinger coefficients to a formula similar to but slightly less general than that in Eq.[4.6] :

$$A_n = -\frac{\sqrt{6}}{\pi^{3/2}}\,\alpha(-3\beta)^n\Gamma(n+\tfrac{1}{2})\{1+\frac{\gamma_1}{n}+\frac{\gamma_2}{n^2}+\frac{\gamma_3}{n^3}+\frac{\gamma_4}{n^4}\}. \tag{6.8}$$

The relevant numerical techniques are discussed in Ref.(3), Appendices D and E.

The numerical predictions for α and β for various values of a, b, and c are given in Table I. The predictions in Eq.[4.55] are as follows:

For $-1\leq c<1$, a=b=1,

$$\alpha=\{-8\pi c/\cos[\tfrac{\pi}{2}(1+8c)^{1/2}]\}^{1/2}\ ,$$
$$\beta=1, \tag{6.9}$$

where we have eliminated ν in favor of c using Eq. [4.33]. For c>1, a=b=1,

$$\alpha=\left\{\frac{8\pi(c-3)}{(1+c)\cos[\pi(25-7c)^{1/2}(4+4c)^{-1/2}]}\right\}^{1/2}$$
$$\beta=(c+1)/2, \tag{6.10}$$

where we have used Eq.[4.4]. For a=b=1 the values of α and β in Table 1 agree to six places with the expressions in Eqs. [6.9] and [6.10]. Note than when the argument of the cosine

function becomes imaginary, cos is replaced by cosh. The function in the curly brackets is always positive.

We have done one further and rather amusing numerical calculation for a=b=1 which does not appear in Table I. We computed A_n for c=-5. This problem has no apparent physical significance because the Hamiltonian is not bounded below, and therefore has no discrete eigenvalues. Nevertheless, the perturbation series is still well defined and we found that β=-2.00000. This result agrees with β in Eq.[6.10].

For a≠b≠1 the relevant formulas are [4.64]-[4.66]. The reader may verify for himself the 6 place agreement is again found.

We now turn to our numerical computations for systems with curved MPEPs. A slight generalization of the argument given below [6.3] shows that for the potential in Eq.[5.1] the Rayleigh-Schroedinger coefficients are given by

$$A_n=(-1)^n(C_{n,1,0}+C_{n,1,0}) \ ,$$

where

$$(2j+2k\sqrt{1+\eta})C_{n,j,k}=(j+1)(2j+1)C_{n,j+1,k}+(k+1)(2k+1)C_{n,j,k+1}$$

$$+C_{n-1,j,k-2}+C_{n-1,j-2,k}+2cC_{n-1,j-1,k-1}$$

$$-\sum_{p=1}^{n-1}(C_{1,1,0}+C_{p,0,1})C_{n-p,j,k}\ . \qquad (6.11)$$

We solved Eq.[6.11] for c=2 and a range of values of $\sqrt{1+\eta}$ and fit the values of A_n near n=65 to the formula[4]

$$A_n \propto -\Gamma(n+\tfrac{1}{2})(-3K)^n. \qquad (6.12)$$

The values of K as a function of η are recorded in Table II.

Several remarks should be made about the entries in Table II. First, when $\sqrt{1+\eta} > \sqrt{c} = \sqrt{2}$, k=1 (to within the expected accuracy) and when $\sqrt{1+\eta} < 1/\sqrt{c} = 1/\sqrt{2}$, $k=(1+\eta)^{-3/2}$. These are just the values that we obtain from straight-line MPEP's passing through the saddle points in Eq.[5.65a] and [5.65b]. That is, the curved path region lies, as it should, for values of $\sqrt{1+\eta}$ between $\sqrt{2}/2$ and $\sqrt{2}$. Second, when η=0 we have an equal-mass potential and the results of Chapter IV imply that k=1.5. This suggests that near $\sqrt{1+\eta}$ = 1 our numerical results in Table II are low by 4 parts in 1.5×10^6. This inaccuracy comes about because we used the approximate

form of [6.11] to compute the numbers in the table.

With this in mind, we proceed to compare our theoretical and numerical data. Our theoretical predictions from Eq. [5.95] take the form of a Taylor series valid near $\eta=0$:

$$K = \frac{3}{2} - \frac{9\eta}{8} + 1.378\eta^2 + 0(\eta^3). \qquad (6.13)$$

Equation [6.13] is obtained by evaluating the expression for I in Eq. [5.94] using $c=2$, $\alpha=3/2$, $\omega=1$, and $\nu=-.5\pm\sqrt{11/12}$.

There is no simple formula that fits the numerical data in Table II for $\sqrt{2} > \sqrt{1+\eta} > \sqrt{2}/2$. We can, however, compute the first three terms in the Taylor series of K about $\sqrt{1+\eta} = 1$. Using the five values for K associated with $\sqrt{1+\eta} = .98, .99, 1.00, 1.01$, and 1.02 and assuming that the 4 parts in 1.5×10^6 discrepancy mentioned above holds for all these values of $\sqrt{1+\eta}$ we obtain the expansion
$K=1.5 - \frac{9}{4}(\sqrt{1+\eta}-1)+4.388(\sqrt{1+\eta}-1)^2+0(\sqrt{1+\eta}-1)^3$. Rewriting this as a power series in , we obtain

$$K = \frac{3}{2} - \frac{9\eta}{8} + 1.378\eta^2 ,$$

in complete agreement with the theoretical predictions in Eq. [6.13].

TABLE CAPTION

Table I. Numerical values of α and β in Eq.[6.8] for various values of a, b, and c in Eq.[4.1]. The theoretical predictions in Eqs.[4.55] and [4.64-66] and the numerical calculations of α and β agree to six figures, which was the available limit of computer accuracy. Some values of α were not computed.

TABLE I

a	b	c	α	β
1	1	-1	-----	1.00000
1	1	-.5	-----	1.00000
1	1	-.25	1.58242	1.00000
1	1	-1/6	1.70541	1.00000
1	1	-.05	1.90378	1.00000
1	1	-.005	1.99002	1.00000
1	1	0	2.00000	1.00000
1	1	.005	2.01002	1.00000
1	1	.05	2.10410	1.00000
1	1	.1	-----	1.00000
1	1	1/6	2.38399	1.00000
1	1	.25	2.62372	1.00000
1	1	1/3	2.90740	1.00000
1	1	.5	3.67206	1.00000
1	1	1.5	4.33836	1.25000
1	1	2	2.90740	1.50000
1	1	2.5	2.32211	1.75000
1	1	3	2.00000	2.00000
1	1	5	1.47228	3.00000
1	1	33	-----	17.0000
1	2	.25	-----	2.00000
1	2	1	-----	2.00000
1	2	5	-----	3.28574
1	3	.25	-----	3.00000
1	3	1	-----	3.00000
1	3	5	-----	3.66667
1	5	.25	-----	5.00000
1	5	1	-----	5.00000

TABLE CAPTION

Table II. Numerical values of K as a function of η in Eq.[6.12]. These values are in good agreement with the theoretical predictions in Eq.[6.13].

TABLE II

$\sqrt{1+\eta}$	η	K
0.4	-.84	15.625004
0.5	-.75	8.000003
0.75	-.4375	2.534419
0.8	-.36	2.216049
0.9	-.19	1.777643
0.98	-.0396	1.546811
0.99	-.0199	1.522942
1.00	0.0	1.499996
1.01	.0201	1.477928
1.02	.0404	1.456695
1.1	.21	1.312745
1.2	.44	1.183367
1.3	.69	1.092641
1.4	.96	1.038811
$\sqrt{2}$	1.0	1.032995
1.5	1.25	1.004064
1.7	1.89	.999954
1.8	2.24	.999994
1.9	2.61	.999999
2.0	3.00	1.000000
2.1	3.41	1.000000
2.9	7.41	1.000000
3.0	8.0	1.000000
3.1	8.61	1.000000

CHAPTER VII

CONCLUSIONS AND SPECULATIONS

In this final chapter I will discuss the problems which arise when one attempts to use the present methods to deal with real quantum field theories. The discussion is divided into three parts. We first discuss the perturbation series for Green's functions. Then the problem of unequal mass oscillators is discussed in the context of field theory. Finally, we make some remarks about Fermion field theories.

A. Perturbation Theory for Green's Functions

The analysis we have presented so far relates to the behavior of the perturbation series for energy levels. The real objects of interest in field theory are Green's functions, and it is natural to wonder what significance our results have for them. Of course, the eigenvalues are the poles of the Green's functions so that the results presented so far do say something about the perturbation series for Green's functions. However, the real issue is how the behavior of these series changes with energy and, in particular, what happens in the limiting case of very high energy. Conventional field theoretic arguments (for example, the method of the renormalization

group) indicate that the rate of growth of the perturbation coefficients increases with increasing energy.

I will now show how the methods of this thesis may be adapted to deal with these questions. Since I will do no explicit calculations I will write all formulas in terms of a single zero dimensional quantum field x(t). (A quantum mechanical system with one degree of freedom.) The two point Green's function is defined for positive coupling constant by

$$G(E) = \int_{-\infty}^{\infty} dt\ e^{iEt}\ \frac{\langle 0|x(t)x(0)|0\rangle}{\langle 0|0\rangle} \tag{7.1}$$

where $|0\rangle$ is the ground state. To analytically continue this formula, we insert a complete set of eigenstates and write a spectral representation

$$G(E) = \sum_n \frac{|\langle 0|x(0)|n\rangle|^2}{\langle 0|0\rangle\langle n|n\rangle(E-(E_n-E_o))} \quad . \tag{7.2}$$

In the representation where x(0) is diagonal, we can write

$$\langle 0|x(0)|n\rangle = \int dx\ x\ \psi_o^*(x)\ \psi_n(x)$$

$$\langle m|n\rangle = \int dx\ \psi_m^*(x)\ \psi_n(x) \tag{7.3}$$

For positive λ the eigenfunctions are real on the real axis and we can replace ψ_n^* by ψ_n in [7.3]. We do this because ψ_n is an analytic function of λ while ψ_n^* is not. We can now analytically continue each term in [7.2] following the procedure outlined in Chapter II. Notice that the integration contour in [7.3] must be rotated as we change λ so that it always goes out to infinity in a sector in which ψ_n vanishes.

Each term in [7.2] will now satisfy a dispersion relation like [2.15]. We assume that the domain of analyticity of the entire sum is qualitatively the same as that of each term. We can thus write a dispersion representation (in λ for fixed E) for the perturbation coefficients of G(E) and conclude that the large order behavior of the perturbation series is determined by the discontinuity of G for small negative λ. Eqs.[7.2] and [7.3] show how to compute this discontinuity in terms of wave functions and eigenvalues. This latter computation can be performed by WKB methods.

There is an important difference between the calculations required by [7.2] and those that we have already done. We have always assumed that the energy eigenvalue E_n was 0(1), but in [7.2] we have an infinite sum containing arbitrarily large eigenvalues. For large n our WKB wave functions are no

longer valid. Nonetheless, a slightly different WKB approximation can be used to compute the eigenfunctions in this domain. (The large eigenvalue region is, of course, the traditional domain of applicability of the WKB method.)

I have done an initial calculation of this type for the one dimensional potential $\frac{x^2}{4} + \frac{\lambda x^4}{4}$ and there appear to be some technical problems with the asymptotic matching procedure. The first order WKB approximation that we have used so far in this thesis does not appear to be sufficiently accurate to produce a consistent match for the matrix elements $\langle 0|x|n\rangle$. This question certainly merits further study. I am sure that a consistent approximation can be produced, if necessary, by going to a higher order WKB approximation.

Even if this question of matching is cleared up, calculations based on [7.2] may become very unwieldy in multidimensional contexts because of the complexity of the set of eigenstates. There is another way to write [7.1] which may be simpler to deal with in N dimensions.

We introduce the eigenstates $|x,t\rangle$ of the Heisenberg operator $x(t)$ and write

$$G(E) = \frac{1}{\langle 0|0\rangle} \int dt e^{iEt} \int dx dx' \langle 0|xt\rangle x\langle xt|x'0\rangle x'\langle x'0|0\rangle$$

$$= \frac{1}{\langle 0|0\rangle} \int dt e^{i(E-E_o)t} \int dx dx' \psi_o^*(x) x\langle xt|x'0\rangle x' \psi_o(x') . \tag{7.4}$$

To evaluate G(E) we thus need to know the ground state wave function ψ_o and the transition amplitude

$$G(xt;x'0) = \langle xt|x'0\rangle$$

G is the Green's function for the Schroedinger equation. It satisfies

$$\left(i \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} - V\right) G(xt,x'0) = \delta(x-x') \delta(t) \tag{7.5}$$

and

$$\lim_{t\to 0} G(xt;x'0) = \delta(x-x') .$$

Introducing the Fourier Transform of G

$G(xtx'0) = \int dE\, d^{-iEt} G(x,x',E)$ we obtain

$$\left[\frac{d^2}{dx^2} - V + E\right] G(x,x',E) = \delta(x-x') \tag{7.6}$$

One would now hope that for small negative coupling constant [7.6] may be solved by WKB methods. The precise form of the WKB solution that must be used will depend on the value of x', and we will have to slice up the two dimensional x,x' space into several regions.

The advantage of [7.4] over [7.2] in multidimensional situations becomes clear if we remember the path integral discussion of Sec. A of Chap. IV. We argued there, that in the tunneling region the dominant contribution to $G(x,x',E)$ came from a small tube surrounding the "classical" path connecting x to x'. Furthermore, the wave function $\psi_o(x)$ vanishes very rapidly as x moves away from the particular classical path which we have called the MPEP. Thus, for the purposes of evaluating [7.4] for small negative $\lambda^{(1)}$, we need only solve [7.6] in a small region surrounding the MPEP.

To conclude, it appears that aside from certain technical complications related to matching[2], the WKB methods that we have introduced are applicable to the problem of computing the large order behavior of the perturbation series for Green's functions. I plan to do some explicit calculations in the near future.

B. Unequal Mass Oscillators and Realistic Field Theories

I have repeatedly emphasized that the problems of unequal mass oscillators and curved MPEP's are unavoidable when one is dealing with the Hamiltonians arising from cut off field theories. I would like to demonstrate this explicity for a cut off 2-dimensional ϕ^4 theory, and to speculate about how these problems may be overcome.

The Hamiltonian of the cut off $(\phi^4)_2$ theory is defined in terms of a scalar field

$$\phi(x) = \frac{1}{\sqrt{2L}}\left\{\sum_{k=1}^{N}\left[(a_k+a^+_{-k})e^{ip_kx}+(a^+_k+a_{-k})e^{-ip_kx}\right]\omega_k^{-1/2} + (a_o+a_o^+)\omega_o^{-1/2}\right\} \tag{7.7}$$

and its conjugate momentum

$$\pi(x) = \frac{-1}{\sqrt{2L}}\left\{\sum_{k=1}^{N}\left[(a_k-a^+_{-k})e^{ip_kx} - (a^+_k-a_{-k})e^{-ip_kx}\right]\omega_k^{1/2} + (a_o-a_o^+)\omega_o^{1/2}\right\} \tag{7.8}$$

In [7.7-8] L is a spatial cutoff, $p_k = \frac{2\pi k}{L}$, $\omega_k = \sqrt{p_k^2+m_o^2}$ $(\equiv\sqrt{p_k^2+\omega_o^2})$ and the upper limit of summation, N, is defined in terms of an ultraviolet cutoff Ω as the largest integer such that

$$\frac{2\pi N}{L} \leq \Omega \tag{7.9}$$

The creation and annihilation operators satisfy

$$[a_k, a^+_{k'}] = \delta_{kk'} \tag{7.10}$$

We now introduce a set of canonical variables

$$Q^k_{e(ven)} = (2\omega_k)^{-1/2}[a_k+a^+_k+a_{-k}+a^+_{-k}] \tag{7.11a}$$

$$Q^k_{o(dd)} = -i(2\omega_k)^{-1/2}[a^+_k-a_k+a_{-k}-a^+_{-k}] \tag{7.11b}$$

$$P^k_e = \frac{-i}{4}(2\omega_k)^{1/2}[a_k-a^+_k+a_{-k}-a^+_{-k}] \tag{7.11c}$$

$$P^k_o + \frac{1}{4}(2\omega_k)^{1/2}[a_k+a^+_k-a_{-k}-a^+_{-k}] \tag{7.11d}$$

$$Q^o = \frac{1}{2}\frac{(a_o+a^+_o)}{\sqrt{\omega_o}} \tag{7.11e}$$

$$P^o = \frac{i\sqrt{\omega_o}}{2}(a^+_o-a_o) \tag{7.11f}$$

It is easy to verify that

$$[Q_e^k , P_e^{k'}] = i\delta_{kk'}$$

$$[Q_o^k , P_o^{k'}] = i\delta_{kk'}$$

$$[Q^o , P^o] = i$$

and that all other commutators vanish. Next we rewrite the field operators in terms of canonical variables

$$\phi(x) = \frac{1}{\sqrt{2L}} \left\{ Q^o + \sum_{k=1}^{N} \frac{Q_-^k}{\sqrt{2}} e^{ip_k x} + \frac{Q_+^k e^{-ip_k x}}{\sqrt{2}} \right\}$$

$$\pi(x) = \frac{1}{\sqrt{2L}} \left\{ 2P^o + \sum_{k=1}^{N} \sqrt{2}\, P_-^k e^{ip_k x} + \sqrt{2}\, P_+^k e^{-ip_k x} \right\}$$

where $Q_\pm^k = Q_e^k \pm iQ_o^k$ and similarly for $P_\pm^k$.

The free Hamiltonian is given by

$$H_o = \frac{1}{2} \int_o^L \left\{ \pi^2(x) + (\nabla\phi)^2(x) + m_o^2 \phi^2(x) \right\} dx$$

We note that

$$\int_0^L e^{inp_k x}\, dx = 0 \text{ if } n \geq 1 \quad k \geq 1 \tag{7.12}$$

so H_o can be rewritten

$$H_o = P^{o^2} + \sum_{k=1}^{N} (P_e{}^{k^2} + P_o{}^{k^2}) + \frac{\omega_o{}^2}{4} Q^{o^2} + \sum_{k=1}^{N} \frac{\omega_k{}^2}{4} (Q_e{}^{k^2} + Q_o{}^{k^2}) \tag{7.13}$$

The interaction Hamiltonian (without Wick ordering) is

$$H_I = \lambda L \int_0^L \phi^4(x)\, dx \tag{7.14}$$

λ has dimensions L^{-3}. It is a straightforward but tedious matter to rewrite [7.14] in terms of canonical variables:

$$H_I = \frac{\lambda}{4} \Bigg\{ Q^{o^4} + 6Q^{o^2} \sum_{k=1}^{N} (Q_e^{k^2} + Q_o^{k^2})$$

$$+6\sqrt{2}\, Q^o \sum_{ij=1}^{N} [Q_e^{i+j}(Q_e^i Q_e^j - Q_o^i Q_o^j) + 2Q_o^{i+j}(Q_e^i Q_o^j)]$$

(cont.)

$$+2\sum_{ijk=1}^{N}\left[Q_e^{i+j+k}Q_e^k(Q_e^iQ_e^j-3Q_o^iQ_o^j) - Q_o^{i+j+k}Q_o^k(Q_o^iQ_o^j-3Q_e^iQ_e^j)\right]$$

$$+3\sum_{ijk=1}^{N}\left[Q_e^{i+j-k}(Q_e^iQ_e^jQ_e^k - Q_o^iQ_o^jQ_e^k + 2Q_o^kQ_e^iQ_e^j)\right.$$

$$\left. + Q_o^{i+j-k}(Q_o^iQ_o^jQ_o^k - Q_e^iQ_e^jQ_o^k + 2Q_e^kQ_o^iQ_o^j)\right] \Big\} \qquad (7.15)$$

We have set $Q_{e,o}^i = 0$ for $i>N$ and $i<0$. The total Hamiltonian given by [7.13] and [7.15] is clearly that of a set of unequal mass coupled anharmonic oscillators. To find the larger order behavior of the perturbation series for this system, we must find its MPEP's.

Is it possible that this system has straight line MPEP's? The discussion of Chap. V suggests (although it does not prove) that any straight line MPEP's must lie in a subspace of the configuration space in which all oscillators have equal mass. Here we will investigate the two most natural(?) possibilities: We will search for straight MPEP's along the Q^o axis and in the Q_e^N-Q_o^N plane.

As in Chap. IV we search for radially directed saddle points. The condition for a critical point on the Q^o axis is:

$$\frac{\omega_o^2}{2} Q^o - \varepsilon Q^{o^3} = 0 \tag{7.16}$$

and the Hessian at the critical point is given by

$$\begin{bmatrix} -\omega_o^2 & & & & \\ & \frac{1}{2}(\omega_1^2-3\omega_o^2) & & & \\ & & \frac{1}{2}(\omega_1^2-3\omega_o^2) & & \\ & & & \ddots & \\ & & & & \frac{1}{2}(\omega_N^2-3\omega_o^2) \end{bmatrix} \tag{7.17}$$

Thus we have a radially directed saddle point iff

$$\omega_1^2 > 3\omega_o^2$$

or, using the definition of ω:

$$L < \frac{\sqrt{2}\pi}{M_o} = \frac{\sqrt{2}\pi}{\omega_o} \tag{7.18}$$

The only way to get a straight MPEP on the Q^o axis is to impose a spatial cutoff on the order of the bare Compton wavelength of the field quanta. Notice that if we try to alleviate this problem by taking the bare mass to zero, then the saddle point moves into the origin and disappears. Since we eventually want to take the limit $L\to\infty$, the bound [7.18] makes the saddle point on the Q^o axis useless.

Turning to the Q_e^N-Q_o^N plane, we note that the only terms in the potential which have non-vanishing second derivative there are contained in

$$V_N = \frac{\omega_o^2}{4} Q^{o2} + \sum_{k=1}^{N} \frac{\omega_k^2}{4} (Q_e^{K2}+Q_o^{K2}) - \frac{\varepsilon}{4} 6Q^o (Q_e^{N2}+Q_o^{N2}) + \frac{3}{2} (Q_e^{N2}+Q_o^{N2})^2 + 6 (Q_e^{N2}+Q_o^{N2}) \sum_{k=1}^{N-1} (Q_e^{K2}+Q_o^{K2})$$

We have critical points in the Q_e^N-Q_o^N plane when

$$\frac{\omega_N^2}{2} - \frac{3\varepsilon}{2} (Q_e^{N2}+Q_o^{N2})$$

and the Hessian at the critical point is

$$\begin{bmatrix} \frac{\omega_o^2-\omega_N^2}{2} & & & & \\ & \frac{\omega_1^2-\omega_N^2}{2} & & & \\ & & \ddots & & \\ & & & -3\varepsilon Q_e{}^N & -3\varepsilon Q_e{}^N Q_o{}^N \\ & & & -3\varepsilon Q_e{}^N Q_o{}^N & -3\varepsilon Q_o{}^{N^2} \end{bmatrix} \tag{7.19}$$

There are two problems with [7.19]. First, since the 2×2 submatrix in the right-hand corner has zero determinant, the critical point is degenerate. This could be cured by going to polar coordinates in the $Q_e{}^N Q_o{}^N$ plane and using separation of variables. (Although the full potential is not separable, it reduces to V_N, which is, in the vicinity of the plane.)

It is not worth doing so however, for the other eigenvalues of [7.19] are always negative! Thus we can never have an MPEP in the Q_e^N-Q_o^N plane.

The foregoing results have convinced me that none of the equal mass substaces contain MPEP's in the limit of large spatial cutoff. Thus, realistic field theories seem to lead inevitably to curved path problems.

It seems hopeless to try to deal directly with the hideously complicated Hamiltonian [7.15], but there appear to be three possible directions in which future progress may be made.

First, we could make the approximation that all masses were equal and then use the perturbative methods of Chapter V. This doesn't appear very promising because the mass differences are not small. Furthermore, taking the masses equal corresponds to dropping the spatial derivative in the configuration space Hamiltonian. We then have an independent oscillator at every point in space. A perturbation theory based on such a highly degenerate system will have extra complications not encountered in Chap. V.

The second possible avenue of progress is a transformation of variables which simplifies [7.15]. It might then become

possible to find the MPEP. I am not at all sure that the straightforward canonical quantization that I have written down is the best mode of approach to the problem at hand. It might be interesting to see for example what happens when we quantize in light-cone variables[3] or use the "radial" quantization scheme of Fubini, Jackiw, and Hanson[4].

The approach in which I place the most hope, however, is based on trying to find the MPEP only in the limit of large momentum cutoff. In this limit the classical equations arising from [7.13-15] are well approximated by the classical ϕ^4 field theory. Since classical field theories are notoriously simpler than large but finite systems of oscillators, one might hope to find the MPEP (in field space!) for the classical field theory and use it to calculate the large order behavior of perturbation theory for the quantum field theory.

C. Theories With Fermions

Although we are still quite far from calculating the large order behavior of the perturbation series in Bose field theories, I would like to say a word about more realistic theories which contain Fermions.

The difference between Fermi and Bose systems arises from the fact that there are no differential operator realizations of operators which obey canonical anti-commutation relations (CAR). In fact, any representation of a finite set of CAR is unitarily equivalent to certain standard finite dimensional matrix representations[5]. Thus, in the study of cutoff Fermi theories, one is naturally led to coupled systems of partial differential equations rather than the single differential equations that we have encountered in Bose theories. For example, if one cuts off the standard γ^5 pion nucleon theory by letting only s wave pions with zero momentum interact with the nucleons, one obtains[6]

$$\left(-\frac{d^2}{dx^2} + x^2 - 2E\right)\chi_1(x) = -2\sqrt{2}gx\chi_2(x)$$

$$\left(-\frac{d^2}{dx^2} + x^2 + \frac{2}{x^2} + 2M - 2E\right)\chi_2 = -2\sqrt{2}gx\chi_1 \qquad (7.20)$$

with

$$\int_0^\infty (\chi_1^2 + \chi_2^2) = 1 \ .$$

Although it is not clear how to extend the methods of this thesis to the equations in [7.20], one important remark can be made. Since for large x, the coupling term in [7.20] is dominated by the unperturbed Hamiltonian, we would expect that the perturbation series for [7.20] has a finite radius of convergence. That this is indeed so has been verified numerically by Ruijgrok[7]. He also showed that the analytic structure of the function $E(g^2)$ in the region outside the radius of convergence was qualitatively the same as that found by Bender and Wu[8] for the anharmonic oscillator.

If we cut off any Yukawa Type theory by restricting it to a finite number of modes, we will obtain a set of coupled partial differential equations with a behavior at large x similar to that of [7.20]. This arises basically because there is only one power of a boson field in the interaction Hamiltonian. Thus, we might expect that the perturbation series for any cut-off Yukawa theory had a finite radius of convergence.

What do we expect to happen to the radius of convergence in the limit as the cutoff goes away? The indications are that it may go to zero in general. For example, renormalization theory tells us that in four dimensions, the $\bar{\psi}\gamma_5\psi\phi$

interaction Hamiltonian is not complete. In order to obtain a finite S matrix, one must add a term $\lambda\phi^4$. If we now cut off the complete Hamiltonian

$$H_I = g\bar{\psi}\gamma_5\psi\phi + \lambda\phi^4$$

we will find that the perturbation series for H_I no longer converges.

Similar speculations are prompted by the work of Simon[9] and Caianello[10] on cut-off Yukawa theories in two and three dimensions. They used diagrammatic techniques to obtain bounds on the Feynman-Dyson series and showed that for finite cutoff, the series had a finite radius of convergence. As the cutoff is taken to infinity, their lower bounds on the radius of convergence go to zero. It would be of great interest to extend the techniques of this thesis to such problems, for in doing so we could establish exact values for the radius of convergence (instead of bounds), and thus make definite statements about its behavior as the cutoff becomes infinite.

D. Conclusion

We have made some progress and we have left some for the future. In retrospect, I think that the most important contribution of this thesis is the understanding we have obtained about how the dispersion relation [2.23] makes Dyson's heuristic remarks about "dominance of the interaction" into a quantitative tool for studying the large order behavior of perturbation theory. We have also shown that this study can be reduced to a problem in classical mechanics and hopefully in classical field theory. For the future we leave the actual realization of this program. And now, as the MPEP tunnels slowly out to infinity, we leave our vibrant friends, the coupled anharmonic oscillators, and come finally to

THE END

APPENDIX A

In this appendix we show that Eq.[5.19] is valid whenever the path $\vec{\phi}(s)$ satisfies the classical equations of motion.

The classical equations are given in Sec. B of Chap. V as

$$2\{V[\vec{\phi}(s)]-E\}\vec{\phi}''(s)+\vec{\phi}'(s)\{\vec{\phi}'(s)\cdot\vec{\nabla}V[\vec{\phi}(s)]\} = \vec{\nabla}V[\vec{\phi}(s)]. \tag{A.1}$$

The functions $V_0(s)$ and $V_1(s)$ which appear in Eq.[5.19] are defined in Eq.[5.13] as

$$V[\vec{\phi}(s)+n\vec{\chi}(s)]-E = V_0(s)+nV_1(s)+0(n^2) , \tag{A.2}$$

where $\vec{\chi}(s)$ is the unit normal vector

$$\vec{\chi} = (-\vec{\phi}_2', \vec{\phi}_1') .$$

Expanding the left-hand side of Eq.[A.2] as a power series in n we find that

$$V_0(s) = V[\vec{\phi}(s)]-E,$$

$$V_1(s) = \vec{\chi}\cdot\vec{\nabla}V[\vec{\phi}(s)] . \tag{A.3}$$

Now we take the scalar product of [A.1] with $\vec{\chi}$ and use $\vec{\chi}\cdot\vec{\phi}' = 0$ to obtain

$$2V_0(s)\vec{\chi}\cdot\vec{\phi}'' = \vec{\chi}\cdot\vec{\nabla}[\vec{\phi}(s)] = V_1(s) \ . \tag{A.4}$$

But,

$$\vec{\chi}\cdot\vec{\phi}'' = \phi_2''\phi_1' - \phi_1''\phi_2' = -\rho \ . \tag{A.5}$$

Hence

$$\rho = -\frac{V_1}{2V_0} \ , \tag{A.6}$$

which is Eq.[5.19]. It is clear then that Eq.[5.19] is just the perpendicular component of the equation of motion. The component of Eq.[A.1] parallel to the path (that is, parallel to $\vec{\phi}'$) is a trivial identity because $\vec{\phi}'^2=1$ and $\vec{\phi}'\cdot\vec{\phi}''=0$. Thus, Eq.[5.19] is valid if and only if $\vec{\phi}$ is a classical path.

APPENDIX B - OSCILLATORS WITH SPHERICAL SYMMETRY

Here we investigate the ground-state-energy perturbation series for spherically coupled oscillators. This special case obtains when we choose a=b=c=1 in Eq.[4.1]. However, a spherically symmetric configuration of oscillators is so easy to treat that we immediately generalize from the two-mode problem of Eq.[4.1] to the N-mode problem, which we define by the equation

$$\{\sum_{i=1}^{N}(-\frac{\partial^2}{\partial x_i^2}+\frac{1}{4}x_i^2)+\frac{\lambda}{4}(\sum_{i=1}^{N}x_i^2)^2-E(\lambda)\}\psi(x_i)=0\ , \tag{B.1}$$

where $\lim_{|x_i|\to\infty}\psi=0$.

We use spherical symmetry to transform Eq.[A.1] to N-dimensional spherical coordinates. Moreover, we seek a wave function ψ which depends only on r, the radial coordinate, because the ground-state wave function has no angular dependence. We thus reduce Eq.[B.1] to the ordinary differential equation

$$\left[-\frac{d^2}{dr^2} - \frac{N-1}{r}\frac{d}{dr} + \frac{r^2}{4} + \frac{\lambda r^4}{4} - E(\lambda)\right]\psi(r) = 0 \tag{B.2}$$

where $r^2 = \sum_{i=1}^{N} x_i^2$.

Our problem is to compute the eigenvalue $E(\lambda)$ perturbatively in large order. We expand $E(\lambda)$ into the perturbation series

$$E(\lambda) = \frac{N}{2} - \sum_{n=1}^{\infty} (-\lambda)^n C_n . \tag{B.3}$$

We could solve this problem by removing the first derivative term from Eq.[B.2] by making a suitable transformation and then using WKB in the same manner as in the body of this paper. However, we prefer a much simpler approach. We will convert Eq.[A.2] into a partial difference equation which has already been solved asymptotically in Ref.(1 of Chap. II).

To transform Eq.[A.2] into a partial difference equation, we substitute the expression

$$\psi(r) = e^{-r^2/4}\{1+\sum_{n=1}^{\infty}(-\lambda)^n \sum_{j=1}^{2n} (\tfrac{1}{2}r^2)^j C_{n,j}\} \tag{B.4}$$

and Eq.[B.3] into Eq.[B.2] and collect powers of $r^2/2$ and $-\lambda$.

The coefficient of $(r^2 2)^j$ and $(-\lambda)^n$ is the desired difference equation:

$$2jC_{n,j} = (j+1)(2j+N)C_{n,j+1}+C_{n-1,j-2} - \sum_{p=1}^{n-1} C_{p,1}C_{n-p,j} \,, \tag{B.5}$$

with initial value $C_{0,0} = 1$ and boundary condition $C_{n,j}\neq 0$ for $n>1$ and $1\leq j\leq 2n$; $C_{n,j}=0$ otherwise. C_n is related to $C_{n,j}$ by

$$C_n = NC_{n,1} \,. \tag{B.6}$$

Following Part A, Sec. VI of Ref.(II-1), we approximate Eq.[B.5] by dropping the nonlinear convolution term. As was argued there, the neglected term does not affect the leading asymptotic behavior of $C_{n,1}$ for large n. Thus, the equation to be solved is

$$2jC_{n,j} = (j+1)(2j+N)C_{n,j+1}+C_{n-1,j-2} \,. \tag{B.7}$$

We put Eq.[B.7] into a more useful form by substituting

$$C_{n,j} = D_{n,j}/[j\Gamma(j+\tfrac{1}{2}N)]. \tag{B.8}$$

The equation satisfied by $D_{n,j}$ is

$$D_{n,j} = D_{n,j+1} + \frac{(j+\frac{1}{2}N-1)(j+\frac{1}{2}N-2)}{2(j-2)} D_{n-1,j-2} . \tag{B.9}$$

Next, we replace Eq.[B.9] by a new approximate equation satisfied by a new dependent variable $E_{n,j}$:

$$E_{n,j} = E_{n,j+1} + \tfrac{1}{2}(j+N-1)E_{n-1,j-2} . \tag{B.10}$$

Equation [B.10] is derived by approximating the coefficient of $D_{n-1,j-2}$ in Eq.[B.9] for large j, keeping terms of orders j and 1 and neglecting terms of order j^{-1}.

We must introduce an extra condition which fixes the multiplicative scale of $E_{n,j}$ because Eq.[B.10] is homogeneous; to wit, we require that

$$\lim_{n\to\infty} E_{n,2n}/D_{n,2n} = 1. \tag{B.11}$$

From Eq.[B.10] we easily deduce that

$$E_{n,2n} = E_{0,0}\Gamma[n+\tfrac{1}{2}(N+1)]/\Gamma[\tfrac{1}{2}(N+1)] . \qquad (B.12)$$

$E_{0,0}$ is the multiplicative factor which adjusts the scale of $E_{n,j}$. Also, from Eqs.[B.7] and [B.8] we have

$$D_{n,2n} = 2n\Gamma(2n+\tfrac{1}{2}N)4^{-n}/n! \ . \qquad (B.13)$$

Combining Eqs.[B.11]-[B.13] gives

$$E_{0,0} = \Gamma[\tfrac{1}{2}(N+1)]2^{N/2}\pi^{1/2} \ . \qquad (B.14)$$

Finally, we recall that Part II, Sec. VI of Ref.(II-1) gives a complete treatment of the asymptotic behavior of solutions for difference equations like that in Eq.[B.10]. It is shown there that for large n

$$E_{n,1} \sim E_{0,0} \frac{3^{n+N/2}\Gamma(n+\frac{1}{2}N)}{2\sqrt{\pi}\Gamma[\frac{1}{2}(N+1)]} \ . \qquad (B.15)$$

Thus, we combine Eqs.[B.6], [B.8], and [B.15] to obtain

$$C_n \sim \Gamma(n+\tfrac{1}{2}N)3^n 6^{N/2}\pi^{-1}/\Gamma(\tfrac{1}{2}N) \ . \qquad (B.16)$$

This is the precise leading asymptotic behavior of C_n for large n and is the general result we have sought.

Two special cases of this equation are noteworthy. For

the one-mode oscillator (N=1), the coefficients of the ground-state-energy perturbation series grow like

$$C_n \sim 6^{\frac{1}{2}} \pi^{-3/2} 3^n \Gamma(n+\tfrac{1}{2}) \ , \tag{B.17}$$

which agrees with Eq.[3.0] for a=1 b=0.

Second, the coefficients for the two-mode oscillator (N=2) diverge like

$$C_n \sim \frac{2}{\pi} 3^{n+1} n! \ , \tag{B.18}$$

as in Eq.[4.55c].

Observe also that the rate of divergence of perturbation theory increases with increasing N: $\Gamma(n+\frac{1}{2}N) \sim \Gamma(n) n^{N/2}$. This is a phenomenon characteristic only of spherically symmetrically coupled oscillators. As is shown in Chap. IV Section D.6, it results physically from a kind of constructive interference.

APPENDIX C - MORE ABOUT N MODE OSCILLATORS

Our discussion of N-dimensional oscillators in Sec. F of Chap. IV was incomplete because we did not show that straight lines through radially directed saddle points were solutions of the classical orbit equations, and we did not give criteria for the existence of saddle points. The purpose of this appendix is to remedy these omissions.

Consider an N-dimensional potential of the form

$$V = \sum_i \frac{x_i^2}{4} + \frac{\lambda}{2M} \sum_{i_1,\ldots,i_{2M}} A_{i_1\ldots i_{2M}} x_{i_1}\ldots x_{i_{2M}} . \qquad (C.1)$$

The interaction term is the most general $2M^{\underline{th}}$ order homogeneous polynomial, subject to the restriction that V be bounded below for $\lambda>0$. Then we have the following theorem: If $\lambda=-\varepsilon$, $\varepsilon>0$, then the radial line through any critical point of V is a solution of the classical equation [4.5]. Furthermore, this radial line is a principal axis of the critical point and V reaches a maximum at the critical point along this line.

Proof: Let $a=(a_1,\ldots,a_n)$ be a critical point of V. Then,

$$\left.\frac{\partial V}{\partial x_i}\right|_{x_i=a_i} = \frac{a_i}{2} - \varepsilon \sum_{i_2,\ldots i_{2M}} A_{ii_2\ldots i_{2M}} a_{i_2}\ldots a_{i_{2M}} = 0 \tag{C.2}$$

The radial line through a_i is

$$x_i(s) = \frac{a_i}{|a|}\, s \ . \tag{C.3}$$

Using Equations [C.2] and [C.3] we can evaluate the expressions in Eq.[4.5] in terms of s:

$$\nabla_i V[x_i(s)] = \frac{a_i}{2|a|s} - \varepsilon \sum_{i_2\ldots i_{2M}} A_{ii_2\ldots i_{2M}} \left(\frac{s}{|a|}\right)^{2M-1}$$

$$= \frac{a_i s}{2|a|s} - \frac{a_i}{s}\left(\frac{s}{|a|}\right)^{2M-1} \tag{C.4}$$

and

$$\sum_i x_i'(s)\nabla_i\, x_i(s) = \frac{s}{2} - \frac{s^{2M-1}}{|a|^{2M-2}} \ . \tag{C.5}$$

Thus, $x_i'(s)\sum_j x_j'(s)\nabla_j\, x_i(s) = \nabla_i V(x_i(s))$; and $x_i(s)$ satisfies Eq.[4.5], the equation of motion.

Next, to show that the radial line $x_i(s)$ is a principal axis of the critical point, we compute the Hessian matrix H_{ij}:

$$H_{ij} = \frac{\partial^2 V}{\partial x_i \partial x_j}(a) = \frac{1}{2}\delta_{ij} - \varepsilon(2M-1) \sum_{i_3,\ldots i_{2M}} A_{iji_3\ldots i_{2M}} a_{i_3}\ldots a_{i_{2M}} \qquad (C.6)$$

Thus,

$$\sum_j H_{ij} x_j(s) = \frac{sa_i}{|a|}(1-M).$$

That is, $x_i(s)$ is an eigenvector of H with eigenvalue 1-M and by definition it is a principal axis of the critical point. Because 1-M is less than 0, the critical point is a maximum along the x_i direction. This completes the proof.

Next we will show that for almost all values of the parameters of V in Eq.[C.1] there is at least one radially directed saddle point.

Let us first review the results that we obtained in two dimensions. For the potential $V=\frac{1}{4}(x^2+y^2) - \frac{1}{4}\varepsilon(ax^4+2cx^2y^2+by^4)$ there are three distinct possibilities:

$$c \neq a,\ c \neq b, \tag{C.7a}$$

$$a=b=c, \tag{C.7b}$$

$$a=c>b. \tag{C.7c}$$

In the first case we found that V always has a radial saddle point. The second case is spherically symmetric and is treated by separation of variables in Appendix B. The third case was not treated in Chap. IV, but it is easy to see that there is a degenerate critical point along the x axis (the Hessian has a zero eigenvalue). We will see that a similar trichotomy occurs for the general potential in Eq.[C.1].

Let us consider an arbitrary unit vector $\hat{u}$ and study the variation of V along the radial line through $\hat{u}$:

$$V = \frac{r^2}{4} - \frac{\varepsilon}{2M} r^{2M} T(\hat{u}). \tag{C.8}$$

where $x_j = r\hat{u}_j$ and $T(\hat{u}) = \sum_{i_1 \cdots i_{2M}} A_{i_1 \cdots i_{2M}} \hat{u}_{i_1} \cdots \hat{u}_{i_{2M}}$. Then

$$\frac{\partial V}{\partial r} = \frac{r}{2} - \varepsilon r^{2M-1} T(\hat{u}), \tag{C.9}$$

and,

$$\frac{\partial^2 V}{\partial r^2} = \frac{1}{2} - \varepsilon(2M-1)r^{2M-2}T(\hat{u}). \tag{C.10}$$

Thus V has a maximum on this radial line at

$$r = \left(\frac{1}{2\varepsilon T(\hat{u})}\right)^{\frac{1}{2M-2}} . \tag{C.11}$$

Eq.[C.11] defines an N-1 dimensional hypersurface B. It is easy to show that V attains a minimum on B. If T(u) does not vanish anywhere, then the hypersurface is compact and, because V is continuous, a minimum exists.

Now let us supposed that $T(\hat{u})$ does vanish somewhere. Since T is a polynomial, the set of $\hat{u}$ for which it vanishes intersects B in a hypersurface B_o of dimension $\leq N-2$. If we cover B_o with a set of "very narrow N-1 dimensional strips" (the reader is advised to think of the case N=3 from which this terminology arises), then the complement of these strips in B will be a compact set, and V will have a minimum there. Since the "strips" covering B_o can be made as "narrow" as we wish, the point where V has its minimum is separated from B_o by a finite distance. (Otherwise the minimum would lie along

one of the directions where T vanishes; but this would be absurd because $V \to +\infty$ in such a direction.)

There are now two possibilities. Either the minimum of V is isolated or else V is constantly equal to its minimum on some nontrivial connected subset of B. The latter case corresponds to Eq.[C.7b] and is exceptional in the sense that we can make the minimum isolated by changing the interaction term $A_{i_1 \cdots i_{2M}}$ infinitesimally and we will therefore consider it further.

If the minimum is isolated, then we have a radially directed saddle point because, by construction, V has a maximum along the radial direction and a minimum in all other directions. As we have shown above, the radial direction is one of the principal axes of the saddle point. The saddle point may, of course, be degenerate as in Eq.[C.7c]. (The WKB methods which we have introduced depend on the existence of a nondegenerate saddle point.) However, since degeneracy is also extremely exceptional, there is almost always a saddle point.

Surprisingly, the analytical techniques of Chap. IV are easily generalized to any potential of the form [C.1],

as long as a radial saddle point exists. We choose a coordinate system which coincides with the principle axes of the saddle point. The x_1 axis is the radial line through the saddle point, whose coordinates are thus P=(P,o,o,...). Since this is a critical point, we have

$$\left.\frac{\partial V}{\partial x_j}\right|_P = 0$$

or

$$A_j \underbrace{111111}_{2M-1} = 0 \qquad j \neq 1 \quad . \tag{C.12}$$

Furthermore

$$\left.\frac{\partial^2 V}{\partial x_i \partial x_j}\right|_P = 0 \qquad i \neq j$$

or

$$A_{ij} \underbrace{1\ldots1}_{2M-2} = 0 \quad i \neq j \tag{C.13}$$

We are now ready to solve

$$(-\nabla^2 + V - E)\psi = 0 \tag{C.14}$$

in a narrow tube surrounding the x_1 axis. For notational convenience we redefine the coefficients $A_{i_1 \cdots i_{2M}}$ so that

$$V = \sum_{i=1}^{N} \frac{x_i^2}{4} + \frac{\lambda}{2^M} \sum_{i_1 \cdots i_{2M}=1}^{N} A_{i_1 \cdots i_{2M}} x_{i_1} \cdots x_{i_{2M}} \qquad \text{(C.15)}$$

and set $\lambda = -\varepsilon^{M-1}$ ($\varepsilon > 0$), and $A_{111111} = 1$. This does not change [C.12] or [C.13].

For x_1 near the origin, [C.14] reduces to the equation for an isotropic harmonic oscillator, and is easily solved. In the tunneling region we write

$$\psi = \chi \left(\frac{x_1^2}{4} - \varepsilon^{M-1} 2^{-M} x_1^{2M-1/2}\right)^{-1/4}$$

$$\times \exp \int_{\sqrt{2}}^{x} \left(\frac{t^2}{4} - \varepsilon^{M-1} 2^{-M} t^{2M-1/2}\right)^{1/2} dt \; . \qquad \text{(C.16)}$$

In the resulting differential equation for χ, we set

$$\varepsilon x_1^2 = Z^2 \qquad \text{(C.17)}$$

and drop all terms which vanish as $\varepsilon \to 0$, obtaining:

$$Z(1-2^{2-M}Z^{2M-2})^{1/2}\chi_Z - \sum_{i=2}^{N} \chi_{x_i x_i} + \sum_{i=2}^{N} \left(\frac{x_i^2}{4} - \frac{c_i}{2^{M-1}} Z^{2M-2} x_i^2 - 1/2\right)\chi = 0. \tag{C.18}$$

In [C.18] we have defined $c_i = A_{ii} \underbrace{111111}_{2M-2}$. This equation, like [4.24] must be solved exactly. To simplify [C.18] we let

$$w = (1-2^{2-M}Z^{2N-2})^{1/2} \tag{C.19}$$

which gives

$$(M-1)(w^2-1)\chi_w - \sum_{i=2}^{N} \chi_{x_i x_i} + \sum_{i=2}^{N} \left[\frac{x_i^2}{4} - \frac{1}{2} - \frac{c_i}{2}(1-w^2)x_i^2\right]\chi = 0 \tag{C.20}$$

The ansatz

$$\chi = A\, e^{-\frac{1}{4}\sum_{i=2}^{N} f_i(w) x_i^2} \tag{C.21}$$

leads us to a set of (N-1) Riccati equations which govern the thickness of the tube of probability current:

$$-(M-1)(w^2-1)f_i'(w) - f_i^2(w) + 1 - 2C_i + 2C_i w^2 = 0 \tag{C.22}$$

To linearize [C.22] we let

$$f_i(w) = (M-1)(w^2-1)u_i'(w)/u_i(w) \tag{C.23}$$

and find that

$$(1-w^2)u_i'' - 2Zu_1' + [\nu_i(\nu_i+1) - \frac{\mu^2}{1-w^2}]u_i = 0 \tag{C.24}$$

where $\nu_i(\nu_i+1) = \frac{2c_i}{(M-1)^2}$ $\mu = \frac{-1}{(M-1)}$.

This is an associated Legendre equation. As in Chap. IV, we choose the solution

$$u_i = P^{\mu}_{\nu_i} \quad . \tag{C.25}$$

When [C.22] is satisfied, the equation for A is much simpler than [C.20]:

$$(M-1)(w^2-1)A_w + \sum_{i=2}^{N}[-A_{x_ix_i} + f_i x_i A_{x_i} + \tfrac{1}{2}(f_i-1)A] = 0 \quad . \tag{C.26}$$

The substitution $(w,x_i)\rightarrow(w,{}^{x_i}/_{u_i(w)})$ reduces [C.26] to a separable equation

$$(M-1)(w^2-1)A_w + \frac{1}{2}\sum_{i=2}^{N}[f_i(w)-1]A = \sum_{i=2}^{N}\frac{1}{u_i^2(w)}A_{s_is_i} \tag{C.27}$$

where $s_i = \frac{x_i}{u_i(w)}$.

Following Sec. IV, D.1, we now require that the wave function in the tunneling region match to the oscillator wave function near the origin, and find that A must be a function of w only. It follows that, up to an arbitrary multiplicative constant β,

$$A = \beta\Big[\prod_{i=2}^{N} u_i(w)\Big]^{-1/2}\,[(1-w)/(1+w)]^{1/4N-4} \tag{C.28}$$

β is determined by matching. In the matching region $w \sim 1$ and we have[1]

$$P_{\nu}^{-\frac{1}{M-1}}(w) \sim \left(\frac{1}{2} - \frac{w}{2}\right)^{1/2(M-1)}/\Gamma(M/M-1) \tag{C.29}$$

Thus, from [C.25] and [C.28],

$$A \sim \beta\,\Gamma^{\frac{N-1}{2}}[M/M-1] \qquad \text{for } w \sim 1$$
$$f_i(w) \sim 1 \ . \tag{C.30}$$

We also have formulas similar to [4.40] and [4.41] which enable us to conclude [see (C.16)] that in the matching region

$$\psi/\chi \sim 2^{1/2}\, e^{1/4}\, e^{-x_i^2/4} \quad . \tag{C.31}$$

Combining Eqs.[C.21, C.30, C.31], we find that in order to obtain a match to the groundstate wave function of an oscillator ($e^{-\sum_{i=1}^{N} x_i^2/4}$), we must have

$$\beta = e^{-1/4}\,[2\Gamma^{N-1}(M/M-1)]^{-1/2} \quad . \tag{C.32}$$

Continuing to follow Section D of Chap. IV, we use the formulae[2]

$$P_\nu^\mu(0) = 2^{\mu+1}\pi^{-1/2}\cos\frac{\pi(\mu+\nu)}{2}\,\frac{\Gamma(\frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2})}{\Gamma(1+\frac{\nu}{2}-\frac{\mu}{2})} \tag{C.33}$$

$$\frac{dP_\nu^\mu}{dx}(0) = 2^{\mu+1}\pi^{-1/2}\sin\frac{\pi(\mu+\nu)}{2}\,\frac{\Gamma(1+\frac{\nu}{2}+\frac{\mu}{2})}{\Gamma(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2})} \tag{C.34}$$

from which it follows that

$$f_i(0) = 2(1-M)\,\frac{\sin\frac{\pi}{2}(\nu_i+\mu)}{\cos\frac{\pi}{2}(\nu_i+\mu)}\,\frac{\Gamma(1+\frac{\nu_i}{2}+\frac{\mu}{2})\,\Gamma(1+\frac{\nu_i}{2}-\frac{\mu}{2})}{\Gamma(\frac{1}{2}+\frac{\nu_i}{2}-\frac{\mu}{2})\,\Gamma(\frac{1}{2}+\frac{\nu_i}{2}+\frac{\mu}{2})} \tag{C.35}$$

Eqs. [C.32-35] may now be used to determine $J(x)$, and then Im E, and finally the large n behavior of A_n. We omit this calculation here because it is a straightforward generalization of the one in Section IV D.4. The result is

$$A_n \sim \prod_{i=2}^{N} \left[\frac{\Gamma(1+\nu_i-\mu)\Gamma(-\nu_i-\mu)}{\Gamma(1-\mu)\Gamma(-\mu)}\right]^{1/2} \frac{M-1}{\pi^{3/2}} (-2)^{-n}$$

$$X\Gamma(nM-n+\frac{1}{2})\left[\frac{\Gamma(\frac{2M}{M-1})}{\Gamma^2(\frac{M}{M-1})}\right]^{nM-M+1/2} \qquad (C.36)$$

The reader may check that this reduces to [4.80] in the limit $M \to 2$.

We conclude finally, that the straight line WKB methods of Chapter IV are applicable to a very wide class of equal mass coupled anharmonic oscillators.

FIGURE CAPTION

Fig. 1 The domain of analyticity of $E(\lambda)$ rigorously established by Simon. $E(\lambda)$ is analytic except on the cut along the negative real axis and in the shaded region. The contour appropriate for proving Eq.[2.4] is indicated by a dashed line.

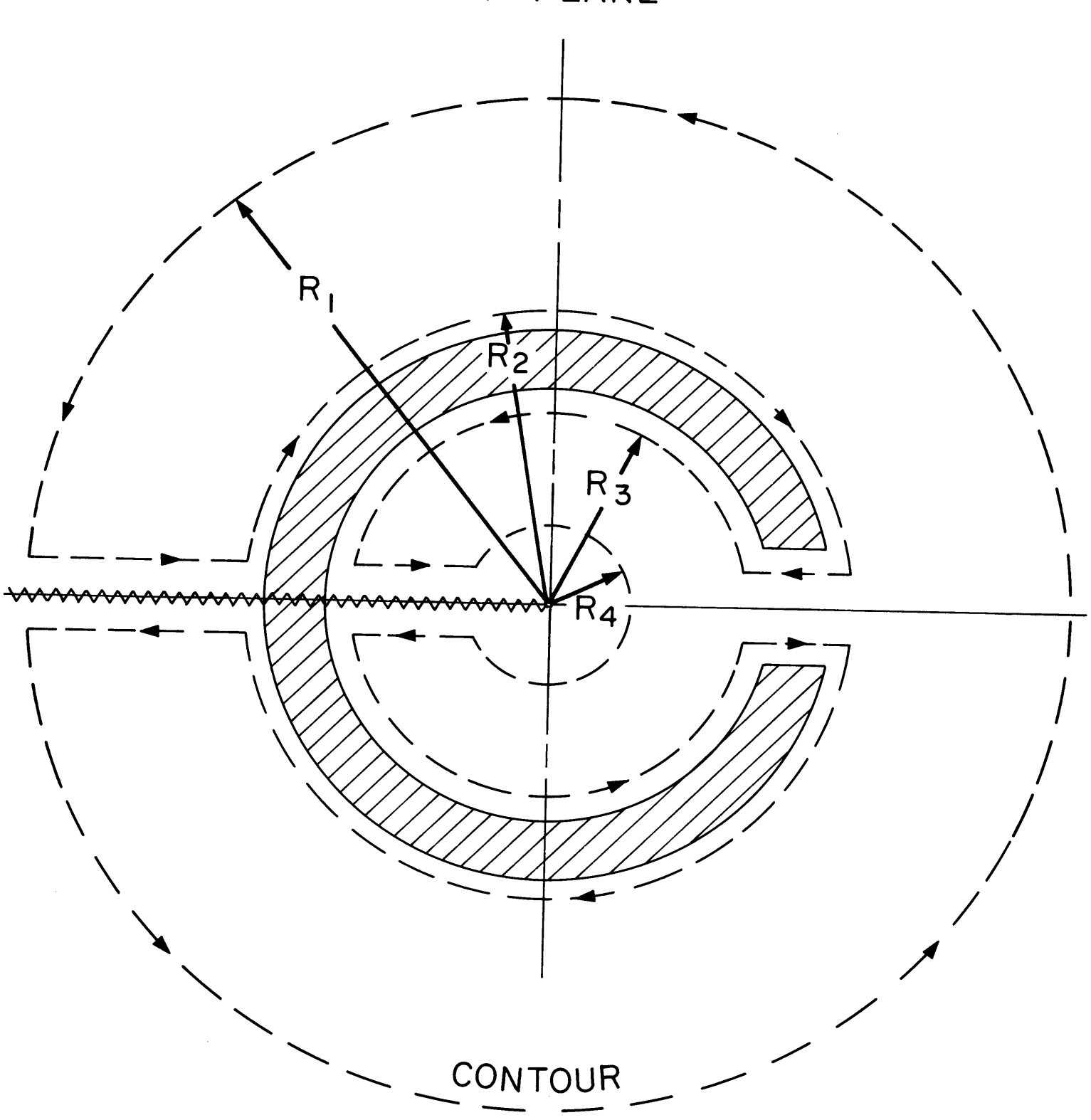

FIGURE 1

FIGURE CAPTION

Fig. 2 The behavior of the potential for small negative λ which corresponds to convergence or divergence. The perturbation series for V_1 will converge while that for V_2 will diverge.

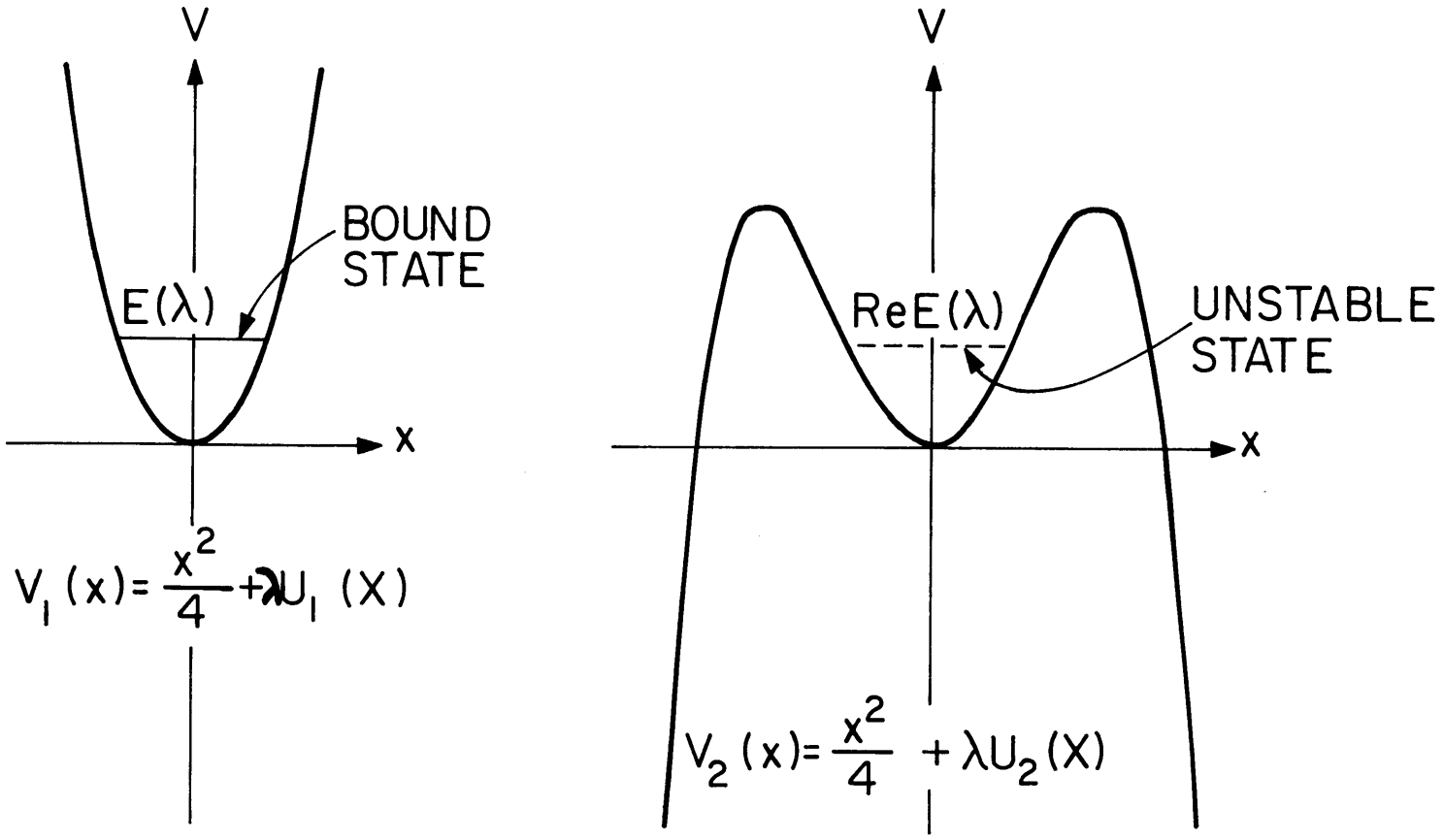
V
BOUND
STATE
E(λ)
x
$V_1(x) = \frac{x^2}{4} + \lambda U_1(X)$
V
ReE(λ)
UNSTABLE
STATE
x
$V_2(x) = \frac{x^2}{4} + \lambda U_2(X)$

FIGURE 2

FIGURE CAPTION

Fig. 3 The behavior of the real and imaginary parts of the wave function for the x^{2N} oscillator.

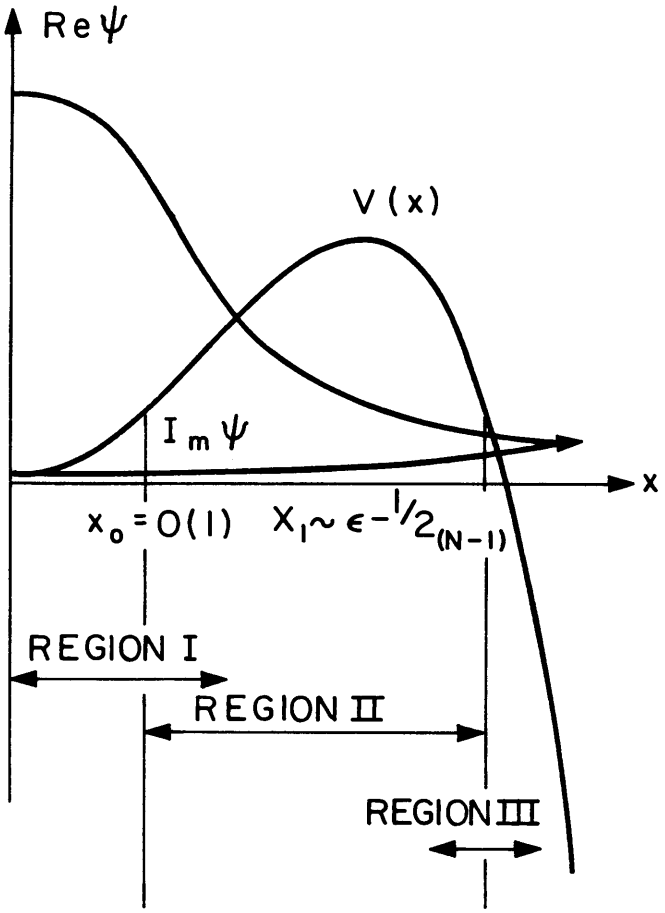

FIGURE 3

FIGURE CAPTION

Fig. 4 The contour Γ which enables one to avoid the upper turning point x_1. If R is large enough, the WKB approximation is good all along Γ.

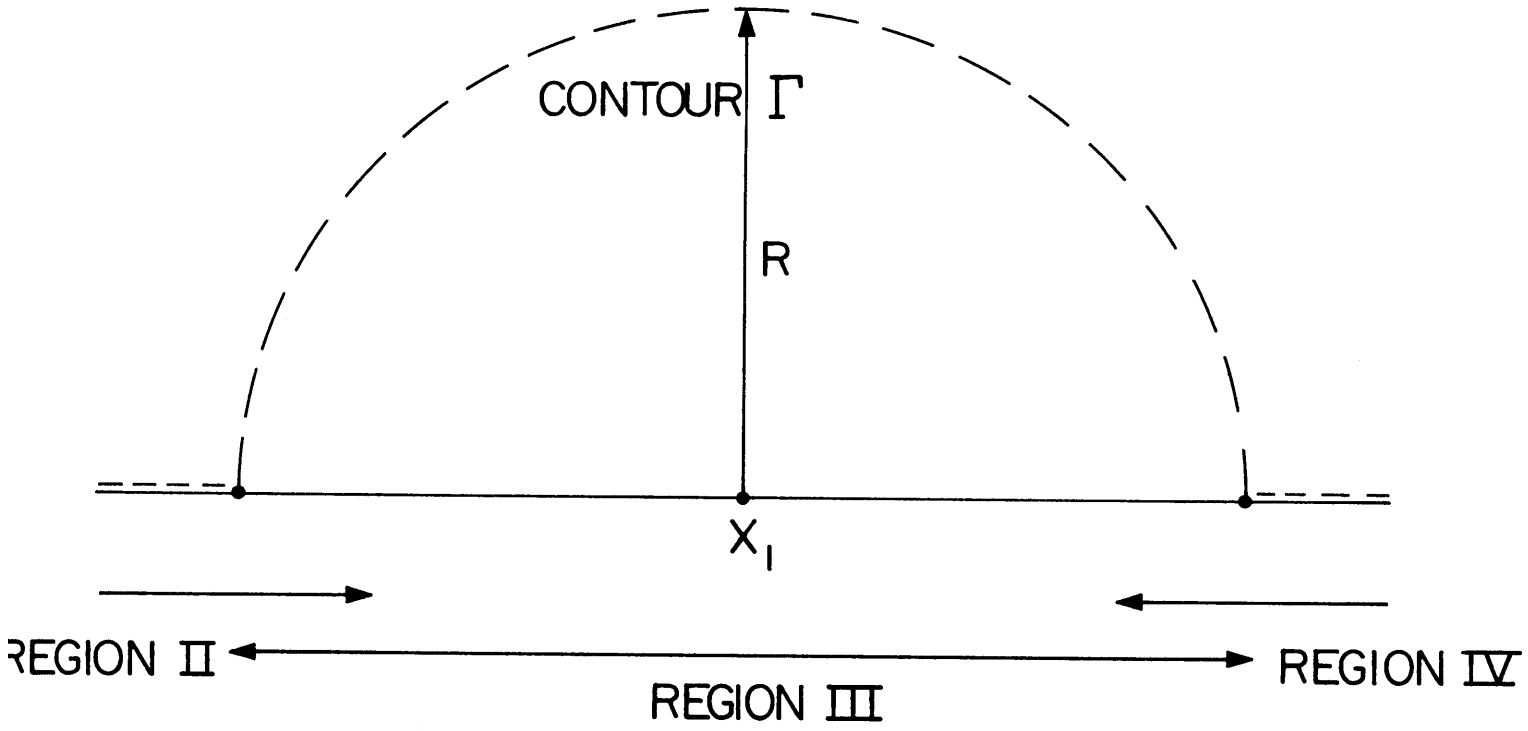

FIGURE 4

FIGURE CAPTION

Fig. 5 The coordinate system suitable for describing the most probable escape path.

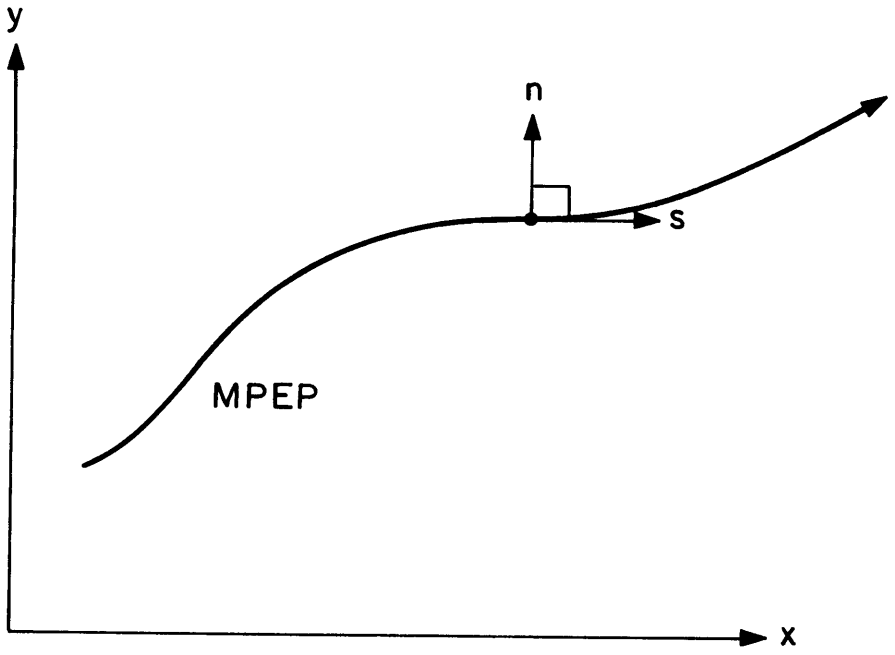

FIGURE 5

REFERENCES

Footnotes for Chapter I

1) This in itself is an indication of the difficulties encountered in field theory. The examination of foundations is usually characteristic of the maturity of a physical theory rather than its infancy.

2) A summary of these important experiments may be found in the contribution of H. W. Kendall to the Proceedings of the International Symposium on Electron and Photon Interactions, 1971. Edited by N. B. Mistry (Cornell U. Press, Ithaca, New York, 1972).

3) See for example S. D. Drell and T. D. Lee, Phys. Rev. D5, 1738 (1972).

4) This can apparently happen only in non-Abelian gauge theories. See D. J. Gross and F. Wilczek, Phys. Rev. Letters 30, 1343 (1973); H. D. Politzer, Same Journal, pg. 1346.

5) M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).

6) S. L. Adler, Phys. Rev. D5, 3021 (1972).

7) A. M. Jaffe, Comm. Math. Phys. 1, 127 (1965).

8) B. Simon, Ann. Phys. (New York) 58, 79 (1970).

9) C. M. Bender and T. T. Wu, Phys. Rev. D7, 1620 (1973).

10) A. M. Jaffe, "Existence Theorems for Cut-Off $\lambda\phi^4$ Field Theories", Proceedings of the Conference on the Mathematical Theory of Elementary Particles. Edited by R. Goodman and I. Segal (M.I.T. Press, Cambridge, Mass., 1966), pg. 45.

Footnotes for Chapter II

1) C. M. Bender and T. T. Wu, Phys. Rev. D7, 1620 (1973).

2) See B. Simon, Ann. Phys. (New York) 58, 79 (1970), Sec. II.2.

3) T. Kato, "Perturbation Theory for Linear Operators", Springer-Verlag, Berlin, 1966.

4) B. Simon, op. cit., Sec. II.1.

5) B. Simon, op. cit., Sec. III.1.

6) F. J. Dyson, Phys. Rev. 85, 631 (1952). See also S. L. Adler, Phys. Rev. D5, 3021 (1972).

Footnotes for Chapter III

1) An alternative treatment of this system using the techniques of Ref. 1, Chap. II has appeared in T. I. Banks and C. M. Bender, J. Math. Phys. 13, 1320 (1972).

2) See the Bateman Manuscript Project, Higher Transcendental Functions, edited by A. Erdelyi (McGraw-Hill Book Co., New York, 1953), Vol. 2, pg. 116. Hereafter, we refer to this work as BMP.

3) BMP, Vol. 2, pg. 122.

4) BMP, Vol. 2, pg. 117.

5) See "Handbook of Mathematical Functions", edited by M. Abramowitz and I. Stegun (Dover Publications, New York), pg. 446. Hereafter, we refer to this reference as HMF.

6) HMF, pp. 448-9.

7) HMF, pp. 448-9.

8) See E. T. Whittaker and G. N. Watson, "A Course of Modern Analysis" (Cambridge University Press, 1962), pg. 240.

9) See reference 1.

10) See for example L. I. Schiff, Quantum Mechanics (McGraw-Hill, New York, 1949), pg. 179 et. seq.

11) Ref. 1 of Chap. II.

Footnotes for Chapter IV

1) A discussion of the WKB approach to tunneling problems may be found in L. D. Landau and E. M. Lifshitz, Quantum Mechanics: Nonrelativistic Theory (Pergamon Press, London, 1958), pg. 171 et. seq. We should point out that considerable formal work has been done on the multidimensional WKB approximation. See for example V. Maslov, Method VKB mnogomermom slychae, prilozhenne k kmge khedinga (Russian), (Matematika Mir, 1965).

2) See for example, L. I. Schiff, Quantum Mechanics (McGraw-Hill, New York, 1949), pg. 179.

3) Equation [3.4] has a classical analogy. It is formally identical to classical equations for the orbit of a particle in the potential V. Of course, for classical motion to occur we must have $E>V$, while in the quantum mechanical tunneling problem $E<V$. Thus, our most probable escape paths are analytic continuations of the classical orbits.

4) In a problem with a continuous symmetry, such as spherical symmetry, the set of MPEP's will, of course, be continuous. This happens in our problem only when $c=1$. When $c\neq 1$ (and $a=b=1$), we have four paths because of reflection symmetry in x and y.

5) It is also easy to show that these are the only straight-line solutions of Eq. [3.4].

6) This transformation is, of course, motivated by the direction of the MPEP's for $c>1$ (they meet the x-axis at 45° angles). See Sec. III.

7) BMP, Vol. 1, pg 121, Eq. (1).

8) We are, of course, free to choose any linear combination of $P_\nu^{-1}(w)$ and $P_\nu^1(w)$. It is most convenient and simplest to make the choice in Eq. [4.21] because it is easy to argue that α^2, the separation constant for Eq. [4.23], vanishes. For choices other than that in Eq. [4.21], there may be an integral over separation constants.

9) BMP, Vol. 1, pg. 163, Eq. 8 .

10) BMP, Vol. 1, pg. 145, Eqs. (20) and (23). There is an error in Eq. (23) which we have corrected.

Footnotes for Chapter V

1) Equation [5.4] is exactly true so long as the lines of constant s, which are normal to the MPEP do not cross. Thus, Eq. [5.4] is valid for $|n|$ less than the radius of curvature ρ^{-1} (see Eq. [5.6]).

2) Recall that the path length $(d\ell)^2$ is given by $dx^\nu dx^\mu g$. $g_{\mu\nu}$ in the (x,y) coordinate system is the 2×2 unit matrix. Thus, $dx^\nu dx^\mu g_{\mu\nu} = (dx)^2 + (dy)^2$. In the (n,s) system $(d\ell)^2 = g_{ss}(ds)^2 + g_{nn}(dn)^2 + 2g_{ns}dsdn$, whence Eq. [5.5], using Eq. [5.3].

3) BMP, Vol. 1, pg. 121, Eq. (3.21).

4) BMP, Vol. 1, pg. 163, Eqs. (8) and (10).

5) BMP, Vol. 1, pg. 145, Eqs. (22) and (23). Equation (23) has a misprint which we have corrected in the text.

6) See BMP, Vol. 1, pg. 169, Eq. (1).

7) See Ref. (4).

8) BMP, Vol. 1, pg. 144, Eq.(12). There is an error in this equation which we have corrected in the text.

9) BMP, Vol. 1, pg. 110, Eq.(12).

10) BMP, Vol. 1, Sec. (1.7.1).

Footnotes for Chapter VI

1) Ref. 1 of Chap. II.

2) We did not discuss Wick ordering explicitly in Chapter III. The details may be found in reference 1 of that chapter.

3) C. M. Bender and T. T. Wu, Phys. Rev. 184, 1231 (1969).

4) The number of entries in the matrix $C_{n,j,k}$ grows as n^3 and the convolution term in Eq.[4.33] requires $C_{n,j,k}$ for all n less than the order of perturbation theory being computed. The limitations of the core memory prevent a calculation of a_n for $n \gtrsim 20$. To go to order 65 in perturbation theory, we retain only the first and last terms in the convolution sum. This is a good approximation for n large and is justified in Ref. 1 of Chap. II. It is this approximation which accounts for the 4 part in 1.5×10^6 error in Table II.

Footnotes for Chapter VII

1) We have not proven a dispersion representation for G, which would enable us to state with certainty that we only need to compute G for small negative λ. Nonetheless, I believe it to be true.

2) This matching difficulty may not appear when we use the second method to compute the two-point function.

3) See for example R. Jackiw, "Dilatation Symmetry and Light Cone Commutators", lectures given at the University of Turin, June, 1971, (Unpublished).

4) S. Fubini, A. Hanson, and R. Jackiw, Phys. Rev. D7, #6 (1972).

5) P. Jordan and E. Wigner, Zeits, fur Phys., 47, 631 (1928).

6) Th. W. Ruijgrok, CERN preprint, TH. 1393.

7) See. Ref. (6).

8) Ref. 3 of Chap. VI.

9) B. Simon, Nuovo Cimento 59, 199 (1969).

10) For an exact reference see 9.

Footnotes for Appendix C

1) See Ref. V-4.

2) See Ref. V-5.

BIOGRAPHICAL NOTE

The author of this thesis was born on April 19, 1949, in Brooklyn, New York. He attended the New York public schools and graduated from Stuyvesant High School in June of 1965. His undergraduate work was done at Reed College in Portland, Oregon, where he received a B.A. in 1969.