A Semi-Infinite Cycle Construction of Floer Homology

by

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Abstract

This dissertation is concerned with the foundations of a new approach to Floer theory. As opposed to the traditional approach, which can be viewed as a generalization of Morse theory to an infinite dimensional setting, our approach is a generalization of bordism to infinite dimensions. The key new insight, based on unpublished work of Tom Mrowka and Peter Ozsváth, is an understanding of how to axiomatize compactness in the infinite dimensional setting. We describe a general axiomatic framework for setting up a Floer theory of a polarized Hilbert space equipped with a functional. The resulting bordism theory can be seen as a refinement of the traditional Floer theory. By introducing cycles with corners, we demonstrate how the bordism theory leads to a geometric description of homology. We relate our geometric construction to the Morse-theoretic approach by indicating how one might compute the Floer homology of the space, if the associated functional is Morse. The general theory is illustrated in two examples: Seiberg-Witten-Floer homology and symplectic Floer theory for loops in $\mathbb{C}^n$. We end by indicating various generalizations of the theory.

Thesis Supervisor: Tomasz Mrowka
Title: Professor of Mathematics

Acknowledgments

Firstly, I owe a tremendous intellectual debt to my advisor Tom Mrowka. The topic of my dissertation stems from ideas developed by him jointly with Peter Ozsváth in the late '90 which he has generously shared with me. Tom has provided encouragement and advice at many crucial moments during my graduate study. He has taught me a great deal about the interaction between geometry and analysis.

I would like to thank Peter Kronheimer for serving on my thesis committee as well as for suggesting the symplectic case of Floer theory as a possible application of the ideas developed in this work. I would also like to thank Katrin Wehrheim for many useful discussions and suggestions as well as for serving on my thesis committee.

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Contents

Chapter 1

Introduction

1.1 Historical Overview

In the mid 80's, Andreas Floer obtained a positive solution to Arnold's conjecture on the minimal number of fixed points of a Hamiltonian symplectomorphism. For this purpose, Floer introduced a new homology theory for the loop space of a symplectic manifold. His theory is an infinite dimensional version of Morse theory applied to the symplectic action functional on the loop space. The critical points of this functional correspond to fixed points of the symplectomorphism, while the boundary operator counts the dimension zero moduli spaces of (perturbed) holomorphic curves connecting the critical points [Flo88]. Under suitable hypothesis, Floer was able to show that the resulting homology theory is well defined and independent of the perturbation data necessary to construct the theory. Moreover, Floer showed that the homology theory was isomorphic to the singular homology of the underlying symplectic manifold and thus proved the the Arnold conjecture. In subsequent work, Floer generalized his theory to other contexts such as the more general problem of Lagrangian intersections as well as an analogous theory for the Chern-Simons invariant for connections on a 3-manifold. In these cases, a simple topological interpretation of the resulting groups is not available. The groups encode deep geometric information about the relevant configuration space which cannot be reduced to the "classical" topology of that space.

From a foundational standpoint, the definition of the Floer homology groups is perhaps not satisfactory. The relevant functionals are usually not Morse-Smale and thus have to be perturbed in some manner to even define the groups. As a result, one has to then show that the groups are indeed independent of the chosen perturbation. Furthermore, a rather delicate analysis of the compactification of the moduli space of trajectories is necessary to establish even the most basic properties of the theory; for instance, the fact that the chain of groups generated by critical points indeed form a complex. The situation is of course analogous to the finite dimensional story. One may take as the definition of homology of a compact manifold the chain complex associated to a Morse-Smale function. However, establishing even the basic properties, such as functoriality under mappings, is quite nontrivial. On the other hand, with singular homology theory at one's disposal, Morse homology becomes an effective and illuminating way of computing the homology groups. The central goal of the present work is to find an appropriate analogue of singular homology in the Floer context. It should be emphasized, however, that while in the finite dimensional situation singular homology provides a way of avoiding the analytic machinery that is necessary for setting up Floer's theory, the theory developed in the current work rests heavily on the use of Sobolev spaces and appropriate nonlinear Fredholm operators between them. This is, perhaps, a reflection of the fact that although many results in Floer theory have purely topological interpretations, ultimately the theory deals with the qualitative behavior of solutions to certain elliptic partial differential equations.

Let's briefly describe, what is to our knowledge, the earliest evidence for the existence of such a theory. In the late '80s, Atiyah [Ati88] and others, observed that, from the point of view of relative Donaldson invariants, one may view Floer's theory as a theory of "semi-infinite cycles". We retell his observation in the language of Seiberg-Witten theory. Consider a closed Riemannian 4-manifold X with a spin$^{\mathbb{C}}$-structure and spinor bundle W (see section 3.1 for the full definitions). Let $\mathcal{M}(X)$ be the moduli space of solutions to the SW equations modulo the gauge group action. As is well known (see for example [KM07]), when $b^+ > 0$ the moduli space is a smooth compact manifold. Now, consider the case when $X = X_+ \sqcup_Y X_-$ decomposes along a 3-manifold Y into two compact 4-manifolds with boundary. Let $\mathfrak{B}(Y)$ denote the configuration space of pairs (B, Ψ), where B is a Clifford connection and Ψ is a section of the spinor bundle over Y, modulo the gauge group action. We have restriction maps

$$\mathcal{R}_\pm : \mathcal{M}(X_\pm) \to \mathfrak{B}(Y)$$

At least on the point-set level, one has

$$\mathcal{M}(X) = \mathcal{M}(X_+) \times_{\mathfrak{B}(Y)} \mathcal{M}(X_-)$$

In other words, modulo the action of the gauge group, solutions on X correspond to solutions on $X_\pm$ that agree on the boundary. Standard elliptic boundary value theory implies that $\mathcal{M}(X_\pm)$ are in fact Hilbert manifolds. Therefore, one might hope to interpret the fibre product $\mathcal{M}(X_+) \times_{\mathfrak{B}(Y)} \mathcal{M}(X_-)$ in the smooth category. Speculating even further, one might hope for the existence of Floer groups $HF^+(Y)$ and $HF^-(Y)$ with an intersection pairing

$$HF^+(Y) \otimes HF^-(Y) \to H_*(\mathfrak{B}(Y))$$

where H_* denote the singular homology functor. Abstracting this situation, given a smooth map

$$\sigma : P \to \mathfrak{B}(Y)$$

where P is some Hilbert manifold, we are led to the following problems:

1. What properties should such maps have to have finite dimensional intersections?
2. What properties should such maps have to have compact intersections?

The answer to the first problem is well known and involves the notion of a polarized Hilbert manifold. Very loosely, one may think of a polarization of a Hilbert manifold such as $\mathfrak{B}(Y)$ as an equivalence class of local splittings of the tangent bundle:

$$T\mathfrak{B}(Y) = T^{+}\mathfrak{B}(Y) \oplus T^{-}\mathfrak{B}(Y)$$

As it turns out, $\mathfrak{B}(Y)$ comes with a natural choice of polarization for which we have

$$\pi^{-} \circ D\mathcal{R}_{+} : T\mathcal{M}(X_{+}) \to T^{-}\mathfrak{B}(Y)$$

Fredholm and

$$\pi^{+} \circ D\mathcal{R}_{+} : T\mathcal{M}(X_{+}) \to T^{+}\mathfrak{B}(Y)$$

compact. Similarly,

$$\pi^{+} \circ D\mathcal{R}_{-} : T\mathcal{M}(X_{-}) \to T^{+}\mathfrak{B}(Y)$$

is Fredholm while

$$\pi^{-} \circ D\mathcal{R}_{-} : T\mathcal{M}(X_{-}) \to T^{-}\mathfrak{B}(Y)$$

is compact. The fact that $\mathcal{M}(X_{+}) \times_{\mathfrak{B}(Y)} \mathcal{M}(X_{-})$ is finite dimensional is an immediate consequence of Fredholm theory. Therefore, to ensure finite dimensional intersections it is reasonable to require our cycles to respect the polarization. The resolution of the second problem is significantly more subtle and is the main subject of this work. As far as we know, the first attempt to do so is due to Tom Mrowka and Peter Ozsvath and the present dissertation owes a considerable debt to their original insight.

1.2 An Outline of the Contents

Here we briefly describe the contents of the thesis providing some motivation for the constructions that follow.

As the basic structure we will be considering a Hilbert space $\mathfrak{B}$ with a polarization $T^{+}\mathfrak{B} \oplus T^{-}\mathfrak{B}$ as well as a functional $\mathcal{L} : \mathfrak{B} \to \mathbb{R}$. In section 2 we lay out the axioms for a map $\sigma : P \to \mathfrak{B}$ from a Hilbert manifold P to define a cycle. The motivation comes from the strong L^2_1 proof of compactness for Seiberg-Witten moduli spaces as presented in [KM07]. In fact, for the case when P is the moduli space of solutions our axioms are an exact restatement of the L^2_1 compactness theorem. On a more technical note, we will define and use the notion of locally cubical (lc) manifolds. It appears to be a useful and technically simple structure to work with. However, it appears that this structure is not satisfactory from a conceptual viewpoint and we will later explore ways of avoiding it.

In section 3.1 we will discuss our main example coming from Seiberg-Witten theory. We are rather brief in our treatment mainly because a complete account of the foundation appears in [KM07]. There is a key technical point that comes up in our treatment that plays no role in [KM07]. We must work with L^2_1 connections and hence with the borderline gauge group of L^2_2 maps. One must be careful since in this case the action of the gauge group, while continuous, is not smooth. Therefore, picking a particular slice for the action specifies a smooth structure on the moduli space. It appears that there is a canonical choice of such a slice (known as the strong Coulomb slice). With any other choice, the corresponding map to $\mathfrak{B}(Y)$, the configuration space on Y appears not to be smooth. Hence, in this work, we will always work in specific slices to ensure we have smooth maps between Hilbert manifolds.

In section 3.3 we construct a family of perturbations for cycles ensuring that intersections can be arranged to be transverse. The construction is quite similar to perturbing manifolds in finite dimensions and also rests on the Sard-Smale theorem. One must simply check that the perturbation does not take us out of the category of cycles. It is perhaps worth remarking that unlike in traditional Floer theory, where perturbations involve changing the metric, complex structure or the hamiltonian function, the perturbations here do not alter the geometric data. Therefore, our theory is defines for a wide class of functionals that can potentially be highly degenerate. This is illustrated in our proof of the existence of periodic orbits for loops in $\mathbb{C}^n$. In this case the functional is degenerate and yet we are able to extract the relevant geometric information.

In section 3.5 we describe how a cobordism W from Y_0 to Y_1 gives rise to a map on the Floer bordism groups. We abstract the situation and define a general class of maps (called correspondences) $Z \to \mathfrak{B}_0 \times \mathfrak{B}_1$ that give rise to maps on the Floer groups via fibre products. The definition is general enough to include not only moduli spaces on a cobordism W but also the diagonal map $\mathfrak{B} \to \mathfrak{B} \times \mathfrak{B}$. The definition is a little technical but is forced on us by the considerations that follow.

Section 3.7 is the technical heart of the theory. Our goal is to prove that the trivial cobordism induces the identity on the Floer groups. This is established by finding a cobordism between the correspondence coming from the moduli space of solutions on the cylinder and the diagonal map. If $\mathcal{M}_t$ denotes the solutions on a cylinder of length t, we can form the disjoint union $\sqcup_{t\in(0,1]}\mathcal{M}_t$. We complete this manifold by adding the diagonal $\mathfrak{B} \to \mathfrak{B} \times \mathfrak{B}$ at $t = 0$. This is achieved by arguing that on a small cylinder, a solution is specified by the appropriate spectral projections to the boundary. For this we need to apply the contraction mapping theorem and, since the nonlinear term is quadratic, need rather precise estimates L^4. The main technical difficulty in establishing this estimate is that the norm of the embedding $L^2_1 \to L^4$ depends size of the cylinder. This establishes that we have a Hilbert manifold that with boundary $\mathcal{M}_1 \sqcup \mathfrak{B}$. However, the restriction maps defined on the cylinder extend only weakly to the diagonal map as we approach the $t = 0$ boundary. Given a cycle $P \to \mathfrak{B}$ we show that by changing coordinates, we can assume that the difference map $\mathcal{R}_0 - \sigma : \sqcup_{t\in(0,1]}\mathcal{M}_t \times P \to \mathfrak{B}$ is C^1 up to the boundary. Such a coordinate change preserves the lc-structure but not the smooth structure on $\sqcup_{t\in(0,1]}\mathcal{M}_t \times P \to \mathfrak{B}$. This is the principal motivation for introducing lc-manifolds. Section 3.8 explains how the topological invariance of the Floer bordism groups is a direct consequence of the previous section.

In section 4 we indicate how one might relate our theory of semi-infinite cycles to the Morse-theoretic approach. The key observation is that the trivial cobordism of length t induces a map on cycles that serves as a substitute for the gradient flow in Floer theory. In particular, there is a rigorous sense in which one might push a cycle up by the gradient flow and push a cocycle down. As a first step, we define a geometric version of homology that can be viewed as a bordism theory with codimension 2 singularities. We show that when $\mathcal{L}$ has an nondegenerate critical point then the localized Floer homology group is $\mathbb{Z}_2$. Thus, as we pass a critical point, the Floer groups pick up at most one $\mathbb{Z}_2$. This already establishes the fact the the groups are finitely generated given that there are finitely many nondegenerate critical points. At this point, we are using that our functional is Morse. If, in addition, we assume that the functional is Morse-Smale, then we can recover the Morse-Smale complex in our setup. The proof that Floer homology defined by geometric cycles is the same as that defined by the Morse-Smale complex is almost identical to the theorem in the finite dimensional setting. I must admit that I have not been able to find this particular proof in the literature even in the finite dimensional case, so our argument can be viewed as a new geometric proof of the Morse homology theorem. Note that at no point do we need to understand how to compactify trajectory spaces, and thus avoid the rather involved gluing techniques.

In section 5 we give our second example of the theory. This one concerns the space of loops in $\mathbb{C}^n$. We prove the required L_1^2 compactness theorem and indicate the changes necessary for the proof that the trivial cobordism induces the identity on the bordism groups. In fact, as we shall see, the proofs are rather simpler in this case. We illustrate the general theory by reproving that for a general class of hamiltonian functions $H : \mathbb{C}^n \to \mathbb{R}$, there exists a nontrivial periodic orbit. Superficially, the proof is similar to the one in [HZ94]. However, our proof is based on the unregularized gradient flow and does not use minimax methods. It gives a rather natural interpretation of the cycles appearing in the proof. In section 6.1 we prove a version of the strong L_1^2 compactness theorem for the blown-up configuration space. As it turns out, this time two functionals are involved in the compactness statement. Thus, one may hope to construct a theory of semi-infinite cycles based on two functionals. The first steps in this direction are taken in the next few sections. While we do not develop the full theory here, we lay down the axioms for cycles and correspondences in this context. We expect to complete this theory in a subsequent work. In this case, the target is a Hilbert manifold with boundary. We obtain a long exact sequence of Floer groups by considering the sequence associated to the pair $(\mathfrak{B}, \partial\mathfrak{B})$. One possible goal is to construct an isomorphism between our groups and the Floer homology as defined by Kronheimer and Mrowka.

We include a short section to illustrate a new approach equivariant Floer theory analogous to a geometric cycle version of the Cartan construction which we have learned from [Jon87]. It is not yet clear what the precise relationship to the theory based on the blownup configuration (aside from the case of S^3). However, this new approach is technically much simpler to construct and deserves to be pursued in a future work.

The thesis ends with several technical appendices.

Chapter 2

Geometric Cycles for Floer Spaces

2.1 lc-Manifolds of Depth ≤ 1

In this work it will be important to work with Hilbert manifolds with corners and some rather weak smoothness between different strata. Later on in the thesis, we will introduce a rather technical notion of locally-cubical manifolds or lc-manifolds for short. For the sake of the reader, in this section we simply write down the definitions for the simplest nontrivial case. In the terminology of lc-manifolds, this is a depth one lc-manifold. For the purposes of defining the bordism groups this is sufficient and illustrates all the essential technical difficulties. Therefore, we decided to first give the definition in this special case. While many of the propositions in the thesis will be stated for general lc-manifolds, on first reading one may simply restrict to the case described below.

Definition 1. *An lc-manifold of depth one is a Hausdorff space P, with a distinguished closed subset called its boundary $\partial P \subset P$. We assume both ∂P and $P - \partial P$ are Hilbert manifolds. Furthermore, each point $p \in \partial P$ has a neighborhood $U \subset \partial P$ and an open embedding $f : U \times [0, \epsilon) \to P$ such $f(u, 0) = u$ while $f_{|U\times\{0\}}$ and $f_{|U\times(0,\epsilon)}$ are diffeomorphisms.*

Let us call such a map $f : U \times [0, \epsilon) \to P$ an lc-chart. Let P be an lc-manifold of depth one and $\mathfrak{B}$ some Hilbert manifold.

Definition 2. *A continuous map $\sigma : P \to \mathfrak{B}$ is lc-smooth if the following hold:*
1. σ is smooth on ∂P and $P - \partial P$
2. Each point $p \in \partial P$ has an lc-chart $U \times [0, \epsilon)$ such that, in the chart coordinates, σ along with its first derivative in the U direction is continuous with respect on $U \times [0, \epsilon)$.

Given an lc-smooth map $\sigma : P \to \mathfrak{B}$, by $D\sigma_p : TP \to T\mathfrak{B}$ we mean the differential restricted to the open stratum on which p lives.

2.2 Basic Definitions

For now, let us only consider the case when $\mathfrak{B}$ is a Hilbert space. We will assume all our Hilbert manifolds to be separable. In this case, we have the following notion of polarization:

Definition 3. *A polarization of a Hilbert space $\mathfrak{B}$ is a direct sum decomposition $T\mathfrak{B} = T^{+}\mathfrak{B} \oplus T^{-}\mathfrak{B}$*

Definition 4. *A **Floer space** $(\mathfrak{B},\mathcal{L})$ is a polarized Hilbert space which in addition to the usual strong topology, is equipped with its usual weak topology as well as a function $\mathcal{L} : \mathfrak{B} \to \mathbb{R}$ which is continuous for the strong topology.*

Definition 5. *A **chain** $\sigma : P \to \mathfrak{B}$ where P is a lc-manifold is an lc-smooth map satisfying the following axioms:*

***Axiom** 1. On $\mathrm{im}(\sigma)$, $\mathcal{L}$ is bounded below and lower semi-continuous for the weak topology.*

***Axiom** 2. Given a weakly converging sequence $\sigma(x_i)$ with limit y, if $\lim(\mathcal{L}(\sigma(x_i)) = \mathcal{L}(y)$ then some subsequence of x_i converges strongly on P.*

***Axiom** 3. Any subset $S \subset \mathrm{im}(\sigma)$ on which $\mathcal{L}$ is bounded is precompact for the weak topology.*

***Axiom** 4. $\Pi^{-} \circ D\sigma_p : TP \to T^{-}\mathfrak{B}$ is Fredholm, $\Pi^{+} \circ D\sigma_p : TP \to T^{+}\mathfrak{B}$ is compact for each $p \in P$.*

Remark. A σ satisfying Axiom 4 is said to be **semi-infinite dimensional**.

Example. Take a Hilbert space $H = H^{+} \oplus H^{-}$ split into two infinite dimensional subspaces with its usual strong/weak topologies and $\mathcal{L}(v^{+}, v^{-}) = |v^{-}|^2 - |v^{+}|^2$. The polarization is given by the splitting. $P = H^{-}$ with the inclusion map defines a cycle.

Definition 6. *A chain $\sigma : P \to \mathfrak{B}$ has **index** k if the linearized map $\Pi^{-} \circ D\sigma : TP \to T^{-}\mathfrak{B}$ has index k at each point of $P - \partial P$.*

Remark. Note that $index(\sigma_{|\partial P}) = index(\sigma) - 1$. Indeed, since in an appropriate lc-chart around a point $p \in P$, σ becomes

$$\sigma : V \times [0, \epsilon) \to \mathfrak{B}$$

the differential of σ in the v-variables is continuous on $V \times [0, \epsilon)$. Therefore, the index of $D\sigma$ on $V \times (0, \epsilon)$ is exactly one greater than the index $D\sigma_{|V}$.

2.3 Definition of Bordism Groups $\Omega F_*(\mathfrak{B}, \mathcal{L})$

The easiest invariant to define is a Floer Bordism Group:

Definition 7. *A cycle is a chain of depth 0.*

Definition 8. *Let $\Omega F_k(\mathfrak{B}, \mathcal{L})$ be the $\mathbb{Z}_2$-vector space generated by isomorphism classes of chains of depth zero and index k. Disjoint union is the additive structure. Furthermore, $[P] = 0$ if $\sigma : P \to \mathfrak{B}$ extends to a chain of depth one $\sigma' : W \to \mathfrak{B}$ with $\partial W = P$ and $\sigma'_{|\partial W} = \sigma$. Let $\Omega F_*(\mathfrak{B}, \mathcal{L}) = \bigoplus_k \Omega F_k(\mathfrak{B}, \mathcal{L})$.*

Definition 9. *Given $(\mathfrak{B}, \mathcal{L})$ as above, let $-\mathfrak{B}$ be the polarized Hilbert space obtained by switching $T^+\mathfrak{B}$ and $T^-\mathfrak{B}$ and let $(-\mathfrak{B}, -\mathcal{L})$ be the Floer space obtained by switching the sign of $\mathcal{L}$.*

Let us make the following perturbation hypothesis which will be verified for examples we consider.
Existence of Perturbations: Given cycles $\sigma : P \to \mathfrak{B}$ and $\tau : Q \to -\mathfrak{B}$ there exists a chain $F : P \times [0,1] \to \mathfrak{B}$ with $F_{|P\times 0} = \sigma$ and $F_{|P\times 1}$ transverse to τ. Furthermore, if σ is already transverse to τ, without changing $F_{P\times 1}$, we may alter F to be transverse to τ as well.

Theorem 1. *Given, cycles σ and τ as above, having transverse intersection, their fibre product $\sigma \times_{\mathfrak{B}} \tau$ is a closed manifold mapping to $\mathfrak{B}$. The fibre product gives a well-defined map*

$$\Omega F_k(\mathfrak{B}, \mathcal{L}) \times \Omega F_l(-\mathfrak{B}, -\mathcal{L}) \to \Omega_{k+l}(\mathfrak{B})$$

where $\Omega_{k+l}(\mathfrak{B})$ denotes ordinary lc-bordism with $\mathbb{Z}_2$-coefficients.

Proof. First, we show that $\sigma \times_{\mathfrak{B}} \tau$ is compact. This part of the proof does not use the fact that the cycles have transverse intersection. On the image, $\mathcal{L}$ is bounded above and below. Furthermore, since on the image $\mathcal{L}$ is both lower and upper semi-continuous, it is continuous in the weak topology. Axiom 2 implies every sequence $x_i \in \sigma \times_{\mathfrak{B}} \tau$ must have a convergent subsequence. Note that we can view $\sigma \times_{\mathfrak{B}} \tau$ as $(\sigma \times \tau)^{-1}(\Delta)$ where Δ is the diagonal in $\mathfrak{B} \times \mathfrak{B}$. By assumption, $\sigma \times \tau$ is transverse to Δ and thus $\sigma \times_{\mathfrak{B}} \tau$ is a smooth finite dimensional lc-manifold. To calculate the dimension, note that locally $\sigma \times_{\mathfrak{B}} \tau$ is $(\sigma - \tau)^{-1}(0)$. Up to compact perturbation, the linearized operator has the form

$$\begin{pmatrix} \Pi^- \circ D\sigma & 0 \\ 0 & -\Pi^+ \circ D\tau \end{pmatrix}$$

Thus, the dimension of $\sigma \times_{\mathfrak{B}} \tau$ is $ind(\Pi^- \circ D\sigma) + ind(\Pi^+ \circ D\tau)$. Finally, if $F : W \to \mathfrak{B}$ is an chain with $\partial F = \sigma$, we have

$$\partial(F \times_{\mathfrak{B}} \tau) = \partial F \times_{\mathfrak{B}} \tau = \sigma \times_{\mathfrak{B}} \tau =$$

when F is transverse to τ. Therefore, the fibre product descends to a map on $\Omega F(\mathfrak{B}, \mathcal{L})$. □

Remark. In the context of Floer theory discussed in this work we will show how to modify the definition of $\Omega F_*(\mathfrak{B}, \mathcal{L})$ so that the fibred product lies in the usual smooth bordism groups, rather than the lc-bordism groups.

Chapter 3

Monopole Moduli Spaces

3.1 Definition of $(\mathfrak{B}_{\mathfrak{s}}(Y), \mathcal{L})$

Here is the main example we will be working with. Given a smooth oriented, closed, connected 3-manifold Y equipped with a metric and *spin*$^{\mathbb{C}}$-structure $\mathfrak{s}$ define the configuration space $\mathcal{C}_{\mathfrak{s}}(Y)$ to be pairs (B, Ψ) where B is a Clifford connection and Ψ is a spinor, both in the $L^2_{1/2}$ topology. Fix a smooth connection B_0. Let $\mathfrak{B}_{\mathfrak{s}}(Y) \subset \mathcal{C}_{\mathfrak{s}}(Y)$ be pairs (B, Ψ) with $B - B_0$ co-closed. This is an affine subspace with tangent space pairs (b, Ψ) where b is co-closed. Note $*d$ is an elliptic self-adjoint operator on co-closed forms. We also have that D_{B_0} is a self-adjoint elliptic operator on spinors. We define a polarization of $T\mathfrak{B}_{\mathfrak{s}}(Y)$ by $(T^+\mathfrak{B}, T^-\mathfrak{B})$ where the bundles correspond to the positive (nonpositive) eigenspaces of the operator $(*d, D_{B_0})$.

We define $\mathcal{L}$ to be minus the Chern-Simons-Dirac functional on $\mathcal{C}_{\mathfrak{s}}(Y)$:

$$\mathcal{L}(B, \Psi) = \frac{1}{8}\int_Y (B^\tau - B_0^\tau) \wedge (F_{B^\tau} + F_{B_0^\tau}) - \frac{1}{2}\langle \Psi, D_B \Psi \rangle_Y \tag{3.1}$$

Here B^τ denotes the induced connection on the determinant bundle of the spinor bundle.

Definition 10. *Let $\Omega F_*(Y, \mathfrak{s})$ ($\widehat{HF}_*(Y, \mathfrak{s})$) be the bordism (homology) theory associated to $(\mathfrak{B}(Y), \mathcal{L})$.*

Remark. For now, we must keep in mind that $\Omega F_*(Y, \mathfrak{s})$ depends on the choice of metric and connection B_0. We will verify later that $\Omega F_*(Y, \mathfrak{s})$ is indeed independent of the choices which justifies our imprecise notation.

Remark. We will assume that Y is a rational homology sphere.

3.2 Moduli Spaces are Cycles

We can repeat the construction of a configuration space now with a smooth oriented, connected 4-manifold X defining the space $\mathfrak{B}_{\mathfrak{s}}(X)$. Given a *spin*$^{\mathbb{C}}$-structure $\mathfrak{s}$ on X, define

$$\mathcal{C}_{\mathfrak{s}}(X) = (A, \Phi)$$

where A is an L^2_1 Clifford connection and $\Phi \in \Gamma(W^+)$, the space of positive L^2_1 spinors. Assume X has a connected boundary. We have the linear restriction map:

$$\mathcal{R} : \mathcal{C}_{\mathfrak{s}}(X) \to \mathcal{C}_{\mathfrak{s}_{|\partial X}}(\partial X), \mathcal{R}(A, \Phi) = (A_{|\partial X}, \Phi_{|\partial X})$$

We will use the notation $\mathcal{R}(A, \Phi) = (B, \Psi)$. Fix a smooth connection A_0.

Definition 11. $\mathcal{G}(X) = \{u \in L^2_2(X, \mathbb{C}) | u : X \to S^1$ *almost everywhere*$\}$

Definition 12. $\mathfrak{B}_{\mathfrak{s}}(X) = \{(A, \Phi) \in \mathcal{C}_{\mathfrak{s}}(X) | d^*_X(A - A_0) = 0, d^*_{\partial X}(B - B_0) = 0\}$

Configurations in $\mathfrak{B}_{\mathfrak{s}}(X)$ are said to be in **strong Coulomb gauge**.

Lemma 1. *Any $(A, \phi) \in \mathcal{C}_{\mathfrak{s}}(X)$ can be put in strong Coulomb gauge by some $u \in \mathcal{G}(X)$.*

Proof. Solve the Dirichlet problem

$$\Delta f = d^*(A - A_0), f_{|\partial X} = h$$

where $d_{|\partial X} h$ is the exact part of $B - B_0$. Note that $h \in L^2_{3/2}$ and $d^*(A - A_0) \in L^2$ thus $f \in L^2_2$ and $u = e^f \in L^2_2$. □

Recall the Seiberg-Witten map:

$$SW(A, \Phi) = (F^+_{A^\tau} - \rho^{-1}(\Phi \otimes \Phi^*), D_A(\Phi))$$

We have:

$$\mathcal{M}_{\mathfrak{s}}(X) = \{(A, \Phi) \in \mathfrak{B}_{\mathfrak{s}}(X) | SW(A, \Phi) = 0\} / \mathcal{G}_h(X)$$

where

$$\mathcal{G}_h(X) = \{u : X \to S^1 | d^*_X(u^{-1}du) = 0, u_{|\partial X} = 1, \}$$

We have the following fundamental result:

Theorem 2. $\mathcal{R} : \mathcal{M}_{\mathfrak{s}}(X) \to \mathfrak{B}_{\mathfrak{s}|Y}(Y)$ *is a cycle for* $(\mathfrak{B}_{\mathfrak{s}|Y}(Y), \mathcal{L})$

Proof. The proof is essentially contained in [KM07] so we simply repeat some steps to set up notation. We write the SW-equations together with the gauge fixing condition as

$$SW \oplus -d^*_{A_0} : \mathcal{A} \oplus W^+ \to i\Lambda^+(X) \oplus W^- \oplus i\Lambda^0(X)$$

As usual, near the boundary, the range and domain may be rewritten as time-dependent sections of

$$i\Lambda^0(Y) \oplus \mathrm{im}(d) \oplus \ker(d^*) \oplus S$$

With this identification, the first order part of the linearization of $SW \oplus -d^*_{A_0}$ becomes

$$\partial_t + \begin{pmatrix} 0 & -d_Y & 0 & 0 \\ -d_Y^* & 0 & 0 & 0 \\ 0 & 0 & *_3 d_Y & 0 \\ 0 & 0 & 0 & D_{B_0} \end{pmatrix}$$

The proof that $\mathcal{M}$ is a Hilbert manifold is no different than the proof in [KM07] with one subtlety: in order to establish the surjectivity of the linearized SW map we must apply a strong form of the unique continuation principle. More precisely, we have an operator of the form

$$\partial_t + L + \phi \cdot ()$$

where L is self adjoint and $\phi \cdot ()$ denotes some pointwise multiplication with an L^2_1 form on a vector bundle over $Y \times (-\infty, 0]$. We are given $\psi \in L^2$ in the cokernel of this operator and we want to conclude that ψ is zero. We may extend ψ by zero to $Y \times (\infty, 1]$ and extend ϕ to an arbitrary L^2_1 form (we denote the extended sections by the same symbols). By hypothesis, the extended ψ is still in the kernel of $\partial_t + L + \phi \cdot ()$. We can apply the strong form of the unique continuation principle (see analysis appendix) once we know that ψ is in L^2_1. This is established in the appendix as well. (In [KM07] this issue does not come up as ϕ has higher regularity and it is therefore easy to conclude that ψ is regular as well). $\square$

3.3 Perturbations

To ensure transverse intersection of cycles we need to be able to perturb them with a sufficiently large parameter space at the same time ensuring that the perturbed map is still a chain. We show how to construct such a space in the SW case.

Definition 13. *Let $\mathfrak{P} \subset \ker(d*)_{i\Lambda^1(Y)} \oplus S$ be the unit ball in the L^2_5 norm. From the compactness of the inclusion $L^2_5 \subset C^1$ we have that every sequence $v_i \in \mathfrak{P}$ has a C^1 convergent subsequence.*

We have the following theorem:

Theorem 3. *Given a chain $\sigma : P \to \mathfrak{B}(Y)$ there is a smooth map $F : P \times \mathfrak{P} \to \mathfrak{B}(Y)$ such that:*
1. $F(x, 0) = \sigma(x)$
2. $DF_{(x,v)}$ has dense image for all (x, v)
3. $\Pi^- \circ D_x F$ is Fredholm, $\Pi^+ \circ D_x F$ and $D_v F$ are compact
4. Given a compact lc-manifold $K \subset \mathfrak{P}$, $F_{|P \times K}$ is again a semi-infinite chain.

Proof. Let ρ be a positive bump function equal to 1 on $[-1,1]$ and $1/x^2$ outside $[-2,2]$. We define the map

$$F : \sigma \times \mathfrak{P} \to \mathfrak{B}(Y)$$

by

$$F(x,v) = \sigma(x) + \rho(||\sigma(x) - (B_0,0)||^2_{L^2_{1/2}})v$$

Claim: F satisfies all the requirements of the theorem.

Part 1: Clear from the construction.

Part 2: Note that

$$DF_{(x,v)}(0,w) = \rho(||\sigma(x) - (B_0,0)||^2_{L^2_{1/2}})w$$

and since $L^2_5(ker(*d)_{i\Lambda^1(Y)} \oplus S) \to L^2_{1/2}(ker(*d)_{i\Lambda^1(Y)} \oplus S)$ is compact with dense image and $\rho(||\sigma(x) - (B_0,0)||^2_{L^2_{1/2}}) > 0$ we have that DF has dense image and D_vF is compact.

Part 3:

$$DF_{(x,v)}(y,0) = D\sigma_x(y) + T(y)v$$

where $T : TP \to \mathbb{R}$ is the linear map given by $T(y) = D\rho(y)$ Therefore,viewed as a map on TP, $DF(y,0)$ differs from $D\sigma$ by an at most rank one map from which part 3 follows.

Part 4: Observe that $\mathcal{L}(F(x,v)) - \mathcal{L}(\sigma(x))$ is bounded independent of x. Indeed, for any $x \in P$ and $(v_1,v_2) \in \mathfrak{P}$ let $F(x,(v_1,v_2)) = \sigma(x) + (cv_1, cv_2)$. For example, we have

$$\langle D_B\Psi, \Psi\rangle - \langle D_{B+cv_1}(\Psi + cv_2), \Psi + cv_2\rangle =$$

$$\langle \rho(cv_1)(\Psi + cv_2), cv_2 + \Psi\rangle + \langle D_{B_0}(cv_2) + \rho(B - B_0)cv_2, cv_2 + \Psi\rangle + \langle \Psi, D_{B_0}(cv_2) + \rho(B - B_0)cv_2\rangle$$

The computation of the other half of $\mathcal{L}$ is similar. Since $c \cdot ||(B_0 - B, \Psi)||^2_{L^2}$ is uniformly bounded the assertion follows. Also, observe that if $\sigma(x_i)$ are weakly $L^2_{1/2}$ convergent $\mathcal{L}(\tilde{F}(x_i,v_i)) - \mathcal{L}(\sigma(x_i))$ is in fact strongly convergent, after passing to a subsequence. This follows from the fact that $(c_iv_{i,1}, c_iv_{i,2})$ is C^1 precompact. This implies $\mathcal{L}(F(x_i,v_i))$ drops exactly when $\mathcal{L}(\sigma(x_i))$ drops. Let us check that $P \times K \to \mathfrak{B}$ satisfies all the axioms. Given a weakly convergent sequence $\sigma(x_i) + c_i \cdot v_i$ note that $c_i \cdot v_i$ converge strongly since K is $L^2_{1/2}$ precompact. Therefore, $\sigma(x_i)$ is weakly convergent and lower semi-continuity follows. If $\mathcal{L}$ does not drop in the limit, we must have x_i precompact and thus (x_i, v_i) precompact as well. $\square$

The following definition will be important for our discussion of homology. Here we investigate how it behaves under perturbations.

Definition 14. *A set $S \subset \mathfrak{B}$ is **k-negligible** if it is contained in the image of a semi-infinite*

Hilbert manifold $Q \to \mathfrak{B}$ of index $< k$.

Definition 15. *A chain $\sigma : P \to \mathfrak{B}$ of index k is said to be* ***negligible*** *if its image is k-negligible.*

Theorem 4. *If $\sigma : P \to \mathfrak{B}$ is k-negligible then $F(\cdot, v) : P \to \mathfrak{B}$ is k-negligible.*

Proof. If given $\sigma(p) = g(q)$ where $g : Q \to \mathfrak{B}$ semi-infinite of smaller index then $F(p, v) = \sigma(p) + \rho(||\sigma(p) - (B_0, 0)||^2_{L^2_{1/2}})v = g(q) + \rho(||g(q) - (B_0, 0)||^2_{L^2_{1/2}})v$ □

We can now use the perturbations to put cycles in general position:

Theorem 5. *Given a cycle $\sigma : P \to (\mathfrak{B}, \mathcal{L})$ and a cycle $\tau : Q \to (-\mathfrak{B}, -\mathcal{L})$ there exists a cobordant cycle $\sigma' : P \to (\mathfrak{B}, \mathcal{L})$ such that σ' is transverse to τ. Furthermore, given two such transverse cycles σ' and σ'' there exists a cobordism $\Sigma : P \times [0, 1] \to (\mathfrak{B}, \mathcal{L})$ transverse to τ, with $\partial\Sigma = \sigma' - \sigma''$.*

Proof. The argument follows the standard route. The map $F \times \tau \to \mathfrak{B} \times \mathfrak{B}$ is transverse to $\Delta \subset \mathfrak{B} \times \mathfrak{B}$. The map $(F \times \tau)^{-1}(\Delta) \to \mathfrak{P}$ is Fredholm. Hence, applying Smale's extension of Sard's theorem [Sma65], we have that for generic $p \in \mathfrak{P}$ the map $F(, p) \times \tau$ is transverse to Δ. Any to such generic values p, q may be connected by an arc $\gamma : [0, 1] \to \mathfrak{P}$ transverse to $(F \times \tau)^{-1}(\Delta) \to \mathfrak{P}$ □

Remark. For some purposes it is necessary to make the perturbations in some small $L^2_{1/2}$ neighborhood of a point. In this case, one would take ρ to be supported in that ball. The rest of the proof remains the same.

3.4 Grading

As discussed above, the polarization defined by the operator $(*d, D_{B_0})$ allows us to introduce an integer grading on $\Omega F_*(Y, \mathfrak{s}) = \bigoplus_k \Omega F_k(Y, \mathfrak{s})$. Note, however, the grading depends on the choice B_0 and will only be an invariant of Y as a $\mathbb{Z}$-set. Given connections B_0 and B_1, let $T_0^{\pm}\mathfrak{B}$ and $T_1^{\pm}\mathfrak{B}$ denote the corresponding polarizations.

Definition 16. *Let $SpecFl(B_0, B_1)$ be the index of the projection $\pi_1^+ : T_0^+\mathfrak{B} \to T_1^+\mathfrak{B}$.*

To avoid the dependence of the grading on the choice of connection we make the following definition:

Definition 17. *Let $\mathfrak{Q}(\mathfrak{B})$ be pairs (n, B) where $n \in \mathbb{Z}$ and B is a Clifford connection with the equivalence relation*

$$(n_1, B_1) \sim (n_0, B_0) \Leftrightarrow n_1 - n_0 = SpecFl(B_0, B_1)$$

In our general discussion of Floer spaces we observed that given a Floer Space $\mathfrak{B}$ we may obtain a Floer space $-\mathfrak{B}$ by reversing the sign of $\mathcal{L}$ and switching the polarization. In the case of configuration spaces associated to 3-manifolds we have $\mathfrak{B}(-Y) = -\mathfrak{B}(Y)$ on the level of spaces. However the polarizations differ. We have

$$T^{-}\mathfrak{B}(-Y) = T^{-}(-\mathfrak{B}) \oplus \ker(*d, D_{B_0})$$

This difference should not be a source of confusion since we use $\Omega F_*(-Y)$ to refer to $\Omega F_*(\mathfrak{B}(-Y))$. We may define a map of $\mathbb{Z}$-sets

$$o : \mathfrak{Q}(\mathfrak{B}(-Y)) \to \mathfrak{Q}(\mathfrak{B}(Y))$$

taking (m, B_0) to $(m + dim(ker(*d, D_{B_0})), B_0)$. Using the perturbations introduced above together with the Sard-Smale theorem we have a well-defined map

$$\Omega F_n(Y, \mathfrak{s}) \otimes \Omega F_m(-Y, -\mathfrak{s}) \to \Omega_{n+o(m)}(\mathfrak{B}_{\mathfrak{s}}(Y), \mathbb{Z}_2)$$

3.5 Correspondences

3.5.1 Definitions

A cobordism W from Y_0 to Y_1 gives rise to a map $\mathcal{M}_W \to \mathfrak{B}(Y_0) \times \mathfrak{B}(Y_1)$. We describe how to associate to such a map a homomorphism $\Omega F(Y_0) \to \Omega F(Y_1)$. More, generally, we describe how certain maps to $\mathfrak{B}_0 \times \mathfrak{B}_1$ give rise to such homomorphisms.

Definition 18. *A correspondence $(Z, f) \in Cor((\mathfrak{B}_0, \mathcal{L}_0), (\mathfrak{B}_1, \mathcal{L}_1))$ is a map $f : Z \to \mathfrak{B}_0 \times \mathfrak{B}_1$ where Z is a Hilbert manifold (possibly with boundary) satisfying the following axioms:*

Axiom $1'$. *On* $\mathrm{im}(f)$, *$\mathcal{L}_1 - \mathcal{L}_0$ is bounded below and lower semi-continuous for the weak topology.*

Axiom $2'$. *If $\mathcal{L}_1(\pi_1(z_i))$ is bounded above and $\pi_0(z_i)$ is a weakly precompact sequence then $\pi_1(z_i)$ weakly precompact.*

Axiom $3'$. *Given $\pi_1(z_i)$ is weakly convergent to x, if $\lim f^*(\mathcal{L}_1 - \mathcal{L}_0)(z_i) = (\mathcal{L}_1 - \mathcal{L}_0)(x)$ and $\pi_0(z_i)$ is converges strongly then z_i converges strongly (up to a subsequence).*

Axiom $4'$. *$(\pi_0^+, \pi_1^-) \circ Df : TZ \to T^+\mathfrak{B}_0 \oplus T^-\mathfrak{B}_1$ is Fredholm. Given a bounded sequence $v_i \in TZ$, if $\pi_0^+(Df)(v_i)$ is weakly convergent, $\pi_1^+(Df)(v_i)$ is precompact.*

Axiom $5'$. *$Df : TZ \to T\mathfrak{B}_0$ is dense. $Df_{|\partial(Z)} : TZ \to T\mathfrak{B}_0$ is also dense.*

Example 1. *The diagonal map $\Delta : \mathfrak{B} \to \mathfrak{B} \times \mathfrak{B}$ is a correspondence.*

Theorem 6. *Given a chain $\sigma : P \to \mathfrak{B}_0$ and a correspondence $f : Z \to \mathfrak{B}_0 \times \mathfrak{B}_1$, the fiber product $\pi_1 \circ f : P \times_{\mathfrak{B}_0} Z \to \mathfrak{B}_1$ is chain in $(\mathfrak{B}_1, \mathcal{L}_1)$.*

Proof. Axiom 1: $-\mathcal{L}_0 + \mathcal{L}_1 > C$ and $\mathcal{L}_0 > C$ imply $\mathcal{L}_1 > 2C$. Given a sequence $(x_i, z_i) \in P \times_{\mathfrak{B}(Y_1)} Z$ with a weakly convergent sequence $f(z_i)$, we have

$$\liminf(-\mathcal{L}_0(f(z_i))) + \mathcal{L}_1(f(z_i)) \geq -\mathcal{L}_0(f(z_\infty)) + \mathcal{L}_1(f(z_\infty))$$

and

$$\liminf \mathcal{L}_0(f(z_i)) \geq \mathcal{L}_0(f(z_\infty))$$

imply $\liminf \mathcal{L}_1(f(z_i)) \geq \mathcal{L}_1(f(z_\infty))$.

Axiom 2: If $\lim \mathcal{L}_1(f(z_i)) = \mathcal{L}_1(f(z_\infty))$ Axiom $1'$ implies that $\mathcal{L}_0$ can only rise in the limit. However, since $\sigma(x_i)$ are assumed weakly convergent as well, we have $\lim \mathcal{L}_0(\sigma(x_i)) = \mathcal{L}_0(\sigma(x_\infty))$ and thus $\sigma(x_i) = \pi_0(f(z_i))$ is strongly convergent so Axiom $3'$ implies z_i strongly convergent as well.

Axiom 3: $\mathcal{L}_1(f(z_i)) < C$ implies $\mathcal{L}_0(f(z_i)) < C$ and thus $\sigma(x_i)$ is weakly precompact. Axiom $2'$ implies that $\pi_1(z_i)$ is weakly precompact as well.

Axiom 4: We use the following lemma

Lemma 2. *Given a linear Fredholm map $T : W \to V_1 \oplus V_2$ such that $\Pi_1 \circ T$ is surjective, $T_{|\ker(\Pi_1 \circ T)} \to V_2$ is Fredholm with the same index.*

Proof. We have $\ker(T) = \ker(T_{|\ker(\Pi_1 \circ T)})$. Surjectivity of $\Pi_1 \circ T$ implies the dimension of cokernel coincides as well. □

Take a polarization of $T\mathfrak{B}_0 \oplus T\mathfrak{B}_1$ with projections (Π_i^+, Π_i^-). We apply this lemma to the map

$$F : TZ \times TP \to T^+\mathfrak{B}_0 \oplus T^-\mathfrak{B}_0 \oplus T^-\mathfrak{B}_1$$

with

$$F(z,p) = (\Pi_0^+(Df(z)) - \Pi_0^+(D\sigma(v))) \oplus (\Pi_0^-(Df(z)) - \Pi_0^-(D\sigma(p))) \oplus \Pi_1^-(Df(z))$$

Axiom $6'$ implies the above lemma applies since $\operatorname{im} Df \oplus D\sigma$ in $T\mathfrak{B}_0$ is closed and dense. To calculate the index deform through Fredholm operators to

$$\tilde{F}(z,p) = \Pi_0^+(Df(z)) \oplus \Pi_0^-(D\sigma(p)) \oplus \Pi_1^-(Df(z))$$

Thus, with respect to the given polarization,

$$\dim(Z \times_{\mathfrak{B}(Y_0)} P) = ind(Df) + ind(D\sigma)$$

□

3.5.2 Map on Bordism Groups

Theorem 7. *A correspondence $F : Z \to \mathfrak{B}_0 \times \mathfrak{B}_1$ of index k without boundary induces a map:*

$$\Omega F_k(F) : \Omega F_*(\mathfrak{B}_0, \mathcal{L}_0) \to \Omega F_{*+k}(\mathfrak{B}_1, \mathcal{L}_1)$$

Proof. Given a cycle $\sigma : P \to \mathfrak{B}_0$ we have $\partial(\sigma \times_{\mathfrak{B}_0} F) = \partial(\sigma) \times_{\mathfrak{B}_0} F$ since $\partial F = \emptyset$ This shows that $\Omega F_k(F)$ commutes with the boundary operator. □

Similarly, we have a version of the homotopy axiom:

Theorem 8. *Given a correspondence $F : Z \to \mathfrak{B}_0 \times \mathfrak{B}_1$ with boundary $(F_0 \sqcup F_1, Z_0 \sqcup Z_1)$ we have:*

$$\Omega F(F_0) = \Omega F(F_1)$$

Proof. Given a cycle $\sigma : P \to \mathfrak{B}$,

$$\partial(\sigma \times_{\mathfrak{B}} F) = \sigma \times_{\mathfrak{B}} \partial(F) = \sigma \times_{\mathfrak{B}} (F_1 \sqcup F_0)$$

□

3.5.3 Cobordisms as Correspondences

As mentioned above, a cobordism from Y_0 to Y_1 gives rise to a correspondence. Before discussing the precise statement let us discuss the definition the moduli space of solutions on a manifold with 2 boundary components. First a lemma:

Lemma 3. *Given a 1-form α with $d_W^*(\alpha) = 0$ we have $\int_{Y_0} \alpha(\vec{n}) = -\int_{Y_1} \alpha(\vec{n})$ where $\vec{n}$ is the unit outward normal. Given a harmonic function f with $f_{|Y_0} = 0$ and $f_{|Y_1} = c$ we have $0 \neq \langle df, df \rangle_W = c \int_{Y_1} \partial f / \partial \nu$*

Proof. For a function g we have the formula

$$\langle \alpha, dg \rangle_W = \langle d^* \alpha, g \rangle_W + \int_{\partial W} \alpha(\vec{n}) g$$

Plugging in $g = 1$ gives the first claim. For harmonic functions f, g we have

$$\langle df, dg \rangle_W = \int_{\partial W} g \partial f / \partial \nu$$

Plug in $f = g$ for the second claim. □

Definition 19. *Given a compact 4-manifold W with boundary $-Y_0 \sqcup Y_1$ let $\tilde{\mathcal{M}}_W$ be the solutions to the SW equations γ in strong Coulomb gauge such that $\int_{Y_0} (B - B_0)(\vec{n}) = 0$.*

Remark. By the lemma above it makes no difference which of the two components of the boundary we choose.

Definition 20. *Let* $\mathcal{G}_h(W) = \{u : W \to S^1 | d_W^*(u^{-1}du) = 0, u_{|\partial W} = 1, \int_{Y_1} u^{-1}du(\vec{n}) = 0\}$

Note that when $e^f \in \mathcal{G}_h(W)$ we have $e^f = 1$. Indeed, since $e^f = 1$ on ∂W we have f a multiple of $2\pi i$ on ∂W. Subtracting a constant multiple of $2\pi i$, we may assume $f_{|Y_0} = 0$. This implies $f = 0$ since, if f is not constant,

$$0 \neq \langle df, df \rangle_W = c \int_{Y_1} \partial f / \partial \nu$$

Lemma 4. *An element of* $\mathcal{C}(W)$ *may be put in* $\tilde{\mathcal{M}}_W$ *by the gauge group.*

Proof. Given $\gamma \in \mathcal{C}(W)$ we may put it in strong Coulomb gauge by solving the Dirichlet problem as above. Take the harmonic function f equal to 0 on Y_0 and 1 on Y_1. We claim that for an appropriate choice of constant c, $e^{cf} \cdot \gamma$ is in $\tilde{\mathcal{M}}_W$. This follows from the fact that $\int_{Y_1} \partial f / \partial \nu \neq 0$. □

Definition 21. *Let* $\mathcal{M}_W = \tilde{\mathcal{M}}_W / \mathcal{G}_h(W)$

We have the following theorem:

Theorem 9. *A cobordism* W *from* Y_0 *to* Y_1 *with a spin structure* $\mathfrak{s}_W$ *gives rise to a correspondence via the restriction map* $\mathcal{R} : \mathcal{M}_W \to \mathfrak{B}(-Y_0, \mathfrak{s}_{|Y_0}) \times \mathfrak{B}(Y_1, \mathfrak{s}_{|Y_1})$

The fact that a cobordism W satisfies Axioms $1'$-$4'$ follows from the L_1^2 compactness proof as well as the APS boundary value theory discussed in [KM07]. Axiom $5'$ requires the following lemma:

Lemma 5. *Consider a smooth compact 4-manifold* W *with (at least) 2 nonempty boundary components* Y_0 *and* Y_1*. Assume we are given a Dirac-type operator* D *which is of the form* $\partial_t + L$ *near the ends where* L *is a first order operator on some vector bundle* E *on* Y*. We assume* $L = L_0 + A$ *where* L_0 *is of Dirac type and* A *is some* L_1^2 *endomorphism. We have* $Q = \{\phi_{|Y_0} | \phi \in L_1^2(W), D(\phi) = 0\}$ *is a dense subspace of* $L_{1/2}^2(Y_0, E)$

Proof. By contradiction, take $w \in L_{1/2}^2(Y_0, E)$ to be $L_{1/2}^2$-orthogonal to the closure of Q. The operator

$$(D, \pi_{\mathbb{C}w} \circ r) : L_1^2(W) \to L^2(W) \oplus \mathbb{C}w$$

where $\pi_{\mathbb{C}w}$ is the $L_{1/2}^2$ projection to $\mathbb{C}w$, has finite dimensional cokernel. This follows from the fact that

$$D : L_1^2(W) \to L^2(W)$$

is surjective. We show that $(D, \pi_{\mathbb{C}w} \circ r)$ is surjective. Take $f = (a, cw)$ orthogonal to the image. By variations in the interior we see that $D^*(a) = 0$. By variations near Y_1 we have that $a_{|Y_1}$ is zero. Thus, the unique continuation principle (UCP) implies $a = 0$ (we need to appeal to the strong form of UCP as in the construction of $\mathcal{M}(X)$). Now, take any L_1^2 extension $c\tilde{w}$ of cw to W. We have,

$$\langle (0, cw), (D(c\tilde{w}), \pi_{\mathbb{C}w} \circ r(c\tilde{w})) \rangle_{(L^2, L_{1/2}^2)} = 0$$

and thus $cw = 0$. From this we conclude that $(D, \pi_{\mathbb{C}w} \circ r)$ has no cokernel and thus $w = \pi_{\mathbb{C}w} \circ r(u)$ where $D(u) = 0$. This implies $\langle w, r(u)\rangle_{L^2_{1/2}} = \langle w, w\rangle_{L^2_{1/2}}$ since

$$\langle w, (1 - \pi_{\mathbb{C}w}) \circ r(u)\rangle_{L^2_{1/2}} = 0$$

Thus, $w = 0$. $\square$

Remark. It is worth noting that no perturbations are used in the definition of the map on chains. This is analogous to the finite dimensional picture where pushing forward a cycle is always defined while pulling back requires perturbations.

We will also need to compose cobordisms and establish that the map from moduli spaces to correspondences is functorial. For this we need to describe other choices of a gauge slice for $\tilde{\mathcal{M}}_W$.

Definition 22. *Given a compact 4-manifold W with boundary $-Y_0 \sqcup Y_1$ let $\tilde{\mathcal{M}}_{W,\alpha}$ be the solutions to the SW equations γ in strong Coulomb gauge such that $\langle \alpha, A - A_0\rangle_W = 0$.*

Definition 23. *Suppose α is a smooth harmonic function with $\int_{Y_1} \partial\alpha/\partial\nu \neq 0$. Let*

$$\mathcal{G}_{h,\alpha}(W) = \{u : W \to S^1 | d^*_W(u^{-1}du) = 0, u_{|\partial W} = 1, \langle \alpha, u^{-1}du\rangle_W = 0\}$$

Note that when $e^f \in \mathcal{G}_{h,\alpha}(W)$, we have $e^f = 1$. Indeed, since $e^f = 1$ on ∂W we have f a multiple of $2\pi i$ on ∂W. Subtracting a constant multiple of $2\pi i$, we may assume $f_{|Y_0} = 0$ and $f_{|Y_1} = c$. This implies $f = 0$ since

$$\langle df, d\alpha\rangle_W = \int_{Y_1} f\partial\alpha/\partial\nu = \int_{Y_1} c\partial\alpha/\partial\nu \neq 0$$

as before, we have:

Lemma 6. *Any element of $\mathcal{C}(W)$ may be put in $\tilde{\mathcal{M}}_{W,\alpha}$ by the gauge group.*

Definition 24. *Let $\mathcal{M}_{W,\alpha} = \tilde{\mathcal{M}}_{W,\alpha}/\mathcal{G}_{h,\alpha}(W)$*

Remark. Let α_0 be the harmonic function equal to 0 on Y_0 and 1 on Y_1. For coclosed $A - A_0$, the formula

$$\langle \alpha, A - A_0\rangle_W = \int_{\partial W} (B - B_0)(\vec{n})$$

implies that the condition $\int_{Y_1} (B - B_0)(\vec{n}) = 0$ may be written as $\langle d\alpha_0, A - A_0\rangle_W = 0$. Therefore, $\mathcal{M}_{W,\alpha_0} = \mathcal{M}_W$.

Lemma 7. *The correspondence $(\mathcal{R}_0, \mathcal{R}_1) : \mathcal{M}_{W,\alpha} \to \mathfrak{B}(Y_0) \times \mathfrak{B}(Y_1)$ is homotopic to*

$$(\mathcal{R}_0, \mathcal{R}_1) : \mathcal{M}_W \to \mathfrak{B}(Y_0) \times \mathfrak{B}(Y_1)$$

Proof. To begin, note that

$$\langle d\alpha_0, d\alpha \rangle_W = \int_{Y_1} \partial\alpha/\partial\nu = \int_{Y_1} 1 \neq 0$$

Rescale α_0 and α so that $||\alpha_0||_W = 1$ and $\langle \alpha, \alpha_0 \rangle = 1$ Let $F : \tilde{\mathcal{M}}_W \to \tilde{\mathcal{M}}_{W,\alpha}$ be the map taking (A, Φ) to $e^{\langle \alpha, A-A_0 \rangle \cdot \alpha_0}(A, \Phi)$. F is an isomorphism with inverse

$$F^{-1}(A, \Phi) = e^{-\langle \alpha_0, A-A_0 \rangle \cdot \alpha_0}(A, \Phi)$$

Since $(\alpha_0)_{Y_0} = 0$ and $(\alpha_0)_{Y_1} = c$,

$$\mathcal{R}_0 \circ F(A, \Phi) = \mathcal{R}_0(A, \Phi)$$

while

$$\mathcal{R}_1 \circ F(A, \Phi) = e^{\langle \alpha, A-A_0 \rangle \cdot c} \mathcal{R}_1(A, \Phi)$$

. The homotopy $H : \mathcal{M}_{W,\alpha} \times [0,1] : \mathfrak{B}(Y_0) \times \mathfrak{B}(Y_1)$ from $\mathcal{R}_0 \times \mathcal{R}_1$ to $\mathcal{R}_0 \times (\mathcal{R}_1 \circ F)$ is given by

$$H(A, \Phi, t) = (\mathcal{R}_0(A, \Phi), e^{\langle \alpha, A-A_0 \rangle \cdot ct} \mathcal{R}_1(A, \Phi))$$

□

Lemma 8. *Consider (spin) cobordisms W_{01} from Y_0 to Y_1 and W_{12} from Y_1 to Y_2. Take a harmonic α_{02} on W_{02} with $\alpha_0 = 0$ on Y_0 and $\alpha_0 = 1$ on Y_2. Let α_{01} and α_{12} denote its restriction to W_{01} and W_{12}. We have a diffeomorphism :*

$$D : \mathcal{M}_{W_{02}} = \mathcal{M}_{W_{01,\alpha_{01}}} \times_{\mathfrak{B}(Y_1)} \mathcal{M}_{W_{12,\alpha_{12}}}$$

compatible with the restriction maps where W_{02} is the connected sum of W_i along Y_1.

Proof. The proof given in [KM07] does not seem to adapt to our situation since the argument uses a gauge transformation to put the connection in temporal gauge that would take us to the space of L^2 connections which we would like to avoid.

The map $D : \tilde{\mathcal{M}}_{W_{02}} = \tilde{\mathcal{M}}_{W_{01,\alpha_{01}}} \times_{\mathfrak{B}(Y_1)} \tilde{\mathcal{M}}_{W_{12,\alpha_{12}}}$ is defined as follows. Given $\gamma \in \tilde{\mathcal{M}}_{W_{02}}$ let γ_1, γ_2 denote the restrictions to $\mathcal{C}(W_{01})$ and $\mathcal{C}(W_{12})$. Since $\gamma \in \tilde{\mathcal{M}}_{W_{02}}$ the γ_i are already in Coulomb gauge. Let $d_{Y_1}(\rho)$ be the exact part of the restriction of γ to Y_1. On the pieces W_{01} and W_{12} we solve the Dirichlet problem for

$$\Delta \tilde{f}_{01} = 0$$

$$\tilde{f}_{01|Y_1} = \rho$$

$$\tilde{f}_{01|Y_0} = 0$$

Note that ρ is only well defined up to a constant. Repeat this for $\tilde{f}_{12}$. We pick the unique ρ

so that $\langle df_{01}, d\alpha_{01}\rangle_{W_{01}} = \langle df_{12}, d\alpha_{12}\rangle_{W_{12}} = 0$. Note that

$$\langle df_{12}, d\alpha_{12}\rangle_{W_{12}} = -\langle df_{01}, d\alpha_{01}\rangle_{W_{01}} = \int_{Y_1} \rho \partial \alpha_{02}/\partial \nu$$

This defines gauge transformations g_{01} and g_{12} which take γ_i to strong Coulomb gauge. Note that D involves multiplication $L^2_2 \times L^2_1 \to L^2_1$ since a priori ρ is only L^2_2. However, by interior regularity estimates discussed in the analysis appendix, the boundary value ρ is in fact in L^2_k for all k with the norms bounded by the L^2_1 norm of γ on W_{02}. This implies that D is smooth.

We show that D is bijective. Let us describe the inverse. Given γ_{01} and γ_{12} with the same boundary value in $\mathfrak{B}(Y_1)$ we would like to gauge transform them to agree on Y_1 and still be in strong Coulomb gauge with respect to the boundaries Y_0 and Y_2. Identify a neighborhood of Y_1 in W_{02} with $[-\epsilon, \epsilon] \times Y$. Let $h_{01}dt$ and $h_{12}dt$ be the dt parts of the 1-forms obtained by restricting γ_{01} and γ_{12} to the boundary Y_1. Given harmonic functions f_{01} and f_{12} with

$$\begin{aligned} f_{01|Y_1} &= f_{12|Y_1} \\ \partial_t f_{01} + h_{01} &= \partial_t f_{12} + h_{12} \\ f_{01|Y_0} &= f_{12|Y_2} = 0 \end{aligned}$$

the gauge transformations $e^{f_{01}}$ and $e^{f_{12}}$ are the ones we seek.
Let us check that $\int_{Y_0} \partial f_{01}/\partial \nu = 0$.
It remains to solve this boundary value problem. We first find local solutions on $[-\epsilon, 0] \times Y_1$ and $[0, \epsilon] \times Y_1$. The Laplacian on functions is $-\partial_t^2 + \Delta_{Y_1}$. Let ϕ_λ be an eigenbasis for Δ_{Y_1}. If

$$h_{12} - h_{01} = \sum_\lambda c_\lambda \phi_\lambda$$

let

$$f'_{01} = \sum_\lambda c_\lambda \lambda^{-1/2} e^{\lambda^{1/2} t} \phi_\lambda / 2$$

and

$$f'_{12} = \sum_\lambda c_\lambda \lambda^{-1/2} e^{-\lambda^{1/2} t} \phi_\lambda / 2$$

Clearly,

$$\partial_t f'_{01} - \partial_t f'_{12} = h_{12} - h_{01}$$

and

$$f'_{01|Y_1} = f'_{12|Y_1} = \sum_\lambda c_\lambda \lambda^{-1/2} \phi_\lambda / 2$$

Given these local solutions use cutoffs supported away from Y_1 to obtain functions $\rho_0 f'_{01}$ and $\rho_1 f'_{12}$ with the right boundary conditions but no longer in Coulomb gauge. By construction $e^{\rho_0 f'_{01}} \cdot \gamma_{01}$ and $e^{\rho_1 f'_{12}} \cdot \gamma_{12}$ agree on Y_1 and have the right boundary conditions at Y_0 and Y_2 so we can glue them to an L_1^2 solution on W_{02}. Finally, put this solution into strong Coulomb gauge by solving a Dirichlet problem with vanishing on the boundary $\partial(W_{02}) = Y_0 \sqcup Y_2$ To establish surjectivity note that the harmonic functions f_{01} and f_{12} are unique. Indeed, given another such pair f_{01}, and f_{12}, we have

$$f_{01} - f'_{01} = f_{12} - f'_{12}$$

and

$$\partial_t f_{01} - \partial_t f'_{01} = \partial_t f_{12} - \partial_t f'_{12}$$

This implies we can glue the functions $f_{01} - f'_{01}$ and $f_{12} - f'_{12}$ to obtain an L_2^2 function F on W_{02} with $\Delta F = 0$ and $F_{|\partial W} = 0$. This implies $F = 0$ as desired. □

The homotopy axiom for correspondences implies:

Theorem 10. *Consider $R_t : (W, g_t, \mathfrak{s}_t) \to \mathfrak{B}_0 \times \mathfrak{B}_1$ where g_t is a smooth family of metrics and $\mathfrak{s}_t$ a family of spin structures for $t \in [0,1]$ on a cobordism W all standard near the boundary. We have:*

$$\Omega F_*(R_0) = \Omega F_*(R_1)$$

Proof. For a given cycle $\sigma : P \to \mathfrak{B}_0$ the chain homotopy between $\sigma \times_{\mathfrak{B}_0} R_0$ and $\sigma \times_{\mathfrak{B}_0} R_1$ is provided by $\cup_{t \in [0,1]} \sigma \times_{\mathfrak{B}_0} R_t$ □

Remark. In case of a trivial cobordism $W_{02} = Y \times [0,T]$, α_{02} is given by t/T. Therefore, we have

$$\mathcal{M}_{W_{02},\alpha_{02}} = \mathcal{M}_{W_{02}}$$

$$\mathcal{M}_{W_{01},\alpha_{01}} = \mathcal{M}_{W_{01}}$$

$$\mathcal{M}_{W_{12},\alpha_{12}} = \mathcal{M}_{W_{12}}$$

Therefore, in this case, the gluing identification

$$\mathcal{M}_{W_{01}} \times_{\mathfrak{B}(Y_1)} \mathcal{M}_{W_{12}} = \mathcal{M}_{W_{02}}$$

works not only up to homotopy but exactly.

3.6 Homology Operations

Observe that Floer spaces have products:

Lemma 9. *Given a finite collection $\{(\mathfrak{B}_i, T^+\mathfrak{B}_i, T^-\mathfrak{B}_i, \mathcal{L}_i)\}$ of Floer spaces, the product*

$$(\sqcap_i \mathfrak{B}, \oplus_i T^+\mathfrak{B}_i, \oplus_i T^-\mathfrak{B}_i, \sum_i \mathcal{L}_i)$$

is a Floer space.

It follows immediately from the axioms that:

Theorem 11. *Given chains, $\sigma_i \in \mathfrak{B}_i$, the product $\sqcap_i \sigma_i$ is a chain in $\sqcap_i \mathfrak{B}_i$. Therefore, we have a chain map $\bigotimes_i CF_*(\mathfrak{B}_i) \to CF_*(\sqcap_i \mathfrak{B}_i)$ that gives rise to a map $\bigotimes_i HF_*(\mathfrak{B}_i) \to HF_*(\sqcup_i \mathfrak{B}_i)$*

Turning now to 3-manifolds we have:

Theorem 12. *There is a natural map $\bigotimes_i \widehat{HF}_*(Y_i, \mathfrak{s}_i) \to \widehat{HF}_*(\sqcup_i Y_i, \sqcup_i \mathfrak{s}_i)$ defined on the chain level.*

Theorem 13. *A cobordism $(W, \mathfrak{s})$ with $\partial W = \sqcup_i Y_i$ gives rise to a cycle in $\mathfrak{B}(\sqcup_i (Y_i, \mathfrak{s}_{Y_i}), \mathcal{L}_i)$*

Proof. This is really no different than the proof in the case of one boundary component. $\square$

Remark. Cobordisms give rise to moduli spaces that can alternatively be viewed as cycles in products or correspondences. Note that a general correspondence does not give rise to a cycle in the product. For this its sufficient to consider the diagonal correspondence in $\mathfrak{B}(Y) \times \mathfrak{B}(Y)$ and note that $\mathcal{L} = 0$ but we do not have weak compactness of the image.

We have:

Theorem 14. *Given a cobordism $(W_k, \mathfrak{s})$ with k incoming 3-manifolds Y_i^- and one outgoing 3-manifold Y^+ we have a (graded) homomorpism*

$$\widehat{HF}(W_k, \mathfrak{s}) : \bigotimes_{i=1}^{k} \widehat{HF}_*(Y_i^-, \mathfrak{s}_{|Y_i^-}) \to \widehat{HF}_*(Y^+, \mathfrak{s}_{|Y^+})$$

3.7 The Trivial Cobordism

In this section we prove that the trivial cobordism $I \times Y$ induces the identity map on $\Omega F(Y)$. The intuition behind the proof is the observation that a trivial cobordism is the analogue of a gradient flow in finite dimensions and letting the flow time shrink to zero induces the identity map on the underlying manifold.

We collect together the analytic tools we need to understand solutions to the SW equations on a short cylinder. This will be used in establishing that the trivial cobordism induced the identity map on ΩF.

3.7.1 APS Boundary Value Problem

Consider $D = \partial_t + L$ where L is a first order elliptic differential operator acting an some vector bundle E over Y. We have the APS boundary value problem:

$$(D_\epsilon, \Pi_L^- \circ r_\epsilon - \Pi_L^+ \circ r_0) : L_1^2([0, \epsilon] \times Y, E) \to L^2([0, \epsilon] \times Y, E) \oplus L_{1/2}^2(Y, E)$$

Thus, given $\alpha \in L^2_1([0,\epsilon]\times Y, E)$ the boundary data is specified by $\beta = \beta_0^+ + \beta_\epsilon^-$ with $\beta_0^+ = -\Pi_L^+ \circ r_0(\alpha)$ and $\beta_\epsilon^- = \Pi_L^- \circ r_\epsilon(\alpha)$ where r denotes restriction. We have the following lemma:

Lemma 10. *D_ϵ is an isomorphism with inverse $P_\epsilon \oplus Q_\epsilon$ where*

$$P_\epsilon \oplus Q_\epsilon : L^2([0,\epsilon]\times Y, E) \oplus L^2_{1/2}(Y,E) \to L^2_1([0,\epsilon]\times Y, E)$$

There exists $C > 0$ such that $||P_\epsilon|| \leq C$ and $||Q_\epsilon|| \leq C$, independent of ϵ.
We have

$$||P_\epsilon(a)_{|\partial([0,\epsilon]\times Y)}||_{L^2_{1/2}} \leq C||a||_{L^2}$$

and

$$||r \circ Q_\epsilon(b) - b||^2_{L^2_{1/2}} \leq 2\int_{[0,\epsilon]\times Y} |\partial_t Q_\epsilon(b)|^2$$

Proof. P_ϵ: Take $\{\phi_\lambda\}$ an orthonormal eigenbasis for $L^2(Y,E)$. We may write any element in $L^2([0,\epsilon]\times Y, E)$ as a sum $\sum_\lambda g_\lambda(t)\phi_\lambda$ with $g_\lambda \in L^2$. For $\lambda > 0$ define

$$P_\epsilon(g_\lambda\phi_\lambda) = \phi_\lambda \int_0^t e^{-\lambda(t-\tau)} g_\lambda(\tau)d\tau = h(t)\phi_\lambda$$

We compute the L^2_1 norm of $h(t)\phi_\lambda$. Since $\lambda > 0$ is bounded below (independently of ϵ) we can use $\lambda^2\int_0^\epsilon |h|^2 + \int_0^\epsilon |\partial_t h|^2$ to compute the square of the L^2_1 norm of $h(t)\phi_\lambda$. We have

$$\int_{[0,\epsilon]\times Y} |D_\epsilon(h\phi_\lambda)|^2 = \lambda^2\int_0^\epsilon |h|^2 + \int_0^\epsilon |\partial_t h|^2 + \lambda|h(\epsilon)|^2$$

This bounds the L^2_1 norm of $h(t)\phi_\lambda$ as well as the $L^2_{1/2}$ norm of $h(\epsilon)\phi_\lambda$ in terms of $||g\phi_\lambda||_{L^2}$.

Q_ϵ: We have an explicit formula for the inverse:

$$Q_\epsilon(\phi_\lambda) = -e^{-t\lambda}\phi_\lambda, \lambda > 0$$
$$Q_\epsilon(\phi_\lambda) = e^{-(t-\epsilon)\lambda}\phi_\lambda, \lambda \leq 0$$

Lets consider the case $\lambda > 0$. We have $D_\epsilon \circ Q_\epsilon = 0$. The L^2-norm of $Q_\epsilon(\phi_\lambda)$ is bounded by 1 since $e^{-t\lambda} \leq 1$. We have $\partial_t(Q_\epsilon(\phi_\lambda)) = -\lambda Q_\epsilon(\phi_\lambda)$ thus,

$$\int_{[0,\epsilon]\times Y} |\partial_t(Q_\epsilon(\phi_\lambda))|^2 = \frac{\lambda(1-e^{-2\lambda\epsilon})}{2}$$

Since $\partial_t \circ Q_\epsilon = L \circ Q_\epsilon$ and the $L^2_{1/2}$-norm of ϕ_λ is $|\lambda|^{1/2}$ this establishes the desired bound. Finally, note that

$$||\phi_\lambda - Q_\epsilon(\phi_\lambda)_{|\{\epsilon\}\times Y}||^2_{L^2_{1/2}} = \lambda(1-e^{-\epsilon\lambda})^2 \leq \lambda(1-e^{-2\epsilon\lambda}) = 2\int |\partial_t(Q_\epsilon(\phi_\lambda))|^2$$

$\square$

In applications, we will have a nonlinear term that is quadratic. We need the following lemmas:

Lemma 11. *Given $\phi \in L^2_1([0,\epsilon] \times Y, E)$, vanishing on one of the ends, we have $||\phi||_{L^4} \leq C||\phi||_{L^2_1}$ with C independent of ϵ.*

Proof. For ϕ on $[0,1]^3 \times [0,\epsilon]$ vanishing on $[0,1]^\epsilon \times \{0\}$ and $\partial[0,1]^3 \times [0,1]$, it is proven (see for example [Tay97]) that

$$||\phi||_{L^4} \leq C\{\Pi_{i=1}^{i=4}||\nabla_i \phi||_{L^2}\}^{1/4}$$

Here C does not depend on ϵ. Now, cover Y by open sets $\{U_\mu\}$ so that we are reduced to the standard situation in each patch. Take a partition of unity $\{\rho_\mu\}$ subordinate to this cover. Note that

$$||\rho_\mu \phi||_{L^4} \leq C\{\Pi_{i=1}^{i=4}(||\rho_\mu \nabla_i \phi||_{L^2} + ||\phi \nabla_i \rho_\mu||_{L^2})\}^{1/4}$$

$\square$

Lemma 12. *Given $\beta \in L^2_{1/2}(Y, E)$ and $v \in L^2([0,\epsilon] \times Y, E)$ we have*

$$||Q_\epsilon(\beta) + P_\epsilon(v)||_{L^4} \leq C||\beta||_{L^2_{1/2}} + C||v||_{L^2}$$

Proof. Decompose $v = v_+ + v_-$ into positive (nonpositive) eigenvectors of L. We have

$$||P_\epsilon(v)||_{L^4} \leq ||P_\epsilon(v_-)||_{L^4} + ||P_\epsilon(v_+)||_{L^4}$$

By construction, each of these two terms vanishes on an end of the cylinder. Thus, since $||P_\epsilon(v_-)||_{L^2_1} \leq C||v||_{L^2}$ and $||P_\epsilon(v_+)||_{L^2_1} \leq C||v||_{L^2}$ the conclusion holds for the v term.

For $Q_\epsilon(\beta)$ we need to investigate terms of 3 types. Let $\beta = \beta_- + \beta_+ + \beta_0$ where the decomposition corresponds to breaking up b into the positive, negative and zero eigenspaces. For β_0, we have $Q_\epsilon(\beta_0) = \beta_0$ Since there are only finitely many eigenvectors of L with zero eigenvalue, we can bound the L^4-norm of $Q_\epsilon(\beta_0)$ by $C||\beta_0||_{L^2_{1/2}}$. For β_+ we note that although $Q_\epsilon(\beta_+) = -\sum_\lambda e^{-t\lambda} c_\lambda \phi_\lambda$ does not vanish on the endpoints it extends to an L^2_1 function on $[0,\infty] \times Y$. In fact, by the calculation on the first lemma, the L^2_1 norm of the extension is bounded by the $L^2_{1/2}$ norm of β_+. The lemma above applies since this extension vanishes at ∞. The argument for β_- is similar. $\square$

Lemma 13. *$||Q_\epsilon||_{(L^2_{1/2}:L^4)}$ is uniformly bounded in ϵ and approaches 0 weakly as $\epsilon \to 0$.*

Proof. Choose any $\delta > 0$. We have $\beta = \sum_{i=1}^k c_i \phi_{\lambda_i} + \beta'$ where $||\beta'||_{L^2_{1/2}} < \delta/C$. As $\epsilon \to 0$, we have $||Q_\epsilon(\phi_{\lambda_i})||_{L^4} \to 0$ since the C^0 norm of $Q_\epsilon(\phi_{\lambda_i})$ is bounded by that of ϕ_{λ_i} the length of the cylinder is going to zero. Thus $||Q_\epsilon(\beta)||_{L^4} \leq \sum_{i=1}^k ||c_i Q_\epsilon(\phi_{\lambda_i})||_{L^4} + \delta$ and we can choose ϵ so small that $\sum_{i=1}^k ||c_i Q_\epsilon(\phi_{\lambda_i})||_{L^4} \leq \delta$ $\square$

Given $\beta \in L^2_{1/2}(Y, E)$, consider the map

$$F^\epsilon(\beta,) : L^2([0, \epsilon] \times Y, E) \to L^2([0, \epsilon] \times Y, E)$$

defined by

$$F^\epsilon(\beta, v) = -(P_\epsilon(v) + Q_\epsilon(\beta))\sharp(P_\epsilon(v) + Q_\epsilon(\beta))$$

where $\sharp$ is some bilinear operation. We investigate when $F^\epsilon(\beta,)$ is a contraction mapping.

Lemma 14. *Fix $\beta \in L^2_{1/2}(Y, E)$. There exists a $C > 0$ such that*

$$|F^\epsilon(\beta, v) - F^\epsilon(\beta, v')| \leq C(|Q_\epsilon(\beta)|_{L^4} + |v| + |v'|) \cdot |v - v'|$$

and

$$|F^\epsilon(\beta, 0)| \leq C|Q_\epsilon(\beta)|^2_{L^4}$$

Thus for ϵ so small that $Q_\epsilon(\beta) < 1/8C$, $F^\epsilon(\beta,)$ is a contraction mapping on a ball of radius $1/8C$.

Proof. If $a = P_\epsilon(v) + Q_\epsilon(\beta)$, $a' = P_\epsilon(v') + Q_\epsilon(\beta)$, we have $a - a' = P_\epsilon(v - v')$.

$$|F^\epsilon(\beta, v) - F^\epsilon(\beta, v')| = |a\sharp a - a'\sharp a'| = |a\sharp a - a\sharp a' + a\sharp a' - a'\sharp a'| \leq$$
$$|a\sharp P_\epsilon(v - v')| + |P_\epsilon(v - v')\sharp a'| \leq C(|Q_\epsilon(\beta)|_{L^4} + |v| + |v'|) \cdot |v - v'|$$

We see that $F^\epsilon(\beta,)$ is a contraction mapping with contraction constant less then $3/8$ so to ensure $F^\epsilon(\beta,)$ maps the ball of radius $1/8C$ to itself we need only check that $F^\epsilon(\beta, 0) < 1/8C$ which follows from our assumptions. □

Slightly more generally, we may consider $F^\epsilon(\beta, v) = g_\epsilon - (P_\epsilon(v) + Q_\epsilon(\beta))\sharp(P_\epsilon(v) + Q_\epsilon(\beta))$ where $g_\epsilon \in L^2([0, \epsilon] \times Y, E)$. $F^\epsilon(\beta,)$ is contraction mapping with the same constants. To ensure $F^\epsilon(\beta,)$ maps a ball of radius $1/8C$ to itself we must suppose that g is sufficiently small.

Consider the map $G^\epsilon(\beta, v) = (\beta, v - F^\epsilon(\beta, v))$. For ϵ small, the previous lemma allows us to conclude that the existence of an inverse $H^\epsilon(\beta, v)$ with $G^\epsilon(\beta, H^\epsilon(\beta, v)) = (\beta, v)$. Furthermore, $H^\epsilon(\beta, 0) \to 0$ as $\epsilon \to 0$. We would like to conclude the same for the derivative:

Lemma 15. *Let $D_1 H^\epsilon_{(\beta,0)}$ be the derivative with respect to the β variable at the point $(\beta, 0)$. We have $|D_1 H^\epsilon_{(\beta,0)}| \to 0$ as $\epsilon \to 0$.*

Proof. Pick $b \in T_\beta L^2_{1/2}([0, \epsilon] \times Y, E)$. $F^\epsilon(\beta, H^\epsilon(\beta, 0)) = H^\epsilon(\beta, 0)$ implies

$$D_1 F^\epsilon_{(\beta, H^\epsilon(\beta,0))}(b) + D_2 F^\epsilon_{(\beta, H^\epsilon(\beta,0))}(D_1 H^\epsilon_{(\beta,0)}(b)) = D_1 H^\epsilon_{(\beta,0)}(b)$$

The desired result will follow if we can estimate the LHS. From the definition,

$$D_1 F^\epsilon_{\beta,v}(b) = -(Q_\epsilon(b))\sharp(P_\epsilon(v) + Q_\epsilon(\beta)) - (P_\epsilon(v) + Q_\epsilon(\beta))\sharp(Q_\epsilon(b))$$

and

$$D_2 F^{\epsilon}_{\beta,v}(w) = -(P_\epsilon(w))\sharp(P_\epsilon(v)+Q_\epsilon(\beta)) - (P_\epsilon(v)+Q_\epsilon(\beta))\sharp(P_\epsilon(w))$$

Plugging in $v = H^{\epsilon}(\beta,0)$ and $w = D_1 H^{\epsilon}(\beta,0)$ and using that $Q_\epsilon(\beta)$ and $P_\epsilon(H^{\epsilon}(\beta,0)) \to 0$ as $\epsilon \to 0$ the conclusion follows. □

3.7.2 Strong Coulomb Boundary Conditions

Our first task is to rewrite the SW equations together with the Coulomb gauge condition as a perturbation of a linear equation described above. Fixing a base connection A_0 we may identify pairs (A,Φ) with

$$(b(t), a(t), c(t)dt, \Psi(t))$$

where $b(t)$ co-closed and $a(t)$ exact. We rewrite the SW equations together with the gauge fixing condition as $\partial_t\gamma + L\gamma + \gamma\sharp\gamma = g$ where $\sharp$ is some bilinear operation on γ, g is some section that does not depend on γ and

$$L = \begin{pmatrix} 0 & -d_Y & 0 & 0 \\ -d_Y^* & 0 & 0 & 0 \\ 0 & 0 & *_3 d_Y & 0 \\ 0 & 0 & 0 & D_{B_0} \end{pmatrix}$$

Let $D = \partial_t + L$. D splits as $\partial_t + L_0 \oplus L_1$ on

$$(i\Lambda^0(Y) \oplus \mathrm{im}(d)) \oplus (\ker(d^*) \oplus S)$$

with

$$L_0 = \begin{pmatrix} 0 & -d_Y \\ -d_Y^* & 0 \end{pmatrix} \qquad L_1 = \begin{pmatrix} *_3 d & 0 \\ 0 & D_{B_0} \end{pmatrix}$$

We have

$$(D, \Pi^-_{*d\oplus D_{B_0}} - \Pi^+_{*d\oplus D_{B_0}}) : \mathcal{D} \oplus (\ker(d^*) \oplus S) \to i\Lambda^1(Y) \oplus L^2_{1/2}(Y, \oplus(\ker(d^*) \oplus S))$$

where $\mathcal{D} = (c(t), a(t)) \in \Lambda^0(Y) \oplus \Lambda^1(Y)$ with $\partial_t c = d^* a$ and $a(y,0) = a(y,\epsilon) = 0$. Thus, $\mathcal{D}$ consists of the elements in strong Coulomb gauge on $[0,\epsilon] \times Y$. In the previous section we examined L_1 with spectral boundary conditions. We need a similar argument to deal with L_0 with strong Coulomb boundary conditions.

Lemma 16. *L_0 is an isomorphism with uniformly bounded inverse P^0_ϵ.*

Proof. Let $\{\phi_\lambda\}$ be an orthonormal eigenbasis of $d_Y^* d_Y$ on $\Lambda^0(Y)$. We have $(c,a) = \sum_\lambda c_\lambda(t)\phi_\lambda dt + \sum_\lambda a_\lambda(t) d_Y(\phi_\lambda)$. The strong Coulomb condition implies $a_\lambda(0) = a_\lambda(\epsilon) = 0$ and $\dot{c_\lambda} = \lambda a_\lambda$. We have

$$\int |\partial_t(c_\lambda, a_\lambda) + L_0(c_\lambda, a_\lambda)|^2 = \int |(\dot{a}_\lambda - c_\lambda) d_Y(\phi_\lambda)|^2$$

Expanding this and using that $\int_Y |d_Y\phi_\lambda|^2 = \lambda$ we get

$$\int_0^\epsilon \lambda|\dot{a}_\lambda|^2 + \lambda|c_\lambda|^2 - \lambda\langle\dot{a}_\lambda, c_\lambda\rangle - \lambda\langle c_\lambda, \dot{a}_\lambda\rangle$$

Use the strong Coulomb condition to rewrite this as

$$\int_0^\epsilon \lambda|\dot{a}_\lambda|^2 + \lambda|c_\lambda|^2 + 2\lambda^2|a_\lambda|^2$$

Observe that the L_1^2 norm of $c_\lambda(t)\phi_\lambda dt + a_\lambda(t)d_Y(\phi_\lambda)$ is

$$\int_0^\epsilon |c_\lambda|^2 + |\dot{c}_\lambda|^2 + \lambda|a_\lambda|^2 + \lambda|\dot{a}_\lambda|^2 + \lambda^2|a_\lambda|^2 = \int_0^\epsilon |c_\lambda|^2 + \lambda|a_\lambda|^2 + \lambda|\dot{a}_\lambda|^2 + 2\lambda^2|a_\lambda|^2$$

Thus, the norm of the image uniformly bounds the norm of (c, a). □

As before, we are interested in obtaining L^4 bounds from the given L_1^2 bounds. Since a vanishes on the endpoints the desired bound follows. In view of lemma 12 it suffices to bound the boundary values of c.

Lemma 17. *We have* $||P_\epsilon^0(v)_{|\partial([0,\epsilon]\times Y)}||_{L^2_{1/2}} \leq \epsilon^{1/2}C||v||_{L^2}$.

Proof. Let $P_\epsilon^{0,spec}$ be the spectral inverse as in the section above. We have good control of the boundary values of $P_\epsilon^{0,spec}(v)$. Thus, we must verify that once we put this solution in strong Coulomb gauge the boundary values change in a controlled way. Fix a λ. We have

$$P_\epsilon^{0,spec}(g(t)\phi_\lambda, g(t)\lambda^{-1/2}d_Y(\phi_\lambda)) = \int_0^t e^{\lambda^{1/2}(\tau - t)}g(\tau)d\tau(\phi_\lambda, \lambda^{-1/2}d_Y(\phi_\lambda))$$

and

$$P_\epsilon^{0,spec}(g(t)\phi_\lambda, -g(t)\lambda^{-1/2}d_Y(\phi_\lambda)) = -\int_t^\epsilon e^{\lambda^{1/2}(t-\tau)}g(\tau)d\tau(\phi_\lambda, -\lambda^{-1/2}d_Y(\phi_\lambda))$$

The boundary values of $P_\epsilon^{0,spec}(2g(t)\phi_\lambda)$ projected to ϕ_λ, are

$$a_1 = \int_0^\epsilon e^{\lambda^{1/2}(\tau-\epsilon)}g(\tau)d\tau$$

and

$$a_0 = -\int_0^\epsilon e^{\lambda^{1/2}(-\tau)}g(\tau)d\tau$$

Since the $P_\epsilon^{0,spec}(v)$ already satisfies the Coulomb condition we modify $P_\epsilon^{0,spec}(v)$ by $d_{[0,\epsilon]\times Y}f\phi_\lambda$ where f is a harmonic function on $[0,\epsilon]\times Y$. We seek a function of the form $(c_0e^{\lambda^{1/2}t} + c_1e^{-\lambda^{1/2}t})\phi_\lambda$ with boundary conditions

$$c_0 + c_1 = a_0$$

and

$$c_0 e^{\lambda^{1/2}\epsilon} + c_1 e^{-\lambda^{1/2}\epsilon} = a_1$$

since

$$d_Y((c_0 e^{\lambda^{1/2}t} + c_1 e^{-\lambda^{1/2}t})\phi_\lambda) = (c_0 e^{\lambda^{1/2}t} + c_1 e^{-\lambda^{1/2}t}) d_Y \phi_\lambda$$

We must bound the boundary values of $\partial_t (c_0 e^{\lambda^{1/2}t} + c_1 e^{-\lambda^{1/2}t})\phi_\lambda$. They are

$$\lambda^{1/2}(c_0 - c_1)$$

and

$$\lambda^{1/2}(c_0 e^{\lambda^{1/2}\epsilon} - c_1 e^{-\lambda^{1/2}\epsilon})$$

We have

$$c_0 = \frac{a_0 - a_1 e^{\epsilon\lambda^{1/2}}}{(1 - e^{2\epsilon\lambda^{1/2}})} = \frac{\int_0^\epsilon (e^{-\lambda^{1/2}\tau} - e^{\lambda^{1/2}\tau}) g(\tau) d\tau}{e^{\epsilon\lambda^{1/2}}(e^{-\epsilon\lambda^{1/2}} - e^{\epsilon\lambda^{1/2}})} = \frac{\int_0^\epsilon \sinh(\lambda^{1/2}\tau) g(\tau) d\tau}{e^{\epsilon\lambda^{1/2}} \sinh(\lambda^{1/2}\epsilon)}$$

Thus $|c_0| \leq \epsilon^{1/2} e^{-\epsilon\lambda^{1/2}} ||g||_{L^2}$. Since $c_1 = a_0 - c_0$ and $|a_0| \leq \epsilon^{1/2} ||g||_{L^2}$ the desired bounds follow. □

3.7.3 Adding the Collar

Lemma 18. *Given $b \in \mathfrak{B}(Y)$, locally decompose b as $b_0^+ + b_\epsilon^-$. There exists sufficiently small $\epsilon > 0$ such that there is a unique small energy solution γ to the SW equations in strong Coulomb gauge with $b_0^+ + b_\epsilon^-$ as the mixed boundary value.*

Proof. This follows immediately from the arguments of the previous subsections. Indeed, let $P_\epsilon = P_\epsilon^0 \oplus P_\epsilon^+$ as above. If v is the unique small fixed point of the map

$$v \mapsto g - (Q_\epsilon(b) + P_\epsilon(v))\sharp(Q_\epsilon(b) + P_\epsilon(v))$$

then

$$D(P_\epsilon(v) + Q_\epsilon(b)) + (Q_\epsilon(b) + P_\epsilon(v))\sharp(Q_\epsilon(b) + P_\epsilon(v)) = g$$

Note that since g is some fixed section with L_1^2 norm and L^4 norm going to zero as $\epsilon \to 0$ the contraction lemma applies. □

We complete $\cup_{t\in(0,1]}\mathcal{M}_t$ to form a lc-manifold $\cup_{t\in[0,1]}\mathcal{M}_t$ as follows. The 0th stratum is $\cup_{t\in(0,1)}\mathcal{M}_t$ while the 1st stratum is $\mathcal{M}_1 \coprod \Delta$. A sequence $z_i \in \cup_{t\in(0,1)}\mathcal{M}_t$ is said to converge to $z \in \Delta$ if $\mathcal{R}(z_i)$ converges in $L_{1/2}^2$ to z. Lemma 18 implies that the completed space has the structure of an lc-manifold. In fact, the completed manifold is smooth, but the extension of the restriction map to the diagonal is not smooth. We will see how to deal with this in a later section.

3.7.4 Verifying the Axioms

In this section we verify that the completed correspondence satisfies the first 3 axioms of a correspondence. This ensures that for a chain σ, we will have that $\sigma \times_{\mathfrak{B}_0} \cup_{t\in[0,1]}\mathcal{M}_t$ satisfies the first 3 axioms of a chain. In other words, those axioms that deal with the convergence properties. The existence of a lc-structure on $\sigma \times_{\mathfrak{B}_0} \cup_{t\in[0,1]}\mathcal{M}_t$ will be handled in a separate section. Note, that it is possible to modify the definition of a correspondence so that $\cup_{t\in[0,1]}\mathcal{M}_t$ is a genuine correspondence and thus $\sigma \times_{\mathfrak{B}_0} \cup_{t\in[0,1]}\mathcal{M}_t$ automatically has such an lc-structure. I choose to avoid this more general definition since it seems to obscure matters and will not be used in the future.
We will first deduce a uniform L_1^2 bound on configurations. Suppose for this section that $\mathcal{E}(\gamma_i)$ is bounded and $\mathcal{R}_0(\gamma_i)$ weakly converges (this hypothesis is satisfied in all of the axioms we need to check). We may also assume that we have a sequence of solutions γ_i on cylinders of shrinking length as all the other cases have been covered. Energy bounds give us uniform bounds on

$$\int_{[0,\epsilon_i]\times Y} |F_{A_i^T}|^2 + |\nabla_{A_i}\Phi_i|^2 + |\Phi_i|^4$$

We use the following lemma:

Lemma 19. *There exists a $C > 0$ such that given $Y \times S^1_\epsilon$ where Y is some Riemannian 3-manifold, S^1_ϵ is a circle of length ϵ and $\alpha \in \Lambda^1(Y \times S^1_\epsilon)$ with $\langle \alpha, dt\rangle_{Y\times S^1_\epsilon} = 0$ we have:*

$$||\alpha||^2_{L^2_1} \leq C(||d^*\alpha||^2 + ||d\alpha||^2 + M(\alpha))$$

where $M(\alpha) = \max_h\{|\int\langle\alpha, h\rangle|^2\}$ and h ranges over harmonic forms of unit L^2-norm on Y.

Proof. Let e_i, g_j, and f_k be orthonormal eigenbasis for Laplacians on $\Lambda^0(S^1_\epsilon)$, $\Lambda^0(Y)$, $\Lambda^1(Y)$ respectively. The zero eigenspaces are spanned by $1_{S^1_\epsilon}$, 1_Y, and $f_1, \ldots f_{b^1}$ which are harmonic 1-forms on Y. Note that first nonzero eigenvalue of $\Lambda^0(S^1_\epsilon)$ is $4\pi^2/\epsilon^2$ which increases as $\epsilon \to 0$. Any $\alpha \in \Lambda^1(Y \times S^1_\epsilon)$ may be written as

$$\sum_{i,j} c_{ij} e_i \cdot g_j dt + \sum_{i,k} d_{ij} e_i \cdot f_k$$

The condition $\langle\alpha, dt\rangle = 0$ implies $d_{00} = 0$. Since

$$\langle d^*\alpha, d^*\alpha\rangle + \langle d\alpha, d\alpha\rangle = \langle\alpha, \Delta\alpha\rangle$$

we have

$$|\alpha|^2 = \sum_{i,j} |c_{ij}|^2 + \sum_{i,k} |d_{ik}|^2$$

and

$$\langle\alpha, \Delta\alpha\rangle = \sum_{i,j}(\lambda_i + \lambda_j)|c_{ij}|^2 + \sum_{i,k}(\lambda_i + \lambda_k)|d_{ik}|^2$$

The desired conclusion follows from the fact that the only terms we need to worry about are those spanned by harmonic 1-forms on Y. $\square$

Corollary 1. *$||A_i||_{L^2_1}$ is uniformly bounded.*

Proof. Write $A_i = c_i dt + b_i + a_i$ where b_i coclosed and a_i is exact. We extend A_i to the double $[0, 2\epsilon] \times Y$ by declaring $c_i(t + \epsilon, y) = c_i(-t + \epsilon, y)$, $a_i(t + \epsilon, y) = -a_i(-t + \epsilon, y)$ and $b_i(t+\epsilon, y) = b_i(-t+\epsilon, y)$. Note the extension is still in Coulomb gauge. Note that $d^*(A_i - A_0)$ and $||dA_i||_{L^2}$ are uniformly bounded. By the definition of the moduli space, $\int_{Y\times[0,\epsilon]} c_i = 0$ which implies $\int_{Y\times[0,2\epsilon]} c_i = 0$ for the double. We can identify the functions on the ends to obtain functions on $Y \times S^1$ and therefore, apply the lemma above. $\square$

Lemma 20. *We have a uniform bound on $||\mathcal{R}_\epsilon(\gamma_i)||_{L^2_{1/2}}$. Assume, $\mathcal{R}_0(\gamma_i)$ is $L^2_{1/2}$ convergent and $\mathcal{E}(\gamma_i) \to 0$. We have $\mathcal{R}_\epsilon(\gamma_i)$ $L^2_{1/2}$-convergent and γ_i L^2_1-convergent.*

Proof. Lets consider $\mathcal{R}_\epsilon(A_i)$ first. As discussed, given the bound $\mathcal{R}_0(A_i)$, bounding $\mathcal{R}_\epsilon(A_i)$ amounts to estimating $||\dot{Q}_\epsilon(b_i)||^2$ where $A_i = P_\epsilon(v_i) + Q_\epsilon(b_i)$. Since $D(A_i) = F^+_{A_i^T} - \rho^{-1}((\Phi_i \otimes \Phi_i^*)_0) = v_i$ we have uniform bounds on $P_\epsilon(v_i)$. This, together with the L^2_1 bound on A_i implies the bound on $||\dot{Q}_\epsilon(b_i)||^2$.

Now, we deal with Φ_i. Note, we have uniform L^4 bounds on A_i. Indeed, by previous sections, this follows from the established bound on the restriction of A_i to the endpoints together with the L^2_1 bound on A_i. By assumption, $\mathcal{R}_0\Phi_i = \Phi_i(0)$ is bounded. With this in place, we can repeat the argument above with Φ_i since in view of the L^4 bounds on A_i we have bounds on $D_{A_0}(\Phi_i) = -\rho(A_i)\Phi_i$ and $||\nabla_{A_0}\Phi||$. This establishes the weak compactness claim.

For strong compactness, we assume $\mathcal{E}(\gamma_i) \to 0$ and $\mathcal{R}_0(\gamma_i)$ converges. By the arguments on weak compactness, convergent of $\mathcal{R}_0(\gamma_i)$ implies the same of $\mathcal{R}_\epsilon(\gamma_i)$. Note, since the energy is approaching zero, eventually the sequence is in the domain of the contraction mapping theorem and thus lies in the collar. Therefore, the endpoints uniquely parameterize the solutions γ_i and strong convergence follows from that of the endpoints. $\square$

Observe that our discussion establishes the following:

Corollary 2. *If $\mathcal{E}(\gamma_i) < C$ and $\mathcal{R}_0(\gamma_i)$ is uniformly bounded, we have $||\gamma_i||_{L^2_1}$ is uniformly bounded*

We are in good shape to verify the axioms:
Axiom $1'$: $\mathcal{L}_0 - \mathcal{L}_1$ is bounded below by 0 since energy $\mathcal{L}$ is nonincreasing on trajectories. To establish lower semi-continuity, we claim that in fact any sequence $(\mathcal{R}_0(\gamma_i), \mathcal{R}_\epsilon(\gamma_i))$ as above weakly converges (up to a subsequence) to a diagonal element. We can assume $(\mathcal{R}_0(\gamma_i), \mathcal{R}_\epsilon(\gamma_i))$ is strongly L^2 convergent. We have:

$$||\mathcal{R}_0(\gamma_i) - \mathcal{R}_\epsilon(\gamma_i)||_{L^2} \leq \int_0^\epsilon ||d\gamma_i/dt||_{L^2} \leq \epsilon^{1/2}||\gamma_i||_{L^2_1} \leq C\epsilon^{1/2}$$

since $||\gamma_i||_{L^2_1}$ is uniformly bounded. This implies the claim and thus the lower semi-continuity of $\mathcal{L}_1 - \mathcal{L}_0$. Axiom $2'$ and $3'$ have been verified in the previous lemma.

3.7.5 Concluding the Proof

Given a cycle $\sigma : P \to \mathfrak{B}(Y)$ we verify that $P \times_{\mathfrak{B}(Y)} \cup_{t\in[0,1]}\mathcal{M}_t$ is a cobordism between P and $P \times_{\mathfrak{B}(Y)} \mathcal{M}_1$. Given that P is transverse to $\cup_{t\in(0,1]}\mathcal{M}_t$, the axioms of a cycle for $P \times_{\mathfrak{B}(Y)} \cup_{t\in[0,1]}\mathcal{M}_t$ were verified in the preceding sections. What needs to be checked is that it has the structure of a lc-manifold. The potential problem occurs near the diagonal where the restriction maps converge in C^0 to the inclusion map. More precisely, in the collar coordinates (b^+, b^-, t), $\mathcal{R}_0(b^+, b^-, t) = b^+ + e^{tL_1}b^- + G_0(t, b)$ where $G_0(t, b)$ as a function of b converges to 0 in C^1 topology as $t \to 0$. Note that $e^{tL_1}_{|B^-}$ is a family of compact operators converging to the identity in the weak topology. Thus, we don't have C^1 convergence for the restriction map.

We will work locally, so assume that P is a ball around the origin in Hilbert space. Let $b_0 = \sigma(0)$. Near b_0, $\mathfrak{B}$ is an affine space modeled on $T^-\mathfrak{B} \oplus T^+\mathfrak{B}$. By assumption, $\pi^- \circ D\sigma$ is Fredholm. Applying the inverse function theorem to $\pi^- \circ \sigma$, we can find coordinates for σ so that $\sigma(p) = b_0 + f(p) + A(p)$ where Df_p is compact at all p and A is an linear Fredholm map

$$A : TP \to T^-\mathfrak{B}$$

In these coordinates, the map $\mathcal{R}_0 - \sigma : P \times \cup_{t\in[0,1]}\mathcal{M}_t \to \mathfrak{B}(Y)$ can written as

$$(p, b^+, b^-, t) \to (e^{tL} - 1)b_0^- + b^+ - A(p) + e^{tL}b^- - f(p) + G_0(t, b_0 + b)$$

Pick a left inverse A^{-1} for A. Thus, $A \circ A^{-1} - I$ has finite rank on $T^-\mathfrak{B}$. Define a change of coordinates by

$$(b^+, b^-, p, t) \mapsto (b^+, b^-, \tilde{p}, t)$$

with $\tilde{p} = p - A^{-1} \circ e^{tL_1}b$. This is a homeomorphism with inverse taking $\tilde{p}$ to $p = \tilde{p} + A^{-1}e^{tL}b^-$. Note that for each fixed $t \geq 0$, the map is a diffeomorphism. With the new coordinates the map $\mathcal{R}_0 - \sigma$ becomes

$$(\tilde{p}, b^+, b^-, t) \to (e^{tL} - 1)b_0^- + b^+ - A(\tilde{p} + A^{-1}e^{tL}b^-) + e^{tL}b^- - f(\tilde{p} + A^{-1}e^{tL}b^-) + G_0(t, b_0 + b)$$

This can be simplified to

$$(\tilde{p}, b^+, b^-, t) \to (e^{tL} - 1)b_0^- + b^+ - A(\tilde{p}) + K \circ e^{tL}b^- - f(\tilde{p} + A^{-1}e^{tL}b^-) + G_0(t, b_0 + b)$$

where $K = I - A \circ A^{-1}$ is a finite rank operator. Since the change of coordinates is a homeomorphism, the continuity of the map up to the boundary still holds. We claim that the differential in the $\tilde{p}$ and b variables converge as $t \to 0$ to the ones for $t = 0$. Computing the differential at $(\tilde{p}, b, t)$ we have:

$$(\delta\tilde{p}, \delta b^+, \delta b^-) \mapsto \delta b^+ - A(\delta\tilde{p}) + K \circ e^{tL}\delta b^- - Df_p(\delta\tilde{p} + A^{-1}e^{tL}\delta b^-) + DG_{0(t,b_0+b)}(\delta b^+ + \delta b^-)$$

We want this differential to converge to

$$(\delta\tilde{p}, \delta b^+, \delta b^-) \mapsto \delta b^+ + K\delta b^- + Df_p(\delta\tilde{p} + A^{-1}\delta b^-)$$

In view of the compactness of Df_p and K as well as the fact that e^{tL} is self-adjoint, the claim is a consequence of the following lemma proved in the appendix:

Lemma 21. *Given a uniformly bounded weakly converging sequence of operators $A_i : V \to W$ between Hilbert spaces and a strongly convergent sequence of compact operators $K_i : W \to U$, $K_i \circ A_i$ converge strongly to $K_\infty \circ A_\infty$ provided A_i^* converge weakly to $(A_\infty)^*$.*

Thus, we have found coordinates where the difference $\mathcal{R}_0 - \sigma$ is lc-smooth so the inverse function theorem with parameter implies $(\mathcal{R}_0 - \sigma)^{-1}(0)$ is a manifold with boundary. Finally, we need to verify that the projection to the other end is smooth in the new coordinates. This time the map is

$$(p, b^+, b^-, t) \mapsto e^{-tL_1}(b^+ + b_0^+) + b^- + b_0^- + G_1(t, b)$$

where again $G_1(t, b)$ as a function of b converges to 0 in C^1 topology as $t \to 0$. Notice, however, restricted to the fiber product $b^+ = \pi^+(\sigma(p))$ and thus, restricted to the fibre product, the map may be written using the $\tilde{p}$ coordinates as

$$(\tilde{p}, b^+, b^-, t) \mapsto e^{-tL_1}(\pi^+(\sigma(\tilde{p} + A^{-1}e^{tL}b^-)) + b_0^+) + b^- + b_0^- + G_1(t, b)$$

The derivative is:

$$(\delta\tilde{p}, \delta b^+, \delta b^-) \mapsto e^{-tL_1}(\pi^+ \circ D\sigma_p(\delta\tilde{p} + A^{-1}e^{tL}\delta b^-)) + \delta b^- + DG_{1(t,b)}(\delta b)$$

Since $\pi^+ \circ D\sigma_p$ is compact for any p, we have $e^{-tL} \circ \pi^+ \circ D\sigma_p$ compact as well. The following lemma (also proved in the appendix) implies $e^{-tL} \circ \pi^+ \circ D\sigma_p$ is converging:

Lemma 22. *Given a uniformly bounded weakly converging sequence of operators $A_i : V \to W$ between Hilbert spaces and a strongly convergent sequence of compact operators $K_i : U \to V$, $A_i \circ K_i$ converge strongly to $A_\infty \circ K_\infty$.*

From the lemma we conclude that $e^{-tL} \circ \pi^+ \circ D\sigma_p$ is a convergent sequence of compact operators, hence we can apply the previous lemma to conclude that $e^{-tL} \circ \pi^+ \circ D\sigma_p \circ A^{-1}e^{tL}$ is converging as well. This completes the proof of the existence of an lc-structure on $P \times_{\mathfrak{B}(Y)} \cup_{t\in[0,1]}\mathcal{M}_t$

3.8 Topological Invariance of $\Omega F_*(Y)$

As an immediate application of the fact that the trivial cobordism gives rise to the identity map on correspondences we have topological invariance of $\Omega F(Y, \mathfrak{s}, g)$:

Theorem 15. *$\Omega F(Y, \mathfrak{s}, g)$ is independent of the choice of Riemannian metric g.*

Proof. Consider a cylinder cobordism $W(g_0, g_1)$ with a metric that interpolates between any g_0 and g_1 on Y. Composing with opposite cobordism $W(g_1, g_0)$ and using the fact that the induced map on homology is metric independent, we have that

$$\Omega F(W(g_0, g_0)) = \Omega F(W(g_0, g_1)) \circ \Omega F(W(g_1, g_0))$$

Therefore, since $W(g_0, g_0)$ induces the identity on ΩF_* we have provided an isomorphism:

$$\Omega F(Y, \mathfrak{s}, g_0) \cong \Omega F(Y, \mathfrak{s}, g_1)$$

□

Remark. The constructed isomorphism is not dependent on the choice of interpolation. Indeed, we can define a correspondence with boundary $W(g_{t,s})$ by interpolating between any two metrics on W.

Chapter 4

Morse-Floer Theory

4.1 Definition of Homology Groups $HF_*(\mathfrak{B},\mathcal{L})$

Using lc-manifolds of arbitrary depth we can define a homology theory.

Definition 25. *Let $\widetilde{CF}_k(\mathfrak{B},\mathcal{L})$ be the $\mathbb{Z}_2$ vector space generated by chains of index k with the disjoint union as the sum operation. Take $\widetilde{CF}_*(\mathfrak{B},\mathcal{L}) = \bigoplus_k \widetilde{CF}_k(\mathfrak{B},\mathcal{L})$*

Definition 26. *Let $\partial : \widetilde{CF}_k(\mathfrak{B},\mathcal{L}) \to \widetilde{CF}_{k-1}(\mathfrak{B},\mathcal{L})$ be defined by $\partial[\sigma] = [\sigma_{|\partial(P)}]$ where $\sigma : P \to \mathfrak{B}$.*

Lemma 23. *$\partial^2 = 0$ in $\widetilde{CF}_*(\mathfrak{B},\mathcal{L})$.*

Proof. By the section on lc-manifolds $\partial^2 P$ is a disjoint sum of two identical lc-manifolds which is 0 in $\widetilde{CF}_*$. □

Definition 27. *A set $S \subset \mathfrak{B}$ is **k-negligible** if it is contained in the image of a semi-infinite Hilbert manifold $Q \to \mathfrak{B}$ of index $< k$.*

Definition 28. *A chain $\sigma : P \to \mathfrak{B}$ of index k is said to be **negligible** if its image is k-negligible.*

Definition 29. *Let $N_k \subset \widetilde{CF}_k$ be the subgroups generated by negligible chains P with ∂P equivalent to a negligible chain in $\widetilde{CF}_{k-1}$*

By construction $\partial : N_k \to N_{k-1}$ since ∂P is equivalent to a negligible chain and $\partial^2 = 0$.

Definition 30. *Let $CF_k = \widetilde{CF}_k/N_k$ and let HF_k be the associated homology group.*

We have the following analogue of the theorem in the last section:

Theorem 16. *Given, $[\sigma] \in HF_k(\mathfrak{B},\mathcal{L})$ and $[\tau] \in HF_l(-\mathfrak{B},-\mathcal{L})$ with transverse intersection, their fiber product $\sigma \times_{\mathfrak{B}} \tau$ is a closed manifold mapping to $\mathfrak{B}$. Since,*

$$\partial(\sigma \times_{\mathfrak{B}} \tau) = \partial\sigma \times_{\mathfrak{B}} \tau + \sigma \times_{\mathfrak{B}} \partial\tau$$

(at least when σ and τ are transverse) the fiber product defines a map

$$HF_k(\mathfrak{B}, \mathcal{L}) \times HF_l(-\mathfrak{B}, -\mathcal{L}) \to H^{ord}_{k+l}(\mathfrak{B})$$

where $H^{ord}_{k+l}(\mathfrak{B})$ denotes ordinary homology with $\mathbb{Z}_2$-coefficients.

4.2 Definition of Flow on Cycles

While there is no local existence of a flow corresponding to the L^2 gradient of $\mathcal{L}$ on $\mathfrak{B}$, as already observed in the section on the trivial cobordism, there is a notion of gradient flow defined for all $t \geq 0$ on the chains.

Definition 31. *Consider a cycle $\sigma : P \to \mathfrak{B}$. For $t > 0$, let $\mathcal{F}_t(\sigma) = \sigma \times_{\mathfrak{B}} Z_t$ where Z_t is cycle corresponding to the moduli space of solutions on the cylinder of length t. For $t = 0$ we put $\mathcal{F}_0(\sigma) = \sigma$.*

To spell out $\mathcal{F}(\sigma)$ as a set note that a point in $\mathcal{F}(\sigma)$ consists of a gradient flow line of length t starting at $f(p)$ for some $p \in P$. The image of this point in $\mathfrak{B}$ is the endpoint of the corresponding flow line.

Definition 32. *Let $\mathfrak{B}^c$ be the subset of $\mathfrak{B}$ where $\mathcal{L} \geq c$.*

Definition 33. *Let $C_k(\mathfrak{B}^c) \subset C_k(\mathfrak{B})$ be the subgroup generated by chains with image in $\mathfrak{B}^c$. Let $CF_k(\mathfrak{B}, \mathfrak{B}^c) = CF_k(\mathfrak{B})/CF_k(\mathfrak{B}^c)$*

Note that the gradient flow on cycles preserves $CF_*(\mathfrak{B}^c)$.

Theorem 17. *There is a long exact sequence:*

$$\cdots \to HF_k(\mathfrak{B}^c) \to HF_k(\mathfrak{B}) \to HF_k(\mathfrak{B}, \mathfrak{B}^c) \to \dots$$

Proof. This follows from the short exact sequence of chain complexes:

$$0 \to C_k(\mathfrak{B}^c) \to C_k(\mathfrak{B}) \to C_k(\mathfrak{B}, \mathfrak{B}^c) \to 0$$

□

Here is a simple vanishing result:

Theorem 18. *Suppose $\mathfrak{B}^c$ contains no critical points of $\mathcal{L}$. Then $HF(\mathfrak{B}^c) = 0$*

Given a cycle $\sigma : P \to \mathfrak{B}^c$, Consider the lc-manifold given by $\cup_{t\in[0,\infty)}\mathcal{F}_t(\sigma)$ with boundary $\sigma \times_{\mathfrak{B}} Z_1$. What needs to be verified is that $\cup_{t\in[0,\infty)}\mathcal{F}_t(\sigma)$ satisfies the axioms of a cycle. In the section on the trivial cobordism we have verified that $\cup_{t\in[0,C]}\mathcal{F}_t(\sigma)$ satisfies the axioms of a cycle for any $C > 0$. The potential difficulty with the compactness axioms occurs as $t \to \infty$. To overcome this, we rule out the possibility that trajectories of arbitrary length starting at $\sigma(P)$ may have uniformly bounded energy. Here and below we use the following easy consequence of the L^2_1 compactness theorem:

Lemma 24. *If a sequence of trajectories γ_i on a cylinder of fixed size has energy approaching 0 then γ_i converge to a constant trajectory and thus the boundary values converge to a critical point of the gradient flow.*

To establish the theorem suppose we have a sequence $x_i \in \mathcal{F}_{t_i}(\sigma)$ with $\mathcal{L}(x_i) < C$ and $t_i \to \infty$. Restricting these trajectories to some unit subintervals produces a sequence of unit length trajectories with energy approaching zero. By the lemma, the boundary values are converging to a critical point. However, these boundary values all lie in $\mathfrak{B}^c$ where no critical points exist. By a similar argument we can prove the following:

Theorem 19. *Take $b > c$. If there are no critical points between the level sets $\mathfrak{B}^b$ and $\mathfrak{B}^c$ then the inclusion $HF(\mathfrak{B}^b) \to HF(\mathfrak{B}^c)$ is an isomorphism.*

Proof. To prove surjectivity we push a cycle $P \in HF(\mathfrak{B}^c)$ into $\mathfrak{B}^b$ via the flow Z_t. What needs to be shown is that for large enough t the cycle will move far enough to be in $\mathfrak{B}^b$. Points in $P \times_{\mathfrak{B}} Z_t$ correspond to trajectories starting at somewhere in $\mathfrak{B}^c$. Suppose we are given a trajectory starting in $\mathfrak{B}^c - \mathfrak{B}^b$. We claim that if t is large the energy is larger then $b - c$ and thus the endpoint lies in $\mathfrak{B}^b$. As before we argue by contradiction. If we have a sequence of trajectories of bounded energy and increasing length all starting and ending in $\mathfrak{B}^c - \mathfrak{B}^b$ we can find trajectories of unit length with energy approaching zero with endpoints also in $\mathfrak{B}^c - \mathfrak{B}^b$. This is a contradiction since they must converge to a critical point. Injectivity is established in a similar manner. If $\sigma \in CF(\mathfrak{B}^b)$ equal to $\partial\tau$ with $\tau \in CF(\mathfrak{B}^c)$ we have $\mathcal{F}_t(\sigma) = \partial\mathcal{F}_t(\tau)$. As before, by taking t large enough $\mathcal{F}_t(\tau) \in CF(\mathfrak{B}^b)$. □

Finally, we must address the case when we pass a critical point. For this we need to assume $\mathcal{L}$ has isolated critical points. In case of SW theory we can always perturb $\mathcal{L}$ to obtain a Morse function. This will be proved in a separate section so for now assume this can be done. We have:

Theorem 20. *Assume the $\mathcal{L}$ has a unique isolated critical point at level c. Choose $c' < c$ so close that $\mathfrak{B}^{c'}$ has no new critical points. Let U be a neighborhood of the critical point y at c. There exists ϵ such that any chain $\sigma : P \to \mathfrak{B}^{c'}$ may be pushed by the flow up to P' so that for $p \in P'$ either $\mathcal{L}(p) \geq c + \epsilon$ or $\sigma(p) \in U$.*

First we prove a lemma:

Lemma 25. *Take a small balls $B_1 \subset B_2 \subset U$ containing y. There exists some $2\epsilon > 0$ such that any sequence of trajectories with lengths at least 1 starting in B_1 and ending in $U - B_2$ must have energy at least 3ϵ.*

Proof. As before, we argue by contradiction. Thus, we have a sequence of trajectories of unit length all ending in $U - B_2$ with energy approaching zero. This is not possible since by compactness they converge to a critical point. □

We return to the proof of the theorem. Firstly, we can push our cycle into $\mathfrak{B}^{c-\epsilon}$. Now, by contradiction, assume there are trajectories of increasing length starting where $c-\epsilon \leq \mathcal{L} \leq c$ and ending with $\mathcal{L} \leq c+\epsilon$ but not in U. For each trajectory γ_i, let a_i be the point when γ_i first enters B_1 and b_i the point when it last leaves B_2. Assume $b_i - a_i \leq 1$ for some subsequence. If so, we can find a sequence of trajectories of unit length with energy going to zero and staying away from B_1. They must converge to the constant trajectory which contradicts the fact that they stay away from B_1. Finally, assume $b_i - a_i \geq 1$. In this case we can find trajectories each of at least unit length which satisfy the hypothesis of the lemma above. This is a contradiction since by construction each such trajectory has energy at most 2ϵ. □

Note that we do not use the fact that the critical point is nondegenerate but only that it is isolated.

4.3 Perturbing the Equations

In Seiberg-Witten theory, as explained in [KM07] it is possible to chose a perturbation of the Chern-Simons-Dirac functional to arrange Morse-Smale transversality. In this section we sketch how one can obtain a Morse perturbation by mimicking the finite-dimensional proof as explained for example in [Mil65]. Fix a $(B_1, \Psi_1) \in \mathfrak{B}(Y)$. Let ρ be a bump function supported near the origin. For any $v \in \mathfrak{B}$, define a function $f_v : \mathfrak{B} \to \mathbb{R}$ by

$$f_v(B, \Psi) = \rho(||(B, \Psi) - (B_1, \Psi_1)||_{L^2})\langle v, (B - B_1, \Psi)\rangle_{L^2}$$

Theorem 21. *For generic choice of v, $\tilde{\mathcal{L}}$ is a Morse function.*

4.4 Morse Lemma

Assume $\tilde{\mathcal{L}}$ has a critical point at y. We will construct a local diffeomorphism from $\mathfrak{B}$ to a neighborhood of the origin in Hilbert space where $\tilde{\mathcal{L}}$ will have a particularly simple form. For this we use the extension of the usual Morse lemma due to Palais. The argument is taken from [Lan83].

Lemma 26. *Consider a function $\tilde{\mathcal{L}}$ defined on a neighborhood U of the origin in a Hilbert space by $\tilde{\mathcal{L}}(x) = \langle x, A(x)x\rangle$ where $A(x)$ is a bounded linear operator for each x with $A_0 = A(0)$ invertible. We assume $A(x)$ is a smooth function of x. Furthermore, $A(x) = A_0 + B(x)$ with $B(x)$ a compact operator for each fixed x. Also, assume that for a fixed u, the operator $v \mapsto D_vB \cdot u$ is compact as well. Then, there exists a local diffeomorphism ϕ, compatible with the polarization, such that in the new coordinates y, $\tilde{\mathcal{L}}(y) = \langle y, A_0 y\rangle$. Finally, assume that the adjoint $B(x)^*$ preserves weak limits in the sense that if $x_i \to x$ weakly, then for any z, $B(x_i)^*z \to B(x)^*z$ strongly. In this case, we have that ϕ preserves weak limits.*

Proof. Only the part about the polarization is not covered in [Lan83]. As suggested, we define $y = (A_0^{-1}A(x))^{1/2}x = (I + A_0^{-1}B(x))^{1/2}x = (I + C(x))x$ where $C(x) = \sum a_i(A_0^{-1}B(x))^i$ and a_i are the binomial expansion coefficients. $C(x)$ is compact for each x as a uniform limit of compact operators. $D_v(C(x))u$ is compact as well. Therefore, at each x, $D\phi$ differs from the identity by a compact operator and hence preserves the polarization. Finally, assume $B(x)^*$ preserves weak limits. Given a weakly convergent $v_i \to v$ and $x_i \to x$, we have that $B(x_i)v_i$ converges to $B(x)v$ since

$$\lim_i \langle B(x_i)v_i, z\rangle = \langle v_i, B(x_i)^* z\rangle = \langle v, B(x)^* z\rangle$$

We conclude that $B(x_i)x_i$ weakly converges to $B(x)x$ and more generally, $B(x_i)^k x_i = B(x_i) \circ B(x_i)^{k-1}x_i$ converges to $B(x)^k x$. In view of the definition of ϕ as an absolutely convergent power series, ϕ preserves weak limits as well. $\square$

For the case of the Chern-Simons-Dirac operator on $\mathfrak{B}$ we have

$$\tilde{\mathcal{L}}(a, \Psi) = c + \langle L(a, \Psi), (a, \Psi)\rangle_{L^2} + \langle \Psi, \rho(a) \cdot \Psi\rangle_{L^2}$$

where $L : L^2_{1/2} \to L^2_{-1/2}$ is an isomorphism. We can rewrite this using an $L^2_{1/2}$ inner product as

$$\langle (a, \Psi), (a, \Psi)\rangle_{L^2_{1/2}} + \langle \Psi, L^{-1}\rho(a)\Psi\rangle_{L^2_{1/2}}$$

To see that $L^{-1}\rho(a) = B(a, \Psi)$ satisfies the hypothesis of the previous lemma we must check that $\rho(a)^*$ preserves weak limits. Given $\Psi \in L^2_{1/2}$, and a weakly $L^2_{1/2}$ convergent sequence a_i note that the operator taking a_i to $\rho(a_i)\Psi$ is compact as an operator from $L^2_{1/2}$ to L^2 as well as an operator from L^2 to $L^2_{-1/2}$. Take a subsequence of $\rho(a_i)\Psi$ with a strong L^2 limit Ψ'. We claim $\Psi' = \rho(a_\infty)\Psi$. Indeed, since the sequence a_i is strongly L^2 convergent to a_∞ we have

$$lim_{i \to \infty} \rho(a_i)\Psi = \rho(a_\infty)\Psi$$

in $L^2_{1/2}$. Therefore, $\rho(a_\infty)\Psi = \Psi'$ as elements of $L^2_{-1/2}$ as desired.
This local decomposition allows us to speak of local stable/unstable manifolds at a critical point. Note, however, they are not defined by the dynamics of the flow.

4.5 Passing a Critical Point

In this section, assume the $\mathcal{L}$ has a unique isolated nondegenerate critical point y at level c.

Definition 34. *The index of a critical point is the index of the local stable manifold at c.*

Let c be an isolated Morse critical point of $\mathcal{L}$ of index k. We will prove:

Theorem 22. *For ϵ small, $HF_k(\mathfrak{B}^{c-\epsilon}, \mathfrak{B}^{c+\epsilon}) = \mathbb{Z}_2$. $HF_j(\mathfrak{B}^{c-\epsilon}, \mathfrak{B}^{c+\epsilon}) = 0$ for $j \neq k$.*

The Morse lemma allows us to identify a neighborhood U of y with a neighborhood of the origin in $V = V^+ \oplus V^-$ and $\mathcal{L} : V \to \mathbb{R}$ with $\mathcal{L}(v) = |v^-|^2 - |v^+|^2$. Let $D^\pm(\epsilon)$ denote the closed disk in $V^\pm$ of radius ϵ. By the lemma above, given a cycle $P \in HF(\mathfrak{B}^{c-\epsilon}, \mathfrak{B}^{c+\epsilon})$ we may assume it lies in $U \cup \mathfrak{B}^{c+\epsilon}$. Thus, we have reduced to the local situation provided by the Morse lemma.

Our first task is to define a correspondence which will locally project a cycle into D^-. For this, pick a smooth function $f : V \to [0,1]$. Consider the map $Z_f : V \to V \oplus V$ with $Z_f(v^-, v^+) = (v^-, v^+) \oplus (v^-, f(v)v^+)$.

Lemma 27. *Z_f is a correspondence. Furthermore, Z_f is cobordant to the identity.*

Proof. Note that, restricted to Z_f, the difference $-\mathcal{L}_1 + \mathcal{L}_2$ is $(1 - f(v^+))|v^+|^2$. This is clearly bounded below. If (v_i^-, v_i^+) is weakly convergent to (v^-, v^+) and $f(v_i)$ converges to c the weak limit of $(v_i^-, f(v_i)v_i^+)$ is $(v^-, c \cdot v^+)$. We have

$$\liminf (1 - f(v_i))|v_i^+|^2 = (1-c) \liminf |v_i^+|^2$$

Since the norm of v^- can only drop in a weak limit $-\mathcal{L}_1 + \mathcal{L}_2$ is lower semicontinuous. The rest of the axioms follow directly from the definitions since Z_f is a graph. We can define a chain homotopy to the identity by

$$Z_{f_t}(v^-, v^+) = (v^-, (t + (1-t)f(v))v^+)$$

□

Note, in particular, taking $f(v) = 0$ near the origin and 1 outside some ball, allows us to deform a cycle to the projection onto the V^- factor near the origin. Given a cycle P we may assume its transverse to $D^-(\epsilon)$. Using Z_f we may also assume its image in contained in $D^-(\epsilon)$ whenever $\tilde{\mathcal{L}}(p) \leq \tilde{\mathcal{L}}(y)$. Thus, points where $\tilde{\mathcal{L}}(p) = \tilde{\mathcal{L}}(y)$ corresponds to those that map to $D^+(\epsilon)$

Claim: For $\epsilon' < \epsilon$ sufficiently small, every point in $D^-(\epsilon')$ is a regular value of the cycle map $\sigma : P \to \mathfrak{B}$.

To see this observe that 0 is a regular value since σ was assumed to be transverse to D^+. By contradiction take a sequence $p_i \in P$ with $\sigma(p_i) \in D^-(1/i)$ and $D\sigma$ not surjective on V^-. Since $\tilde{\mathcal{L}}(0) = \lim \tilde{\mathcal{L}}(\sigma(p_i))$ the p_i converge strongly in P to a point in $\sigma^{-1}(y)$. But $\sigma^{-1}(y)$ is compact and thus there exists an open set where $D\sigma$ is surjective. This is a contradiction.

With the claim at our disposal, we may cut our cycle into two pieces by taking the preimage of a small sphere around the origin in V^-. Here is argument bifurcates depending on the dimension of the cycle.

Case 1: $dim(P) < k$. In this case $\sigma^{-1}(y)$ is empty and thus P is contained in the region where $\tilde{\mathcal{L}} < c$. It may be pushed to lie in $\mathfrak{B}^{c+\epsilon}$.

Case 2: $dim(P) = k$. Cutting by a small sphere as above decomposes the cycle into chain where $\tilde{\mathcal{L}} < c$ and a finite collection of disks mapping diffeomorphically to $D^-(\epsilon)$. We see that $D^-(\epsilon)$ generates the homology in this dimension. Its a nontrivial generator of the homology since its intersection with $D^+(\epsilon)$ is 1.

Case 3: $dim(P) > k$. In this final case, cutting by a small sphere decomposes the cycle into a chain lying above the critical set, and a chain that is trivial in homology since it factors through a map of strictly smaller dimension.

To summarize we have verified that the homology is nontrivial in a single degree and is generated by the disk D^-.

We have the following easy corollary:

Theorem 23. *Suppose $\mathfrak{B}^c$ contains no critical points of index k. We have $HF_k(\mathfrak{B}^c, \mathcal{L}) = 0$.*

Proof. Given a cycle $P \in \mathfrak{B}^c$, we try to map it by the flow up as far as possible. If y is the first critical point of the functional in $\mathfrak{B}^c$ say at level d, by the lemmas above we can assume P maps to $\mathfrak{B}^{d-\epsilon}$. By the results of this section, P is in fact homologous to a cycle in $\mathfrak{B}^{d+\epsilon}$, in view of the assumption on the index. Repeating this argument finitely many times we see that P is cobordant to a cycle lying above all critical points and is therefore zero in homology. ☐

4.6 Morse Homology for a Self-Indexing $\mathcal{L}$

In the previous section, we saw that if there is a unique nondegenerate critical point y at $\tilde{\mathcal{L}} = c$ then $HF_{-j}(\mathfrak{B}^{c-\epsilon}, \mathfrak{B}^{c+\epsilon}) = \mathbb{Z}_2$ when $j = index(y)$ and 0 else. Suppose that the next isolated nondegenerate critical point z is at d with $index(z) = index(y) + 1$. Consider the composite map:

$$HF_*(\mathfrak{B}^{c-\epsilon}, \mathfrak{B}^{c+\epsilon}) \xrightarrow{\alpha} HF_{*-1}(\mathfrak{B}^{c+\epsilon}) \xrightarrow{\beta} HF(\mathfrak{B}^{c+\epsilon}, \mathfrak{B}^{d+\epsilon})$$

As computed above, $HF(\mathfrak{B}^{c+\epsilon}, \mathfrak{B}^{d+\epsilon}) = HF(\mathfrak{B}^{d-\epsilon}, \mathfrak{B}^{d+\epsilon}) = \mathbb{Z}_2$. We compute $\beta \circ \alpha$. Since $HF(\mathfrak{B}^{c-\epsilon}, \mathfrak{B}^{c+\epsilon})$ is generated by $[D^-(\epsilon)]$, $\beta \circ \alpha([D^-(\epsilon)]) = [\partial D^-(\epsilon)]$ viewed as an element of $HF(\mathfrak{B}^{c+\epsilon}, \mathfrak{B}^{d+\epsilon})$. Using the gradient flow, we may push this cycle down to $\mathfrak{B}^{d-\epsilon}$. As discussed above, the image in $HF(\mathfrak{B}^{d-\epsilon}, \mathfrak{B}^{d+\epsilon})$ is computed by counting intersections of the cycle with the disk $D^+(\epsilon)$ in a neighborhood of d.

The situation is quite analogous to the finite dimensional case where the boundary operator may be computed by counting the intersections of attaching/belt spheres. In this section, we show that in the special case of a self-indexing Morse functional one may obtain a chain complex generated by critical points whose homology coincides with $HF(\mathfrak{B}, \mathcal{L})$. Assume that the critical points of index i are on level i of $\mathcal{L}$. We call such a functional self-indexing.

We have a filtration $\mathfrak{B}^{n+\epsilon} \subset \mathfrak{B}^{n-1+\epsilon} \subset \cdots \mathfrak{B}^{n-2+\epsilon} \subset \mathfrak{B}$ with $HF(\mathfrak{B}^{n-\epsilon}, \mathfrak{B}^{n+1-\epsilon}) = \mathbb{Z}_2^j$ where j is the number of critical points of index n. We have:

Definition 35. *Let $C_*^M(\mathfrak{B}, \mathcal{L}) = (\oplus_k C_k^M, \partial^M)$ be the $\mathbb{Z}_2$ chain complex*

$$C_{-k} = HF(\mathfrak{B}^{k-\epsilon}, \mathfrak{B}^{k+1-\epsilon})$$

The differential $\partial^M : C_*^M \to C_{*-1}^M$, $\partial^M = \beta \circ \alpha$ arises from the connecting homomorphism:

$$\alpha : HF_{-k}(\mathfrak{B}^k, \mathfrak{B}^{k+1}) \to HF_{-k-1}(\mathfrak{B}^{k+1})$$

and the map induces by the projection:

$$\beta : HF_{-k-1}(\mathfrak{B}^{k+1}) \to HF_{-k-1}(\mathfrak{B}^{k+1}, \mathfrak{B}^{k+2})$$

Let $HF_*^M(\mathfrak{B}, \mathcal{L})$ be the homology of the chain complex $(\oplus C_k^M, \partial^M)$.

Theorem 24. *Given a self-indexing Morse function $\mathcal{L}$ as above, we have $HF_*(\mathfrak{B}, \mathcal{L}) \cong HF_*^M(\mathfrak{B}, \mathcal{L})$*

Proof. The identical to the proof of the cellular homology theorem (see for example [Hat02]). $\square$

4.7 Morse Homology in the General Case

The results of the previous section are not sufficient for applications in Floer theory. The main issue is that the assumption that the Morse function is self-indexing is too restrictive. Indeed, the perturbations in Floer theory are usually taken to be small and it is not clear to us how to change $\mathcal{L}$ to obtain a self-indexing function. In this section we explain how under certain technical assumptions we can still define the Morse chain complex and show that its homology coincides with the geometric cycle homology.

Assume we are given a Morse functional $\mathcal{L}$ with a finite number of nondegenerate critical points. Furthermore, assume that each critical point x has an open neighborhood U_x such that there is no gradient flow from U_x to U_y if $index(x) \geq index(y)$. This condition is satisfied in Floer theory under the assumption that the stable/unstable manifolds intersect transversely.

Definition 36. *A chain σ is said to be $k-small$ if each critical point of index $j < k$ has a neighborhood U such that $U \cap \mathcal{F}_t(\sigma) = \emptyset$ for all $t \leq 0$.*

Definition 37. *Let $C_*^k \subset C_*(\mathfrak{B}, \mathcal{L})$ be the subcomplex generated by k-small chains. We have a filtration:*

$$\cdots \subset C_*^{k+1} \subset C_*^k \subset \ldots C_*(\mathfrak{B}, \mathcal{L})$$

Note that the filtration stabilizes since we assume there are only finitely many critical points.

Theorem 25. *We have $HF_j(C^k_*) = 0$ when $j > -k$.*

Proof. Given a cycle $\sigma : P \to \mathfrak{B}$ in C^k_j, let us see what happens as we try to push it up by the flow. As we attempt to push it past a critical point x two things can happen. If the flow lines stay away from the critical point, by taking t large, we have $\mathcal{F}_t(\sigma)$ lies strictly above the critical point. Thus, we can focus on the case where $\mathcal{F}_t(\sigma)$ gets arbitrary close to the critical point. By assumption, this happens only when $index(x) \geq k$. First, we perturb P to P' in a small neighborhood of x to ensure it intersects the stable manifold of x transversely. Here we are simply using the local unstable manifold provided by the Morse lemma. A crucial point is that this perturbation does not take us out of C^k_*. Indeed, since σ changes only on points mapping to a small neighborhood of x, and no gradient lines from a small neighborhood of x can flow arbitrary close to critical points of $index < k$, $[P'] \in C^k_*$. By assumption, we have $j + k > 0$,therefore P' intersects the stable manifold in a manifold of dimension at least 1. By the arguments in the section on passing critical points, we can modify P' to P'' locally around x so that $[P''] = [P]$ and $P'' \in C^k_*$. Thus, without changing the homology class, we can modify a cycle to lie below any critical point. $\square$

Theorem 26. *We have $HF_{-k}(C^k_*, C^{k+1}_*) = \mathbb{Z}_2^n$, where n is the number of critical points of index k and $HF_j(C^k_*, C^{k+1}_*) = 0$ for all other j.*

Proof. The argument is quite similar to the proof of the previous theorem. We are given a chain σ with $[\sigma] \in C^k_j$ and $\partial[\sigma] \in C^{k+1}_{j-1}$. As before, we are trying to push our chain past a critical point x of $index(x) \geq k$. First we perturb the chain in a neighborhood of x to be transverse to the stable manifold. The perturbation gives a chain $F : P \times [0,1] \to \mathfrak{B}$ with $F_{|P\times 0} = \sigma$ and $\partial F = F_{|P\times 1} - \sigma + F_{\partial(P)\times[0,1]}$. Note that $F_{\partial(P)\times[0,1]} \in C^{k+1}_*$ since the perturbation is supported in a neighborhood of x and thus is zero in C^k_*/C^{k+1}_*. Therefore, we can perturb the cycle to be transverse without changing its homology class. Next, we project the cycle to $T^-\mathfrak{B}_x$. By the same argument this does not change our class. Now, assume $j \neq -k$. If $j < -index(x)$ the intersection with the stable manifold is empty so we can flow past it. Otherwise we cut our σ to produce $\sigma_a + \sigma_b$ with σ_a strictly below the critical point. If $j > -index(x)$, σ_b is negligible and thus we can flow past the critical point. Otherwise, $j = -index(x)$, σ_b is a collection of disks all isomorphic to the unstable manifold at x. However, in this case $\sigma_b \in C^{k+1}_*$ since the index of x is at least $k+1$. Thus, $\sigma = \sigma_a$ in C^k_*/C^{k+1}_* so again we can flow past x.
Finally, we deal with the case $j = -k = index(x)$. Take $D^-(x_m)$ to be a small disk in the unstable manifold around the critical point x_m of index k. Note that $D^-(x_m)$ is a cycle in C^k_{-k}/C^{k+1}_{-k}. Indeed, the boundary of such a disk can converge by the flow only to a critical point of higher index and thus is zero in $C^k_{-k-1}/C^{k+1}_{-k-1}$. We need only verify that there is no linear dependence among these disks. We define a linear map

$$Ev : HF_{-k}(C^k, C^{k+1}) \to \mathbb{Z}_2^n$$

which will be an isomorphism. Take a cycle $\sigma \in HF_{-k}(C^k, C^{k+1})$. Consider $\mathcal{F}_T(\sigma)$ for T very large. We claim that if T is sufficiently large, the cycle maps to $\cup_m U_{x_m} \bigcup V$, where U_{x_m}

are arbitrarily small neighborhoods of the critical points x_m and V are points that can only converge to critical points of index more than k. In other words, if γ is a trajectory starting in V it stays away from some neighborhood of critical points of index less than $k+1$. This claim is easy to verify. Indeed, by contradiction assume there are large times T_l and trajectories starting at points of σ and ending outside $\cup_m U_{x_n} \bigcup V$. By compactness, this means that the trajectories spend some time arbitrary close to some critical points. However, these critical points can only be of index k or more by the hypothesis on σ. By the assumptions in the beginning of this section, this cannot happen since trajectories that come close to critical points of index k or more cannot get arbitrarily close to critical points of index less than k. Notice that $\partial\sigma$ does not map to U_{x_n} at it lies in C^{k+1}_*. We claim that this gives rise to a well defined intersection with $D^+(x_m)$, the stable manifolds at critical points x_m. To see this take two representatives,P and P' of some homology class in $HF_{-k}(C^k, C^{k+1})$. Assume they are both in $\cup_m U_{x_n} \bigcup V$. Consider $W \in C^k_{-k+1}$ with $\partial W = P - P'$. The point is that by considering $\mathcal{F}_T(W)$ for large T we can assume W lies in $\cup_m U_{x_n} \bigcup V$ as well. Note that for P, composing with the flow $\mathcal{F}_T(P)$ does not change the intersections with $D^+(x_m)$ since the cobordism connecting P to $\mathcal{F}_T(P)$ does not touch $\partial D^+(x_m)$. Therefore, we are free to assume W lies in $\cup_m U_{x_n} \bigcup V$ as well. This implies P and P' have the same intersection with $D^+(x_m)$ since they are related by a cobordism that stays away from $\partial D^+(x_m)$. At last, note that the intersections $D^-(x_a) \cap D^+(x_b) = \delta_{ab}$. $\square$

With these two theorems in place we can proceed to define

$$C_i^M(\mathfrak{B}, \mathcal{L}) = HF_i(C^{-i}, C^{-i+1})$$

. The differential $\partial^M : C_*^M \to C_{*-1}^M$, $\partial^M = \beta \circ \alpha$ arises from the connecting homomorphism:

$$\alpha : HF_{-k}(C^k, C^{k+1}) \to HF_{-k-1}(C^{k+1})$$

and the projection:

$$\beta : HF_{-k-1}(C^{k+1}) \to HF_{-k-1}(C^{k+1}, C^{k+2})$$

Geometrically, the differential may be interpreted as follows. Take generators for $HF_*(C^k, C^{k+1})$. These can be taken to be small disks $D^-_{x_m}$ in the unstable manifolds around the critical points as above. Let $D^+_{y_l}$ be the stable disks around critical points of index $k+1$. Take the intersections $\mathcal{F}_T(\partial D^-(x_m)) \cap D^+(y_l)$ for T large. These give you the coefficients of the incidence matrix. By the same arguments as in the previous section, we have

Theorem 27. *There exists an isomorphism*

$$HF_*(\mathfrak{B}, \mathcal{L}) \cong HF_*^M(\mathfrak{B}, \mathcal{L})$$

Chapter 5

Floer Theory for Loops in $\mathbb{C}^n$

5.1 L_1^2 Compactness for a Holomorphic Cylinder

Take S^1 to be the standard circle of length 2π. Let $Z_T = [0,T] \times S^1$ be the cylinder with coordinates (t,θ) and complex structure $j(\partial_t) = \partial_\theta$. Let $H : \mathbb{C}^n \times S^1 \to \mathbb{R}$ be a hamiltonian with associated vector field $X_H = J \circ \nabla H$. Given an L_1^2-map $u : Z_T \to \mathbb{R}^{2n}$ we define the energy to be

$$E(u) = \frac{1}{2}\int_0^T \int_0^{2\pi} |u_t|^2 + |u_\theta - X_H(u,t)|^2 d\theta ds$$

Consider $X_H(u,t) = c \cdot u + P(u,t)$ where $P : \mathbb{R}^{2n} \times S^1 \to \mathbb{R}^{2n}$ is C^1 and compactly supported and $c \in \sqrt{-1}\mathbb{R}$.

Lemma 28. *$P(u,t)$ is continuous in u for the L^2 topology.*

Proof. Since $P(u,t)$ has compact support, we have

$$|P(v,t) - P(v',t)| \leq C|v - v'|$$

Therefore,

$$|P(v,t) - P(v',t)|^2 \leq C^2|v - v'|^2$$

Given $u_1, u_2 : S^1 \to \mathbb{R}^{2n}$, we have

$$\int |P(u_1,t) - P(u_2,t)|^2 dt \leq C^2 \int |u_1 - u_2|^2 dt = C^2||u_1 - u_2||_{L^2}^2$$

□

Theorem 28. *Assume $c \notin \sqrt{-1}\mathbb{Z}$. Given a sequence with $E(u_i) < C$, the u_i are uniformly L_1^2 bounded and thus weakly precompact. Given a weakly convergent sequence u_i, we have $E(u_\infty) \leq \liminf E(u_i)$. If $\lim E(u_i) = E(u_\infty)$, the u_i converge strongly in L_1^2 to u_∞. Finally, if $c \in \sqrt{-1}\mathbb{Z}$, the theorem applies if we furthermore assume $||u_i(0,\theta)||_{L^2} < C'$.*

Proof. We first prove the theorem when $P = 0$. When $c \notin \sqrt{-1}\mathbb{Z}$ we have an isomorphism

$$\partial_\theta + c : L_1^2(S^1) \to L^2(S^1)$$

Thus,

$$\frac{1}{2}\int_0^T \int_0^{2\pi} |u_t|^2 + |u_\theta - X_H(u,t)|^2 d\theta ds = const \cdot ||u||_{L_1^2}^2$$

Thus, the energy is equivalent to the L_1^2 norm from which everything follows. When $c \in \sqrt{-1}\mathbb{Z}$ a special argument is needed. Since we have

$$\int_0^T \int_0^{2\pi} |u_t|^2 d\theta dt < C$$

we get

$$||u(0,\theta)|_{L^2} - |u(\tau,\theta)|_{L^2}| \le \tau^{1/2}||u_t||_{L^2}$$

for all $\tau \in [0,T]$. This, together with the L^2 bound on $u(0,\theta)$, implies an L^2 bound on u. The bound on u_θ follows since we have bounds on u_t and $X_H(u) = c \cdot u$. Now, assume u_i are weakly convergent. We may rewrite the energy as

$$\frac{1}{2}\int_0^T \int_0^{2\pi} |u_{i,t}|^2 + |u_{i,\theta}|^2 + |c \cdot u_i|^2 + 2\langle u_{i,\theta}, c \cdot u_i\rangle d\theta ds$$

Weak L_1^2 convergence of u_i implies $\int_0^T \int_0^{2\pi} \langle u_{i,\theta}, c\cdot u_i\rangle d\theta ds$ converges to $\int_0^T \int_0^{2\pi} \langle u_{\infty,\theta}, c\cdot u_\infty\rangle d\theta ds$ and thus lower semicontinuity follows from that of the L_1^2 norm. The case $P \neq 0$ is a slight modification of the argument. We observe that

$$|(u_\theta - c\cdot u) - P(u,t)|^2 = |u_\theta - c \cdot u|^2 + |P(u,t)|^2 + 2\langle u_\theta - c\cdot u, P(u,t)\rangle$$

Hence

$$|(u_\theta - c\cdot u) - P(u,t)|^2 \ge |u_\theta - c \cdot u|^2 + |P(u,t)|^2 - 2|\langle u_\theta - c\cdot u, P(u,t)\rangle| \ge$$

$$|u_\theta - c \cdot u|^2 + |P(u,t)|^2 - |u_\theta - c\cdot u|^2/2 - 2|P(u,t)|^2 = 1/2|u_\theta - c\cdot u|^2 - |P(u,t)|^2$$

Since we have an L^∞ bound on $P(u,t)$, a bound on $E(u)$ is the same as a bound on

$$\frac{1}{2}\int_0^T \int_0^{2\pi} |u_t|^2 + |u_\theta - c\cdot u|^2 d\theta ds$$

In addition, since $P(u,t)$ is continuous for the L^2 topology, the $\lim_{i\to\infty} E(u_i) = E(u_\infty)$ exactly when

$$\lim_{i\to\infty} \frac{1}{2}\int_0^T \int_0^{2\pi} |u_{i,t}|^2 + |u_{i,\theta} - c\cdot u_i|^2 d\theta ds = \frac{1}{2}\int_0^T \int_0^{2\pi} |u_{\infty,t}|^2 + |u_{\infty,\theta} - c\cdot u_\infty|^2 d\theta ds$$

$\square$

Remark. In the case $c \in \sqrt{-1}\mathbb{Z}$ the assumption on u may seem artificial. However, this assumption is exactly met when describing the axioms of a correspondence. Therefore, the holomorphic maps on a cylinder will always give rise to a correspondence under the above assumptions.

5.2 Semi-Infinite Cycles for the Action Functional on $L^2_{1/2}(S^1, \mathbb{C}^n)$

We explain how the existence of a critical point for the action functional (which is proved using mini-max techniques in [HZ94]) may be established using the framework of semi-infinite cycles. We will follow [HZ94] quite closely and will avoid repeating arguments explained there. The basic setup is as follows: Let $\mathfrak{B}$ be the space of $L^2_{1/2}$ loops on $\mathbb{C}^n$ given a smooth function $H : \mathbb{C}^n \to \mathbb{R}$ we define the action functional by:

$$\mathcal{L}_H(\gamma) = \int \frac{1}{2}\langle -J\dot{\gamma}(\theta), \gamma\rangle - H(\gamma(\theta))d\theta$$

where J is the standard complex structure in $\mathbb{C}^n$. The formal L^2 gradient of $\mathcal{L}_H$ is:

$$\nabla\mathcal{L}_H(\gamma) = -J\partial_\theta\gamma - \nabla H(\gamma)$$

Therefore, the upward gradient flow of $\nabla\mathcal{L}_H$ is:

$$\partial_t u(t,\theta) = \nabla\mathcal{L}_H(u(t,\theta)) = -J\partial_\theta u(t,\theta) - \nabla H(u(t,\theta))$$

We write this as a perturbed J-holomorphic curve equation as:

$$\partial_t u(t,\theta) + J(\partial_\theta u(t,\theta) - J \circ \nabla H(u(t,\theta))) = 0$$

Therefore, for a $u : [0,T] \times S^1 \to \mathbb{C}^n$ satisfying the perturbed holomorphic curve equation we have:

$$\mathcal{L}_H(u(T,\cdot)) - \mathcal{L}_H(u(0,\cdot)) = \int_0^T ||u(t,\cdot)||_{L^2} dt = E(u)$$

The polarization is defined by the splitting $T\mathfrak{B} = T^+\mathfrak{B} \oplus T^-\mathfrak{B}$ where $T^+\mathfrak{B}$ is spanned by the positive eigenvectors of $-J\partial_\theta$ and $T^-\mathfrak{B}$ by the nonpositive eigenvectors of $-J\partial_\theta$. We have constructed a Floer space and thus have an associated Floer homology group $HF(\mathfrak{B}, \mathcal{L}_H)$.

5.3 The Existence of a Critical Point

One of the central themes in [HZ94] is how the existence of a critical point of the function $\mathcal{L}_H$ leads to a variety of applications in symplectic geometry. For example, the celebrated nonsqueezing theorem is a rather direct consequence of the existence of a symplectic capacity which in turn is defined crucially using the existence of critical points of $\mathcal{L}_H$. In [HZ94] this is established by using compactness properties of the regularized gradient flow of $\mathcal{L}_H$ as well as the Leray-Schauder degree theory. In this section we will demonstrate how the existence of a critical point can be established using the unregularized gradient by appealing to the theory developed in this work. Recall that we have the decomposition $T\mathfrak{B} = T^+\mathfrak{B} \oplus T^-\mathfrak{B}$. We may identify $T\mathfrak{B}$ with $\mathfrak{B}$. Fix a unit vector $e^+ \in T^+\mathfrak{B}$. We assume H is smooth with $H = 0$ near 0 and $H(x) = (1+\epsilon)|x|^2$ for $|x|$ large. Following [HZ94], we have distinguished subsets:

$$\Sigma_\tau = \{\gamma | \gamma^- + se^+, ||\gamma^-||_{L^2_{1/2}} \leq \tau, 0 \leq s \leq \tau\}$$

and

$$\Gamma_\alpha = \{\gamma \in T\mathfrak{B}^+, ||\gamma||_{L^2_{1/2}} = \alpha\}$$

It is elementary to show (see [HZ94]) that for $\tau \gg 1$, $\mathcal{L}_{H|\partial\Sigma_\tau} \leq 0$ and there exists $\alpha > 0$ and $\beta > 0$ such that $\mathcal{L}_{H|\Gamma_\alpha} \geq \beta$. Note that $\Sigma_\tau \cap \Gamma_\alpha = \{\alpha e^+\}$ transversely.

Lemma 29. *Σ_τ is a cycle for $(-\mathfrak{B}, -\mathcal{L}_H)$ and Γ_α is a cycle for $(\mathfrak{B}, \mathcal{L}_H)$.*

Proof. The proofs are nearly identical so let us focus on Γ_α. The key observation is that

$$\mathcal{L}_H(\gamma^+) = \int \frac{1}{2}\langle -J\dot{\gamma}^+(\theta), \gamma^+\rangle - H(\gamma^+(\theta))d\theta = ||\gamma^+||^2_{L^2_{1/2}} - \int H(\gamma^+(\theta))d\theta$$

From this it follows that the action functional on γ^+ essentially coincides with the $L^2_{1/2}$ norm. Indeed, we may write $H(x) = H_c(x) + (1+\epsilon)|x|^2$ where H_c has compact support. Given $\gamma^+ \in \Gamma_\tau$ we have $||\gamma^+||_{L^2_{1/2}}$ bounded uniformly. Therefore, $\mathcal{L}_H$ is bounded. Lower semicontinuity follows from the fact that the $L^2_{1/2}$ norm can only drop in a weak limit and the fact that H_c is continuous for the weak topology. Given that $\mathcal{L}_H$ does not drop implies the $L^2_{1/2}$ does not drop which in turn implies convergence. □

As in the Seiberg-Witten case there is a notion of gradient flow given by the moduli space of perturbed holomorphic cylinders of a given length t, denotes Z_t. Given a cycle $\sigma : P \to \mathfrak{B}$, let $\mathcal{F}(\sigma) = \sigma \times_{\mathfrak{B}} Z_t$. The shrinking argument applies in this case to show that the new cycle is cobordant to the original one. In fact, the argument in this case is simpler. The uniform L^2_1 estimates follow directly from the L^2_1 compactness theorem. We indicate how to deal with the nonlinear term in this situation. First, we indicate how to bound the L^4 norm in two dimensions:

Lemma 30. *There exists uniform constant $C > 0$ such that given a function f on $S^1 \times [0, \epsilon]$*

vanishing on one of the ends, we have:

$$||f||_{L^4} \leq C||f||_{L^2_1}$$

Proof. As in the 4-dim case the argument reduces to that of a function on R^2 with support in the rectangle $[0,1] \times [0,\epsilon]$. We assume f vanishes on say $\{0\} \times [0,\epsilon]$ and $[0,1] \times \{0\}$. We have

$$|f(x,y)| \leq \int |\partial_1 f(x',y)|dx' \leq \epsilon^{1/2}(\int |\partial_1 f(x',y)|^2 dx')^{1/2}$$

Similar estimate with ∂_2 implies

$$|f(x,y)|^4 \leq \epsilon \int |\partial_1 f(x',y)|^2 dx' \int |\partial_2 f(x,y')|^2 dy'$$

Integrating, gives

$$\int |f(x,y)|^4 dxdy \leq \epsilon \int |\partial_1 f(x,y)|^2 dxdy \int |\partial_2 f(x,y)|^2 dydx \leq \epsilon(\int |\nabla f(x,y)|^2 dxdy)^2$$

□

Lemma 31. *We may write $J \circ \nabla H(x)$ as $F(x) \cdot x$ where F is a function with $F(0) = 0$ and $|F(x)| \leq C|x|$. Furthermore, $|F(x) \cdot x - F(y) \cdot y| \leq 2C(|x| + |y|)|x - y|$.*

Proof. Let $G(x) = J \circ \nabla H(x)$. Since $H(x) = 0$ when x is near 0, we have

$$G(x) = \int_0^1 DG(tx)dt \cdot x$$

Let $F(x) = \int_0^1 DG(tx)dt$. Since $DG(0) = 0$ and $|DG(tx)| \leq C|t||x|$, we have $F(0) = 0$ and $|F(x)| \leq C|x|$ as desired. Note that

$$|F(x) - F(y)| \leq C|x - y|$$

since

$$|DG(tx) - DG(ty)| \leq C|tx - ty|$$

We estimate:

$$|F(x) \cdot x - F(y) \cdot y| \leq |F(x) \cdot x - F(x) \cdot y| + |F(x) \cdot y - F(y) \cdot y| \leq 2C(|x| + |y|)|x - y|$$

□

Lemma 32. *Given functions α, β on the cylinder $S^1 \times [0,\epsilon]$, we have*

$$||J \circ \nabla H(\alpha) - J \circ \nabla H(\beta)||_{L^2} \leq 2C(||\alpha||_{L^4} + ||\beta||_{L^4})||\alpha - \beta||_{L^4}$$

where C is independent of ϵ.

Proof. Integrating the inequality of the previous lemma we get:

$$\int |J \circ \nabla H(\alpha) - J \circ \nabla H(\beta)|^2 d\theta dt \leq$$

$$C^2 \int (|\alpha|^2 + |\beta|^2)|\alpha - \beta|^2 d\theta dt \leq 4C^2(||\alpha||^2_{L^4} + ||\beta||^2_{L^4})||\alpha - \beta||^2_{L^4}$$

□

Borrowing notation from the section on the shrinking cylinder, we deduce that $J \circ \nabla H(Q_\epsilon(\beta) + P_\epsilon(v))$ is a contraction mapping for small enough ϵ and a fixed β.

Theorem 29. *There exists a critical point x of $\mathcal{L}_H$ with $\mathcal{L}_H(x) \geq \beta$.*

Proof. We argue by contradiction. Assume no such critical point exists. Then $\mathcal{L}_H(\mathcal{F}_t(\Gamma_\alpha)) > C$ for any $C > 0$ given that t is sufficiently large. This follows as in the section on the gradient flow. However, we then would have

$$\mathcal{F}_t(\Gamma_\alpha) \times_{\mathfrak{B}} \Sigma_\tau = \emptyset$$

since $\mathcal{L}_H$ restricted to Σ_τ is bounded above. This is impossible since $\mathcal{F}_t(\Gamma_\alpha)$ is cobordant to Γ_α by a cobordism staying away from points where $\mathcal{L}_H \leq 0$ and thus from $\partial(\Sigma_\tau)$, while Γ_α intersects Σ_τ transversely in a point. □

Chapter 6

Further Extensions of the Theory

6.1 Compactness for the Blown-Up Configuration Space

Define $\mathcal{B}^\sigma(X)$ as:

$$\mathcal{B}^\sigma(X) = \{(A, s, \Phi) \in L_1^2 | d^*(A - A_0) = 0, d^*_{\partial X}(B - B_0) = 0, |\Phi|_{L^2} = 1, s \in \mathbb{R}^{\geq 0}\}/\mathcal{G}_{h,X}$$

Similarly, we define $\mathcal{B}^\sigma(Y)$ as:

$$\mathcal{B}^\sigma(Y) = \{(B, t, \Psi) \in L_{1/2}^2 | d^*(B - B_0) = 0, |\Psi|_{L^2} = 1, t \in \mathbb{R}^{\geq 0}\}/\mathcal{G}_{h,Y}$$

Let $\pi^\sigma : \mathfrak{B}^\sigma \to \mathfrak{B}$ denote the blowdown map taking (A, s, Φ) to $(A, s\Phi)$.
We have the blown-up analog of the SW equations:

$$\mathcal{F}^\sigma(A, s, \Phi) = (\frac{1}{2}F^+_{A^\tau} - s^2(\Phi \otimes \Phi^*)_0, D^+_A\Phi)$$

We will at times refer to pairs $(A, s, \Phi) \in \mathcal{B}^\sigma(X)$ as γ^σ. We define:

$$\mathcal{M}^\sigma(X) = \{\gamma \in \mathcal{B}^\sigma(X) | \mathcal{F}^\sigma(\gamma^\sigma) = 0\}$$

Define $\mathcal{B}^\sigma_{red}(X)$ (resp. $\mathcal{B}^\sigma_{red}(Y)$) by setting s (resp. t) equal to zero.

Note that an L_1^2-bounded (or $L_{1/2}^2$-bounded) sequence converges strongly in L^2 and thus our spaces are closed under weak limits. We need the following unique continuation theorem:

Theorem 30. *If $(A, \Phi) \in \mathcal{M}^\sigma(X)$ then $\Psi \neq 0$. In fact, Φ cannot vanish on any component of the boundary.*

Proof. Suppose Φ vanishes on some end Y of X. We may extend X by gluing in a metric cylinder $[0, 1] \times Y$ to form X' and extend Φ by zero to the new manifold to form Φ'. Note that since Φ was assumed to vanish on $0 \times Y$ it extends to a L_1^2 section on X'. We have $D_{A_0}\Phi' = -\rho(A' - A_0)\Phi'$ where A' is any extension of A to an L_1^2 connection on X'. We now appeal to the weak unique continuation theorem [Man94] since Φ' is an L^2_1 configuration which vanishes on an open subset of X'. $\square$

In the blown-up context it is natural to introduce, in addition to $\mathcal{L}$,

$$\Lambda(A, \Phi) = -\langle \Psi, D_B \Psi \rangle / ||\Psi||^2_{L^2}$$

Here and below, Ψ will denote $\Phi_{|\partial X}$.
Notice Λ is scale invariant.

Our aim is to prove the following analog of the standard compactness theorem:

Theorem 31. *Given a sequence $(A_i, s_i, \Phi_i) \in \mathcal{M}^\sigma(X)$ with A_i converging in L^2_1 to A and s_i converging to s there exists a constant C such that:*

1. *$\Lambda(A_i, s_i, \Phi_i) > -C$*

2. *If $\Lambda(A_i, s_i, \Phi_i) < C'$ for some C' the Φ_i are weakly convergent to Φ (possibly after passing to a subsequence). Λ is lower semi-continuous for the weak topology.*

3. *If $\Lambda(A, \Phi) = \lim \Lambda(A_i, \Phi_i)$ the Φ_i converge strongly.*

Proof. The case not covered by the usual compactness theorem is when $s \neq 0$. We make use of the following key formula [KM07] valid for all configurations (A, Φ):

$$\int_X |D^+_A \Phi|^2 = \int_X |\nabla_A \Phi|^2 + \frac{1}{2} \int_X \langle \Phi, \rho_X(F^+_{A^\tau}) \Phi \rangle + \frac{1}{4} \int_X r_X |\Phi|^2 + \int_{\partial X} \langle \Phi, D_B \Phi \rangle$$

Where r_X is the scalar curvature of X. Note that for $(A, \Phi) \in \mathcal{M}^\sigma(X)$ the formula becomes:

$$0 = \int_X |\nabla_A \Phi|^2 + \frac{1}{4} \int_X r_X |\Phi|^2 + \frac{s_i^2}{2} \int_X |\Phi|^4 + \int_{\partial X} \langle \Psi, D_B \Psi \rangle$$

We also use the following:

Claim: $\int_X |\nabla_{A_i} \Phi_i|^2$ bounded implies $||\Phi_i||_{L^2_1}$ bounded.

To see this observe that in view of Kato's inequality we have a uniform L^4 bound and thus a bound on $\int_X |\nabla_{A_0} \Phi_i|^2$.

Proof of 1. By contradiction, choose Φ_i with $\Lambda(A_i, \Phi_i) < -i$. Since

$$-\int_{\partial X} \langle \Psi_i, D_{B_i} \Psi_i \rangle \leq 0$$

we must have $\int_X |\nabla_{A_i}\Phi_i|^2$ bounded. Thus, we have a weakly L^2_1-convergent Φ_i which implies strong L^2 convergence for both Φ_i and Ψ_i. Since $||\Phi||_{L^2} = 1$, Ψ cannot be zero by UCP. This bounds $|\Psi_i|^2_{L^2}$ from below and since $-\int_{\partial X}\langle\Psi_i, D_{B_i}\Psi_i\rangle$ is bounded so is Λ.

Proof of 2 and 3. To establish an L^2_1 bound on Φ_i we need to bound $-\int_{\partial X}\langle\Psi_i, D_{B_i}\Psi_i\rangle$. Since we assume $\Lambda(A_i, \Phi_i) < C$ the potential problem occurs when $|\Psi_i|^2_{L^2} \to \infty$. We rule this out by this following argument. Rescale Φ_i to obtain Φ'_i with $|\Psi'_i|^2_{L^2} = 1$. We have $\Phi'_i \to 0$ in L^2. Since $\Lambda(A_i, \Phi_i) = \Lambda(A_i, \Phi'_i)$ we have $-\int_{\partial X}\langle\Psi'_i, D_{B_i}\Psi'_i\rangle < C$. This implies a bound on $\int_X |\nabla_{A_i}\Phi'_i|^2$ which in turn gives us an L^2_1 bound on Φ'_i. Thus, Ψ_i converge strongly in L^2 to a Ψ' with $||\Psi'||_{L^2} = 1$. This is a contradiction since $\Phi' = 0$ and $\Phi'_{|\partial X} = \Psi'$.

With the L^2_1 bound in place weak convergence follows. Notice since both Φ_i and Ψ_i are strongly converging in L^2 and $||\Phi||_{L^2} = 1$, $||\Psi_i||_{L^2}$ is bounded below by UCP. Since

$$\int_X |\nabla_A\Phi|^2 \leq \liminf \int_X |\nabla_{A_i}\Phi_i|^2$$

we have

$$\int_{\partial X}\langle\Psi, D_B\Psi\rangle \geq \limsup \int_{\partial X}\langle\Psi_i, D_{B_i}\Psi_i\rangle$$

Note that this implies the lower semi-continuity of Λ. Finally, if $\Lambda(A, \Phi) = \lim\Lambda(A_i, \Phi_i)$ we can conclude $\int_{\partial X}\langle\Psi, D_B\Psi\rangle = \lim\int_{\partial X}\langle\Psi_i, D_{B_i}\Psi_i\rangle$ and thus

$$\int_X |\nabla_A\Phi|^2 = \int_X |\nabla_{A_i}\Phi_i|^2$$

so the (A_i, Φ_i) converge strongly. □

6.2 Chains in the Blown-Up Configuration Space

Definition 38. *A chain $\sigma : P \to \mathcal{B}^\sigma$ where P is a lc-manifold is a proper map satisfying the following axioms:*

Axiom 1^σ. *On* $\mathrm{im}(\pi^\sigma \circ \sigma)$, *$\mathcal{L}$ is bounded below and lower semi-continuous for the weak topology.*

Axiom 2^σ. *Given a weakly converging sequence $\pi^\sigma \circ \sigma(x_i)$ with* $\lim(\mathcal{L}(\pi^\sigma \circ \sigma(x_i)) = \mathcal{L}(y)$ *where y is the limit of $\pi^\sigma \circ \sigma(x_i)$, $\pi^\sigma \circ \sigma(x_i)$ converge strongly.*

Axiom 3^σ. *Any subset $S \subset$* $\mathrm{im}(\pi^\sigma \circ \sigma)$ *on which $\mathcal{L}$ is bounded is precompact for the weak topology.*

Axiom 4^σ. *$\Pi^- \circ D\sigma : TP \to T^-\mathcal{B}^\sigma$ is Fredholm, $\Pi^+ \circ D\sigma : TP \to T^+\mathcal{B}^\sigma$ is compact.*

Furthermore, given $\pi^\sigma \circ \sigma(x_i)$ converges strongly, we have:

Axiom 5^σ. *On $\mathrm{im}(\sigma(x_i))$, Λ is bounded below and lower semi-continuous for the weak topology.*

Axiom 6^σ. *Given a weakly converging sequence $\sigma(x_i)$ with $\lim(\Lambda(\sigma(x_i)) = \Lambda(y)$ where y is the limit of $\sigma(x_i)$, x_i converge strongly.*

Axiom 7^σ. *Any subset $S \subset \mathrm{im}(\sigma)$ on which Λ is bounded is precompact for the weak topology.*

6.3 Correspondences in the BlownUp Configuration Space

Let $\mathcal{B}^\sigma(-Y_0) \times^\sigma \mathcal{B}^\sigma(Y_1)$ be $(A_0, A_1, t, \Psi_0, \Psi_1)$ with $t \geq 0$ and $|\Psi_0|^2 + |\Psi_1|^2 = 1$. We have the blowdown map

$$\pi^\sigma : \mathcal{B}^\sigma(-Y_0) \times^\sigma \mathcal{B}^\sigma(Y_1) \to \mathfrak{B}(-Y_0) \times \mathfrak{B}(Y_1)$$

given by

$$(A_0, A_1, t, \Psi_0, \Psi_1) \mapsto (A_0, A_1, t\Psi_0, t\Psi_1)$$

We also have the partially defined projection maps

$$\pi_i((A_0, A_1, t, \Psi_0, \Psi_1)) = (A_i, t||\Psi_i||_{L^2}, \Psi_i/||\Psi_i||_{L^2})$$

6.3.1 Definitions

Definition 39. *A correspondence $(Z, f) \in Cor(Y_0, Y_1)$ is a proper map $f : Z \to \mathcal{B}^\sigma(-Y_0) \times^\sigma \mathcal{B}^\sigma(Y_1)$ where Z is a Hilbert manifold with boundary satisfying the following axioms:*

Axiom $1'^\sigma$ *On $\mathrm{im}(\pi^\sigma \circ f)$, $\mathcal{L}_1 - \mathcal{L}_0$ is bounded below and lower semi-continuous for the weak topology.*

Axiom $2'^\sigma$ *If $\mathcal{L}_1 \circ \pi^\sigma \circ \pi_1(f(z_i))$ is bounded above and $\pi^\sigma \circ \pi_0(f(z_i))$ is a weakly convergent sequence then $\pi^\sigma \circ \pi_1(z_i)$ converges weakly.*

Axiom $4'^\sigma$ *Given $\pi^\sigma \circ \pi(z_i)$ is weakly convergent to x, if $\lim f^*(\mathcal{L}_1 - \mathcal{L}_0)(z_i) = (\mathcal{L}_1 - \mathcal{L}_0)(x)$ and $\pi^\sigma \circ \pi_0(z_i)$ is converges strongly then $\pi^\sigma \circ f(z_i)$ converges strongly.*

Axiom $5'^\sigma$ *$Df : TZ \to T^+\mathcal{B}^\sigma_{Y_0} \oplus T^-\mathcal{B}^\sigma_{Y_1}$ is Fredholm. $Df : TZ \to T\mathcal{B}^\sigma(Y_0)$ is dense. Given a bounded sequence $v_i \in TZ$, if $\pi_0(Df)(v_i)$ is weakly convergent, so is $\pi_1(Df)(v_i)$.*

Furthermore, if $\pi^\sigma \circ f(z_i)$ converges strongly, we have:

Axiom $6'^\sigma$ *If $\Lambda \circ \pi_0 \circ f(x_i) > -C$ then $\Lambda \circ \pi_1 \circ f(x_i) > -C'$ for some C'.*

Axiom $7'^\sigma$ *Λ is bounded below and lower semi-continuous on $f(x_i)$. If $\Lambda(f(x_i)) < C$ for some C $||\Psi_1|| > C'$ and $||\Psi_0|| > C'$ for some $C' > 0$. Furthermore, $\pi_0 \circ f(x_i)$ converges weakly then $\pi_1 \circ f(x_i)$ converges weakly.*

Axiom $8'^\sigma$ *If z is the weak limit of $f(x_i)$ with $\lim(\Lambda(f(x_i))) = \Lambda(z)$ and $\pi_0 \circ f(x_i)$ converge strongly then x_i converge strongly in Z*

Axiom $9'^\sigma$ *If $(A_0, A_1, s, \Psi_0, \Psi_1)$ is in the weak closure of $\mathrm{im}(f)$, we have $\Psi_i \neq 0$*

Theorem 32. *Given a chain P and a correspondence Z, $P \times_{\mathcal{B}^\sigma(Y_0)} Z$ is a chain for $\mathcal{B}^\sigma(Y_1)$.*

Proof. Most axioms follow immediately from the definitions. We verify Axioms $5^\sigma - 7^\sigma$. For 5^σ we verify lower-semicontinuity. We have

$$\Lambda(\Psi_0^i, \Psi_1^i) = -\Lambda(\Psi_0^i)||\Psi_0^i||^2 + \Lambda(\Psi_1^i)||\Psi_1^i||^2$$

From the bound $\Lambda(\Psi_1^i)||\Psi_1^i||^2 < C$ and $\Lambda(\Psi_0^i, \Psi_1^i) > -C$ we have $\Lambda(\Psi_0^i)||\Psi_0^i||^2 < C'$ and $\Lambda(\Psi_0^i, \Psi_1^i) < C'$. Since $||\Psi_0^i|| > C'$ we conclude that $\Lambda(\Psi_0^i) < C''$ and thus $\pi_0(\Psi_0, \Psi_1)$ converges weakly which implies $\pi_1(\Psi_0, \Psi_1)$ converges weakly as well. Therefore, the lower-semicontinuity of $\Lambda(\Psi_0^i)$ together with that of $\Lambda(\Psi_0^i, \Psi_1^i)$ implies the same of $\Lambda(\Psi_1^i)$. □

6.3.2 Cobordisms

A cobordism W from Y_0 to Y_1 gives rise to a correspondence. This follows from our compactness proof on the blowup. The only nontrivial point that needs to be addressed is Axiom $6'^\sigma$. Indeed, let Φ_i be the spinor component of $f(x_i)$ and let Ψ_i^0 and Ψ_i^1 be the restrictions to the boundary. we have $\Lambda(\Psi_i^0) > -C$ and we want to conclude $\Lambda(\Psi_i^1) > -C'$. We have

$$\Lambda(\Phi_i) = \frac{-\Lambda(\Psi_i^0)||\Psi_i^0||^2 + \Lambda(\Psi_i^1)||\Psi_i^1||^2}{||\Psi_i^0||^2 + ||\Psi_i^1||^2} > -C''$$

and thus

$$\frac{\Lambda(\Psi_i^1)}{1 + ||\Psi_i^0||^2/||\Psi_i^1||^2} > -C'' + \frac{\Lambda(\Psi_i^0)}{1 + ||\Psi_i^1||^2/||\Psi_i^0||^2} > -C'''$$

so the only potential difficulty occurs when $\Lambda(\Psi_i^1) < 0$ and $||\Psi_i^1||/||\Psi_i^0|| \to 0$. First assume $\Lambda(\Phi_i)$ is bounded. Then, by the compactness theorem, Φ_i converge weakly in L_1^2 and thus Φ_i, Ψ_i^1 and Ψ_i^0 converge strongly in L^2. Since $||\Phi_\infty||_{L^2} = 1$ we have Ψ_i^1 and Ψ_i^0 both nonzero.

This rules out $||\Psi_i^1||/||\Psi_i^0|| \to 0$. Finally, assume $\Lambda(\Phi_i) =\!\!\rightarrow \infty$. We have

$$\frac{\Lambda(\Psi_i^1)}{1 + ||\Psi_i^0||^2/||\Psi_i^1||^2} = \Lambda(\Phi_i) + \frac{\Lambda(\Psi_i^0)}{1 + ||\Psi_i^1||^2/||\Psi_i^0||^2}$$

Since $\Lambda(\Psi_i^0) > -C$ this implies $\Lambda(\Psi_i^1) > 0$ for i large.

6.4 Trivial Cobordism

As in the "hat" version it is crucial to show that the trivial cobordism induces the identity map in homology. While the main ideas are the same, special care needs to be taken to treat the blown-up case. We begin by verifying the convergence properties as the size of the cylinder shrinks to zero. We start with a lemma:

Lemma 33. *There exists $C > 0$ such that on $Y \times [0, \epsilon]$ the we have*

$$||f||_{L^4} \leq \epsilon^{-1/4} C ||f||_{L^2_1}$$

Proof. On $Y \times [0, \epsilon]$ we have $||\tilde{f}||_{L^4} \leq C||\tilde{f}||_{L^2_1}$ Now, let $f = \tilde{f}(y, t/\epsilon)$ viewed as a function on $Y \times [0, \epsilon]$. We have $||\tilde{f}||_{L^4} = \epsilon^{-1/4}||f||_{L^4}$ and $||\tilde{f}||_{L^2_1} \leq \epsilon^{-1/2}||f||_{L^2_1}$. Therefore, we have

$$||f||_{L^4} = \epsilon^{1/4}||\tilde{f}||_{L^4} \leq C||\tilde{f}||_{L^2_1} \leq C\epsilon^{-1/4}||f||$$

□

As our setup, we have a sequence (A_i, s_i, Φ_i) on cylinders of shrinking length with $\pi^\sigma(A_i, s_i\Phi_i)$ strongly convergent. This implies that the A_i are uniformly L^4 bounded and $s_i \to 0$. We need to rescale to determine whether the limiting configuration is reducible. We use the following lemma.

Lemma 34. *We have $s_i/\epsilon_i^{1/2}$ bounded. Let s_∞ be a limit of this sequence. If $s_\infty > 0$ the $s_i\Phi_i$ converge to an irreducible configuration.*

First, we verify axiom $6'^\sigma$. By the argument in the previous section, when $\Lambda(\Phi_i) \to \infty$ we have $\Lambda(\Psi_i^2) > C$. We need only consider the case when $\Lambda(\Phi_i) < C$. Rescale Φ_i so that its restriction to each slice $t \times Y$ has $||\Psi_i^t||_{L^2} \leq 1$ with equality on some slice. We have $||\Phi_i||_{L^2} \leq \epsilon^{1/2}$ We have

$$\int_X |\nabla_{A_i}\Phi_i|^2 \leq \epsilon C$$

and using the estimates from the lemma above together with Kato's inequality

$$\int_X |\nabla_A \Phi_i|^2 \leq (1 + \epsilon^{-1/2}C) \int_X |\nabla_{A_i}\Phi_i|^2 \leq \epsilon^{1/2}C'$$

With this uniform L_1^2 norm in place observe that

$$||\Psi_i^1||^2 - ||\Psi_i^0||^2 \leq \int_0^1 ||\dot{\Phi}(t)||_{L^2} dt \leq ||\Phi||_{L_1^2} \epsilon^{1/2} \leq C\epsilon^{3/2}$$

6.5 Equivariant Theory

The theory of geometric cycles discussed in this work lends itself well to various equivariant formulations. Below, we sketch some of them.

6.5.1 Cartan Construction

Definition 40. *Let I be the map taking $f : P \to \mathfrak{B}$ to $F : P \times S^1 \to \mathfrak{B}$ with $F(p, e^{i\theta}) = f(p)e^{i\theta}$*

Definition 41. *Let $HFC_*^+(\mathfrak{B}, \mathcal{L})$ be the homology theory associated with the complex $(CF_*(\mathfrak{B}) \otimes \mathbb{Z}_2[u], \partial_I)$ where u has degree -2. The differential of the complex is $\partial_I(P \otimes u^k) = \partial(P)u^k + I(P)u^{k+1}$.*

Lemma 35. $\partial_I^2 = 0$

Proof. We compute

$$\partial_I^2(P) = \partial_I(\partial(P) + I(P)u) = \partial^2(P) + 2I(\partial P)u + I^2(P)u^2$$

Observe that $I(P)$ is already S^1 invariant and thus $I^2(P)$ has the same image as $I(P)$. From our construction of CF_*, this implies $I^2(P)$ is negligible. □

The new group fits into a long exact sequence with the original group:

Theorem 33. *There is a long exact sequence:*

$$\cdots \to HFC_{k+2}^+(\mathfrak{B}) \to HFC_k^+(\mathfrak{B}) \to HF_k(\mathfrak{B}) \to \dots$$

Proof. We have an exact sequence of complexes:

$$0 \to CF_{k+2}(\mathfrak{B}) \otimes \mathbb{Z}_2[u] \xrightarrow{u} CF_k(\mathfrak{B}) \otimes \mathbb{Z}_2[u] \to CF_k(\mathfrak{B}) \otimes \mathbb{Z}_2[u]/u \cdot CF_{k+2}(\mathfrak{B}) \otimes \mathbb{Z}_2[u] \to 0$$

Since $CF_k(\mathfrak{B}) = CF_k(\mathfrak{B}) \otimes \mathbb{Z}_2[u]/u \cdot CF_{k+2}(\mathfrak{B}) \otimes \mathbb{Z}_2[u]$ the desired conclusion follows. □

The group HFC^+ also fits into a long exact sequence resembling the one in monopole Floer homology.

Definition 42. *Let $HFC_*^\infty(\mathfrak{B}, \mathcal{L})$ be the homology theory associated with the complex $(CF_*(\mathfrak{B}) \otimes \mathbb{Z}_2[u, u^{-1}], \partial_I)$. Let $HFC_*^-(\mathfrak{B}, \mathcal{L})$ be the homology of the complex $CF_*(\mathfrak{B}) \otimes \mathbb{Z}_2[u, u^{-1}]/u \cdot CF_*(\mathfrak{B}) \otimes \mathbb{Z}_2[u]$*

Theorem 34. *There is a long exact sequence:*

$$\cdots \to HFC^{+}_{k+2}(\mathfrak{B}) \to HFC^{\infty}_{k}(\mathfrak{B}) \to HFC^{-}_{k}(\mathfrak{B}) \to \dots$$

We can define a pairing $HFC^{-}_{*}(\mathfrak{B}, \mathcal{L}) \otimes HFC^{+}_{*}(\mathfrak{B}, \mathcal{L}) \to H^{sing}_{*}(\mathfrak{B})$

Appendix A

Geometric Preliminaries

A.1 Spaces Stratified By Hilbert Manifolds

Definition 43. *A second countable Hausdorff space P has a* ***stratification by Hilbert manifolds of depth*** *k, if $P^k \subset P^{k-1} \dots P^0 = P$ where for each i, P^i is closed in P and the open stratum $P^i - P^{i+1}$ is Hilbert manifold. A* ***stratum smooth map*** *$f : P \to X$ where X is a Hilbert manifold is a continuous map smooth on each open stratum. Such a map is said to be* ***transverse*** *to a submanifold $Y \subset X$ if it is transverse on each stratum.*

Note that the product of P and Q, for any two such spaces, is also stratified by Hilbert manifolds.

A.2 Locally Cubical Hilbert Manifolds

Let $\vec{t}(k) = [0,1)^k$. We view $\vec{t}(k)$ is a stratified space in the natural way. Let $\vec{t}$ denote a typical coordinate in $\vec{t}(k)$. At times, by abuse of notation, we let $\vec{t}(k)$ denote a neighborhood of the origin in $[0,1)^k$

Definition 44. *Given a space P stratified by Hilbert manifolds and an open set V in $P^i - P^{i+1}$, a* ***locally cubical Hilbert manifold*** *chart about V is an embedding (of stratified Hilbert manifolds)*

$$f : V \times \vec{t}(i) \to U \subset P$$

where U is open in P and $f(v,0) = v$.

Definition 45. *A* ***locally cubical Hilbert manifold*** *(or lc-manifold for short) is a stratified Hilbert space P with a cover by locally cubical charts as above with no further compatibility assumptions (other than those impossed by the being embeddings of stratified Hilbert manifolds).*

Lemma 36. *The product of two lc-manifolds is a lc-manifold.*

Proof. Since $\vec{t}(k) \times \vec{t}(k') = \vec{t}(k+k')$ in a canonical way, a cover is specified by charts of the form $V \times V' \times \vec{t}(k+k')$. $\square$

Definition 46. *A* ***smooth map*** *σ from a lc-manifold P to a Hilbert manifold X is a stratum smooth map such that each point has at least one chart $V \times \vec{t}(i)$ where σ has the form $\sigma(v, \vec{t})$ with σ is smooth in the v coordinates and, along with its v-derivative, continuous in the $\vec{t}$ coordinates.*

Remark. The reason for restricting to lc-maps as opposed to say smooth maps from manifolds with corners will become apparent when dealing the shrinking cylinder argument in the section "The Trivial Cobordism".

Remark. Given a smooth map $\sigma : P \to X$ as in the previous lemma and a smooth map $f : X \to Y$ of Hilbert manifolds the composition $f \circ \sigma : P \to Y$ is also smooth.

Lemma 37. *Given a smooth map $\sigma : P \to X$ as in the previous lemma and a closed submanifold $Y \subset X$ such that σ is transverse to Y, $\sigma^{-1}(\Delta)$ is an lc-manifold.*

Proof. Note that $\sigma^{-1}(\Delta)$ is naturally a space stratified by Hilbert manifolds. In a chart, we are reduced to the following local situation. Given $\sigma : V \times \vec{t}(k) \to W$ where V, W are Hilbert spaces and f is smooth in the v variables and, along with its first v-derivative, continuous in the $\vec{t}$ variables. Assume $\sigma_0(v) = \sigma(v, 0)$ is has 0 as a regular value. Then, locally there is a stratum preserving smooth homeomorphism $\sigma_0^{-1}(0) \times \vec{t}(k) \to \sigma^{-1}(0)$. This, in turn, follows from the inverse function theorem with dependence on a parameter. $\square$

A.3 Boundary Operator

Note that a stratified space such as $V \times \vec{t}(k)$ has a natural order on its open $k-1$ dimensional stratum since $(V \times \vec{t}(k))^{k-1} = V \times \cup_i(\vec{t}(k-1), f_i)$ where $f_i : \vec{t}(k-1) \to \vec{t}(k)$ are the face inclusion maps obtained by omitting the ith coordinate.

Definition 47. *An lc-manifold P is said to be* ***ordered*** *if we are given a maximal atlas of lc-charts so that the overlap maps preserve the natural ordering on the strata.*

Definition 48. *Define $\partial\vec{t}(k) = \coprod_i(\vec{t}(k-1), f_i)$*

Definition 49. *Given a lc-manifold P we construct ∂P in steps. As the top stratum take $P^1 - P^2$. Given a chart of the type $V \times \vec{t}(2)$ glue in $V \times \partial\vec{t}(2)$ to $P^1 - P^2$ using the inclusion maps. In general, to obtain the kth stratum, glue in $V \times \partial\vec{t}(k+1)$.*

Remark. ∂P has a natural lc-manifold structure and inherits an ordering if P is ordered.

Lemma 38. *If P is ordered, then $\partial^2 P$ is naturally a disjoint union of two identical lc-manifolds P_1 and P_2.*

Proof. Locally, $\partial^2(V \times \vec{t}(k)) = V \times \partial^2\vec{t}(k)$. Now, $\partial^2\vec{t}(k) = \coprod_{i \neq j}(\vec{t}(k-2), \sigma_{ij})$ where σ_{ij} is the inclusion of the $(k-2)$ face obtained by ignoring the i^{th} and j^{th} factor. Thus $\partial^2\vec{t}(k)$ is naturally a disjoint union of two spaces corresponding to when $i < j$ and $i > j$. $\square$

A.4 Cutting by a Hypersurface

We will need to cut a lc-manifold by a hypersurface obtaining a decomposition of the manifold into two parts. Here is the main result:

Theorem 35. *Consider a hypersurface $H \subset M$ separating M into closed regions H^+ and H^- with common boundary H. Let $\sigma : P \to M$ be a lc-map transverse to H. We have that $\sigma^{-1}(H^\pm)$ are lc-manifolds. Furthermore,*

$$\partial(\sigma^{-1}(H^\pm)) = (\sigma_{|\partial(P)})^{-1}(H^\pm) \sqcup \sigma^{-1}(H)$$

Proof. Locally we have $\sigma : V \times \vec{t}(k) \to M$. Applying the inverse function theorem, we can represent $\sigma^{-1}(H^\pm)$ as $(\sigma_{|V})^{-1}(H^\pm) \times \vec{t}(k)$, where $(\sigma_{|V})^{-1}(H^\pm)$ is of depth one. Note that, given that P is ordered, we can order $\sigma^{-1}(H^\pm)$ by taking the new stratum to be first. The formula for the boundary follows since

$$\partial(\sigma_{|V}^{-1}(H^\pm) \times \vec{t}(k)) = \partial(\sigma_{|V}^{-1}(H^\pm)) \times \vec{t}(k) \sqcup \sigma_{|V}^{-1}(H^\pm) \times \partial(\vec{t}(k))$$

and

$$\partial(\sigma_{|V}^{-1}(H^\pm)) = \sigma_{|V}^{-1}(H)$$

□

Finally, let us record the fact, (useful in defining homology) that $\sigma^{-1}(H^+) \sqcup \sigma^{-1}(H^-)$ and P are cobordant:

Theorem 36. *Given a P with $\partial P = \emptyset$, there exists an lc-manifold W with $\partial W = \sigma^{-1}(H^+) \sqcup \sigma^{-1}(H^-) \sqcup P$*

Proof. Let us prove the local result. Let $P = V$ and $\sigma^{-1}(H^\pm) = V^\pm$. Locally, we may write V^+ as $V^0 \times [0, 1)$ and V^- as $V^0 \times (-1, 0]$. In complex coordinates, the map $z \mapsto z^2$ identifies $[0, 1)i$ with $[0, -1)$ and $[0, 1)$ with $[0, 1)$. This gives us an identification of neighborhoods of the origin in $[0, 1) \times [0, 1)$ with that of $(-1, 1) \times [0, 1)$. As a topological space, let $W = V \times [0, 1]$. Near $V \times 1$, let W have the induced stratified structure. Thus, $V \times (0, 1)$ is the top stratum and $V \times 1$ is a codimension 1 stratum. Near 0, let the codimension 1 stratum be $V - V^0$ and the codimension 2 stratum be V^0. The lc-chart near a point in V^0 is provided by

$$V^0 \times [0, 1) \times [0, 1) \to V^0 \times (-1, 1) \times [0, 1)$$

where we identify $V^0 \times (-1, 1)$ with V as discussed. We have:

$$\partial(W) = V \sqcup V^+ \sqcup V^-$$

□

Remark. More generally, if P is allowed to have a boundary, the same construction implies that there exists an lc-manifold W such that $\partial W = \sigma^{-1}(H^+) \sqcup \sigma^{-1}(H^-) \sqcup P + Q$ where Q only depends on ∂P and in particular has the same image as ∂P.

A.5 Extension to Manifolds with Boundary

In applications, we will consider the case when the target manifold is a Hilbert manifold with boundary. Therefore, we propose the following extension of lc-manifolds to the boundary case.

Definition 50. *A second countable Hausdorff space P is* ***stratified by Hilbert manifolds with boundary*** *if $P^k \subset P^{k-1} \dots P^0 = P$ where for each i, $P^i - P^{i-1}$ is Hilbert manifold with boundary and P^i is closed in P. Furthermore, we assume the subspace formed by the boundaries $\partial(P^i - P^{i-1})$ is closed in P. A stratum smooth map $f : P \to X$ where X is a Hilbert manifold with boundary is said to be proper if on each component V_i of $P^i - P^{i-1}$ its a map $(V_i, Bd(V_i)) \to (X, Bd(X))$ with its differential an inclusion on the normal bundle of the boundary or, if $Bd(V_i) = \emptyset$, it maps V_i either to $X - Bd(X)$ or to $Bd(X)$.*

Let us discuss how to modify the previous discussion to take the presence of a boundary into account. An lc-chart is of the form $V \times \vec{t}(k)$ where V is now a manifold with boundary. Given two transverse proper maps $f : P \to X$ and $g : Q \to X$ we may take their fiber product which is again a lc-manifold with boundary.

Definition 51. *Given P, an lc-manifold with boundary $Bd(P)$ is the lc-manifold obtained by restricting to the boundary in each stratum.*

In view of this extra boundary component, we must modify the boundary operator:

Definition 52. *For $V \times \vec{t}(k)$, let $\partial(V \times \vec{t}(k)) = Bd(V) + V \times \partial\vec{t}(k)$*

As before, this definition extends readily to the case of a ordered lc-manifold with boundary. Note that $\partial(V)$ is the sum of $Bd(V)$ and the boundary operator defined in the previous section. We have as before:

Lemma 39. *If P is ordered, then $\partial^2 P$ is naturally a disjoint union of two identical lc-manifolds P_1 and P_2.*

Proof. Locally, $\partial^2(V \times \vec{t}(k)) = \partial(V \times \partial\vec{t}(k)) + \partial(Bd(V) \times \vec{t}(k)) = V \times \partial^2\vec{t}(k) + Bd(V) \times \partial\vec{t}(k) + Bd(V) \times \partial\vec{t}(k)$. □

Appendix B

Pseudohomology

B.1 Definition of Pseudocycle Homology $\mathcal{H}F_*(\mathfrak{B}, \mathcal{L})$

As in the finite dimensional story it is possible to define a geometric version of homology similar in spirit to bordism without introducing manifolds with corners (or in our case, lc-manifolds of arbitrary depth). In the finite dimensional case this is known as pseudohomology.

Definition 53. *A* ***pseudochain*** *of index k is a map $\sigma : P \to \mathfrak{B}$ where P is a lc-manifold satisfying the following axioms:*

Axiom 1. *On* $\mathrm{im}(\sigma)$*, $\mathcal{L}$ is bounded below and lower semi-continuous for the weak topology.*

Axiom 2^p. *There exists a $(k-2)$-negligible set $N \subset \mathfrak{B}$ such that given a weakly converging sequence $\sigma(x_i)$ with limit y, if $\lim(\mathcal{L}(\sigma(x_i)) = \mathcal{L}(y)$ then either x_i (up to a subsequence) converge strongly on P or $y \in N$.*

Axiom 3. *Any subset $S \subset \mathrm{im}(\sigma)$ on which $\mathcal{L}$ is bounded is precompact for the weak topology.*

Axiom 4. $\Pi^- \circ D\sigma : TP \to T^-\mathfrak{B}$ *is Fredholm of index k,* $\Pi^+ \circ D\sigma : TP \to T^+\mathfrak{B}$ *is compact.*

Definition 54. *Let $\mathcal{H}F_k(\mathfrak{B}, \mathcal{L})$ be the $\mathbb{Z}_2$ vector space generated by pseudochains of depth 0 and index k subject to the following relation: $[P] = 0$ if there exists a pseudochain W of depth 1 with $\partial W = P$*

In the definition of pseudochains we restrict to lc-manifolds of depth at most 1 and thus avoid talking about corners.

Theorem 37. *Given, $\sigma \in \mathcal{H}F_k(\mathfrak{B}, \mathcal{L})$ and $\tau \in \mathcal{H}F_l(-\mathfrak{B}, -\mathcal{L})$ with representatives σ and τ having transverse intersection, their fiber product $\sigma \times_{\mathfrak{B}} \tau$ is a closed manifold mapping to $\mathfrak{B}$.*

The fiber product gives a well-defined map

$$\mathcal{H}F_k(\mathfrak{B}, \mathcal{L}) \times \mathcal{H}F_l(-\mathfrak{B}, -\mathcal{L}) \to \mathcal{H}F_{k+l}(\mathfrak{B})$$

where $\mathcal{H}F_{k+l}(\mathfrak{B})$ *denotes ordinary pseudohomology with* $\mathbb{Z}_2$*-coefficients.*

Appendix C

Analytic Lemmas

C.1 The Borderline Gauge Group

First we prove a lemma about convergence:

Lemma 40. *Consider two sequences of positive L^1-functions f_i, g_i on a finite measure space X. Assume $f_i \to f$, $g_i \to g$ in L^1 and $||g_i||_\infty \leq C$. Then $\lim_i \int_X f_i g_i = \int_X fg$*

Proof. First we show that

$$\lim_i \int_X f g_i = \int_X fg$$

Given an $\epsilon > 0$, write $f = f^b + f^u$ as a sum of a positive functions with f^b bounded and f^u unbounded. Furthermore, we can choose this decomposition so that $\int_X f^u \leq \epsilon / C$. We have

$$\int_X f|g - g_i| = \int_X f^b|g - g_i| + \int_X f^u|g - g_i| \leq ||f^b||_\infty \int_X |g - g_i| + ||g - g_i||_\infty \cdot \int_X f^u$$

Since $||g - g_i||_\infty \cdot \int_X f^u \leq 2\epsilon$ and $||f^b||_\infty \int_X |g - g_i| \to 0$ the claim follows. To finish the lemma observe that

$$\int_X |(f_i - f)g_i| \leq C \int_X |f_i - f|$$

we have

$$\lim_i \int_X f_i g_i = \lim_i \int_X f g_i$$

since

$$\int_X |f - f_i| g_i \leq C \int_X |f - f_i| \to 0$$

□

Given a compact 4-manifold X, let $\mathcal{G}(X)$ be the set of L^2_2 maps $g : X \to \mathbb{C}$ with image on $S^1 \subset \mathbb{C}$ almost everywhere.

Lemma 41. *$\mathcal{G}$ is a topological group with the subspace topology*

Let $W \to X$ be some complex vector bundle. Let $\Gamma(W)$ be the space of sections with the L_1^2 topology.

Lemma 42. *The multiplication map $\mathcal{G} \times \Gamma(W) \to \Gamma(W)$ is continuous.*

If we take gauge transformations with higher Sobolev norms we get smoothness of the action:

Lemma 43. *Let $\mathcal{G}_{L_3^2}$ denote the group of L_3^2 configurations. The map $\mathcal{G}_{L_3^2} \times \Gamma(W) \to \Gamma(W)$ is smooth.*

C.2 Weakly Convergent Operators

Definition 55. *A sequence of operators $A_i : V \to W$ between Hilbert spaces is said to converge weakly, if there exists a bounded operator $A_\infty : V \to W$ such that for any $v \in V$ we have $A_i(v) \to A_\infty(v)$.*

Lemma 44. *Given a uniformly bounded weakly converging sequence of operators $A_i : W \to U$ between Hilbert spaces and a compact operator $K : V \to W$, $A_i \circ K$ converge strongly to $A_\infty \circ K$.*

Proof. Taking the new sequence $A_i - A_\infty$ we can assume $A_\infty = 0$. By contradiction, suppose there exists a sequence v_i with $|v_i| = 1$ and $|A_i \circ K(v_i)| \geq C > 0$. Since K is compact, the elements $w_i = K(v_i)$ have a converging subsequence w_j with limit w_∞. By assumption, $\lim_i(A_i(w_\infty)) = 0$. Since A_i are uniformly bounded we have

$$A_j(w_j) = A_j(w_j - w_\infty) + A_j(w_\infty)$$

Since A_j are uniformly bounded and we have

$$\lim_j |A_j(w_j - w_\infty)| \leq const \cdot \lim_j |w_j - w_\infty| = 0$$

and thus $\lim_j A_j(w_j) = 0$ contradicting the fact that $|A_j(w_j)| = |A_j \circ K(v_j)| \geq C > 0$ □

Similarly we have:

Lemma 45. *Given a uniformly bounded weakly converging sequence of operators $A_i : V \to W$ between Hilbert spaces and a compact operator $K : W \to U$. Assume A_i^* is also weakly converging with limit $(A_\infty)^*$. We have, that $K \circ A_i$ converge strongly to $K \circ A_\infty$.*

Proof. Apply the previous first lemma to A_i^* and K^*. □

Finally, combining the previous two lemmas we obtain:

Lemma 46. *Given a uniformly bounded weakly converging sequence of operators $A_i : V \to W$ and $A'_i : V' \to W'$ and a strongly convergent sequence of compact operators $K_i : W \to V'$, $A'_i \circ K_i \circ A_i$ converge strongly to $A'_\infty \circ K_\infty \circ A_\infty$.*

Proof. We have

$$A'_i \circ K_i \circ A_i - A'_i \circ K_\infty \circ A_i = A'_i \circ (K_i - K_\infty) \circ A_i$$

and

$$|A'_i \circ (K_i - K_\infty) \circ A_i| \leq |A'_i||K_i - K_\infty||A_i|$$

Thus, since A_i and A'_i are uniformly bounded, it suffices to assume $K_i = K_\infty$. On the one hand, the previous lemmas imply that the uniformly bounded sequence of compact operators $T_i = K \circ A_i$ is strongly convergent. Now, apply the same argument to $A'_i \circ T_i$. □

C.3 Regularity for L^2_1 Configurations

In this section we establish higher regularity and compactness for the Seiberg-Witten equations. The method is proof is inspired by the treatment of the self-dual equations in [FU91]. For clarity, we abstract the situation somewhat. Consider a vector bundle E over a closed oriented Riemannian 4-manifold X. To shorten notation let L^2_k denote L^2_k sections of E. Assume E has some fibrewise multiplication $\sharp$. Given a section ϕ we have a linear operator $Q_\phi(\psi) = \phi \sharp \psi$. Recall the following consequences of the Sobolev multiplication and embedding theorems:

Lemma 47. *If $\phi \in L^2_1$, it induces a compact operator Q_ϕ on $L^2 \to L^2_{-1}$ and $L^2_k \to L^2$ for $k \geq 1$. If $\phi \in L^2_k$ with $k > 2$, it induces a compact operator Q_ϕ on $L^2_i \to L^2_i$.*

Let $\Delta = \nabla \circ \nabla^*$ be the Laplacian on E. Let K_1, K_2 be first order differential operators on X.

Theorem 38. *Given $\phi \in L^2_1$ and $g \in L^2$ assume they satisfy the equation*

$$\Delta\phi + K_1\phi + (K_2\phi)\sharp\phi = g$$

Then $\phi \in L^2_2$. Furthermore, if $g \in C^\infty$ then $\phi \in C^\infty$.

Proof. Observe that ϕ satisfies the equation

$$(\Delta + c)\phi + K_1\phi + (K_2\phi)\sharp\phi = g + c\phi$$

Consider $\psi \in L^2_2$ satisfying

$$(\Delta + c)\psi + K_1\psi + (K_2\psi)\sharp\phi = g + c\phi$$

We claim that such a solution exists if c is chosen sufficiently large. Indeed, for $c > 0$ the operator $\Delta + c$ is invertible. Furthermore, $(\Delta + c)^{-1}$ converges weakly to zero as $c \to \infty$.

Composing the previous equation with $(\Delta+c)^{-1}$ we get

$$\psi+(\Delta+c)^{-1}K_1\psi+(\Delta+c)^{-1}(K_2\psi)\sharp\phi=(\Delta+c)^{-1}(g+c\phi)$$

Lemma 44 implies that $T(\psi)=(\Delta+c)^{-1}K_1\psi+(\Delta+c)^{-1}(K_2\psi)\sharp\phi$ converges to 0 strongly as $c\to\infty$. Thus, for sufficiently large c, the solution ψ exists. Finally, we claim $\phi=\psi$. This argument is similar. Consider $u=\phi-\psi$ which satisfies

$$(\Delta+c)u+K_1u+(K_2\phi)\sharp u=0$$

now viewed as an operator $L^2_1\to L^2_{-1}$. For large c, the operator is invertible and thus $u=0$. Given that g is smooth we can repeat this argument now with an operator $L^2_3\to L^2_1$ to conclude that $\phi\in L^2_3$ and so on. □

We can also deduce higher convergence from initial L^2_1 convergence of such a sequence.

Theorem 39. *Given a sequence ϕ_i all satisfying the equation in the theorem (with the same g, K_1 and K_2) assume ϕ_i are converging to ϕ in L^2_1. If $g\in C^\infty$, then ϕ_i are converging in all Sobolev norms.*

Proof. The argument is similar. The difference $\phi_i-\phi$ satisfies the equation

$$(\Delta+c)(\phi_i-\phi)+K_1(\phi_i-\phi)+K_2(\phi_i-\phi)\sharp(\phi_i-\phi)+K_2(\phi_i-\phi)\sharp\phi+K_2\phi\sharp(\phi_i-\phi)=g+c(\phi_i-\phi)$$

Consider the equation for ψ_i

$$(\Delta+c)(\psi_i)+K_1(\psi_i)+K_2(\psi_i)\sharp(\phi_i-\phi)+K_2(\psi_i)\sharp\phi+K_2\phi\sharp(\psi_i)=g+c(\phi_i-\phi)$$

As before, for an appropriate c we can find such $\psi_i\in L^2_2$ solving the equation and $\psi_i=\phi_i-\phi$. However, this time we can do slightly better and pick the constant c independently of i for i sufficiently large. This follows from the fact that the compact operators in question are either i independent or have a term like $\phi_i-\phi$ which goes to zero in L^2_1 and hence in the strong operator topology for i large. As a result, we have uniform control over the L^2_2 norm of ψ_i. Repeating the argument with higher norms gives us uniform bounds for all Sobolev norms and thus the desired convergence. □

One may use the method of the theorem to prove regularity on a noncompact manifold as well as a convergence result on the interior.

Theorem 40. *Given a open set U in a 4-manifold and $\phi\in L^2_1(U)$, $g\in L^2(U)$ assume they satisfy the equation*

$$\Delta\phi+K_1\phi+(K_2\phi)\sharp\phi=g$$

We have $\phi\in L^2_2$ and if $g\in C^\infty$ then $\phi\in C^\infty$. Furthermore, on any compact $B\subset U$ the L^2_k norm of ϕ is bounded by a multiple of its L^2_1 norm on U.

Proof. Let ρ be a bump function equal to 1 on B. We have that $u = \rho \cdot \phi$ solves:

$$\Delta u + K_1 u + (K_2 u)\sharp\phi = h$$

with $h \in L^2$. We can proceed as in the theorem above. Indeed, restrict U to a compact manifold with boundary X still containing B. By choosing ρ we can assume u vanishes on the boundary of X. Now, use the invertibility of the operator $\Delta : L^2_{2,0} \to L^2$ where $L^2_{2,0}$ consists of sections vanishing on the boundary. The rest of the proof proceeds as above. $\square$

We will also need a regularity theorem for a linearized version of the equations. Consider a 4-manifold X with boundary and an elliptic first order operator $D : E \to F$ on a vector bundles E, F defined on X. By considering $D \oplus D^* : E \oplus F \to E \oplus F$ we may assume D to be self-adjoint.

Theorem 41. *Given $u \in L^2_{loc}$, if u satisfies the equation*

$$Du + \phi\sharp u = g$$

where $g \in L^2_{loc}$ and $\phi \in L^2_{1,loc}$ then $u \in L^2_{1,loc}$.

Proof. Take a bump function ρ and let $v = \rho \cdot u$. We have $Dv + \phi\sharp v = h$, with $h \in L^2$. Let $\psi \in L^2_1$ solve

$$(D + c)\psi + \phi\sharp\psi = h + cu$$

for a large imaginary c. This is possible since the spectrum of D is real and $(D + c)^{-1}$ is converging to 0 weakly as an operator $L^2 \to L^2_1$. We conclude that $\psi = u$ since the difference solves

$$(D + c)(\psi - u) + \phi\sharp(\psi - u) = 0$$

this time viewed as a map $L^2 \to L^2_{-1}$ $\square$

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