Essays on Mechanism Design

by

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Abstract

This thesis is a collection of three essays on mechanism design. In Chapter 1, we consider a general Informed Principal Problem in the context of procurement. Both the potential suppliers and the buyer hold some private information: each supplier knows his cost of production, the buyer knows how much each supplier's product fits her technical requirements. We derive the optimal auction in this environment, and analyze the implementation problem with special emphasis on three aspects that are particularly relevant in real practice: privacy protection, dynamic consistency, and simplicity. We design a dynamic mechanism, the Modified English Auction, that implements the optimal auction outcome, is privacy preserving, dynamically consistent, and simple. Chapter 2 is a joint work with Joao Leao. How do mechanims like hotwire.com work? What is their economic impact on the existing markets of hotel rooms, airplane tickets, and rental cars? We address these questions by investigating whether lotteries over the basic goods can be profitably used by any of the market participants. We consider lotteries in which the buyers win a prize for sure, but they do not know which one. Our main finding is that the perfect cartel always uses lotteries to maximize its profits. Moreover, under specific conditions, the entry of a lottery provider in a competitive market may bring the existing firms closer to the cartel solution. The introduction of lotteries has two effects. First, the firms can use them to price-discriminate their consumers. Second, the firms can use lotteries to cover a larger part of the market. Indeed, the consumers who find the basic goods too expensive may still want to buy cheaper lottery tickets. In Chapter 3 we initiate the formal analysis of the First Price-First Score Auction in a general context where the auctioneer is a seller and two bidders compete to buy one indivisible good. The auctioneer's preferences are assumed to directly depend on the identity of the buyer to whom the good is allocated. In this auction, the bidders submit monetary bids, and then the seller decides which bid to accept after comparing the bidders' scores. A particular class of auction we focus on have simple scoring functions: each bidder's score is given by the summation of his bid and a bidder-specific additional parameter. Our main goal is to obtain the specification of the problem that generates a closed-form analytical solutions for the bidding strategies. The task is complicated as there are at least two sources of asymmetries inherent to the problem that can quickly lead to intractable formulas. The main contribution of this work is to provide closed formulas for the inverse bidding functions. Our results generalize the comparison of bidding strategies in asymmetric first price auctions obtained by Maskin and Riley (2002). Even if the asymmetry between the bidders is exogenously introduced by the auctioneer, in equilibrium the disadvantaged bidder bids more aggressively. We are also able to determine the ranges of bids that can be submitted by the two bidders. They are actually different, and their extremes depend on the extra-bid

parameter.

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Chapter 1

A Modified English Auction for an informed buyer

* I'm deeply indebted to Kemal Guler. He introduced me to the "Informed Auctioneer Problem." Some of the contents in Section 2 and 3.1 appear in the internship report [3] I wrote under his wise and competent guidance at HP Lab in August 2006. His advice and friendship have been the most precious to me. I'd like to thank Fereydoon Safai for that wonderful experience as well.

1.1 Introduction

In procurement, when a company wants to buy a good or hire a service, it's rarely the case that just the price matters. On the contrary, several other factors are usually considered, such as quality, capacity constraints, time of delivery, the costs of switching to a new potential supplier, the value of the existing business relationships, etc.[1] Even though some of these elements enter in the range of matters that are the subject of bargaining between the buyer and the potential suppliers, there are others that are left outside. This happens for a variety of reasons: to reduce the complexity of a multi-party negotiation over multiple dimensions, because some factors are non-contractible,[2] and also because the buyer does not always fully reveal what really matters to her. In other words, the actual choice of

[1] For instance, in the IT sector it's a common practice to compare the potential suppliers' offers on the base of their respective Total Cost of Ownership (TCO). TCO is a synthetic measure that is calculated taking in account all direct and indirect costs attributed to owning and managing an IT product.

[2] For example, there may be technical characteristics of the good that cannot be appropriately customized for a lack of time.

the buyer among the potential suppliers may be affected by external factors that are not controlled by the suppliers, and that are buyer's private information. For this reason, in practice, it's often the case that "the Auctioneer reserves the right to award the work to a bidder other than the one who submitted the lowest bid if it deems such action to be in her best interest".[3]

In an environment where both buyer and suppliers hold some private information about their preferences, both determining the maximum expected utility the buyer can achieve and designing a mechanism that implements it are important problems. The contribution of our work is both theoretical and applied. We derive the optimal auction, and we offer a dynamic mechanism that implements the optimal auction outcome in a very practical way.

We consider a setting where both the buyer and the suppliers have quasi linear utility functions and independent private values.[4] Each potential supplier knows privately his cost of production, and the buyer knows privately the *suitability value* of each potential supplier, that measures how much each supplier's product fits the buyer's requirements.

We formalize the problem as a generalized Informed Principal Problem (Myerson (83), Maskin Tirole (90)). We show that, for each type of the buyer, the optimal auction is the same as the optimal auction in an environment with common knowledge of the buyer's type. Then, we introduce three mechanisms that implement the optimal auction outcome: the Modified First Price Auction, the Modified Second Score Auction, and the Modified English Auction.

In the first format the allocation rule is such that for each supplier there is a scoring function that integrates the buyer's information with the supplier's bid. The winner is the supplier with the highest score. Like in the standard first price auction, the winner is the only bidder who gets paid, and he receives a payment equal to his bid.

The second format is a variation of the second price auction. The allocation rule uses a simple scoring function, the same for all suppliers, where the score is given by the difference between the suitability value and the bid. The winner is the supplier with the highest score. The payment rule is such that the winner is the only one who gets paid, and the amount he receives is a function of his suitability value and the score of the second best supplier.

[3]City of New Haven Purchasing Policies

[4]It's important to notice that, even if we are considering a procurement setting, our analysis and our results can be applied also to the standard auction setting, where a seller faces a number of potential buyers.

The Modified English Auction is a dynamic mechanism. At the first round, the buyer makes an individual take-it-or-leave-it offer to each potential supplier. If a supplier refuses the offer, he drops out of the auction; if he accepts the offer, he stays active and passes to the following round. From the second round on, out of all the active suppliers, the buyer chooses one as the temporary winner, and makes a new offer to all the others. Whenever the buyer makes a new offer, he decreases the corresponding standing offer from the previous round by a small, discrete amount. Each offer is made directly to the supplier who is the intended recipient, and it is not disclosed to anyone else. When the temporary winner is the only active supplier left, the auction ends. The temporary winner becomes the auction winner, and he receives a payment equivalent to the last offer he accepted.

When considering the implementation problem, we focus on three aspects that are particularly critical in real business practice: privacy protection, mechanism dynamic consistency,[5] and mechanism simplicity. Companies that run procurement auctions are often very reluctant to fully disclose the criteria on the base of which the suppliers' offers are compared. This happens because these evaluations are based on technical and strategic considerations whose privacy is perceived by the management as critical for the company's future success. However, the choice of keeping secret many details of the mechanism may end up in compromising the functioning of the mechanism itself. For example, considering the three optimal mechanisms we introduced, if the buyer's private information is not disclosed, we show that the Modified First Price Auction fails to implement the optimal auction outcome, and that the Modified Second Score Auction becomes dynamically inconsistent. The strongest result of our work is to have designed a mechanism that is optimal, privacy preserving, dynamically consistent, and simple: the Modified English Auction.

It may seem surprising that a dynamic mechanism is the solution. At each round, indeed, the auctioneer can reveal some more of her private information and she can change her strategy. However, we show that the flow of information the buyer receives is such that she never deviates from the optimal strategy she adopts from the first round. On top of that, the confidential manner with which every offer is disclosed just to the intended recipient guarantees that any supplier never gets enough information to infer the buyer's type.

[5] A mechanism is dynamically consistent if, over time, the decision-maker may want to change the mechanism rules. For example, in an environment with asymmetric bidder, if an auctioneer chose to run a second-price auction, after learning the bids, he may be willing to renege on the auction rules.

Notwithstanding the complex setting we are considering, the Modified English Auction is also a detail-free mechanism. It inherits from the English auction the dynamic procedure, and the properties of universality and anonymity.[6] In the English auction format often used by auction houses, at each round the auctioneer announces the standing price to the public; in our Modified version, at each round the buyer extends her offers that are personalized and confidential.

1.2 Optimal Auction

1.2.1 The model

There is a buyer who wants to purchase one indivisible good, and there are n potential suppliers.[7] Let $N = \{1, ..., n\}$ be the set of the potential suppliers, and let $i \in N$ be the generic one.

Both suppliers and buyer hold some private information. Each supplier i is privately informed about his cost of production c_i, that is distributed independently according to a cumulative distribution function F_i over the support $C_i = [\underline{c}_i, \overline{c}_i]$. Let $C = C_1 \times ... \times C_i \times ... \times C_n$. The buyer's private information is represented by a n+1-dimensional vector $e = (e_0, e_1, ..., e_n)$. Its components, called the *suitability values*, measure how much the outside option, e_0, or each potential supplier's product, e_i, fits her requirements. Each e_i is distributed independently according to a cumulative distribution function G_i over the support $E_i = [\underline{e}_i, \overline{e}_i]$. Let $E = E_0 \times E_1 \times ... \times E_i \times ... \times E_n$.

Both suppliers and buyer are risk neutral and have quasi-linear utilities. If a buyer of any type $e \in E$ chooses to buy the good from supplier i and pay each supplier $j \in N$ an amount of money x_j, she gets utility $U_e = e_i - \sum_{j \in N} x_j$, supplier i's gets utility $U_i = x_i - c_i$, any supplier $k \neq i$ gets utility $U_k = x_k$. If the buyer chooses not to buy the good from any supplier, she gets $e_0 - c_0$, and the suppliers get zero utility. Let c_0 be a market price that is common knowledge.

Given the information structure and the utility functions specification, we refer to the above environment as *Independent Private Values*: each player's type is statistically inde-

[6] Its rules are neither specific to the object of the procurement (universality) nor specific to the identity of the suppliers (anonymity) (Wilson, 1987).

[7] Throughout the paper feminine pronouns refer to the buyer, whereas masculine pronouns refer to the potential suppliers.

pendent from any other player's type and does not enter directly in any other player's utility function.

1.2.2 Informed Buyer Problem

We consider the problem of a buyer who first learns her type, and then designs a mechanism to maximize her (ex-interim) expected utility.

A mechanism specifies the actions that the buyer and the potential suppliers can take, an *allocation rule*, and a *payment rule.*

Applying a generalized version of the Direct Revelation Principle (Myerson 79), without loss of generality, we can restrict the set of feasible mechanisms to the set of direct mechanisms M. The Direct Revelation Principle states that any given equilibrium in the original mechanism can be replicated by a direct mechanism $\mu \in M$, in which (i) it is an equilibrium for each player to report truthfully her or his type, and (ii) the outcomes are the same as in the given equilibrium of the original game. In a direct mechanism each player reports her or his type to a fictitious mediator, who then determines the final outcome according to an allocation rule and a payment rule. The allocation rule is a function $p : C \times E \rightarrow [0,1]^n$ that maps each combination of players' reports into a n-dimensional vectors of probabilities p. Each component p_i of p is the probability that supplier i is the one selected by the buyer. It has to be that, for any i, $p_i \geq 0$, and $\sum_{j \in N} p_j \leq 1$.[8] The payment rule is a function $x : C \times E \rightarrow \mathbb{R}^n$ that maps each combination of players' reports into a n-dimensional vector of payments x. Each component x_i of x is the payment supplier i receives.

Given a truthful equilibrium (e, c) and the resulting $p(c, e)$ and $x(c, e)$, the utility of a type e buyer is $U_e = \sum_{i \in N} (p_i e_i - x_i) + (1 - \sum_i p_i)(e_0 - c_0)$, and the utility of a generic potential supplier i is $U_i = x_i - p_i c_i$.

The problem we consider is a variation of the Informed Principal Problem (Myerson 1983, Maskin Tirole 1990), and we refer to it as the Informed Buyer Problem. The buyer, like the Principal, chooses the mechanism after learning her type, and her choice may reveal something about what she has learnt. If that is the case, the suppliers, like the Agents, update their beliefs about the buyer's type, and play consistently. The mechanism becomes

[8] Notice that the fact that the probabilities p are not necessariliy degenerate, assigning probability one to at most one supplier and zero to all the others, means that we let the mechanism to have random outcome.

a communication device that the buyer uses to send some (or no) information about her type to the suppliers.[9]

We first determine the maximum expected utility the buyer can achieve, when she fully reveals her private information, i.e. e is common knowledge.

Let $u_i(c,e) = \underset{c_{-i}}{E}\left[x_i(e,c_i,c_{-i}) - p_i(e,c_i,c_{-i})\, c_i\right]$ be the expected utility of a supplier i when he reports his true cost c_i, and let $u_i(\widehat{c}_i,c,e) = \underset{c_{-i}}{E}\left[x_i(e,\widehat{c}_i,c_{-i}) - p_i(e,\widehat{c}_i,c_{-i})\, c_i\right]$ be his expected utility when he reports $\widehat{c}_i$.

The common knowledge optimal auction problem of a buyer of type e is

$$\begin{aligned}
&\max_{p,x}\underset{c}{E}\left[\sum_i \left(p_i(c,e)\, e_i - x_i(c,e)\right) + \left(1-\sum_i p_i(c,e)\right)(e_0 - c_0)\right] && \\
&\text{s.t. } u_i(c,e) \geq 0 \;\forall i && \text{(IR)}\\
&u_i(c,e) \geq u_i(\widehat{c}_i,c,e) \quad \forall \widehat{c}_i, i && \text{(IC)}\\
&\sum_i p_i(c,e) \leq 1 \;\text{ and }\; p_i(c,e) \geq 0 \;\; \forall i &&
\end{aligned}$$

where the first constraints are the suppliers' individual rationality constraints, the second ones are the supplier's incentive compatibility constraints, an the last ones guarantee that the probabilities are well defined.

We use μ_e^{CK} to refer to the common knowledge optimal auction when the buyer's type is e. $p^{CK}(c,e)$ and $x^{CK}(c,e)$ are, respectively, the allocation and payment rule.

Let $J_i(c_i) = c_i + \frac{F_i(c_i)}{f_i(c_i)}$ be i's virtual cost function, and assume that it is strictly increasing $\forall i$[10].

Proposition 1 *In the common knowledge optimal auction, the allocation rule is*

$$\begin{aligned}
p_i^{CK}(c,e) = 1 \quad & \textit{if } e_i - J_i(c_i) > e_0 - c_0 \\
& \textit{and } e_i - J_i(c_i) > e_j - J_j(c_j) \;\; \forall j \neq i \\
p_i^{CK}(c,e) = 0 \quad & \textit{otherwise}^{11}
\end{aligned}$$

[9] Unlike in Myerson's and Maskin Tirole's versions of the Informed Principal Problem, in our setting the potential suppliers (the Agents) do not choose between accepting or refusing the mechanism. Before the mechanism is implemented, the buyer does not get any new information to update her beliefs about the potential suppliers' types.

[10] If $J_i(c_i)$ is not strictly increasing, we can "iron" it (Myerson 81).

[11] *In the case of ties, the allocation is decided randomly.*

The payment rule is

$$
\begin{aligned}
x_i^{CK}(c,e) &= c_i^* \textit{ if } p_i(c,e) = 1 \\
x_i^{CK}(c,e) &= 0 \textit{ if } p_i(c,e) = 0
\end{aligned}
$$

where c_i^* is the maximum cost level at which bidder i would still be able to win the auction

$$
e_i - J_i(c_i^*) = \max\left\{e_0 - c_0, \max_{j \neq i}\{e_j - J_j(c_j)\}\right\}
$$

Proof. *The common knowledge optimal auction is a generalization of the standard optimal auction a' la Myerson. For each potential supplier, the suitability value is integrated with the cost parameter according to a virtual utility function. The suppliers are ranked on the base of their corresponding virtual utilities. The supplier with the highest virtual utility is the winner, and he gets paid the equivalent of the maximum cost realization that would have let him win the auction.*

The proof is in Appendix A. ■

Given that the suitability values are private information of the buyer, she may also choose mechanisms that do not reveal the value of e and how the e_i's enter in the allocation and payment rules, leaving the potential suppliers uncertain about it.

A buyer of any type e is assumed to be always able to select her common knowledge optimal mechanism μ_e^{CK}. This mechanism is derived maximizing the buyer's ex-interim expected utility subject to the potential suppliers' individual rationality and incentive compatibility constraint at a given buyer's type value (e). Once the potential suppliers are uncertain about the buyer's type, these constraints do not have to hold pointwise anymore. They have to hold in expectation over all the types the potential suppliers believe as likely. Now, at a specific value of e, there may be slack in the constraints.

Every type of the buyer is expected to choose her optimal mechanism. If, given a direct mechanism $\widetilde{\mu}$, the potential suppliers are not able to infer if the realized buyer's type is e or e', then $\widetilde{\mu}$ has to be the optimal mechanism for both of them. Indeed, if the mechanism was optimal just for one of them, for example e, it would not be rational to expect type e' to have chosen a sub-optimal mechanism.

In order to characterize the set of rationalizable choices of mechanisms that pool together different buyer's types, we borrow an insightful market representation from Maskin Tirole (90). The idea is that the optimal allocation of the slacks can be derived as a competitive market equilibrium where all the buyer's types participate. Buying and selling slack, each buyer's type has the chance to relax the constraints of her original common knowledge optimal mechanism problem and get a higher expected utility. Moreover, the Pareto Optimality of the competitive market equilibrium guarantees that any outcome derived is a rationalizable choice for all the buyer's types.

Given the Independent Private Values setting and the quasi-linear utility specification, we show that the marginal rate of substitution between any two types of slack is the same for all buyer's types: no trade is an equilibrium of this market. All other equilibria are payoff equivalent.

Proposition 2 *For any e, the common knowledge optimal mechanism* μ_e^{CK} *is the optimal mechanism* μ^* *for an informed buyer of type e.*

Proof. *In Appendix B.* ■

1.3 Implementation Problem

1.3.1 Two static optimal mechanisms

The direct optimal mechanism is not a practical solution. Its rules are too dependent on the specific detailed of the environment. Moreover, in reality, it may be very complicated to convince the suppliers to directly reveal their cost of production. Designing a mechanism that is outcome-equivalent to the optimal direct mechanism and that can be actually used in practice is an open issue. We call this the implementation problem.

A natural way to proceed is to tailor procurement auction formats that are already widely used in different contexts to our specific needs.

We first consider the first price auction.[12] From this format, we keep the simple payment rule, i.e. the supplier who wins gets paid his own bid. In order to implement the optimal mechanism outcome, we modify appropriately the allocation rule.

[12] In a standard setting, with an uninformed buyer, in a first price auction the supplier who wins is the one who bid the lowest price. The winner gets paid his own bid, all the other potential suppliers do not receive anything.

Proposition 3 *The optimal auction solution μ^* can be implemented by a buyer who first reveals her type, and then runs a modified first price auction. The rules of this auctions are: (i) all the potential suppliers bid simultaneously; (ii) the allocation rule is*

$$
\begin{aligned}
q_i^{MFP}(b_i, e) = 1 \qquad & \text{if } e_i - T_i(b_i) \geq e_0 - c_0 \\
& \text{and } e_i - T_i(b_i) \geq e_j - T_j(b_j) \ \forall j \\
q_i^{MFP}(b_i, e) = 0 \qquad & \text{otherwise}
\end{aligned}
$$

where b_i is bidder i's bid, q_i is the probability that bidder i wins the auction, and T_i is such that $T_i(b_i) = J_i\left(\beta_i^{-1}(b_i)\right)$ where $J_i(.)$ is i's virtual cost function, β_i^{-1} is the inverse bid function of $\beta_i(\alpha) = \alpha + \frac{\int_\alpha^{\bar{c}} \tilde{p}_i^{CK}(s,e)ds}{\tilde{p}_i^{CK}(\alpha,e)}$, and $\tilde{p}_i^{CK}(\alpha, e) = \underset{c_{-i}}{E}\, p_i^{CK}(\alpha, c_{-i}, e)$ with $\alpha \in [\underline{c}_i, \bar{c}_i]$; (iii) the payment rule is such that the winner receives a transfer equal to his bid, and the others do not get anything.

Notice that, unlike in the standard format, in this modified version of the first price auction the supplier who bids the lowest price does not necessarily win the auction. Indeed, the allocation rule is such that the suppliers are ranked on the base of their respective scores.[13] Each score is determined by a supplier-specific scoring rule, where the bid b_i is integrated with the supplier's suitability value e_i. The allocation rule is designed in order to induce a symmetric equilibrium bidding strategy in which each supplier bids the equivalent of what he would expect to get according to the common knowledge optimal auction payment rule conditional to the event of winning the auction. In order to choose that particular equilibrium bidding function, the suppliers need to know all the suitability values e_i.

Proof. First, notice that it is optimal for the buyer to report truthfully his type as long as, doing that, the modified first price auction implements the common knowledge optimal auction. Indeed, we proved in the previous section that the common knowledge optimal auction is the optimal solution also for the more general informed buyer problem. Bidder i's problem is

$$\max_{\beta} \underset{c_{-i}}{E} \left[(\beta_i(c_i) - c_i)\, q_i(\beta_i, e) \right]$$

[13]Notice that, unlike in the *scoring auctions* literature (Che (93), Asker Cantillon (06)), in our setting the bidders do not control all the elements that enter in the scoring function. The suitability values, indeed, do not enter in the bids.

where $\beta_i(c_i)$ is supplier i's bidding strategy. In order to implement the optimal outcome, for each potential supplier with cost c_i and suitability value e_i, the expected probability of winning in the first score auction has to be the same than the one in the optimal mechanism. Let $\widetilde{p_i}^{CK}(c_i, e) = \underset{c_{-i}}{E} p_i^{CK}(c_i, c_{-i}, e)$, and $\widetilde{q_i}(\beta_i, e) = \underset{c_{-i}}{E} q_i(\beta_i, e)$.

$$\begin{aligned}
&\widetilde{q_i}(\beta_i, e) = \Pr[\{e_i - T_i(b_i) \geq e_j - T_j(b_j) \quad \forall j \neq i \mid e_i - T_i(B_i) \geq e_0 - c_0\}] = \\
&= \Pr[e_i - J_i(c_i) > e_j - J_j(c_j) \;\; \forall j \neq i \mid e_i - J_i(c_i) > e_0 - c_0] = \widetilde{p_i}^{CK}(c_i, e)
\end{aligned}$$

Notice that it has to be that $T_i(b_i) = J_i(c_i)$. We look for a symmetric equilibrium with a monotone bidding strategy $b_i = \beta_i(c_i)$ for each i, such that $c_i = \beta_i^{-1}(b_i)$ is the corresponding inverse bidding function. We can write $T_i(b_i) = J_i\left(\beta_i^{-1}(b_i)\right)$. We can express the maximization problem as

$$\max_{\beta_i} (\beta_i - c_i)\, \widetilde{p_i}^{CK}\left(\beta_i^{-1}(\beta_i), e\right)$$

The first order condition is

$$\begin{aligned}
\frac{\widetilde{p_i}^{CK\prime}\left(\beta_i^{-1}(\beta_i), e\right)}{\beta_i'\left(\beta_i^{-1}(\beta_i)\right)}(\beta_i - c_i) + \widetilde{p_i}^{CK}\left(\beta_i^{-1}(\beta_i), e\right) &= 0 \\
\frac{\widetilde{p_i}^{CK\prime}(c_i, e)}{\beta_i'(c_i)}(\beta_i - c_i) + \widetilde{p_i}^{CK}(c_i, e) &= 0
\end{aligned}$$

Rearranging the terms we get

$$\begin{aligned}
&\widetilde{p_i}^{CK}(c_i, e)\, \beta_i'(c_i) + \widetilde{p_i}^{CK\prime}(c_i, e)\, \beta_i(c_i) = \widetilde{p_i}^{CK\prime}(c_i, e)\, c_i \\
&\tfrac{d}{dc_i}\left[\widetilde{p_i}^{CK}(c_i, e)\, \beta_i(c_i)\right] = \widetilde{p_i}^{CK\prime}(c_i, e)\, c_i \\
&\widetilde{p_i}^{CK}(c_i, e)\, \beta_i(c_i) = \overline{c}_i \widetilde{p_i}^{CK}(e, \overline{c}_i) - \textstyle\int_{c_i}^{\overline{c}} s \widetilde{p_i}^{CK\prime}(s, e)\, ds
\end{aligned}$$

$$\beta_i(c_i) = \frac{\overline{c}_i \widetilde{p_i}^{CK}(\overline{c}_i, e) - \int_{c_i}^{\overline{c}} s \widetilde{p_i}^{CK\prime}(s, e)\, ds}{\widetilde{p_i}^{CK}(c_i, e)} = c_i + \frac{\int_{c_i}^{\overline{c}} \widetilde{p_i}^{CK}(s, e)\, ds}{\widetilde{p_i}^{CK}(c_i, e)}$$

The expected payoff of bidder i is

$$\begin{aligned}
&\Pi(\beta_i, c_i) = \underset{c_{-i}}{E} p_i^{CK}(c_i, c_{-i}, e)\left[\beta_i(c_i) - c_i\right] = \\
&= c_i \widetilde{p_i}^{CK}(\overline{c}_i, e) + \textstyle\int_{c_i}^{\overline{c}} \widetilde{p_i}^{CK}(s, e)\, ds - c_i \widetilde{p_i}^{CK}(c_i, e) = \int_{c_i}^{\overline{c}} \widetilde{p_i}^{CK}(s, e)\, ds
\end{aligned}$$

To check that $\beta_i(c_i)$ is a best response when the true cost is c_i, we have to compare

$\Pi(\beta(c), c)$ to the expected profit $\Pi(\beta(z), c)$ that a bidder would get bidding $\beta(z)$.

$$\Pi(\beta(z_i), c_i) = z_i \widetilde{p_i}^{CK}(\overline{c}_i, e) + \int_{z_i}^{\overline{c}} \widetilde{p_i}^{CK}(s, e)\, ds - c_i \widetilde{p_i}^{CK}(c_i, e)$$

$\Pi(\beta_i(c_i), c_i) - \Pi(\beta_i(z_i), c_i) = \widetilde{p_i}^{CK}(\overline{c}_i, e)(c_i - z_i) - \int_{c_i}^{z_i} \widetilde{p_i}^{CK}(s, e)\, ds \geq 0$. Ergo $T_i(b_i) = J_i\left(\beta_i^{-1}(b_i)\right)$. ■

Scoring rules are widely used in practice. However, unlike in the *modified first price auction*, the scoring functions are usually very simple. We design an auction format that implements the optimal auction outcome starting from a very intuitive scoring function, i.e. the score is equal to the difference between the suitability value and the bid. In this case, we adjust the payment rule in order to implement the optimal mechanism outcome.

Proposition 4 *The optimal auction solution μ^* can be implemented by a buyer who first reveals her type, and then runs a modified second score auction. The rules of this auctions are: (i) all the potential suppliers bid simultaneously, (ii) the allocation rule is*

$$\begin{aligned} q_i^{MSS}(b_i, e) &= 1 \text{ if } e_i - b_i > e_j - b_j,\ \forall j \\ q_i^{MSS}(b_i, e) &= 0 \text{ otherwise} \end{aligned}$$

where b_i is bidder i's bid, (iii) the payment rule is such that the winner (w) gets

$$x_w(e_w, e_{2h}, b_{2h}) = J_w^{-1}(e_w - e_{2h} + b_{2h})$$

where J_i^{-1} is the inverse of $J_i(c_i) = c_i + \frac{F_i(c_i)}{f_i(c_i)}$, e_{2h} and b_{2h} are respectively the suitability and the bid of the second highest score bidder. The losing bidders do not get anything.

Proof. *Like in case of the modified first price auction, it's optimal for the buyer to reveal truthfully her type, as long as, doing that, the modifified second score auction implements the common knowledge optimal auction. The weakly dominant bidding strategy for a bidder i in a symmetric equilibrium is*

$$b_i = J_i(c_i)$$

Indeed, if everyone use that bidding strategy and bidder i wins the auction, he gets

$$x_i = J_i^{-1}(e_i - e_{2h} + J_{2h}(c_{2h}))$$

His bid does not affect the payment he gets, it affects only his probability of winning. There are only two situations in which bidding a price b_i other than $J_i(c_i)$ changes the final outcome:

1) if $e_i - b_i > e_{2h} - J_{2h}(c_{2h}) > e_i - J_i(c_i)$ bidder i wins, but what he gets is $J_i^{-1}(e_i - e_2 + J_2(c_2)) < c_i$.

2) if $e_i - J_i(c_i) > e_2 - J_2(c_2) > e_i - B$, then bidder i loses and misses the opportunity of getting a positive return, given that $J_i^{-1}(e_i - e_2 + J_2(c_2)) - c_i > 0$. Given the equilibrium bidding strategy $b_i = J_i(c_i)$, the modified second score auction allocation rule is equivalent to the allocation rule of the common knowledge optimal auction; the payment to the winner $x_w(e_w, e_{2h}, b_{2h})$ is equal to c_w^*. ■

We refer to this auction as *modified second score auction*[14], because the payment rule is such that the winner gets an amount that is a function of the second best score. However, unlike in the standard second score auction, the payment is also affected by the winner's suitability parameter. Higher is the the value of the winning supplier, e_w, compare to the value of the second best supplier, e_{2h}, higher is the payment to the winner. More efficient is the second best supplier (lower c_{2h}), lower is the payment to the winner.

1.3.2 Simplicity, Privacy, Dynamic Consistency

Even though we built the modified first price auction and the modified second score auction starting respectively from a simple payment rule and a very intuitive scoring functions, still the two formats present characteristics that make them impractical. Indeed, analyzing their drawbacks, we would like to bring attention on three issues that are very important in real business practice: simplicity, privacy protection, and dynamic consistency.

Following Wilson[15], we define a procurement mechanism as *simple*, if it is anonymous

[14] *Mechanisms similar to the modified first price auction and the modified second score auction appear in Naegelen (2002). Naegelen analyzes the problem of implementing the optimal procurement auction in an environment with symmetric suppliers, where each supplier's product is identified with an exogenous commonly known quality level.*

[15] Wilson (1987) writes, "The optimal trading rule for a direct revelation game is specialized to a particular environment. For example, the rule typically depends on the agents' probability assessments about each other's private information. Changing the environment requires changing the trading rule. If left in this form, therefore, the theory is mute on one of the most basic problems challenging the theory. I refer to the problem of explaining the prevalence of a few simple trading rules in most of the commerce conducted va organized exchanges. A short list -including auctions, double auctions, bid-ask markets, and specialists trading- accounts for most organized exchange. ... The rules of these markets are not changed daily as the environment changes; rather they persist as stable, viable institutions.

and universal. It's anonymous if its rules do not depend on the identity of the bidders; it is universal if they do not depend on the object of the procurement. Simplicity and practicability are intertwined properties. Indeed, it seems reasonable to believe that a mechanism is more likely to be used in practice if it is detail-free.

Neither the modified first price auction nor the modified second score auction is *simple.* In both mechanisms, potential suppliers with different suitability values are treated differently because they are subject to different scoring functions; moreover, the object of the procurement determines which kind of information enters in the auction rules.

In both formats the buyer has to reveal her type. In that way, the allocation and the payment rule are functions only of the bids. However, in real practice, the private information that is used in procurement processes is often related to strategic and technical factors that the buyers want to keep secret.

We adapt the previous formats taking in consideration these privacy concerns, and we consider the case in which the suitability values enter into the mechanisms simultaneously with the bids. In other words, the supplier are not informed about the values of the e_i's when they are asked to bid.

In this case, the modified first price auction fails to implement the optimal auction outcome. Indeed, the true values of the e_i's are necessary information for the suppliers to bid in equilibrium in such a way that the optimal mechanism outcome is implemented. Uninformed about the suitability values, each supplier would bid according to his beliefs about them. In the modified first price auction, to each supplier corresponds a score $\widehat{e}_i - J_i\left(\beta_i^{-1}(b_i)\right)$. In order for each supplier's score to be equivalent to the corresponding virtual utility $e_i - J_i(c_i)$, it's necessary that the the reported value of $\widehat{e}_i$ that enters in the scoring function and the one that enters in the bidding function are both the true one, e_i.

On the contrary, the modified second score auction is still optimal. As long as the buyer can commit to report the values of the e_i's simultaneously with the bids and never change them, each supplier i still bids $b_i = J_i(c_i)$, and the buyer reports the true e.

A mechanism in which the decision maker would like to change its rules at different points of time is called *dynamically inconsistent.* The *modified second score auction* is an example of dynamic inconsistency: after learning the potential suppliers' bids, if she has the possibility, the buyer would change his report about the e_i's in order to maximize her utility. Pretending that the the second best supplier had a very high suitability value e_{2h}

she can decrease her payment to the winner.

The performance of a dynamic inconsistent mechanism depends critically on the ability of the decision maker to commit to its rules. In our case, the optimality of the *modified second score auction* is undermined also by the fact that the suitability parameters are non-observable and non-verifiable. No one would be able to notice if the buyer manipulated her report about the e_i's after collecting the bids. Finding the way to credibly commit to the rules may be very difficult or expensive.

In the next section, we present a dynamic mechanism that is optimal, simple, privacy-protecting, and dynamically consistent.

1.4 Modified English Auction

We offer a procurement auction format that is a particularly appropriate mechanism to implement the optimal auction outcome in an environment where the buyer has some private information about her preferences.

We call this auction *Modified English Auction.*

The auction unravels in several rounds. At the first round the buyer makes an offer to each potential supplier. Those who accept the starting offer participate to the auction. From the second round on, the buyer chooses a temporary winner, and makes new offer to the other active suppliers (temporary losers). The new offers decrease by a constant, small, discrete amount ε at each round. If a potential supplier accepts the offer, he moves to the following round; if he refuses, he drops out of the auction. Each potential supplier gets to know privately his personal offer, and if he is a temporary winner or not. Every potential supplier knows how many competitors are still active in the auction, but not their identity. The auction ends when no temporary losing bidder accepts to drop his standing offer. In that case, the standing temporary winner becomes the final winner and gets paid the last offer he received.

The mechanism inherits from the English auction the fact the auctioneer makes a sequence of bids. However, the bids are not open and direct to the general audience, but, on the contrary, they are confidential and specific to each single potential supplier.

We first prove that this auction format is optimal.

Proposition 5 *For any e, the modified English auction implements the corresponding com-*

mon knowledge optimal auction outcome.

Proof. *The equilibrium in the modified English auction is defined by the equilibrium strategies of the active players, who are the auctioneer and the temporary losing suppliers. Each temporary losing bidder chooses if accepting or not the offers he receives. Given the auction rules are such that whoever refuses an offer drops out of the auction, it's weakly dominant for each supplier to accept all offers that are weakly higher than their cost of production* c_i.

The equilibrium strategy of the auctioneer is more complex. He has to choose the starting offer for each supplier and the temporary winner at each round. The offers at the second and at all following rounds are not part of the buyer's strategy but are instead determined by the rules of the auction that set exogenously the marginal decrement ε.

The auctioneer plays like a monopsonist making a take-it-or-leave-it offer. Let' b_i^1 *be the starting offer (at the first round) to supplier* i. *Supplier* i *will accept the offer if and only if its cost* c_i *is lower than* b_i^1: *this event has probability* $q_i = F_i\left(b_i^1\right)$*.Given this, we can express the starting bid as a function of its probability of being accepted:* $b_i^1 = F_i^{-1}\left(q_i\right)$. *Considering each supplier separately, the buyer chooses his first offer in such a way that her marginal utility (with respect to the probability of awarding supplier* i*) is equal to its outside option value* $e_0 - c_0$.

The utility of the buyer is

$$U^B\left(b_i^1\right) = \left(e_i - b_i^1\right) F_i\left(b_i^1\right)$$

or equivalently

$$V^B\left(q_i\right) = \left(e_i - F_i^{-1}\left(q_i\right)\right) q_i$$

Her marginal utility is

$$\frac{dV^B\left(q_i\right)}{dq_i} = e_i - F_i^{-1}\left(q_i\right) - \frac{q_i}{F_i'\left(F_i^{-1}\left(q_i\right)\right)}$$

or

$$MU^B = e_i - b_i^1 - \frac{F_i\left(b_i^1\right)}{f_i\left(b_i^1\right)}$$

The first offer b_i^1 to each supplier i is such that

$$e_i - b_i^1 - \frac{F_i\left(b_i^1\right)}{f_i\left(b_i^1\right)} = e_0 - c_0$$

or, using the function $J_i(x) = x + \frac{F_i(x)}{f_i(x)}$, and assuming that is invertible, we can write

$$b_i^1 = J_i^{-1}\left(e_i - e_0 + c_0\right)$$

This is equivalent to the reserve price for supplier i in the common knowledge optimal auction.

Starting from the second round on, the auctioneer's action is to choose the temporary winner. In order to maximize her expected utility, the buyer should compare the marginal utilities that she would derive increasing the probabilities q_i for each supplier i winning the auction. At each round t, he should choose the supplier who guarantees hers the highest marginal utility:

$$i : \arg\max_{i \in N} e_i - b_i^t - \frac{F_i\left(b_i^t\right)}{f_i\left(b_i^t\right)}$$

where b_i^t is the offer that supplier i receives at round t, and make a lower offer to all other suppliers still competing. Notice that $e_i - b_i^t - \frac{F_i\left(b_i^t\right)}{f_i\left(b_i^t\right)} = e_i - J_i\left(b_i^t\right)$.

The combination of suppliers' acceptance best response, and auctioneer's first offer and temporary winner selection rule is such that the optimal common knowledge auction outcome is implemented. At the last round T, the second best supplier drops out because the standing bid is by ε below his cost: $b_{2h}^T = c_{2h} + \varepsilon$. In the previous round $T-1$, $b_{2h}^{T-1} = c_{2h}$, and $e_w - J_w\left(b_w^{T-1}\right) \geq e_{2h} - J_{2h}\left(c_{2h}\right)$. The payment the winner is going to get is equal to his standing offer $b_w^{T-1} \leq J_w^{-1}\left(e_w - e_{2h} - J_{2h}\left(c_{2h}\right)\right) = c_w^*$ ■

The mechanism is simple. Its allocation rule and its payment rule do not treat differently the potential suppliers, and do not depend on the object of the procurement. The bias as embedded in the optimal auction rules is now shifted to the equilibrium strategy of the buyer, who is an active player inside the mechanism. The rules are anonymous and universal, the buyer's choices at each round are biased by her information about the potential suppliers' suitability values and cost distributions.

The mechanism protects the buyer's privacy. The fact that each take-it-or-leave-it offer is made in a confidential manner only to the intended recipient prevent any potential supplier to infer the type of the buyer. The fact that each potential supplier does not know e_0 implies that he is not able to infer his e_i from the starting offer (the personalized reserve price). Even though everyone knows how many people are participating in the auction, no one but the buyer knows their identities. For example, when just two bidders are active, each one of them does not know who is the other, and cannot infer the difference between $e_i - e_j$.

The mechanism is dynamic consistent. At any point of time there is no incentive for the buyer to deviate from the rules of the mechanism. The modified English auction is the optimal mechanism at each round of the bidding process. Moreover, the fact that everyone is informed about the number of active players at each stage makes everyone able to check that the buyer abides by the rules of the auction.

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1.5 Appendix A

Proposition 1 Proof The Proof follows Myerson (81)

The problem of an auctioneer of type e is

$$\begin{aligned}
&\max_{p,x}\sum_i \left(p_i\left(e,c_i,c_{-i}\right)e_i - x_i\left(e,c_i,c_{-i}\right)\right) + \left(1-\sum_i p_i\left(e,c_i,c_{-i}\right)\right)\left(e_0-c_0\right)\\
&\text{s.t. } U_i\left(e,c_i\right) = \underset{c_{-i}}{E}\left[x_i\left(e,c_i,c_{-i}\right) - p_i\left(e,c_i,c_{-i}\right)c_i\right] \geq \underset{c_{-i}}{E}\left[x_i\left(e,\widehat{c}_i,c_{-i}\right) - p_i\left(e,\widehat{c}_i,c_{-i}\right)c_i\right] = U_i\left(e,\widehat{c}_i\right) \ \ \forall\widehat{c}_i, i\\
&\qquad U_i\left(e,c_i,c_{-i}\right) \geq 0 \ \forall c_i, \forall i\\
&\qquad \sum_i p_i\left(e,c_i,c_{-i}\right) \leq 1 \ \text{ and } \ p_i\left(e,c_i,c_{-i}\right) \geq 0 \ \ \forall i
\end{aligned}$$

where we use a "hat" to distinguish the reported type from the true one.

The first constraint is the incentive compatibility constraint: reporting truthfully his type is convenient for each supplier. The second constraint is the individual rationality constraint: each supplier's expected utility from the mechanism has to be higher than his outside option. The third constraint requires that the probabilities of winning the procurement auction are well defined. Define

$$\overline{x}_i(e, c_i) = \underset{c_{-i}}{E}\, x_i(e, c_i, c_{-i})$$

and

$$\overline{p}_i(e, c_i) = \underset{c_{-i}}{E}\, p_i(e, c_i, c_{-i})$$

We can rewrite $U_i(e, c_i) = \overline{x}_i(e, c_i) - \overline{p}_i(e, c_i)\, c_i$ and $U_i(e, \widehat{c}_i) = \overline{x}_i(e, \widehat{c}_i) - \overline{p}_i(e, \widehat{c}_i)\, c_i$.

From the incentive compatibility constraints, we have that $U(e, \widehat{c}_i)$ is maximized when $\widehat{c}_i = c_i$.

$$\frac{\partial U(e, \widehat{c}_i)}{\partial \widehat{c}_i} = \frac{\partial \overline{x}_i(e, \widehat{c}_i)}{\partial \widehat{c}_i} - \frac{\partial \overline{p}_i(e, \widehat{c}_i)}{\partial \widehat{c}_i} c_i = 0 \text{ if } \widehat{c}_i = c_i$$

and

$$\frac{\partial U(e, c_i)}{\partial c_i} = \frac{\partial \overline{x}_i(e, c_i)}{\partial c_i} - \frac{\partial \overline{p}_i(e, c_i)}{\partial c_i} c_i - \overline{p}_i(e, c_i) = -\overline{p}_i(e, c_i) \leq 0$$

The second order conditions of the problem $\max U(e, \widehat{c}_i)$ are

$$\frac{\partial^2 U(e, \widehat{c}_i)}{\partial \widehat{c}_i^2} = \frac{\partial^2 \overline{x}_i(e, \widehat{c}_i)}{\partial \widehat{c}_i^2} - \frac{\partial^2 \overline{p}_i(e, \widehat{c}_i)}{\partial \widehat{c}_i^2} c_i < 0 \ \forall \widehat{c}_i$$

Differentiating $\frac{\partial U(e, \widehat{c}_i)}{\partial \widehat{c}_i}$ at $\widehat{c}_i = c_i$, we get

$$\frac{\partial^2 \overline{x}_i(e, c_i)}{\partial c_i^2} - \frac{\partial^2 \overline{p}_i(e, c_i)}{\partial c_i^2} c_i - \frac{\partial \overline{p}_i(e, c_i)}{\partial c_i} = 0$$

We can prove that $\frac{\partial p}{\partial c} \leq 0$,combining $\frac{\partial^2 U(e,c_i)}{\partial c_i^2} = \frac{\partial^2 \overline{x}_i(e,c_i)}{\partial c_i^2} - \frac{\partial^2 \overline{p}_i(e,c_i)}{\partial c_i^2} c_i < 0$ and $\frac{\partial^2 \overline{x}_i(e,c_i)}{\partial c_i^2} - \frac{\partial^2 \overline{p}_i(e,c_i)}{\partial c_i^2} c_i - \frac{\partial \overline{p}_i(e,c_i)}{\partial c_i} = 0$.

The auctioneer's maximization problem can be rewritten as

$$\max_{p} \underset{c}{E} \left\{ \sum_i p_i(e, c_i, c_{-i}) [e_i - c_i] - \sum_i \int_{\underline{c}}^{\overline{c}} U_i(e, c_i) f(c_i)\, dc_i + \left(1 - \sum_i p_i(e, c_i, c_{-i})\right)(e_0 - x_0) \right\}$$

Integrating by parts the second term of the summation and using the fact that $\frac{\partial U_i(e,c_i)}{\partial c_i} =$

$-\overline{p}_i(e, c_i)$, we get

$$\begin{aligned}\int_{\underline{c}}^{\overline{c}} U_i(e, c_i) f(c_i) dc_i &= U_i(e, c_i) F_i(c_i) |_{\underline{c}_i}^{\overline{c}_i} + \int_{\underline{c}}^{\overline{c}} \overline{p}_i(e, s_i) F_i(c_i) ds_i \\ &= U(e, \overline{c}_i) + \underset{c}{E}\left[p_i(e, c_i, c_{-i}) \frac{F_i(c_i)}{f_i(c_i)}\right]\end{aligned}$$

The objective function becomes

$$\underset{c}{E}\left\{\sum_i p_i(e, c_i, c_{-i})\left[e_i - c_i - \frac{F_i(c_i)}{f_i(c_i)}\right] + \left(1 - \sum_i p_i(e, c_i, c_{-i})\right)(e_0 - x_0)\right\} - \sum_i U_i(e, \overline{c}_i)$$

Ergo

$$\max_p \underset{c}{E}\left\{\sum_i p_i(e, c_i, c_{-i})\left[e_i - c_i - \frac{F_i(c_i)}{f_i(c_i)} - (e_0 - x_0)\right]\right\} - \sum_i U_i(e, \overline{c}_i) + (e_0 - x_0)$$

The objective function is maximized when $U_i(e, \overline{c}_i) = 0 \ \forall i$ and p_i is defined as in the Proposition.

Given that $U_i(e, \overline{c}_i) = 0$ and $\frac{\partial U(e,c_i)}{\partial c_i} = -\overline{p}_i(e, c_i) \ \forall i$, we have

$$U_i(e, c_i) = \overline{x}_i(e, c_i) - \overline{p}_i(e, c_i) c_i = -\int_{\underline{c}_i}^{c_i} \overline{p}_i(e, s_i) ds_i = \int_{c_i}^{\overline{c}_i} \overline{p}_i(e, s_i) ds_i$$

$$\underset{c_{-i}}{E}[x_i(e, c_i, c_{-i}) - p_i(e, c_i, c_{-i}) c_i] = \int_{c_i}^{\overline{c}_i} \underset{c_{-i}}{E}[p_i(e, s_i, c_{-i})] ds_i$$

For a given c_{-i}

$$x_i(e, c_i, c_{-i}) = p_i(e, c_i, c_{-i}) c_i + \int_{c_i}^{\overline{c}_i} p_i(e, s_i, c_{-i}) ds_i$$

Let

$$c_i^* = \sup\left\{c_i \mid e_i - c_i - \frac{F_i(c_i)}{f_i(c_i)} > e_0 - c_0 \text{ and } e_i - c_i - \frac{F_i(c_i)}{f_i(c_i)} > e_j - c_j - \frac{F_j(c_j)}{f_j(c_j)} \ \forall j\right\}$$

be the maximum cost level at which bidder i wins the auction. Using the assumption that J_i is invertible, we can rewrite

$$c_i^* = \sup\left\{c_i \mid e_i - c_i - \frac{F_i(c_i)}{f_i(c_i)} > e_0 - c_0 \text{ and } c_i = J_i^{-1}(e_i - e_j + J_j(c_j)) \ \forall j\right\}$$

Given that

$$p_i(e, s_i, c_{-i}) = 1 \quad \text{if} \quad s_i < c_i^*(e, c_{-i})$$
$$p_i(e, s_i, c_{-i}) = 0 \quad \text{if} \quad s_i > c_i^*(e, c_{-i})$$

then

$$\int_{c_i}^{\overline{c}_i} p_i(e, s_i, c_{-i})\, ds_i = c_i^*(e, c_{-i}) - c_i \quad \text{if} \quad c_i < c_i^*(e, c_{-i})$$
$$\int_{c_i}^{\overline{c}_i} p_i(e, s_i, c_{-i})\, ds_i = 0 \quad \text{if} \quad c_i > c_i^*(e, c_{-i})$$

and the payment rule is

$$x_i(e, c_i, c_{-i}) = c_i^*(e, c_{-i}) \quad \text{if} \quad p_i(e, c_i, c_{-i}) = 1$$
$$x_i(e, c_i, c_{-i}) = 0 \quad \text{if} \quad p_i(e, c_i, c_{-i}) = 0$$

1.6 Appendix B

Proposition 2 Proof The proof borrows techniques from Skreta (2007)[16] and adapt them to the context we are considering.

Let $U_i(c_i) = \underset{e}{E}\underset{c_{-i}}{E}[x_i(c, e) - p_i(c, e)\, c_i]$ be the expected utility of supplier i with cost realization c_i over all the other player types, and let $U_i(\widehat{c}_i, c_i) = \underset{e}{E}\underset{c_{-i}}{E}[x_i(\widehat{c}_i, c_{-i}, e) - p_i(\widehat{c}_i, c_{-i}, e)\, c_i]$ be the expected utility of supplier i when it's cost is c_i and his report is $\widehat{c}_i$.

In the informed buyer problem, when the buyer does not reveal her type, the incentive compatibility and the individual rationality constraint of the buyer have to hold in expectation over all the buyer's types for whom it is optimal to choose the selected mechanism. By the Inscrutability Principle[17] (Myerson 83), without loss of generality, we can consider mechanisms that are chosen with probability one by all types of the player.

[16] Skreta (2007) considers an environment in which the auctioneer observes a vector of signls correlated with bidder's valuations. Each bidder knows only the signal that the auctioneer observes about him but not the signals she observes about other buyers.

[17] It may be the case that different types of the buyer maximize their utility using different mechanisms. For example, we can imagine that there is a partition of the buyer's type space $E^A \cup E^B = E$ and $E^A \cap E^B = \emptyset$, such that if $e \in E^A$ the buyer optimally chooses a mechanism μ^A; if $e \in E^B$ the buyer optimally chooses a mechanism μ^B. Facing a mechanism μ^A or μ^B, the potential suppliers learn if the buyer's type belongs to E^A or to E^B, and, after updating their beliefs, they play. This is equivalent to having the mediator to choose an "inscrutable" compound mechanism $\mu = \begin{smallmatrix} \mu^A \text{ if } \theta \in E^A \\ \mu^B \text{ if } \theta \in E^B \end{smallmatrix}$. Notice that if revealing truthfully the type is an incentive compatible and individually rational strategy for the suppliers in both sub-mechanisms μ^A and μ^B separately, it is incentive compatible and individually rational in the compound mechanism μ as well.

The suppliers' constraints become

$$\begin{aligned} U_i(c_i) &\geq 0 \ \forall i \\ U_i(c_i) &\geq U_i(\widehat{c}_i, c_i) \ \forall i, \widehat{c}_i \end{aligned}$$

Let us define the *slack* $m_i(c_i, e)$ as the difference between the expected utility of supplier i over the other supplier's types when the buyer is type of type e $(U_i(c_i, e) = \underset{c_{-i}}{E}[x_i(c, e) - p_i(c, e)\, c_i])$ and $U_i(c_i)$.

$$m_i(c_i, e) = U_i(c_i, e) - U_i(c_i)$$

Similarly, we define $m_i(\widehat{c}_i, c_i, e)$ the difference between these two expected utilities when supplier i's type is c_i, but he reported $\widehat{c}_i$.

$$m_i(\widehat{c}_i, c_i, e) = U_i(\widehat{c}_i, c_i, e) - U_i(\widehat{c}_i, c_i)$$

Now we can reformulate the informed buyer problem as a fictitious common knowledge optimal mechanism problem with slacks in the suppliers' constraints. Individual rationality and incentive compatibility constraint can be rewritten as

$$\begin{aligned} U_i(c_i, e) - m_i(c_i, e) &\geq 0 \ \forall i \\ U_i(c_i, e) - m_i(c_i, e) &\geq U_i(\widehat{c}_i, c_i, e) - m_i(\widehat{c}_i, c_i, e) \ \forall i, \widehat{c}_i, \end{aligned}$$

Given this transformation, for given slack values $m_i(c_i, e)$ and $m_i(\widehat{c}_i, c_i, e)$, we can derive the corresponding common knowledge optimal auction

The optimal allocation rule is not affected by the presene of the slacks:

$$\begin{aligned} p_i^{CKS}(c, e, m_i(c_i, e), m_i(\widehat{c}_i, c_i, e)) &= 1 && \begin{array}{l} \text{if } e_i - c_i - \frac{F_i(c_i)}{f_i(c_i)} > e_0 - c_0 \\ \text{and } e_i - c_i - \frac{F_i(c_i)}{f_i(c_i)} > e_j - c_j - \frac{F_j(c_j)}{f_j(c_j)} \ \forall j \neq i \end{array} \\ p_i^{CKS}(c, e, m_i(c_i, e), m_i(\widehat{c}_i, c_i, e)) &= 0 && \text{otherwise} \end{aligned}$$

The optimal payment rule is

$$x_i^{CKS}(c, e, m_i(c_i, e), m_i(\widehat{c}_i, c_i, e)) = p_i(c, e)\, c_i^* + m_i(\overline{c}_i, e) - m_i(c_i, e)$$

Once we have the optimal common knowledge mechanism for given values of the slacks, we need to find the optimal allocation of the slacks between the buyer's types.

There is a feasibility constraint: for each supplier i and each realization of c_{-i}, the expectation of the slack value over the buyer's types has to be zero

$$\underset{e}{E} m_i (c_i, e) = 0 \ \ \forall i, c_i$$

Following Maskin Tirole (90), we derive the optimal slacks' allocation as a competitive equilibrium of a fictitious market where all types of the buyer exchange slacks.

Plugging $\{p_i^{CKS}\}_{i=1}^n$ and $\{x_i^{CKS}\}_{i=1}^n$ in type e buyer's utility $U_e = \sum\limits_{i \in N} (p_i e_i - x_i) + (1 - \sum\limits_i p_i)(e_0 - c_0)$, we get type e buyer's indirect utility, a function of the slacks $\{m_i\}_{i=1}^n$.

$$U_e \left(p^{CKS}, x^{CKS}\right) = U_e^I \left(\{m_i\}_{i=1}^n\right)$$

Each type of the buyer can guarantee himself the common knowledge optimal auction outcome. This is equivalent to say that each type of the buyer partecipates to the market with an endowment of zero slack. Like in a standard income constraint, the slacks purchased and the slacks sold have to sum up to the value of the initial endowment.

$$\sum_i \sum_{c_i} w_i (c_i) \, m_i (c_i) = 0$$

However, in this market, there are no gains from trade to be exploited: the marginal rate of substitution between any two slacks $m_i (c_i)$ and $m_j (c_j)$ is the same for all types e of the buyer.

Chapter 2

Price discrimination through lotteries

* This is a joint-work with Joao Leao.

2.1 Introduction

The online travel market has recently observed the entry of innovative websites like hotwire.com or priceline.com. The impact of these new players on the market is interesting because of the peculiarity of their business practice. While popular websites like expedia.com or travelocity.com offer valuable services such as search, certification and advertising[1], hotwire and priceline operate very differently, and the economic value of their service is less evident. For instance, apart of the standard menu of hotel rooms and corresponding prices, hotwire offers the option of paying a lower price for a room in a hotel that is not revealed until after the payment is made. The only information the consumer has when deciding if taking or leaving such a "blind offer" is the hotel's category and the indication of a more or less precise area where the hotel is located[2]. In other words, the consumer can choose to pay

[1] Caillaud and Jullien (2003) study the role of informational intermediaries like expedia.com or amazon.com in a digital economy. There, the emphaisis is on the efficiency improvement due to a better information provision.

[2] The mechanism used by priceline.com is more complex. The consumer selects the hotel category and the area he is interested in. Then the consumer makes a bid that may be accepted or refused by a hotel that fits the charachteristics specified. Still, the good offered has an uncertainty embedded of the same kind as in the hotwire-mechanism.

less for an uncertain outcome. This innovative selling strategy appears to be successful[3] in the market for hotel rooms, airplane tickets[4] and rental cars[5], and seems to be easily exportable to many other markets with substitute goods.

How do hotwire and priceline make money? What is the economic impact of these innovative entrants on the existing market?

Without going into the details of the specific business practices, we address these questions by investigating wheter lotteries over the basic goods (hotel rooms, airplane tickets,...) can be profitably used by any of the market partecipants. We consider lotteries in which the buyers win a prize for sure, but they do not know which one. We consider these mechanisms, because they offer a simple representation for the kind of uncertainty an hotwire or priceline customer is exposed to: buying a lottery ticket implies paying in advance for an outcome that is determined later.

Our main finding is that the perfect cartel always uses lotteries to maximize its profits.[6] Moreover, under specific conditions, the entry of a lottery provider in a competitive market may bring the existing firms closer to the cartel solution. The introduction of lotteries has two effects. First, the firms can use them to price-discriminate their consumers. The consumers who are indifferent between the alternatives prefer to buy a cheaper lottery ticket, the ones who have a stronger preference towards one particular good want to pay more and have that good for sure. Second, the firms can use lotteries to cover a larger part of the market. New consumers, indeed, can be reached selling lottery tickets for a cheaper price.

In our analysis we use a generalization of the Hotelling model with horizontally differen-

[3]Hotwire has been bought by InterActive Corp (expedia.com) in 2003 for $665million (BusinessWeek, September 22 2003). Priceline has quickly became the fourth company in the online travel market in terms of revenues. (Harris Interactive study)

[4]In the case of airplane tickets, the flight data are revealed only after the payment. The offer specifies the number of stops and a frame of 6 or 12 hours for the departure time.

[5]Priceline offers the possibility for the consumer to choose the category of the car, an area where to pick up and leave the car, the starting day and the duration of the rental period. Then the consumer is asked to make a bid. If the bid is accepted, the consumer is notified with the details about the rental company and about the car assigned to him.

[6]Our result is in striking contrast with Riley and Zeckhauser (1983). They are the first to examine the use of uncertainties as an attempt to discriminate consumers. They consider the case of a monopolist that produces one good, and the uncertainty embedded in their lotteries is about the good's delivery. They conclude that no lotteries are used in the monopolist optimal mechanism. Thanassoulis (2004) considers a setting with two goods and two-dimensional consumer types, and he presents numerical examples in which lotteries can increase the monopolist's profit.

tiated goods. We first look to an environment with two symmetric firms, each one producing one good, that are located at the extremes of a segment of lenght one. Consumers are assumed to be distributed uniformly along the segment. Their preferences towards each good are represented by the transportation costs of reaching the extreme where the corresponding firm is located. Each consumer is privatly informed about his location.

In this environment we derive the optimal mechanism for the cartel. We find that lotteries are always used. If the transportation costs are concave the optimal mechanism is such that just one lottery is enough to maximize the cartel's profits. If the transportation costs are convex a continuum of lotteries are used to perfectly price discriminate the consumers who are almost indifferent between the two goods.

We then investigate the use of lotteries in a market where the two firms compete against each other. We examine the possibility for an external agent to enter in the market and package lotteries for the firms in exchange of a commission (or a fee). In other words, we check if the agent can design any mechanism such that both firms find optimal to join a lottery providing their goods as prizes, and receiving part of the lottery's revenues.

We distinguish two cases: we call *saturated market* the situation in which, in the market with no lotteries, all consumers are already served by the two firms; we call *market in expansion* the opposite case. We show that, if the market is saturated, it is more complicated for the external agent to enter: he cannot use naive mechanisms with just one lottery, but he needs to design more sophisticated mechanisms with more than one lottery and appropriately designed revenues distribution rules. In a competitive environment, indeed, the introduction of a lottery brings a trade-off to the firms that the external agent has to carefully consider. From the firms point of view, on one side, a lottery represents a price-discrimination device, and a tool to reach new consumers; on the other side, it is a new different good that enters in the market, and this raises the competitive pressure.

Both in a saturated market and in a market in expansion, if there is competition between entrants, we show that the mechanisms that appear in equilibrium are the ones that increase the firms profits. As a limit case, we present a mechanism that implements the maximum profit for the industry when the transportation costs are concave.

The intuition is the following: if the firms are two, each one of them holds a veto power on the lottery, and, equivalently, on the agent's entry. If there is competition between lottery sellers, it is reasonable to expect that, any new entrant would have to increase more and more each firm's profit to convince each of them to join the new lottery. The only way to make more profits is to move closer to the perfect cartel solution.

If the surplus of the firms always increases, the welfare analysis results for the consumers are ambiguous. In a saturated market, the consumer surplus decreases; in a market in expansion, if we restrict the mechanism space to mechanisms in which the lottery price is set before the firms are asked to join, the consumer surplus may increase.

It is interesting to stress that, if there are more than two firms, then the external agent may be able to enter even in saturated markets with simple mechanisms, and play aggressively. Now no firm holds veto power on the lottery. Under some conditions, when asked to partecipate to the lottery, each firm consider as outside option the expected profit in a market with a lottery involving only other firms, and this may be lower than the profit in a market with no lotteries at all. The external agent does not necessarily has to increase the firms' profit to secure his entry. We present an interesting example using an extension of the linear Hotelling model to three firms, where the firms are left worse off in equilibrium.

Our work is complementary to a growing literature that analyzes the role of internet intermediaries as match-makers (Caillaud and Jullien (2003), Jullien (2001), Rochet and Tirole (2003)). The focus of these studies are the efficiency gains provided by the appearance of the intermediaries, and their "chicken and egg" problem, the necessity to convince both sides of the markets (sellers and buyers) to use them. We analyze these intermediaries as mechanism providers that may enter in the market helping the firms to implement the perfect cartel solution, or, on the contrary, playing aggressively leaving them with a lower profit.

2.2 Model

We consider a variation of the Hotelling model of horizontal differentiation. There are two firms, indexed by $i = \{A, B\}$, located at the two endpoints of a segment $[0, 1]$. Each firm produces one good. They have identical and constant marginal cost of production, which we assume to be equal to 0 without loss of generality. Good A is produced by firm A, located in 0, and good B is produced by firm B, located in 1.

There is a continuum of consumers with unit demand. We assume that the consumers are distributed uniformly along the segment.[7] Each consumer's preference over the goods A and B is represented as a function of his location on the segment, that is his private information. If a consumer is located in x, with $x \in [0,1]$, his utility from buying good A is

$$U(x) = V - c(x) - p_A$$

where p_A is the price of good A, V is a positive constant value, and $c(x)$ is a generic transportation cost function. We assume that $c(\cdot)$ and $c'(\cdot)$ are continuous functions, $c(0) = 0$, and $c'(x) > 0$.

Similarly, if a consumer in x buys good B, his utility is

$$U(x) = V - c(1-x) - p_B$$

If a consumer does not buy any good, he gets 0 utility.

When setting the price p_A the profit of firm A is

$$\pi_A = \int_{x \in D(A)} p_A dx$$

where $D(A)$ is the set of costumers buying good A. The profit of firm B is defined similarly.

2.3 Perfect Cartel

We consider the case in which both firms A and B can collude and commit to maximize their joint profit. This is equivalent to consider the profit maximization problem of a monopolist who produces both goods A and B.

The problem is perfectly symmetric for the two firms located at the extremes of the segment. We can find the profit maximizing mechanism for the cartel dividing the segment $[0,1]$ in two sub-segments $\left[0, \frac{1}{2}\right]$, $\left[\frac{1}{2}, 1\right]$, and maximize the profit of firm B over the second sub-segment. The profit maximizing solution for firm A is symmetric.

Applying the Direct Revelation Principle, without loss of generality we can restrict to consider direct mechanisms, where each consumer reveals his location x to a fictitious

[7] Different assumptions about the transportation cost function and the types distribution along the line can be made. To preserve our results, we need that the total consumer surplus is decreasing moving away from the endpoints of the segment.

mediator who then selects an outcome for him. The set of feasible outcome we consider includes stochastic outcomes, or, in other words, probability distribution over the two goods. In general, we define an outcome as a combination of a probability q of getting good A[8] (that may be degenerate) and a corresponding price $p(q)$. We refer to this probabilistic outcomes as lotteries.

The profit maximization problem of firm B can be formalized in the following way

$$\max_{p,q} \int_{\frac{1}{2}}^{1} p(x)\, dx$$

$$s.t.$$

$$V - q(x)\, c(x) - (1 - q(x))\, c(1-x) - p(x) \geq 0 \quad \forall x$$

$$\begin{array}{l} V - q(x)\, c(x) - (1 - q(x))\, c(1-x) - p(x) \geq \\ \geq V - q(y)\, c(x) - (1 - q(y))\, c(1-x) - p(y) \end{array} \quad \forall y, x$$

$$q(x) \geq 0; \quad \forall x$$

The program is subject to the individual rationality constraint, and incentive compatibility constraint of each consumer. The last constraint is a feasibility constraint: the probabilities of all lotteries have to be well defined.

The solution to the problem turns out to be different depending on the shape of the transportation cost function.

2.3.1 Concave costs

Proposition 6 *If $V \geq c\left(\frac{1}{2}\right)$[9] and the transportation cost function $c(x)$ is concave, the optimal mechanism for the perfect cartel is such that only one lottery with probability $q = \frac{1}{2}$ is sold at a price $p = V - c\left(\frac{1}{2}\right)$. Good A and good B are sold at a price $p_A = p_B = V + \frac{1}{2}\left(c(x^*) - c(1-x^*)\right) - c\left(\frac{1}{2}\right)$, where $x^* = 1 - \frac{c(x^*) - c(1-x^*)}{c'(x^*) + c'(1-x^*)}$.*

Proof. We start considering the sub-segment $\left[\frac{1}{2}, 1\right]$ of the segment of length one. Let

[8] The probability $1 - q$ is the probability of getting good B.

[9] *If $V \leq c\left(\frac{1}{2}\right)$, the optimal contract for the monopolist is to sell the two goods at prices $p_A = p_B = V + \frac{1}{2}\left(c(1-x^*) - c(x^*)\right) - c\left(\frac{1}{2}\right)$, when c is a concave cost function; $p_A = p_B = V - c(x^{**})$ when c is a convex.*

$W(x)$ be the utility of consumer x.

$$\begin{aligned} W(x) &= V - q(x)\, c(x) - (1 - q(x))\, c(1-x) - p(x) \\ &= \max_y V - q(y)c(x) - (1 - q(y))c(1-x) - p(y) \end{aligned}$$

By the envelope theorem

$$\frac{\partial W(x)}{\partial x} = W'(x) = -qc'(x) + (1-q)c'(1-x) \tag{2.1}$$

To apply the Mirrlees solution, we need the utility $W(x)$ to be an increasing function of x. If $W(x)$ is increasing in x, then

$$q < \frac{c'(1-x)}{c'(x) + c'(1-x)} \tag{2.2}$$

Given that $x \geq \frac{1}{2}$ and $c(\cdot)$ is concave, $c'(1-x) \geq c'(x)$. This implies

$$\frac{c'(1-x)}{c'(x) + c'(1-x)} \geq 0.5 \ \forall x$$

where $\frac{c'(1-x)}{c'(x)+c'(1-x)} = \frac{1}{2}$ only at $x = \frac{1}{2}$. Since q must be weakly lower than $\frac{1}{2}$, we have

$$q < \frac{c'(1-x)}{c'(x) + c'(1-x)} \ \forall x \in \left(\frac{1}{2}, 1\right]$$

Hence $W(x)$ is increasing in x. Integrating equation (2.1), we get

$$W(x) = \int_{\frac{1}{2}}^{x} \left(-qc'(z) - (1-q)c'(1-z)\right) dz + W\left(\frac{1}{2}\right) \tag{2.3}$$

By the individual rationality constraint of type $x = \frac{1}{2}$, $W\left(\frac{1}{2}\right) = 0$. Rearranging the terms in 2.3, we can express each price as

$$p(x) = V - qc(x) - (1-q)\, c(1-x) - \int_{0.5}^{x} \left(-qc'(z) - (1-q)c'(1-z)\right) dz$$

Now we can reformulate the problem as

$$\max_{p,q} \pi = \int_{\frac{1}{2}}^{1} p(x)\, dx$$

$$\max_{q} \int_{\frac{1}{2}}^{1} \left\{ V - qc(x) - (1-q)\,c\,(1-x) - \int_{\frac{1}{2}}^{x} \left(-qc'(z) - (1-q)c'(1-z)\right)\right) dz \right\} dx$$

After integrating by parts $\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{x} \left(-qc'(z) - (1-q)c'(1-z)\right)\right) dz dx$[10] we obtain

$$\max_{q} \int_{\frac{1}{2}}^{1} \left\{ V - qc(x) - (1-q)\,c\,(1-x) - \left(-qc'(x) - (1-q)c'(1-x)\right) (1-x) \right\} dx$$

The maximization of π with respect to the schedule $q\,(.)$ requires the term under the integral to be maximized with respect to each $q\,(x)$ for any x. Taking the first order conditions we get

$$-c(x) + c\,(1-x) + \left[c'(x) + c'(1-x)\right] (1-x)\,dx\ \forall x \tag{2.4}$$

This first derivative does not depend on q. Hence, for each type x it is either positive or negative. If it is positive, the optimal solution is to assign to type x a lottery with the highest possible probability q, $q = 0.5$. If it is negative, the solution is $q = 0$. Setting equation (2.4) equal to zero, we can determine the threshold value x^*. In the optimal mechanism, all types $x \geq x^*$ get their most preferred good (good B) for sure (lottery with $q = 0$); all types $x < x^*$ get a lottery with $q = \frac{1}{2}$.

$$-c(x^*) + c\,(1-x^*) + \left[c'(x^*) + c'(1-x^*)\right] (1-x^*) = 0$$

$$x^* = 1 - \frac{c(x^*) - c\,(1-x^*)}{c'(x^*) + c'(1-x^*)} \tag{2.5}$$

Notice that at $x^* = 0.5$, the left side of the equation is smaller than the right side of the equation, (0.5<1). For $x^* = 1$ the right side of the equation is smaller. Therefore, given that c and c' are continuous functions, by the Bolzano's theorem, there is a solution x^* in the segment $(0.5, 1)$. Similarly, we solve the problem for the sub-segment $\left[0, \frac{1}{2}\right]$. Once we have determined that the optimal mechanism is a set of three lotteries $q = \frac{1}{2}$, $q^B = 0$, and $q^A = 1$, we need to determine their prices in such a way that types $x \in \left[1-x^*, x^*\right]$ choose lottery $q = \frac{1}{2}$, all types $x \in [0, x^*]$ choose lottery $q^A = 1$, all types $x \in [x^*, 1]$ choose lottery

[10] $\int_{\frac{1}{2}}^{1} \left(\int_{\frac{1}{2}}^{x} \left(-qf'(z) - (1-q)f'(1-z)\right)\right) dz \right) dx = \left(\int_{\frac{1}{2}}^{x} \left(-qf'(z) - (1-q)f'(1-z)\right)\right) dz \right) x \;\;|_{\frac{1}{2}}^{1}$
$- \int_{\frac{1}{2}}^{1} \left(-qf'(x) - (1-q)f'(1-x)\right)\right) x dx =$
$= \int_{\frac{1}{2}}^{1} \left(-qf'(x) - (1-q)f'(1-x)\right)\right) dx - \int_{\frac{1}{2}}^{1} \left(-qf'(x) - (1-q)f'(1-x)\right)\right) x dx =$
$\int_{\frac{1}{2}}^{1} \left(-qf'(x) - (1-q)f'(1-x)\right)\right) (1-x)\,dx$

$q^B = 0$. Notice that lotteries q^A and q^B give prize good A and good B with certainty, so their prices correspond to the price of good A (p_A) and the price of good B (p_B). The price p of lottery $q = \frac{1}{2}$ is pinned down by the individual rationality constraint of the lowest type ($x = \frac{1}{2}$):

$$p = V - c\left(\frac{1}{2}\right)$$

The price p_B (or p_A) is determined by making the incentive compatibility constraint of type $x = x^*$ (or $x = 1 - x^*$) binding.

$$\begin{aligned} p_B &= V - c(1-x^*) - \left(V - \frac{1}{2}c(x^*) - \frac{1}{2}c(1-x^*) - \left(V - c\left(\frac{1}{2}\right)\right)\right) \\ &= V + \frac{1}{2}\left(c(x^*) - c(1-x^*)\right) - c\left(\frac{1}{2}\right) \end{aligned}$$

■

Interestingly, the just one lottery result with concave costs can be seen as extension of the no lottery result of Riley and Zeckhauser [2] to the multiproduct case. When the transportation costs are concave, all consumers derive a higher utility of a lottery with probability $\frac{1}{2}$ than the consumer located in the middle of the segment, $x = 0.5$. Intuitively, we can think as if the monopolist sells a lottery with probability $\frac{1}{2}$ to all consumers. Then, on top of this, he sells to some consumers an additional option of trading the lottery for their favorite good. In each half of the segment line, the demand for this option is like a demand for a monopolist with one good. Consumers located at the extreme of the segments have the highest willingness to pay for this extra option, while the consumer located at $x = 0.5$ is willing to pay zero. Consider now the possibility of offering a lottery with probability q of delivering this option. Here, we can apply the Riley and Zeckhauser [2] result to show that the monopolist does not offer lotteries of this extra option, hence $q = 0$. Offering to deliver this extra option with probability q on top of lottery with probability $\frac{1}{2}$ of getting each good is formally equivalent to offering a lottery with probability $q + (1-q)\frac{1}{2}$ of getting each of the two goods. Hence, the optimal selling strategy is to just offer one lottery with probability $\frac{1}{2}$.

2.3.2 Convex costs

Proposition 7 *If $V \geq \frac{1}{2}$ and the transportation cost function $c(x)$ is convex, the optimal mechanism for the perfect cartel is such that each good is sold at $p_A = p_B = V - c(1 - x^{**})$, where $x^{**} = 1 - \frac{c(x^{**}) - c(1-x^{**})}{c'(x^{**}) + c'(1-x^{**})}$, and a continuum of lotteries with probabilities $q = \frac{c'(1-x)}{c'(1-x)+c'(x)}$ where $x \in \left[\frac{1}{2}, x^{**}\right]$ are sold at prices $p(q) = V - q + q^2$.*

Proof. Notice that, if both IR and local IC constraints bind for each type x in the monopolist problem, from the local IC constraints we get

$$p'(x) = -q'(x)(c(x) - c(1-x))$$

from the IR constraints

$$p(x) = V - q(x)c(x) - (1 - q(x))c(1-x)$$

Both conditions determine a sequence of lotteries and corresponding prices

$$q(x) = \frac{c'(1-x)}{c'(1-x) + c'(x)}$$

$$p(x) = V - \frac{c'(1-x)}{c'(1-x) + c'(x)}c(x) - \frac{c'(x)}{c'(1-x) + c'(x)}c(1-x) \tag{2.6}$$

To determine the optimal mechanism, we restrict our analysis to the half-segment $\left[\frac{1}{2}, 1\right]$ [11] and we proceed through three stages: first, we show that the local IC constraint has to bind for all types and the IR constraint has to bind for at least one type; second, we prove that, given c convex, if the IR constraint binds for a type x^{**},in the optimal mechanism it has to bind for all types $x \leq x^{**}$; third, we determine x^{**}, the threshold type such that, in the optimal mechanism, all $x < x^{**}$ buy the type contingent lottery $q(x)$ at price $p(x)$, and all type $x > x^{**}$ buy good B at price $p_B = V - c(x^{**})$. See Appendix. ■

Hence, the optimal selling strategy for a multiproduct monopolist implies offering at least one lottery with probability $\frac{1}{2}$. This result is in stark contrast with the no lottery result for a single product monopolist in Riley and Zeckhauser [2]. The intuition is as follows. The single product monopolist by offering a lottery with probability q of delivering the good (and $1 - q$ of not delivering the good) is damaging the good in the same proportion to all

[11] Same analysis has to be repeated for the complementary sub-segment $\left[0, \frac{1}{2}\right]$.

consumers. In contrast, the multiproduct monopolist by offering a lottery with probability $\frac{1}{2}$ of getting each differentiated good is damaging the good mostly for consumers with strong preferences for a particular good. If these consumers have also higher willingness to pay, the monopolist can increase its profits by offering a lottery with probability $\frac{1}{2}$.

2.4 Competition

2.4.1 Saturated market

Now we look to the case in which the two firms, A and B, compete against each other and no lotteries are sold. Each firm can only sell his own good, and choose its price to maximize his profit. We first analyze the case in which the value of V is high enough that in the symmetric equilibrium all consumers buy a good[12]. We call this case *saturated market.*

Proposition 8 *If the market is saturated, and the firms compete against each other, in the symmetric equilibrium each firm sets his price equal to*

$$p_A = p_B = \frac{g^{-1}(0)}{g^{-1\prime}(0)}$$

where the function $g^{-1}(\cdot)$ *is the inverse of the function* $g(x) = c(x) - c(1-x)$*.*

Proof. See Appendix B ∎

We may expect that, as in the monopolist case, the firms would want to use lotteries, as a tool to price discriminate. We introduce now the possibility that an external agent can enter in the market offering to arrange lotteries for the two firms. We examine different mechanisms the third agent may propose.

The simplest mechanism is to set a price p_L for a lottery with probability $\frac{1}{2}$ of getting each good, and ask the two firms to join. If they accept, the firms share equally the lottery's profit, leaving a commission fee C to the external agent. If the firms do not accept, the status quo with no lotteries continues.

The problem of the agent is to choose the right price p_L, that convinces both firms to join the lottery. Interestingly, we show that, under general conditions, the firms would

[12]This is equivalent to require that $V \geq f\left(\frac{1}{2}\right) + \frac{g^{-1}(0)}{g^{-1\prime}(0)}$, where $g^{-1}(\cdot)$ is the inverse of the function $g(k) = f(k) - f(1-k)$.

always refuse to join the lottery if the market is saturated.

Proposition 9 *If* $\frac{[g^{-1\prime\prime}(\cdot)][g^{-1}(\cdot)]}{[g^{-1\prime}(\cdot)]^2} \leq \frac{3}{2}$[13]*, an agent offering to package a lottery with equal probabilities of getting each good cannot enter in the market.*

Proof. A consumer is indifferent between buying good A directly from the seller and buying the lottery if

$$V - c(x) - p_A = V - \frac{1}{2}(c(x) - c(1-x)) - p_L$$

Which simplifies to

$$p_L - p_A = \frac{1}{2}(c(x) - c(1-x))$$

Substituting $c(x) - c(1-x)$ with $g\,(x)$, we get

$$g(x) = 2(p_L - p_A)$$

Demand for good A is given by

$$x = g^{-1}(2(p_L - p_A))$$

To show that there is no space for lotteries in equilibrium, we consider the most favorable case for the firms: the third agent does not charge any commission. The firms split the profits from the lottery in two equal parts. Firm A maximizes

$$\max_{p_A} p_A g^{-1}(2(p_L - p_A)) + \frac{p_L}{2}\left[1 - g^{-1}(2(p_L - p_A)) - g^{-1}(2(p_L - p_B))\right]$$

From the first order condition, we get

$$g^{-1}(2(p_L - p_A)) - p_A 2 g^{-1\prime}(2(p_L - p_A)) + p_L\left[g^{-1\prime}(2(p_L - p_A))\right] = 0$$

$$p_A = \frac{1}{2}\left(\frac{g^{-1}(2(p_L - p_A))}{g^{-1\prime}(2(p_L - p_A))} + p_L\right)$$

[13] *This is a general condition that guarantees that the price of a good and the price of the lottery are strategical complements. It's easy to verify that, for example, linear, quadratic and square root transportation costs satisfy the condition.*

or, equivalently

$$p_A - \frac{1}{2}\left(\frac{g^{-1}(2(p_L - p_A))}{g^{-1\prime}(2(p_L - p_A))} + p_L\right) = 0 \qquad (2.7)$$

In order to determine the relation between p_A and p_L, we define the function

$$d\left(p_L, p_A\right) = p_A - \frac{1}{2}\left(\frac{g^{-1}(2(p_L - p_A))}{g^{-1\prime}(2(p_L - p_A))} + p_L\right) = 0$$

and we apply the Implicit Function Theorem[14]. We look for a function $p_A = h\left(p_L\right)$ such that $d\left(p_L, h\left(p_L\right)\right) = 0$. From Dini's Implicit Function Theorem we get

$$\begin{aligned} h'\left(p_L\right) &= -\frac{\frac{\partial d}{\partial p_L}}{\frac{\partial d}{\partial p_A}} = -\frac{-\frac{1}{2}\left(\frac{2[g^{-1\prime}(\cdot)][g^{-1\prime}(\cdot)]-2[g^{-1\prime\prime}(\cdot)][g^{-1}(\cdot)]}{[g^{-1\prime}(\cdot)]^2}+1\right)}{1-\frac{1}{2}\left(\frac{-2[g^{-1\prime}(\cdot)][g^{-1\prime}(\cdot)]+2[g^{-1\prime\prime}(\cdot)][g^{-1}(\cdot)]}{[g^{-1\prime}(\cdot)]^2}\right)} \\ &= \frac{\left(1+\left(2-\frac{2[g^{-1\prime\prime}(\cdot)][g^{-1}(\cdot)]}{[g^{-1\prime}(\cdot)]^2}\right)\right)}{\left(2+\left(2-\frac{2[g^{-1\prime\prime}(\cdot)][g^{-1}(\cdot)]}{[g^{-1\prime}(\cdot)]^2}\right)\right)} \end{aligned}$$

that implies $h'\left(p_L\right) < 1$. Notice that $h'\left(p_L\right) \geq 0$, if $\frac{[g^{-1\prime\prime}(\cdot)][g^{-1}(\cdot)]}{[g^{-1\prime}(\cdot)]^2} \leq \frac{3}{2}$. Suppose that the firms are in equilibrium, each one charging a price $p_A = p_B = \frac{g^{-1}(0)}{g^{-1\prime}(0)}$, as derived in section 2.1. An agent appears and offers to sell a lottery with probability $\frac{1}{2}$ of getting each good $i = \{A, B\}$ at a price equal to the equilibrium price in the market with no agent, $p_L = \frac{g^{-1}(0)}{g^{-1\prime}(0)}$.Substituting p_L in equation (??) we obtain that firms charge the same price as the lottery price, $p_A = p_L = \frac{g^{-1}(0)}{g^{-1\prime}(0)}$. Hence, if the agent sells a lottery at $p_L = \frac{g^{-1}(0)}{g^{-1\prime}(0)}$, the lottery would have no demand. Consider now the case in which the intermediary sells a lottery at a lower price $p_L < \frac{g^{-1}(0)}{g^{-1\prime}(0)}$. If $\frac{[g^{-1\prime\prime}(\cdot)][g^{-1}(\cdot)]}{[g^{-1\prime}(\cdot)]^2} \leq \frac{3}{2}$, each firm's price and the lottery price are strategic complements ($h'\left(p_L\right) \geq 0$). Ergo, if the agent sets a lower price than $\frac{g^{-1}(0)}{g^{-1\prime}(0)}$, then each firm would decrease its price as well. This results in a drop in the industry profits. Therefore, firms do not sell the goods through the lottery in equilibrium. If the agent instead chooses to set a price above $\frac{g^{-1}(0)}{g^{-1\prime}(0)}$, each firm has a profitable devation

[14] In order to apply the Implicit function theorem, we need $\frac{\partial d}{\partial p_A} = 1 + \left(\frac{[g^{-1\prime}(2(p_L-p_A))][g^{-1\prime}(2(p_L-p_A))]-[g^{-1\prime\prime}(2(p_L-p_A))][[g^{-1}(2(p_L-p_A))]]}{[g^{-1\prime}(2(p_L-p_A))]^2}\right) \neq 0$, that is equivalent to $\frac{[g^{-1\prime\prime}(2(p_L-p_A))][[g^{-1}(2(p_L-p_A))]]}{[g^{-1\prime}(2(p_L-p_A))]^2} \neq -2$

in keeping the price at $\frac{g^{-1}(0)}{g^{-1'}(0)}$. Again, in equilibrium, there is no demand for the lottery.[15]

■

The firms do not accept to join lottery because of the competitive pressure the firms still suffer in this mechanism. Given that the lottery profits are shared, each firm compete against the lottery in order to expand their own exclusive demand.

In order to enter in the market the agent has to make sure that the firms can increase their profits compare to the status quo. For that to be possible, the mechanism should reduce the competitive pressure, and, at the same time, emphasize the price discrimination opportunity value of introducing a lottery.

We provide here a simple example in a context with linear transportation costs that illustrates the idea.

Proposition 10 *An external agent can enter in a saturated market charging the firms a positive commission C for implementing the following mechanism: two lotteries α and β with probability of getting good A, $q^{\alpha} = \frac{1}{2} + \varepsilon$ and $q^{\beta} = \frac{1}{2} - \varepsilon$ respectively ($\varepsilon > 0$ small) are sold at price $p_L = 1$. The revenues of lottery α are entirely transferred to firm A, and the revenues from lottery β to firm B. In equilibrium the price of the goods are $p_A = p_B = \frac{5}{4}$, and the external agent receives a fixed fee $C \in [0, (\pi - \pi^{comp})]$ from each firm, where $\pi = \frac{9}{16}$ is each firm's profit in this mechanism, and $\pi^{comp} = \frac{1}{2}$ is each firm's profit in the competitive equilibrium with no lotteries.*

The fact that two lotteries are used and full revenues of each lottery go directly to the firm for which the lottery would have represented the closest competitor[16] reduces the competition pressure on each firm. At the same time the lottery price set at the competitive equilibrium level guarantees price discrimination revenues.

2.4.2 Market in expansion

In this section we consider the case in which the two firms in competition do not sell to all consumers in the equilibrium without lotteries. This happens if the value V that enters in

[15] Notice that the fact that $0 \leq h'(p_L) < 1$ guarantees that the the agent would not be able to enter in the market, even offering a lottery with equal probabilities and leaving the price of the lottery to be determined in equilibrium simultaneously with the goods' prices. There is no an equilibrium value for p_L such that the lottery has positive demand and the firms increse their profit.

[16] In other words, each lottery may be interpreted as a substitute good of the two basic goods A and B. Higher is the probability of getting A in the lottery, more competition firm A would suffer from the lottery.

the consumers utility function is such that, given the equilibrium prices of the goods, the consumers located in the middle of the segment do not buy any good. We call this case *market in expansion.* The equilibrium with no lotteries is the following.

Proposition 11 *If the market is in expansion, in the symmetric equilibrium each firm sets his price equal to*

$$p_A = p_B = \widehat{p}$$

where $\widehat{p}$ solves $c^{-1}(V - \widehat{p}) - \widehat{p}c^{-1\prime}((V - \widehat{p})) = 0$.

Proof. See Appendix C ∎

The market is *in expansion if* $V < V^*$, where V^* is defined as $V^* - c\left(\frac{1}{2}\right) - \widehat{p}(V^*) = 0$. In this case by improving price discrimination the introduction of lotteries may also increase the number of consumers that buy a good.

We examine the possibility for a third agent to enter in the market introducing a lottery with probability $\frac{1}{2}$ of getting each good at a price p_L. Let's assume that $V > V^{**}$, where $V^{**} = c\left(\frac{1}{2}\right)$. In that case, if the agent wants to maximize the market expansion effect, he sets

$$p_L^* = V - c(1/2) \tag{2.8}$$

The optimal price for the firm becomes

Each firm optimal price is given by equation 2.7: if we replace p_L using equation 2.8 we get

$$p_A = p_B = \frac{g^{-1}(2(V - c(1/2) - p_A))}{2g^{-1\prime}(2(V - c(1/2) - p_A))} + \frac{V - c(1/2)}{2}$$

Choosing a specific transportation cost function and a value for V such that $V^{**} < V < V^*$, we can build examples in which the agent successfully enter in the market.

Proposition 12 *If $c(x) = x$ and $V = \frac{4}{5}$, with no lottery each firm set a price $p_A = p_B = 0.4$, and get a profit $\pi^{no\ lottery} = 0.16$. If an agent offers to sell lotteries with $\left(\frac{1}{2}, \frac{1}{2}\right)$ probabilities at a price $p_L = 0.3$, then the firms set prices $p_A = p_B = 0.475$, and get a profit $\pi^{lottery} = 0.2068 - C$. The agent can charge a commission $C \in [0, 0.046\,8]$ for his service.*

2.5 Extensions

2.5.1 Competition between lottery sellers

If a competition race between external agents willing to enter in the market arises, the only winner would be the agent that convinces both firms to join exclusively his mechanism promising them the highest possible profits.

Considering the concave transportation cost, we can use the optimal mechanism as a benchmark. In that case, indeed, the optimal mechanism is simple and is implementable: the industry profit is maximized introducing a $\left(\frac{1}{2}, \frac{1}{2}\right)$ lottery in the market with the two goods. So far the mechanisms we have considered are such that the lotteries' prices are set in advance. Now we revert the procedure: the lotteries prices are function of the goods' prices.

Proposition 13 *If $c(x)$ is concave, an external agent can enter in the market and implement the perfect cartel optimal mechanism outcome through the following mechanism: two lotteries α and β are sold. The probability of getting good A are $q^{\alpha} = \frac{1}{2} + \varepsilon$ and $q^{\beta} = \frac{1}{2} - \varepsilon$ respectively. The price of lottery α is $p_{L\alpha} = p_B - \left(c\left(\frac{1}{2}\right) - c(x^*)\right)$, and its revenue are transferred to firm A. The price of lottery α is $p_{L\alpha} = p_B - \left(c\left(\frac{1}{2}\right) - c(x^*)\right)$, and its revenue are transferred to firm A. The external agent receives a fixed fee $C \approx 0$.*

The fact that the prices of the lotteries depend on the prices of the goods and are interswitched (the lottery price that depends on p_B is the price of the lottery whose revenue goes to firm A, and viceversa) takes care that there are no profitable strategies where one firm drops his price to undercut the lotteries and the competitor.

If the costs are convex, the optimal mechanism is more difficult to implement because of the presence of type contingent lotteries. This is a topic for future research.

2.5.2 Model with three hotels

When there are more than two firms in the market, the conditions under which an external agent can enter and offer lotteries change significantly. From one side, lotteries still represent an opportunity for the firms to price discriminate the consumers on the base of their preferences. However, now no firm detains veto power on the lottery. If there are three firms in the market, for the agent it's enough to convince two of them. If the third one will not accept to join the lottery, she will still face a "new market" with a lottery in place. In other words, the outside option for each firm is not the status quo without any lottery, but a "new market" with the other firms able to price discriminate.

Consider a simple mechanisms where the external agent set a price p_L for a lottery with equal probabilities of getting each good, and ask the firms to join it. For each good sold through the lottery the external agent gets a commission fee C.

We want to give an example where the firms find optimal to join the lottery, even though their profits after joining the lottery are lower than the ones in the status quo with no lotteries at all. This example is provided in a generalized Hotelling linear city model with three firms, each located at one extreme of three segments, each one long a third of a unit, that are joint together at the other extreme. We call this "the star model".

In order to have an equilibrium without lottery in the star model, we need to slightly modify the linear transportation cost function $c(x) = tx$ in the following way:

$$c(x) = \begin{cases} x \text{ if } x \leq \frac{1}{3} \\ 2x + \frac{1}{3} \text{ if } x > \frac{1}{3} \end{cases}$$

Proposition 14 *In a market with no lotteries each firm charge a price* $p_A = p_B = p_C = \frac{2}{3}$ *for his own good, and each firm gets a profit* $\pi^{no\ lottery} = \frac{2}{9} = 0.222\,22$.

Consider now an external agent that offer to the firms the following mechanism: the goods can be sold through a lottery with equal probabilities of getting each good at a price p_L. For each good sold through the lottery the external agent gets a commission fee C.

Proposition 15 *The external agent can enter in the market setting a price* $p_L = 1 - \frac{1}{6}\sqrt{6} = 0.591\,75$ *for the lottery, a commission fee* $C = \frac{1}{3}$, *and getting a profit* $\pi^{agent} = \frac{1}{12}$. *The firms find optimal to join the lottery setting pries* $p_A = p_B = p_C = \frac{7}{6} - \frac{1}{6}\sqrt{6} = 0.758\,42$, *and getting a profit* $\pi^{lottery} = 0.211\,14$. *This is the only strong Nash equilibrium of the game.*

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2.6 Appendix A

Proof Proposition 2 The Spence-Mirlees condition is satisfied $\frac{\partial}{\partial x}\left(\frac{\frac{\partial}{\partial q}U}{\frac{\partial}{\partial p}U}\right) > 0$: this implies we can consider only the local incentive compatibility constraints for all types. For a fixed price, any lottery provides each type a different level of utility. Given that c is assumed to be convex, the function that relates the utility from a lottery and the types

x is not monotonic in x. Still, for a given lottery q, the types can be ranked in terms of the utility they can get from that lottery: We call the *lowest type for a lottery* q the type that gets the lowest utility from that lottery. In order to maximize the profit, at least one lottery used in the optimal mechanism is going to be priced in such a way that the IR constraint of its lowest type binds[17]. Define x^{**} this type. What we show now is that, in the optimal mechanism, the IR constraints of all types $x \leq x^{**}$ have to bind. We prove that by contradiction. If both the IC and IR constraints bind for all types $x \leq x^{**}$, the optimal mechanism would be a continuum of lotteries $q(x)$ and corresponding prices $p(x)$. Let's assume that there is a type $\widetilde{x} < x^{**}$ for which the IR constraint does not bind. Consider the case[18] in which $\widetilde{x}$ buys a lottery q such that $q < q(x)$. Given that we are assuming that $\widetilde{x}$ IR constraint does not bind, the price of that lottery has to be $p(q) < V - qc(x) - (1-q)c(1-x)$. Given that x^{**}'s IR binds, $q < q(x^{**})$, and there is a type x'', $\widetilde{x} < x'' \leq x^{**}$, such that $p(q) = V - qc(x'') - (1-q)c(1-x'')$. Define x''' as $p(x''') = p(q)$. Notice that x''' has to be lower than the value of x corresponding to the type that gets the highest utility from lottery q. Compare to the mechanism with $q(x)$ and $p(x)$, the new mechanism brings a gain equal to

$$\begin{aligned}&(x''' - x')\left(V - qc(x'') - (1-q)c(1-x'')\right) + \\ &-\int_{x'}^{x'''}\left(V - \frac{c'(1-x)}{c'(1-x)+c'(x)}c(x) - \frac{c'(x)}{c'(1-x)+c'(x)}c(1-x)\right)dx\end{aligned}$$

and a loss

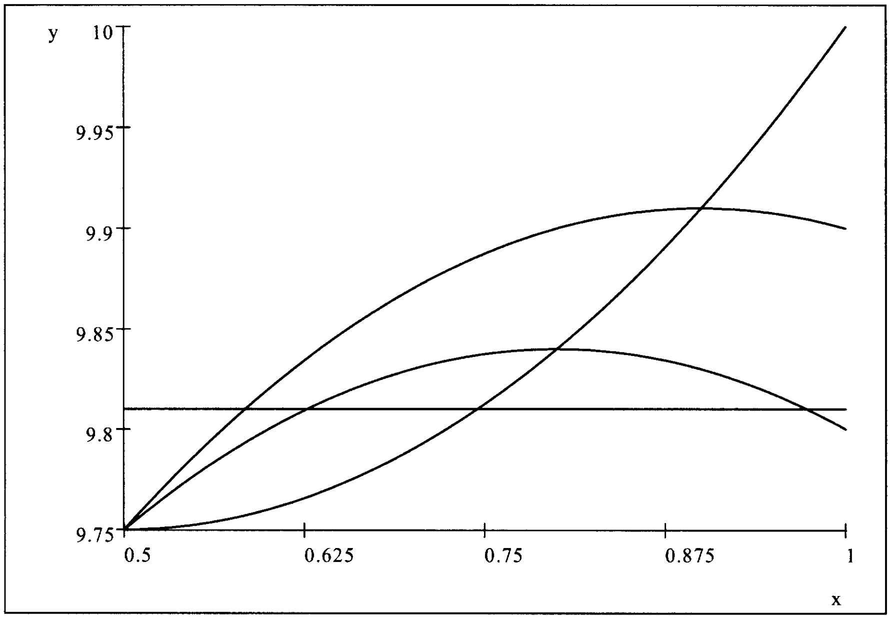
$$\begin{aligned}&\int_{x'''}^{x''}\left(V - \frac{c'(1-x)}{c'(1-x)+c'(x)}c(x) - \frac{c'(x)}{c'(1-x)+c'(x)}c(1-x)\right)dx + \\ &-(x'' - x''')\left(V - qc(x'') - (1-q)c(1-x'')\right)\end{aligned}$$

where $x' = \max\left\{x : p(q) = V - qc(x) - (1-q)c(1-x) \text{ and } x < \widetilde{x}, \frac{1}{2}\right\}$. The gain comes from the fact that there is a subset of types that pays a higher price than $p(x)$ in order to buy a lottery with a probability q higher than $q(x)$. The loss comes from the fact there is also a subset of types that given the price $p(q)$ gets a level of utility such that it's not incentive compatible to charge them a price $p(x)$ for any lottery q' without violating the IR constraint

[17]Notice that, if f is concave, the lowest type fr any lottery q is $x = \frac{1}{2}$. If f is convex, the lowest type is a function of q: it varies with the lotteries we consider.

[18]This is the only relevant case.

of type x^{**}. If the function $\frac{\partial^2}{\partial x^2}\left(V - \frac{c'(1-x)}{c'(1-x)+c'(x)}c(x) - \frac{c'(x)}{c'(1-x)+c'(x)}c(1-x)\right) \geq 0$ in a neighborhood of $x = \frac{1}{2}$[19], then it's immediate to verify that the loss is bigger than the gain for any q if x^{**} is close enough to the neighborhood of $x = \frac{1}{2}$ where the function $V - \frac{c'(1-x)}{c'(1-x)+c'(x)}c(x) - \frac{c'(x)}{c'(1-x)+c'(x)}c(1-x)$ is convex.



For consumers located in the segment $[x^{**}, 1]$ only their IC constraint bind. Hence, in this region the price of a lottery with probability $q(x)$ of getting good A is

$$p(x) = V - [qc(x) + (1-q)\,c(1-x)] - \int_{x^{**}}^{x} \left(-qc'(x) - (1-q)c'(1-x)\right) dz$$

The monopolist maximizes

$$\pi = \int_{\frac{1}{2}}^{1} p(x)\, dx$$

Which is given by

$$\max_{x^{**},q} \int_{\frac{1}{2}}^{x^{**}} \left(V - \frac{c'(1-x)}{c'(x)+c'(1-x)}c(x) - (1 - \frac{c'(1-x)}{c'(x)+c'(1-x)})c(1-x)\right) dx +$$
$$+ \int_{x^{**}}^{1} \left\{V - [qc(x) + (1-q)\,c(1-x)] - \int_{x^{**}}^{x} \left(c'(1-z) - q(c'(z) + c'(1-z))\right) dz\right\} dx$$

[19]For example, if $f(x) = x^2$, then 2.6 is convex over all support $\left[\frac{1}{2}, 1\right]$.

After simplifying and integrating by parts, we obtain

$$\max_{x^{**},q} \int_{\frac{1}{2}}^{x^{**}} \left(V - \frac{c'(1-x)\,c(x) + c'(x)c(1-x)}{c'(x) + c'(1-x)} \right) dx +$$
$$+ \int_{x^{**}}^{1} \left\{ V - [qc(x) + (1-q)\,c(1-x)] - \left(c'(1-x) - q(c'(x) + c'(1-x))\right)(1-x) \right\} dx$$

Notice that now only the second integral depends on $q(x)$. Maximizing the term under the second integral with respect to $q(x)$ for all $x > x^{**}$, we obtain

$$-c(x) + c(1-x) + \left[c'(x) + c'(1-x)\right](1-x) \tag{2.9}$$

Expression (2.9) does not depend on q. Hence, for each $x \in [x^{**}, 1]$ the expression is either positive or negative. If, for a given x, it is negative, then $q(x) = 0$. If it is positive, then $q(x)$ is equal to the highest possible q, $q = q(x^{**})$. The first-order condition with respect to x^{**} is

$$V - \frac{c'(1-x^{**})\,c(x^{**}) + c'(x^{**})c(1-x^{**})}{c'(x^{**}) + c'(1-x^{**})} - V + q(x^{**})c(x^{**}) + \tag{2.10}$$
$$+ (1 - q(x^{**}))\,c(1-x^{**}) + \left(c'(1-x^{**}) - q(x^{**})(c'(x^{**}) + c'(1-x^{**}))\right)(1-x^{**}) = 0$$

We guess and verify that expression (2.9) is negative for all $x \in [x^{**}, 1]$, and is equal to zero for $x = x^{**}$. In that case, $q(x) = 0$ for all $x \in [x^{**}, 1]$, and equation (2.10) simplifies to

$$1 - x^{**} = \frac{1}{c'(1-x^{**})} \frac{c'(1-x^{**})\,c(x^{**}) + c'(x^{**})c(1-x^{**})}{c'(x^{**}) + c'(1-x^{**})} - \frac{c(1-x^{**})}{c'(1-x^{**})}$$
$$1 - x^{**} = \frac{c(x^{**}) - c(1-x^{**})}{c'(x^{**}) + c'(1-x^{**})} \tag{2.11}$$

If we evaluate expression (2.9) at $x = x^{**}$ and substitute $1 - x^{**}$ by expression (2.11) we verify that expression (2.9) is indeed equal to zero.

2.7 Appendix B

Proof Proposition 3 The consumer type x who is indifferent between buying good A or good B is determined by the equation

$$V - c(x) - p_A = V - c(1-x) - p_B \tag{2.12}$$

Let $g(x) = c(x) - c(1-x)$ be the difference between the transportation costs of type x. Equation (2.12) becomes

$$g(x) = p_B - p_A$$

Hence, the indifferent consumer is given by

$$x = g^{-1}(p_B - p_A)$$

The profit maximization problem of firm A is

$$\max \pi = \; p_A g^{-1}(p_B - p_A)$$

Taking the first order conditions we get

$$g^{-1}(p_B - p_A) - p_A g^{-1'}(p_B - p_A) = 0$$
$$p_A = \frac{g^{-1}(p_B - p_A)}{g^{-1'}(p_B - p_A)}$$

In the symmetric equilibrium, where $p_A = p_B$, we have

$$p_A = p_B = \frac{g^{-1}(0)}{g^{-1'}(0)}$$

2.8 Appendix C

Proof Proposition 5 Let us consider the profit maximization problem of firm A, located at 0. The consumer x who is indifferent between buying good A and not buying any good is

$$V - c(x) - p_A = 0$$
$$x = c^{-1}(V - p_A)$$

The value x defines the demand for good A, all types below x prefer good A than no good.Firm A maximizes his profit

$$\max \pi = p_A \left(c^{-1}(V - p_A)\right)$$

The first order condition gives

$$c^{-1}(V - p_A) - p_A c^{-1\prime}(V - p_A) = 0$$

Chapter 3

A First Price-First Score Auction

3.1 Introduction

In many contexts the allocation problem of a seller who faces a group of potential buyers is not limited to a simple comparison of prices. Other factors may play a crucial role in the seller's decision process. For instance, if a specific selling event represents only one episode of a long-lasting business relationship with different buyers, intertemporal strategic considerations become important: it may be worthy to select a buyer with not the best offer today, if this is a necessary condition to receive an advantageous deal with the same buyer tomorrow. As factors change from one allocation problem to another, the emergence and success of specific selling mechanisms can be attributed to their flexibility: the ability to adapt to different seller's demands.

In many businesses, as procurement and real estate, it has become a widespread practice to use a hybrid auction format that builds on the first price auction and allows to consider extra-bid factors. The bidders are asked to submit their bids. The payment rule is such that the selected bidder is paid an amount equal to his bid, as in the first price auction, but the allocation rule need not to select the highest/lowest bid as the winner. In fact, the allocation rule is at the discretion of the seller. The bidders may not even know the exact allocation rule. The bidders are actually ranked on the base of a *score* comparison, and the winner is the one with the highest/lowest score.[1] For each bidder, the *score* is the synthetic value that takes in account the bid and all the other factors that matter to the auctioneer.

[1] *In procurement, the score is equal to the summation of all the costs related to buying a good or service from a specific supplier: price, switching osts, technology adaptation costs...*

We call this format First Price-First Score Auction (FPFSA).[2] It's immediate to realize the reason for the great appeal of this format: it allows to take in account extra-bid factors, and it leaves several degrees of freedom to the auctioneer in defining the *score.*

Auction theorists have so far studied a particular kind of procurement score auctions. The bidders are assumed to be in control of all the elements that enter in the score, and, if a bidder wins, he is committed to fulfill some score level. For example, when the score is based on quality of service provided and payment[3], then, as long as a certain score level is guaranteed, a bidder is free to choose any combination of these two parameters. In real business practice, however, the bidders do not control all the factors the auctioneer cares about, and the price the auctioneer has to pay to buy any bidder's good or service is usually defined at the moment of the bid.

As in Chapter 1, we are interested in a setting where bid and extra-bid components matter. In Chapter we derived the optimal auction format. However, that format has not found practical applications yet. On the contrary, the FPFSA is widely used. As Wilson (1987) writes "... one of the most basic problems challenging the theory (is) the problem of explaining the prevalence of a few simple trading rules in most of he commerce conducted via organized exchanges. ... I believe that practice advances before theory, and the task of theory is to explain how is that practitioners are (usually) right..."

In this paper, we initiate the formal analysis of the First Price-First Score Auction in a general context where the auctioneer is a seller and two bidders compete to buy one indivisible good. The auctioneer's preferences are assumed to directly depend on the identity of the buyer to whom the good is allocated. In this auction, the bidders submit monetary bids, and then the seller decides which bid to accept after comparing the bidders' scores. A particular class of auction we focus on have simple scoring functions: each bidder's score is given by the summation of his bid and a bidder-specific additional parameter.

Our main goal is to obtain the specification of the problem that generates a closed-form analytical solutions for the bidding strategies. The task is complicated as there are at least two sources of asymmetries inherent to the problem that can quickly lead to intractable formulas. These are the distributions of the bidders' evaluations of the good, and the

[2] This format is used by major companies (Purchasing.com, September 13, 2007) to buy inputs and services. In the jargon of procurement managers the *score* is called Total Cost of Ownership, and the auction frmat is denominated TCO-Auction.

[3] See Asker Cantillon (2007)

extra-bid component of the score. As for first price auctions with asymmetric bidders, it is difficult to obtain from the bidders' first order conditions differential equations that can be easily solved. To simplify as much as possible our analysis, we consider the case in which the bidders' valuations are identically and independently distributed according to a uniform distribution function on a common support. The asymmetry comes only from the extra-bid component of the score that penalizes one of the two Bidders.

The main contribution of this work is to provide closed formulas for the inverse bidding functions. Our results generalize the comparison of bidding strategies in asymmetric first price auctions obtained by Maskin and Riley (2002). Even if the asymmetry between the bidders is exogenously introduced by the auctioneer, in equilibrium the disadvantaged bidder bids more aggressively. We are also able to determine the ranges of bids that can be submitted by the two bidders. They are actually different, and their extremes depend on the extra-bid parameter.

We present a new challenging problem: the optimal choice of the extra-bid parameter. We consider an auctioneer who suffers a disutility from selling the good to a specific bidder, and we ask if the auctioneer should design a FPFSA using a score function that adds the true disutility value. The score function and its components enter in the toolbox of the auction designer and can be used strategically. The closed form of the inverse bidding function offers some insight into the nature of the problem and provides useful tools for numerical experiments.

Our work is complementary to the literature that searches for closed solutions in asymmetric first price auctions. Griesmer et alii (1967) derive an expression for the inverse bidding functions of two bidders with values uniformly distributed over the supports $[0,1]$ and $[0,\beta]$, which is used later by Lebrun (1998, 1999), Maskin and Riley (2002) and Cantillon (2008) in examples and comparative statics exercises. Plum (1992) extends that result to the power distribution $F_1(x) = x^{\mu}$ and $F_2(x) = \left(\frac{x}{\beta}\right)^{\mu}$. The most recent and relevant work in this literature is by Kaplan, Zamir (2007), who provide general closed form solutions of the inverse bidding functions for the case of two bidders with values uniformly distributed over generic supports $[\underline{v}_1, \overline{v}_1]$ and $[\underline{v}_2, \overline{v}_2]$. These results do not easily extend to the FPFSA since the probabilities of winning, in addition to the bids of the others, are now also affected by the extra-bid components of the score.

Our study is also related to the literature about biased auctioneers. So far, the problem that has been addressed is the optimal auction design for a biased auctioneer. McAfee and McMillan (1989), Branco (1994, and Mougeot and Naegelen (1998) study the problem of a government running a procurement auction in an international trade environment. The optimal procurement auction is such that the domestic firms are advantaged. Naegelen (2002) derives the optimal procurement auction for a buyer who faces different potential suppliers, and where each supplier offers a good with a different quality level. Our work is the first to consider a biased auctioneer in a specific mechanism, a variation of the First Price Auction, and consider how this bias affects the equilibrium bidding strategies.

3.2 The model

We analyze an environment in which two bidders, bidder 1 and bidder 2, compete in order to buy one indivisible good. Each bidder i's valuation x_i for the good is distributed according to a uniform distribution function F on the interval $[0, 1]$. Each bidder is privately informed about his valuation. The auctioneer has some a priori preferences over the two bidders: independently from the standing bid, he would get some extra utility ε_1 from selling the good to bidder 1 and ε_2 from selling the good to bidder 2. Without loss of generality, we consider the case in which the auctioneer has some a preference bias towards one bidder, bidder 2. At parity of prices, the auctioneer receives an extra utility of $\varepsilon = \varepsilon_2 - \varepsilon_1 > 0$ from selling to bidder 2 instead to bidder 1. The value of ε is common knowledge.

A First Price-First Score Auction is implemented. The rules of this auction format are the following. Each bidder submits a sealed bid b_i. Bidder 1's bid is integrated with the additional parameter ε. Bidder 1 score is $s_1 = b_1 - \varepsilon$, bidder 2 score is $s_2 = b_2$. The scores are then compared, and the bidder with the highest score wins the auction and pays his bid. We assume that the auctioneer sets a common reserve price equal to zero: no negative bids are accepted.

The reason to consider the case in which the true value ε enters in the allocation rule is that, if the auctioneer cannot commit to a specific scoring rule, after collecting the bids, he compares the bidders' offers in terms of the utility that they can provide him.

In the last section, we consider the auction design problem of choosing the reserve price and of the penalty $\widehat{\varepsilon}$ in a strategic way, assuming that the auctioneer can commit to a specific scoring rule.

3.3 The Equilibrium Analysis

Because of the penalty ε that decreases the score of bidder 1 the two bidders are asymmetric and we cannot expect symmetric bidding functions to arise in equilibrium

In order to find the equilibrium bidding strategies, we analyze each bidder maximization problem.

We derive the equilibrium inverse bidding functions in few steps: first, we determine the implications of the first order condition for the bidders' maximization problems; second, we derive the maximum and minimum equilibrium bids for each bidder; third, we obtain a system of differential equations, and we solve them up to a constant; fourth, we derive the value of the constant using the information we have on each bidder's maximum bid.

Step one. Let $\beta_i : x_i \to b$ be Bidder i's bidding strategy that maps valuations x_i to bids b, and define $\phi_i(b) = \beta^{-1}(b)$ to be the inverse bidding function.

The expected utility of bidder 1 with value x_1 and bid b is

$$\Pr\left(b > \beta_2(x_2) - \varepsilon\right)(x_1 - b) = F\left(\phi_2(b - \varepsilon)\right)(x_1 - b)$$

Bidder 1 chooses his bid in order to maximize his expected utility

$$\max_b F\left(\phi_2(b - \varepsilon)\right)(x_1 - b)$$

The first order condition gives us

$$\begin{aligned} f\left(\phi_2(b-\varepsilon)\right)\phi_2'(b-\varepsilon)(x_1-b) - F\left(\phi_2(b-\varepsilon)\right) &= 0 \\ \phi_2'(b-\varepsilon) &= \frac{F\left(\phi_2(b-\varepsilon)\right)}{f\left(\phi_2(b-\varepsilon)\right)}\frac{1}{\left(\phi_1(b)-b\right)} \end{aligned}$$

Since the valuation x_1 is uniformly distributed over $[0,1]$, the first order condition becomes

$$\phi_2'(b-\varepsilon) = \frac{\phi_2(b-\varepsilon)}{\left(\phi_1(b)-b\right)} \tag{3.1}$$

Similarly, bidder 2 solves

$$\max_b F\left(\phi_1(b - \varepsilon)\right)(x_2 - b)$$

which brings a first order condition

$$\phi_1'(b+\varepsilon) = \frac{\phi_1(b+\varepsilon)}{(\phi_2(b)-b)} \tag{3.2}$$

Both conditions can be rewritten as

$$\begin{aligned}
\left(\phi_2'(b-\varepsilon)-1\right)(\phi_1(b)-b) &= \phi_2(b-\varepsilon)-(\phi_1(b)-b) \\
\left(\phi_1'(b+\varepsilon)-1\right)(\phi_2(b)-b) &= \phi_1(b+\varepsilon)-(\phi_2(b)-b)
\end{aligned}$$

Next, evaluate the first condition at a bid value $c = b+\varepsilon$

$$\left(\phi_2'(b)-1\right)(\phi_1(b+\varepsilon)-(b+\varepsilon)) = \phi_2(b)-(\phi_1(b+\varepsilon)-(b+\varepsilon))$$

and then add the two conditions together

$$\begin{gathered}
\left(\phi_1'(b+\varepsilon)-1\right)(\phi_2(b)-b)+\left(\phi_2'(b)-1\right)(\phi_1(b+\varepsilon)-(b+\varepsilon)) = \\
= 2b+\varepsilon
\end{gathered}$$

$$\frac{d}{db}(\phi_1(b+\varepsilon)-(b+\varepsilon))(\phi_2(b)-b) = 2b+\varepsilon \tag{3.3}$$

Integrating, we get[4]

$$(\phi_1(b+\varepsilon)-(b+\varepsilon))(\phi_2(b)-b) = b^2+b\varepsilon \tag{3.4}$$

The constant of integration is equal to 0, given that that $\beta_2(0)=0$.

Step two. Notice that, even though the two bidders are asymmetric, the maximum and the minimum score they submit in equilibrium having some positive probability of winning are the same.

Proposition 16 *The maximum and the minimum equilibrium score such that a bidder has a positive probability of winning are the same for both bidders:* $\overline{s}_1 = \overline{s}_2$, *and* $\underline{s}_1 = \underline{s}_2$.

[4]Note that if $\varepsilon = 0$, we obtain

$$(\phi_1(b)-b)(\phi_2(b)-b) = b^2$$

It is well known that in FPA with two bidders with iid uniform values distributed over a common support $[0,1]$, $\beta_i(x) = \frac{1}{2}x$. Thus $\phi(b) = 2b$ and (1) holds.

Proof. It's immediate to verify that, if $s_i > \overline{s}_j$, bidder i can increase his expected utility decreasing his bid without affecting his probability of winning the auction. Assuming that bidders with zero probability of winning submit their value as bid, bidder 1 has some positive probability if and only if his value x_1 is such that $x_1 \geq \varepsilon$. At $x_1 = \varepsilon$, bidder 1 has positive probability of winning if he bids $b_1 = \varepsilon$. In that case $\underline{s}_1 = \underline{b}_2 = \underline{s}_2 = 0$. Given that the reserve price is zero, this is the minimum score in equilibrium. ■

Notice that $\overline{s}_1 = \overline{b}_1 - \varepsilon = \overline{s}_2 = \overline{b}_2$. The highest bid the two bidders can submit in equilibrium is different. Bidder 1 overbids bidder 2 exactly by the amount of his penalty.

The minimum score is $\underline{s}_2 = \beta_2(0)$ that is equal to $\underline{s}_1 = \underline{b}_1 - \varepsilon$. Since $\beta_2(0) = 0$, we get that $\underline{b}_1 = \varepsilon$.

Corollary 17 *Bidder 1's minimum bid is higher than bidder 2's: $\underline{b}_1 = \varepsilon > 0 = \underline{b}_2$. Any type $x_1 < \varepsilon$ of bidder 1 does not have any positive probability of winning the auction.*

Proof. The first statement is true given that $\varepsilon > 0$. The second one is proved by the bidders' individual rationality constraint. For each bidder i it is not rational to bid more then the value x_i. ■

From equation (3.4), we can derive the explicit values for the maximum bids.

$$\begin{aligned}
\left(\phi_1\left(\overline{b}_2+\varepsilon\right)-\left(\overline{b}_2+\varepsilon\right)\right)\left(\phi_2\left(\overline{b}_2\right)-\overline{b}_2\right) &= \overline{b}_2^2+\overline{b}_2\varepsilon \\
\left(1-\left(\overline{b}_2+\varepsilon\right)\right)\left(1-\overline{b}_2\right) &= \overline{b}_2\left(\overline{b}_2+2\varepsilon\right) \\
\overline{s} = \overline{b}_2 &= \frac{1-\varepsilon}{2} \\
\overline{b}_1 &= \frac{1+\varepsilon}{2}
\end{aligned}$$

Step three. We can rewrite (3.2) as

$$\phi_1'(b+\varepsilon) = \frac{\phi_1(b+\varepsilon)\left(\phi_1(b+\varepsilon)-(b+\varepsilon)\right)}{b^2+b\varepsilon}$$

and similarly (3.1) as

$$\phi_2'(b) = \frac{\phi_2(b)\left(\phi_2(b)-b\right)}{b^2+b\varepsilon}$$

These are two Riccardi differential equations, and we can solve them with the appropriate change of variables.

For bidder 2, we have

$$\phi_2'(b) = \frac{1}{b^2 + b\varepsilon}(\phi_2(b))^2 - \frac{1}{b+\varepsilon}\phi_2(b)$$
$$\phi_2'(b)(\phi_2(b))^{-2} = \frac{1}{b^2 + b\varepsilon} - \frac{1}{b+\varepsilon}(\phi_2(b))^{-1}$$

Define $z = \phi_2{}^{-1}$, then $z' = -\phi_2'\phi_2^{-2}$. Changing variables, the differential equation becomes

$$z' = \frac{1}{b+\varepsilon}z - \frac{1}{b^2 + b\varepsilon}$$

Its solution is

$$C_2(b+\varepsilon) + \left(1 + \frac{b}{\varepsilon}\right)\left(\frac{1}{\varepsilon}\ln\frac{(b+\varepsilon)}{b} - \frac{1}{(b+\varepsilon)}\right)$$

The inverse bidding function for bidder 2 is

$$\phi_2(b) = \frac{1}{C_2(b+\varepsilon) + \left(1 + \frac{b}{\varepsilon}\right)\left(\frac{1}{\varepsilon}\ln\frac{(b+\varepsilon)}{b} - \frac{1}{(b+\varepsilon)}\right)}$$

Step fourth. The value of the constant C_2 can be obtained from the equation $\phi_2(\overline{b}_2) = 1$, where $\overline{b}_2 = \frac{1-\varepsilon}{2}$

$$C_2 = \frac{2}{1+\varepsilon}\left(\frac{2}{1+\varepsilon} + \frac{1+\varepsilon}{2\varepsilon^2}\ln\left(\frac{1-\varepsilon}{1+\varepsilon}\right) + \frac{1+\varepsilon^2}{\varepsilon(1+\varepsilon)}\right) \tag{3.5}$$

Applying the same analysis for the case of Bidder 1, we get an inverse bidding function

$$\phi_1(b) = \frac{1}{C_1(b-\varepsilon) + \left(1 - \frac{b}{\varepsilon}\right)\left(\frac{1}{\varepsilon}\ln\frac{b}{b-\varepsilon} - \frac{1}{b-\varepsilon}\right)}$$

The constant C_1 is derived from the equation $\phi_1(\overline{b}_1) = 1$, where $\overline{b}_1 = \frac{1+\varepsilon}{2}$, and it is

$$C_1 = \frac{2}{1-\varepsilon} - \frac{1}{\varepsilon}\left(\frac{2}{1-\varepsilon} - \frac{1}{\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon}\right) \tag{3.6}$$

We summarize our findings in Proposition 3.

Proposition 18 *The equilibrium inverse bidding function of the two Bidders are given by:*

$$\phi_1(b) = \frac{1}{C_1(b-\varepsilon) + \left(1 - \frac{b}{\varepsilon}\right)\left(\frac{1}{\varepsilon}\ln\frac{b}{b-\varepsilon} - \frac{1}{b-\varepsilon}\right)}$$

$$\phi_2(b) = \frac{1}{C_2(b+\varepsilon) + \left(1+\frac{b}{\varepsilon}\right)\left(\frac{1}{\varepsilon}\ln\frac{(b+\varepsilon)}{b} - \frac{1}{(b+\varepsilon)}\right)}$$

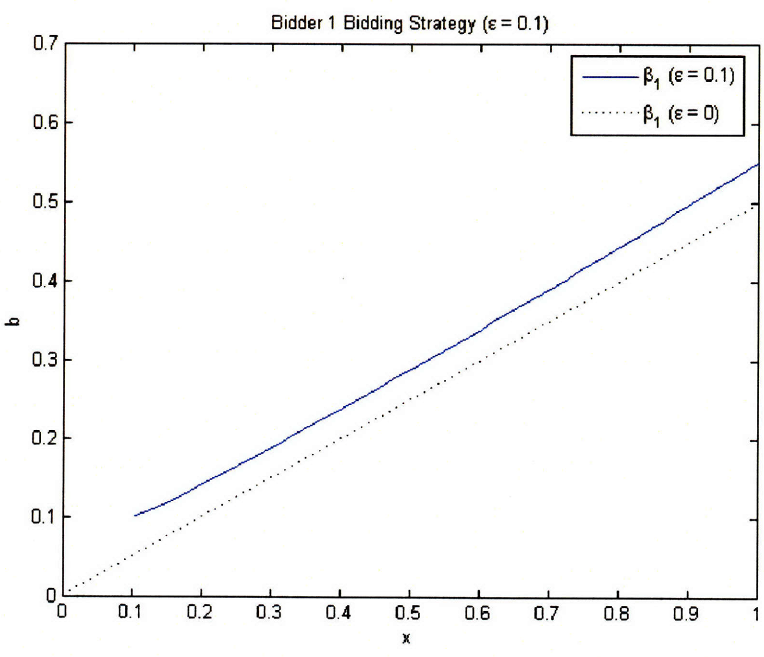
where C_1 and C_2 are given, respectively, by (3.6) and by (3.5).

Note that the equilibrium inverse bidding functions of both bidders are continuous and strictly increasing on the relevant intervals $[\varepsilon, \bar{b}_1]$ for Bidder 1 and on the interval $[0, \bar{b}_2]$ for Bidder 2. This implies that they are invertible on the intervals of interest for any relevant value of ε, $\varepsilon \leq \frac{1}{3}$.[5]

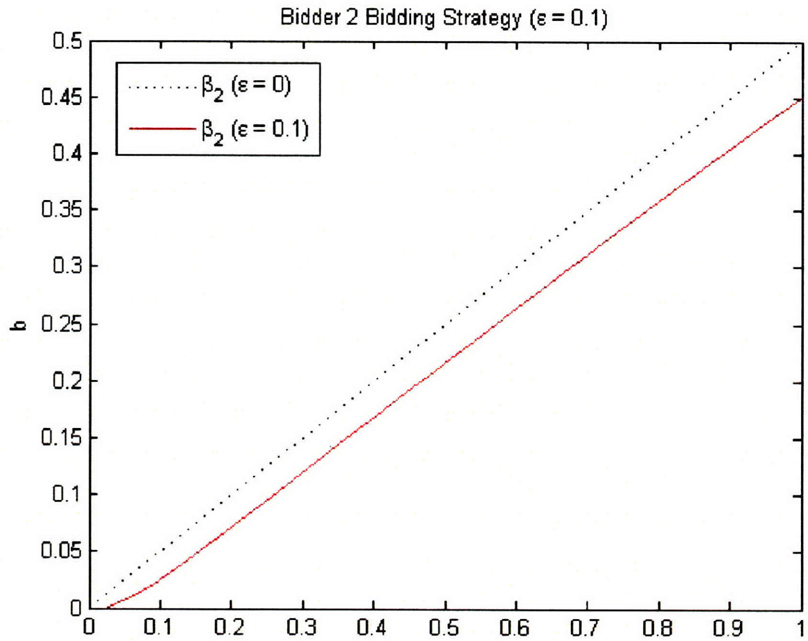
3.4 Comparative Statics

In this section we want to evaluate the effect of setting different values for the penalty ε n the bidding strategies of the two bidders.

Here we select a specific value of ε ($\varepsilon = \frac{1}{10}$), and we compare the equilibrium bidding strategies to the ones of the symmetric case. If $\varepsilon = 0$, indeed, the two bidders $i = 1, 2$ are perfectly symmetric, and their equilibrium bidding function is $\beta_i(x_i) = \frac{1}{2}x_i$.



[5]Otherwise bidder 1 never wins the auction.



Notice that the Maskin and Riley (2000) result on asymmetric first price auctions is confirmed: the disadvantaged bidder bids more aggressively than the advantaged one. In Maskin and Riley's environment a different distribution of the bidders' valuations determines their asymmetry. We depart from that setting in assuming that the asymmetry comes only from an exogenous penalty imposed by the Auctioneer.

If ε was zero, the bidders would be perfectly symmetric and they would use the same bidding strategy in equilibrium. From the Figures, we can immediately see that the disadvantaged bidder, bidder 1, bids consistently more than what he would have done if there was no penalty. The opposite is true for bidder 2.

The fact that we have closed solutions for the inverse bidding functions allow us to evaluate the level of "aggressiveness", measuring the value of the difference $\phi_2(b) - \phi_1(b)$.

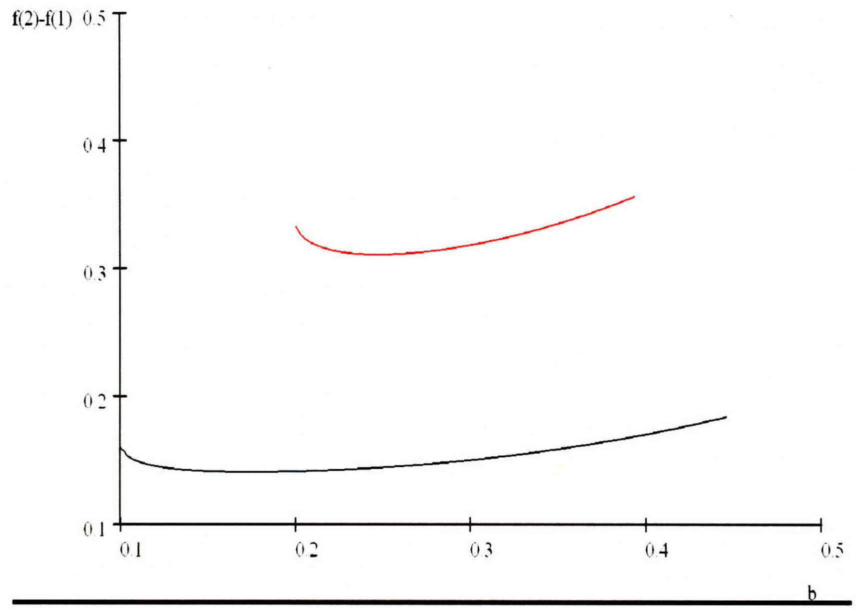


Figure 1: Aggressiveness ($\varepsilon = 0.1$ vs $\varepsilon = 0.2$)

Proposition 19 *For any given value of* b^6*, the difference* $\phi_2(b) - \phi_1(b)$ *is increasing in* ε.

Given that we consider the inverse functions, the difference $\phi_2(b) - \phi_1(b)$ gives us the difference in the valuations x_2 and x_1 at which, for each bidder, it's optimal to bid b in equilibrium. Given that bidder 2 can indirectly benefit from bidder 1's penalty, his value is higher than bidder 1's to bid b in equilibrium. Higher is bidder 2 value $x_2 = \phi_2(b)$ compare to bidder 1's value $x_1 = \phi_1(b)$ for any b, more "aggressive" is bidder 1's strategy. Looking at Figure 1, we can easily compare the difference between inverse functions for two different values of ε. The line on top represents $\phi_2(b) - \phi_1(b)$ when $\varepsilon = 0.2$, the line on the bottom represents the case of $\varepsilon = 0.1$. Notice that for a higher the value of ε is, the interval of bids' values that both bidders submit in equilibrium is shorter. This happens because $\underline{b}_1 = \varepsilon$ increases, and, at the same time $\underline{b}_2 = \frac{1-\varepsilon}{2}$ decreases. The higher is the value of ε, the higher is the probability of bidder 2 to win with a zero bid, and the lower is the probability that bidder 2 submits high bids.

[6] *Notice that for different values of* ε*, the range of bids changes. Indeed, both* $\overline{b}_1$ *and* $\overline{b}_2$ *are functions of* ε*, and* $\underline{b}_1 = \varepsilon$*. What we do is comparing* $\phi_2 - \phi_1$ *over the overlapping range of admissable bids* $B(\varepsilon)$ *and* $B(\varepsilon')$.

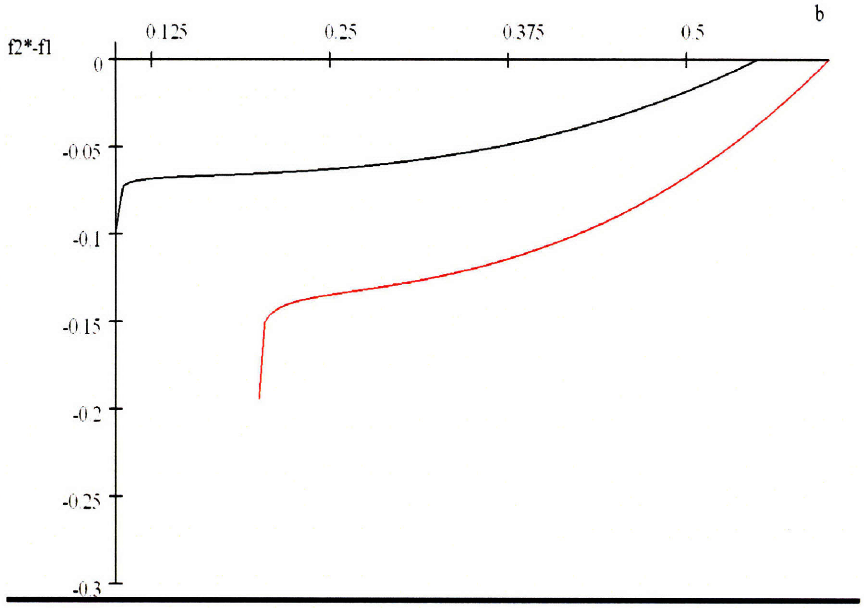


Figure 2: $\phi_2(b-\varepsilon)-\phi_1(b)$ for $\varepsilon=0.1$ and $\varepsilon=0.2$.

Proposition 20 *For a given value of ε, the difference between the inverse bidding functions is such that $-\varepsilon \leq \phi_2(b-\varepsilon)-\phi_1(b) \leq 0$ and it is equal to ε only at $b=\underline{b}_1$, and equal to 0 only at $b=\overline{b}_1$.*

In Figure 2 the difference between the inverse functions is calculated at different bids' values for bidder 1 and bidder 2 in the case $\varepsilon=0.1$ (line on the bottom), and in the case $\varepsilon=0.2$ (line on top). The bids are such that the corresponding *scores* of the two bidders are equal. In this way, we discount for the penalty ε of bidder 1, and we may expect to have zero difference between the bidders valuations $\phi_2(b-\varepsilon)$ and $\phi_1(b)$. This is true only at $b=\overline{b}_1$. For any lower bid, the difference $\phi_2(b-\varepsilon)-\phi_1(b)$ is negative. This implies that the valuation x_2 at which bidder 2 bids $b-\varepsilon$ is lower than the valuation x_1 at which bidder 1 bids x_1. Another way to look at the same result is the following.

Proposition 21 *For a given value of ε, $\phi_2(b)-\phi_1(b)>\varepsilon$.*

Comparing the values at which the two bidders submit the same bid, the value of bidder 2 is higher than bidder 1's value by more than ε.

Both Propositions indicates that the effect of introducing a penalty ε on the score of bidder 1 increases the probabilities of bidder 2 winning the auction by more than ε.

3.5 Extensions

3.5.1 Reserve prices

So far we have considered a specific mechanism with a given reserve price $r = 0$. It is important to check how other reserve price levels may affect the bidding behavior in equilibrium.

In the case with reserve price equal to zero, the equilibrium is such that if bidder 1 has a value below ε, he does not have any positive probability of winning the auction. On the contrary, bidder 2 has ε probability of winning the auction with a bid equal to zero.

Setting a different common reserve price $\overline{r} \neq 0$ implies that bidder 2 cannot win the auction bidding zero anymore; on the other hand, with positive probability $(F(\overline{r}))^2$, neither one of the two bidders wins the auction. Moreover, if $\overline{r} < \varepsilon$, with probability $F(\overline{r})$ bidder 1 wins the auction bidding $\overline{r}$ and leaving the auctioneer with a disutility $\overline{r} - \varepsilon < 0$. Instead, if $\overline{r} > \varepsilon$, the auctioneer never incurs in any auction outcome with negative utility.

By the individual rationality constraint of bidder 2, we have that $\phi_2(\overline{r}) = \overline{r}$. This implies that the condition (3.3)

$$\frac{d}{db}\left(\phi_1(b+\varepsilon) - (b+\varepsilon)\right)\left(\phi_2(b) - b\right) = 2b + \varepsilon$$

can be still integrated as (3.4)

$$\left(\phi_1(b+\varepsilon) - (b+\varepsilon)\right)\left(\phi_2(b) - b\right) = b^2 + b\varepsilon$$

with a constant of integration equal to zero. The derivation of the equilibrium bidding strategies does not change.

Our results are also robust with respect of bidder specific reserve prices.

An interesting and challenging task is to derive the optimal reserve prices. Given that there is an asymmetry between the bidders, it is optimal to set them at a different level for each bidder. The difficulty of the derivation is due to the fact we do not have explicit solutions for the bidding functions.

3.5.2 Scoring parameters

If the auctioneer can commit to a specific scoring rule, the shape of the scoring rule itself and the elements that enter in it become tools that the auctioneer can use to increase his utility. An auctioneer who has a disutility ε from selling the good to bidder 1 and runs the FPFSA has the following expected utility from the mechanism

$$\Pr(1 \text{ wins})(b_1 - \varepsilon) + \Pr(2 \text{ wins})(b_2)$$

Restricting the analysis to linear scoring rules, a new task for future research is to derive the optimal penalty $\widehat{\varepsilon}$, that the auctioneer sets in the auction rules in order to maximize his utility given a disutility parameter ε.

TNow, given the closed form solutions for the inverse bidding strategies, we have a better understanding of how different choices of the parameter $\widehat{\varepsilon}$ affect the equilibrium behavior of the bidders. Using the true ε the auctioneer induces bidder 1 to bid more aggressively and bidder 2 more relaxedly than in the symmetric case. As we have shown in the previous section, the probability of bidder 2 to win the auction increases by more than ε.

Let's analyze the trade-off that the auctioneer faces when he considers if decreasing the penalty to $\varepsilon' < \varepsilon$. A lower ε' exposes the auctioneer to the risk of selling to bidder 1 for bids lower than the true ε. Moreover, bidder 2's probabilities of winning decreases, even though his bidding strategy becomes more aggressive. In other words, reducing the value of ε has a positive effect in terms of revenues (increasing the competition between the two bidders) and a negative effect in terms of allocation outcomes (the most preferred bidder, bidder 2, wins the auction less often).

The trade-off can be solved maximizing the expected utility of the auctioneer with respect of $\widehat{\varepsilon}$.

$$\max_{\widehat{\varepsilon}} \int_{\widehat{\varepsilon}}^{1} \phi_2\left(\beta_1\left(x_1, \widehat{\varepsilon}\right), \widehat{\varepsilon}\right)\left(\beta_1\left(x_1, \widehat{\varepsilon}\right) - \varepsilon\right) dx_1 + \int_{\widehat{\varepsilon}}^{1} \phi_1\left(\beta_2\left(x_2, \widehat{\varepsilon}\right), \widehat{\varepsilon}\right) \beta_2\left(x_2, \widehat{\varepsilon}\right) dx_2$$

It's not trivial to derive an analytical formula for the bidding functions, and this creates a big obstacle to the analysis. However, for any given value of $\widehat{\varepsilon}$, given that we have a closed solution for the inverse bidding function, we know how the bidding strategies look like. For each value of b, indeed, we can recover the corresponding bidder value x. Once obtained a graph for the bidding functions, we can use quadratic or even linear interpolations to describe their relation and solve the auctioneer's design problem numerically.

3.6 Conclusion

We initiate the formal analysis of an auction format, the first price-first score auction, that is widely used in business practice, especially in procurements. Its success is due to a combination of simplicity and flexibility. The auction rules are the following: the payment rule is such that the winner pays his bid; the allocation is determined ranking the bidders in terms of their score, where each bidder's score is a synthetic value that represents bid and extra-bid factors that matter to the auctioneer.

Using this auction format the auctioneer can redefine the scoring function case by case in order to adapt it to his specific requirements.

Our main contribution is to provide closed form solutions for the inverse bidding functions in a setting with two bidders. This enables us to perform comparative statics exerises in order to measure the impact of the extra-bid components on the bidding functions.

We enlight the possibility for the auctioneer to use strategically the extra-bid components of the score in order to increase his expected utility from the auction. Future research on the strategic aspects of the design of the first price-first score auction will help to understand its success in business practice.

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