Combinatorics of determinantal identities

by

Matjaž Konvalinka

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Author . . . .
Department of Mathematics
April 25, 2008

Certified by . . . .
Igor Pak
Associate Professor of Applied Mathematics
Thesis Supervisor

Accepted by . . . .
David Jerison
Chairman, Department Committee on Graduate Students

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Abstract

In this thesis, we apply combinatorial means for proving and generalizing classical determinantal identities. In Chapter 1, we present some historical background and discuss the algebraic framework we employ throughout the thesis. In Chapter 2, we construct a fundamental bijection between certain monomials that proves crucial for most of the results that follow. Chapter 3 studies the first, and possibly the best-known, determinantal identity, the matrix inverse formula, both in the commutative case and in some non-commutative settings (Cartier-Foata variables, right-quantum variables, and their weighted generalizations). We give linear-algebraic and (new) bijective proofs; the latter also give an extension of the Jacobi ratio theorem. Chapter 4 is dedicated to the celebrated MacMahon master theorem. We present numerous generalizations and applications. In Chapter 5, we study another important result, Sylvester's determinantal identity. We not only generalize it to non-commutative cases, we also find a surprising extension that also generalizes the master theorem. Chapter 6 has a slightly different, representation theory flavor; it involves representations of the symmetric group, and also Hecke algebras and their characters. We extend a result on immanants due to Goulden and Jackson to a quantum setting, and reprove certain combinatorial interpretations of the characters of Hecke algebras due to Ram and Remmel.

Thesis Supervisor: Igor Pak
Title: Associate Professor of Applied Mathematics

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Contents

Chapter 1

Introduction

1.1 Historical background

Determinants have a long and illustrious history. Since their first recorded use in 3rd century BCE in the Chinese math textbook The Nine Chapters on the Mathematical Art, they have been one of the most studied mathematical objects, with innumerable applications in diverse areas of mathematics and other sciences.

Their proper definition in non-commutative settings has remained elusive. The star-studded collection of people who tried to remedy that includes Cayley, Dyson, Frobenius, Schur, Wedderburn and Richardson. The "best" definition of the determinant of a matrix with completely non-commutative entries appeared only in the 1990s. We are referring, of course, to the beautiful theory of quasideterminants, which originated in the works of Gelfand and Retakh [GR91]. For a definitive treatment of quasideterminants and a comprehensive historical account, see [GGRW05].

In this thesis, however, we restrict our attention to matrices whose entries satisfy certain conditions. As a result, we do not need general quasideterminants (which are rational functions of entries) and can use much simpler polynomial determinants (e.g. quantum determinants). For some results in "quantum linear algebra", see [Man88] and [KL95].

Considering how well established determinants and determinantal identities are (in both commutative and non-commutative contexts), it is surprising how little work has been done in trying to understand them combinatorially. Zeilberger's article [Zei85] discussed some simple proofs of determinantal identities, but only for commutative variables. Foata's paper [Foa79] is closer to the concept of this thesis, as it proves a non-commutative linear algebraic result bijectively, however, the proof is rather complicated and does not seem to extend to more general non-commutative settings. See also Cartier and Foata's version of MacMahon master theorem [CF69].

The determinantal identities treated in this thesis in detail are the matrix inverse formula, MacMahon master theorem, Sylvester's determinantal identity, and Goulden-Jackson's determinantal expression for an immanant. See each particular chapter for historical background on the particular identity.

The general framework described in this and the following chapter is used for proofs of results in Chapters 3–5. Chapter 6 stands apart somewhat, both in terms of the non-commutative setting (it deals with quantum matrices, not Cartier-Foata or right-quantum matrices) and the methods of proof.

Let us conclude this overview with a description of papers that are the source for this thesis, in chronological order. The general algebraic framework discussed in Section 1.2, the fundamental transformation from Chapter 2, non-combinatorial proofs of the non-commutative matrix inverse formulas in Section 3.2, and most of the results on MacMahon master theorem in Chapter 4 are from [KP07] (joint work with Igor Pak). Chapter 5 is entirely based on [Kon07]. The short paper [Konb] was the basis for Sections 6.1 and 6.2. Parts of Chapter 2, Sections 3.3–3.7, and Section 4.7 are found in [Kon08], Sections 2.7, 6.3 and 6.4 are from [KS] (joint work with Mark Skandera), and Sections 6.5 and 6.6 are from [Kona].

1.2 Algebraic framework

In this section, we describe the way we want to think about determinantal identities, and introduce some basic terminology. Let us start with two well-known examples.

The most famous determinantal identity of all is the matrix inverse formula.

Theorem 1.2.1 (matrix inverse formula) *For a complex invertible matrix* $A = (a_{ij})_{m \times m}$*, we have*

$$\left(A^{-1}\right)_{ij} = (-1)^{i+j} \frac{\det A^{ji}}{\det A}. \qquad \square$$

We know that (under certain conditions, e.g. if $\|A\| < 1$ or if the entries of A are variables)

$$(I - A)^{-1} = I + A + A^2 + \ldots,$$

so

$$\left((I - A)^{-1}\right)_{ij} = \delta_{ij} + a_{ij} + \sum_k a_{ik} a_{kj} + \ldots,$$

and we can rephrase the matrix inverse formula as follows:

$$\det(I - A) \cdot \left(\delta_{ij} + a_{ij} + \sum_k a_{ik} a_{kj} + \ldots \right) = (-1)^{i+j} \det(I - A)^{ji}.$$

In other words, the matrix inverse formula says that two power series in variables a_{ij} are the same, provided that the variables commute.

Another famous determinantal identity is the following result due to MacMahon. In the statement of this theorem, as well as throughout the thesis, a bold letter denotes a vector of variables (e.g. $\mathbf{x}$ stands for $(x_1, \ldots, x_m)$), and the meaning of expressions such as $\mathbf{x}^{\mathbf{r}}$ and $\mathbf{r}!$ should be obvious from the context ($x_1^{r_1} \cdots x_m^{r_m}$ and $r_1! \cdots r_m!$ for these examples).

Theorem 1.2.2 (MacMahon master theorem) *Let $A = (a_{ij})_{m\times m}$ be a complex matrix, and let $x_1, \ldots, x_m$ be a set of variables. Denote by $G(\mathbf{r})$ the coefficient of $\mathbf{x}^\mathbf{r}$ in*

$$\prod_{i=1}^{m}(a_{i1}x_1 + \ldots + a_{im}x_m)^{r_i}.$$

Let $t_1, \ldots, t_m$ be another set of variables, and $T = \operatorname{diag}\mathbf{t}$. Then

$$\sum_{\mathbf{r}\geq\mathbf{0}} G(\mathbf{r})\mathbf{t}^\mathbf{r} = \frac{1}{\det(I - TA)}. \qquad \square$$

Example 1.2.3 The coefficient of $x^2y^0z^2$ in $(y+z)^2(x+z)^0(x+y)^2$ is 1, and the coefficient of $x^2y^3z^1$ in $(y+z)^2(x+z)^3(x+y)^1$ is 3. On the other hand, for

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} t & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & v \end{pmatrix},$$

we have

$$\frac{1}{\det(I - TA)} = \frac{1}{1 - tu - tv - uv - 2tuv} =$$
$$= 1 + \ldots + t^2u^0v^2 + \ldots + 3t^2u^3v^1 + \ldots \qquad \diamondsuit$$

We can take a_{ij} to be variables; each $G(\mathbf{r})$ is then a finite sum of monomials in a_{ij}. By taking $t_1 = \ldots = t_m = 1$, MacMahon master theorem gives

$$\sum_{\mathbf{r}\geq\mathbf{0}} G(\mathbf{r}) = \frac{1}{\det(I - A)}.$$

Since $\det(I - A) = 1 - a_{11} - \ldots - a_{mm} + a_{11}a_{22} - a_{21}a_{12} + \ldots$, the right-hand side is also a power series in a_{ij}'s.

Therefore the MacMahon master theorem also says that two power series in variables a_{ij} are the same, provided that the variables commute.

This thesis tries to find answers to the following natural questions.

- Do (versions of) these (and other) determinantal identities hold when the variables are not commutative, provided we define properly the concept of non-commutative determinants?
- Can we find combinatorial proofs of these (commutative and non-commutative) identities?
- Can we add parameters and find natural q-analogues?

We denote by $\mathcal{A}$ the $\mathbb{C}$-algebra of formal power series in non-commuting variables a_{ij}, $1 \leq i, j \leq m$. Elements of $\mathcal{A}$ are infinite linear combinations of words in variables

a_{ij} (with coefficients in $\mathbb{C}$). In most cases we take elements of $\mathcal{A}$ modulo some ideal $\mathcal{I}$ generated by a finite number of quadratic relations. For example, if $\mathcal{I}_{\text{comm}}$ is generated by $a_{ij}a_{kl} = a_{kl}a_{ij}$ for all i, j, k, l, then $\mathcal{A}/\mathcal{I}_{\text{comm}}$ is the symmetric algebra (the free commutative algebra with variables a_{ij}). See Section 1.3 for examples and further definitions.

We abbreviate the product $a_{\lambda_1\mu_1} \cdots a_{\lambda_\ell\mu_\ell}$ to $a_{\lambda,\mu}$ for $\lambda = \lambda_1 \cdots \lambda_\ell$ and $\mu = \mu_1 \cdots \mu_\ell$, where λ and μ are regarded as words in the alphabet $\{1, \ldots, m\}$. For example, we write $a_{413225433314,132254333144}$ for $a_{41}a_{13}a_{32}a_{22}a_{25}a_{54}a_{43}a_{33}a_{33}a_{31}a_{14}a_{44}$. For a word $\nu = \nu_1 \cdots \nu_\ell$, define the *set of inversions*

$$\mathcal{I}(\nu) \ = \ \{(i, j)\colon i < j, \nu_i > \nu_j\},$$

and let $\operatorname{inv}(\nu) = |\mathcal{I}(\nu)|$ be the *number of inversions*.

We often consider *lattice steps* of the form $(x, i) \to (x + 1, j)$ for some $x, i, j \in \mathbb{Z}$, $1 \leq i, j \leq m$. We think of x being drawn along the x-axis, increasing from left to right, and refer to i and j as the *starting height* and *ending height*, respectively. We identify the step $(x, i) \to (x + 1, j)$ with the variable a_{ij}. Similarly, we identify a finite sequence of steps with a word in the alphabet $\{a_{ij}\}$, $1 \leq i, j \leq m$, i.e. with an element of the algebra $\mathcal{A}$. If each step in a sequence starts at the ending point of the previous step, we call such a sequence a *lattice path*. A lattice path with starting height i and ending height j is called *a path from i to j*.

Example 1.2.4 Figure 1-1 represents a path from 4 to 4. $\diamondsuit$

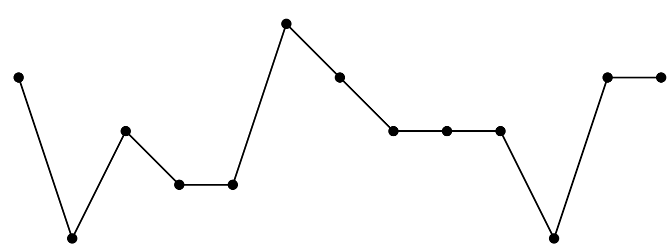

Figure 1-1: Representation of the word $a_{41}a_{13}a_{32}a_{22}a_{25}a_{54}a_{43}a_{33}a_{33}a_{31}a_{14}a_{44}$.

Recall that the (i, j)-th entry of A^k is the sum of all paths of length k from i to j. Since

$$(I - A)^{-1} = I + A + A^2 + \ldots,$$

the (i, j)-th entry of $(I - A)^{-1}$ is the sum of all paths (of any length) from i to j.

As mentioned in the historical background, determinants play a vital role in this thesis. Definitions in various non-commutative contexts are discussed in Section 1.4.

Throughout the thesis, we work over the field of complex numbers. However, any field with characteristic $\neq 2$ would do.

1.3 Ideals and the "$1 = q$ principle"

Throughout the thesis, we use the following definitions and notation.

(1) The ideal $\mathcal{I}_{\text{comm}}$ of $\mathcal{A}$ generated by relations

$$a_{jl}a_{ik} = a_{ik}a_{jl} \text{ for all } i,j,k,l \tag{1.3.1}$$

is called the *commutative ideal*; the algebra $\mathcal{A}_{\text{comm}} = \mathcal{A}/\mathcal{I}_{\text{comm}}$ is the *commutative algebra*; and a matrix $(a_{ij})_{1\le i,j\le m}$ satisfying these relations is called a *commutative matrix.*

(2) The ideal $\mathcal{I}_{\text{cf}}$ of $\mathcal{A}$ generated by relations

$$a_{jl}a_{ik} = a_{ik}a_{jl} \text{ for all } i,j,k,l,\ i \ne j \tag{1.3.2}$$

is called the *Cartier-Foata ideal*; the algebra $\mathcal{A}_{\text{cf}} = \mathcal{A}/\mathcal{I}_{\text{cf}}$ is the *Cartier-Foata algebra*; and a matrix $(a_{ij})_{1\le i,j\le m}$ satisfying these relations is called a *Cartier-Foata matrix.*

(3) The ideal $\mathcal{I}_{\text{rq}}$ of $\mathcal{A}$ generated by relations

$$a_{jk}a_{ik} = a_{ik}a_{jk} \text{ for all } i,j,k, \tag{1.3.3}$$
$$a_{ik}a_{jl} - a_{jk}a_{il} = a_{jl}a_{ik} - a_{il}a_{jk} \text{ for all } i,j,k,l \tag{1.3.4}$$

is called the *right-quantum ideal*; the algebra $\mathcal{A}_{\text{rq}} = \mathcal{A}/\mathcal{I}_{\text{rq}}$ is the *right-quantum algebra*; and a matrix $(a_{ij})_{1\le i,j\le m}$ satisfying these relations is called a *right-quantum matrix.*

(4) The ideal $\mathcal{I}_{q-\text{cf}}$ of $\mathcal{A}$ generated by relations

$$a_{jl}a_{ik} = a_{ik}a_{jl} \text{ for all } i<j, k<l, \tag{1.3.5}$$
$$a_{jl}a_{ik} = q^2\ a_{ik}a_{jl} \text{ for all } i<j, k>l, \tag{1.3.6}$$
$$a_{jk}a_{ik} = q\ a_{ik}a_{jk} \text{ for all } i,j,k,\ i<j \tag{1.3.7}$$

for $q \in \mathbb{C}\backslash\{0\}$ is called the *q-Cartier-Foata ideal*; the algebra $\mathcal{A}_{q-\text{cf}} = \mathcal{A}/\mathcal{I}_{q-\text{cf}}$ is the *q-Cartier-Foata algebra*; and a matrix $(a_{ij})_{1\le i,j\le m}$ satisfying these relations is called a *q-Cartier-Foata matrix.*

(5) The ideal $\mathcal{I}_{q-\text{rq}}$ of $\mathcal{A}$ generated by relations

$$a_{jk}a_{ik} = q\ a_{ik}a_{jk} \text{ for all } i,j,k,\ i<j, \tag{1.3.8}$$
$$a_{ik}a_{jl} - q^{-1}\ a_{jk}a_{il} = a_{jl}a_{ik} - q\ a_{il}a_{jk} \text{ for all } i<j, k<l; \tag{1.3.9}$$

for $q \in \mathbb{C}\backslash\{0\}$ is called the *q-right-quantum ideal*; the algebra $\mathcal{A}_{q-\text{rq}} = \mathcal{A}/\mathcal{I}_{q-\text{rq}}$ is the *q-right-quantum algebra*; and a matrix $(a_{ij})_{1\le i,j\le m}$ satisfying these relations is called a *q-right-quantum matrix.*

(6) The ideal $\mathcal{I}_{\mathbf{q}-\mathrm{cf}}$ of $\mathcal{A}$ generated by relations

$$\begin{aligned} a_{jl}a_{ik} &= q_{kl}^{-1}q_{ij}\ a_{ik}a_{jl} \quad \text{for all} \quad i<j, k<l, & (1.3.10)\\ a_{jl}a_{ik} &= q_{ij}q_{lk}\ a_{ik}a_{jl} \quad \text{for all} \quad i<j, k>l, & (1.3.11)\\ a_{jk}a_{ik} &= q_{ij}\ a_{ik}a_{jk} \quad \text{for all} \quad i,j,k,\ i<j & (1.3.12) \end{aligned}$$

for $q_{ij} \in \mathbb{C} \setminus \{0\}$, $1 \le i < j \le m$, is called the **q**-*Cartier-Foata ideal*; the algebra $\mathcal{A}_{\mathbf{q}-\mathrm{cf}} = \mathcal{A}/\mathcal{I}_{\mathbf{q}-\mathrm{cf}}$ is the **q**-*Cartier-Foata algebra*; and a matrix $(a_{ij})_{1\le i,j\le m}$ satisfying these relations is called a **q**-*Cartier-Foata matrix.*

(7) The ideal $\mathcal{I}_{\mathbf{q}-\mathrm{rq}}$ of $\mathcal{A}$ generated by relations

$$\begin{aligned} a_{jk}a_{ik} &= q_{ij}\ a_{ik}a_{jk} \quad \text{for all } i,j,k,\ i<j, & (1.3.13)\\ a_{ik}a_{jl} - q_{ij}^{-1}\ a_{jk}a_{il} &= q_{kl}q_{ij}^{-1}\ a_{jl}a_{ik} - q_{kl}\ a_{il}a_{jk} \quad \text{for all } i<j, k<l & (1.3.14) \end{aligned}$$

for $q_{ij} \in \mathbb{C} \setminus \{0\}$, $1 \le i < j \le m$, is called the **q**-*right-quantum ideal*; the algebra $\mathcal{A}_{\mathbf{q}-\mathrm{rq}} = \mathcal{A}/\mathcal{I}$ is the **q**-*right-quantum algebra*; and a matrix $(a_{ij})_{1\le i,j\le m}$ satisfying these relations is called a **q**-*right-quantum matrix.*

(8) The ideal $\mathcal{I}_q$ of $\mathcal{A}$ generated by relations

$$\begin{aligned} a_{il}a_{ik} &= q\ a_{ik}a_{il} \quad \text{for all} \quad i,k,l,\ k<l, & (1.3.15)\\ a_{jk}a_{ik} &= q\ a_{ik}a_{jk} \quad \text{for all} \quad i,j,k,\ i<j, & (1.3.16)\\ a_{jk}a_{il} &= a_{il}a_{jk} \quad \text{for all} \quad i<j, k<l, & (1.3.17)\\ a_{jl}a_{ik} &= a_{ik}a_{jl} + (q-q^{-1})\ a_{il}a_{jk} \quad \text{for all} \quad i<j, k<l & (1.3.18) \end{aligned}$$

is called the *quantum ideal*; the algebra $\mathcal{A}_q = \mathcal{A}/\mathcal{I}_q$ is the *quantum algebra*; and a matrix $(a_{ij})_{1\le i,j\le m}$ satisfying these relations is called a *quantum matrix.*

We have the following implications:

$$\begin{array}{ccccc} & & (8) & \Rightarrow & (1)\\ & & \Uparrow & & \\ (7) & \Rightarrow & (5) & \Rightarrow & (3)\\ \Downarrow & & \Downarrow & & \Downarrow\\ (6) & \Rightarrow & (4) & \Rightarrow & (2)\\ & & & & \Downarrow\\ & & & & (1) \end{array}$$

For example, by (7) $\Rightarrow$ (6) we mean that if a statement is true for all **q**-right-quantum matrices, it is also true for all **q**-Cartier-Foata matrices. equivalently, every **q**-Cartier-Foata matrix is also **q**-right-quantum.

Note that if we write $q_{ii} = 1$ for all i and $q_{ji} = q_{ij}^{-1}$ for $i < j$, we can write the conditions of (6) more concisely as

$$q_{kl}\ a_{jl}a_{ik} = q_{ij}\ a_{ik}a_{jl}, \tag{1.3.19}$$

for all i, j, k, l, and $i \neq j$, and we can write the conditions of (7) as

$$a_{ik}a_{jl} - q_{ij}^{-1}\ a_{jk}a_{il} \ = \ q_{kl}q_{ij}^{-1}\ a_{jl}a_{ik} - q_{kl}\ a_{il}a_{jk} \tag{1.3.20}$$

Remark 1.3.1 In [GLZ06], the term *right-quantum* was used for what we call q-right-quantum. In [CF], the authors call our right-quantum matrices *Manin matrices.* $\diamondsuit$

Example 1.3.2 Let $x_{ij}, y_{i,j}$ be complex variables, and write $X = (x_{ij})_{m\times m}$, $Y = (y_{ij})_{m\times m}$ (where the entries are operators of multiplication by x_{ij} or y_{ij}), $\partial_X = (\partial/\partial x_{ij})_{m\times m}$, $\partial_Y = (\partial/\partial y_{ij})_{m\times m}$. Since x_{ij} and $\partial/\partial x_{ij}$ (or y_{ij} and $\partial/\partial y_{ij}$) do not commute, the matrix $\begin{pmatrix} X & \partial Y \\ Y & \partial X \end{pmatrix}$ is not Cartier-Foata. However,

$$(x_{ij}\cdot \partial/\partial x_{ij} - \partial/\partial x_{ij}\cdot x_{ij})f = -f = (y_{ij}\cdot \partial/\partial y_{ij} - \partial/\partial y_{ij}\cdot y_{ij})f,$$

so it is right-quantum. $\diamondsuit$

Foata and Han introduced ([FH08, Section 3]) the so-called "$1 = q$ principle" to derive identities in the algebra $\mathcal{A}_{q-\mathrm{rq}}$ from those in the algebra $\mathcal{A}_{\mathrm{rq}}$.

Lemma 1.3.3 ("$1 = q$ principle") *Let $\phi\colon \mathcal{A} \to \mathcal{A}$ denote the linear map induced by*

$$\phi\left(a_{\lambda,\mu}\right) = q^{\mathrm{inv}(\mu)-\mathrm{inv}(\lambda)} a_{\lambda,\mu}.$$

Then:

(a) ϕ maps $\mathcal{I}_{\mathrm{rq}}$ into $\mathcal{I}_{q-\mathrm{rq}}$;

(b) Call $a_{\lambda,\mu}$ balanced *if λ is a rearrangement of μ (i.e. if λ and μ contain the same letters with the same multiplicities). Then $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$ for α, β linear combinations of balanced sequences.*

Proof. (a) It suffices to prove the claim for elements of the form

$$\alpha = a_{\lambda,\mu}(a_{ik}a_{jk} - a_{jk}a_{ik})a_{\lambda',\mu'}$$

and

$$\beta = a_{\lambda,\mu}(a_{ik}a_{jl} - a_{jk}a_{il} - a_{jl}a_{ik} + a_{il}a_{jk})a_{\lambda',\mu'}$$

with $i < j$ (and $k < l$). Note that the sets of inversions of the words $\lambda ij\lambda'$ and $\lambda ji\lambda'$ differ only in one inversion. Therefore $\phi(\alpha)$ is a multiple of

$$a_{ik}a_{jk} - q^{-1}a_{jk}a_{ik}.$$

For β the proof is analogous. (b) It suffices to prove the claim for $\alpha = a_{\lambda,\mu}$ with λ a rearrangement of μ and $\beta = a_{\lambda',\mu'}$ with λ' a rearrangement of μ'. The number of inversions of $\lambda\lambda'$ is equal to the number of inversions of λ, plus the number of inversions of λ', plus the number of pairs (i, j) with $\lambda_i > \lambda'_j$. Similarly, the number of inversions of $\mu\mu'$ is equal to the number of inversions of μ, plus the number of inversions of μ', plus the number of pairs (i, j) with $\mu_i > \mu'_j$. Since λ is a rearrangement of μ and λ' is a rearrangement of μ', $\text{inv}(\mu\mu') - \text{inv}(\lambda\lambda') = (\text{inv}(\mu) - \text{inv}(\lambda)) + (\text{inv}(\mu') - \text{inv}(\lambda'))$, which concludes the proof. □

Lemma 1.3.4 ("$1 = q_{ij}$ principle") *Let $\phi\colon \mathcal{A} \to \mathcal{A}$ denote the linear map induced by*

$$\phi\left(a_{\lambda,\mu}\right) = \left(\prod_{(i,j)\in I(\mu)} q_{\mu_j\mu_i} \prod_{(i,j)\in I(\lambda)} q^{-1}_{\lambda_j\lambda_i}\right) a_{\lambda,\mu}.$$

Assume that every element of a set $\mathcal{S}$ is of the form $\sum_{i\in\mathcal{J}} c_i a_{\lambda^i,\mu^i}$, where λ^i is a rearrangement of λ^j and μ^i is a rearrangement of μ^j.for all $i, j \in \mathcal{J}$. Denote by $\mathcal{I}$ the ideal of $\mathcal{A}$ generated by $\mathcal{S}$, and by $\mathcal{I}'$ the ideal of $\mathcal{A}$ generated by $\phi(\mathcal{S})$. Then

(a) ϕ maps $\mathcal{I}$ into $\mathcal{I}'$;

(b) $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$ for α, β linear combinations of balanced sequences.

Proof. The proof of lemma follows verbatim the proof of Lemma 1.3.3. We omit the details. □

Many of the results in this thesis involving q- (or $\mathbf{q}$-) right-quantum variables can be achieved in one of the following (fundamentally equivalent) ways. We can first prove the result in the case when $q = 1$ (respectively, $q_{ij} = 1$) by combinatorial means, and then use the "$1 = q$ principle" (respectively, "$1 = q_{ij}$ principle") described above. Alternatively, we can adapt every step of the proof by keeping track of powers of q (respectively, monomials in q_{ij}) attached to monomials in the non-commutative variables a_{ij}. In both cases, the main difficulty is proving the result in the case without parameter, and the rest of the proof is relatively straightforward.

This is in sharp contrast with the case of *quantum* variables. If we put $q = 1$, quantum variables become commutative, and proofs are fundamentally simpler. Compare Chapter 6 with Chapters 2–5.

1.4 Non-commutative determinants

Let $B = (b_{ij})_{n\times n}$ be a square matrix with entries in $\mathcal{A}$, i.e. b_{ij}'s are linear combinations of words in $\mathcal{A}$. To define the determinant of B, expand the terms of

$$\sum_{\sigma\in S_n} (-1)^{\text{inv}(\sigma)} b_{\sigma_1 1} \cdots b_{\sigma_n n},$$

and weight a word $a_{\lambda,\mu}$ with a certain weight $w(a_{\lambda,\mu})$. The resulting expression is called the *determinant* of B (with respect to $\mathcal{A}$). In the usual commutative case, all weights are equal to 1.

In all cases we consider we have $w(\varepsilon) = w(a_{\varnothing,\varnothing}) = 1$. Therefore

$$\frac{1}{\det(I - A)} = \frac{1}{1 - \Sigma} = 1 + \Sigma + \Sigma^2 + \dots,$$

where Σ is a certain finite linear combination of words in a_{ij}, and both the left and the right inverse of $\det(I - A)$ are equal to the infinite sum on the right. We can use the fraction notation as above in non-commutative situations.

Throughout the thesis, we use the following determinants.

(1) The usual (non-weighted) determinant det. Here the weight $w(a_{\lambda,\mu})$ is equal to 1 for every λ and μ. This determinant appears in settings (1), (2) and (3) from Section 1.3. Note that in this case,

$$\det B = \sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} b_{\sigma_1 1} \cdots b_{\sigma_n n}$$

for every matrix $B = (b_{ij})_{n \times n}$.

(2) The q-determinant $\det_q$. Here the weight $w(a_{\lambda,\mu})$ is equal to $q^{\mathrm{inv}(\mu) - \mathrm{inv}(\lambda)}$. This determinant appears in settings (4), (5) and (8) from Section 1.3.

(3) The $\mathbf{q}$-determinant $\det_{\mathbf{q}}$. Here the weight $w(a_{\lambda,\mu})$ is equal to

$$\left(\prod_{(i,j) \in \mathcal{I}(\mu)} q_{\mu_j \mu_i} \right) \left(\prod_{(i,j) \in \mathcal{I}(\lambda)} q^{-1}_{\lambda_j \lambda_i} \right).$$

This determinant appears in settings (6) and (7) from Section 1.3.

Example 1.4.1 For

$$B = \begin{pmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{pmatrix},$$

we have

$$\det B = a_{12} a_{24} - a_{22} a_{14},$$
$$\det_q B = a_{12} a_{24} - q^{-1} a_{22} a_{14},$$
$$\det_{\mathbf{q}} B = a_{12} a_{24} - q^{-1}_{12} a_{22} a_{14}.$$

◇

Chapter 2

The fundamental transformation

2.1 The transformation and the Cartier-Foata case

The *type* of a sequence $a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}$ is defined to be $(\mathbf{p};\mathbf{r})$ for $\mathbf{p}=(p_1,\ldots,p_m)$ and $\mathbf{r}=(r_1,\ldots,r_m)$, where p_k (respectively r_k) is the number of k's among $i_1,\ldots,i_n$ (respectively $j_1,\ldots,j_n$). If $\mathbf{p}=\mathbf{r}$, we call the sequence *balanced*.

Take non-negative integer vectors $\mathbf{p}=(p_1,\ldots,p_m)$ and $\mathbf{r}=(r_1,\ldots,r_m)$ with $\sum p_i = \sum r_i = n$, and a permutation $\pi \in S_m$. An *ordered sequence of type* $(\mathbf{p};\mathbf{r})$ *with respect to* π (o-sequence for short) is a sequence $a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}$ of type $(\mathbf{p};\mathbf{r})$ such that $\pi^{-1}(i_k)\leq\pi^{-1}(i_{k+1})$ for $k=1,\ldots,n-1$. Denote the set of ordered sequence of type $(\mathbf{p};\mathbf{r})$ with respect to π by $\mathbf{O}^\pi(\mathbf{p};\mathbf{r})$. Clearly, there are $\binom{n}{r_1,\ldots,r_m}$ elements in $\mathbf{O}^\pi(\mathbf{p};\mathbf{r})$.

A *back-ordered sequence of type* $(\mathbf{p};\mathbf{r})$ *with respect to* π (or bo-sequence) is a sequence $a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}$ of type $(\mathbf{p};\mathbf{r})$ such that $\pi^{-1}(j_k)\geq\pi^{-1}(j_{k+1})$ for $k=1,\ldots,n-1$. Denote the set of back-ordered sequences of type $(\mathbf{p};\mathbf{r})$ with respect to π by $\overline{\mathbf{O}}^\pi(\mathbf{p};\mathbf{r})$. There are $\binom{n}{p_1,\ldots,p_m}$ elements in $\overline{\mathbf{O}}^\pi(\mathbf{p};\mathbf{r})$.

Example 2.1.1 For $m=3$, $n=4$, $\mathbf{p}=(2,1,1)$, $\mathbf{r}=(0,3,1)$ and $\pi=231$, $\mathbf{O}^\pi(\mathbf{p};\mathbf{r})$ is

$$\{a_{22}a_{32}a_{12}a_{13}, a_{22}a_{32}a_{13}a_{12}, a_{22}a_{33}a_{12}a_{12}, a_{23}a_{32}a_{12}a_{12}\}.$$

For $m=3$, $n=4$, $\mathbf{p}=(2,2,0)$, $\mathbf{r}=(1,2,1)$ and $\pi=132$, $\overline{\mathbf{O}}^\pi(\mathbf{p};\mathbf{r})$ is

$$\{a_{12}a_{12}a_{23}a_{21}, a_{12}a_{22}a_{13}a_{21}, a_{12}a_{22}a_{23}a_{11}, a_{22}a_{12}a_{13}a_{21}, a_{22}a_{12}a_{23}a_{11}, a_{22}a_{22}a_{13}a_{11}\}.$$

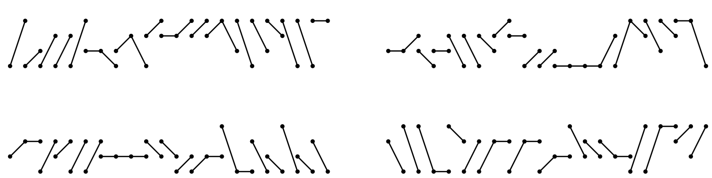

Figure 2-1: Some ordered and back-ordered sequences.

Figure 2-1 shows ordered sequences with respect to 1234 and 2314, and back-ordered sequences with respect to 1234 and 4231. $\diamondsuit$

We abbreviate $\mathbf{O}^\pi(\mathbf{p};\mathbf{p})$ and $\overline{\mathbf{O}}^\pi(\mathbf{p};\mathbf{p})$ to $\mathbf{O}^\pi(\mathbf{p})$ and $\overline{\mathbf{O}}^\pi(\mathbf{p})$, respectively; and if $\pi =$ id, we write simply $\mathbf{O}(\mathbf{p};\mathbf{r})$ and $\overline{\mathbf{O}}(\mathbf{p};\mathbf{r})$.

Take non-negative integer vectors $\mathbf{p} = (p_1, \ldots, p_m)$ and $\mathbf{r} = (r_1, \ldots, r_m)$ with $\sum p_i = \sum r_i = n$, and a permutation $\pi \in S_m$. Define a *path sequence of type* $(\mathbf{p};\mathbf{r})$ *with respect to* π (p-sequence for short) to be a sequence $a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}$ of type $(\mathbf{p};\mathbf{r})$ that is a concatenation of lattice paths with starting heights i_{k_s} and ending heights j_{l_s} so that $\pi^{-1}(i_{k_s}) \leq \pi^{-1}(i_t)$ for all $t \geq k_s$, and $i_t \neq j_{l_s}$ for $t > l_s$. Denote the set of all path sequences of type $(\mathbf{p};\mathbf{r})$ with respect to π by $\mathbf{P}^\pi(\mathbf{p};\mathbf{r})$.

Similarly, define a *back-path sequence of type* $(\mathbf{p};\mathbf{r})$ *with respect to* π (or bp-sequence) to be a sequence $a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}$ of type $(\mathbf{p};\mathbf{r})$ that is a concatenation of lattice paths with starting heights i_{k_s} and ending heights j_{l_s} so that $\pi^{-1}(j_{k_s}) \leq \pi^{-1}(j_t)$ for all $t \leq k_j$, and $j_t \neq i_{k_s}$ for $t < k_s$. Denote the set of all back-path sequences of type $(\mathbf{p};\mathbf{r})$ by $\overline{\mathbf{P}}^\pi(\mathbf{p};\mathbf{r})$.

Example 2.1.2 Figure 2-2 shows some path sequences with respect to 2341 and 3421, and back-path sequences with respect to 1324 and 4321. The second path sequence and the second back-path sequence are balanced. $\diamondsuit$

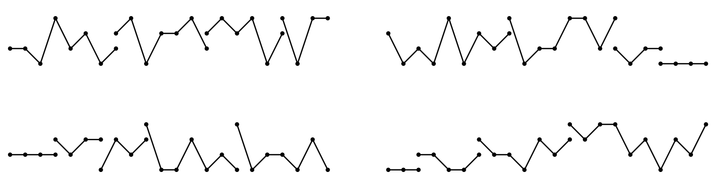

Figure 2-2: Some path and back-path sequences.

We abbreviate $\mathbf{P}^\pi(\mathbf{p};\mathbf{p})$ and $\overline{\mathbf{P}}^\pi(\mathbf{p};\mathbf{p})$ to $\mathbf{P}^\pi(\mathbf{p})$ and $\overline{\mathbf{P}}^\pi(\mathbf{p})$; and if $\pi =$ id, we write simply $\mathbf{P}(\mathbf{p};\mathbf{r})$ and $\overline{\mathbf{P}}(\mathbf{p};\mathbf{r})$. Note that a (back-)path sequence of type $(\mathbf{p};\mathbf{p})$ is a concatenation of lattice paths with the same starting and ending height, with no ending height of a lattice path appearing to the right of the lattice path.

Recall that for a word $w = i_1i_2\ldots i_n$, we say that (k,l) is an inversion of u if $k < l$ and $i_k > i_l$, and we write $\mathrm{inv}(w)$ for the number of inversions of w and $\mathcal{I}$ for the set of inversion of w. For $\alpha = a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}$, write $\mathrm{inv}(\alpha) = \mathrm{inv}(j_1j_2\ldots j_n) - \mathrm{inv}(i_1i_2\ldots i_n)$. Furthermore, define

$$O^\pi(\mathbf{p};\mathbf{r}) = \sum_{\alpha\in\mathbf{O}^\pi(\mathbf{p};\mathbf{r})} \alpha, \quad \overline{O}^\pi(\mathbf{p};\mathbf{r}) = \sum_{\alpha\in\overline{\mathbf{O}}^\pi(\mathbf{p};\mathbf{r})} (-1)^{\mathrm{inv}(\alpha)}\,\alpha,$$

$$P^\pi(\mathbf{p};\mathbf{r}) = \sum_{\alpha\in\mathbf{P}^\pi(\mathbf{p};\mathbf{r})} \alpha, \quad \overline{P}^\pi(\mathbf{p};\mathbf{r}) = \sum_{\alpha\in\overline{\mathbf{P}}^\pi(\mathbf{p};\mathbf{r})} (-1)^{\mathrm{inv}(\alpha)}\,\alpha,$$

Let us construct a natural map

$$\varphi\colon \mathbf{O}^{\pi}(\mathbf{p};\mathbf{r}) \longrightarrow \mathbf{P}^{\sigma}(\mathbf{p};\mathbf{r}). \tag{2.1.1}$$

Without loss of generality, we assume $\pi = \mathrm{id}$, since this is just relabeling of the variables a_{ij} according to π.

Take an o-sequence $\alpha = a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}$, and interpret it as a concatenation of steps. Among the steps $i_k \to j_k$ with the lowest $\sigma^{-1}(i_k)$, take the leftmost one. Continue switching this step with the one on the left until it is at the beginning of the sequence. Then take the leftmost step to its right that begins with j_k, move it to the left until it is the second step of the sequence, and continue this procedure while possible. Now we have a concatenation of a lattice path and a (shorter) o-sequence. Clearly, continuing this procedure on the remaining o-sequence, we are left with a p-sequence with respect to σ.

Example 2.1.3 Figure 2-3 shows the transformation of

$$a_{14}a_{12}a_{13}a_{13}a_{14}a_{22}a_{21}a_{23}a_{31}a_{34}a_{33}a_{34}a_{34}a_{34}a_{42}a_{41}a_{42}a_{43}a_{41}a_{41}a_{44}$$

into

$$a_{22}a_{21}a_{14}a_{42}a_{23}a_{31}a_{12}a_{34}a_{41}a_{13}a_{33}a_{34}a_{42}a_{34}a_{43}a_{34}a_{41}a_{13}a_{41}a_{14}a_{44}$$

with respect to $\sigma = 2341$. In the first five drawings, the step that must be moved to the left is drawn in bold. In the next three drawings, all the steps that will form a lattice path in the p-sequence are drawn in bold. ♢

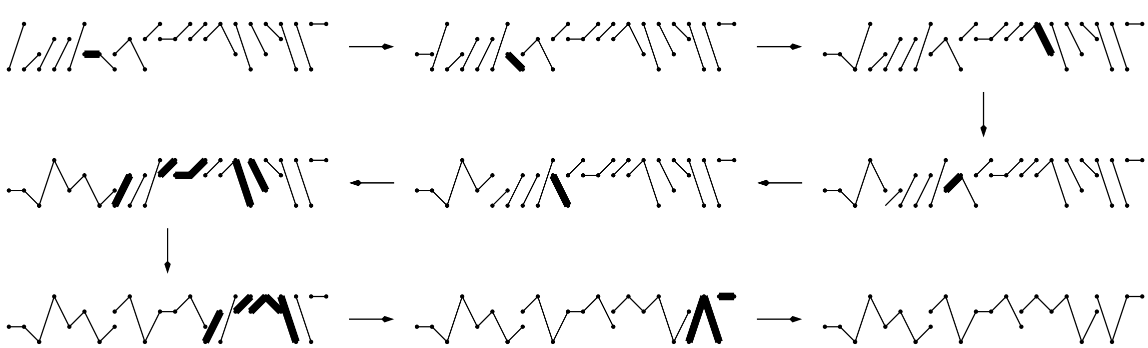

Figure 2-3: The transformation φ.

Lemma 2.1.4 *The map* $\varphi\colon \mathbf{O}(\mathbf{p};\mathbf{r}) \to \mathbf{P}^{\sigma}(\mathbf{p};\mathbf{r})$ *constructed above is a bijection.*

Proof. Since the above procedure never switches two steps that begin at the same height, there is exactly one o-sequence that maps into a given p-sequence: take all steps starting at height 1 in the p-sequence in the order they appear, then all the steps starting at height 2 in the p-sequence in the order they appear, etc. Clearly, this map preserves the type of the sequence. □

Lemma 2.1.4 has the following immediate application.

Theorem 2.1.5 *Take non-negative integer vectors* $\mathbf{p}, \mathbf{r}$ *with* $\sum p_i = \sum r_i$*, and permutations* $\pi, \sigma \in S_m$*. If the matrix* $A = (a_{ij})_{m \times m}$ *is Cartier-Foata, i.e. if it satisfies*

$$a_{jk} a_{il} \;=\; a_{il} a_{jk} \;\; \textit{for all} \;\; i \neq j, \tag{2.1.2}$$

then

$$O^{\pi}(\mathbf{p}; \mathbf{r}) = P^{\sigma}(\mathbf{p}; \mathbf{r}). \tag{2.1.3}$$

Proof: The equation follows from the fact that the map φ never switches two steps that start at the same height. That means that modulo the ideal $\mathcal{I}_{\mathrm{cf}}$ generated by the relation (2.1.2), we have $\varphi(\alpha) = \alpha$ for each $\alpha \in \mathbf{O}^{\pi}(\mathbf{p}; \mathbf{r})$; in other words, if A is Cartier-Foata, $\varphi(\alpha) = \alpha$ for each $\alpha \in \mathbf{O}^{\pi}(\mathbf{p}; \mathbf{r})$ and so

$$O^{\pi}(\mathbf{p}; \mathbf{r}) = \sum_{\alpha \in \mathbf{O}^{\pi}(\mathbf{p}; \mathbf{r})} \alpha = \sum_{\alpha \in \mathbf{O}^{\pi}(\mathbf{p}; \mathbf{r})} \varphi(\alpha) = \sum_{\alpha \in \mathbf{P}^{\pi}(\mathbf{p}; \mathbf{r})} \alpha = P^{\pi}(\mathbf{p}; \mathbf{r}). \qquad \square$$

We can construct an analogous map

$$\overline{\varphi} \colon \overline{\mathbf{O}}^{\pi}(\mathbf{p}; \mathbf{r}) \longrightarrow \overline{\mathbf{P}}^{\sigma}(\mathbf{p}; \mathbf{r}), \tag{2.1.4}$$

where we move steps to the right instead of to the left. Again, $\overline{\varphi}$ is a bijection, and it implies the following result. Note that the condition on A is weaker than in Theorem 2.1.5, but we assume an extra condition on $\mathbf{p}$.

Theorem 2.1.6 *Take non-negative integer vectors* $\mathbf{p}, \mathbf{r}$ *with* $\sum p_i = \sum r_i$*, and permutations* $\pi, \sigma \in S_m$*. Assume that the matrix* $A = (a_{ij})_{m \times m}$ *satisfies*

$$a_{jk} a_{il} \;=\; a_{il} a_{jk} \;\; \textit{for all} \;\; i \neq j, k \neq l, \tag{2.1.5}$$

and that $\mathbf{p} \leq \mathbf{1}$*. Then*

$$\overline{O}^{\pi}(\mathbf{p}; \mathbf{r}) = \overline{P}^{\sigma}(\mathbf{p}; \mathbf{r}). \tag{2.1.6}$$

Proof: The equation follows from the fact that the map $\overline{\varphi}$ never switches two steps that end at the same height. Furthermore, since $p_i \leq 1$, we never switch two steps that begin at the same height, so we do not need (2.1.5) for $i = j$. That means that modulo the ideal generated by the relation (2.1.5), we have $\overline{\varphi}(\alpha) = \alpha$ for each $\alpha \in \overline{\mathbf{O}}^{\pi}(\mathbf{p}; \mathbf{r})$; in other words, if A satisfies (2.1.5), $\varphi(\alpha) = \alpha$ for each $\alpha \in \overline{\mathbf{O}}^{\pi}(\mathbf{p}; \mathbf{r})$ and

$$\overline{O}^{\pi}(\mathbf{p}; \mathbf{r}) = \sum_{\alpha \in \overline{\mathbf{O}}^{\pi}(\mathbf{p}; \mathbf{r})} \alpha = \sum_{\alpha \in \overline{\mathbf{O}}^{\pi}(\mathbf{p}; \mathbf{r})} \varphi(\alpha) = \sum_{\alpha \in \overline{\mathbf{P}}^{\pi}(\mathbf{p}; \mathbf{r})} \alpha = \overline{P}^{\pi}(\mathbf{p}; \mathbf{r}). \qquad \square$$

It is not immediately clear why we need Theorems 2.1.5 and 2.1.6 (and their right-quantum, weighted and quantum analogues proved in Sections 2.2–2.7). They are, however, crucial for the remaining chapters. They will give us concise proofs of the matrix inverse formulas and Jacobi ratio theorem in Chapter 3, of MacMahon master theorem and various extensions of it in Chapter 4, and, in Chapter 5, of the fact that a certain matrix is ($q-$ or $\mathbf{q}-$)right-quantum, which will lead to non-commutative extensions of Sylvester's determinantal identity.

The transformation was found independently, but is related to the *first fundamental transformation* from [Foa65].

2.2 The right-quantum case

Theorems 2.1.5 and 2.1.6 have the following natural extension to right-quantum (and "almost" right-quantum) matrices.

Theorem 2.2.1 *Take a matrix $A = (a_{ij})_{m\times m}$, non-negative integer vectors $\mathbf{p}, \mathbf{r}$ with $\sum p_i = \sum r_i$, and permutations $\pi, \sigma \in S_m$.*

(1) Assume that A is right-quantum, i.e. that it has the properties

$$\begin{aligned} a_{jk}a_{ik} &= a_{ik}a_{jk}, & (2.2.1)\\ a_{ik}a_{jl} - a_{jk}a_{il} &= a_{jl}a_{ik} - a_{il}a_{jk} \quad \text{for all} \quad k \neq l. & (2.2.2)\end{aligned}$$

Then

$$O^{\pi}(\mathbf{p};\mathbf{r}) = P^{\sigma}(\mathbf{p};\mathbf{r}). \tag{2.2.3}$$

(2) Assume that A satisfies the property (2.2.2) above, and that $\mathbf{p} \leq \mathbf{1}$. Then

$$\overline{O}^{\pi}(\mathbf{p};\mathbf{r}) = \overline{P}^{\sigma}(\mathbf{p};\mathbf{r}). \tag{2.2.4}$$

Define a *q-sequence* to be a sequence we get in the transformation of o-sequences into p-sequences with the above procedure (including the o-sequence and the p-sequence).

A sequence $a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}$ is a q-sequence if and only if it is a concatenation of

- some lattice paths with starting heights i_{k_s} and ending heights j_{l_s} so that $\sigma^{-1}(i_{k_s}) \leq \sigma^{-1}(i_t)$ for all $t \geq k_s$, and $i_t \neq j_{l_s}$ for $t > l_s$;
- a lattice path with starting height i_k and ending height j_k so that $\sigma^{-1}(i_{k_s}) \leq \sigma^{-1}(i_t)$ for all $t \geq k_s$; and
- a sequence that is an o-sequence except that the leftmost step with starting height j_k can be before some of the steps with starting height i, $\sigma^{-1}(i) \leq \sigma^{-1}(j_k)$.

For a q-sequence α, denote by $\psi(\alpha)$ the q-sequence we get by performing the switch described above; for a p-sequence α (where no more switches are needed), define $\psi(\alpha) = \alpha$. By construction, the map ψ always switches steps with different starting heights.

For a sequence $a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}$, define the *rank* as $\mathrm{inv}(i_1i_2\ldots i_n)$ (more generally, the rank with respect to π is $\mathrm{inv}(\pi^{-1}(i_1)\pi^{-1}(i_2)\ldots\pi^{-1}(i_n))$). Clearly, o-sequences are exactly the sequences of rank 0. It is important to note that the map ψ increases by 1 the rank of sequences that are not p-sequences.

Write $\mathbf{Q}_n^\sigma(\mathbf{p};\mathbf{r})$ for the union of two sets of sequences of type $(\mathbf{p},\mathbf{r})$: the set of all q-sequences of type $(\mathbf{p},\mathbf{r})$ with rank n and the set of p-sequences of type $(\mathbf{p},\mathbf{r})$ (with respect to σ) with rank $< n$; in particular, $\mathbf{O}(\mathbf{p};\mathbf{r}) = \mathbf{Q}_0^\sigma(\mathbf{p};\mathbf{r})$ and $\mathbf{P}^\sigma(\mathbf{p};\mathbf{r}) = \mathbf{Q}_N^\sigma(\mathbf{p};\mathbf{r})$ for N large enough.

Lemma 2.2.2 *The map $\psi : \mathbf{Q}_n^\sigma(\mathbf{p};\mathbf{r}) \to \mathbf{Q}_{n+1}^\sigma(\mathbf{p};\mathbf{r})$ is a bijection for all n.*

Proof. A q-sequence of rank n which is not a p-sequence is mapped into a q-sequence of rank $n+1$, and ψ is the identity map on p-sequences. This proves that ψ is indeed a map from $\mathbf{Q}_n^\sigma(\mathbf{p};\mathbf{r})$ to $\mathbf{Q}_{n+1}^\sigma(\mathbf{p};\mathbf{r})$. It is easy to see (and it also follows from Lemma 2.1.4) that ψ is injective and surjective. □

As an illustration of how the proof of Theorem 2.2.1 works, consider $\mathbf{p} = \mathbf{r} = (3,1,1)$; there are $\binom{5}{3,1,1} = 20$ o-sequences of type $\mathbf{r}$. The sum of all the q-sequences in a line of Figure 2.2 is, modulo $\mathcal{I}_{\mathrm{rq}}$, equal to the sum of all q-sequences in the next line; if only one q-sequence in the line is drawn in bold, it means that we are using the relation (2.2.1), and if two q-sequences in the line are drawn in bold, it means that we are using the relation (2.2.2).

Proof of Theorem 2.2.1. (1) Recall that we are assuming that A is right-quantum. Take a q-sequence α. If α is a p-sequence, then $\psi(\alpha) = \alpha$. Otherwise, assume that $(x-1,i) \to (x,k)$ and $(x,j) \to (x+1,l)$ are the steps to be switched in order to get $\psi(\alpha)$. If $k = l$, then $\psi(\alpha) = \alpha$ by (2.2.1). Otherwise, denote by β the sequence we get by replacing these two steps with $(x-1,i) \to (x,l)$ and $(x,j) \to (x+1,k)$. The crucial observation is that β is also a q-sequence, and that its rank is equal to the rank of α. Furthermore, $\alpha + \beta = \psi(\alpha) + \psi(\beta)$ because of (2.2.2). This implies that $\sum \psi(\alpha) = \sum \alpha$ with the sum over all sequences in $\mathbf{Q}_n^\sigma(\mathbf{p};\mathbf{r})$. Repeated application of this shows that

$$\sum \varphi(\alpha) = \sum \alpha$$

with the sum over all $\alpha \in \mathbf{O}(\mathbf{p};\mathbf{r})$. Because φ is a bijection, this finishes the proof of (2.2.3).

(2) For the proof of (2.2.4), we use (analogously defined) bq-sequences, back-rank, the sets $\overline{\mathbf{Q}}_n^\sigma(\mathbf{p};\mathbf{r})$, and the bijections $\overline{\psi}\colon \overline{\mathbf{Q}}_n^\sigma(\mathbf{p};\mathbf{r}) \to \overline{\mathbf{Q}}_{n+1}^\sigma(\mathbf{p};\mathbf{r})$ and $\overline{\varphi}\colon \overline{\mathbf{O}}^\pi(\mathbf{p};\mathbf{r}) \to \overline{\mathbf{P}}^\sigma(\mathbf{p};\mathbf{r})$, for which we move steps to the right instead of to the left. Assume that $(x-1,j) \to (x,l)$ and $(x,i) \to (x+1,k)$ are the steps in α we want to switch. The condition $p_i \leq 1$ guarantees that $i \neq j$. Denote by β the sequence we get by replacing these two steps with $(x-1,i) \to (x,l)$ and $(x,j) \to (x+1,k)$; β is also a q-sequence of the same rank, and because $i \neq j$, its number of inversions differs from α by ± 1. The relation (2.2.2) implies $\alpha - \beta = \overline{\psi}(\alpha) - \overline{\psi}(\beta)$, and this means that $\sum (-1)^{\mathrm{inv}(\psi(\alpha))}\overline{\psi}(\alpha) = \sum (-1)^{\mathrm{inv}(\alpha)}\alpha$ and hence also

$$\sum (-1)^{\mathrm{inv}(\overline{\varphi}(\alpha))}\overline{\varphi}(\alpha) = \sum (-1)^{\mathrm{inv}(\alpha)}\alpha$$

with the sum over all $\alpha \in \overline{\mathbf{O}}(\mathbf{p};\mathbf{r})$. □

Figure 2-4: Graphical illustration of $O(3,1,1) = P(3,1,1)$ in the right-quantum case.

2.3 The q-Cartier-Foata case

Choose $q \in \mathbb{C} \setminus \{0\}$, and define

$$O_q^{\pi}(\mathbf{p};\mathbf{r}) = \sum_{\alpha \in \mathbf{O}^{\pi}(\mathbf{p};\mathbf{r})} q^{\mathrm{inv}(\alpha)} \alpha, \quad \overline{O}_q^{\pi}(\mathbf{p};\mathbf{r}) = \sum_{\alpha \in \overline{\mathbf{O}}^{\pi}(\mathbf{p};\mathbf{r})} (-q)^{-\mathrm{inv}(\alpha)}\ \alpha,$$

$$P_q^\pi(\mathbf{p};\mathbf{r}) = \sum_{\alpha \in \mathbf{P}^\pi(\mathbf{p};\mathbf{r})} q^{\mathrm{inv}(\alpha)}\alpha, \quad \overline{P}_q^\pi(\mathbf{p};\mathbf{r}) = \sum_{\alpha \in \overline{\mathbf{P}}^\pi(\mathbf{p};\mathbf{r})} (-q)^{-\mathrm{inv}(\alpha)}\ \alpha.$$

Recall that $\mathrm{inv}(\alpha)$ is defined to be $\mathrm{inv}(\mu) - \mathrm{inv}(\lambda)$ for $\alpha = a_{\lambda,\mu}$.

Example 2.3.1 For the p-sequence

$$\alpha = a_{13}a_{32}a_{24}a_{43}a_{31}a_{11}a_{22}a_{34}a_{44}a_{43}$$

shown in Figure 2-5, we have

$$\mathrm{inv}(1324312344) = 0+3+1+4+2+0+0+0+0+0 = 10$$

and

$$\mathrm{inv}(3243112443) = 4+2+5+3+0+0+0+1+1+0 = 16.$$

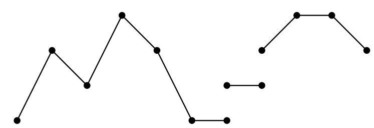

Figure 2-5: A p-sequence with weight q^6.

Therefore, the p-sequence α is weighted by q^6 in $P_q(2,2,3,3)$. $\diamondsuit$

Theorem 2.3.2 *Take a matrix $A = (a_{ij})_{m\times m}$, non-negative integer vectors $\mathbf{p}, \mathbf{r}$ with $\sum p_i = \sum r_i$, and permutations $\pi, \sigma \in S_m$.*

(1) Assume that A is q-Cartier-Foata, i.e. that it has the properties

$$a_{jl}a_{ik} = a_{ik}a_{jl} \quad \textit{for all} \quad i<j, k<l, \tag{2.3.1}$$
$$a_{jl}a_{ik} = q^2\ a_{ik}a_{jl} \quad \textit{for all} \quad i<j, k>l, \tag{2.3.2}$$
$$a_{jk}a_{ik} = q\ a_{ik}a_{jk} \quad \textit{for all} \quad i<j \tag{2.3.3}$$

Then

$$O_q^\pi(\mathbf{p};\mathbf{r}) = P_q^\sigma(\mathbf{p};\mathbf{r}). \tag{2.3.4}$$

(2) Assume that A satisfies (2.3.1) *and* (2.3.2) *above, and that $\mathbf{p} \le \mathbf{1}$. Then*

$$\overline{O}_q^\pi(\mathbf{p};\mathbf{r}) = \overline{P}_q^\sigma(\mathbf{p};\mathbf{r}). \tag{2.3.5}$$

The proof of the theorem is a weighted analogue of the proof of Theorems 2.1.5 and 2.1.6. The main technical difference is essentially bookkeeping of the powers of q which appear after switching the letters a_{ij} (equivalently, the lattice steps in the q-sequences).

Proof of Theorem 2.3.2. (1) Choose a q-sequence $\alpha = a_{\lambda,\mu}$ and let $\psi(\alpha) = a_{\lambda',\mu'}$. Assume that the switch we perform is between steps $(x-1,i) \to (x,k)$ and $(x,j) \to$

$(x+1,l)$; write $\lambda = \lambda_1 ij\lambda_2$, $\mu = \mu_1 kl\mu_2$, $\lambda' = \lambda_1 ji\lambda_2$, $\mu' = \mu_1 lk\mu_2$. If $i < j$ and $k < l$, we have $\mathrm{inv}(\lambda') = \mathrm{inv}(\lambda) + 1$, $\mathrm{inv}(\mu') = \mathrm{inv}(\mu) + 1$. By (2.3.1), $\psi(\alpha) = \alpha$,

$$q^{\mathrm{inv}(\mu')-\mathrm{inv}(\lambda')}\psi(\alpha) = q^{\mathrm{inv}(\mu)-\mathrm{inv}(\lambda)}\alpha \mod \mathcal{I}_{q-\mathrm{cf}}\,. \tag{2.3.6}$$

Similarly, if $i < j$ and $k > l$, we have $\mathrm{inv}(\lambda') = \mathrm{inv}(\lambda) + 1$, $\mathrm{inv}(\mu') = \mathrm{inv}(\mu) - 1$. By (2.3.2), we have $\psi(\alpha) = q^2\alpha$, which implies equation (2.3.6). If $i < j$ and $k = l$, we have $\mathrm{inv}(\lambda') = \mathrm{inv}(\lambda) + 1$, $\mathrm{inv}(\mu') = \mathrm{inv}(\mu)$. By (2.3.3), we have $\psi(\alpha) = q\alpha$, which implies (2.3.6) again. The cases when $i > j$ are analogous.
Iterating equation (2.3.6), we conclude that if $\alpha = a_{\lambda,\mu}$ is an o-sequence and $\varphi(\alpha) = a_{\lambda',\mu'}$ is the corresponding p-sequence, then

$$q^{\mathrm{inv}(\mu')-\mathrm{inv}(\lambda')}\ \varphi(\alpha) = q^{\mathrm{inv}(\mu)-\mathrm{inv}(\lambda)}\ \alpha.$$

This implies (2.2.4). Part (2) is analogous. □

Before continuing with the q-right-quantum case, let us make an important observation about the weight of balanced paths with respect to the identity permutation. Let us call a sequence $a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ni_1}$ a *primitive path* if it has the property $i_j > i_1$ for $j = 2,\ldots,n$. A p-sequence $\alpha \in \mathbf{P}(\mathbf{r})$ decomposes into a product of primitive paths; for example, the p-sequence in Example 2.3.1 is a product of four primitive paths: $(a_{13}a_{32}a_{24}a_{43}a_{31})(a_{11})(a_{22})(a_{34}a_{44}a_{43})$.

Proposition 2.3.3 *If a p-sequence α of length n decomposes into p primitive paths, then α is weighted by q^{n-p} in $P_q(\mathbf{r})$.*

Proof. By definition, $\alpha = a_{\lambda,\mu}$ is weighted by $q^{\mathrm{inv}(\mu)-\mathrm{inv}(\lambda)}$ as a term of $P_q(\mathbf{r})$. Assume first that α is primitive, $\alpha = a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ni_1}$ with $i_j > i_1$ for $j = 2,\ldots,n$. We have $\lambda = i_1i_2\cdots i_{n-1}i_n$ and $\mu = i_2i_3\cdots i_ni_1$, which means that μ has all the inversions that λ has, plus the inversions $(1,n),(2,n),\ldots,(n-1,n)$. In other words, $q^{\mathrm{inv}(\mu)-\mathrm{inv}(\lambda)} = q^{n-1}$.
Now take $\alpha = a_{\lambda,\mu} = \alpha_1\cdots\alpha_p$, where every α_k is a primitive path. Every primitive path is balanced, which means that inversions corresponding to different primitive paths in λ and μ cancel each other out (cf. the proof of the "$1 = q$ principle", Proposition 1.3.3); in other words,

$$q^{\mathrm{inv}(\mu)-\mathrm{inv}(\lambda)} = q^{\mathrm{inv}(\mu_1)-\mathrm{inv}(\lambda_1)}\cdots q^{\mathrm{inv}(\mu_p)-\mathrm{inv}(\lambda_p)} = q^{n_1-1}\cdots q^{n_p-1},$$

where $\alpha_k = a_{\lambda_k,\mu_k}$ has length n_k. Since $n_1 + \ldots + n_p = n$, this finishes the proof. □

The p-sequence from Example 2.3.1 has length 10 and is a product of 4 primitive paths. Therefore it is weighted by q^6 in $P_q(2,2,3,3)$, agreeing with the direct calculation.

2.4 The q-right-quantum case

Let us retain the notation of Section 2.3.

Theorem 2.4.1 *Take a matrix* $A = (a_{ij})_{m\times m}$*, non-negative integer vectors* $\mathbf{p}, \mathbf{r}$ *with* $\sum p_i = \sum r_i$*, and permutations* $\pi, \sigma \in S_m$.

(1) Assume that A is q-right-quantum, i.e. that it has the properties

$$\begin{aligned} a_{jk}a_{ik} &= q\ a_{ik}a_{jk} \quad \textit{for all} \quad i<j,, \qquad (2.4.1)\\ a_{ik}a_{jl} - q^{-1}\ a_{jk}a_{il} &= a_{jl}a_{ik} - q\ a_{il}a_{jk} \quad \textit{for all} \quad i<j, k<l. \qquad (2.4.2)\end{aligned}$$

Then

$$O_q^\pi(\mathbf{p};\mathbf{r}) = P_q^\sigma(\mathbf{p};\mathbf{r}). \qquad (2.4.3)$$

(2) Assume that A satisfies (2.4.2) *above, and that* $\mathbf{p} \le \mathbf{1}$*. Then*

$$\overline{O}_q^\pi(\mathbf{p};\mathbf{r}) = \overline{P}_q^\sigma(\mathbf{p};\mathbf{r}). \qquad (2.4.4)$$

Proof: The proof of the theorem is almost identical to the one given in Section 2.2, with some (bookkeeping) modifications similar to those in the proof of Theorem 2.3.2. We only prove part (1), as the proof of (2) is completely analogous. Take a q-sequence α, and assume that $(x-1,i) \to (x,k)$ and $(x,j) \to (x+1,l)$ are the steps to be switched in order to get $\psi(\alpha)$. If $i<j$ and $k=l$, then $q\psi(\alpha) = \alpha$ by (2.4.1) and so

$$q^{\mathrm{inv}(\psi(\alpha))}\psi(\alpha) = q^{\mathrm{inv}(\alpha)}\alpha.$$

If $i<j$ and $k<l$, denote by β the sequence we get by replacing these two steps with $(x-1,i) \to (x,l)$ and $(x,j) \to (x+1,k)$. Then β is also a q-sequence, and its rank is equal to the rank of α. Furthermore, $\psi(\alpha) + q^{-1}\psi(\beta) = \alpha + q\beta$ by (2.4.2), and so

$$q^{\mathrm{inv}(\psi(\alpha))}\psi(\alpha) + q^{\mathrm{inv}(\psi(\beta))}\psi(\beta) = q^{\mathrm{inv}(\alpha)}\alpha + q^{\mathrm{inv}(\beta)}\beta.$$

This implies that $\sum q^{\mathrm{inv}(\psi(\alpha))}\psi(\alpha) = \sum q^{\mathrm{inv}(\alpha)}\alpha$ with the sum over all sequences in $\mathbf{Q}_n^\sigma(\mathbf{p};\mathbf{r})$, and so

$$O_q(\mathbf{p};\mathbf{r}) = \sum_{\alpha\in\mathbf{O}(\mathbf{p};\mathbf{r})} q^{\mathrm{inv}(\alpha)}\alpha = \sum_{\alpha\in\mathbf{O}(\mathbf{p};\mathbf{r})} q^{\mathrm{inv}(\varphi(\alpha))}\varphi(\alpha) = \sum_{\alpha\in\mathbf{P}^\pi(\mathbf{p};\mathbf{r})} q^{\mathrm{inv}(\alpha)}\alpha = P_q^\sigma(\mathbf{p};\mathbf{r}). \qquad \square$$

2.5 The q-Cartier-Foata case

For $1 \le i < j \le m$, choose $q_{ij} \in \mathbb{C} \setminus \{0\}$, and write

$$\mathbf{q}^{\mathbf{inv}\ \alpha} = \left(\prod_{(i,j)\in\mathcal{I}(\mu)} q_{\mu_j\mu_i}\right)\left(\prod_{(i,j)\in\mathcal{I}(\lambda)} q^{-1}_{\lambda_j\lambda_i}\right).$$

for $\alpha = a_{\lambda,\mu}$. Define

$$O_{\mathbf{q}}^\pi(\mathbf{p};\mathbf{r}) = \sum_{\alpha\in\mathbf{O}^\pi(\mathbf{p};\mathbf{r})} \mathbf{q}^{\mathbf{inv}\ \alpha}\alpha, \quad \overline{O}_{\mathbf{q}}^\pi(\mathbf{p};\mathbf{r}) = \sum_{\alpha\in\overline{\mathbf{O}}^\pi(\mathbf{p};\mathbf{r})} (-\mathbf{q})^{-\mathbf{inv}\ \alpha}\ \alpha,$$

$$P_{\mathbf{q}}^{\pi}(\mathbf{p};\mathbf{r}) = \sum_{\alpha \in \mathbf{P}^{\pi}(\mathbf{p};\mathbf{r})} \mathbf{q}^{\mathbf{inv}\ \alpha}\alpha, \quad \overline{P}_{\mathbf{q}}^{\pi}(\mathbf{p};\mathbf{r}) = \sum_{\alpha \in \overline{\mathbf{P}}^{\pi}(\mathbf{p};\mathbf{r})} (-\mathbf{q})^{-\mathbf{inv}\ \alpha}\ \alpha.$$

EXAMPLE 2.5.1 For the p-sequence

$$\alpha = a_{13}a_{32}a_{24}a_{43}a_{31}a_{11}a_{22}a_{34}a_{44}a_{43}$$

shown in Figure 2-5, we have

$$\mathcal{I}(1324312344) = \{(2,3),(2,6),(2,7),(3,6),(4,5),(4,6),(4,7),(4,8),(5,6),(5,7)\}$$

and

$$\begin{aligned}\mathcal{I}(3243112443) &= \{(1,2),(1,5),(1,6),(1,7),(2,5),(2,6),(3,4),(3,5),\\ &\quad (3,6),(3,7),(3,10),(4,5),(4,6),(4,7),(8,10),(9,10)\}\end{aligned}$$

Therefore, the p-sequence α is weighted by

$$(q_{23}q_{13}q_{13}q_{23}q_{12}q_{12}q_{34}q_{14}q_{14}q_{24}q_{34}q_{13}q_{13}q_{23}q_{34}q_{34})(q_{23}q_{13}q_{23}q_{12}q_{34}q_{14}q_{24}q_{34}q_{13}q_{23})^{-1} =$$

$$= q_{12}q_{13}^2q_{14}q_{34}^2$$

in $P_{\mathbf{q}}(2,2,3,3)$. ♢

Theorem 2.5.2 *Take a matrix* $A = (a_{ij})_{m\times m}$*, non-negative integer vectors* $\mathbf{p}, \mathbf{r}$ *with* $\sum p_i = \sum r_i$*, and permutations* $\pi, \sigma \in S_m$*.*

(1) Assume that A is $\mathbf{q}$*-Cartier-Foata, i.e. that it has the properties*

$$a_{jl}a_{ik} = q_{kl}^{-1}q_{ij}\ a_{ik}a_{jl} \quad \text{for all } i<j, k<l \tag{2.5.1}$$
$$a_{jl}a_{ik} = q_{ij}q_{lk}\ a_{ik}a_{jl} \quad \text{for all } i<j, k>l \tag{2.5.2}$$
$$a_{jk}a_{ik} = q_{ij}\ a_{ik}a_{jk} \quad \text{for all } i<j \tag{2.5.3}$$

Then

$$O_{\mathbf{q}}^{\pi}(\mathbf{p};\mathbf{r}) = P_{\mathbf{q}}^{\sigma}(\mathbf{p};\mathbf{r}). \tag{2.5.4}$$

(2) Assume that A satisfies (2.5.1) *and* (2.5.2) *above, and that* $\mathbf{p} \leq \mathbf{1}$*. Then*

$$\overline{O}_{\mathbf{q}}^{\pi}(\mathbf{p};\mathbf{r}) = \overline{P}_{\mathbf{q}}^{\sigma}(\mathbf{p};\mathbf{r}). \tag{2.5.5}$$

The proof of the theorem is a multiparameter analogue of the proof of Theorem 2.3.2; instead of counting the number of inversions, we have to keep track of the actual inversions.

Proof of Theorem 2.5.2. (1) Choose a q-sequence $\alpha = a_{\lambda,\mu}$ and let $\psi(\alpha) = a_{\lambda',\mu'}$. Assume that the switch we perform is between the n-th step $(x-1,i) \to (x,k)$ and the $(n+1)$-th step $(x,j) \to (x+1,l)$; write $\lambda = \lambda_1 ij\lambda_2$, $\mu = \mu_1 kl\mu_2$, $\lambda' = \lambda_1 ji\lambda_2$, $\mu' = \mu_1 lk\mu_2$. If $i<j$ and $k<l$, we have $\mathcal{I}(\lambda') = \mathcal{I}(\lambda) \cup \{(n,n+1)\}$, $\mathcal{I}(\mu') =$

$\mathcal{I}(\mu) \cup \{(n, n+1)\}$. By (2.5.1), $q_{kl}q_{ij}^{-1}\psi(\alpha) = \alpha$ modulo $\mathcal{I}_{\mathbf{q}-\mathrm{cf}}$ and

$$\mathbf{q}^{\mathbf{inv}\ \psi(\alpha)}\psi(\alpha) = \mathbf{q}^{\mathbf{inv}\ \alpha}\alpha. \tag{2.5.6}$$

We prove this relation similarly in other cases, using (2.5.2) and (2.5.3). We conclude that if $\alpha = a_{\lambda,\mu}$ is an o-sequence and $\varphi(\alpha) = a_{\lambda',\mu'}$ is the corresponding p-sequence, then

$$\mathbf{q}^{\mathbf{inv}\ \varphi(\alpha)}\varphi(\alpha) = \mathbf{q}^{\mathbf{inv}\ \alpha}\alpha$$

This implies (2.5.4). Part (2) is analogous. □

We also need a multiparameter analogue of Proposition 2.3.3.

Proposition 2.5.3 *The weight of a p-sequence α in $P_{\mathbf{q}}(\mathbf{r})$ is*

$$\prod q_{ij},$$

where the product runs over all pairs (i, j) with i the starting height of a primitive path of α, and j another stasrting height of the same primitive path.

Proof. Take a primitive path $\alpha = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}$ first. Then $\mu = i_2 i_3 \cdots i_n i_1$ has all the inversions that $\lambda = i_1 i_2 \cdots i_{n-1} i_n$ has, plus the inversions $(1, n), (2, n), \ldots, (n-1, n)$, and therefore

$$\mathbf{q}^{\mathbf{inv}\ \alpha} = q_{i_1 i_2} q_{i_1 i_3} \cdots q_{i_1 i_n}.$$

Now take $\alpha = a_{\lambda,\mu} = \alpha_1 \cdots \alpha_p$, where every α_k is a primitive path. Every primitive path is balanced, which means that inversions corresponding to different primitive paths in λ and μ cancel each other out; in other words,

$$\mathbf{q}^{\mathbf{inv}\ \alpha} = \mathbf{q}^{\mathbf{inv}\ \alpha_1} \cdots \mathbf{q}^{\mathbf{inv}\ \alpha_p}.$$

This finishes the proof. □

Note that if we take $q_{ij} = q_{ji}^{-1}$ for $i > j$, Proposition 2.5.3 also holds for weights of sequences α in $P_{\mathbf{q}}^{\pi}(\mathbf{r})$ for any permutation π.

The p-sequence from Example 2.5.1 has four primitive paths with starting heights $1, 1, 2, 3$, and is by Proposition 2.5.3 weighted by

$$(q_{13}q_{12}q_{14}q_{13})()()(q_{34}q_{34}) = q_{12}q_{13}^2 q_{14} q_{34}^2$$

in $P_{\mathbf{q}}(2, 2, 3, 3)$, agreeing with the direct calculation in Example 2.5.1.

2.6 The q-right-quantum case

Let us deal with the **q**-right-quantum case. We retain the notation of Section 2.5.

Theorem 2.6.1 *Take a matrix $A = (a_{ij})_{m \times m}$, non-negative integer vectors $\mathbf{p}, \mathbf{r}$ with $\sum p_i = \sum r_i$, and permutations $\pi, \sigma \in S_m$.*

(1) Assume that A is q-right-quantum, i.e. that it has the properties

$$a_{jk}a_{ik} = q_{ij}a_{ik}a_{jk} \quad \text{for} \quad i<j, \tag{2.6.1}$$
$$a_{ik}a_{jl} - q_{ij}^{-1}a_{jk}a_{il} = q_{kl}q_{ij}^{-1}a_{jl}a_{ik} - q_{kl}a_{il}a_{jk} \quad \text{for} \quad i<j, k<l. \tag{2.6.2}$$

Then

$$O_{\mathbf{q}}^{\pi}(\mathbf{p};\mathbf{r}) = P_{\mathbf{q}}^{\sigma}(\mathbf{p};\mathbf{r}). \tag{2.6.3}$$

(2) Assume that A satisfies the property (2.6.2) *above, and that* $\mathbf{p} \le \mathbf{1}$. *Then*

$$\overline{O}_{\mathbf{q}}^{\pi}(\mathbf{p};\mathbf{r}) = \overline{P}_{\mathbf{q}}^{\sigma}(\mathbf{p};\mathbf{r}). \tag{2.6.4}$$

Proof. The proof of the theorem is a combination of the proofs of Theorems 2.4.1 and 2.5.2. The details are left to the reader. □

2.7 The quantum case

For $\mathbf{p} = \mathbf{r} = \mathbf{1}$, we can prove a more general statement than Theorem 2.4.1 when the variables are quantum. To simplify the presentation slightly, we only prove the case when $\pi = \sigma = \mathrm{id}$.

Denote the permutation with cycle notation $(1,\ldots,\mu_1)(\mu_1+1,\ldots,\mu_2)\cdots$ for some composition $\mu = (\mu_1,\ldots,\mu_r)$ of m by γ_μ. A permutation is of such form if and only if $w(i) \le i+1$ for all i. For $w \in S_m$, denote by $\mathbf{a}_w$ the o-sequence $a_{\mathrm{id},w}$, by $\mathbf{a}^w$ the corresponding p-sequence $\varphi(a_{\mathrm{id},w})$, and by $\mu(w)$ the cycle type of w (treated as a composition). Write γ_w for $\gamma_{\mu(w)}$. For example, for $w = 532461$, we have

$$\mathbf{a}_w = a_{15}a_{23}a_{32}a_{44}a_{56}a_{61}, \quad \mathbf{a}^w = a_{15}a_{56}a_{61}a_{23}a_{32}a_{44},$$

$$\mu(w) = (3,2,1), \quad \gamma_w = (1,2,3)(4,5)(6).$$

Theorem 2.7.1 *Take a matrix $A = (a_{ij})_{m\times m}$, and assume that A is quantum, i.e. that it has the properties*

$$a_{il}a_{ik} = q\ a_{ik}a_{il} \quad \text{for all} \quad i,k,l,\ k<l, \tag{2.7.1}$$
$$a_{jk}a_{ik} = q\ a_{ik}a_{jk} \quad \text{for all} \quad i,j,k,\ i<j, \tag{2.7.2}$$
$$a_{jk}a_{il} = a_{il}a_{jk} \quad \text{for all} \quad i<j, k<l, \tag{2.7.3}$$
$$a_{jl}a_{ik} = a_{ik}a_{jl} + (q-q^{-1})\ a_{il}a_{jk} \quad \text{for all} \quad i<j, k<l. \tag{2.7.4}$$

Assume that we are given complex coefficients $c_w, w \in S_m$. For a permutation w, choose the smallest i that satisfies $w(i) = j+1 > i+1$ (if such an i exists). Denote the transposition $(j, j+1)$ by s_j. The permutation s_jws_j either has the same number of inversions as w, or it has 2 fewer. Assume that:

- *if* $\mathrm{inv}(s_jws_j) = \mathrm{inv}(w)$, *then* $c_w = c_{s_jws_j}$;
- *if* $\mathrm{inv}(s_jws_j) = \mathrm{inv}(w) - 2$, *then* $c_w = c_{s_jws_j} + (q-q^{-1})c_{ws_j}$.

Then

$$\sum_{w\in S_m} c_w \mathbf{a}_w = \sum_{w\in S_m} c_{\gamma_w}\mathbf{a}^w.$$

EXAMPLE 2.7.2 Take $m=3$, and assume that $c_{312}=c_{231}$ and $c_{321}=c_{213}+(q-q^{-1})c_{312}=c_{213}+(q-q^{-1})c_{231}$. Then

$$c_{312}a_{21}a_{32}+c_{321}a_{22}a_{31}=c_{231}a_{21}a_{32}+(c_{213}+(q-q^{-1})c_{231})a_{22}a_{31}=$$

$$=c_{231}(a_{21}a_{32}+(q-q^{-1})a_{22}a_{31})+c_{213}a_{22}a_{31}=c_{231}a_{32}a_{21}+c_{213}a_{31}a_{22}$$

and so

$$c_{123}a_{11}a_{22}a_{33}+c_{132}a_{11}a_{23}a_{32}+c_{213}a_{12}a_{21}a_{33}+c_{231}a_{12}a_{23}a_{31}+c_{312}a_{13}a_{21}a_{32}+c_{321}a_{13}a_{22}a_{31}$$

is equal to

$$c_{123}a_{11}a_{22}a_{33}+c_{132}a_{11}a_{23}a_{32}+c_{213}a_{12}a_{21}a_{33}+c_{231}a_{12}a_{23}a_{31}+c_{231}a_{13}a_{32}a_{21}+c_{213}a_{13}a_{31}a_{22},$$

as predicted by the theorem. ♢

Proof of Theorem 2.7.1. Choose a permutation $v\in S_m$, and write $\alpha_0^v=\mathrm{id}$, $\beta_0^v=v$. Assume that after k steps of the fundamental transformation, we have the q-sequence $a_{\alpha_k^v,\beta_k^v}$ for permutations α_k^v and β_k^v. Write $\mathbf{a}^w=\varphi(\mathbf{a}_w)=a_{\alpha^v,\beta^v}$. Furthermore, denote $(\alpha_k^v)^{-1}\beta_k^v$ by π_k^v and $(\alpha^v)^{-1}\beta^v$ by π^v. The crucial observation is the connection between the fundamental transformation and the transformation $w\mapsto s_jws_j$, where i is minimal with $w(i)>i+1$, and $j=w(i)-1$. Take the q-sequence $a_{\alpha_k^v,\beta_k^v}$ of rank k. Then $\pi^v(i)=j+1$ with i the minimal position which satisfies $\pi^v(i)>i+1$ if and only if we have to switch steps at positions j and $j+1$ in order to get $a_{\alpha_{k+1}^v,\beta_{k+1}^v}$ from $a_{\alpha_k^v,\beta_k^v}$. In other words, $\alpha_{k+1}^v=\alpha_k^v s_j$, $\beta_{k+1}^v=\beta_k^v s_j$ and $\pi_{k+1}^v=s_j\pi_k^v s_j$. We also have $a_{\alpha^v,\beta^v}=\mathbf{a}^v$ and $\pi^v=\gamma_v$. As an example, take $v=45132$; then we get

k	0	1	2	3
α_k^v	12345	12435	14235	14325
β_k^v	45132	45312	43512	43152
π_k^v	(143)(25)	(134)(25)	(124)(35)	(123)(45)
i	1	1	2	
j	3	2	3	

If we show that

$$\sum_{w\in S_m} c_{\pi_k^w} a_{\alpha_k^w,\beta_k^w} = \sum_{w\in S_m} c_{\pi_{k+1}^w} a_{\alpha_{k+1}^w,\beta_{k+1}^w}, \tag{2.7.5}$$

for every k, then

$$\sum_{w\in S_m} c_w\mathbf{a}_w=\sum_{w\in S_m} c_{\pi_0^w}a_{\alpha_0^w,\beta_0^w}=\sum_{w\in S_m} c_{\pi_1^w}a_{\alpha_1^w,\beta_1^w}=\ldots=\sum_{w\in S_m} c_{\pi^w}a_{\alpha^w,\beta^w}=\sum_{w\in S_m} c_{\gamma_w}\mathbf{a}^w$$

and the proof of the theorem is complete.

To prove (2.7.5), take $v \in S_m$, k and the corresponding j, and note that both $a_{\alpha_k^v,\beta_k^v}$ and $a_{\alpha_k^v,\beta_k^v \cdot s_j}$ appear in

$$\sum_{w \in S_m} c_{\pi_k^w} a_{\alpha_k^w,\beta_k^w},$$

one with coefficient $c_{\alpha^{-1}\beta}$ and the other with coefficient $c_{\alpha^{-1}\beta s_j}$, where we write $\alpha = \alpha_k^v$ and $\beta = \beta_k^v$. We have $\alpha(j) < \alpha(j+1)$. Also, we can assume without loss of generality that $\beta(j) < \beta(j+1)$ (otherwise we reverse the roles of β and βs_j). Then

$$a_{\alpha(j+1)\beta(j+1)} a_{\alpha(j)\beta(j)} = a_{\alpha(j)\beta(j)} a_{\alpha(j+1)\beta(j+1)} + (q - q^{-1}) a_{\alpha(j)\beta(j+1)} a_{\alpha(j+1)\beta(j)}$$

and

$$a_{\alpha(j+1)\beta(j)} a_{\alpha(j)\beta(j+1)} = a_{\alpha(j)\beta(j+1)} a_{\alpha(j+1)\beta(j)},$$

so

$$a_{\alpha s_j,\beta s_j} = a_{\alpha,\beta} + (q - q^{-1}) a_{\alpha,\beta s_j}$$

and

$$a_{\alpha s_j,\beta} = a_{\alpha,\beta s_j}.$$

Note that $\operatorname{inv}(s_j \alpha^{-1} \beta s_j) = \operatorname{inv}(\alpha^{-1}\beta)$ and $\operatorname{inv}(\alpha^{-1}\beta s_j) = \operatorname{inv}(s_j \alpha^{-1}\beta) - 2$. This implies

$$c_{\alpha^{-1}\beta} a_{\alpha,\beta} + c_{\alpha^{-1}\beta s_j} a_{\alpha,\beta s_j} = c_{\alpha^{-1}\beta} a_{\alpha,\beta} + \left(c_{s_j\alpha^{-1}\beta} + (q - q^{-1}) c_{\alpha^{-1}\beta}\right) a_{\alpha,\beta s_j} =$$

$$= c_{\alpha^{-1}\beta} \left(a_{\alpha,\beta} + (q - q^{-1}) a_{\alpha,\beta s_j}\right) + c_{s_j\alpha^{-1}\beta} a_{\alpha,\beta s_j} = c_{s_j\alpha^{-1}\beta s_j} a_{\alpha s_j,\beta s_j} + c_{s_j\alpha^{-1}\beta} a_{\alpha s_j,\beta}.$$

This completes the induction step and the proof of the lemma. □

EXAMPLE 2.7.3 The functions $c_w = q^{\operatorname{inv}(w)}$ and $c_w = (-q)^{-\operatorname{inv}(w)}$ both satisfy the condition of the lemma. This implies Theorem 2.4.1 for quantum variables and $\mathbf{p} = \mathbf{r} = \mathbf{1}$ (for part (2), we have to use some symmetry). ♢

Chapter 3

The matrix inverse formulas

3.1 Introduction

As mentioned in the introduction, non-commutative versions of the matrix inverse formula play an important role in this thesis; they are crucial for the proofs of MacMahon master theorem and Sylvester's determinantal identity, as well as for various extensions, generalizations and related results.

A common pattern for proofs in the following chapters will be to use (a variant of) the fundamental transformation discussed in Chapter 2, and then to finish the proof by invoking the matrix inverse formula.

The classical matrix inverse formula is such a simple linear algebraic result that it is to be expected that there are also simple linear algebraic proofs of its non-commutative analogues. However, since we are interested in combinatorial aspects of determinantal identities, we also present combinatorial, if more complicated proofs.

Not surprisingly, these proofs also strongly rely on the fundamental transformation. Also not surprisingly, they are easily adaptable to other settings; for example, the Jacobi ratio theorem (Section 3.7) in the right-quantum case requires almost no extra effort, while a direct linear algebraic proof would be more difficult.

We start off the chapter by stating the matrix inverse formulas, and proving them non-combinatorially in Section 3.2. In Section 3.3, we give a combinatorial proof in the Cartier-Foata case; in Section 3.4, we extend this to right-quantum variables; and in Sections 3.5 and 3.6, we prove the results with parameters. We close the chapter by proving the right-quantum Jacobi ratio theorem in Section 3.7.

Theorem 3.1.1 (Cartier-Foata matrix inverse formula) *If $A = (a_{ij})_{m\times m}$ is a Cartier-Foata matrix, we have*

$$\left(\frac{1}{I-A}\right)_{ij} = (-1)^{i+j} \cdot \frac{1}{\det(I-A)} \cdot \det\,(I-A)^{ji}$$

for all i, j.

Theorem 3.1.2 (right-quantum matrix inverse formula) *If $A = (a_{ij})_{m\times m}$ is a right-quantum matrix, we have*

$$\left(\frac{1}{I-A}\right)_{ij} = (-1)^{i+j} \cdot \frac{1}{\det(I-A)} \cdot \det{(I-A)}^{ji}$$

for all i, j.

Theorem 3.1.3 (weighted matrix inverse formula) *If $A = (a_{ij})_{m\times m}$ is a q-Cartier-Foata or a q-right-quantum matrix, we have*

$$\left(\frac{1}{I-A_{[i]}}\right)_{ii} = \frac{1}{\det_q(I-A)} \cdot \det_q{(I-A)}^{ii}$$

for all i, where

$$A_{[i]} = \begin{pmatrix} q^{-1}a_{11} & \cdots & q^{-1}a_{1i} & a_{1,i+1} & \cdots & a_{1m} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ q^{-1}a_{i-1,1} & \cdots & q^{-1}a_{i-1,i} & a_{i-1,i+1} & \cdots & a_{i-1,m} \\ a_{i1} & \cdots & a_{ii} & qa_{i,i+1} & \cdots & qa_{im} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mi} & qa_{m,i+1} & \cdots & qa_{mm} \end{pmatrix}. \qquad (3.1.1)$$

Theorem 3.1.4 (multiparameter matrix inverse formula) *If $A = (a_{ij})_{m\times m}$ is a* **q**-*Cartier-Foata matrix or a* **q**-*right-quantum matrix, we have*

$$\left(\frac{1}{I-A_{[i]}}\right)_{ii} = \frac{1}{\det_{\mathbf{q}}(I-A)} \cdot \det_{\mathbf{q}}{(I-A)}^{ii}$$

for all i, where

$$A_{[i]} = \begin{pmatrix} q_{1i}^{-1}a_{11} & \cdots & q_{1i}^{-1}a_{1i} & q_{1i}^{-1}q_{i,i+1}a_{1,i+1} & \cdots & q_{1i}^{-1}q_{im}a_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ q_{i-1,i}^{-1}a_{i-1,1} & \cdots & q_{i-1,i}^{-1}a_{i-1,i} & q_{i-1,i}^{-1}q_{i,i+1}a_{i-1,i+1} & \cdots & q_{i-1,i}^{-1}q_{im}a_{i-1,m} \\ a_{i1} & \cdots & a_{ii} & q_{i,i+1}a_{i,i+1} & \cdots & q_{im}a_{im} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mi} & q_{i,i+1}a_{m,i+1} & \cdots & q_{im}a_{mm} \end{pmatrix}. \qquad (3.1.2)$$

A reader not interested in combinatorial proofs of these formulas and in the right-quantum version of the Jacobi ratio theorem (which will not be needed in subsequent chapters) is advised to skim over Section 3.2, and then proceed to Chapter 4.

Similar results are also important (and well-known) in quantum algebra, see for example the results of Manin on quantum determinants [Man89, Man88] and advanced technical results of Etingof and Retakh who found analogues of this result for all twisted quantum groups [ER99].

3.2 Non-combinatorial proofs of Theorems 3.1.1 – 3.1.4

Lemma 3.2.1 *Let $B = (b_{ij})_{m\times m}$ be a matrix.*

(1) If B is Cartier-Foata and if B' denotes the matrix we get by interchanging adjacent columns of B, then B' is also Cartier-Foata, and $\det B' = -\det B$.

(2) If B is Cartier-Foata and has two equal columns, then $\det B = 0$.

(3) For every B, we have

$$\det B = \sum_{k=1}^{m} (-1)^{m+k} (\det B^{km}) b_{km}.$$

Proof: (1) Clearly B' is also Cartier-Foata. By definition,

$$\det B = \sum_{\pi \in S_m} (-1)^{\mathrm{inv}(\pi)} b_{\pi(1)1} \cdots b_{\pi(i)i} b_{\pi(i+1),i+1} \cdots b_{\pi(m)m},$$

and because $b_{\pi(i)i}$ and $b_{\pi(i+1),i+1}$ commute and because $\mathrm{inv}(\pi') = \mathrm{inv}(\pi) \pm 1$ for $\pi' = \pi \cdot (i, i+1)$, this is equal to $-\det B'$. (2) By (1), it is enough to prove that $\det B = 0$ if adjacent columns are equal. But for such a matrix, $\det B = -\det B$ by (1). (3) We have

$$\det B = \sum_{\pi \in S_m} (-1)^{\mathrm{inv}(\pi)} b_{\pi(1)1} \cdots b_{\pi(m)m} = \sum_{k=1}^{m} \sum_{\pi(m)=k} (-1)^{\mathrm{inv}(\pi)} b_{\pi(1)1} \cdots b_{\pi(m)m} =$$

$$= \sum_{k=1}^{m} \left(\sum_{\pi(m)=k} (-1)^{\mathrm{inv}(\pi)} b_{\pi(1)1} \cdots b_{\pi(m-1),m-1} \right) b_{km} = \sum_{k=1}^{m} (-1)^{m+k} (\det B^{km}) b_{km}. \qquad \square$$

Proof of Theorem 3.1.1. Take $B = I - A$ and recall that B is invertible. Note that

$$(\delta_{ik} - a_{ik})(\delta_{jl} - a_{jl}) = \delta_{ik}\delta_{jl} - \delta_{jl} a_{ik} - \delta_{ik} a_{jl} + a_{ik} a_{jl} =$$

$$= \delta_{ik}\delta_{jl} - \delta_{jl} a_{ik} - \delta_{ik} a_{jl} + a_{jl} a_{ik} = (\delta_{jl} - a_{jl})(\delta_{ik} - a_{ik})$$

for $k \neq l$, so B is also Cartier-Foata. The j-th coordinate of the matrix product

$$((-1)^{i+1} \det B^{1i}, (-1)^{i+2} \det B^{2i}, \ldots, (-1)^{i+m} B^{mi}) \cdot B$$

is $\sum_{k=1}^{m} (-1)^{i+k} \det B^{ki} b_{kj}$; which is by Lemma 3.2.1, part (3), equal to

$$(-1)^{m-i} \det \begin{pmatrix} b_{11} & \cdots & b_{1,i-1} & b_{1,i+1} & \cdots & b_{1m} & b_{1j} \\ b_{21} & \cdots & b_{2,i-1} & b_{2,i+1} & \cdots & b_{2m} & b_{2j} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{m1} & \cdots & b_{m,i-1} & b_{m,i+1} & \cdots & b_{mm} & b_{mj} \end{pmatrix}.$$

The last matrix is also Cartier-Foata, so parts (1) and (2) of the lemma yield

$$((-1)^{i+1}\det B^{1i}, (-1)^{i+2}\det B^{2i}, \ldots, (-1)^{i+m}B^{mi}) = \det B \cdot \varepsilon_i \cdot B^{-1}$$

and

$$\left(B^{-1}\right)_{ij} = (-1)^{i+j}\frac{1}{\det B}\cdot \det B^{ji},$$

which finishes the proof. □

Lemma 3.2.2 *Let $B = (b_{ij})_{m\times m}$ be a matrix.*

(1) If B is right-quantum and if B' denotes the matrix we get by interchanging adjacent columns of B, then B' is also right-quantum, and $\det B' = -\det B$.

(2) If B is right-quantum and has two columns equal, then $\det B = 0$.

Proof. (1) Clearly B' is also right-quantum. For a permutation $\pi \in S_m$, denote by π' the permutation $\pi \cdot (i, i+1)$. Then

$$(-1)^{\mathrm{inv}(\pi)} b_{\pi(1)1}\cdots b_{\pi(m)m} + (-1)^{\mathrm{inv}(\pi')} b_{\pi'(1)1}\cdots b_{\pi'(m)m} =$$

$$= (-1)^{\mathrm{inv}(\pi)} b_{\pi(1)1}\cdots b_{\pi(i-1),i-1}(b_{\pi(i)i}b_{\pi(i+1),i+1} - b_{\pi(i+1)i}b_{\pi(i),i+1})b_{\pi(i+2),i+2}\cdots b_{\pi(m)m} =$$

$$= (-1)^{\mathrm{inv}(\pi)} b_{\pi(1)1}\cdots b_{\pi(i-1),i-1}(b_{\pi(i+1),i+1}b_{\pi(i)i} - b_{\pi(i),i+1}b_{\pi(i+1)i})b_{\pi(i+2),i+2}\cdots b_{\pi(m)m} =$$

$$= -\left((-1)^{\mathrm{inv}(\pi)} b'_{\pi(1)1}\cdots b'_{\pi(m)m} + (-1)^{\mathrm{inv}(\pi')} b'_{\pi'(1)1}\cdots b'_{\pi'(m)m}\right),$$

so $\det B' = -\det B$. Part (2) follows. □

Proof of Theorem 3.1.2. Because

$$(\delta_{jk} - a_{jk})(\delta_{ik} - a_{ik}) = \delta_{jk}\delta_{ik} - \delta_{jk}a_{ik} - \delta_{ik}a_{jk} + a_{jk}a_{ik} =$$

$$= \delta_{jk}\delta_{ik} - \delta_{jk}a_{ik} - \delta_{ik}a_{jk} + a_{ik}a_{jk} = (\delta_{ik} - a_{ik})(\delta_{jk} - a_{jk})$$

and

$$(\delta_{ik} - a_{ik})(\delta_{jl} - a_{jl}) - (\delta_{jk} - a_{jk})(\delta_{il} - a_{il}) =$$

$$= \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} - \delta_{ik}a_{jl} - \delta_{jl}a_{ik} + \delta_{jk}a_{il} + \delta_{il}a_{jk} + a_{ik}a_{jl} - a_{jk}a_{il} =$$

$$= \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} - \delta_{ik}a_{jl} - \delta_{jl}a_{ik} + \delta_{jk}a_{il} + \delta_{il}a_{jk} + a_{jl}a_{ik} - a_{il}a_{jk} =$$

$$= (\delta_{jl} - a_{jl})(\delta_{ik} - a_{ik}) - (\delta_{il} - a_{il})(\delta_{jk} - a_{jk}),$$

the matrix $B = I - A$ is also right-quantum. Now we can repeat the proof of Theorem 3.1.1 verbatim, using Lemma 3.2.2 instead of parts (1) and (2) of Lemma 3.2.1. □

Proof of Theorem 3.1.3. We have proved that (for A either Cartier-Foata or right-quantum)

$$\det(I - A)\cdot\left(\frac{1}{I - A}\right)_{ii} - \det\,(I - A)^{ii}$$

is an element of $\mathcal{I}_{\text{rq}}$. Apply the map ϕ defined in the statement of the "$1 = q$ principle" (Lemma 1.3.3); we get

$$\phi\left(\det(I-A)\cdot(I-A)^{-1}_{ii} - \det(I-A)^{ii}\right) =$$

$$= \phi(\det(I-A))\cdot\phi\left((I-A)^{-1}_{ii}\right) - \phi\left(\det(I-A)^{ii}\right) \in \mathcal{I}_{q-\text{rq}},$$

where we used the fact that all terms of $\det(I-A)$ and $(I-A)^{-1}_{ii}$ are balanced. Clearly,

$$\phi(\det(I-A)) = \phi\left(\sum_{J\subseteq[m]}(-1)^{|J|}\det A_J\right) = \sum_{J\subseteq[m]}(-1)^{|J|}\det{}_q A_J = \det{}_q(I-A)$$

and similarly

$$\phi\left(\det(I-A)^{ii}\right) = \det{}_q(I-A)^{ii},$$

so it remains to evaluate

$$\phi\left((I-A)^{-1}_{ii}\right).$$

A term in $(I-A)^{-1}_{ii}$ is a path from i to i. For $a_{\lambda,\mu} = a_{ii_1}a_{i_1i_2}\cdots a_{i_ni}$, $\operatorname{inv}(\mu)-\operatorname{inv}(\lambda)$ is equal to $|\{j\colon i_j > i\}| - |\{j\colon i_j < i\}|$. On the other hand, if a step a_{kl} is weighted by q^{-1} for $k < i, l \leq i$ and by q for $k \geq i, l > i$, then $a_{ii_1}a_{i_1i_2}\cdots a_{i_ni}$ is weighted by $q^{|\{j\colon i_j>i\}|-|\{j\colon i_j<i\}|}$ (as can be proved by induction on the length of the path). This means that

$$\phi\left((I-A)^{-1}_{ii}\right) = (I-A_{[i]})^{-1}_{ii},$$

where $A_{[i]}$ is given by (3.2.2). □

Proof of Theorem 3.1.4. The proof is very similar to the proof of Theorem 3.1.3 and we omit some of the details. We know that

$$\det(I-A)\cdot\left(\frac{1}{I-A}\right)_{ii} - \det(I-A)^{ii} \in \mathcal{I}_{\text{rq}},$$

and we can apply the map ϕ defined in the statement of the "$1 = q_{ij}$ principle" (Lemma 1.3.4) with $\mathcal{S}$ the set of relations (1.3.3)–(1.3.4) (and $\mathcal{I} = \mathcal{I}_{\text{rq}}$, $\psi(\mathcal{S})$ the set of relations (1.3.13)–(1.3.14), $\mathcal{I}' = \mathcal{I}_{\mathbf{q}-\text{rq}}$). We get

$$\phi\left(\det(I-A)\cdot(I-A)^{-1}_{ii} - \det(I-A)^{ii}\right) =$$

$$= \phi(\det(I-A))\cdot\phi\left((I-A)^{-1}_{ii}\right) - \phi\left(\det(I-A)^{ii}\right) \in \mathcal{I}_{\mathbf{q}-\text{rq}},$$

where we used the fact that all terms of $\det(I-A)$ and $(I-A)^{-1}_{ii}$ are balanced. Then

$$\phi(\det(I-A)) = \det{}_{\mathbf{q}}(I-A), \qquad \phi\left(\det(I-A)^{ii}\right) = \det{}_{\mathbf{q}}(I-A)^{ii},$$

and

$$\phi\left((I-A)^{-1}_{ii}\right) = (I - A_{[i]})^{-1}_{ii}$$

for $A_{[i]}$ given by (3.1.2). □

3.3 Cartier-Foata case

As promised, this section is devoted to the combinatorial proof of Theorem 3.1.1. We prove the equivalent formula

$$\det(I-A)\cdot\left(\frac{1}{I-A}\right)_{ij} \;=\; (-1)^{i+j}\det\left(I-A\right)^{ji} \tag{3.3.1}$$

for a Cartier-Foata matrix $A = (a_{ij})_{m\times m}$.

First note that

$$\det(I-A) = \sum_{J\subseteq[m]} (-1)^{|J|}\det A_J. \tag{3.3.2}$$

In other words, $\det(I-A)$ is the weighted sum of $a_{\pi(j_1)j_1}\cdots a_{\pi(j_k)j_k}$ over all permutations π of all subsets $\{j_1,\ldots,j_k\}$ of $[m]$, with $a_{\pi(j_1)j_1}\cdots a_{\pi(j_k)j_k}$ weighted by $(-1)^{\operatorname{cyc}\pi}$. Also note that every term of

$$\det A_J = \sum_{\pi\in S_J}(-1)^{\operatorname{inv}(\pi)} a_{\pi(j_1)j_1}a_{\pi(j_2)j_2}\cdots a_{\pi(j_k)j_k}$$

is a product of commuting variables; that means that the variables can be written in any order.

Proof of Theorem 3.1.1. When $i=j$, the right-hand side of (3.3.1) is simply the right-hand side of (3.3.2), with $[m]$ replaced by $[m]\setminus\{i\}$, and we can use the map $\overline{\varphi}$ with $\pi=(m,\ldots,1)$ and $\sigma=\mathrm{id}$ defined in Section 2.1 (see (2.1.4) and the paragraph following it) to transform all sequences into bp-sequences with respect to id. Figure 3-1 shows the right-hand side of (3.3.1) for $m=4$, $i=j=3$. If $i\neq j$, use $\overline{\varphi}$ with respect to $\pi=(m,\ldots,1)$ and $\sigma=(j,1,\ldots,j-1,j+1,\ldots,m)$ to transform every sequence into a bp-sequence with distinct starting and ending heights, with the last lattice path being a path from i to j, and with a weight of such a lattice path being 1 if the number of lattice paths is odd, and -1 otherwise. Figure 3-2 shows this for $m=4$, $i=2$, $j=3$.

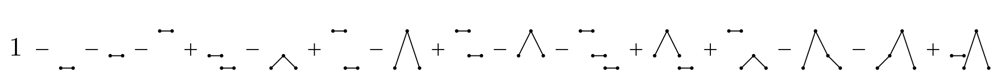

Figure 3-1: A representation of $\det\left(I-A\right)^{33}$.

Cycles of odd length are even permutations, and cycles of even length are odd permutations. For a permutation π of $J\subseteq[m]$ with $|J|=k$, the product $(-1)^{\operatorname{inv}(\pi)}\cdot(-1)^k$

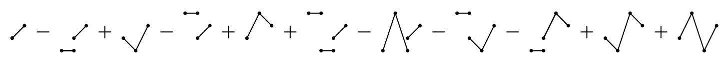

Figure 3-2: A representation of $-\det(I-A)^{32}$.

is therefore equal to $(-1)^{\operatorname{cyc}\alpha}$, and the left-hand side of (3.3.1) is equal to the sum

$$\sum(-1)^{\operatorname{cyc}\alpha}\ \alpha\cdot\beta, \tag{3.3.3}$$

where the sum runs over all pairs (α,β) with the following properties:

- $\alpha = a_{\pi(j_1)j_1}\cdots a_{\pi(j_k)j_k}$ for some $j_1 < \ldots < j_k$, and π is a permutation of $\{j_1,\ldots,j_k\}$; $\operatorname{cyc}\alpha$ denotes the number of cycles of π;
- β is a lattice path from i to j.

Our goal is to cancel most of the terms and get the right-hand side of (3.3.1).
Let us divide the pairs (α,β) in two groups.

- $(\alpha,\beta)\in\mathcal{G}_1$ if no starting or ending height is repeated in $\alpha\cdot\beta$, or the first height that is repeated in $\alpha\cdot\beta$ is a starting height;
- $(\alpha,\beta)\in\mathcal{G}_2$ if the first height to be repeated in $\alpha\cdot\beta$ is an ending height.

The sum (3.3.3) splits into two sums S_1 and S_2. Let us discuss each of these in turn.

(1) Note that if the first height that is repeated in $\alpha\cdot\beta$ is a starting height, this starting height must be i, either as the starting height of the first step of β if α contains i, or the second occurrence of i as a starting height of β if α does not contain i.
Apply the map $\overline{\varphi}$ with $\pi=(m,\ldots,1)$ and $\sigma=(i,1,\ldots,i-1,i+1,\ldots,m)$ to every α in S_1. The terms $(-1)^{\operatorname{cyc}\alpha}\ \overline{\varphi}(\alpha)\cdot\beta$ that do not include i as a starting height sum up to the right-hand side of (3.3.1). The terms that do include i as a starting height either have it in α (and possibly in β) or they have it only in β. There is an obvious sign-reversing involution between the former and the latter – just move the cycle of α containing i over to β or vice versa. This means that S_1 is equal to the right-hand side of (3.3.1).

(2) Note that the first height k that is repeated in $\alpha\cdot\beta$ as an ending height cannot be i. Write $\beta=\beta'\beta''$, where β' is a path from i to k with no repeated heights. Apply $\overline{\varphi}$ with respect to $\pi=(m,\ldots,1)$ and $\sigma=(k,1,\ldots,k-1,k+1,\ldots,m)$ to α. The height k can either appear in α (then it appears only as an ending height in β') or not (then it appears once as a starting height and twice as an ending height in β'). There exists an obvious involution between the sets of pairs with either of these properties: move the cycle in $\overline{\varphi}(\alpha)$ starting with k to the end of β' if k appears as a height in α, and move the cycle starting with k from β' to the end of $\overline{\varphi}(\alpha)$ otherwise. We can make this move without changing the product $\overline{\varphi}(\alpha)\cdot\beta$ modulo $\mathcal{I}_{\mathrm{cf}}$ because all the steps in $\overline{\varphi}(\alpha)\cdot\beta'$ have different

starting heights. Also note that the involution changes the number of cycles in $\overline{\varphi}(\alpha)$ by 1; since $\alpha \cdot \beta$ is weighted by $(-1)^{\mathrm{cyc}\,\alpha}$ in (3.3.3), the involution is sign-reversing, and the sum S_2 is equal to 0.

This completes the proof. □

EXAMPLE 3.3.1 Figure 3-3 shows some pairs that are canceled by the involutions described in (1) and (2) above for $m = 4$, $i = 2$ and $j = 3$.

Figure 3-3: Some pairs that cancel in (3.3.3).

The top two pairs belong to $\mathcal{G}_1$, and the bottom two pairs belong to $\mathcal{G}_2$. The sequence α in the pair (α, β) is drawn in bold. ♢

3.4 Right-quantum case

The involutions that prove the theorem in the right-quantum case are the same, but we have to be more careful because we have to move the steps simultaneously. This will be done by repeated use of Theorem 2.2.1.

Again, we have

$$\det(I - A) = \sum_{J \subseteq [m]} (-1)^{|J|} \det A_J. \tag{3.4.1}$$

Proof of Theorem 3.1.2. We deal with the equivalent identity

$$\det(I - A) \cdot \left(\frac{1}{I - A}\right)_{ij} = (-1)^{i+j} \det (I - A)^{ji}. \tag{3.4.2}$$

When $i = j$, the right-hand side of (3.4.2) is the right-hand side of (3.3.1), with $[m]$ replaced by $[m] \setminus \{i\}$, and since

$$\det A = \overline{O}^{w_0}(\mathbf{1}),$$

for $w_0 = (m, \ldots, 1)$, equation (2.2.4) for $\pi = w_0$ and $\sigma = \mathrm{id}$ tells us that we can transform all bo-sequences in $\det (I - A)^{ii}$ into bp-sequences with respect to id, weighted by 1 if the number of lattice paths is even and by -1 if it is odd. If $i \neq j$, use (2.2.4) with $\pi = (m, \ldots, 1)$ and $\sigma = (j, 1, \ldots, j-1, j+1, \ldots, m)$ to transform every sequence into a bp-sequence with the last lattice path being a path from i to j, and with a weight of such a lattice path being 1 if the number of lattice paths is odd, and -1

otherwise.
The description of the left-hand side of (3.4.2) is the same as in the Cartier-Foata case, and we divide the sum $\sum(-1)^{\operatorname{cyc}\alpha}\ \alpha\cdot\beta$ into S_1 and S_2 in the same fashion. Again we discuss these sums separately.

(a) If the first height that is repeated in $\alpha\cdot\beta$ is a starting height, this starting height is i, either as the starting height of the first step of β if α contains i, or the second occurrence of i as a starting height of β if α does not contain i. For each β, we can apply (2.2.4) with respect to $\sigma=(i,1,\ldots,i-1,i+1,\ldots,m)$ to the sum

$$\sum(-1)^{\operatorname{cyc}\alpha}\alpha$$

over all α with $(\alpha,\beta)\in\mathcal{G}_1$. The terms $(-1)^{\operatorname{cyc}\alpha}\ \alpha\cdot\beta$ that do not include i sum up to the right-hand side of (3.4.2). The terms that do include i either have it in α (and possibly in β) or they have it only in β. The same sign-reversing involution as in the Cartier-Foata case proves that S_1 is equal to the right-hand side of (3.4.2).

(b) The first height k that is repeated in $\alpha\cdot\beta$ as an ending height cannot be i. Fix k and a path γ from k to j. For each path γ' from i to k without repeated heights, use (2.2.4) with respect to $\sigma=(k,1,\ldots,k-1,k+1,\ldots,m)$ on the sum

$$\sum(-1)^{\operatorname{cyc}\alpha}\ \alpha$$

over all α such that $(\alpha,\gamma'\gamma)\in\mathcal{G}_2$ and the only repeated height in $\alpha\cdot\gamma'$ is the ending height k. The sum of

$$\sum(-1)^{\operatorname{cyc}\alpha}\ \alpha\cdot\beta$$

over $(\alpha,\beta)\in\mathcal{G}_2$, $\beta=\gamma'\gamma$, and k the only repeated (ending) height in $\alpha\cdot\gamma'$, is therefore equal to

$$\left(\sum\overline{P}^{\sigma}(\mathbf{p};\mathbf{r})\right)\cdot\gamma$$

with

- $\mathbf{p}$ a vector of 1's and 0's, with 1 in the i-th entry and the k-th entry, and
- $\mathbf{r}$ equal to $\mathbf{p}$ except that the i-th entry is 0 and the k-th entry is 2.

Equation (2.2.4) of Theorem 2.2.1 yields

$$\overline{P}^{\sigma}(\mathbf{p};\mathbf{r})=\overline{O}(\mathbf{p};\mathbf{r}),$$

and this is clearly equal to 0 since $\alpha_1a_{i_1k}a_{i_2k}\alpha_2$ and $\alpha_1a_{i_2k}a_{i_1k}\alpha_2$ have opposite signs in $\overline{O}(\mathbf{p};\mathbf{r})$, and since $a_{i_1k}a_{i_2k}=a_{i_2k}a_{i_1k}$.

This completes the proof. □

3.5 Weighted cases

Proof of Theorem 3.1.3. Assume that A is q-right-quantum. The proof consists of a careful analysis of the powers of q we get if we repeat the steps of the proof of Theorem 3.1.2. We want to prove the identity

$$\det_q(I - A) \cdot \left(\frac{1}{I - A_{[i]}}\right)_{ii} = \det_q (I - A)^{ii}. \tag{3.5.1}$$

We have $\det_q(I - A) = \sum_{J \subseteq [m]} (-1)^{|J|} \det_q A_J$, so every sequence $\alpha = a_{\lambda',\mu'}$ that appears in $\det_q(I - A)$ is weighted by $(-1)^{\operatorname{cyc}\alpha} q^{-\operatorname{inv}(\lambda')} = (-1)^{\operatorname{cyc}\alpha} q^{\operatorname{inv}(\mu') - \operatorname{inv}(\lambda')}$. A term $\beta = a_{\lambda'',\mu''} = a_{ii_1} a_{i_1 i_2} \cdots a_{i_n i}$ in $(I - A_{[i]})^{-1}_{ii}$, on the other hand, is weighted by $q^{|\{j \colon i_j > i\}| - |\{j \colon i_j < i\}|}$ (see the non-combinatorial proof of the same theorem on page 38), which is equal to $q^{\operatorname{inv}(\mu')' - \operatorname{inv}(\lambda')'}$. Furthermore, both α and β are balanced, so

$$\operatorname{inv}(\mu'\mu'') - \operatorname{inv}(\lambda'\lambda'') = \operatorname{inv}(\mu') - \operatorname{inv}(\lambda') + \operatorname{inv}(\mu'') - \operatorname{inv}(\lambda''),$$

cf. the proof of part (2) of Lemma 1.3.3. Therefore a term $\alpha \cdot \beta = a_{\lambda,\mu}$ on the left-hand side of (3.5.1) is weighted by $(-1)^{\operatorname{cyc}\alpha} q^{\operatorname{inv}(\mu) - \operatorname{inv}(\lambda)}$.
The switches we perform always involve relations (1.3.8) and (1.3.9), and we have already verified (in the proof of Theorem 2.4.1) that this implies that every term $a_{\lambda,\mu}$ at each step of the proof is weighted by $\pm q^{\operatorname{inv}(\mu) - \operatorname{inv}(\lambda)}$, and the same cancellations occur as in the proof of Theorem 3.1.2. We are left with the terms of $\det_q(I - A)$ that do not involve i, and this is exactly the right-hand side of (3.5.1). □

REMARK 3.5.1 Theorems 3.1.1 and 3.1.2 do not seem to have nice q-analogues when $i \neq j$. The reason is that the terms of $(1 - A)^{-1}_{ij}$ (with the entries of A possibly multiplied by some power of q, as in $A_{[i]}$) are (weighted) paths from i to j and are therefore not balanced sequences. ♢

3.6 Multiparameter cases

Proof of Theorem 3.1.4. This is almost identical to the proof of Theorem 3.1.3 in the previous section. Assume that A is **q**-right-quantum, we want to prove the identity

$$\det_{\mathbf{q}}(I - A) \cdot \left(\frac{1}{I - A_{[i]}}\right)_{ii} = \det_{\mathbf{q}} (I - A)^{ii}. \tag{3.6.1}$$

We continue as before, with the weight

$$q^{\operatorname{inv}(\mu) - \operatorname{inv}(\lambda)}$$

of the sequence $a_{\lambda,\mu}$ replaced by

$$\prod_{(i,j) \in I(\mu)} q_{\mu_j \mu_i} \prod_{(i,j) \in I(\lambda)} q^{-1}_{\lambda_j \lambda_i},$$

and relations (1.3.13) and (1.3.14) playing the role of (1.3.8) and (1.3.9). □

REMARK 3.6.1 It Chapter 5, we will need the conclusion of Theorem 3.1.4 for $i = m$ under the weaker assumption that $q_{kl}a_{jl}a_{ik} = q_{ij}a_{ik}a_{jl}$ for $k, l \neq m$ (or that $a_{ik}a_{jl} - q_{ij}^{-1}a_{jk}a_{il} = q_{kl}q_{ij}^{-1}a_{jl}a_{ik} - q_{kl}a_{il}a_{jk}$ for $k, l \neq m$), i.e. that only the variables in the first $m-1$ columns of A satisfy the **q**-Cartier-Foata (**q**-right-quantum) relations. It is easy to modify either the proof given in this section or in Section 3.2 to prove this more general statement.
Namely, the variable $a_{\pi(m)m}$ appearing in α with $(\alpha, \beta) \in \mathcal{G}_1$ must be at the end of α, and we do not have to swap this variable with any other when we apply the map $\overline{\varphi}$ with respect to $\pi = (m, \ldots, 1)$ and $\sigma = (m, 1, \ldots, m-1)$. On the other hand, the steps that we switch in the case $(\alpha, \beta) \in \mathcal{G}_2$ involve exactly one step starting at m, and no steps ending at m, so we never use the **q**-Cartier-Foata (or **q**-right-quantum) relation for variables that are in the last column of A.
Alternatively, take a look at the proof on page 37. If the first $m-1$ columns of A are Cartier-Foata (or right-quantum) and $B = I - A$, the j-th coordinate of

$$((-1)^{m+1} \det B^{1m}, (-1)^{m+2} \det B^{2m}, \ldots, B^{mm}) \cdot B$$

is $\sum_{k=1}^{m} (-1)^{m+k} \det B^{km} b_{kj}$, which is equal to

$$\det \begin{pmatrix} b_{11} & \cdots & b_{1,m-1} & b_{1j} \\ b_{21} & \cdots & b_{2,m-1} & b_{2j} \\ \vdots & \ddots & \vdots & \vdots \\ b_{m1} & \cdots & b_{m,m-1} & b_{mj} \end{pmatrix}.$$

For $j = m$, this is clearly $\det B$, and for $j \neq m$, the determinant involves no variables a_{*m}, so Lemma 3.2.1 (or Lemma 3.2.2) implies that the determinant is 0 in this case. Therefore

$$((-1)^{m+1} \det B^{1m}, (-1)^{m+2} \det B^{2m}, \ldots, B^{mm}) = \det B \cdot \varepsilon_m \cdot B^{-1}$$

and

$$\left(B^{-1}\right)_{mm} = \frac{1}{\det B} \cdot \det B^{mm},$$

which is what we wanted to prove. ◇

3.7 Jacobi ratio theorem

The proofs in Sections 3.3–3.4 are not only the simplest combinatorial proof of the matrix inverse formula (see [Foa79] for an alternative combinatorial proof in the Cartier-Foata case), but also generalize easily to the proof of Jacobi ratio theorem. This result appears to be new (for either Cartier-Foata or right-quantum matrices), although a variant was proved for general non-commutative variables in [GR91] and for quantum matrices in [KL95].

We need the following proposition.

Proposition 3.7.1 *If the matrix* $A = (a_{ij})_{m\times m}$ *is right-quantum, so is the matrix* $C = (c_{ij})_{m\times m} = (I - A)^{-1}$.

Proof. First note that

$$c_{ij} = (I - A)^{-1}_{ij}$$

is the sum of all lattice paths from i to j.
We need some notation:

- let O denote the sum of $O(\mathbf{p})$ over all $\mathbf{p} \geq 0$, and let P denote the sum of $P(\mathbf{p})$ over all $\mathbf{p} \geq 0$;

- the superscript i in front of an expression E means that E contains no variable a_{i*}; for example, ${}^{i}c_{ji}$ denotes the sum of all paths from j to i that reach i exactly once, and ${}^{ij}O$ is the sum of $O(\mathbf{p})$ over all $\mathbf{p} \geq 0$ with $p_i = p_j = 0$;

- O_i^j for $i \neq j$ means the sum of $O(\mathbf{p}; \mathbf{r})$ with $p_i = r_i + 1$, $p_j = r_j - 1$.

- O_{ij}^k for different i, j, k means the sum of $O(\mathbf{p}; \mathbf{r})$ with $p_i = r_i + 1$, $p_j = r_j + 1$, $p_k = r_k - 2$.

- O_{ij}^{kl} for different i, j, k, l means the sum of $O(\mathbf{p}; \mathbf{r})$ with $p_i = r_i + 1$, $p_j = r_j + 1$, $p_k = r_k - 1$, $p_l = r_l - 1$.

Let us prove that $c_{ik}c_{jk} = c_{jk}c_{ik}$. There is nothing to prove when $i = j$, so we have to consider two possible cases:

- Take $i \neq j = k$. First let us prove that $c_{jj}\,{}^{j}c_{ij} = c_{ij}$. To see this, use (2.2.3) on O_i^j twice, once with respect to the permutation $\pi = (i, j, 1, 2, \ldots, m)$ and once with respect to the permutation $\sigma = (j, i, 1, 2, \ldots, m)$. We get

  $$O_i^j = c_{ij}\,{}^{j}P^{\pi} \qquad \text{and} \qquad O_i^j = c_{jj}\,{}^{j}c_{ij}\,{}^{j}P^{\pi}$$

  and since ${}^{j}P^{\pi} = {}^{j}P$ is invertible in $\mathcal{A}$ (its constant term is 1), we have

  $$c_{ij} = c_{jj}\,{}^{j}c_{ij}. \tag{3.7.1}$$

  But every path from i to j splits into a path from i to j with no step starting at j, and a path from j to j. Therefore multiplying (3.7.1) by c_{jj} yields

  $$c_{ij}c_{jj} = c_{jj}({}^{j}c_{ij}c_{jj}) = c_{jj}c_{ij},$$

  which is the relation we had to prove.

- Assume that $i \neq k \neq j$. Note that using (2.2.3) with respect to π and σ gives

  $$O_{ij}^k = c_{ik}\,{}^{k}c_{jk}\,{}^{k}P = c_{jk}\,{}^{k}c_{ik}\,{}^{k}P$$

and so

$$c_{ik}\ {}^{k}c_{jk} = c_{jk}\ {}^{k}c_{ik}; \tag{3.7.2}$$

when we multiply this by c_{kk}, we get

$$c_{ik}c_{jk} = c_{jk}c_{ik}.$$

We also have to prove $c_{ik}c_{jl} + c_{il}c_{jk} = c_{jk}c_{il} + c_{jl}c_{ik}$ for $k \neq l$. Let us investigate three possible cases:

- Take $i = k$, $j = l$. By using (2.2.3) on O twice, once with respect to π and once with respect to σ, we get

$$O = c_{ii}\ {}^{i}c_{jj}\ {}^{ij}P = c_{jj}\ {}^{j}c_{ii}\ {}^{ij}P. \tag{3.7.3}$$

  Furthermore, (3.7.1) yields

$$c_{ii}\ {}^{i}c_{jj} = c_{ii}\ (c_{jj} - {}^{i}c_{ji}\ c_{ij}) = c_{ii}\ c_{jj} - (c_{ii}\ {}^{i}c_{ji})\ c_{ij} = c_{ii}\ c_{jj} - \ c_{ji}c_{ij}$$

  and similarly

$$c_{jj}\ {}^{j}c_{ii} = c_{jj}\ c_{ii} - c_{ij}\ c_{ji},$$

  so (3.7.3) gives

$$c_{ii}\ c_{jj} - c_{ji}\ c_{ij} = c_{jj}\ c_{ii} - c_{ij}\ c_{ji}.$$

- Take $i \neq k$, $j = l$ and use (2.2.3) on O_j^k for different permutations, we get

$$O_j^k = c_{ii}\ {}^{i}c_{jk}\ {}^{ik}P + c_{ik}\ {}^{k}c_{ji}\ {}^{ik}P \qquad \text{and} \qquad O_j^k = c_{jk}\ {}^{k}c_{ii}\ {}^{ik}P.$$

  But then

$$c_{ii}\ {}^{i}c_{jk} + c_{ik}\ {}^{k}c_{ji} = c_{jk}\ {}^{k}c_{ii}$$

  implies

$$c_{ii}\ (c_{jk} - {}^{i}c_{ji}\ c_{ik}) + c_{ik}\ (c_{ji} - {}^{k}c_{jk}\ c_{ki}) = c_{jk}\ (c_{ii} - {}^{k}c_{ik}\ c_{ki})$$

  and

$$c_{ii}\ c_{jk} + c_{ik}\ c_{ji} = c_{jk}\ c_{ii} + c_{ii}\ {}^{i}c_{ji}\ c_{ik} + (c_{ik}\ {}^{k}c_{jk} - c_{jk}\ {}^{k}c_{ik})\ c_{ki},$$

  and so (3.7.1) and (3.7.2) imply

$$c_{ii}\ c_{jk} + c_{ik}\ c_{ji} = c_{jk}\ c_{ii} + c_{ji}\ c_{ik}.$$

- Assume that $i, j \neq k, l$. Then the equalities

$$O_{ij}^{kl} = c_{ik}\ {}^{k}c_{jl}\ {}^{kl}P + c_{il}\ {}^{l}c_{jk}\ {}^{kl}P = c_{jk}\ {}^{k}c_{il}\ {}^{kl}P + c_{jl}\ {}^{l}c_{ik}\ {}^{kl}P,$$

$$c_{ik}\ (c_{jl} - {}^{k}c_{jk}\ c_{kl}) + c_{il}\ (c_{jk} - {}^{l}c_{jl}\ c_{lk}) = c_{jk}\ (c_{il} - {}^{k}c_{ik}\ c_{kl}) + c_{jl}\ (c_{ik} - {}^{l}c_{il}\ c_{lk}),$$

and (3.7.2) imply

$$c_{ik}\ c_{jl} + c_{il}\ c_{jk} = c_{jk}\ c_{il} + c_{jl}\ c_{ik}.$$

This completes the proof. □

Theorem 3.7.2 (right-quantum Jacobi ratio theorem) *Take $I, J \subseteq [m]$ with $|I| = |J|$. If $A = (a_{ij})_{m\times m}$ is right-quantum and $C = (c_{ij})_{m\times m} = (I-A)^{-1}$, then*

$$\det C_{I,J} = (-1)^{\sum_{i\in I} i + \sum_{j \in J} j} \cdot \frac{1}{\det(I-A)} \cdot \det(I-A)^{J,I}.$$

In particular,

$$\det\left(\frac{1}{I-A}\right) = \frac{1}{\det(I-A)}.$$

Proof. We only sketch the proof as it is very similar to the proof of Theorem 3.1.2 given in Section 3.4 once we have Proposition 3.7.1, and we assume that $I = J$ as this makes the reasoning slightly simpler. Use (2.2.4) for $\sigma = \mathrm{id}$ (we can do that because of Proposition 3.7.1) on $\det C_I$; for a permutation π of I with cyclic structure $(i_1^1 i_2^1 \dots i_{k_1}^1)(i_1^2 i_2^2 \dots i_{k_2}^2)\cdots(i_1^l i_2^l \dots i_{k_l}^l)$ (where the first element of each cycle is the smallest, and where starting elements of cycles are increasing), we get the term

$$(-1)^{\mathrm{inv}(\pi)}\ (c_{i_1^l, i_2^l} \cdots c_{i_{k_l}^l, i_1^l}) \cdots (c_{i_1^2, i_2^2} \cdots c_{i_{k_2}^2, i_1^2})\ (c_{i_1^1, i_2^1} \cdots c_{i_{k_1}^1, i_1^1}).$$

For each selection of paths (in variables a_{ij})

$$i_1^l \to i_2^l, \dots, i_{k_l}^l \to i_1^l, \dots, i_1^2 \to i_2^2, \dots, i_{k_2}^2 \to i_1^2, i_1^1 \to i_2^1, \dots, i_{k_1}^1 \to i_1^1,$$

this yields a concatenation of (possibly empty) lattice paths from i_1^l to i_1^l, i_1^{l-1} to i_1^{l-1}, etc., with exactly one starting height i_s^t for every s marked on each path $i_1^t \to i_1^t$. For example, take $m = 5$, $I = \{1,2,4\}$, and $\pi = \binom{124}{421}$. The term of $\det C_I$ corresponding to π is $-c_{22}c_{14}c_{41}$, and some of the sequences (without the minus sign) corresponding to this term are depicted in Figure 3-4. Note the empty path corresponding to c_{22} in the second example. When we multiply $\det C_I$ on the left by $\det(I-A)$, we get the

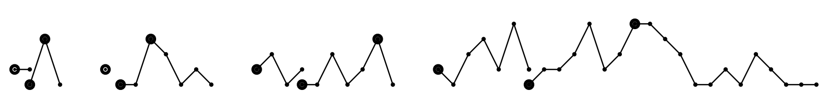

Figure 3-4: Some sequences in $c_{22}c_{14}c_{41}$.

sum

$$\sum (-1)^{\mathrm{cyc}\,\alpha + \mathrm{inv}(\beta)}\ \alpha \cdot \beta, \tag{3.7.4}$$

where the sum runs over all pairs (α, β) with the following properties:

- $\alpha = a_{\pi(j_1)j_1} \cdots a_{\pi(j_k)j_k}$ for some $j_1 < \ldots < j_k$, and π is a permutation of $\{j_1, \dots, j_k\}$; $\mathrm{cyc}\,\alpha$ denotes the number of cycles of π;

- β is a concatenation of lattice path from i_1^l to i_l^1, i_1^{l-1} to i_1^{l-1} with exactly one starting height i_s^t marked on each path $i_1^t \to i_1^t$, where

$$\sigma = (i_1^1 i_2^1 \dots i_{k_1}^1)(i_1^2 i_2^2 \dots i_{k_2}^2) \cdots (i_1^l i_2^l \dots i_{k_l}^l)$$

and $\mathrm{inv}(\beta)$ denotes the number of inversions of σ.

The cancellation process described in the proof of the matrix inverse formula applies here almost verbatim, and this shows that $\det(I-A) \cdot \det C_I$ is equal to $\det(I-A)^I$. $\square$

Chapter 4

MacMahon master theorem

4.1 Introduction

The MacMahon master theorem is one of the jewels in enumerative combinatorics, and it is as famous and useful as it is mysterious. Most recently, a new type of algebraic generalization was proposed in [GLZ06] and was further studied in [FH08, FH07a, FH07b, HL07]. In this chapter we present further generalizations of the MacMahon master theorem and several other related results. While our generalizations are algebraic in statement, the heart of our proofs is completely bijective, unifying all generalizations. The approach seems to be robust enough to allow further generalizations in this direction.

Let us begin with a brief outline of the history of the subject. The master theorem was discovered in 1915 by Percy MacMahon in his landmark two-volume *"Combinatory Analysis"*, where he called it "a master theorem in the Theory of Partitions" [Mac16, page 98]. Much later, in the early sixties, the real power of the master theorem was discovered, especially as a simple tool for proving binomial identities (see [GJ83]). The proof of the MacMahon master theorem using Lagrange inversion is now standard, and the result is often viewed in the analytic context [Goo62, GJ83].

An algebraic approach to the MacMahon master theorem goes back to Foata's thesis [Foa65], parts of which were later expanded in [CF69] (see also [Lal79]). The idea was to view the theorem as a result on "words" over a (partially commutative) alphabet, so one can prove it and generalize it by means of simple combinatorial and algebraic considerations. This approach became highly influential and led to a number of new related results (see e.g. [Kob92, Min01, Vie86, Zei85]).

While the MacMahon master theorem continued to be extended in several directions (see [FZ88, KS99]), the "right" q- and non-commutative analogues of the results evaded discovery until recently. This was in sharp contrast with the Lagrange inversion, whose q- and non-commutative analogues were understood fairly well [Gar81, GR92, Ges80, GS83, PPR, Sin95]. Unfortunately, no reasonable generalizations of the master theorem followed from these results.

An important breakthrough was made by Garoufalidis, Lê and Zeilberger (GLZ), who introduced a new type of q-analogue, with a technical proof [GLZ06]. In a series of papers, Foata and Han first modified and extended the Cartier-Foata combinatorial approach to work in this algebraic setting, obtaining a new (involutive) proof of the GLZ-theorem [FH08]. Then they developed a beautiful "1 = q" principle which gives perhaps the most elegant explanation of the results [FH07a] (see also Section 1.3). They also analyze a number of specializations in [FH07b]. Most recently, Hai and Lorenz gave an interesting algebraic proof of the GLZ-theorem, opening yet another direction for exploration.

In this chapter, we present a number of generalizations of the classical MacMahon master theorem in the style of Cartier-Foata and Garoufalidis-Lê-Zeilberger. Our approach is bijective and is new even in the classical cases.

We begin with the statement of the commutative MacMahon master theorem due to Percy MacMahon [Mac16] and show two classical applications. We give a short combinatorial proof in Section 4.2. In Section 4.3, we show a non-commutative generalization of the theorem: the right-quantum master theorem. We see that the theorem is an easy corollary of the work in Chapters 2 and 3. In Section 4.4 we prove the Garoufalidis-Lê-Zeilberger's theorem (q-right-quantum master theorem). In Section 4.5, we generalize the theorem to the $\mathbf{q}$-right-quantum case. In later sections, we present some applications and extensions of this result:

- a β-extension in Section 4.6;
- a "non-balanced" version of the master theorem in Section 4.7;
- a concise proof of a q-extension of MacMahon master theorem due to Krattenthaler and Schlosser in Section 4.8.

We close the chapter with some remarks in Section 4.9. Note that Theorem 5.8.1 is a generalization of both the β-extension of the master theorem and of Sylvester's identity. There are also connections between the master theorem and Goulden-Jackson's immanant formula (6.1.3). See the appropriate sections for more details.

Theorem 4.1.1 (MacMahon master theorem) *Let $A = (a_{ij})_{m\times m}$ be a complex matrix, and let $x_1, \ldots, x_m$ be a set of variables. Denote by $G(\mathbf{r})$ the coefficient of $\mathbf{x}^{\mathbf{r}}$ in*

$$\prod_{i=1}^{m}(a_{i1}x_1 + \ldots + a_{im}x_m)^{r_i}. \tag{4.1.1}$$

Let $t_1, \ldots, t_m$ be another set of variables, and $T = \operatorname{diag}\mathbf{t}$. Then

$$\sum_{\mathbf{r}\geq\mathbf{0}} G(\mathbf{r})\mathbf{t}^{\mathbf{r}} = \frac{1}{\det(I - TA)}, \tag{4.1.2}$$

where the summation is over all non-negative integer vectors $\mathbf{r}$.

By taking $\mathbf{t} = \mathbf{1}$ we get

$$\sum_{\mathbf{r}\geq\mathbf{0}} G(\mathbf{r}) = \frac{1}{\det(I - A)}, \tag{4.1.3}$$

whenever both sides of the equation are well defined, for example when all a_{ij} are formal variables. Moreover, replacing a_{ij} in (4.1.3) with $a_{ij}\,t_i$ shows that (4.1.3) is actually equivalent to (4.1.2). We will use this observation throughout the chapter.

MacMahon master theorem is classically used to prove binomial identities. The following is a typical example.

EXAMPLE 4.1.2 Let us derive Dixon's identity. Denote the sum

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}^3.$$

by $S(n)$. Since

$$\left(1-\frac{x}{y}\right)^n\left(1-\frac{y}{z}\right)^n\left(1-\frac{z}{x}\right)^n=\sum_{0\le i,j,k\le n}\binom{n}{i}\binom{n}{j}\binom{n}{k}(-1)^{i+j+k}x^{i-k}y^{j-i}z^{k-j},$$

we have

$$S(n)=[x^0y^0z^0]\left(1-\frac{x}{y}\right)^n\left(1-\frac{y}{z}\right)^n\left(1-\frac{z}{x}\right)^n=[x^ny^nz^n](y-x)^n(z-y)^n(x-z)^n$$

and by Theorem 4.1.1,

$$S(n)=[t^nu^nv^n]\frac{1}{\det(I-TA)},$$

where

$$T=\begin{pmatrix}t&0&0\\0&u&0\\0&0&v\end{pmatrix},\qquad A=\begin{pmatrix}0&-1&1\\1&0&-1\\-1&1&0\end{pmatrix}.$$

Therefore

$$S(n)=[t^nu^nv^n](1+tu+uv+vt)^{-1}=\sum_{p,q,r\ge 0}(-1)^{p+q+r}\binom{p+q+r}{p,q,r},$$

where the summation is over (p,q,r) with $p+r=p+q=r+q=n$. In other words,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}^3=0$$

for n odd and, for $n=2l$,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}^3=(-1)^l\binom{3l}{l,l,l}.$$

$\diamondsuit$

EXAMPLE 4.1.3 Let us find the generating function for the number of multiderangements, i.e. permutations of the multiset $\{1^{r_1},2^{r_2},\ldots,m^{r_m}\}$ for which no i is in a

position previously occupied by (a copy of) i. We get the usual derangements for $r_1 = r_2 = \ldots = 1$.
Denote the number of multiderangement of $\{1^{r_1}, 2^{r_2}, \ldots, m^{r_m}\}$ by $a(\mathbf{r})$. Then

$$a(\mathbf{r}) = [\mathbf{x}^{\mathbf{r}}](x_2 + x_3 + \ldots + x_m)^{r_1}(x_1 + x_3 + \ldots + x_m)^{r_2} \cdots (x_1 + x_2 + \ldots + x_{m-1})^{r_m},$$

which is, by the master theorem, equal to $[\mathbf{t}^{\mathbf{r}}] \det^{-1}(I - TA)$, where

$$T = \begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_m \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix},$$

in other words,

$$\sum_{\mathbf{r} \geq \mathbf{0}} a(\mathbf{r}) \mathbf{t}^{\mathbf{r}} = \begin{vmatrix} 1 & -t_1 & \cdots & -t_1 \\ -t_2 & 1 & \cdots & -t_2 \\ \vdots & \vdots & \ddots & \vdots \\ -t_m & -t_m & \cdots & 1 \end{vmatrix}^{-1}.$$

By definition,

$$D_m = \sum_{\sigma \in S_m} \left(\operatorname{sign}(\sigma)(-1)^{m - |\operatorname{Fix}(\sigma)|} \prod_{i \notin \operatorname{Fix}(\sigma)} t_i \right),$$

where $\operatorname{Fix}(\sigma)$ is the set of fixed points of σ. Since D_m is symmetric in $t_1, \ldots, t_m$, we have $D_m = \sum_{j=0}^{m} (-1)^j Q_j e_j(\mathbf{t})$, where e_j is the elementary symmetric function and $Q_j = \sum_{\pi \in S_j, \operatorname{Fix}(\pi) = \emptyset} \operatorname{sign}(\pi)$. Define

$$P_j = \sum_{\pi \in S_j} \operatorname{sign}(\pi) = \left\{ \begin{array}{lcl} 1 & : & j = 0, 1 \\ 0 & : & \text{otherwise} \end{array} \right. .$$

Then $P_j = \sum_{i=0}^{j} \binom{j}{i} Q_i$ and $Q_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} P_i = (-1)^{j-1}(j-1)$ by the inclusion-exclusion principle. Therefore

$$\sum_{\mathbf{r} \geq \mathbf{0}} a(\mathbf{r}) \mathbf{t}^{\mathbf{r}} = \frac{1}{1 - \sum_{i<j} t_i t_j - 2 \left(\sum_{i<j<k} t_i t_j t_k \right) - \ldots - (m-1) t_1 \cdots t_m}. \qquad \diamondsuit$$

4.2 A combinatorial proof of the classical MacMahon master theorem

Observe that choosing a term of $\prod_{i=1}^{m} (a_{i1} x_1 + \ldots + a_{im} x_m)^{r_i}$ means choosing a term $a_{1*} x_*$ r_1 times, then choosing a term $a_{2*} x_*$ r_2 times, etc., and then multiplying all these terms. Moreover, if we are interested in the coefficient of $\mathbf{x}^{\mathbf{r}}$, $*$ has to represent 1 r_1 times, 2 r_2 times, etc. In other words, each term on the left-hand side of (4.1.2) corresponds to an o-sequence (with respect to id) in $\mathbf{O}(\mathbf{r})$ for a unique vector $\mathbf{r}$.

On the other hand, let $B = (b_{ij})$ be an invertible $m \times m$ matrix over $\mathbb{C}$. Denote by B^{11} the matrix B without the first row and the first column, by $B^{12,12}$ the matrix B without the first two rows and the first two columns, etc. For the entries of the inverse matrix we have

$$\left(B^{-1}\right)_{11} = \frac{\det B^{11}}{\det B}. \tag{4.2.1}$$

Substituting $B = I - A$ and iterating (4.2.1), we obtain

$$\frac{1}{\det(I-A)} = \frac{\det\left(I - A^{11}\right)}{\det(I-A)} \cdot \frac{\det\left(I - A^{12,12}\right)}{\det(I-A^{11})} \cdot \frac{\det\left(I - A^{123,123}\right)}{\det(I-A^{12,12})} \cdots \frac{1}{1-a_{mm}} =$$

$$= \left(\frac{1}{I-A}\right)_{11} \left(\frac{1}{I-A^{11}}\right)_{22} \left(\frac{1}{I-A^{12,12}}\right)_{33} \cdots \frac{1}{1-a_{mm}},$$

provided that all the minors are invertible. Now let a_{ij} be commuting variables. We obtain that the right-hand side of equation (4.1.3) is the product of entries in the inverses of matrices.

Since $(I-A)^{-1} = I + A + A^2 + \ldots$, we get a combinatorial interpretation of the (11)-entry:

$$\left(\frac{1}{I-A}\right)_{11} = \sum a_{1j_1} a_{j_1 j_2} \cdots a_{j_n 1}, \tag{4.2.2}$$

where the summation is over all finite sequences $(j_1, \ldots, j_n)$ with $j_r \in [m]$ for all r. Similarly,

$$\left(\frac{1}{I-A^{11}}\right)_{22} = \sum a_{2j_1} a_{j_1 j_2} \cdots a_{j_n 2},$$

where the summation is over all finite sequences $(j_1, \ldots, j_n)$ with $j_r \in \{2, \ldots, m\}$ for all r. A combinatorial interpretation of the other terms of the product is analogous. But that means that each term on the right-hand side of (4.1.2) corresponds to a p-sequence (with respect to id) in $\mathbf{P}(\mathbf{r})$ for a unique vector $\mathbf{r}$.

This means that Theorem 4.1.1 is equivalent to the existence of a bijection

$$\varphi \colon \mathbf{O}(\mathbf{r}) \longrightarrow \mathbf{P}(\mathbf{r})$$

for which $\varphi(\alpha)$ is a rearrangement of α for every o-sequence α. Of course, this is precisely what we did in Section 2.1. As a reminder, φ in this special case (where the sequences are balanced and $\pi = \sigma = \mathrm{id}$) is constructed as follows. Take an o-sequence α, and let $[0, x]$ be the maximal interval on which it is part of a p-sequence, i.e. the maximal interval $[0, x]$ on which the o-sequence has the property that if a step ends at level i, and the following step starts at level $j > i$, the o-sequence stays on or above height j afterwards. Let i be the height at x. Choose the step $(x', i) \to (x'+1, i')$ in the o-sequence that is the first to the right of x that starts at level i (such a step exists because we have a balanced sequence). Continue switching this step with the one to the left until it becomes the step $(x, i) \to (x+1, i')$. The new object is part of a p-sequence at least on the interval $[0, x+1]$. Continuing this procedure we get a p-sequence $\varphi(\alpha)$.

EXAMPLE 4.2.1 Figure 4-1 shows the switches for an o-sequence of type $(3,1,1)$, and the p-sequence

$$a_{13}a_{32}a_{22}a_{23}a_{31}a_{11}a_{12}a_{22}a_{21}a_{13}a_{31}a_{23}a_{33}a_{32}a_{22}a_{23}a_{32}a_{33}a_{33}$$

in Figure 4-2 is the result of applying this procedure to the o-sequence

$$a_{13}a_{11}a_{12}a_{13}a_{22}a_{23}a_{22}a_{21}a_{23}a_{22}a_{23}a_{32}a_{31}a_{31}a_{33}a_{32}a_{32}a_{33}a_{33}$$

shown in the same figure (we need 33 switches). ♢

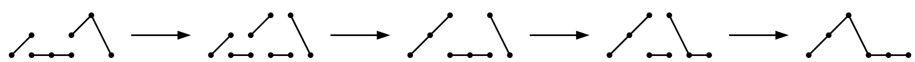

Figure 4-1: Transforming an o-sequence into a p-sequence.

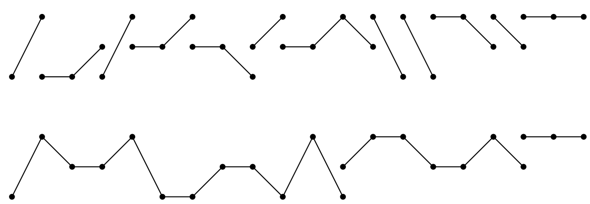

Figure 4-2: An o-sequence and the corresponding p-sequence of type $(4,7,8)$.

4.3 The right-quantum case

In this section, we assume that the variables $x_1, \ldots, x_m$ commute with each other and with all a_{ij}. Note that the notion of the coefficient of $\mathbf{x}^{\mathbf{r}}$ in

$$\prod_{i=1}^{m} (a_{i1}x_1 + \ldots + a_{im}x_m)^{r_i} \tag{4.3.1}$$

still makes sense, since we can move the x_i's in each term to the right and order them.

Theorem 4.3.1 (right-quantum master theorem) *Assume that $A = (a_{ij})_{m\times m}$ is a right-quantum matrix. Denote the coefficient of $\mathbf{x}^{\mathbf{r}}$ in (4.3.1) by $G(\mathbf{r})$. Then*

$$\sum_{\mathbf{r}\geq\mathbf{0}} G(\mathbf{r}) = \frac{1}{\det(I-A)}, \tag{4.3.2}$$

where the summation is over all non-negative integer vectors $\mathbf{r} = (r_1, \ldots, r_m)$.

Proof. Exactly as in the commutative case, $\sum G(\mathbf{r})$ is the sum of all o-sequences, and by (2.2.1),

$$\sum_{\mathbf{r}\geq\mathbf{0}} G(\mathbf{r}) = \sum_{\mathbf{r}\geq\mathbf{0}} O(\mathbf{r}) = \sum_{\mathbf{r}\geq\mathbf{0}} P(\mathbf{r}).$$

On the other hand, we have

$$\frac{1}{\det(I-A)} =$$

$$= \left(\frac{1}{\det(I-A)} \cdot \det\left(I - A^{11}\right)\right) \cdot \left(\frac{1}{\det(I-A^{11})} \cdot \det\left(I - A^{12,12}\right)\right) \cdots \frac{1}{1-a_{mm}},$$

which is, by the right-quantum matrix inverse formula (Theorem 3.1.2), equal to

$$\left(\frac{1}{I-A}\right)_{11} \cdot \left(\frac{1}{I-A^{11}}\right)_{22} \cdot \left(\frac{1}{I-A^{12,12}}\right)_{31} \cdots \left(\frac{1}{I-a_{mm}}\right). \tag{4.3.3}$$

It follows that (4.3.3) is the sum over all p-sequences. This finishes the proof. □

4.4 The q-right-quantum case

In this section, we assume that the variables $x_1, \ldots, x_m$ q-commute, $x_j x_i = q\, x_i x_j$ for $i < j$, where $q \in \mathbb{C} \setminus \{0\}$ is fixed. Suppose also that $x_1, \ldots, x_m$ commute with all a_{ij}. Again, it makes sense to talk about the coefficient of $\mathbf{x}^{\mathbf{r}}$ in

$$\prod_{i=1}^{m} \left(a_{i1}x_1 + \ldots + a_{im}x_m\right)^{r_i}. \tag{4.4.1}$$

In Section 1.4, we defined the concept of the q-determinant of a matrix $B = (b_{ij})_{m\times m}$ with entries in $\mathcal{A}$. Recall that we have

$$\det_q(I-A) = \sum_{J\subseteq[m]} (-1)^{|J|} \det_q A_J,$$

where

$$\det_q A_J = \sum_{\sigma\in S_J} (-q)^{-\operatorname{inv}(\sigma)} a_{\sigma(j_1)j_1} \cdots a_{\sigma(j_k)j_k}$$

for $J = \{j_1 < j_2 < \ldots < j_k\}$. The following theorem was first proved in [GLZ06].

Theorem 4.4.1 (q-right-quantum master theorem) *Assume that $A = (a_{ij})_{m\times m}$ is a q-right-quantum matrix. Denote the coefficient of $\mathbf{x}^{\mathbf{r}}$ in* (4.4.1) *by* $G(\mathbf{r})$*. Then*

$$\sum_{\mathbf{r}\geq\mathbf{0}} G(\mathbf{r}) = \frac{1}{\det_q(I-A)}, \tag{4.4.2}$$

where the sum is over all $\mathbf{r} = (r_1, \ldots, r_m) \geq \mathbf{0}$*, and* $\det_q$ *denotes the q-determinant.*

Proof: When we expand the product (4.4.1), move the x_i's to the right and order them, the coefficient at $a_{\lambda,\mu}$ is equal to $q^{\mathrm{inv}(\mu)}$. This means that $\sum G(\mathbf{r})$ is a weighted sum of o-sequences, with an o-sequence $\alpha = a_{\lambda,\mu}$ weighted by $q^{\mathrm{inv}(\mu)} = q^{\mathrm{inv}(\alpha)}$. So

$$\sum_{\mathbf{r}\geq\mathbf{0}} G(\mathbf{r}) = \sum_{\mathbf{r}\geq\mathbf{0}} O_q(\mathbf{r}) = \sum_{\mathbf{r}\geq\mathbf{0}} P_q(\mathbf{r}),$$

where the last equality follows from Theorem 2.4.1. On the other hand, we have

$$\frac{1}{\det_q(I-A)} =$$

$$= \left(\frac{1}{\det_q(I-A)} \cdot \det_q\left(I - A^{11}\right)\right) \cdot \left(\frac{1}{\det_q(I-A^{11})} \cdot \det_q\left(I - A^{12,12}\right)\right) \cdots \frac{1}{1-a_{mm}},$$

which is, by the q-right-quantum matrix inverse formula (Theorem 3.1.3), equal to

$$\left(\frac{1}{I-A_{[1]}}\right)_{11} \cdot \left(\frac{1}{I-A_{[2]}}\right)_{22} \cdot \left(\frac{1}{I-A_{[3]}}\right)_{33} \cdots \left(\frac{1}{I-A_{[m]}}\right)_{mm}, \tag{4.4.3}$$

where

$$A_{[i]} = \begin{pmatrix} a_{ii} & qa_{i,i+1} & \cdots & qa_{i,m} \\ a_{i+1,i} & qa_{i+1,i+1} & \cdots & qa_{i+1,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,i} & qa_{m,i+1} & \cdots & qa_{mm} \end{pmatrix}.$$

An entry in (4.4.3) is a p-sequence, and every step contributes q to the weight, except for the last step of every primitive sequence. A p-sequence α of length n which decomposes into p primitive sequences is therefore weighted by q^{n-p} in (4.4.3), and, by Proposition 2.3.3,

$$\frac{1}{\det_q(I-A)} = \sum_{\mathbf{r}\geq\mathbf{0}} P_q(\mathbf{r}).$$

This finishes the proof. □

4.5 The q-right-quantum case

Fix complex numbers $q_{ij} \neq 0$, where $1 \leq i < j \leq m$. Suppose that the variables $x_1, \ldots, x_m$ are **q**-commuting, $x_j x_i = q_{ij}\ x_i x_j$ for $i < j$, and that they commute with all a_{ij}. Again, we can talk about the coefficient of $\mathbf{x}^{\mathbf{r}}$ in

$$\prod_{i=1}^{m} \left(a_{i1}x_1 + \ldots + a_{im}x_m\right)^{r_i}. \tag{4.5.1}$$

In Section 1.4, we defined the concept of the **q**-determinant. We have

$$\det_{\mathbf{q}}(I-A) = \sum_{J\subseteq[m]} (-1)^{|J|} \det_{\mathbf{q}} A_J,$$

where

$$\det_{\mathbf{q}} A_J = \sum_{\sigma \in S_J} \left(\prod_{p<r:\, j_p>j_r} (-q_{j_r j_p})^{-1} \right) a_{\sigma(j_1)j_1} \cdots a_{\sigma(j_k)j_k}$$

for $J = \{j_1 < j_2 < \ldots < j_k\}$.

Theorem 4.5.1 *Assume that $A = (a_{ij})_{m\times m}$ is a $\mathbf{q}$-right-quantum matrix. Denote the coefficient of $\mathbf{x}^{\mathbf{r}}$ in (4.5.1) by $G(\mathbf{r})$. Then*

$$\sum_{\mathbf{r}\geq \mathbf{0}} G(\mathbf{r}) = \frac{1}{\det_{\mathbf{q}}(I-A)}, \tag{4.5.2}$$

where the he sum is over all $\mathbf{r} = (r_1, \ldots, r_m) \geq \mathbf{0}$, and $\det_{\mathbf{q}}$ denotes the $\mathbf{q}$-determinant.

Proof. When we expand the product (4.5.1), move the x_i's to the right and order them, the coefficient at $a_{\lambda,\mu}$ is equal to $\prod_{(i,j)\in\mathcal{I}(\mu)} q_{\mu_j\mu_i}$. This means that $\sum G(\mathbf{r})$ is a weighted sum of o-sequences, with an o-sequence $\alpha = a_{\lambda,\mu}$ weighted by

$$\prod_{(i,j)\in\mathcal{I}(\mu)} q_{\mu_j\mu_i} = \prod_{(i,j)\in\mathcal{I}(\mu)} q_{\mu_j\mu_i} \prod_{(i,j)\in\mathcal{I}(\lambda)} q^{-1}_{\lambda_j\lambda_i} = \mathbf{q}^{\mathbf{inv}\ \alpha}.$$

In other words, we have

$$\sum_{\mathbf{r}\geq\mathbf{0}} G(\mathbf{r}) = \sum_{\mathbf{r}\geq\mathbf{0}} O_{\mathbf{q}}(\mathbf{r}) = \sum_{\mathbf{r}\geq\mathbf{0}} P_{\mathbf{q}}(\mathbf{r}),$$

where the last equality follows from Theorem 2.6.1. On the other hand, we have

$$\frac{1}{\det_{\mathbf{q}}(I-A)} =$$

$$= \left(\frac{1}{\det_{\mathbf{q}}(I-A)} \cdot \det_{\mathbf{q}}\left(I - A^{11}\right)\right) \cdot \left(\frac{1}{\det_{\mathbf{q}}(I-A^{11})} \cdot \det_{\mathbf{q}}\left(I - A^{12,12}\right)\right) \cdots \frac{1}{1-a_{mm}},$$

which is, by the $\mathbf{q}$-right-quantum matrix inverse formula (Theorem 3.1.4), equal to

$$\left(\frac{1}{I-A_{[1]}}\right)_{11} \cdot \left(\frac{1}{I-A_{[2]}}\right)_{22} \cdot \left(\frac{1}{I-A_{[3]}}\right)_{33} \cdots \left(\frac{1}{I-A_{[m]}}\right)_{mm}, \tag{4.5.3}$$

where

$$A_{[i]} = \begin{pmatrix} a_{ii} & q_{i,i+1}a_{i,i+1} & \cdots & q_{i,m}a_{i,m} \\ a_{i+1,i} & q_{i,i+1}a_{i+1,i+1} & \cdots & q_{i,m}a_{i+1,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,i} & q_{i,i+1}a_{m,i+1} & \cdots & q_{i,m}a_{mm} \end{pmatrix}.$$

An entry in (4.5.3) is a p-sequence, with a step that is not the first step of a primitive path contributing q_{ij} to the weight, where j is the step's ending height and i is the starting height of the primitive path that the step is part of. A p-sequence α therefore has, as a term of (4.5.3), weight

$$\prod q_{ij},$$

where the product runs over all pairs (i, j) with i the starting height of a primitive path of α, and j another height of the same primitive path. By Proposition 2.3.3,

$$\frac{1}{\det_{\mathbf{q}}(I - A)} = \sum_{\mathbf{r} \geq \mathbf{0}} P_{\mathbf{q}}(\mathbf{r}).$$

This finishes the proof. □

4.6 The β-extension

In this section we first present an extension of MacMahon master theorem due to Foata and Zeilberger. Their theorem does not generalize to our non-commutative settings, but we find a variant that does.

First, assume that a_{ij} are commutative variables and let $\beta \in \mathbb{N}$ be a non-negative integer. For $\mathbf{r} = (r_1, \ldots, r_m)$, let $\mathcal{P}(\mathbf{r})$ denote the set of all permutations of the set

$$\{(1, 1), \ldots, (1, r_1), (2, 1), \ldots, (2, r_2), \ldots, (m, 1), \ldots, (m, r_m)\}. \tag{4.6.1}$$

For a permutation $\pi \in \mathcal{P}(\mathbf{r})$, we define $\pi_{ij} = i'$ whenever $\pi(i, j) = (i', j')$. Define the *weight* $v(\pi)$ by the word

$$v(\pi) = \prod_{i=1}^{m} \prod_{j=1}^{r_i} a_{i,\pi_{ij}}$$

and the *β-weight* $v_\beta(\pi)$ by the product

$$v_\beta(\pi) = \beta^{\operatorname{cyc} \pi} v(\pi),$$

where $\operatorname{cyc} \pi$ is the number of cycles of the permutation π. For example, if

$$\pi = \begin{pmatrix} (1,1) & (1,2) & (1,3) & (2,1) & (3,1) \\ (2,1) & (1,2) & (1,1) & (3,1) & (1,3) \end{pmatrix} \in \mathcal{P}(3, 1, 1),$$

then $v(\pi) = a_{12}a_{11}a_{11}a_{23}a_{31}$ and $v_\beta(\pi) = \beta^2 \, a_{12}a_{11}a_{11}a_{23}a_{31}$.

By definition, the word $v(\pi)$ is always an o-sequence of type $\mathbf{r}$. Note now that the word $\alpha \in \mathbf{O}(\mathbf{r})$ does not determine the permutation π uniquely, since the second coordinate j' in $(i', j') = \pi(i, j)$ can take any value between 1 and $r_{i'}$. From here it follows that there are exactly $r_1! \cdots r_m!$ permutations $\pi \in \mathcal{P}(\mathbf{r})$ corresponding to a given o-sequence $\alpha \in \mathbf{O}(\mathbf{r})$. The (usual) MacMahon master theorem can be restated as

$$\frac{1}{\det(I - A)} = \sum_{\mathbf{r} \geq \mathbf{0}} \frac{1}{\mathbf{r}!} \sum_{\pi \in \mathcal{P}(\mathbf{r})} v(\pi).$$

Foata and Zeilberger proved in [FZ88, §3] the following extension of the master theorem:

$$\left(\frac{1}{\det(I-A)}\right)^{\beta} = \sum_{\mathbf{r}\geq\mathbf{0}} \frac{1}{\mathbf{r}!} \sum_{\pi\in\mathcal{P}(\mathbf{r})} v_\beta(\pi). \tag{4.6.2}$$

Proof: For the sake of completeness, we include Foata and Zeilberger's proof.
Call a permutation $\pi \in \mathcal{P}(\mathbf{r})$ *connected* if it has only one cycle, and write $\mathcal{C}(\mathbf{r})$ for the set of connected permutations of the set (4.6.1). Note that $v_\beta(\pi) = \beta v(\pi)$ $\pi \in \mathcal{C}(\mathbf{r})$ and that v_β is multiplicative in the sense that if π is the product of cycles $\pi_1, \ldots, \pi_k$ then $v_\beta(\pi) = v_\beta(\pi_1)\cdots v_\beta(\pi_k)$. Every permutation decomposes into a product of connected permutations, so by the exponential formula (see e.g. [Sta99, Corollary 5.1.6])

$$\frac{1}{\det(I-A)} = \sum_{\mathbf{r}\geq\mathbf{0}} \frac{1}{\mathbf{r}!} \sum_{\pi\in\mathcal{P}(\mathbf{r})} v(\pi) = \exp\left(\sum_{\mathbf{r}\geq\mathbf{0}} \frac{1}{\mathbf{r}!} \sum_{\pi\in\mathcal{C}(\mathbf{r})} v(\pi)\right)$$

and

$$\left(\frac{1}{\det(I-A)}\right)^{\beta} = \exp\left(\sum_{\mathbf{r}\geq\mathbf{0}} \frac{1}{\mathbf{r}!} \sum_{\pi\in\mathcal{C}(\mathbf{r})} \beta v(\pi)\right) =$$

$$= \exp\left(\sum_{\mathbf{r}\geq\mathbf{0}} \frac{1}{\mathbf{r}!} \sum_{\pi\in\mathcal{C}(\mathbf{r})} v_\beta(\pi)\right) = \sum_{\mathbf{r}\geq\mathbf{0}} \frac{1}{\mathbf{r}!} \sum_{\pi\in\mathcal{P}(\mathbf{r})} v_\beta(\pi). \qquad \square$$

Take a word μ in the alphabet $\{1,\ldots,m\}$, and let λ denote its non-decreasing rearrangement. Define

$$c_\mu(\beta) = \frac{1}{\mathbf{r}!}\sum_{\pi} \beta^{\mathrm{cyc}(\pi)},$$

where $\mathbf{r}$ is the type of the o-sequence $a_{\lambda,\mu}$ (i.e. r_i is the number of i's in μ) and the sum runs over all $\pi \in \mathcal{P}(\mathbf{r})$ with $v(\pi) = a_{\lambda,\mu}$. Another way to phrase (4.6.2) is to say that

$$\left(\frac{1}{\det(I-A)}\right)^{\beta} = \sum_{\mu} c_\mu(\beta) a_{\lambda,\mu} \tag{4.6.3}$$

where μ runs over all words in the alphabet $\{1,\ldots,m\}$ and λ is the non-decreasing rearrangement of μ.

Example 4.6.1 Take $a_{11}a_{12}a_{21}a_{22}$ (so $r_1 = r_2 = 2$). The relevant permutations are

$$\begin{pmatrix}(1,1) & (1,2) & (2,1) & (2,2)\\(1,1) & (2,1) & (1,2) & (2,2)\end{pmatrix}, \quad \begin{pmatrix}(1,1) & (1,2) & (2,1) & (2,2)\\(1,1) & (2,2) & (1,2) & (2,1)\end{pmatrix},$$

$$\begin{pmatrix}(1,1) & (1,2) & (2,1) & (2,2)\\(1,2) & (2,1) & (1,1) & (2,2)\end{pmatrix}, \quad \begin{pmatrix}(1,1) & (1,2) & (2,1) & (2,2)\\(1,2) & (2,2) & (1,1) & (2,1)\end{pmatrix},$$

and therefore

$$c_{1212}(\beta) = (\beta^3 + 2\beta^2 + \beta)/4. \qquad \diamondsuit$$

Denote by $S_{\mathbf{r}}$ the natural embedding of $S_{r_1} \times \cdots \times S_{r_m}$ into $S_{r_1+\ldots+r_m}$. Since the variables a_{ij} commute, $a_{\lambda,\mu}$ remains the same if we apply $\pi \in S_{\mathbf{r}}$ to μ. For example, $a_{12}a_{11}a_{22}a_{21} = a_{11}a_{12}a_{21}a_{22}$. Furthermore, it is easy to see that c_μ is also invariant with respect to the action of $S_{\mathbf{r}}$.

Write $\mu = \mu_1\mu_2\cdots\mu_m$, where μ_i is of length r_i. Denote by r_i^j the number of j's in μ_i; clearly, $r_i = r_i^1 + \ldots + r_i^m$. The orbit of μ with respect to the action of $S_{\mathbf{r}}$ has

$$\binom{r_1}{r_1^1, r_1^2 \ldots, r_1^m}\binom{r_2}{r_2^1, r_2^2, \ldots, r_2^m}\cdots\binom{r_m}{r_m^1, r_m^2, \ldots, r_m^m}$$

elements. This means that if we rearrange the terms on the right-hand size of (4.6.2) lexicographically (so a_{ij} never appears before a_{ik} for $j > k$), the coefficient of $a_{\lambda,\mu}$ is

$$\frac{1}{\prod_{i,j} r_i^j!}\sum_\pi \beta^{\mathrm{cyc}(\pi)},$$

where the sum runs over all $\pi \in \mathcal{P}(\mathbf{r})$ with $v(\pi) = a_{\lambda,\mu}$.

Assume now that the matrix A is Cartier-Foata. Choosing a term of $(\det(I - A))^{-\beta}$ means choosing β terms in $(\det(I - A))^{-1}$ and multiplying them from left to right. Since the variables with different left indices commute, we can write each term of $(\det(I - A))^{-\beta}$ as an o-sequence, and there must exist non-negative integers $d_\mu(\beta)$ so that

$$\left(\frac{1}{\det(I - A)}\right)^\beta = \sum_\mu d_\mu(\beta)a_{\lambda,\mu}$$

where μ runs over all words in the alphabet $\{1, \ldots, m\}$ and λ is the non-decreasing rearrangement of μ.

Even though commutative variables are also Cartier-Foata, that does not mean that $c_\mu = d_\mu$ for all μ; in fact, $c_\mu(\beta)$ is not always an integer. For example, we have (by Example 4.6.1)

$$c_{1212}(2) = c_{1221}(2) = c_{2112}(2) = c_{2121}(2) = 9/2,$$

while

$$d_{1212}(2) = d_{2121}(2) = 4, \quad d_{1221}(2) = d_{2112}(2) = 5.$$

Note, however, that we do have

$$c_{1212}(2) + c_{1221}(2) + c_{2112}(2) + c_{2121}(2) = d_{1212}(2) + d_{2121}(2) + d_{1221}(2) + d_{2112}(2)$$

since the values on both sides represent the coefficient of $a_{11}a_{12}a_{21}a_{22}$ in $(\det(I-A))^{-2}$ when we rearrange commuting variables a_{ij} in lexicographic order. In general, we have

$$\sum_\mu c_\mu(\beta) = \sum_\mu d_\mu(\beta), \tag{4.6.4}$$

where both sums run over the orbit of a word of type $\mathbf{r}$ under the action of $S_{\mathbf{r}}$.

Let us find a combinatorial interpretation of d_μ. For an o-sequence $a_{\lambda,\mu}$, take the corresponding p-sequence $a_{\lambda',\mu'} = \varphi(a_{\lambda,\mu})$ (see Section 4.2), and interpret it as a sequence of steps.

In what follows, we call a lattice path from i to i with each height appearing at most once as the starting height a *disjoint cycle*. For example, $a_{12}a_{25}a_{53}a_{31}$ is a disjoint cycle while $a_{12}a_{23}a_{33}a_{31}$ is not.

If the first repeated height in $a_{\lambda',\mu'}$ is the starting height of the sequence, the sequence starts with a disjoint cycle; remove it and repeat the algorithm. If the first repeated height in $a_{\lambda',\mu'}$ is not the starting height of the sequence, we have λ' starting with $i_1 i_2 \cdots i_p i_{p+1} i_{p+2} \cdots i_{p+r-1}$ and μ' starting with $i_2 i_3 \cdots i_{p+1} i_{p+2} \cdots i_p$ for different indices $i_1, \ldots, i_{p+r-1}$. Then we can move the disjoint cycle $i_p \to i_{p+1} \to \ldots \to i_{p+r-1} \to i_p$ to the beginning, remove it, and repeat the algorithm with the rest of the sequence. The resulting sequence is a concatenation of disjoint cycles, and we call it the *disjoint cycle decomposition* of the o-sequence $a_{\lambda,\mu}$.

EXAMPLE 4.6.2 Take the o-sequence

$$a_{13}a_{11}a_{12}a_{13}a_{22}a_{23}a_{22}a_{21}a_{23}a_{22}a_{23}a_{32}a_{31}a_{31}a_{33}a_{32}a_{32}a_{33}a_{33}$$

from Example 4.2.1. The corresponding disjoint cycle decomposition is

$$a_{22}a_{32}a_{23}a_{13}a_{31}a_{11}a_{22}a_{12}a_{21}a_{13}a_{31}a_{33}a_{23}a_{32}a_{22}a_{23}a_{32}a_{33}a_{33}.$$

and has 13 cycles. See Figure 4-3. ♢

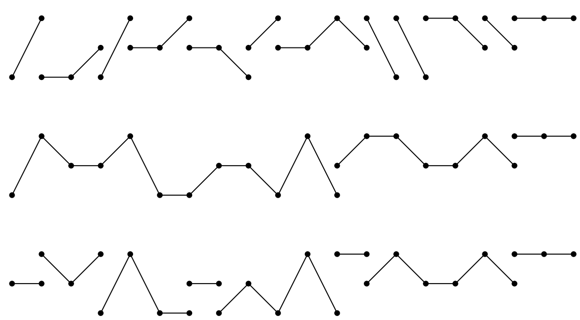

Figure 4-3: Disjoint cycle decomposition of an o-sequence.

REMARK 4.6.3 Note that the word "disjoint" means that the starting heights in each cycle are different, not that the starting heights of different cycles are disjoint as sets. If μ is a permutation of $\{1, 2, \ldots, m\}$, the disjoint cycle decomposition of $a_{12\ldots m,\mu}$ is a canonically chosen disjoint cycle decomposition of the permutation μ. ♢

Recall that we are trying to calculate the β-th power of the sum of all o-sequences. Imagine we have β (linearly ordered) slots, and that we are given a word μ and its non-decreasing rearrangement λ. The number of ways of putting each step $\lambda_i \to \mu_i$ in one of the slots so that the steps within each slot form an o-sequence, and so that the resulting sequence is $a_{\lambda,\mu}$, is precisely the coefficient $d_\mu(\beta)$.

The following lemma will reduce the calculation of d_μ to a simple combinatorial argument.

Lemma 4.6.4 *All the steps in a cycle of the disjoint cycle decomposition must be placed in the same slot.*

Proof: Assume by induction that we have proved this for the first $p-1$ cycles, with the base of induction being $p=1$. Take the p-th cycle $i_1 \to i_2 \to \ldots \to i_l \to i_1$. If $l=1$, there is nothing to prove, otherwise assume that $i_1 \to i_2$ is placed in slot r and that $i_2 \to i_3$ is placed in slot $r' > r$. The step $i_1 \to i_2$ in slot r ends with i_2, and since the sequence in each slot is balanced, there must be a step with starting height i_2 in slot r. If this step is in one of the first $p-1$ cycles, there is another step with ending height i_2 in slot r by the induction hypothesis. This would mean that in slot r there is a step with starting height i_2 that belongs to one of the cycles $p+1, p+2, \ldots$, which is a contradiction since this step should be to the right of $i_2 \to i_3$.
The contradiction proves that $i_2 \to i_3$ must be in a slot $r' \leq r$. But if $r' < r$, the same reasoning as above shows that $i_3 \to i_4$, $i_4 \to i_5$ etc. are in a slot strictly to the left of r. In particular, this would hold for the step $i_1 \to i_2$, which is a contradiction. This shows that $i_2 \to i_3$ is in the same slot as $i_1 \to i_2$, and the same proof shows that the whole cycle $i_1 \to i_2 \to \ldots \to i_l \to i_1$ is in the same slot. □

We will say that two cycles in the disjoint cycle decomposition are *disjoint* if the sets of their starting heights are disjoint. Furthermore, let $\operatorname{des}(\pi)$ denote the number of *descents* of a permutation π, i.e. the cardinality of the set $\{i \colon \pi(i) > \pi(i+1)\}$.

Theorem 4.6.5 *Assume $A = (a_{ij})_{m \times m}$ is a Cartier-Foata matrix. For a word μ in the alphabet $\{1, \ldots, m\}$, denote by λ its non-decreasing rearrangement, and denote by $d_\mu(\beta)$ the coefficient of $a_{\lambda,\mu}$ in $(\det(I-A))^{-\beta}$.*
Let $u_1 u_2 \cdots u_k$ be the disjoint cycle decomposition of the o-sequence $a_{\lambda,\mu}$. Then

$$d_\mu(\beta) = \sum_\pi \binom{\beta + k - 1 - \operatorname{des}(\pi)}{k}, \tag{4.6.5}$$

where the sum is over $\pi \in S_k$ for which $i < j, \pi(i) > \pi(j)$ implies disjoint $u_{\pi(i)}, u_{\pi(j)}$.

Example 4.6.6 The sequence $a_{11}a_{12}a_{21}a_{22}$ is already written as a product of three disjoint cycles, $u_1 = a_{11}$, $u_2 = a_{12}a_{21}$, $u_3 = a_{22}$, so u_1 has to appear before u_2, and u_2 has to appear before u_3. The only permutation in the sum (4.6.5) is therefore id, and

$$d_{1212}(\beta) = \binom{\beta+2}{3} = \frac{\beta(\beta+1)(\beta+2)}{6}.$$

The disjoint cycle decomposition of $a_{11}a_{12}a_{22}a_{21}$ is $(a_{11})(a_{22})(a_{12}a_{21})$, so u_1 and u_2

have to appear before u_3. The permutations in the sum (4.6.5) are 123 and 213, and

$$d_{1221}(\beta) = \binom{\beta+2}{3} + \binom{\beta+1}{3} = \frac{\beta(\beta+1)(2\beta+1)}{6}.$$

We also get $d_{2112}(\beta) = d_{1221}(\beta)$ and $d_{2121}(\beta) = d_{1212}(\beta)$. Note that indeed

$$c_{1212}(\beta) + c_{1221}(\beta) + c_{2112}(\beta) + c_{2121}(\beta) = d_{1212}(\beta) + d_{2121}(\beta) + d_{1221}(\beta) + d_{2112}(\beta),$$

as predicted by (4.6.4). ♢

EXAMPLE 4.6.7 It is interesting to note what (4.6.4) gives for special cases of μ. For $\mu = 11\ldots1$, we get

$$\frac{1}{r!}\sum_{\pi\in S_r} \beta^{\operatorname{cyc}\pi} = \binom{\beta+r-1}{r},$$

for $\mu = 12\ldots r$, we get

$$\beta^r = \sum_{\pi\in S_r} \binom{\beta+r-1-\operatorname{des}(\pi)}{r},$$

while for $\mu = 11\ldots122\ldots2\ldots mm\ldots m$ of type $\mathbf{r}$, we get

$$\frac{1}{\mathbf{r}!}\sum_{\pi_i\in S_{r_i}} \beta^{\operatorname{cyc}\pi_1+\ldots+\operatorname{cyc}\pi_m} = \prod_{i=1}^{m}\binom{\beta+r_i-1}{r_i} = \sum_{\pi}\binom{\beta+r-1-\operatorname{des}(\pi)}{r},$$

where π runs over permutations in S_r, $r = r_1 + \ldots + r_m$, with $\pi^{-1}(i) < \pi^{-1}(j)$ whenever $r_1 + \ldots + r_p < i < j \leq r_1 + \ldots + r_{p+1}$ for some p. In other words, we get a combinatorial interpretation of the coefficients e_d in the expansion

$$\prod_{i=1}^{m}\binom{\beta+r_i-1}{r_i} = \sum_{d=1}^{r} e_d\binom{\beta+r-d}{r}:$$

e_d is the number of permutations $\pi \in S_n$ with d descents and with the property $\pi^{-1}(i) < \pi^{-1}(j)$ whenever $r_1 + \ldots + r_p < i < j \leq r_1 + \ldots + r_{p+1}$ for some p. ♢

EXAMPLE 4.6.8 It is easy to see that the coefficient $d_\mu(\beta)$ of the o-sequence $a_{\lambda,\mu}$ in $(\det(I-A))^{-\beta}$ is a polynomial in β for $\beta \in \mathbb{C}$. It follows that the formula (4.6.5) holds for all $\beta \in \mathbb{C}$. For example, take $\beta = -1$. If the disjoint cycle decomposition of $a_{\lambda,\mu}$ has r cycles, the binomial coefficients appearing in (4.6.5) are $\binom{r-2}{r}, \binom{r-3}{r}, \ldots, \binom{-1}{r}$. These are all zero except $\binom{-1}{r} = (-1)^r$. In other words, $d_\mu(-1) = 0$ unless the permutation $r\ldots21$ is permissible, i.e. unless all r cycles are disjoint, in which case $d_\mu(-1) = (-1)^r$. But this is exactly $\det(I-A) = \sum_{J\subseteq[m]}(-1)^{|J|}\det A_J$. ♢

Proof (of Theorem 4.6.5)*:* By the lemma, each cycle has to lie in one of the slots. So we have to find the number of ways to place the k cycles in β slots so that their product is $a_{\lambda,\mu}$. Note that two cycles commute if and only if they are either the same or disjoint. That means that a permutation π of the cycles gives $a_{\lambda,\mu}$ if and only if the inversions of π correspond to pairs of disjoint cycles. Take the identity permutation, which does not have inversions and therefore certainly satisfies this condition. We have to find the number of ways of placing these k linearly ordered cycles in β linearly ordered slots. Of course, there are $\binom{\beta+k-1}{k}$ ways to do that. Now assume that the first two cycles are disjoint, so that the permutation $2134\dots k$ satisfies the condition. There are again $\binom{\beta+k-1}{k}$ ways to place the cycles in the slots, but since the first two cycles commute, placing them in the same slot would give the same term as the corresponding placement for the identity permutation. Basic enumeration shows that the number of ways to place k linearly ordered cycles in β linearly ordered slots so that the first two cycles are in different slots is $\binom{\beta+k-2}{k}$. Similarly, for any permutation π whose inversions correspond to pairs of disjoint cycles, a descent i of π corresponds to commuting cycles $u_{\pi(i)}, u_{\pi(i+1)}$, so we should not place $\pi(i)$ and $\pi(i+1)$ in the same slot to avoid double counting. In other words, $d_\mu(\beta) = \sum_\pi \binom{\beta+k-1-\mathrm{des}(\pi)}{k}$, where the sum is over all permutations $\pi \in S_k$ such that $i < j, \pi(i) > \pi(j)$ implies that the cycles $u_{\pi(i)}, u_{\pi(j)}$ are disjoint. □

Finally, assume that A is right-quantum. By the right-quantum MacMahon theorem (Theorem 4.3.1), each of the β factors of $(\det(I-A))^{-\beta}$ is a sum of o-sequences. After multiplication, we get a sum over all concatenations of β o-sequences. However, this cannot be transformed into a sum of o-sequences, as shown by the following.

Example 4.6.9 The sum over all sequences of type $(1,1)$ in $(\det(I-A))^{-2}$ is

$$a_{11}a_{22}\cdot 1 + a_{11}\cdot a_{22} + a_{22}\cdot a_{11} + 1\cdot a_{11}a_{22} + a_{12}a_{21}\cdot 1 + 1\cdot a_{12}a_{21} = 3a_{11}a_{22} + a_{22}a_{11} + 2a_{12}a_{21},$$

which is not equal to a weighted sum of o-sequences. ♢

Extensions to weighted cases in the spirit of Section 2.3 and 2.5 are possible, however.

Theorem 4.6.10 *Assume $A = (a_{ij})_{m\times m}$ is a $\mathbf{q}$-Cartier-Foata matrix. For a word μ in the alphabet $\{1,\dots,m\}$, denote by λ its non-decreasing rearrangement, and denote by $d^{\mathbf{q}}_\mu(\beta)$ the coefficient of $a_{\lambda,\mu}$ in $(\det_{\mathbf{q}}(I-A))^{-\beta}$. Let $u_1u_2\cdots u_k$ be the disjoint cycle decomposition of $a_{\lambda,\mu}$. Then*

$$d^{\mathbf{q}}_\mu(\beta) = \left(\prod_{(i,j)\in\mathcal{I}(\mu)} q_{\mu_j\mu_i}\right)\cdot\sum_\pi \binom{\beta+k-1-\mathrm{des}(\pi)}{k},$$

with the sum over $\pi \in S_k$ for which $u_{\pi(i)}, u_{\pi(j)}$ are disjoint if $i<j, \pi(i)>\pi(j)$. □

4.7 Non-balanced master theorem

For commuting variables $t_1,\dots,t_m$, write $T = \mathrm{diag}\,\mathbf{t}$, and choose $\mathbf{d}$ with $\sum d_i = 0$. Denote by $\mathcal{M}$ the multiset of all i with $d_i < 0$, with each i appearing $-d_i$ times, and by $S(\mathcal{M})$ the set of all permutations of the multiset $\mathcal{M}$; denote by $\mathcal{N} = (N_1, N_2, \dots, N_d)$

the multiset of all i with $d_i > 0$, with each i appearing d_i times. Note that $\mathcal{M}$ and $\mathcal{N}$ have the same cardinality δ; write M (respectively N) for the sum of all elements of $\mathcal{M}$ (respectively $\mathcal{N}$) counted with their multiplicities. For $\pi = \pi_1 \cdots \pi_\delta \in S(\mathcal{M})$, let I_π^k (for $1 \leq k \leq \delta$) be the set $\{\pi_1, \ldots, \pi_k\}$, write ε_π^k for the size of the intersection of I_π^{k-1} and the open interval between π_k and N_k, and write $J_\pi^k = (I_\pi^k \setminus \{\pi_k\}) \cup \{N_k\}$.

Theorem 4.7.1 *Choose a right-quantum matrix $A = (a_{ij})_{m \times m}$, and let $x_1, \ldots, x_m$ be commuting variables that commute with a_{ij}. For $\mathbf{p}, \mathbf{r} \geq \mathbf{0}$, denote the coefficient of $\mathbf{x}^\mathbf{r}$ in $\prod_{i=1}^m (a_{i1}x_1 + \ldots + a_{im}x_m)^{p_i}$ by $G(\mathbf{p}; \mathbf{r})$. Then*

$$F_{A,\mathbf{d}}(\mathbf{t}) = \sum_{\mathbf{p}=\mathbf{r}+\mathbf{d}} G(\mathbf{p}; \mathbf{r}) \mathbf{t}^\mathbf{p} =$$

$$= \frac{(-1)^{M+N}}{\det(I - TA)} \sum_{\pi \in S(\mathcal{M})} \prod_{k=1}^{\delta} (-1)^{\varepsilon_\pi^k} \cdot \det(I - TA)^{I_\pi^k, J_\pi^k} \cdot \frac{1}{\det(I - TA)^{I_\pi^k, I_\pi^k}}. \qquad (4.7.1)$$

Proof. Fix non-negative integer vectors $\mathbf{p}, \mathbf{r}$ with $\mathbf{p} = \mathbf{r} + \mathbf{d}$, and use (2.1.3) on $\mathbf{O}(\mathbf{p}; \mathbf{r})$ with respect to the permutation $\sigma = i_1 \cdots i_s j_1 \cdots, j_t$, where $i_1 < i_2 < \ldots < i_s$ form the underlying set of $\mathcal{N}$ (in other words, $\{i_1, \ldots, i_s\} = \{i \colon d_i > 0\}$) and $j_1 < j_2 < \ldots < j_t$ are the remaining elements of $[m]$.
A path sequence in $\mathbf{P}^\sigma(\mathbf{p}; \mathbf{r})$ has the following structure. The first path starts at $N_1 = i_1$ and ends at one of the heights in $\mathcal{M}$; the second path starts at N_2 (which is i_1 if $d_{i_1} > 1$, and i_2 if $d_{i_1} = 1$), and ends at one of the heights in $\mathcal{M}$, and it does not include the ending height of the previous path except possibly as the ending height. In general, the k-th path starts at N_k and ends at one of the heights in $\mathcal{M}$, and does not contain any of the ending heights of previous paths except possibly as the ending height. All together, the ending heights of these δ paths form a permutation of $\mathcal{M}$, which explains why $F_{A,\mathbf{d}}(\mathbf{t})$ is written as a sum over $\pi \in S(\mathcal{M})$. After these paths, we have a balanced path sequence that does not include any height in $\mathcal{M}$.
Now choose $\pi = \pi_1 \cdots \pi_\delta \in S(\mathcal{M})$, and look at all the p-sequences in $\mathbf{P}^\sigma(\mathbf{p}; \mathbf{r})$ (for all $\mathbf{p}, \mathbf{r} \geq \mathbf{0}$ with $\mathbf{p} = \mathbf{r} + \mathbf{d}$) whose first δ ending heights of paths are $\pi_1, \ldots, \pi_\delta$ (in this order). The k-th path is a path from N_k to π_k, and it does not include $\pi_1, \ldots, \pi_{k-1}$ except possibly as an ending height. By the matrix inverse formula (Theorem 3.1.2), such paths, weighted by $\mathbf{t}^{\mathbf{p_k}}$, where $(\mathbf{p_k}, \mathbf{r_k})$ is the path's type, are enumerated by

$$\pm \frac{1}{\det(I - TA)^{I_\pi^{k-1}, I_\pi^{k-1}}} \cdot \det(I - TA)^{I_\pi^k, J_\pi^k},$$

and a simple consideration shows that the sign is $(-1)^{N_k + \pi_k + \varepsilon_\pi^k}$. The balanced path sequences that do not include heights from $\mathcal{M}$ are enumerated by

$$\left(\frac{1}{\det(I - TA)^{I_\pi^\delta}} \cdot \det(I - TA)^{I_\pi^\delta \cup \{j_1\}} \right) \cdot \left(\frac{1}{\det(I - A)^{I_\pi^\delta \cup \{j_1\}}} \cdot \det\left(I - A\right)^{I_\pi^\delta \cup \{j_1, j_2\}} \right) \cdots \frac{1}{1 - a_{j_t j_t}} =$$

$$= \frac{1}{\det(I - TA)^{I_\pi^\delta}},$$

where we wrote $\det(I-TA)^I$ instead of $\det(I-TA)^{I,I}$. Formula (4.7.1) follows. □

EXAMPLE 4.7.2 For $\mathbf{d}=\mathbf{0}$, we have $\mathcal{M}=\mathcal{N}=\varnothing$, $(\mathcal{M})=\{\varnothing\}$, $M=N=0$, $\delta=0$,

$$F_{A,\mathbf{d}}(\mathbf{t})=\frac{1}{\det(I-TA)},$$

which is the right-quantum master theorem. ♢

EXAMPLE 4.7.3 Take

$$A=\begin{pmatrix}2&1&4&2\\3&2&4&3\\3&4&1&1\\1&3&5&5\end{pmatrix},\qquad \mathbf{d}=(1,-2,2,-1),$$

$$F=F_{A,\mathbf{d}}(\mathbf{t})=40tv^2+262t^2v^2+128tuv^2+312tv^3+251tv^2w+\dots.$$

where we write t,u,v,w instead of t_1,t_2,t_3,t_4. We have $\mathcal{M}=(2,2,4)$, $S(\mathcal{M})=\{224,242,422\}$, $\mathcal{N}=(1,3,3)$, $\delta=3$, $M=8$, $N=7$, $I^1_{224}=\{2\}$, $I^2_{224}=\{2\}$, $I^3_{224}=\{2,4\}$, $I^1_{242}=\{2\}$, $I^2_{242}=\{2,4\}$, $I^3_{242}=\{2,4\}$, $I^1_{422}=\{4\}$, $I^2_{422}=\{2,4\}$, $I^3_{422}=\{2,4\}$, $\varepsilon^i_\pi=0$ for all π and i, $J^1_{224}=\{1\}$, $J^2_{224}=\{3\}$, $J^3_{224}=\{3,4\}$, $J^1_{242}=\{1\}$, $J^2_{242}=\{2,3\}$, $J^3_{242}=\{3,4\}$, $J^1_{422}=\{1\}$, $J^2_{422}=\{3,4\}$, $J^3_{422}=\{3,4\}$. Therefore

$$\begin{aligned}F=-\frac{1}{\det(I-TA)}\Bigg(&\frac{\det(I-TA)^{2,1}}{\det(I-TA)^{2,2}}\,\frac{\det(I-TA)^{2,3}}{\det(I-TA)^{2,2}}\,\frac{\det(I-TA)^{24,34}}{\det(I-TA)^{24,24}}+\\&\frac{\det(I-TA)^{2,1}}{\det(I-TA)^{2,2}}\,\frac{\det(I-TA)^{24,23}}{\det(I-TA)^{24,24}}\,\frac{\det(I-TA)^{24,34}}{\det(I-TA)^{24,24}}+\\&\frac{\det(I-TA)^{4,1}}{\det(I-TA)^{4,4}}\,\frac{\det(I-TA)^{24,34}}{\det(I-TA)^{24,24}}\,\frac{\det(I-TA)^{24,34}}{\det(I-TA)^{24,24}}\Bigg)=\end{aligned}$$

$$F=-\frac{D_{24,34}\left(D_{2,1}D_{2,3}D_{4,4}D_{24,24}+D_{2,1}D_{2,2}D_{4,4}D_{24,23}D_{24,34}+D_{4,1}D_{2,2}^2D_{24,34}\right)}{DD_{2,2}^2D_{4,4}D_{24,24}^2},$$

where

$$\begin{aligned}D&=\det(I-TA)&&=1-2t-2u-v-5w+tu-10tv+8tw-14uv+\\&&&\quad uw-5tuv-4tuw+28tvw+17uvw+46tuvw,\\D_{2,1}&=\det(I-TA)^{2,1}&&=-t-15tv-tw+34tvw,\\D_{2,2}&=\det(I-TA)^{2,2}&&=1-2t-v-5w-10tv+8tw+28tvw,\\D_{2,3}&=\det(I-TA)^{2,3}&&=-4v+5tv+17vw-30tvw,\\D_{4,1}&=\det(I-TA)^{4,1}&&=-2t+tu-2tv-13tuv,\\D_{4,4}&=\det(I-TA)^{4,4}&&=1-2t-2u-v+tu-10tv-14uv-5tuv,\\D_{24,23}&=\det(I-TA)^{24,23}&&=-v-4tv,\\D_{24,24}&=\det(I-TA)^{24,24}&&=1-2t-v-10tv,\\D_{24,34}&=\det(I-TA)^{24,34}&&=-4v+5tv.\end{aligned}$$

Since we are dealing with complex variables, we do not have to worry about the order of multiplication. $\diamondsuit$

Corollary 4.7.4 *For every n, we have*

$$\sum_{i=1}^{n-1}(-1)^i\binom{n}{i-1}\binom{n}{i}\binom{n}{i+1} = \begin{cases} 2(-1)^m\binom{2m}{m-1}\binom{3m}{m-1} & : \quad n=2m \\ 0 & : \quad n=2m-1 \end{cases}.$$

Proof. Let us denote the sum we are trying to calculate by $S(n)$. Clearly,

$$[x^{n+1}y^{n+1}z^{n-2}](z-y)^n(x-z)^n(y-x)^n = [xyz^{-2}](1-\tfrac{y}{z})^n(1-\tfrac{z}{x})^n(1-\tfrac{x}{y})^n =$$

$$= [xyz^{-2}]\sum_{i,j,k}(-1)^{i+j+k}\binom{n}{i}\binom{n}{j}\binom{n}{k}\left(\frac{x}{y}\right)^i\left(\frac{y}{z}\right)^j\left(\frac{z}{x}\right)^k = S(n),$$

and so we have to use Theorem 4.7.1 for

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \qquad \mathbf{d} = (-1,-1,2).$$

We get

$$\sum_{\mathbf{p}=\mathbf{r}+\mathbf{d}} G(\mathbf{p};\mathbf{r})t^{p_1}u^{p_2}v^{p_3} = \frac{-v^2(1+t)}{(1+tv)(1+tu+tv+uv)} + \frac{-v^2(1-u)}{(1+uv)(1+tu+tv+uv)}$$

and

$$S(n) = [t^nu^nv^n]\left(\frac{-v^2(1+t)}{(1+tv)(1+tu+tv+uv)} + \frac{-v^2(1-u)}{(1+uv)(1+tu+tv+uv)}\right) =$$

$$= 2[t^nu^nv^n]\left(\frac{-v^2}{(1+tv)(1+tu+tv+uv)}\right),$$

where we used some obvious symmetry. Then

$$S(n) = 2\sum_{i,j,k,l}(-1)^{l+1}(tv)^lv^2\binom{i+j+k}{i,j,k}(-1)^{i+j+k}(tu)^i(tv)^j(uv)^k,$$

with the sum over all $i,j,k,l \geq 0$ with $l+i+j=n$, $i+j=n$, $l+2+j+k=n$, i.e. $S(n)=0$ if n is odd and

$$S(2m) = \frac{2(-1)^m}{(m+1)!(m-1)!}\sum_{l=0}^{m-1}\frac{(3m-1-l)!}{(m-1-l)!} = 2(-1)^m\binom{2m}{m-1}\sum_{l=0}^{m-1}\binom{3m-1-l}{m-1-l};$$

therefore $S(2m) = 2(-1)^m\binom{2m}{m-1}\binom{3m}{m-1}$, since every $(m-1)$-subset of $[3m]$ consists of elements $1,\ldots,l$ and an $(m-1-l)$-subset of $\{l+2,\ldots,3m\}$ for a uniquely determined l. $\square$

Remark 4.7.5 It is easy to find the "dual" version of Theorem 4.7.1, namely to calculate the generating function

$$\widetilde{F}_{A,\mathbf{d}}(\mathbf{t}) = \sum_{\mathbf{p}=\mathbf{r}+\mathbf{d}} G(\mathbf{p};\mathbf{r})\mathbf{t}^{\mathbf{r}} :$$

paths from N_k to π_k that do not include $\pi_1, \dots, \pi_{k-1}$ except possibly as an ending height, weighted by $\mathbf{t}^{\mathbf{r_k}}$, where $(\mathbf{p_k}, \mathbf{r_k})$ is the type of the path, are enumerated by

$$\pm \frac{1}{\det(I - AT)^{I_\pi^{k-1}, I_\pi^{k-1}}} \cdot \det(I - AT)^{I_\pi^k, J_\pi^k}.$$

Therefore

$$\widetilde{F}_{A,\mathbf{d}}(\mathbf{t}) = \frac{(-1)^{M+N}}{\det(I-AT)} \sum_{\pi \in S(\mathcal{M})} \prod_{k=1}^{\delta} (-1)^{\varepsilon_\pi^k} \cdot \det(I-AT)^{I_\pi^k, J_\pi^k} \cdot \frac{1}{\det(I-AT)^{I_\pi^k, I_\pi^k}}$$

with the same notation as in Theorem 4.7.1. ♢

Remark 4.7.6 It would be nice to use Theorem 4.7.1 to prove

$$\sum_{i=k}^{n-k} (-1)^i \binom{n}{i-k}\binom{n}{i}\binom{n}{i+k} = \frac{(-1)^m (2m)!^2 (3m)!}{m!(m-k)!(m+k)!(2m-k)!(2m+k)!}, \quad (4.7.2)$$

which can be established by the WZ method [PWZ96] (see also Remark 4.7.7). The author proved the left-hand side is equal to 0 if n is odd and, if $k \geq 1$, is equal to

$$2 \sum_{j=1}^{k} \binom{2k-j-1}{k-1} \sum_{i=0}^{\lfloor j/2 \rfloor} (-1)^{m-i} \binom{j}{2i} \binom{3m-i+j-k}{m-i} \binom{2m}{m+k-i},$$

and it is easy to see for small k that this is equal to the right-hand side of (4.7.2).

Remark 4.7.7 It was pointed out to the author by Christian Krattenthaler that the identity (4.7.2) is a special case of

$${}_3F_2\,(a,b,c;1+a-b,1+a-c;1) \;=\; \frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma\,(1+a-b)\,\Gamma\,(1+a-c)\,\Gamma\left(1+\frac{a}{2}-b-c\right)}{\Gamma\,(1+a)\,\Gamma\left(1+\frac{a}{2}-b\right)\Gamma\left(1+\frac{a}{2}-c\right)\Gamma\,(1+a-b-c)},$$

a more general form of Dixon's identity. ♢

4.8 Krattenthaler-Schlosser's q-analogue

In the context of multidimensional q-series, an interesting q-analogue of the MacMahon master theorem was obtained in [KS99, Theorem 9.2]. In this section we place the result in our non-commutative framework and deduce it from Theorem 4.3.1.

We start with some basic definitions and notations. Let z_i, b_{ij}, $1 \le i, j \le m$, be commutative variables, and let $q_1, \ldots, q_m \in \mathbb{C}$ be fixed complex numbers. Denote by $\mathcal{E}_i$ the q_i-shift operator

$$\mathcal{E}_i : \mathbb{C}[z_1, \ldots, z_m] \longrightarrow \mathbb{C}[z_1, \ldots, z_m]$$

that replaces each occurrence of z_i by $q_i z_i$. We assume that $\mathcal{E}_r$ commutes with b_{ij}, for all $1 \le i, j, r \le m$. For a non-negative integer vector $\mathbf{k} = (k_1, \ldots, k_m)$, denote by $[\mathbf{z}^{\mathbf{k}}]F$ the coefficient of $z_1^{k_1} \cdots z_m^{k_m}$ in the series F. Denote by $\mathbf{1}$ the constant polynomial 1. Finally, let

$$(a;q)_k \,=\, (1-a)(1-aq)\cdots(1-aq^{k-1}).$$

Theorem 4.8.1 (Krattenthaler-Schlosser) *Let $A = (a_{ij})_{m\times m}$, where*

$$a_{ij} \,=\, z_i \delta_{ij} - z_i b_{ij} \mathcal{E}_i \,, \quad \text{for all} \;\; 1 \le i, j \le m.$$

Then, for non-negative integer vector $\mathbf{k}$, we have:

$$[\mathbf{z}^{\mathbf{0}}] \prod_{i=1}^{m} \left(\sum_{j=1}^{m} b_{ij} z_j / z_i ; q_i \right)_{k_i} \;=\; \left[\mathbf{z}^{\mathbf{k}}\right] \left(\frac{1}{\det(I-A)} \cdot \mathbf{1} \right). \tag{4.8.1}$$

Note that the right-hand side of (4.8.1) is non-commutative and (as stated) does not contain q_i's, while the left-hand side contains only commutative variables and q_i's. It is not immediately obvious and was shown in [KS99] that the theorem reduces to the MacMahon master theorem. Here we give a new proof of the result.

Proof of Theorem 4.8.1. Think of variables z_i and b_{ij} as operators acting on polynomials by multiplication. Then a matrix entry a_{ij} is an operator as well. Note that multiplication by z_i and the operator $\mathcal{E}_j$ commute for $i \neq j$. This implies that the equation (1.3.2) holds, i.e. that A is a Cartier-Foata (and therefore also a right-quantum) matrix. Let $x_1, \ldots, x_m$ be formal variables that commute with each other and with a_{ij}'s. By Theorem 4.3.1, we have:

$$\frac{1}{\det(I-A)} \;=\; \sum_{\mathbf{r} \ge \mathbf{0}} G(r_1, \ldots, r_m),$$

where

$$G(r_1, \ldots, r_m) \;=\; [\mathbf{x}^{\mathbf{r}}] \prod_{i=1}^{m} \left(a_{i1}x_1 + \ldots + a_{im}x_m\right)^{r_i}.$$

Recall that $a_{ij} = z_i(\delta_{ij} - b_{ij}\mathcal{E}_i)$. Now observe that every coefficient $G(r_1, \ldots, r_m) \cdot \mathbf{1}$ is equal to $\mathbf{z}^{\mathbf{r}}$ times a polynomial in b_{ij} and q_i. Therefore, the right-hand side of (4.8.1) is equal to

$$\left[\mathbf{z}^{\mathbf{k}}\right] \left(\frac{1}{\det(I-A)} \cdot \mathbf{1} \right) \;=\; \left[\mathbf{z}^{\mathbf{k}}\right] \left(\sum_{\mathbf{r}} G(\mathbf{r}) \cdot \mathbf{1} \right) \;=\; \left[\mathbf{z}^{\mathbf{k}}\right] \left(G(\mathbf{k}) \cdot \mathbf{1} \right).$$

This is, of course, a sum of $[\mathbf{z}^{\mathbf{k}}](\alpha \cdot \mathbf{1})$ over all o-sequences α of type $\mathbf{k}$. Define

$$c_{ij}^k = z_i\delta_{ij} - z_i b_{ij} q_i^{k-1} \quad \text{and} \quad d_{ij}^k = z_j\delta_{ij} - z_j b_{ij} q_i^{k-1}.$$

It is easy to prove by induction that

$$a_{i\lambda_1} a_{i\lambda_2} \cdots a_{i\lambda_\ell} \cdot \mathbf{1} \,=\, c_{i\lambda_1}^{\ell} c_{i\lambda_2}^{\ell-1} \cdots c_{i\lambda_\ell}^{1}.$$

Therefore, for every o-sequence

$$\alpha \,=\, a_{1\lambda_1^1} a_{1\lambda_2^1} \cdots a_{1\lambda_{k_1}^1} a_{2\lambda_1^2} a_{2\lambda_2^2} \cdots a_{2\lambda_{k_2}^2} \cdots a_{m\lambda_1^m} a_{m\lambda_2^m} \cdots a_{m\lambda_{k_m}^m} \tag{4.8.2}$$

we have:

$$\begin{aligned}\alpha \cdot \mathbf{1} \,&=\, c_{1\lambda_1^1}^{k_1} c_{1\lambda_2^1}^{k_1-1} \cdots c_{1\lambda_{k_1}^1}^{1}\; c_{2\lambda_1^2}^{k_2} c_{2\lambda_2^2}^{k_2-1} \cdots c_{2\lambda_{k_2}^2}^{1}\; \cdots c_{m\lambda_1^m}^{k_m} c_{m\lambda_2^m}^{k_m-1} \cdots c_{m\lambda_{k_m}^m}^{1} \\ &=\, d_{1\lambda_1^1}^{k_1} d_{1\lambda_2^1}^{k_1-1} \cdots d_{1\lambda_{k_1}^1}^{1}\; d_{2\lambda_1^2}^{k_2} d_{2\lambda_2^2}^{k_2-1} \cdots d_{2\lambda_{k_2}^2}^{1}\; \cdots d_{m\lambda_1^m}^{k_m} d_{m\lambda_2^m}^{k_m-1} \cdots d_{m\lambda_{k_m}^m}^{1},\end{aligned}$$

where the second equality holds because α is a balanced sequence. On the other hand,

$$\left[\mathbf{z}^{\mathbf{0}}\right] \prod_{i=1}^{m} \left(\sum_{j=1}^{m} b_{ij} z_j / z_i ; q_i \right)_{k_i} \;=\; \left[\mathbf{z}^{\mathbf{k}}\right] \prod_{i=1}^{m} \prod_{j=1}^{k_i} (d_{i1}^{j} + \ldots + d_{im}^{j})$$

is equal to the sum of

$$\left[\mathbf{z}^{\mathbf{k}}\right] \left(d_{1\lambda_1^1}^{k_1} d_{1\lambda_2^1}^{k_1-1} \cdots d_{1\lambda_{k_1}^1}^{1}\; d_{2\lambda_1^2}^{k_2} d_{2\lambda_2^2}^{k_2-1} \cdots d_{2\lambda_{k_2}^2}^{1}\; \cdots d_{m\lambda_1^m}^{k_m} d_{m\lambda_2^m}^{k_m-1} \cdots d_{m\lambda_{k_m}^m}^{1} \right)$$

over all o-sequences α of form (4.8.2). This completes the proof. □

4.9 Final remarks

From our presentation, one may assume that the choice of a (q_{ij})-analogue was a lucky guess or a carefully chosen deformation designed to make the technical lemmas work. This was not our motivation, of course. These quadratic algebras are well known generalizations of the classical quantum groups of type A (see [Man87, Man89, Man88]). They were introduced and extensively studied by Manin, who also proved their Koszulity and defined the corresponding (generalized) quantum determinants. While our proof is combinatorial, the Hai-Lorenz approach works in the $\mathbf{q}$-case as well.

The relations studied in this paper always lead to quadratic algebras. While the deep reason lies in the Koszul duality, the fact that Koszulity can be extended to non-quadratic algebras is suggestive [Ber01]. The first such effort is made in [EP] where an unusual algebraic extension of MacMahon master theorem is obtained.

In the previous papers [FH08, FH07a, FH07b, GLZ06] the authors used $Bos(\cdot)$ and $Fer(\cdot)$ notation for the left- and the right-hand side of (4.1.3). While the implied connection is not unjustified, it might be misleading when the results are generalized. Indeed, in view of Koszul duality connection, the algebras can be interchanged, while giving the same result with notions of *Boson* and *Fermion* summations switched. On the other hand, we should point out that in the most interesting cases the *Fermion* summation is finite, which makes it special from combinatorial point of view.

Even though the statement of Theorem 4.7.1 appears rather intricate even in the commutative case, a computer algebra program finds the generating function easily. A Mathematica package `genmacmahon.m` that calculates $F_{A,\mathbf{d}}(\mathbf{t})$ for a commutative matrix $A = (a_{ij})_{m\times m}$ and an integer vector $\mathbf{d} = (d_1, \ldots, d_m)$ with $\sum d_i = 0$ is available at `http://www-math.mit.edu/~konvalinka/genmacmahon.m` (read in the package with `<< genmacmahon.m` and write `F[A,d,t]`), and it would be easy to adapt this to the non-commutative situation.

The Krattenthaler-Schlosser's q-analogue (Theorem 4.8.1) is essentially a byproduct of the authors' work on q-series. It was pointed out by Michael Schlosser that the Cartier-Foata matrices routinely appear in the context of "matrix inversions" for q-series (see [KS99, Sch97]). It would be interesting to see if our extensions (such as the q_{ij}-analogue in Section 4.5) can be used to obtain new results, or to give new proofs of existing results.

Chapter 5

Sylvester's determinantal identity

5.1 Introduction

Sylvester's identity is a classical determinantal identity that is usually written in the form used by Bareiss [Bar68]. We present his proof for the sake of completeness.

Theorem 5.1.1 (Sylvester's identity) *Let A denote a matrix $(a_{ij})_{m\times m}$; take $n < i, j \le m$ and define*

$$A_0 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad a_{i*} = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix}, \quad a_{*j} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix},$$

$$b_{ij} = \det \begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij} \end{pmatrix}, \quad B = (b_{ij})_{n+1\le i,j\le m}$$

Then

$$\det A \cdot (\det A_0)^{m-n-1} = \det B.$$

Proof: By continuity, it is enough to prove the identity in the case when A_0 is invertible. Write A in block form

$$A = \begin{pmatrix} A_0 & A_1 \\ A_2 & A_3 \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ A_2 & I \end{pmatrix} \begin{pmatrix} I & A_0^{-1}A_1 \\ 0 & A_3 - A_2A_0^{-1}A_1 \end{pmatrix},$$

by taking determinants we obtain

$$\det A = \det A_0 \cdot \det(A_3 - A_2A_0^{-1}A_1). \tag{5.1.1}$$

By applying the same formula to the matrix

$$\begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij} \end{pmatrix},$$

for $n < i, j \le m$, we get

$$b_{ij} = \det A_0 \cdot (a_{ij} - a_{i*} A_0^{-1} a_{*j}).$$

This implies

$$\det B = (\det A_0)^{m-n} \det(A_3 - A_2 A_0^{-1} A_1),$$

which is by (5.1.1) equal to

$$\det A \cdot (\det A_0)^{m-n-1}. \qquad \square$$

See [MG85], [AAM96] for other proofs and some mild generalizations.

Example 5.1.2 If we take $n = 1$ and $m = 3$, the Sylvester's identity says that

$$(a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13})a_{11} =$$

$$= \begin{vmatrix} a_{11}a_{22} - a_{21}a_{12} & a_{11}a_{23} - a_{21}a_{13} \\ a_{11}a_{32} - a_{31}a_{12} & a_{11}a_{33} - a_{31}a_{13} \end{vmatrix}. \qquad \diamondsuit$$

The Sylvester's identity has been intensely studied, mostly in algebraic rather than combinatorial contexts. In 1991, a generalization to quasideterminants, essentially equivalent to our Theorem 5.2.1, was found by Gelfand and Retakh [GR91]. Krob and Leclerc [KL95] used their result to prove the following quantum version.

Theorem 5.1.3 (Krob, Leclerc) *For a quantum matrix $A = (a_{ij})_{m \times m}$, take n, A_0, a_{i*} and a_{*j} as before, and define*

$$b_{ij} = \det_q \begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij} \end{pmatrix}, \quad B = (b_{ij})_{n+1 \le i,j \le m}.$$

Then

$$\det_q A \cdot (\det_q A_0)^{m-n-1} = \det_q B. \qquad \square$$

Krob and Leclerc's proof consists of an application of the so-called quantum Muir's law of extensible minors to the expansion of a minor.

Since then, Molev found several far-reaching extensions to Yangians, including other root systems [Mol02, Mol06]; see also [HM06].

In this chapter, we prove the following multiparameter right-quantum analogue of Sylvester's identity.

Theorem 5.1.4 (**q**-right-quantum Sylvester's determinant identity) *Suppose that $A = (a_{ij})_{m \times m}$ is a **q**-right-quantum matrix, and choose $n < m$. Let A_0, a_{i*}, a_{*j} be defined as above, and let*

$$c_{ij}^{\mathbf{q}} = -\det_{\mathbf{q}}^{-1}(I - A_0) \cdot \det_{\mathbf{q}} \begin{pmatrix} I - A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix}, \quad C^{\mathbf{q}} = (c_{ij}^{\mathbf{q}})_{n+1 \le i,j \le m}.$$

Suppose $q_{ij} = q_{i'j'}$ for all $i, i' \le n$ and $j, j' > n$. Then

$$\det_{\mathbf{q}}^{-1}(I - A_0) \cdot \det_{\mathbf{q}}(I - A) = \det_{\mathbf{q}}(I - C^{\mathbf{q}}).$$

The determinant $\det_{\mathbf{q}}(I - A_0)$ does not commute with other determinants in the definition of $c_{ij}^{\mathbf{q}}$, so the identity cannot be written in a form analogous to Theorem 5.1.1. See Remark 5.7.6 for a discussion of the necessity of the condition $q_{ij} = q_{i'j'}$ for $i, i' \le n$, $j, j' > n$.

The proof roughly follows the pattern of the proof of the main theorem in Chapter 4. First we show a combinatorial proof of the classical Sylvester's identity (Sections 5.2 and 5.3). Then we adapt the proof to simple non-commutative cases – the Cartier-Foata case (Section 5.4) and the right-quantum case (Section 5.5). We extend the results to cases with a weight (Section 5.6) and to multiparameter weighted cases (Section 5.7). We also present a β-extension of Sylvester's identity (Section 5.8).

5.2 Non-commutative Sylvester's identity

As in Section 5.1, choose $n < m$, and denote the matrix $(a_{ij})_{m\times m}$ by A and $(a_{ij})_{n\times n}$ by A_0.

We show a combinatorial proof of the non-commutative Sylvester's identity due to Gelfand and Retakh, see [GR91].

Theorem 5.2.1 (Gelfand-Retakh) *Consider the matrix $C = (c_{ij})_{n+1\le i,j\le m}$, where*

$$c_{ij} = a_{ij} + a_{i*}(I - A_0)^{-1}a_{*j}.$$

Then

$$(I - A)^{-1}_{ij} = (I - C)^{-1}_{ij}.$$

Proof. Take a lattice path $a_{ii_1}a_{i_1i_2}\cdots a_{i_{\ell-1}j}$ with $i, j > n$. Clearly it can be uniquely divided into paths $P_1, P_2, \ldots P_p$ with the following properties:

- the ending height of P_i is the starting height of P_{i+1}
- the starting and the ending heights of all P_i are strictly greater than n
- all intermediate heights are less than or equal to n

Next, note that

$$c_{ij} = a_{ij} + a_{i*}(I - A_0)^{-1}a_{*j} = a_{ij} + \sum_{k,l\le n} a_{ik}(I + A_0 + A_0^2 + \ldots)_{kl}a_{lj}$$

is the sum over all non-trivial paths with starting height i, ending height j, and intermediate heights $\le n$. This decomposition hence proves the theorem. □

Example 5.2.2 The following figure depicts the path from Example 1.2.4 with a dotted line between heights n and $n+1$, and the corresponding decomposition, for $n = 3$. $\diamondsuit$

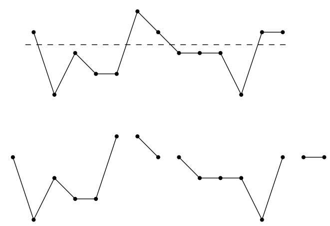

Figure 5-1: The decomposition $(a_{41}a_{13}a_{32}a_{22}a_{25})(a_{54})(a_{43}a_{33}a_{33}a_{31}a_{14})(a_{44})$.

The theorem implies that

$$(I-A)^{-1}_{n+1,n+1}(I-A^{n+1,n+1})^{-1}_{n+2,n+2}\cdots\left(I-\begin{pmatrix}A_0 & a_{*m}\\ a_{m*} & a_{mm}\end{pmatrix}\right)^{-1}_{mm} = \tag{5.2.1}$$

$$= (I-C)^{-1}_{n+1,n+1}(I-C^{n+1,n+1})^{-1}_{n+2,n+2}\cdots(1-c_{mm})^{-1}.$$

In all the cases we consider in the following sections, both the left-hand side and the right-hand side of this equation can be written in terms of determinants, as in the classical Sylvester's identity.

5.3 The commutative case

Recall that if D is an invertible matrix with commuting entries, we have

$$\left(D^{-1}\right)_{ij} = (-1)^{i+j}\frac{\det D^{ji}}{\det D},$$

where D^{ji} denotes the matrix D without the j-th row and the i-th column. Apply this to (5.2.1): the numerators (except the last one on the left-hand side) and denominators (except the first one on both sides) cancel each other, and we get

$$\frac{\det(I-A_0)}{\det(I-A)} = \frac{1}{\det(I-C)}. \tag{5.3.1}$$

Proposition 5.3.1 *For $i, j > n$ we have*

$$\delta_{ij} - c_{ij} = \frac{\det\begin{pmatrix}I-A_0 & -a_{*j}\\ -a_{i*} & \delta_{ij}-a_{ij}\end{pmatrix}}{\det(I-A_0)}. \tag{5.3.2}$$

Proof: Clearly we have

$$(1-c_{ij})^{-1} = \left(\left(I - \begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij} \end{pmatrix} \right)^{-1} \right)_{ij},$$

and by (5.3.1), this is equal to

$$\frac{\det(I-A_0)}{\det\left(I - \begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij} \end{pmatrix}\right)}.$$

This finishes the proof for $i = j$, and for $i \neq j$ we have

$$1 - c_{ij} = \frac{\det\begin{pmatrix} I-A_0 & -a_{*j} \\ -a_{i*} & 1-a_{ij} \end{pmatrix}}{\det(I-A_0)} = \frac{\det\begin{pmatrix} I-A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix} + \det\begin{pmatrix} I-A_0 & 0 \\ -a_{i*} & 1 \end{pmatrix}}{\det(I-A_0)} =$$

$$= \frac{\det\begin{pmatrix} I-A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix} + \det(I-A_0)}{\det(I-A_0)} = \frac{\det\begin{pmatrix} I-A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix}}{\det(I-A_0)} + 1. \qquad \square$$

Proof of Theorem 5.1.1. The proposition, together with (5.3.1), implies that

$$\frac{\det(I-A)}{\det(I-A_0)} = \det(I-C) = \det(I-A_0)^{n-m} \det B$$

for

$$b_{ij} = \det\begin{pmatrix} I-A_0 & -a_{*j} \\ -a_{i*} & \delta_{ij}-a_{ij} \end{pmatrix}, \quad B = (b_{ij})_{n+1 \leq i,j \leq m},$$

which is Theorem 5.1.1 for the matrix $I - A$. $\square$

While this proof is clearly more complicated than the one presented in Section 5.1, it has a fairly straightforward extension to non-commutative cases.

5.4 The Cartier-Foata case

In this section, we extend Sylvester's identity to Cartier-Foata matrices. The crucial step is the following lemma.

Lemma 5.4.1 *If* $A = (a_{ij})_{m\times m}$ *is a Cartier-Foata matrix, then* $C = (c_{ij})_{n+1\leq i,j\leq m}$, *where*

$$c_{ij} = a_{ij} + a_{i*}(I-A_0)^{-1}a_{*j},$$

is a right-quantum matrix.

Proof. Choose $i, j, k > n$, $i \neq j$. The product $c_{ik}c_{jk}$ is the sum of terms of the form

$$a_{ii_1}a_{i_1i_2}\cdots a_{i_pk}a_{jj_1}a_{j_1j_2}\cdots a_{j_rk}$$

for $p, r \geq 0$, $i_1, \ldots, i_p, j_1, \ldots, j_r \leq n$. Note that with the (possible) exception of i, j, k, all other terms appear as starting heights exactly as many times as they appear as ending heights.

Identify this term with a sequence of steps, as described in Section 1.2. We will perform a series of switches of steps that will transform such a term into a term of $c_{jk}c_{ik}$.

The variable a_{jj_1} (or a_{jk} if $r = 0$) commutes with all variables that appear before it. In other words, in the algebra $\mathcal{A}$, the expressions $a_{ii_1}a_{i_1i_2}\cdots a_{i_pk}a_{jj_1}a_{j_1j_2}\cdots a_{j_rk}$ and $a_{jj_1}a_{ii_1}a_{i_1i_2}\cdots a_{i_pk}a_{j_1j_2}\cdots a_{j_rk}$ are the same modulo the ideal $\mathcal{I}_{\mathrm{cf}}$ generated by $a_{ik}a_{jl} - a_{jl}a_{ik}$ for $i \neq j$. Graphically, we can keep switching the step $j \to j_1$ with the step to its left until it is at the beginning of the sequence.

If $r = 0$, we are already done. If not, take the first step to the right of a_{jj_1} that has starting height j_1; such a step certainly exists – for example $j_1 \to j_2$. Without changing the expression modulo $\mathcal{I}_{\mathrm{cf}}$, we can switch this step with the ones to the left until it is just right of $j \to j_1$. Continue this procedure; eventually, our sequence is transformed into an expression of the form

$$a_{jj'_1}a_{j'_1j'_2}\cdots a_{j'_{r'}k}a_{ii'_1}a_{i'_1i'_2}\cdots a_{i'_{p'}k}$$

which is equal modulo $\mathcal{I}_{\mathrm{cf}}$ to the expression we started with.

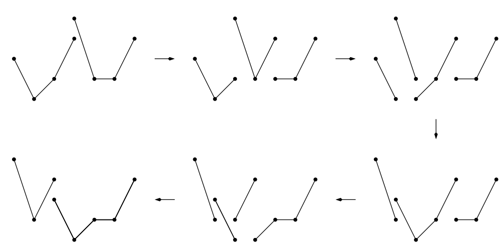

Figure 5-2: Transforming $a_{31}a_{12}a_{24}a_{52}a_{22}a_{24}$ into $a_{52}a_{24}a_{31}a_{12}a_{22}a_{24}$.

As an example, take $m = 5$, $n = 2$, $i = 3$, $j = 5$, $k = 4$ and $a_{31}a_{12}a_{24}a_{52}a_{22}a_{24}$. The steps shown in Figure 5-2 transform it into $a_{52}a_{24}a_{31}a_{12}a_{22}a_{24}$.

It is clear that applying the same procedure to the result, but with the roles of i's and j's interchanged, gives the original sequence. This proves that indeed $c_{ik}c_{jk} = c_{jk}c_{ik}$.

The proof of the other relation (1.3.3) is similar and we only sketch it. Choose $i, j, k, l > n$, $i \neq j$, $k \neq l$. Then $c_{ik}c_{jl} + c_{il}c_{jk}$ is the sum of terms of the form

$$a_{ii_1}a_{i_1i_2}\cdots a_{i_pk}a_{jj_1}a_{j_1j_2}\cdots a_{j_rl}$$

and of the form

$$a_{ii_1}a_{i_1i_2}\cdots a_{i_pl}a_{jj_1}a_{j_1j_2}\cdots a_{j_rk}$$

for $p, r \geq 0$, $i_1, \ldots, i_p, j_1, \ldots, j_r \leq n$. Applying the same procedure as above to the first term yields either

$$a_{jj_1'} a_{j_1'j_2'} \cdots a_{j_{r'}'k} a_{ii_1'} a_{i_1'i_2'} \cdots a_{i_{p'}'l}$$

or

$$a_{jj_1'} a_{j_1'j_2'} \cdots a_{j_{r'}'l} a_{ii_1'} a_{i_1'i_2'} \cdots a_{i_{p'}'k},$$

this procedure is reversible and it yields the desired identity.

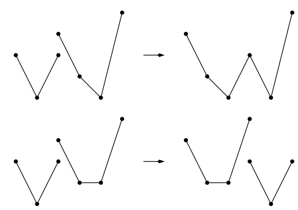

Figure 5-3: Transforming $a_{31}a_{13}a_{42}a_{21}a_{15}$ and $a_{31}a_{13}a_{42}a_{22}a_{25}$.

See Figure 5-3 for examples with $m = 5$, $n = 2$, $i = 3$, $j = 4$, $k = 3$, $l = 5$. □

Remark 5.4.2 It is of interest to note that the matrix C is not necessarily Cartier-Foata when A is Cartier-Foata. For example, take $m = 5$ and $n = 1$. the sum of the terms of type $(2, 1, 1, 0, 0; 2, 0, 0, 1, 1)$ in $c_{24}c_{35}$ is

$$(a_{24})(a_{31}a_{11}a_{15}) + (a_{21}a_{14})(a_{31}a_{15}) + (a_{21}a_{11}a_{14})(a_{35}) =$$

$$= a_{11}a_{15}a_{24}a_{31} + a_{14}a_{15}a_{21}a_{31} + a_{11}a_{14}a_{21}a_{35},$$

while the sum of the terms of the same type in $c_{35}c_{24}$ is

$$(a_{35})(a_{21}a_{11}a_{14}) + (a_{31}a_{15})(a_{21}a_{14}) + (a_{31}a_{11}a_{15})(a_{24}) =$$

$$= a_{11}a_{14}a_{21}a_{35} + a_{15}a_{14}a_{21}a_{31} + a_{11}a_{15}a_{24}a_{31}. \qquad \diamondsuit$$

If A is Cartier-Foata, the matrix inverse formula (Theorem 3.1.1) implies

$$(I - A)^{-1}_{n+1,n+1} (I - A^{n+1,n+1})^{-1}_{n+2,n+2} \cdots = \det{}^{-1}(I - A) \cdot \det(I - A_0).$$

By Lemma 5.4.1, C is right-quantum, so by Theorem 3.1.2

$$(I - C)^{-1}_{n+1,n+1} (I - C^{n+1,n+1})^{-1}_{n+2,n+2} \cdots = \det{}^{-1}(I - C),$$

and hence

$$\det{}^{-1}(I - A_0) \cdot \det(I - A) = \det(I - C).$$

In the classical Sylvester's identity, the entries of $I - C$ are also expressed as determinants. The following is an analogue of Proposition 5.3.1.

Proposition 5.4.3 *If A is Cartier-Foata, then*

$$c_{ij} = -\det{}^{-1}(I - A_0) \cdot \det \begin{pmatrix} I - A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix}. \tag{5.4.1}$$

Proof: We can repeat the proof of Proposition 5.3.1 almost verbatim. We have

$$(1 - c_{ij})^{-1} = \left(\left(I - \begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij} \end{pmatrix} \right)^{-1} \right)_{ij},$$

and because the matrix

$$\begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij} \end{pmatrix}$$

is still Cartier-Foata, Theorem 3.1.1 shows that this is equal to

$$\det{}^{-1} \left(I - \begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij} \end{pmatrix} \right) \cdot \det(I - A_0).$$

We get

$$1 - c_{ij} = \det{}^{-1}(I - A_0) \cdot \det \left(I - \begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij} \end{pmatrix} \right) =$$

$$= \det{}^{-1}(I - A_0) \cdot \left(\det \begin{pmatrix} I - A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix} + \det \begin{pmatrix} I - A_0 & 0 \\ -a_{i*} & 1 \end{pmatrix} \right) =$$

$$= \det{}^{-1}(I - A_0) \cdot \left(\det \begin{pmatrix} I - A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix} + \det(I - A_0) \right) =$$

$$= \det{}^{-1}(I - A_0) \cdot \det \begin{pmatrix} I - A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix} + 1. \qquad \square$$

We have proved the following.

Theorem 5.4.4 (Cartier-Foata Sylvester's identity) *Let $A = (a_{ij})_{m \times m}$ be a Cartier-Foata matrix, and choose $n < m$. Let A_0, a_{i*}, a_{*j} be defined as above, and let*

$$c_{ij} = -\det{}^{-1}(I - A_0) \cdot \det \begin{pmatrix} I - A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix}, \quad C = (c_{ij})_{n+1 \leq i,j \leq m}.$$

Then

$$\det{}^{-1}(I - A_0) \cdot \det(I - A) = \det(I - C). \qquad \square$$

5.5 The right-quantum case

The right-quantum version of the Sylvester's identity is very similar; we prove a right-quantum version of Lemma 5.4.1 and Proposition 5.4.3, and a right-quantum version of Theorem 5.4.4 follows.

Lemma 5.5.1 *If $A = (a_{ij})_{m\times m}$ is a right-quantum matrix, so is $C = (c_{ij})_{n+1\le i,j\le m}$, where*

$$c_{ij} = a_{ij} + a_{i*}(I - A_0)^{-1}a_{*j}.$$

Proof. Choose $i, j, k > n$, $i \neq j$. Instead of dealing directly with the equality $c_{ik}c_{jk} = c_{jk}c_{ik}$, we prove an equivalent identity.
Denote by $\mathbf{O}_{ij}^k(\mathbf{r})$ the set $\mathbf{O}(\mathbf{r} + \varepsilon_i + \varepsilon_j; \mathbf{r} + 2\varepsilon_k)$, where ε_l denotes the vector with 1 in l-th entry and zeros elsewhere. In other words, $\mathbf{O}_{ij}^k(\mathbf{r})$ is the set of sequences of $r_1 + \ldots + r_n + 2$ steps with the following properties:

- starting heights form a non-decreasing sequence;
- each s between 1 and n appears exactly r_s times as a starting height and exactly r_s times as an ending height;
- i and j appear exactly once as starting heights;
- k appears exactly twice as an ending height.

For $m = 5, n = 2, i = 3, j = 5, k = 4, r_1 = 1, r_2 = 1$, all such sequences are shown in Figure 5-4.

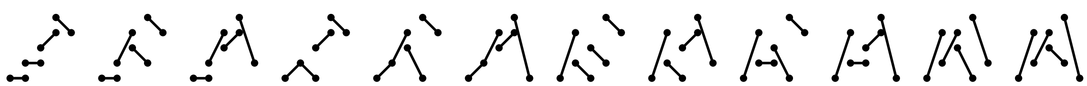

Figure 5-4: Sequences in the set $\mathbf{O}_{35}^4(1,1)$.

Now we use the relation (2.2.3) (with $\pi = \sigma = \mathrm{id}$). It tells us that modulo the ideal $\mathcal{I}_{\mathrm{rq}}$, the sum of all sequences of $\mathbf{O}_{ij}^k(\mathbf{r})$ is equal to the sum of all sequences of the form $P_1P_2P_3$, where:

- P_1 is a path from i to k with all intermediate heights $\le n$;
- P_2 is a path from j to k with all intermediate heights $\le n$;
- P_3 is a sequence of steps with non-decreasing heights, with all heights $\le n$, and with the number of steps with starting height s equal to the number of steps with ending height s for all s.

For example, the sequence $a_{11}a_{24}a_{34}a_{52}$ is transformed into $a_{34}a_{52}a_{24}a_{11}$, see Figure 5-5.

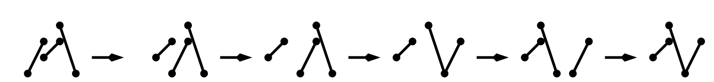

Figure 5-5: Transforming $a_{11}a_{24}a_{34}a_{52}$ into $a_{34}a_{52}a_{24}a_{11}$.

This means that the sum of all elements of $\mathbf{O}_{ij}^k(\mathbf{r})$ over all $\mathbf{r} \geq \mathbf{0}$ is modulo $\mathcal{I}_{\mathrm{rq}}$ equal to

$$c_{ik}c_{jk}S,$$

where S is the sum over all sequences of steps with the following properties:

- starting heights form a non-decreasing sequence;
- starting and ending heights are all between 1 and n;
- each s between 1 and n appears as many times as a starting height as an ending height.

Of course, we can also reverse the roles of i and j, and this proves that the sum of all elements of $\mathbf{O}_{ij}^k(\mathbf{r})$ is modulo $\mathcal{I}_{\mathrm{rq}}$ also equal to

$$c_{jk}c_{ik}S.$$

Hence, modulo $\mathcal{I}_{\mathrm{rq}}$,

$$c_{ik}c_{jk}S = c_{jk}c_{ik}S. \tag{5.5.1}$$

But $S = 1 + a_{11} + \ldots + a_{nn} + a_{11}a_{22} + a_{12}a_{21} + \ldots$ is an *invertible* element of $\mathcal{A}$ – and actually it is equal to $1/\det(I - A_0)$ modulo $\mathcal{I}_{\mathrm{rq}}$ by the right-quantum master theorem – so (5.5.1) implies

$$c_{ik}c_{jk} = c_{jk}c_{ik},$$

provided A is a right-quantum matrix.
The proof of the other relation is almost completely analogous. Now we take $i \neq j$, $k \neq l$, and define $\mathbf{O}_{ij}^{kl}(\mathbf{r})$ to be $\mathbf{O}(\mathbf{r} + \varepsilon_i + \varepsilon_j; \mathbf{r} + \varepsilon_k + \varepsilon_l)$, i.e. the set of sequences of $r_1 + \ldots + r_n + 2$ steps with the following properties:

- starting heights form a non-decreasing sequence;
- each s between 1 and n appears exactly r_s times as a starting height and exactly r_s times as an ending height;
- i and j appear exactly once as starting heights;
- k and l appear exactly once as ending heights.

A similar reasoning shows that the sum over all elements of $\mathbf{O}_{ij}^{kl}(\mathbf{r})$ is equal both to $(c_{ik}c_{jl} + c_{il}c_{jk})S$ and to $(c_{jl}c_{ik} + c_{jk}c_{il})S$ modulo $\mathcal{I}_{\mathrm{rq}}$, which implies $c_{ik}c_{jl} + c_{il}c_{jk} = c_{jl}c_{ik} + c_{jk}c_{il}$. □

Note that the method of proof is very similar to the proof of Proposition 3.7.1.

Proposition 5.5.2 *If A is right-quantum, then*

$$c_{ij} = -\det{}^{-1}(I - A_0) \cdot \det \begin{pmatrix} I - A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix}. \tag{5.5.2}$$

Proof: The proof is exactly the same as the proof of Proposition 5.4.3. □

Theorem 5.5.3 (right-quantum Sylvester's identity) *Let $A = (a_{ij})_{m\times m}$ be a right-quantum matrix, and choose $n < m$. Let A_0, a_{i*}, a_{*j} be defined as above, and let*

$$c_{ij} = -\det{}^{-1}(I - A_0)\cdot\det\begin{pmatrix} I - A_0 & -a_{*j} \\ -a_{i*} & -a_{ij}\end{pmatrix}, \quad C = (c_{ij})_{n+1\le i,j\le m}.$$

Then

$$\det{}^{-1}(I - A_0)\cdot\det(I - A) = \det(I - C).$$ □

5.6 Weighted cases

In this section, we extend the results of the previous two sections to weighted cases.

Suppose that the matrix $A = (a_{ij})_{m\times m}$ is q-Cartier-Foata. We will use Theorem 5.2.1 for the matrix

$$A_{[ij]} = \begin{pmatrix} q^{-1}a_{11} & \cdots & q^{-1}a_{1j} & a_{1,j+1} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ q^{-1}a_{i-1,1} & \cdots & q^{-1}a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,m} \\ a_{i1} & \cdots & a_{ij} & qa_{i,j+1} & \cdots & qa_{i,m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & qa_{m,j+1} & \cdots & qa_{mm}\end{pmatrix}.$$

Let us find the corresponding $C = (c'_{i'j'})_{n+1\le i',j'\le m}$. Denote

$$a_{i'j'} + q^{-1}a_{i'*}(I - q^{-1}A_0)^{-1}a_{*j'}$$

by $c_{i'j'}$ for $i', j' > n$. If $i' < i, j' \le j$, we have

$$c'_{i'j'} = q^{-1}a_{i'j'} + (q^{-1}a_{i'*})(I - q^{-1}A_0)^{-1}(q^{-1}a_{*j'}) = q^{-1}c_{i'j'};$$

if $i' < i, j' > j$, we have

$$c'_{i'j'} = a_{i'j'} + (q^{-1}a_{i'*})(I - q^{-1}A_0)^{-1}a_{*j'} = c_{i'j'};$$

if $i' \ge i, j' \le j$, we have

$$c'_{i'j'} = a_{i'j'} + a_{i'*}(I - q^{-1}A_0)^{-1}(q^{-1}a_{*j'}) = c_{i'j'};$$

and if $i' \ge i, j' > j$, we have

$$c'_{i'j'} = qa_{i'j'} + a_{i'*}(I - q^{-1}A_0)^{-1}a_{*j'} = qc_{i'j'}.$$

We have proved the following.

Proposition 5.6.1 *With* $A_{[ij]}$ *as above and with* $C = (c_{i'j'})_{n+1 \leq i',j' \leq m}$ *for*

$$c_{i'j'} = a_{i'j'} + a_{i'*}(I - q^{-1}A_0)^{-1}(q^{-1}a_{*j'}),$$

we have

$$(I - A_{[ij]})^{-1}_{i'j'} = (I - C_{[ij]})^{-1}_{i'j'}. \qquad \square$$

REMARK 5.6.2 Let us present a slightly different proof of the proposition. Another way to characterize $A_{[ij]}$ is to say that the entry a_{kl} has weight q to the power of

$$\begin{cases} 1 \colon l > j \\ 0 \colon l \leq j \end{cases} - \begin{cases} 1 \colon k < i \\ 0 \colon k \geq i \end{cases}.$$

That means that in $\left(A^{\ell}_{[ij]}\right)_{i_1 i_\ell}$, $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{\ell-1} i_\ell}$ has weight $q^{|\{r \colon i_r > j\}| - |\{r \colon i_r < i\}|}$. Assume that we have a decomposition of a path of length ℓ from i' to j', $i', j' > n$, as in Section 5.2, say $a_{\lambda,\mu} = a_{i'\lambda_1,\lambda_1 i_1} a_{i_1\lambda_2,\lambda_2 i_2} \cdots a_{i_{p-1}\lambda_p,\lambda_p j'}$, with all elements of λ_r at most n, $i_r > n$, and the length of λ_r equal to ℓ_r. Put $i_0 = i'$, $i_{p+1} = j'$.The number of indices of $\lambda = i'\lambda_1 \ldots \lambda_p$ that are strictly smaller than i is clearly $\sum_{r=1}^{p} \ell_r + |\{r \colon i_r < i\}| = \ell - p + |\{r \colon i_r < i\}|$, and the number of indices of $\mu = \lambda_1 \ldots \lambda_p j'$ that are strictly greater than j is $|\{r \colon i_r > j\}|$. Therefore the path $a_{\lambda,\mu}$ is weighted by $q^{-\ell+p+|\{r \colon i_r > j\}| - |\{r \colon i_r < i\}|}$. On the other hand, take a term $a_{\lambda,\mu} = a_{i'\lambda_1,\lambda_1 i_1} a_{i_1\lambda_2,\lambda_2 i_2} \cdots a_{i_{p-1}\lambda_p,\lambda_p j'}$ (with λ_r, i_r, ℓ_r as before) of $(C^{\ell}_{[ij]})_{i'j'}$. Each $a_{i_{r-1}\lambda_r,\lambda_r i_r}$ has weight $q^{-\ell_r}$ as an element of C, and $a_{\lambda,\mu}$ has the additional weight $q^{|\{r \colon i_r > j\}| - |\{r \colon i_r < i\}|}$ as a term of $(C^{\ell}_{[ij]})_{i'j'}$. The proposition follows. $\diamondsuit$

Lemma 5.6.3 *If* $A = (a_{ij})_{m \times m}$ *is a q-Cartier-Foata matrix, then* $C = (c_{ij})_{n+1 \leq i,j \leq m}$*, where*

$$c_{ij} = a_{ij} + a_{i*}(I - A_0)^{-1} a_{*j},$$

is a q-right-quantum matrix.

Proof. We adapt the proof of Lemma 5.4.1. Choose $i, j, k > n$, $i < j$. The product $c_{ik} c_{jk}$ is the sum of terms of the form

$$q^{-p-r} a_{ii_1} a_{i_1 i_2} \cdots a_{i_p k} a_{jj_1} a_{j_1 j_2} \cdots a_{j_r k}$$

for $p, r \geq 0$, $i_1, \ldots, i_p, j_1, \ldots, j_r \leq n$. Without changing the expression modulo $\mathcal{I}_{q-\mathrm{cf}}$, we can repeat the procedure in the proof of Lemma 5.4.1, keeping track of weight changes. The resulting expression

$$a_{jj'_1} a_{j'_1 j'_2} \cdots a_{j'_{r'} k} a_{ii'_1} a_{i'_1 i'_2} \cdots a_{i'_{p'} k}$$

has, by the discussion preceding the lemma, weight $q^{-1-r'-p'}$ (the extra -1 comes from the fact that the step with starting height j is now to the left of the step with starting height i). In other words,

$$c_{jk} c_{ik} = q c_{ik} c_{jk}.$$

The proof of the other relation is completely analogous. □

If A is q-Cartier-Foata, Theorem 3.1.3 implies

$$(I-A_{[n+1,n+1]})^{-1}_{n+1,n+1}(I-\left(A^{n+1,n+1}\right)_{[n+2,n+2]})^{-1}_{n+2,n+2}\cdots = \mathrm{det}_q{}^{-1}(I-A)\cdot\mathrm{det}_q(I-A_0).$$

By Lemma 5.6.3, C is q-right-quantum, so by Theorem 3.1.3

$$(I-C_{[n+1,n+1]})^{-1}_{n+1,n+1}(I-\left(C^{n+1,n+1}\right)_{[n+2,n+2]})^{-1}_{n+2,n+2}\cdots = \mathrm{det}_q{}^{-1}(I-C),$$

and hence

$$\mathrm{det}_q{}^{-1}(I-A_0)\cdot\mathrm{det}_q(I-A)=\mathrm{det}_q(I-C).$$

The final step is to write entries of C as quotients of q-determinants.

Proposition 5.6.4 *If A is q-Cartier-Foata, then*

$$c_{ij}=-\mathrm{det}_q{}^{-1}(I-A_0)\cdot\mathrm{det}_q\begin{pmatrix} I-A_0 & -a_{*j} \\ -a_{i*} & -a_{ij}\end{pmatrix}.$$

Proof. Again,

$$(1-c_{ij})^{-1}=\left(\left(I-\begin{pmatrix} q^{-1}A_0 & q^{-1}a_{*j} \\ a_{i*} & a_{ij}\end{pmatrix}\right)^{-1}\right)_{ij},$$

and because the matrix

$$\begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij}\end{pmatrix}$$

is still q-Cartier-Foata, Theorem 3.1.3 shows that this is equal to

$$\mathrm{det}_q{}^{-1}\left(I-\begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij}\end{pmatrix}\right)\cdot\mathrm{det}_q(I-A_0).$$

The rest of the proof is exactly the same as in Proposition 5.4.3, with det_q playing the role of det. □

We have proved the following.

Theorem 5.6.5 (q-Cartier-Foata Sylvester's identity) *Let $A=(a_{ij})_{m\times m}$ be a q-Cartier-Foata matrix, and choose $n<m$. Let A_0, a_{i*}, a_{*j} be defined as above, and let*

$$c^q_{ij}=-\mathrm{det}_q{}^{-1}(I-A_0)\cdot\mathrm{det}_q\begin{pmatrix} I-A_0 & -a_{*j} \\ -a_{i*} & -a_{ij}\end{pmatrix},\quad C^q=(c^q_{ij})_{n+1\le i,j\le m}.$$

Then

$$\mathrm{det}_q{}^{-1}(I-A_0)\cdot\mathrm{det}_q(I-A)=\mathrm{det}_q(I-C^q).$$ □

The results easily extend to a q-right-quantum Sylvester's identity.

Lemma 5.6.6 *If $A = (a_{ij})_{m\times m}$ is a q-right-quantum matrix, so is $C = (c_{ij})_{n+1\le i,j\le m}$, where*

$$c_{ij} = a_{ij} + a_{i*}(I - A_0)^{-1}a_{*j}.$$

Proof. This is a weighted analogue of Lemma 5.5.1. By equation (2.4.3), the sum over all elements of $\mathbf{O}_{ij}^k(\mathbf{r})$ with $a_{\lambda,\mu}$ weighted by $q^{\mathrm{inv}(\mu)-\mathrm{inv}(\lambda)} = q^{\mathrm{inv}(\mu)}$, is modulo $\mathcal{I}_{q-\mathrm{rq}}$ equal to both $c_{ik}c_{jk}S$ and $q^{-1}c_{jk}c_{ik}S$; this implies the relation (1.3.8) for elements of C, and the proof of (1.3.8) is completely analogous. □

Proposition 5.6.7 *If A is q-right-quantum, then*

$$c_{ij} = -{\det}_q^{-1}(I - A_0)\cdot{\det}_q\begin{pmatrix} I - A_0 & -a_{*j} \\ -a_{i*} & -a_{ij}\end{pmatrix}.$$

Proof. The proof is exactly the same as the proof of Proposition 5.6.4. □

Proposition 5.6.1, Lemma 5.6.6 and Proposition 5.6.7 imply the following theorem.

Theorem 5.6.8 (q-right-quantum Sylvester's identity) *Let $A = (a_{ij})_{m\times m}$ be a q-right-quantum matrix, and choose $n < m$. Let A_0, a_{i*}, a_{*j} be defined as above, and let*

$$c_{ij}^q = -{\det}_q^{-1}(I - A_0)\cdot{\det}_q\begin{pmatrix} I - A_0 & -a_{*j} \\ -a_{i*} & -a_{ij}\end{pmatrix}, \quad C^q = (c_{ij}^q)_{n+1\le i,j\le m}.$$

Then

$${\det}_q^{-1}(I - A_0)\cdot{\det}_q(I - A) = {\det}_q(I - C^q).$$ □

5.7 Multiparameter cases

In this section, we prove a multiparameter extension of Theorems 5.6.5 and 5.6.8. We will use Theorem 5.2.1 for the matrix

$$A_{[ij]} = \begin{pmatrix} q_{1i}^{-1}a_{11} & \cdots & q_{1i}^{-1}a_{1j} & q_{1i}^{-1}q_{j,j+1}a_{1,j+1} & \cdots & q_{1i}^{-1}q_{jm}a_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ q_{i-1,i}^{-1}a_{i-1,1} & \cdots & q_{i-1,i}^{-1}a_{i-1,j} & q_{i-1,i}^{-1}q_{j,j+1}a_{i-1,j+1} & \cdots & q_{i-1,i}^{-1}q_{jm}a_{i-1,m} \\ a_{i1} & \cdots & a_{ij} & q_{j,j+1}a_{i,j+1} & \cdots & q_{jm}a_{i,m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & q_{j,j+1}a_{m,j+1} & \cdots & q_{jm}a_{mm} \end{pmatrix}.$$

Assume that $q_{ij} = q_{i'j'}$ for $i, i' \le n$, $j, j' > n$; denote this value by q. We will use Theorem 5.2.1 for the matrix $A_{[ij]}$ and the corresponding $C = (c'_{i'j'})_{n+1\le i',j'\le m}$. Define

$$c_{i'j'} = a_{i'j'} + q^{-1}a_{i'*}(I - q^{-1}A_0)^{-1}a_{*j'}$$

for $i', j' > n$. If $i' < i, j' \le j$, we have

$$c'_{i'j'} = q_{i'i}^{-1}a_{i'j'} + (q_{i'i}^{-1}a_{i'*})(I - q^{-1}A_0)^{-1}(q^{-1}a_{*j'}) = q_{i'i}^{-1}c_{i'j'};$$

if $i' < i, j' > j$, we have

$$c'_{i'j'} = q_{i'i}^{-1} q_{jj'} a_{i'j'} + (q_{i'i}^{-1} a_{i'*})(I - q^{-1}A_0)^{-1}(q^{-1} q_{jj'} a_{*j'}) = q_{i'i}^{-1} q_{jj'} c_{i'j'};$$

if $i' \geq i, j' \leq j$, we have

$$c'_{i'j'} = a_{i'j'} + a_{i'*}(I - q^{-1}A_0)^{-1}(q^{-1} a_{*j'}) = c_{i'j'};$$

and if $i' \geq i, j' > j$, we have

$$c'_{i'j'} = q_{jj'} a_{i'j'} + a_{i'*}(I - q^{-1}A_0)^{-1}(q^{-1} q_{jj'} a_{*j'}) = q_{jj'} c_{i'j'}.$$

We have proved the following.

Proposition 5.7.1 *With $A_{[ij]}$ as defined above and with $C = (c_{i'j'})_{n+1 \leq i',j' \leq m}$ for*

$$c_{i'j'} = a_{i'j'} + a_{i'*}(I - q^{-1}A_0)^{-1}(q^{-1} a_{*j'}),$$

we have

$$(I - A_{[ij]})^{-1}_{i'j'} = (I - C_{[ij]})^{-1}_{i'j'}. \qquad \square$$

Remark 5.7.2 Another way to characterize $A_{[ij]}$ is to say that the entry a_{kl} has weight

$$\begin{cases} q_{jl} \colon l > j \\ 1 \colon l \leq j \end{cases} \cdot \begin{cases} q_{ki}^{-1} \colon k < i \\ 1 \colon k \geq i \end{cases}.$$

That means that in $\left(A_{[ij]}^{\ell}\right)_{i_1 i_\ell}$, $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{\ell-1} i_\ell}$ has weight

$$\prod_{i_r > j} q_{j i_r} \cdot \prod_{i_r < i} q_{i_r i}^{-1}.$$

An alternative way to prove the proposition is analogous to the proof of Proposition 5.6.1 outlined in Remark 5.6.2. $\diamondsuit$

Lemma 5.7.3 *If $A = (a_{ij})_{m \times m}$ is a $\mathbf{q}$-Cartier-Foata matrix, then $C = (c_{ij})_{n+1 \leq i,j \leq m}$, where*

$$c_{ij} = a_{ij} + a_{i*}(I - A_0)^{-1} a_{*j}.$$

is a $\mathbf{q}$-right-quantum matrix.

Proof. We adapt the proof of Lemma 5.6.3. Choose $i, j, k > n$, $i < j$. The product $c_{ik} c_{jk}$ is the sum of terms of the form

$$q^{-p-r} a_{ii_1} a_{i_1 i_2} \cdots a_{i_p k} a_{jj_1} a_{j_1 j_2} \cdots a_{j_r k}$$

for $p, r \geq 0$, $i_1, \ldots, i_p, j_1, \ldots, j_r \leq n$.
Note that since

$$q^{-p-r} = q_{j_1 k} \cdots q_{j_r k} q_{i_1 i}^{-1} \cdots q_{i_p i}^{-1} q_{j_1 i}^{-1} \cdots q_{j_r i}^{-1} q_{j_1 j}^{-1} \cdots q_{j_r j}^{-1},$$

the weight of $a_{ii_1}a_{i_1i_2}\cdots a_{i_pk}a_{jj_1}a_{j_1j_2}\cdots a_{j_rk}$ is of the form

$$\prod_{(i,j)\in\mathcal{I}(\mu)} q_{\mu_j\mu_i} \prod_{(i,j)\in\mathcal{I}(\lambda)} q^{-1}_{\lambda_j\lambda_i}$$

for $\lambda = ii_1\dots i_pjj_1\dots j_r$ and $\mu = i_1\dots i_pkj_1\dots j_rk$. Without changing the expression modulo $\mathcal{I}_{\mathbf{q}-\mathrm{cf}}$, we can repeat the procedure in the proof of Lemma 5.4.1, but changing the weight at each switch. The resulting expression

$$a_{jj'_1}a_{j'_1j'_2}\cdots a_{j'_{r'}k}a_{ii'_1}a_{i'_1i'_2}\cdots a_{i'_{p'}k}$$

has, by the discussion preceding the lemma, weight

$$q_{i'_1k}\cdots q_{i'_{p'}k}q^{-1}_{j'_1j}\cdots q^{-1}_{j'_{r'}j}q^{-1}_{i'_1j}\cdots q^{-1}_{i'_{p'}j}q^{-1}_{i'_1i}\cdots q^{-1}_{i'_{p'}i}q^{-1}_{ij} = q^{-r'-p'}q^{-1}_{ij}$$

(the extra q^{-1}_{ij} comes from the fact that the step with starting height j is now to the left of the step with starting height i), In other words,

$$c_{jk}c_{ik} = q_{ij}c_{ik}c_{jk}.$$

The proof of the other relation is completely analogous. □

If A is $\mathbf{q}$-Cartier-Foata, Theorem 3.1.4 implies

$$(I-A_{[n+1,n+1]})^{-1}_{n+1,n+1}(I-\left(A^{n+1,n+1}\right)_{[n+2,n+2]})^{-1}_{n+2,n+2}\cdots = \det{}^{-1}_{\mathbf{q}}(I-A)\cdot\det{}_{\mathbf{q}}(I-A_0).$$

By Lemma 5.6.3, C is $\mathbf{q}$-right-quantum, so by Theorem 3.1.4

$$(I-C_{[n+1,n+1]})^{-1}_{n+1,n+1}(I-\left(C^{n+1,n+1}\right)_{[n+2,n+2]})^{-1}_{n+2,n+2}\cdots = \det{}^{-1}_{\mathbf{q}}(I-C),$$

and hence

$$\det{}^{-1}_{\mathbf{q}}(I-A_0)\cdot\det{}_{\mathbf{q}}(I-A) = \det{}_{\mathbf{q}}(I-C).$$

So far, the extension to the multiparameter case has been straightforward. However, we need something extra for the proof of the analogue of Proposition 5.6.4 since the matrix

$$\begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij}\end{pmatrix}$$

is in general not $\mathbf{q}$-Cartier-Foata – only the variables in the first n columns satisfy the relation (1.3.19).

However, by Remark 3.6.1, we still have

$$(1-c_{ij})^{-1} = \left(\left(I-\begin{pmatrix} q^{-1}A_0 & q^{-1}a_{*j} \\ a_{i*} & a_{ij}\end{pmatrix}\right)^{-1}\right)_{ij} =$$

$$= \det_{\mathbf{q}}^{-1}\left(I - \begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij} \end{pmatrix}\right) \cdot \det_{\mathbf{q}}(I - A_0).$$

Proposition 5.7.4 *If A is $\mathbf{q}$-Cartier-Foata, then*

$$c_{ij} = -\det_{\mathbf{q}}^{-1}(I - A_0) \cdot \det_{\mathbf{q}} \begin{pmatrix} I - A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix}.$$

Proof: This follows from the previous proposition, using the same technique as in the proof of Proposition 5.4.3. □

We have proved the following.

Theorem 5.7.5 ($\mathbf{q}$-Cartier-Foata Sylvester's theorem) *Let $A = (a_{ij})_{m\times m}$ be a $\mathbf{q}$-Cartier-Foata matrix, and choose $n < m$. Let A_0, a_{i*}, a_{*j} be defined as above, and let*

$$c_{ij}^{\mathbf{q}} = -\det_{\mathbf{q}}^{-1}(I - A_0) \cdot \det_{\mathbf{q}} \begin{pmatrix} I - A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix}, \quad C^{\mathbf{q}} = (c_{ij}^{\mathbf{q}})_{n+1\le i,j\le m}.$$

Suppose $q_{ij} = q_{i'j'}$ for all $i, i' \le n$ and $j, j' > n$. Then

$$\det_{\mathbf{q}}^{-1}(I - A_0) \cdot \det_{\mathbf{q}}(I - A) = \det_{\mathbf{q}}(I - C^{\mathbf{q}}).$$

Remark 5.7.6 It is important to note that the determinant $\det_{\mathbf{q}}(I - C^{\mathbf{q}})$ is with respect to $\mathcal{C}$, the algebra generated by c_{ij}'s, not with respect to $\mathcal{A}$. For example, for $n = 2$ and $m = 4$, we have

$$\det_{\mathbf{q}}(I - C^{\mathbf{q}}) = 1 - c_{33}^{\mathbf{q}} - c_{44}^{\mathbf{q}} + c_{33}^{\mathbf{q}} c_{44}^{\mathbf{q}} - q_{34}^{-1} c_{43}^{\mathbf{q}} c_{34}^{\mathbf{q}}.$$

The condition $q_{ij} = q_{i'j'}$ whenever $i, i' \le n$, $j, j' > n$ is indeed necessary, as shown by the following. Take $n = 1$ and $m = 3$. In $\det_{\mathbf{q}}^{-1}(I - A_0) \cdot \det_{\mathbf{q}}(I - A)$ we have the term

$$-q_{12}^{-1} q_{13}^{-1} a_{21} a_{32} a_{13},$$

while in $\det_{\mathbf{q}}(I - C^{\mathbf{q}})$ we have

$$-q_{23}^{-1}(-a_{32})(-q_{12}^{-1} a_{21} a_{13}) = -q_{12}^{-2} a_{21} a_{32} a_{13}. \qquad \diamondsuit$$

Assume we have a $\mathbf{q}$-right-quantum matrix, with $q_{ij} = q$ for $i \le n$, $j > n$. In the notation of the previous section, we have the following.

Lemma 5.7.7 *If $A = (a_{ij})_{m\times m}$ is a $\mathbf{q}$-right-quantum matrix, so is $C = (c_{ij})_{n+1\le i,j\le m}$, where*

$$c_{ij} = a_{ij} + a_{i*}(I - A_0)^{-1} a_{*j}.$$

Proof. We use a combination of proofs of Lemmas 5.6.6 and 5.7.3. □

Proposition 5.7.8 *If A is $\mathbf{q}$-right-quantum, then*

$$c_{ij} = -\det_{\mathbf{q}}^{-1}(I - A_0) \cdot \det_{\mathbf{q}} \begin{pmatrix} I - A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix}.$$

Proof. We use the same technique as in the proof of Proposition 5.7.4. □

This finishes the proof of Theorem 5.1.4.

5.8 The β-extension

Theorem 5.1.1 trivially implies that

$$(\det B)^\beta = (\det A)^\beta \cdot (\det A_0)^{\beta(m-n-1)}$$

for any $\beta \in \mathbb{Z}$, where a_{ij} are commutative variables and

$$b_{ij} = \det \begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij} \end{pmatrix}, \quad B = (b_{ij})_{n+1 \le i,j \le m}.$$

It is not immediately clear what the non-commutative extension of this could be. Of course, Theorem 5.4.4 implies that

$$(\det(I-C))^\beta = \left(\det{}^{-1}(I-A_0) \cdot \det(I-A)\right)^\beta$$

for

$$c_{ij} = -\det{}^{-1}(I-A_0) \cdot \det \begin{pmatrix} I-A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix}, \quad C = (c_{ij})_{n+1 \le i,j \le m},$$

where A is a Cartier-Foata or right-quantum matrix, but this does not tell us how to calculate the terms of $(\det(I-C))^\beta$. However, a technique similar to the proof of the β-extension of the non-commutative MacMahon master theorem, Theorem 4.6.5, gives a reasonable interpretation of $(\det(I-C))^\beta$ for $\beta \in \mathbb{Z}$ when A is a Cartier-Foata matrix.

We use the terminology of Section 4.6.

Recall that for a Cartier-Foata matrix A, the matrix $C = (c_{ij})_{n+1 \le i,j \le m}$ with

$$c_{ij} = -\det{}^{-1}(I-A_0) \cdot \det \begin{pmatrix} I-A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix}$$

is right-quantum by Lemma 5.4.1 and Proposition 5.4.3, so

$$\begin{aligned} \det{}^{-1}(I-C) &= (I-C)^{-1}_{n+1,n+1}(I-C^{n+1,n+1})^{-1}_{n+2,n+2}\cdots = \\ &= (I-A)^{-1}_{n+1,n+1}(I-A^{n+1,n+1})^{-1}_{n+2,n+2}\cdots \end{aligned} \tag{5.8.1}$$

by Theorem 5.2.1. The last expression is the sum over all sequences which are concatenations of a lattice path from $n+1$ to $n+1$, a lattice path from $n+2$ to $n+2$, etc.

Theorem 5.8.1 (β-extension of Cartier-Foata Sylvester's identity) *Assume* $A =$

$(a_{ij})_{m\times m}$ *is a Cartier-Foata matrix. Write*

$$C = (c_{ij})_{n+1\le i,j\le m} \qquad \textit{for} \qquad c_{ij} = -\det{}^{-1}(I-A_0)\cdot \det\begin{pmatrix} I-A_0 & -a_{*j} \\ -a_{i*} & -a_{ij}\end{pmatrix}.$$

For each $\beta \in \mathbb{Z}$, *the expression*

$$\left(\frac{1}{\det(I-C)}\right)^{\beta}$$

is equal to

$$\sum e_\mu(\beta) a_{\lambda,\mu},$$

where μ *runs over all words in the alphabet* $\{1,\dots,m\}$, λ *is the non-decreasing rearrangement of* μ, *and* $e_\mu(\beta)$ *is a polynomial function of* β *that is calculated as follows. Let* $u_1u_2\dots u_k$ *be the disjoint cycle decomposition of* $a_{\lambda,\mu}$. *Let* $\mathcal{J}$ *be the set of* i *for which* u_i *contains a height* $> n$. *Then*

$$e_\mu(\beta) = \sum_\pi \binom{\beta + l - 1 - \operatorname{des}'(\pi)}{l}, \tag{5.8.2}$$

where the sum is over all permutations $\pi \in S_k$ *with the following properties:*

- *if* $i<j, \pi(i)>\pi(j)$, *then* $u_{\pi(i)}, u_{\pi(j)}$ *are disjoint;*
- *for each* $i \notin \mathcal{J}$ *there exists* $j>i$ *such that* $u_{\pi(i)}$ *and* $u_{\pi(j)}$ *are not disjoint and such that* $\{\pi(i+1),\dots,\pi(j-1)\}\cap \mathcal{J} = \emptyset$;
- *if* $\pi(i) > \pi(i+1)$ *then* $\pi(i)\in\mathcal{J}$.

Here $\operatorname{des}'(\pi)$ *denotes the number of descents of the subword of* $(\pi(1),\pi(2),\dots,\pi(k))$ *composed of* $\pi(i)\in\mathcal{J}$, *and* $l = |\mathcal{J}|$.

Example 5.8.2 Take $m=5$, $n=2$, $\mu = 132521421325$. The disjoint cycle decomposition of the o-sequence $a_{11}a_{13}a_{12}a_{25}a_{22}a_{21}a_{24}a_{32}a_{31}a_{43}a_{52}a_{55}$ is

$$u_1u_2u_3u_4u_5u_6 = (a_{11})(a_{25}a_{52})(a_{22})(a_{13}a_{32}a_{21})(a_{12}a_{24}a_{43}a_{31})(a_{55}).$$

We have $\mathcal{J} = \{2,4,5,6\}$, the only permutations in S_6 that appear in the sum (5.8.2) are $213456, 213465, 261345$ with $\operatorname{des}'(213456)=0$, $\operatorname{des}'(213465)=1$, $\operatorname{des}'(261345) = 1$. Therefore

$$e_\mu(\beta) = \binom{\beta+3}{4} + 2\binom{\beta+2}{4} = \frac{\beta^4}{8} + \frac{5\beta^3}{12} + \frac{3\beta^2}{8} + \frac{\beta}{12}. \qquad \diamondsuit$$

Example 5.8.3 Take $n=0$. In this case $\mathcal{J} = [k]$, only the first condition is not vacuously true on π, and we get the β-extension of MacMahon master theorem, Theorem 4.6.5. $\diamondsuit$

It is clear that each term of $(\det(I - C))^{-\beta}$ is an o-sequence modulo $\mathcal{I}_{\mathrm{cf}}$, and that the coefficients of o-sequences are polynomial functions of β. Therefore it is enough to prove the theorem for $\beta \in \mathbb{N}$, and this is an enumerative problem. We are given an o-sequence $a_{\lambda,\mu}$ and β slots, and we have to calculate in how many ways we can choose terms of $(\det(I-C))^{-1}$ in each slot so that their product is, modulo $\mathcal{I}_{\mathrm{cf}}$, equal to $a_{\lambda,\mu}$. We start the proof with a lemma.

Lemma 5.8.4 *All the steps in a cycle of the disjoint cycle decomposition must be placed in the same slot.*

Proof: This is proved in exactly the same way as the proof of Lemma 4.6.4, since all we used there was that the sequence chosen in each slot must be balanced, which is also true in our case. □

Proof of Theorem 5.8.1. We call cycles with all heights $\leq n$ *low cycles*, and cycles containing at least one height $> n$ *high cycles*.
The lemma tells us that we must choose a permutation $\pi \in S_k$ such that $u_1 \cdots u_k = u_{\pi(1)} \cdots u_{\pi(k)}$ modulo $\mathcal{I}_{\mathrm{cf}}$, and place the cycles $u_{\pi(1)}, \ldots, u_{\pi(k)}$ in the β slots so that the cycles in each slot give a term appearing in $(\det(I - C))^{-1}$.
Two cycles commute if and only if they are disjoint. That means that the condition $u_1 \cdots u_k = u_{\pi(1)} \cdots u_{\pi(k)}$ is equivalent to

- if $i < j, \pi(i) > \pi(j)$, then $u_{\pi(i)}, u_{\pi(j)}$ are disjoint,

which is the first condition in Theorem 5.8.1.
Take a low cycle $u_{\pi(i)}$, and assume that it is disjoint with all $u_{\pi(j)}$ for $j > i$. That means we can push it to the end of the chosen slot without changing the sequence modulo $\mathcal{I}_{\mathrm{cf}}$. But then the sequence in the slot is not equal modulo $\mathcal{I}_{\mathrm{cf}}$ to a sequence in (5.8.1). Therefore

- for each $i \notin \mathcal{J}$ there exists $j > i$ such that $u_{\pi(i)}$ and $u_{\pi(j)}$ are not disjoint,

which is part of the second condition in Theorem 5.8.1. Furthermore, if $j > i$ is the lowest integer with this property, then $u_{\pi(i)}, \ldots, u_{\pi(j)}$ must all be placed in the same slot. If there is a $\pi(p) \in \mathcal{J}$ with $i < p < j$, then this placement of cycles is also counted in the permutation when $(\pi(1), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(p), \pi(i), \pi(p+1), \ldots, \pi(k))$. That means that it is enough to take permutations which satisfy

- for each $i \notin \mathcal{J}$ there exists $j > i$ such that $u_{\pi(i)}$ and $u_{\pi(j)}$ are not disjoint and such that $\{\pi(i+1), \ldots, \pi(j-1)\} \cap \mathcal{J} = \emptyset$,

which is the second condition in Theorem 5.8.1.
Finally, assume that we have $\pi(i) > \pi(i+1)$ with $\pi(i) \notin \mathcal{J}$. Then $\pi(i)$ must be placed in the same slot as a high cycle $\pi(j)$ for $j > i$, and so $\pi(i+1)$, which commutes with $\pi(i)$, must be placed in the same slot as well. But then this placement of cycles in slots is already counted in the permutation where $\pi(i)$ and $\pi(i+1)$ are switched. Therefore we have

- if $\pi(i) > \pi(i+1)$ then $\pi(i) \in \mathcal{J}$,

which is the third condition in Theorem 5.8.1.
We have described all permutations that give $a_{\lambda,\mu}$, and now we have to find the number of ways to place $u_{\pi(1)}, \ldots, u_{\pi(k)}$ in the β slots so that the cycles in each slot give a term appearing in $(\det(I-C))^{-1}$. All cycles between two consecutive high cycles must appear in the same slot as the right-hand high cycle. Therefore placing the cycles in slots is the same as placing $\beta-1$ dividers after (some of the) high cycles. Of course, there are $\binom{\beta-1+l}{l}$ ways of doing this, but we can get the same terms several times: if we take two consecutive high cycles $u_{\pi(i)}, u_{\pi(j)}$ with $i<j, \pi(i)>\pi(j)$, then $u_{\pi(i)}$ must necessarily commute with $u_{\pi(j)}$ *and* with all the low cycles between them, we can move $u_{\pi(j)}$ to the right of $u_{\pi(i)}$, possibly move some of the low cycles before $u_{\pi(j)}$ to the right of $u_{\pi(i)}$, and we see that this term has already been counted for a different π. In order to avoid overcounting, we have to place a divider after $u_{\pi(i)}$. Therefore the number of unique placements in slots corresponding to π is $\binom{\beta-1+l-\operatorname{des}'(\pi)}{l}$, and this finishes the proof of Theorem 5.8.1. □

Remark 5.8.5 Theorem 5.8.1 is an extension of Theorem 4.6.5 and is therefore not true for general right-quantum matrices, see Example 4.6.9. ♢

Chapter 6

Goulden-Jackson's immanant formula

6.1 Introduction

In this chapter, we give two new proofs of a beautiful determinantal formula due to Goulden and Jackson [GJ92b], and generalize this formula to the quantum setting. The formula involves determinants of the terms of $\det^{-1}(I-A)$ and is a generalization of a part of the MacMahon master theorem, see the paragraph following Theorem 6.1.1. Goulden and Jackson gave a straightforward but non-combinatorial proof. In Section 6.2, we present two simple combinatorial proofs, prove a β-extension in the spirit of Sections 4.6 and 5.8, and give an interesting determinantal expression for the value of irreducible characters of the symmetric group. In Section 6.3, we give a brief background on Hecke algebras of type A, which is needed in Section 6.4, where a quantum extension of the formula is stated and proved. In Section 6.5, we use the results of the previous sections to (re)prove beautiful combinatorial representations of characters of Hecke algebras of type A. The necessary bijective proofs appear in Section 6.6.

Let us start with a discussion of symmetric functions of eigenvalues of a matrix.

Take a complex matrix $A = (a_{ij})_{i,j=1}^m$ for complex variables a_{ij}, its characteristic polynomial $\chi_A(t) = \det(A - tI)$ and its eigenvalues $\omega_1, \ldots, \omega_m$. Vieta's formulas tell us that the elementary symmetric functions

$$e_i = e_i(\omega_1, \ldots, \omega_m) = \sum_{j_1 < \ldots < j_i} \omega_{j_1} \cdots \omega_{j_i}$$

are easily expressible in terms of a_{ij}:

$$e_0 t^m - e_1 t^{m-1} + \ldots + (-1)^m e_m = (t - \omega_1) \cdots (t - \omega_m) =$$

$$= \det(tI - A) = \sum_{i=0}^{m} (-1)^i t^{m-i} \sum_{J \in \binom{[m]}{i}} \det A_J,$$

with $A_J = (a_{ij})_{i,j\in J}$, i.e.

$$e_i = \sum_{J\in\binom{[m]}{i}} \det A_J.$$

The complete homogeneous symmetric functions

$$h_i = h_i(\omega_1, \ldots, \omega_m) = \sum_{j_1\le\ldots\le j_i} \omega_{j_1}\cdots\omega_{j_i}$$

are also easy. We know that

$$\sum h_i t^i = \frac{1}{\sum(-1)^i e_i t^i} = \frac{1}{t^m \sum(-1)^i e_i t^{i-m}} = \frac{1}{t^m \det(t^{-1}I - A)} = \frac{1}{\det(I - tA)} \tag{6.1.1}$$

and by MacMahon master theorem (Theorem 4.1.1)

$$h_i(\omega_1, \ldots, \omega_m) = \sum_\lambda a_{\overline{\lambda}_1\lambda_1} a_{\overline{\lambda}_2\lambda_2} \cdots a_{\overline{\lambda}_i\lambda_i}, \tag{6.1.2}$$

where:

- $\lambda = \lambda_1 \cdots \lambda_i$ runs over all sequences of i letters from $\{1, \ldots, m\}$, and
- $\overline{\lambda} = \overline{\lambda}_1 \cdots \overline{\lambda}_i$ is the non-decreasing rearrangement of λ.

Goulden and Jackson proved the following.

Theorem 6.1.1 *Abbreviate $h_i(\omega_1, \ldots, \omega_m)$ to h_i, $e_i(\omega_1, \ldots, \omega_m)$ to e_i, choose a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ of m, and write $\mathbf{a}_\pi = a_{1\pi(1)}a_{2\pi(2)}\cdots a_{m\pi(m)}$ for a permutation $\pi \in S_m$. Then*

$$[\mathbf{a}_\pi] \det(h_{\lambda_i - i + j})_{p\times p} = [\mathbf{a}_\pi] \det(e_{\lambda'_i - i + j})_{\lambda_1\times\lambda_1} = \chi^\lambda(\pi), \tag{6.1.3}$$

where $[\mathbf{a}_\pi]E$ is the coefficient of the basis element $\mathbf{a}_\pi$ of E, and χ^λ is the irreducible character of the symmetric group S_m corresponding to λ.

For $\lambda = (m)$, we have $\det(h_{\lambda_i - i + j})_{p\times p} = h_m$ and $\chi^\lambda(\pi) = 1$ for all $\pi \in S_m$, so (6.1.3) is a special case of the MacMahon master theorem.

The characters of the symmetric group are class functions in the sense that $\chi^\lambda(\pi)$ depends only on the cycle type of π. That means that it makes sense to define $\chi^\lambda(\mu)$ for partitions λ, μ of the same size.

Note that since $h_{\lambda_i - i + j} = h_{\lambda_i - i + j}(\omega_1, \ldots, \omega_m)$, $\det(h_{\lambda_i - i + j})$ and $\det(e_{\lambda'_i - i + j})$ can both be expressed as

$$s_\lambda(\omega_1, \ldots, \omega_m)$$

by the (dual) Jacobi-Trudi identity (see for example [Sta99, Theorem 7.16.1 and Corollary 7.16.2]). Here s_λ is the Schur symmetric function corresponding to the partition λ.

Goulden and Jackson's result is stated in the (clearly equivalent) language of immanants. Theirs is one of the many papers in the early 1990's that brought about fascinating conjectures and results on immanants; see for example [GJ92a], [Gre92], [SS93], [Ste91], [Hai93] for details and further references.

We give two more proofs of this result. The first gives a recursion that specializes to Murnaghan-Nakayama's rule, and the second is a simple combinatorial proof of a statement equivalent to (6.1.3).

6.2 Two proofs of Theorem 6.1.1

First, let us give a proof that uses recursion; more specifically, let us give an expression for the left-hand side of (6.1.3) which expresses the desired coefficient in terms of coefficients of determinants of smaller matrices.

Note first that all the terms $\mathbf{a} = a_{1*} \cdots a_{1*} a_{2*} \cdots$ of

$$s_\lambda(\omega_1, \ldots, \omega_m) = \det(e_{\lambda'_i - i + j})$$

are balanced (each i appears as many times among a_{i*} as among a_{*i}).

Suppose we want to find the coefficient of $\mathbf{a} = a_{1*} \cdots a_{1*} a_{2*} \cdots$ in $s_\lambda(\omega_1, \ldots, \omega_m)$. Assume $C = \{1, \ldots, k\}$, $D = \{k+1, \ldots, m\}$ and that $\mathbf{a}$ does not contain a_{ij} for $i \in C, j \in D$ or $i \in D, j \in C$. The coefficient of $\mathbf{a}$ does not change if we set all a_{ij} that do not appear in $\mathbf{a}$ equal to 0; we may therefore assume that the matrix A has a block diagonal form $A_1 \oplus A_2$, and if $\xi_1, \ldots, \xi_k$ are the eigenvalues of A_1 and $\zeta_{k+1}, \ldots, \zeta_m$ are the eigenvalues of A_2, the eigenvalues of A are $(\omega_1, \ldots, \omega_m) = (\xi_1, \ldots, \xi_k, \zeta_{k+1}, \ldots, \zeta_m)$. By definition,

$$s_\lambda(\omega_1, \ldots, \omega_m)$$

is the sum of $\omega^T = \omega_1^{\alpha_1(T)} \omega_2^{\alpha_2(T)} \cdots$ over all semistandard Young tableaux (SSYT) T of shape λ; here $\alpha_i(T)$ is the number of i's in T. See [Sta99, §7.10] for definitions and details. In every such T, the numbers $1, \ldots, k$ form a SSYT of some shape $\nu \subseteq \lambda$, and the numbers $k+1, \ldots, m$ form a SSYT of shape λ/ν. Therefore

$$[\mathbf{a}] s_\lambda(\omega_1, \ldots, \omega_m) = \sum_{\nu \subseteq \lambda} [\mathbf{a}_1] s_\nu(\xi_1, \ldots, \xi_k) [\mathbf{a}_2] s_{\lambda/\nu}(\zeta_{k+1}, \ldots, \zeta_m), \tag{6.2.1}$$

where $\mathbf{a}_1$ (respectively $\mathbf{a}_2$) is the product of the terms of $\mathbf{a}$ with indices in C (respectively D). Since $s_{\lambda/\nu}$ is homogeneous of degree $|\lambda| - |\nu|$, we can restrict the sum to partitions $\nu \vdash k$.

The factors in this sum can either be calculated explicitly (for example using the dual Jacobi-Trudi identity) or recursively.

Example 6.2.1 Let us use this formula to find the coefficient of $a_{11} a_{12} a_{21} a_{22}^2 a_{34} a_{43}$ in $s_{322}(\omega_1, \ldots, \omega_4)$. By (6.2.1), we have to find the coefficient of $a_{34} a_{43}$ in $s_{\lambda/\nu}$ for

$\nu = 32, 311, 221$. We have

$$s_{322/32}(\zeta_3, \zeta_4) = s_2(\zeta_3, \zeta_4) = h_2(\zeta_3, \zeta_4) = a_{33}^2 + a_{33}a_{44} + a_{34}a_{43} + a_{44}^2,$$

$$s_{322/311}(\zeta_3, \zeta_4) = s_{11}(\zeta_3, \zeta_4) = e_2(\zeta_3, \zeta_4) = a_{33}a_{44} - a_{34}a_{43},$$

$$s_{322/221}(\zeta_3, \zeta_4) = s_{21/1}(\zeta_3, \zeta_4) = e_1^2(\zeta_3, \zeta_4) = a_{33}^2 + 2a_{33}a_{44} + a_{44}^2$$

and therefore

$$[a_{11}a_{12}a_{21}a_{22}^2a_{34}a_{43}]s_{322}(\omega_1, \dots, \omega_4) = [a_{11}a_{12}a_{21}a_{22}^2]\left(s_{32}(\zeta_1, \zeta_2) - s_{311}(\zeta_1, \zeta_2)\right).$$

Furthermore,

$$s_{32}(\zeta_1, \zeta_2) = \begin{vmatrix} e_2 & 0 & 0 \\ e_1 & e_2 & 0 \\ 0 & e_0 & e_1 \end{vmatrix} = e_2^2 e_1 = (a_{11}a_{22} - a_{12}a_{21})^2(a_{11} + a_{22})$$

and

$$s_{311}(\zeta_1, \zeta_2) = \begin{vmatrix} 0 & 0 & 0 \\ 1 & e_1 & 0 \\ 0 & 1 & e_1 \end{vmatrix} = 0.$$

Therefore

$$[a_{11}a_{12}a_{21}a_{22}^2a_{34}a_{43}]s_{322}(\omega_1, \dots, \omega_4) = -2. \qquad \diamondsuit$$

Let us prove that this recursion specializes to the Murnaghan-Nakayama's rule. In order to do that, assume that $\pi = \pi_1 \cdot (k+1, k+2, \dots, m)$ for $\pi_1 \in S_k$. In this case, $\mathbf{a}_2$ is of the form $a_{k+1,k+2}a_{k+2,k+3} \cdots a_{m,k+1} = b_1 \cdots b_l$. The corresponding matrix A_2 is

$$\begin{pmatrix} 0 & b_1 & 0 & \dots & 0 \\ 0 & 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{l-1} \\ b_l & 0 & 0 & \dots & 0 \end{pmatrix} \tag{6.2.2}$$

and its characteristic polynomial is $(-1)^l(t^l - b_1b_2 \cdots b_l)$. If the zeros are denoted $\eta_1, \dots, \eta_l$, then $e_0(\eta_1, \dots, \eta_l) = 1$, $e_l(\eta_1, \dots, \eta_l) = (-1)^{l-1}b_1 \cdots b_l$, $e_i(\eta_1, \dots, \eta_l) = 0$ for $i \neq 0, l$.

For a partition $\lambda = (\lambda_1, \dots, \lambda_p)$, define the conjugate partition λ' by $\lambda'_i = |\{j \colon \lambda_j \geq i\}|$. A border strip is a connected skew shape with no 2×2 square, and the height ht of a border strip is defined to be one less than the number of rows. The following result is well known; we include the proof for the sake of completeness. It will imply that the only non-zero terms in the sum (6.2.1) are indexed by border strips.

Lemma 6.2.2 *If $\lambda = (\lambda_1, \dots, \lambda_p)$ is a partition and $\nu \subseteq \lambda$ is a subpartition with $|\lambda| - |\nu| = l$, then:*

1. *λ/ν is a border strip if and only if λ'/ν' is a border strip;*

2. λ/ν *is a border strip if and only if* $\lambda_1 + p = l + 1$;

3. λ/ν *is a border strip if and only if* $\lambda_i = \nu_{i-1} + 1$ *for* $2 \leq i \leq p$.

Proof. Let (a, b) be the entry in row a and column b. (1) is obvious. (2) The squares of a border strip λ/μ form a NE-path from $(1, p)$ to $(\lambda_1, 1)$. Each such path has $(p-1) + (\lambda_1 - 1) + 1$ squares. Conversely, a partition containing $(1, p)$, $(\lambda_1, 1)$ and $\lambda_1 + p - 3$ other squares must be a NE-path and hence its squares form a border strip. (3) A 2×2 square in rows $i-1, i$ implies that $\lambda_i \geq \nu_{i-1} + 2$. It is clear that if the squares of a shape λ/μ form a NE-path, we have $\lambda_i = \nu_{i-1} + 1$. □

Proposition 6.2.3 *If the zeros of* $t^l - b_1 b_2 \cdots b_l$ *are* $\eta_1, \ldots, \eta_l$, *then* $s_{\lambda/\nu}(\eta_1, \ldots, \eta_l) = 0$ *unless* λ/ν *is a border strip, and* $s_{\lambda/\nu}(\eta_1, \ldots, \eta_l) = (-1)^{\mathrm{ht}(\lambda/\nu)} b_1 b_2 \cdots b_l$ *for a border strip* λ/ν.

Proof. We can assume that $\lambda_i \neq \nu_i$ and that $\nu_p = 0$. The indices of the entries of the matrix

$$\det(e_{\lambda'_i - \nu'_j - i + j})_{\lambda_1 \times \lambda_1}$$

are strictly increasing in rows; the largest possible index is $\lambda_1 + p - 1 = l$; and the indices of the diagonal elements are $\lambda'_i - \nu'_i$ with $1 \leq \lambda'_i - \nu'_i \leq l$ (and $\lambda'_i - \nu'_i = l$ only in the trivial case $\lambda_1 = l$, $\nu = \emptyset$). Note that this determinant is equal to $s_{\lambda/\nu}$. By the lemma and the fact that $e_k = 0$ for $k \neq 0, l$, the matrix is 0 on and above the diagonal unless λ/ν is a border strip. If it is a border strip, the matrix entries in positions (i, j), $i \leq j < \lambda_1$, are 0 (in particular, all the entries in the first row are 0 except the last, which is e_l), and by the lemma, the subdiagonal entries are 1 and the element $(1, \lambda_1)$ is $(-1)^{l-1} b_1 \cdots b_l$. Therefore $s_{\lambda/\nu}(\eta_1, \ldots, \eta_l) = (-1)^{l-1+\lambda_1-1} b_1 b_2 \cdots b_l = (-1)^{p-1} b_1 b_2 \cdots b_l$, with $p - 1 = \mathrm{ht}(\lambda/\nu)$. □

In other words, if $\pi = \pi_1 \cdot (k+1, k+2, \ldots, m)$ for $\pi_1 \in S_k$, then by (6.2.1),

$$[\mathbf{a}_\pi] s_\lambda(\omega_1, \ldots, \omega_m) = \sum_\nu (-1)^{\mathrm{ht}(\lambda/\nu)} [\mathbf{a}_1] s_\nu(\xi_1, \ldots, \xi_k),$$

where $\mathbf{a}_1 = \mathbf{a}_{\pi_1}$ and the sum is over all partitions $\nu \subseteq \mu$ for which μ/ν is a border strip. This is precisely the Murnaghan-Nakayama's rule, see [Sag01, Theorem 4.10.2]. Together with the fact that Murnaghan-Nakayama's rule completely determines the irreducible characters χ^λ, this shows that

$$[\mathbf{a}_\pi] s_\lambda(\omega_1, \ldots, \omega_m) = \chi^\lambda(\pi).$$

Note that this also gives us the coefficient of $\mathbf{a}_\pi = a_{1\pi(1)} \cdots a_{m\pi(m)}$ in $p_\lambda(\omega_1, \ldots, \omega_m)$: we know that

$$p_\lambda = \sum_\mu \chi^\mu(\lambda) s_\mu$$

and hence

$$[\mathbf{a}_\pi] p_\lambda(\omega_1, \ldots, \omega_m) = \sum_\mu \chi^\mu(\lambda) \chi^\mu(\pi)$$

is (by the orthogonality of the columns of the table of characters) equal to

$$z_\lambda = 1^{j_1} j_1! 2^{j_2} j_2! \cdots$$

if the cycle type of π is the partition $\lambda = \langle 1^{j_1} 2^{j_2} \cdots \rangle$, and 0 otherwise; see e.g. [Sta99, Proposition 7.17.6].

An even simpler proof uses the scalar product in the space of symmetric function. Let us find the coefficient of $\mathbf{a}_\pi$ in $e_\lambda(\omega_1, \ldots, \omega_m) = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_p}$. If we pick $a_{i\pi(i)}$ from e_{λ_1}, we must also pick $a_{\pi(i)j}$ from e_{λ_1} because every term in e_{λ_1} is balanced. But since $a_{\pi(i)\pi^2(i)}$ is the only term of $\mathbf{a}_\pi$ with $\pi(i)$ as the first index, we must have $j = \pi^2(i)$. In other words, each of the cycles of π must be chosen from one of the e_{λ_i}'s. We know that for $J = \{j_1 < j_2 < \ldots < j_i\}$ and τ a permutation of $j_1, \ldots, j_i$, $a_{j_1\tau(j_1)} \cdots a_{j_i\tau(j_i)}$ appears in e_i with coefficient equal to the sign of τ.

This reasoning implies that the coefficient of $\mathbf{a}_\pi = a_{1\pi(1)} \cdots a_{m\pi(m)}$ with π of cycle type $\mu = (\mu_1, \ldots, \mu_q)$ in e_λ is equal to $\sigma_\mu R_{\mu\lambda}$, where

- σ_μ is equal to $(-1)^{j_2+j_4+\cdots}$ for $\mu = \langle 1^{j_1} 2^{j_2} \cdots \rangle$, and
- $R_{\mu\lambda}$ is the number of ordered partitions $(B_1, \ldots, B_p)$ of the set $\{1, \ldots, q\}$ such that
$$\lambda_j = \sum_{i \in B_j} \mu_i$$
for $1 \leq j \leq p$.

But we know that if $\langle \cdot, \cdot \rangle$ is the standard scalar product in the space of symmetric functions defined by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$ and ω is the standard (scalar product preserving) involution given by $\omega(h_\lambda) = e_\lambda$, then $p_\mu = \sum_\nu R_{\mu\nu} m_\nu$ and $\omega(p_\mu) = \sigma_\mu p_\mu$ (see [Sta99, §7.4 – §7.9]), do

$$\langle e_\lambda, p_\mu \rangle = \langle \omega(e_\lambda), \omega(p_\mu) \rangle = \sigma_\mu \langle h_\lambda, p_\mu \rangle = \sigma_\mu R_{\mu\lambda}.$$

Since e_λ form a vector-space basis of the space of symmetric functions and since both the scalar product with a fixed function and the operator $[\mathbf{a}_\pi]$ are linear, we have proved the following.

Proposition 6.2.4 *For any symmetric function f and for π a permutation of cycle type μ, we have*

$$[\mathbf{a}_\pi] f(\omega_1, \ldots, \omega_m) = \langle f, p_\mu \rangle$$

In particular,

$$[\mathbf{a}_\pi] s_\lambda(\omega_1, \ldots, \omega_m) = \langle s_\lambda, p_\mu \rangle = \chi^\lambda(\mu). \qquad \square$$

The proposition of course also implies that

$$[\mathbf{a}_\pi] p_\lambda(\omega_1, \ldots, \omega_m) = \langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu},$$

which we already proved above.

The β-extension of the MacMahon master theorem, equation (4.6.2), yields the following extension of Theorem 6.1.1.

Theorem 6.2.5 *Denote by h_i^β the coefficient of t^i in*

$$\left(\frac{1}{\det(I-tA)}\right)^\beta.$$

Choose a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ of m, and write $\mathbf{a}_\pi = a_{1\pi(1)}a_{2\pi(2)}\cdots a_{m\pi(m)}$ for a permutation $\pi \in S_m$. Then

$$[\mathbf{a}_\pi]\det(h^\beta_{\lambda_i-i+j})_{p\times p} = \beta^{\operatorname{cyc}\pi}\chi^\lambda(\pi), \tag{6.2.3}$$

where χ^λ is the irreducible character of the symmetric group S_m corresponding to λ, and $\operatorname{cyc}\pi$ is the number of cycles of π. In particular, for $\beta=-1$, the formula (6.2.3) *gives the relation*

$$\chi^{\lambda'}(\pi) = (-1)^{\operatorname{cyc}\pi+|\lambda|}\chi^\lambda(\pi). \tag{6.2.4}$$

Proof. Each term in the expansion of $\det(h^\beta_{\lambda_i-i+j})_{p\times p}$ is a product of certain h_i^β's. Since the terms of h_i^β are also balanced, the same reasoning as on page 102 shows that for each cycle $(j, \pi(j), \pi^2(j), \ldots)$, the variables $a_{j,\pi(j)}, a_{\pi(j),\pi^2(j)}, \ldots$ must be chosen from the same h_i^β. Assume that the cycle decomposition of π is

$$(j_1, \pi(j_1), \pi^2(j_1), \ldots)(j_2, \pi(j_2), \pi^2(j_2), \ldots)\cdots(j_s, \pi(j_s), \pi^2(j_s), \ldots,),$$

and let us find the coefficient of

$$\left(a_{j_1,\pi(j_1)}a_{\pi(j_1),\pi^2(j_1)}\cdots\right)\left(a_{j_2,\pi(j_2)}a_{\pi(j_2),\pi^2(j_2)}\cdots\right)\cdots\left(a_{j_s,\pi(j_s)}a_{\pi(j_s),\pi^2(j_s)}\cdots\right)$$

in h_i^β. We use the equation (4.6.3). In the notation of Section 4.6, there is exactly one permutation in $\sigma \in \mathcal{P}(\mathbf{1})$ with $v(\sigma) = \mathbf{a}_\pi$; it is defined by $\sigma(i,1) = (\pi(i), 1)$. Clearly, $\operatorname{cyc}\sigma = \operatorname{cyc}\pi$, so the appropriate coefficient is β^s. It follows that the coefficient of $\mathbf{a}_\pi$ in $\det(h^\beta_{\lambda_i-i+j})_{p\times p}$ is equal to $\beta^{\operatorname{cyc}\pi}$ times the coefficient of $\mathbf{a}_\pi$ in $\det(h_{\lambda_i-i+j})_{p\times p}$, which is, by Theorem 6.1.1, precisely (6.2.3).
We saw in (6.1.1) that

$$\det(I-tA) = \sum_i (-1)^i e_i t^i,$$

where $e_i = e_i(\omega_1, \ldots, \omega_m)$, so, for $\beta=-1$, we have $h_i^\beta = (-1)^i e_i$ and

$$\det(h^\beta_{\lambda_i-i+j})_{p\times p} = \det((-1)^{\lambda_i-i+j}e_{\lambda_i-i+j})_{p\times p} = (-1)^{|\lambda|}s_{\lambda'}(\omega_1, \ldots, \omega_m)$$

by the dual Jacobi-Trudi identity, and by taking the coefficient of $\mathbf{a}_\pi$ on both sides and using Theorem 6.1.1 and (6.2.3) we get $(-1)^{\operatorname{cyc}\pi}\chi^\lambda(\pi) = (-1)^{|\lambda|}\chi^{\lambda'}(\pi)$, which is equivalent to (6.2.4). $\square$

It is worthwhile to note the following determinantal description of irreducible characters of the symmetric group.

Corollary 6.2.6 *Let λ, μ be partitions of m, and define $f_0, \ldots, f_m$ via the formula*

$$(t^{\mu_1} - u_1)(t^{\mu_2} - u_2) \cdots (t^{\mu_q} - u_q) = f_0 t^m - f_1 t^{m-1} + \ldots + (-1)^m f_m.$$

Then

$$\chi^\lambda(\mu) = [u_1 \cdots u_q] \det(f_{\lambda'_i - i + j}) = (-1)^{|\lambda|+q} [u_1 \cdots u_q] \det(f_{\lambda_i - i + j}).$$

EXAMPLE 6.2.7 Let $\lambda = (2, 2, 2, 1)$ and $\mu = (3, 2, 2)$. Then

$$(t^3 - u_1)(t^2 - u_2)(t^2 - u_3) = t^7 - (u_2 + u_3)t^5 - u_1 t^4 + u_2 u_3 t^3 + (u_1 u_2 + u_1 u_3)t^2 - u_1 u_2 u_3$$

and so $f_0 = 1$, $f_1 = 0$, $f_2 = -u_2 - u_3$, $f_3 = u_1$, $f_4 = u_2 u_3$, $f_5 = -u_1 u_2 - u_1 u_3$, $f_6 = 0$, $f_7 = u_1 u_2 u_3$. Hence

$$\chi^{2221}(322) = [u_1 u_2 u_3] \begin{vmatrix} f_4 & f_5 \\ f_2 & f_3 \end{vmatrix} = [u_1 u_2 u_3](f_4 f_3 - f_2 f_5) = 1 - 2 = -1. \qquad \diamondsuit$$

Proof of Corollary 6.2.6. Take $\pi = (1, 2, \ldots, \mu_1)(\mu_1 + 1, \mu_1 + 2, \ldots, \mu_1 + \mu_2) \cdots$, and form the block diagonal matrix $A = A_1 \oplus \ldots \oplus A_q$ with blocks of the form (6.2.2) corresponding to cycles of π. For example, when $\mu = (3, 2, 2)$, the permutation is $(123)(45)(67)$ and the matrix is

$$\begin{pmatrix} 0 & a_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{23} & 0 & 0 & 0 & 0 \\ a_{31} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{45} & 0 & 0 \\ 0 & 0 & 0 & a_{54} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{67} \\ 0 & 0 & 0 & 0 & 0 & a_{76} & 0 \end{pmatrix}.$$

The characteristic polynomial of A is $(t^{\mu_1} - u_1)(t^{\mu_2} - u_2) \cdots (t^{\mu_q} - u_q)$ for $u_1 = a_{12} a_{23} \cdots a_{\mu_1 1}$, $u_2 = a_{\mu_1+1, \mu_1+2} a_{\mu_1+2, \mu_1+3} \cdots a_{\mu_1+\mu_2, \mu_1+1}$, etc. In other words, if the eigenvalues of A are $\omega_1, \ldots, \omega_m$, then $e_i(\omega_1, \ldots, \omega_m) = f_i$. But then $\det(f_{\lambda'_i - i + j}) = s_\lambda(\omega_1, \ldots, \omega_m)$ and equation (6.1.3) implies

$$[u_1 \cdots u_q] \det(f_{\lambda'_i - i + j}) = [\mathbf{a}_\pi] s_\lambda(\omega_1, \ldots, \omega_m) = \chi^\lambda(\mu).$$

The second equality is a corollary of (6.2.4). $\square$

6.3 Some background on Hecke algebras of type A

A beautiful quantization of the symmetric group is $H_m(q)$, the Hecke algebra of type A. In this section, we define and give some basic information about this algebra; in the next section, we prove a q-generalization of Theorem 6.1.1; and in Sections 6.5 and 6.6, we use these results to give combinatorial formulas for three families of characters of $H_m(q)$.

The symmetric group S_m is generated by transpositions $s_i = (i, i+1)$, $1 \le i \le m-1$, which satisfy the relations:

$$\begin{aligned} s_i^2 &= 1 \qquad \text{for } i = 1, \ldots, m-1, && (6.3.1)\\ s_i s_j s_i &= s_j s_i s_j \qquad \text{if } |i-j| = 1, && (6.3.2)\\ s_i s_j &= s_j s_i \qquad \text{if } |i-j| \ge 2. && (6.3.3) \end{aligned}$$

The Hecke algebra of type A, $H_m(q)$, for $q \in \mathbb{C} \setminus \{0\}$ is the $\mathbb{C}$-algebra generated by the elements $\{T_{s_i} \colon 1 \le i \le m-1\}$ satisfying the following relations:

$$\begin{aligned} T_{s_i}^2 &= (q^2-1)T_{s_i} + q^2 \qquad \text{for } i = 1, \ldots, m-1, && (6.3.4)\\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j} \qquad \text{if } |i-j| = 1, && (6.3.5)\\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} \qquad \text{if } |i-j| \ge 2. && (6.3.6) \end{aligned}$$

If $s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for w of length $\operatorname{inv}(w) = \ell$, we define $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$. Every reduced expression can be obtained from another reduced expression by using only the relations (6.3.2) and (6.3.3) (see [Man01, Proposition 2.1.6]), so (6.3.5) and (6.3.6) imply that T_w is well defined. This also follows from Matsumoto's theorem (see [GP00, Theorem 1.2.2]). The elements T_w, $w \in S_m$, form a basis of the algebra $H_m(q)$ by Bourbaki's theorem, see for example [GP00, Theorem 4.4.6].

We also define $\{\widetilde{T}_{s_i} \colon 1 \le i \le m-1\}$ with $\widetilde{T}_{s_i}(q) = q^{-1}T_{s_i}$ and $\widetilde{T}_w = q^{-\operatorname{inv}(w)}T_w = \widetilde{T}_{s_{i_1}} \cdots \widetilde{T}_{s_{i_\ell}}$. The generators $\widetilde{T}_w$ satisfy the relations

$$\begin{aligned} \widetilde{T}_{s_i}^2 &= (q-q^{-1})\widetilde{T}_{s_i} + 1 \qquad \text{for } i = 1, \ldots, m-1, && (6.3.7)\\ \widetilde{T}_{s_i}\widetilde{T}_{s_j}\widetilde{T}_{s_i} &= \widetilde{T}_{s_j}\widetilde{T}_{s_i}\widetilde{T}_{s_j} \qquad \text{if } |i-j| = 1, && (6.3.8)\\ \widetilde{T}_{s_i}\widetilde{T}_{s_j} &= \widetilde{T}_{s_j}\widetilde{T}_{s_i} \qquad \text{if } |i-j| \ge 2. && (6.3.9) \end{aligned}$$

Remark 6.3.1 Usually, we take $T_{s_i}^2 = (q-1)T_{s_i} + q$ instead of (6.3.4) in the definition of $H_m(q)$, and $\widetilde{T}_w = q^{-\operatorname{inv}(w)/2}T_w$. However, in this context, the quantum algebra is assumed to satisfy relations (1.3.15)–(1.3.18) with q replaced by $q^{1/2}$. ♢

If $\operatorname{inv}(s_i w) = \operatorname{inv}(w) - 1$, we have $w = s_i(s_i w)$ with $\operatorname{inv}(s_i w) = \operatorname{inv}(w) - 1$. Then

$$\widetilde{T}_{s_i}\widetilde{T}_w = \begin{cases} \widetilde{T}_{s_i w} & : \ \operatorname{inv}(s_i w) = \operatorname{inv}(w) + 1 \\ \widetilde{T}_{s_i w} + (q-q^{-1})\widetilde{T}_w & : \ \operatorname{inv}(s_i w) = \operatorname{inv}(w) - 1 \end{cases} \tag{6.3.10}$$

by (6.3.7), and similarly

$$\widetilde{T}_w\widetilde{T}_{s_i} = \begin{cases} \widetilde{T}_{w s_i} & : \ \operatorname{inv}(w s_i) = \operatorname{inv}(w) + 1 \\ \widetilde{T}_{w s_i} + (q-q^{-1})\widetilde{T}_w & : \ \operatorname{inv}(w s_i) = \operatorname{inv}(w) - 1 \end{cases}. \tag{6.3.11}$$

The following lemma is a generalization of the equivalence $v = wu \iff w^{-1} = uv^{-1}$ and turns out to be very useful.

Lemma 6.3.2 *For every $u, v, w \in S_m$, we have*

$$[\widetilde{T}_v]\widetilde{T}_w\widetilde{T}_u = [\widetilde{T}_{w^{-1}}]\widetilde{T}_u\widetilde{T}_{v^{-1}}.$$

Proof. Let us first prove the statement for $v = \mathrm{id}$. Then we have to prove that

$$[1]\widetilde{T}_w\widetilde{T}_u = \begin{cases} 1 & : \quad u = w^{-1} \\ 0 & : \quad \text{otherwise} \end{cases}$$

Indeed, it is clear from equations (6.3.10) and (6.3.11) that there is no term $\widetilde{T}_{\mathrm{id}}$ in $\widetilde{T}_w\widetilde{T}_u$ if $\mathrm{inv}(w) < \mathrm{inv}(u)$ or $\mathrm{inv}(w) > \mathrm{inv}(u)$. Furthermore, there is no term $\widetilde{T}_{\mathrm{id}}$ in $\widetilde{T}_w\widetilde{T}_u$ if $\mathrm{inv}(w) = \mathrm{inv}(u)$ unless the reduced words for u is the reduced word for w in reverse order; in other words, unless $u = w^{-1}$. Clearly, $[1]\widetilde{T}_w\widetilde{T}_{w^{-1}} = 1$.
For every u, v, w, we can write

$$\widetilde{T}_w\widetilde{T}_u = \sum_x c_x\widetilde{T}_x, \qquad \widetilde{T}_u\widetilde{T}_{v^{-1}} = \sum_y d_y\widetilde{T}_y$$

for some $c_x, d_y \in \mathbb{C}$. Then

$$[1]\widetilde{T}_w\widetilde{T}_u\widetilde{T}_{v^{-1}} = [1]\left(\sum_x c_x\widetilde{T}_x\right)\widetilde{T}_{v^{-1}} = c_v,$$

but also

$$[1]\widetilde{T}_w\widetilde{T}_u\widetilde{T}_{v^{-1}} = [1]\widetilde{T}_w\left(\sum_y d_y\widetilde{T}_y\right) = d_{w^{-1}}.$$

Therefore $[\widetilde{T}_v]\widetilde{T}_w\widetilde{T}_u = [\widetilde{T}_{w^{-1}}]\widetilde{T}_u\widetilde{T}_{v^{-1}}$. □

Representations and characters. Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. A left $\mathcal{A}$-module $\mathcal{V}$ is called a *representation* of $\mathcal{A}$. We can also represent an algebra representation as an algebra homomorphism $\varphi = \varphi_{\mathcal{V}} \colon \mathcal{A} \to \mathrm{End}(\mathcal{V})$, where $\mathrm{End}(\mathcal{V})$ is the algebra of endomorphisms of $\mathcal{V}$.

Example 6.3.3 Let us prove that the map $\eta_q \colon H_m(q) \to \mathbb{C}$ defined by $\eta_q(\widetilde{T}_w) = q^{\mathrm{inv}(w)}$ is a representation, i.e. that we have $\eta_q(\widetilde{T}_w\widetilde{T}_v) = q^{\mathrm{inv}(w)+\mathrm{inv}(v)}$ for all w, v. This is obviously true if $v = \mathrm{id}$, assume that it holds for all w, v with $\mathrm{inv}(v) = k - 1$, and assume $\mathrm{inv}(v) = k$. We have $v = s_iv'$ for some $i \in \{1, \ldots, m-1\}$, $\mathrm{inv}(v') = k - 1$. If $\mathrm{inv}(ws_i) = \mathrm{inv}(w) + 1$, then

$$\eta_q(\widetilde{T}_w\widetilde{T}_v) = \eta_q(\widetilde{T}_w\widetilde{T}_{s_i}\widetilde{T}_{v'}) = \eta_q(\widetilde{T}_{ws_i}\widetilde{T}_{v'}) = q^{\mathrm{inv}(ws_i)+\mathrm{inv}(v')} = q^{\mathrm{inv}(w)+\mathrm{inv}(v)},$$

and if $\operatorname{inv}(ws_i) = \operatorname{inv}(w) - 1$, then

$$\eta_q(\widetilde{T}_w\widetilde{T}_v) = \eta_q(\widetilde{T}_w\widetilde{T}_{s_i}\widetilde{T}_{v'}) = \eta_q((\widetilde{T}_{ws_i} + (q-q^{-1})\widetilde{T}_w)\widetilde{T}_{v'}) =$$
$$= \eta_q(\widetilde{T}_{ws_i}\widetilde{T}_{v'}) + (q-q^{-1})\eta_q(\widetilde{T}_w\widetilde{T}_{v'}) = q^{\operatorname{inv}(ws_i)+\operatorname{inv}(v')} + (q-q^{-1})q^{\operatorname{inv}(w)+\operatorname{inv}(v')} =$$
$$= q^{\operatorname{inv}(w)+\operatorname{inv}(v)-2} + (q-q^{-1})q^{\operatorname{inv}(w)+\operatorname{inv}(v)-1} = q^{\operatorname{inv}(w)+\operatorname{inv}(v)}.$$

The representation is called the *trivial representation*. We can similarly prove that $\varepsilon_q\colon H_m(q) \to \mathbb{C}$ defined by $\varepsilon_q(\widetilde{T}_w) = (-q)^{-\operatorname{inv}(w)}$ is a representation, we call it the *sign representation*. ♢

For an $\mathcal{A}$-module $\mathcal{V}$ which is finitely generated and free over $\mathbb{C}$, the *character* of $\mathcal{V}$ is the linear map $\chi_{\mathcal{V}}\colon \mathcal{A} \longrightarrow \mathbb{C}$, $a \mapsto \operatorname{tr}(\varphi_{\mathcal{V}}(a))$.

First note the following. For a representation φ and the corresponding character φ, we have

$$\chi(ab) = \operatorname{tr}(\varphi(ab)) = \operatorname{tr}(\varphi(a)\varphi(b)) = \operatorname{tr}(\varphi(b)\varphi(a)) = \operatorname{tr}(\varphi(ba)) = \chi(ba). \qquad (6.3.12)$$

for every $a, b \in \mathcal{A}$.

The algebra $H_m(q)$ is finitely generated and free. Therefore it makes sense to talk about characters of its representations.

Equation (6.3.12) implies the following relation for characters of the Hecke algebra.

Theorem 6.3.4 *Take $w \in S_m$, $i \in \{1, \dots, m-1\}$, $s = s_i$, and a character χ of $H_m(q)$. Then:*

- *if $\operatorname{inv}(sws) = \operatorname{inv}(w)$, then $\chi(\widetilde{T}_{sws}) = \chi(\widetilde{T}_w)$;*
- *if $\operatorname{inv}(sws) = \operatorname{inv}(w) + 2$, then $\chi(\widetilde{T}_{sws}) = \chi(\widetilde{T}_w) + (q-q^{-1})\chi(\widetilde{T}_{sw}) = \chi(\widetilde{T}_w) + (q-q^{-1})\chi(\widetilde{T}_{ws})$;*
- *if $\operatorname{inv}(sws) = \operatorname{inv}(w) - 2$, then $\chi(\widetilde{T}_{sws}) = \chi(\widetilde{T}_w) - (q-q^{-1})\chi(\widetilde{T}_{sw}) = \chi(\widetilde{T}_w) - (q-q^{-1})\chi(\widetilde{T}_{ws})$;*

Proof. Assume that $\operatorname{inv}(sw) = \operatorname{inv}(w) - 1$ and $\operatorname{inv}(sws) = \operatorname{inv}(w)$. Then

$$\chi(\widetilde{T}_s\widetilde{T}_w\widetilde{T}_s) = \chi((\widetilde{T}_{sw} + (q-q^{-1})\widetilde{T}_w)\widetilde{T}_s) = \chi(\widetilde{T}_{sws}) + (q-q^{-1})\chi(\widetilde{T}_w\widetilde{T}_s)$$

and, by (6.3.12),

$$\chi(\widetilde{T}_s\widetilde{T}_w\widetilde{T}_s) = \chi(\widetilde{T}_w\widetilde{T}_s\widetilde{T}_s) = \chi(\widetilde{T}_w(1 + (q-q^{-1})\widetilde{T}_s)) = \chi(\widetilde{T}_w) + (q-q^{-1})\chi(\widetilde{T}_w\widetilde{T}_s),$$

so $\chi(\widetilde{T}_{sws}) = \chi(\widetilde{T}_w)$. If $\operatorname{inv}(sw) = \operatorname{inv}(w) + 1$ and $\operatorname{inv}(sws) = \operatorname{inv}(w)$, then

$$\chi(\widetilde{T}_s\widetilde{T}_w\widetilde{T}_s) = \chi(\widetilde{T}_{sw}\widetilde{T}_s) = \chi(\widetilde{T}_{sws}) + (q-q^{-1})\chi(\widetilde{T}_{sw})$$

and, by (6.3.12),

$$\chi(\widetilde{T}_s\widetilde{T}_w\widetilde{T}_s) = \chi(\widetilde{T}_s\widetilde{T}_s\widetilde{T}_w) = \chi((1+(q-q^{-1})\widetilde{T}_s)\widetilde{T}_w) = \chi(\widetilde{T}_w) + (q-q^{-1})\chi(\widetilde{T}_{sw}).$$

This proves (a). Let us prove (b). If $\mathrm{inv}(sws) = \mathrm{inv}(w)+2$, then $\mathrm{inv}(sw) = \mathrm{inv}(ws) = \mathrm{inv}(w)+1$, and so

$$\chi(\widetilde{T}_s\widetilde{T}_w\widetilde{T}_s) = \chi(\widetilde{T}_{sws}) = \chi(\widetilde{T}_w\widetilde{T}_s\widetilde{T}_s) =$$
$$= \chi(\widetilde{T}_w(1+(q-q^{-1})\widetilde{T}_s)) = \chi(\widetilde{T}_w) + (q-q^{-1})\chi(\widetilde{T}_{ws}),$$

and since $\chi(\widetilde{T}_s\widetilde{T}_w) = \chi(\widetilde{T}_w\widetilde{T}_s)$, we have $\chi(\widetilde{T}_{sws}) = \chi(\widetilde{T}_w) + (q-q^{-1})\chi(\widetilde{T}_{sw})$. Swapping the roles of w and sws, we get (c) from (b). □

In particular, we have the following; γ_μ, $\mu(w)$, $\mathbf{a}_w$ and $\mathbf{a}^w$ were defined in Section 2.7.

Corollary 6.3.5 *Every character χ of $H_m(q)$ has the following property. For a permutation $w \neq \gamma_\mu$, choose the smallest i that satisfies $w(i) = j+1 > i+1$. Then:*

- *if* $\mathrm{inv}(s_j w s_j) = \mathrm{inv}(w)$, *then* $\chi(\widetilde{T}_w) = \chi(\widetilde{T}_{s_j w s_j})$;
- *if* $\mathrm{inv}(s_j w s_j) = \mathrm{inv}(w) - 2$, *then* $\chi(\widetilde{T}_w) = \chi(\widetilde{T}_{s_j w s_j}) + (q-q^{-1})\chi(\widetilde{T}_{w s_j})$. □

Note that this is precisely the recursion from Theorem 2.7.1. That means that for any character χ of $H_m(q)$, the *quantum immanant*

$$\mathrm{Imm}_\chi A = \sum_{w \in S_m} \chi(\widetilde{T}_w)\mathbf{a}_w$$

is also given by

$$\mathrm{Imm}_\chi A = \sum_{w \in S_m} \chi(\widetilde{T}_{\gamma_{\mu(w)}})\mathbf{a}^w.$$

One important quantum immanant is the *quantum permanent*

$$\mathrm{per}_q A = \sum_{w \in S_m} q^{\mathrm{inv}(w)}\mathbf{a}_w = \sum_{w \in S_m} q^{m-\mathrm{cyc}(w)}\mathbf{a}^w,$$

which corresponds to the Hecke algebra trivial character. Furthermore, we have the *quantum determinant*,

$$\det_q A = \sum_{w \in S_m} (-q)^{-\mathrm{inv}(w)}\mathbf{a}_w = \sum_{w \in S_m} (-q)^{-(m-\mathrm{cyc}(w))}\mathbf{a}^w,$$

which corresponds to the Hecke algebra sign character.

Remark 6.3.6 Our definition of the quantum determinant in Section 1.4 was different: instead of $\mathbf{a}_w = a_{1w_1}a_{2w_2}\cdots a_{mw_m}$, we had $a_{w_1 1}a_{w_2 2}\cdots a_{w_m m}$. However, these terms are equal in the quantum algebra: we can move a_{1*} all the way to the left using (1.3.17), then move a_{2*} to the slot immediately to the right of a_{1*} using (1.3.17), and continue. ♢

Representations induced from parabolic subalgebras. For a subset J of $\{1,\dots,m-1\}$, we call the subgroup $S_J = \langle s_i \colon i \in J\rangle$ of S_m the *parabolic subgroup* of S_m, and the subalgebra H_J of $H_m(q)$ generated by $\{\widetilde{T}_{s_i} \colon i \in J\}$ the *parabolic subalgebra* of $H_m(q)$. Say that we are given a representation of H_J, i.e. a H_J-module $\mathcal{V}$. The tensor product $H_m(q) \otimes_{H_J} \mathcal{V}$ is naturally an $H_m(q)$-module with the action $h'(h \otimes v) = h'h \otimes v$. This is the *induced representation.* See [GP00, page 287].

Define

$$X_J = \{x \in S_m \colon \operatorname{inv}(xs_i) > \operatorname{inv}(x) \text{ for all } i \in J\} = \{x \in S_m \colon \operatorname{des} x \cap J = \emptyset\}.$$

Clearly, every $x \in X_J$ is the unique element of minimal length in the coset xS_J of S_m. Furthermore, for every $w \in S_m$, there exist unique $x \in X_J$ and $v \in S_J$ satisfying $w = xv$; we also have $\operatorname{inv}(xv) = \operatorname{inv}(vx^{-1}) = \operatorname{inv}(x) + \operatorname{inv}(v)$. See [GP00, Proposition 2.1.1] for the proof for a general Coxeter group.

Lemma 6.3.7 *For $x \in X_J$, $u \in S_J$ and $w \in S_m$, we have*

$$[\widetilde{T}_{xu}]\widetilde{T}_w = [\widetilde{T}_u]\widetilde{T}_{x^{-1}}\widetilde{T}_w.$$

Proof. The left hand side is equal to 1 if $w = xu$ and 0 otherwise. Write $w = yv$ for unique $y \in X_J$, $v \in S_J$. Then $\widetilde{T}_{x^{-1}}\widetilde{T}_w = \widetilde{T}_{x^{-1}}\widetilde{T}_y\widetilde{T}_v$, and $\widetilde{T}_{x^{-1}}\widetilde{T}_y$ is a linear combination of $\widetilde{T}_z$ with $z \in S_J$. But $\widetilde{T}_z\widetilde{T}_v = \widetilde{T}_{zv}$, so we can only get a non-zero coefficient in $\widetilde{T}_u$ if $v = u$. Furthermore,

$$[\widetilde{T}_u]\widetilde{T}_{x^{-1}}\widetilde{T}_y\widetilde{T}_u = [1]\widetilde{T}_{x^{-1}}\widetilde{T}_y,$$

and this is 1 if $y = x$ and 0 otherwise by Lemma 6.3.2. Therefore $[\widetilde{T}_u]\widetilde{T}_{x^{-1}}\widetilde{T}_w$ is 1 if $w = xu$ and 0 otherwise. □

The lemma is the Hecke algebra analogue of the statement $xu = w \iff u = x^{-1}w$.

Theorem 6.3.8 *If φ is a representation of H_J, the induced representation Φ has the block matrix form*

$$\Phi(\widetilde{T}_\pi) = \begin{pmatrix} F(\widetilde{T}_{x_1^{-1}}\widetilde{T}_\pi\widetilde{T}_{x_1}) & F(\widetilde{T}_{x_1^{-1}}\widetilde{T}_\pi\widetilde{T}_{x_2}) & \dots & F(\widetilde{T}_{x_1^{-1}}\widetilde{T}_\pi\widetilde{T}_{x_d}) \\ F(\widetilde{T}_{x_2^{-1}}\widetilde{T}_\pi\widetilde{T}_{x_1}) & F(\widetilde{T}_{x_2^{-1}}\widetilde{T}_\pi\widetilde{T}_{x_2}) & \dots & F(\widetilde{T}_{x_2^{-1}}\widetilde{T}_\pi\widetilde{T}_{x_d}) \\ \vdots & \vdots & \ddots & \vdots \\ F(\widetilde{T}_{x_d^{-1}}\widetilde{T}_\pi\widetilde{T}_{x_1}) & F(\widetilde{T}_{x_d^{-1}}\widetilde{T}_\pi\widetilde{T}_{x_2}) & \dots & F(\widetilde{T}_{x_d^{-1}}\widetilde{T}_\pi\widetilde{T}_{x_d}) \end{pmatrix},$$

where $X_J = \{x_1,\dots,x_d\}$, and F is the linear map that satisfies

$$F(\widetilde{T}_\pi) = \begin{cases} \varphi(\widetilde{T}_\pi) & : \ \pi \in S_J \\ 0 & : \ \pi \notin S_J \end{cases}.$$

Sketch of proof: The basis of the induced module is $\{T_y \otimes v_i\}$, where y runs over X_J and $\{v_i\}$ form the basis of V. Write $\widetilde{T}_\pi\widetilde{T}_y = \sum_w c_w\widetilde{T}_w$. Then every w can be written as xu for $x \in X_J$ and $u \in S_J$, and $\widetilde{T}_w = \widetilde{T}_x \widetilde{T}_u$. Then

$$\widetilde{T}_\pi(\widetilde{T}_y \otimes v_i) = \left(\sum_w c_w \widetilde{T}_w \right) \otimes v_j = \sum_{x,u} c_{xu} \widetilde{T}_x \widetilde{T}_u \otimes v_j =$$

$$= \sum_{x,u} c_{xu} \widetilde{T}_x \otimes \widetilde{T}_u v_j = \sum_{x,u} c_{xu} \widetilde{T}_x \otimes \varphi(\widetilde{T}_u) v_j.$$

But

$$c_{xu} = [\widetilde{T}_{xu}] \widetilde{T}_\pi \widetilde{T}_y = [\widetilde{T}_u] \widetilde{T}_{x^{-1}} \widetilde{T}_\pi \widetilde{T}_y$$

by the last lemma. That implies the theorem. □

Irreducible characters and Kostka numbers. We will use the following fact about irreducible characters of $H_m(q)$ without proof. In order to motivate it, let us return to the characters of the symmetric group for a moment. We already discussed its irreducible characters in Sections 6.1 and 6.2. Let us describe two more families of characters. The trivial characters of Young subgroups $S_{\lambda_1} \times \cdots \times S_{\lambda_p}$ of S_m induce the characters $\{\eta^\lambda : \lambda \vdash m\}$. They are given by

$$\eta^\lambda(\pi) = R_{\mu\lambda}, \tag{6.3.13}$$

where $\mu = (\mu_1, \ldots, \mu_r)$ is the type of the permutation π, and $R_{\mu\lambda}$ was defined on page 102.

The sign characters of Young subgroups induce the characters $\{\varepsilon^\lambda : \lambda \vdash m\}$, which are given by

$$\varepsilon^\lambda(\pi) = \sigma_\mu R_{\mu\lambda}, \tag{6.3.14}$$

where $\sigma_\mu = \operatorname{sign} \pi = (-1)^{j_2 + j_4 + \cdots}$ for $\mu = \langle 1^{j_1} 2^{j_2} \cdots \rangle$.

It is well known that the inverse Kostka numbers $K^{-1}_{\mu,\lambda} = \langle s_\lambda, m_\mu \rangle$ describe the expansions of irreducible S_m characters in terms of induced sign and trivial characters of S_m, i.e.

$$\chi^\lambda = \sum_\mu K^{-1}_{\mu,\lambda} \eta^\mu = \sum_\mu K^{-1}_{\mu,\lambda'} \varepsilon^\mu.$$

Somewhat surprisingly, these numbers also describe the expansions of irreducible $H_m(q)$ characters in terms of induced sign and trivial characters of $H_m(q)$ no "quantum analogue" of inverse Kostka numbers is needed for this purpose. See [GP00, §9.1.9]. In other words, we have

$$\chi_q^\lambda = \sum_\mu K^{-1}_{\mu,\lambda} \eta_q^\mu = \sum_\mu K^{-1}_{\mu,\lambda'} \varepsilon_q^\mu. \tag{6.3.15}$$

6.4 Quantum Goulden-Jackson immanant formula

The purpose of this section is to prove the following Hecke algebra analogue of Theorem 6.1.1.

Define the sequences $(H_k)_{k\in\mathbb{Z}}$, $(E_k)_{k\in\mathbb{Z}}$ in $\mathcal{A}$ by

$$\frac{1}{\det_q(I-tA)} = \sum_{k=0}^{\infty} H_k t^k, \qquad \det_q(I+tA) = \sum_{k=0}^{m} E_k t^k,$$

and by requiring that polynomials with indices not appearing in these sums be zero.

Theorem 6.4.1 *Take a partition $\lambda \vdash m$ and a permutation $\pi \in S_m$. If the matrix $A = (a_{ij})_{m\times m}$ is quantum, then*

$$[\mathbf{a}_\pi]\det(H_{\lambda_i-i+j})_{p\times p} = [\mathbf{a}_\pi]\det(E_{\lambda'_i-i+j})_{\lambda_1\times\lambda_1} = \chi_q^\lambda(\widetilde{T}_\pi). \tag{6.4.1}$$

Note that det in the theorem does *not* contain powers of q, i.e. it is not the quantum determinant.

Before proving the theorem, let us notice that equation (6.3.13) is equivalent to the following. For a commutative matrix $A = (a_{ij})_{m\times m}$, the immanant

$$\mathrm{Imm}_{\eta^\lambda} A = \sum_{\pi\in S_m} \eta^\lambda(\pi) a_{1\pi(1)}\cdots a_{m\pi(m)}$$

can be expressed in terms of permanents of submatrices $A_I = (a_{ij})_{i,j\in I}$ as

$$\mathrm{Imm}_{\eta^\lambda} A = \sum_{(I_1,\ldots,I_p)} \mathrm{per}(A_{I_1})\cdots\mathrm{per}(A_{I_p}), \tag{6.4.2}$$

where the sum is over all p-tuples $(I_1,\ldots,I_p)$ of disjoint I_j, $|I_j| = \lambda_j$, with union $[m]$.

Similarly, equation (6.3.14) is equivalent to the following. For a commutative matrix $A = (a_{ij})_{m\times m}$, the immanant

$$\mathrm{Imm}_{\varepsilon^\lambda} A = \sum_{\pi\in S_m} \varepsilon^\lambda(\pi) a_{1\pi(1)}\cdots a_{m\pi(m)}$$

can be expressed in terms of determinants as

$$\mathrm{Imm}_{\varepsilon^\lambda} A = \sum_{(I_1,\ldots,I_p)} \det(A_{I_1})\cdots\det(A_{I_p}), \tag{6.4.3}$$

where the sum is over all p-tuples $(I_1,\ldots,I_p)$ of disjoint I_j, $|I_j| = \lambda_j$, with union $[m]$.

Formulas (6.4.2) and (6.4.3) are called Merris-Watkins formulas, see [MW85]. Theorem 6.4.1 will follow from (6.3.15) and the quantum analogues of these formulas (Theorem 6.4.4).

Lemma 6.4.2 *For a composition $\lambda = (\lambda_1,\ldots,\lambda_p) \vdash m$, define the subset J of generators of S_m by*

$$J = [m-1] \setminus \{\lambda_1, \lambda_1+\lambda_2, \ldots, \lambda_1+\ldots+\lambda_{p-1}\}.$$

Then we have

$$\sum_{v\in X_J, u\in S_J} q^{\operatorname{inv}(u)}\widetilde{T}_v\widetilde{T}_{u^{-1}}\widetilde{T}_{v^{-1}} = \sum_{w\in S_m}\eta_q^\lambda(\widetilde{T}_w)\widetilde{T}_{w^{-1}},$$

$$\sum_{v\in X_J, u\in S_J} (-q)^{-\operatorname{inv}(u)}\widetilde{T}_v\widetilde{T}_{u^{-1}}\widetilde{T}_{v^{-1}} = \sum_{w\in S_m}\varepsilon_q^\lambda(\widetilde{T}_w)\widetilde{T}_{w^{-1}}.$$

Proof. By Theorem 6.3.8, we have

$$\eta_q^\lambda(\widetilde{T}_w) = \sum_{v\in X_J}\sum_{u\in S_J} q^{\operatorname{inv}(u)}[\widetilde{T}_u]\widetilde{T}_{v^{-1}}\widetilde{T}_w\widetilde{T}_v,$$

and so

$$\sum_{w\in S_m}\eta_q^\lambda(\widetilde{T}_w)\widetilde{T}_{w^{-1}} = \sum_{w\in S_m}\left(\sum_{v\in X_J}\sum_{u\in S_J} q^{\operatorname{inv}(u)}[\widetilde{T}_u]\widetilde{T}_{v^{-1}}\widetilde{T}_w\widetilde{T}_v\right)\widetilde{T}_{w^{-1}}.$$

But by Lemma 6.3.2,

$$[\widetilde{T}_u]\widetilde{T}_{v^{-1}}\widetilde{T}_w\widetilde{T}_v = [\widetilde{T}_v]\widetilde{T}_w\widetilde{T}_v\widetilde{T}_{u^{-1}} = [\widetilde{T}_{w^{-1}}]\widetilde{T}_v\widetilde{T}_{u^{-1}}\widetilde{T}_{v^{-1}}$$

and so

$$\sum_{w\in S_m}\eta_q^\lambda(\widetilde{T}_w)\widetilde{T}_{w^{-1}} = \sum_{v\in X_J}\sum_{u\in S_J} q^{\operatorname{inv}(u)}\sum_{w\in S_m}\left([\widetilde{T}_{w^{-1}}]\widetilde{T}_v\widetilde{T}_{u^{-1}}\widetilde{T}_{v^{-1}}\right)\widetilde{T}_{w^{-1}} =$$

$$= \sum_{v\in X_J}\sum_{u\in S_J} q^{\operatorname{inv}(u)}\widetilde{T}_v\widetilde{T}_{u^{-1}}\widetilde{T}_{v^{-1}}.$$

The proof for the sign character is completely analogous. □

Remark 6.4.3 It is actually possible to prove the lemma directly from the definition of an induced character and Lemma 6.3.2, without proving Theorem 6.3.8. ♢

Define the subspace $\mathcal{A}_0$ of $\mathcal{A}$ as the space generated by $\{\mathbf{a}_w : w \in S_m\}$. Define an action of $H_m(q)$ on $\mathcal{A}_0$ by

$$\widetilde{T}_{s_i}\circ\mathbf{a}_w = a_{s_i,w} = \begin{cases}\mathbf{a}_{ws_i} & \text{if } \operatorname{inv}(ws_i) > \operatorname{inv}(w)\\ \mathbf{a}_{ws_i} + (q-q^{-1})\mathbf{a}_w & \text{if } \operatorname{inv}(ws_i) < \operatorname{inv}(w)\end{cases}.$$

Theorem 6.4.4 (quantum Merris-Watkins formulas) *Let $\lambda = (\lambda_1, \dots, \lambda_p)$ be a composition of m. Then we have*

$$\operatorname{Imm}_{\eta_q^\lambda} A = \sum_{(I_1,\dots,I_p)} \operatorname{per}_q(A_{I_1})\cdots\operatorname{per}_q(A_{I_p}), \tag{6.4.4}$$

$$\operatorname{Imm}_{\varepsilon_q^\lambda} A = \sum_{(I_1,\dots,I_p)} \det_q(A_{I_1})\cdots\det_q(A_{I_p}), \tag{6.4.5}$$

where the sums are over all sequences $(I_1, \dots, I_p)$ of pairwise disjoint subsets of $[m]$

satisfying $|I_j| = \lambda_j$, with union $[m]$.

Proof. Let us prove first that

$$\widetilde{T}_{s_i} \circ a_{v,w} = a_{s_i v,w} \text{ if } s_i v > v. \tag{6.4.6}$$

This is true by definition for $v = \mathrm{id}$, now assume that it is true for all v with k inversions. We want to prove that $\widetilde{T}_{s_i} \circ a_{vs_j,w} = a_{s_i vs_j,w}$ for $s_i vs_j > vs_j > v$. Note that $s_i v > v$ and so $\widetilde{T}_{s_i} \circ a_{v,w} = a_{s_i v,w}$ for all permutations w. We have

$$a_{vs_j,w} = \begin{cases} a_{v,ws_j} & : \quad w(j) < w(j+1) \\ a_{v,ws_j} + (q - q^{-1}) a_{v,w} & : \quad w(j) > w(j+1) \end{cases}$$

and therefore $\widetilde{T}_{s_i} \circ a_{vs_j,w} = \widetilde{T}_{s_i} \circ a_{v,ws_j} = a_{s_i v,ws_j} = a_{s_i vs_j,w}$ if $w(j) < w(j+1)$, and $\widetilde{T}_{s_i} \circ a_{vs_j,w} = \widetilde{T}_{s_i} \circ \big(a_{v,ws_j} + (q-q^{-1})a_{v,w}\big) = a_{s_i v,ws_j} + (q-q^{-1})a_{s_i v,w} = a_{s_i vs_j,w}$ if $w(j) > w(j+1)$. This proves the induction step for (6.4.6).
For a permutation π, let σ be the permutation we get by rearranging $\pi(\lambda_1 + \ldots + \lambda_{k-1} + 1), \ldots, \pi(\lambda_1 + \ldots + \lambda_k)$ in increasing order. For example, if $\lambda = (3,2,2)$ and $\pi = 6742135$, then $\sigma = 4671235$. Note that $\sigma \in X_J$. Furthermore, we get π from σ by permuting only positions $1, \ldots, \lambda_1$, $\lambda_1 + 1, \ldots, \lambda_2$, etc. In other words, $\pi = \sigma u$ for a permutation $u \in S_J$. Note that this is the unique decomposition of π as a product of an element of X_J and an element of S_J. In our example, $u = 2315467$. Since $\mathrm{inv}(u^{-1}\sigma^{-1}) = \mathrm{inv}(u^{-1}) + \mathrm{inv}(\sigma^{-1})$, we have $\widetilde{T}_{u^{-1}}\widetilde{T}_{\sigma^{-1}} \circ \mathbf{a}_{\mathrm{id}} = \mathbf{a}_{\sigma u} = \mathbf{a}_\pi$. By equation (6.4.6),

$$\widetilde{T}_\sigma \widetilde{T}_{u^{-1}} \widetilde{T}_{\sigma^{-1}} \circ \mathbf{a}_{\mathrm{id}} = a_{\sigma,\pi}.$$

Note that $\pi \mapsto (\sigma, u)$ is a bijection between S_m and $X_J \times S_J$, and that $\mathrm{inv}(u) = \mathrm{inv}(\pi) - \mathrm{inv}(\sigma)$. Therefore

$$\left(\sum_{v \in X_J, u \in S_J} q^{\mathrm{inv}(u)} \widetilde{T}_v \widetilde{T}_{u^{-1}} \widetilde{T}_{v^{-1}} \right) \circ \mathbf{a}_{\mathrm{id}} = \sum_{(I_1,\ldots,I_p)} \mathrm{per}_q(A_{I_1}) \cdots \mathrm{per}_q(A_{I_p}).$$

where the sum is over sequences $(I_1, \ldots, I_p)$ of pairwise disjoint subsets of $[m]$ satisfying $|I_j| = \lambda_j$, with union $[m]$. Similarly

$$\left(\sum_{v \in X_J, u \in S_J} (-q)^{-\mathrm{inv}(u)} \widetilde{T}_v \widetilde{T}_{u^{-1}} \widetilde{T}_{v^{-1}} \right) \circ \mathbf{a}_{\mathrm{id}} = \sum_{(I_1,\ldots,I_p)} \det_q(A_{I_1}) \cdots \det_q(A_{I_p}).$$

The result now follows from Lemma 6.4.2 and the fact that $T_{w^{-1}} \circ \mathbf{a}_{\mathrm{id}} = \mathbf{a}_w$. □

Corollary 6.4.5 *Define the elements*

$$\alpha_k = \sum_{|I|=k} \mathrm{per}_q(A_I), \qquad \beta_k = \sum_{|I|=k} \det_q(A_I)$$

for all k. Then α_k's (respectively, β_k's) commute modulo $\mathcal{A}_0$.

Proof. The product $\alpha_k\alpha_j$ is, modulo $\mathcal{A}_0$, equal to $\sum_{(I_1,I_2)} \mathrm{per}_q(A_{I_1}) \cdot \mathrm{per}_q(A_{I_2})$ with the sum over disjoint subsets I_1 and I_j of m, with $|I_1| = k$ and $|I_2| = j$. That is, by Theorem 6.4.4, equal to $\sum_I \mathrm{Imm}_{\eta_q^{(k,j)}} A_I$, with the sum over all $(k+j)$-subsets I of $[m]$. The result follows since $\eta_q^{(k,j)} = \eta_q^{(j,k)}$. The other proof is analogous. □

Proof of Theorem 6.4.1. All expression in the proof are taken modulo $\mathcal{A}_0$. Because quantum matrices are also q-right-quantum, we can use the q-right-quantum master theorem (Theorem 4.4.1) to see that $H_k = \alpha_k$, where α_k was defined in Corollary 6.4.5. The elements H_k commute; therefore we have

$$\det(H_{\lambda_i-i+j})_{p\times p} = \sum_\mu K^{-1}_{\mu,\lambda} H_{\mu_1}\cdots H_{\mu_p} = \sum_\mu K^{-1}_{\mu,\lambda} \mathrm{Imm}_{\eta_q^\lambda}(A) = \mathrm{Imm}_{\chi_q^\lambda} A.$$

The last equality is (6.3.15).
The second identity is even simpler, as we do not need the master theorem. It is obvious that $E_k = \beta_k$ (even in the algebra $\mathcal{A}$), and the rest of the proof is analogous to the one above. □

6.5 Combinatorial interpretations of characters of Hecke algebras

Theorem 6.4.4 can be used to (re)prove the combinatorial interpretation of the characters $\eta_q^\lambda(\widetilde{T}_{\gamma_\mu})$ (given implicitly by Ram [Ram91, Theorem 4.1]), $\varepsilon_q^\lambda(\widetilde{T}_{\gamma_\mu})$ and $\chi_q^\lambda(\widetilde{T}_{\gamma_\mu})$ (given by Ram and Remmel [RR97, Theorem 3]).

Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ and a composition $\mu = (\mu_1, \ldots, \mu_r)$ of m, and write $w = \gamma_\mu$. For a permutation $\pi \in S_m$, say that i is in kth λ-slot of π if $\lambda_1 + \ldots + \lambda_{k-1} < \pi^{-1}(i) \le \lambda_1 + \ldots + \lambda_k$ for some k. For example, for $\lambda = (4, 2, 1, 1)$ and $\pi = 56483721$, $4, 5, 6, 8$ are in the first λ-slot, and 3 and 7 are in the second λ-slot.

We proved in Theorem 6.4.4 that

$$\eta_q^\lambda(\widetilde{T}_{\gamma_\mu}) = [\mathbf{a}_w]\, \mathrm{Imm}_{\eta_q^\lambda} A = \sum_{(I_1,\ldots,I_p)} [\mathbf{a}_w]\mathrm{per}_q(A_{I_1})\cdots\mathrm{per}_q(A_{I_p}) =$$

$$= \sum_{\pi\in S_m} q^{\mathrm{inv}(\pi)-\mathrm{inv}(\sigma)}[\mathbf{a}_w]a_{\sigma_1,\pi_1}a_{\sigma_2,\pi_2}\cdots a_{\sigma_n,\pi_n} = \sum_{\pi\in S_m} q^{\mathrm{inv}(\pi)-\mathrm{inv}(\sigma)}[\mathbf{a}_w]a_{\sigma,\pi},$$

where we obtain σ from π by placing the elements in the same λ-slot into increasing order. For example, if $\lambda = (4, 2, 1, 1)$ and $\pi = 56483721$, then $\sigma = 45683721$.

Take a permutation $\pi \in S_m$. Denote by $\mathcal{J}$ the r-tuple $(J_1, \ldots, J_r)$ of multisets J_k so that J_k contains a copy of j for every i with $\lambda_1 + \ldots + \lambda_{j-1} < i \le \lambda_1 + \ldots + \lambda_j$ and $\mu_1 + \ldots + \mu_{k-1} < \pi(i) \le \mu_1 + \ldots + \mu_k$. For example, for $\pi = 56483721$, $\lambda = (4, 2, 1, 1)$ and $\mu = (3, 3, 2)$, we get $\mathcal{J} = ((2, 3, 4), (1, 1, 1), (1, 2))$. Clearly, J_k has μ_k elements, and there are λ_j copies of j in $\cup J_k$.

Theorem 6.5.1 *Take $w = \gamma_\mu$ for some composition μ. The map $\varphi = \varphi_{\lambda,\mu} \colon \pi \mapsto \mathcal{J}$ is a bijection between the set of π with*

$$[\mathbf{a}_w]a_{\sigma,\pi} \neq 0$$

and the set of r-tuples $\mathcal{J} = (J_1, \ldots, J_r)$ of multisets J_k for which $|J_k| = \mu_k$ and for which $\cup J_k$ contains λ_j copies of j. Furthermore, in this case

$$q^{\mathrm{inv}(\pi)-\mathrm{inv}(\sigma)}[\mathbf{a}_w]a_{\sigma,\pi} = q^{r-m}q^{2\sum_j N_=(J_j)}(q^2-1)^{\sum_j N_<(J_j)},$$

where $N_=(J) = |\{j < r \colon i_j = i_{j+1}\}|$, $N_<(J) = |\{j < r \colon i_j < i_{j+1}\}|$ for $J = (i_1, \ldots, i_r)$.

The theorem is proved in Section 6.6.

Define *quantum power symmetric function* $\overline{p}_\mu(x_1, x_2, \ldots)$ for a composition μ by

$$\overline{p}_\mu = \overline{p}_{\mu_1} \cdots \overline{p}_{\mu_s},$$

where

$$\overline{p}_r = \sum_J q^{2N_=(J)}(q^2-1)^{N_<(J)} x_{i_1}x_{i_2}\cdots x_{i_n};$$

here J runs over multisets $(i_1, \ldots, i_r)$ with $1 \leq i_1 \leq \ldots \leq i_r \leq m$. For example,

$$\overline{p}_3 = q^4 m_3 + q^2(q^2-1)m_{21} + (q^2-1)^2 m_{111}.$$

If $q = 1$, we get ordinary power symmetric functions. Furthermore, define

$$\widetilde{p}_\mu = \widetilde{p}_{\mu_1} \cdots \widetilde{p}_{\mu_s},$$

where

$$\widetilde{p}_r = \sum_J (-1)^{N_=(J)}(q^2-1)^{N_<(J)} y_{i_1}y_{i_2}\cdots y_{i_n};$$

here J runs over multisets $(i_1, \ldots, i_r)$ with $1 \leq i_1 \leq \ldots \leq i_r \leq m$. For example,

$$\overline{p}_3 = m_3 - (q^2-1)m_{21} + (q^2-1)^2 m_{111}.$$

If $q = 1$, we get $\sigma_\mu p_\mu$.

Theorem 6.5.1 immediately yields a combinatorial description of the characters η_q^λ and ε_q^λ, which we give in Theorem 6.5.3. In order to prove the combinatorial description of the irreducible characters χ_q^λ, we need the following Theorem. Note that this result was already proved in [RR97], see equation (22) and the remark following it. Our proof, though essentially equivalent, is completely direct and elementary (in particular, it does not need Littlewood-Richardson rule, Pieri rule or λ-ring manipulations). See also [RRW96].

Recall that a border strip is a connected skew shape with no 2×2 square. Equivalently, a skew shape λ/μ is a border strip if and only if $\lambda_i = \mu_{i-1} + 1$ for $i \geq 2$. The height ht of a border strip is one less than the number of rows, and the width wt is one less than the number of columns. The ordinary Murnaghan-Nakayama rule states that for any partition μ and $r \in \mathbb{N}$, we have

$$s_\mu \cdot p_r = \sum_\lambda (-1)^{\mathrm{ht}(\lambda/\mu)} s_\lambda, \tag{6.5.1}$$

where the sum is over all partitions $\lambda \supseteq \mu$ for which λ/μ is a border strip of size r. See [Sta99, Theorem 7.17.1].

Define a *broken border strip* to be a (not necessarily connected) skew shape with no 2×2 square. Equivalently, a skew shape λ/μ is a broken border strip if and only if $\lambda_i \leq \mu_{i-1} + 1$ for $i \geq 2$. A broken border strip λ/μ is a union of a finite number, $\mathrm{st}(\lambda/\mu)$, of border strips. Denote by $\mathrm{ht}(\lambda/\mu)$ the sum of heights of these strips and by $\mathrm{wt}(\lambda/\mu)$ the sum of widths of these tableaux. The following generalizes (6.5.1).

Theorem 6.5.2 (quantum Murnaghan-Nakayama rule) *For any partition μ and $r \in \mathbb{N}$ we have*

$$s_\mu \cdot \overline{p}_r = \sum_\lambda (-1)^{\mathrm{ht}(\lambda/\mu)} q^{2\,\mathrm{wt}(\lambda/\mu)} (q^2 - 1)^{\mathrm{st}(\lambda/\mu)-1} s_\lambda, \tag{6.5.2}$$

where the sum runs over all partitions $\lambda \supseteq \mu$ for which λ/μ is a broken border strip of size r.

The proof appears in Section 6.6.

A broken border strip tableau of shape λ/μ and type α is an assignment of positive integers to the squares of λ/μ such that:

- every row and column is weakly increasing,
- the integer i appears α_i times, and
- the set of squares occupied by i forms a broken border strip or is empty.

The weight of a broken border strip b is

$$\mathrm{weight}\, b = (-1)^{\mathrm{ht}(b)} q^{2\,\mathrm{wt}(b)} (q^2 - 1)^{\mathrm{st}(b)-1},$$

and the weight $\mathrm{weight}\, T$ of a broken border strip tableau T is the product of weights of its non-empty broken border strips.

Theorem 6.5.3 *Theorems 6.5.1 and 6.5.2 imply the following combinatorial descriptions of characters of the Hecke algebra.*

1. *We have*

$$\eta_q^\lambda(T_{\gamma_\mu}) = \sum q^{2\sum_j N_=(J_j)} (q^2 - 1)^{\sum_j N_<(J_j)}$$

and

$$\varepsilon_q^\lambda(T_{\gamma_\mu}) = \sum (-1)^{\sum_j N_=(J_j)} (q^2 - 1)^{\sum_j N_<(J_j)},$$

where the sums are over all r-tuples $(J_1, \ldots, J_r)$ of multisets J_j with $|J_j| = \mu_j$ and for which $\cup J_k$ contains λ_j copies of j.

2. We have

$$\chi_q^\lambda(T_{\gamma_\mu}) = \sum_T \text{weight}\, T,$$

where the sum is over all broken border strip tableaux T of shape λ and type μ.

Proof. (1) The first formula follows immediately from

$$\eta_q^\lambda(\widetilde{T}_{\gamma_\mu}) = \sum_{\pi\in S_m} q^{\mathrm{inv}(\pi)-\mathrm{inv}(\sigma)}[\mathbf{a}_w]a_{\sigma,\pi},$$

Theorem 6.5.1, and relation $T_{\gamma_\mu} = q^{m-r}\widetilde{T}_{\gamma_\mu}$ for $\mu = (\mu_1,\ldots,\mu_r)\vdash m$.
The second formula follows from

$$\varepsilon_q^\lambda(\widetilde{T}_{\gamma_\mu}) = \sum_{\pi\in S_m} (-q^{-1})^{\mathrm{inv}(\pi)-\mathrm{inv}(\sigma)}[\mathbf{a}_w]a_{\sigma,\pi},$$

Theorem 6.5.1, and relation $T_{\gamma_\mu} = q^{m-r}\widetilde{T}_{\gamma_\mu}$ for $\mu = (\mu_1,\ldots,\mu_r)\vdash m$.
(2) By equation (6.3.15),

$$\chi_q^\lambda(T_{\gamma_\mu}) = \sum_\nu K_{\nu,\lambda}^{-1}\eta_q^\nu(T_{\gamma_\mu}) = \sum_\nu K_{\nu,\lambda}^{-1}[m_\nu]\overline{p}_\mu =$$

$$= \sum_\nu K_{\nu,\lambda}^{-1}\langle\overline{p}_\mu, h_\nu\rangle = \left\langle \overline{p}_\mu, \sum_\nu K_{\nu,\lambda}^{-1}h_\nu\right\rangle = \langle\overline{p}_\mu, s_\lambda\rangle.$$

The result follows by Theorem 6.5.2 and induction on the length of μ. □

Example 6.5.4 For $\lambda = (3,2,1)$ and $\mu = (2,2,1,1)$, we have the following 4-tuples.

J_1	J_2	J_3	J_4	weight	J_1	J_2	J_3	J_4	weight
11	12	2	3	$q^2(q^2-1)^1$	11	12	3	2	$q^2(q^2-1)^1$
11	13	2	2	$q^2(q^2-1)^1$	11	22	1	3	$q^4(q^2-1)^0$
11	22	3	1	$q^4(q^2-1)^0$	11	23	1	2	$q^2(q^2-1)^1$
11	23	2	1	$q^2(q^2-1)^1$	12	11	2	3	$q^2(q^2-1)^1$
12	11	3	2	$q^2(q^2-1)^1$	12	12	1	3	$q^0(q^2-1)^2$
12	12	3	1	$q^0(q^2-1)^2$	12	13	1	2	$q^0(q^2-1)^2$
12	13	2	1	$q^0(q^2-1)^2$	12	23	1	1	$q^0(q^2-1)^2$
13	11	2	2	$q^2(q^2-1)^1$	13	12	1	2	$q^0(q^2-1)^2$
13	12	2	1	$q^0(q^2-1)^2$	13	22	1	1	$q^2(q^2-1)^1$
22	11	1	3	$q^4(q^2-1)^0$	22	11	3	1	$q^4(q^2-1)^0$
22	13	1	1	$q^2(q^2-1)^1$	23	11	1	2	$q^2(q^2-1)^1$
23	11	2	1	$q^2(q^2-1)^1$	23	12	1	1	$q^0(q^2-1)^2$

That means that

$$\nu_q^{(3,2,1)}(T_{214356}) = 4q^4 + 12q^2(q^2-1) + 8(q^2-1)^2 = 24q^4 - 28q^2 + 8.$$

Similarly we get

$$\varepsilon_q^{(3,2,1)}(T_{214356}) = 4 - 12(q^2-1) + 8(q^2-1)^2 = 8q^4 - 28q^2 + 24.$$

Finally, the following are all broken border strip tableaux of shape λ and type μ:

$$\begin{array}{llll} \begin{matrix}1&1&3\\2&2&\\4&&\end{matrix} & \begin{matrix}1&1&4\\2&2&\\3&&\end{matrix} & \begin{matrix}1&1&3\\2&4&\\2&&\end{matrix} & \begin{matrix}1&1&4\\2&3&\\2&&\end{matrix} \\ \\ \begin{matrix}1&2&2\\1&3&\\4&&\end{matrix} & \begin{matrix}1&2&2\\1&4&\\3&&\end{matrix} & \begin{matrix}1&2&3\\1&2&\\4&&\end{matrix} & \begin{matrix}1&2&4\\1&2&\\3&&\end{matrix} \\ \\ \begin{matrix}1&1&2\\2&3&\\4&&\end{matrix} & \begin{matrix}1&1&2\\2&4&\\3&&\end{matrix} & \begin{matrix}1&2&3\\1&4&\\2&&\end{matrix} & \begin{matrix}1&2&4\\1&3&\\2&&\end{matrix} \end{array}$$

Note that the first 8 tableaux are actually border strip tableaux. We have

$$\chi^{(3,2,1)}(T_{214356}) = 2q^4 - 4q^2 + 2 + 2q^2(q^2-1) - 2(q^2-1) = 4q^4 - 8q^2 + 4. \qquad \diamondsuit$$

Theorem 6.5.3 implies the following interesting fact.

Corollary 6.5.5 *The endomorphism ω defined on the algebra of symmetric functions by $\omega(e_r) = h_r$ satisfies $\omega(\overline{p}_\mu) = \widetilde{p}_\mu$.*

Proof. It is enough to prove that $\omega(\overline{p}_r) = \widetilde{p}_r$ for all r. We have

$$\langle \omega(\widetilde{p}_r), s_\lambda\rangle = \langle \widetilde{p}_r, \omega(s_\lambda)\rangle = \langle \widetilde{p}_r, s_{\lambda'}\rangle = \langle \widetilde{p}_r, \sum_\nu K^{-1}_{\nu,\lambda'} h_\nu\rangle =$$

$$= \sum_\nu K^{-1}_{\nu,\lambda'}[m_\nu]\widetilde{p}_r = \sum_\nu K^{-1}_{\nu,\lambda'}\varepsilon_q^\nu(T_{\gamma_r}) = \chi_q^\lambda(T_{\gamma_r}) = \langle \overline{p}_r, s_\lambda\rangle$$

and the claim follows. □

Remark 6.5.6 In the preparation of this thesis, an error was found in [Ram91, §8]. The entry for $\lambda = (3,2,1)$ and $\mu = (2,2,2)$ should be $2q^3 - 6q^2 + 6q - 2$ (or, in our language, $2q^6 - 6q^4 + 6q^2 - 2$) instead of $q^3 - 5q^2 + 5q - 1$. ♢

6.6 Proofs of Theorems 6.5.1 and 6.5.2

Proof of Theorem 6.5.1. We want to write $a_{\sigma,\pi}$ in the basis $\{\mathbf{a}_v : v \in S_m\}$. In order to do that, take the variable a_{1r} and move it to the left, all the while using relations (1.3.17), (1.3.18). Write $R = q - q^{-1}$. For example, when $\lambda = (2,2)$ and $\pi = 3421$, we have

$$a_{33}a_{44}a_{12}a_{21} = a_{33}(a_{12}a_{44} + Ra_{14}a_{42})a_{21} = a_{12}a_{33}a_{44}a_{21} + Ra_{13}a_{32}a_{44}a_{21} + Ra_{14}a_{33}a_{42}a_{21}.$$

In each of the new terms choose the variable $a_{2r'}$ and move it to the left, then the variable $a_{3r''}$, etc. In the end, we get a linear combination of certain $\mathbf{a}_v$ for $v \in S_m$, with each $\mathbf{a}_v$ having coefficient R^k for some integer k. Of course, if we are only interested in the coefficient of $\mathbf{a}_w$ for $w = \gamma_\mu$, we can disregard every term that cannot give $\mathbf{a}_w$ in the end. As an example, take $\lambda = (4,2,1,1)$, $\mu = (3,3,2)$ (so $w = 23156487$) and $\pi = 56483721$. We have $a_{45}a_{56}a_{64}a_{88}a_{33}a_{77}a_{22}a_{11} = a_{45}a_{56}a_{64}a_{88}a_{33}a_{77}(a_{11}a_{22} + Ra_{12}a_{21})$, but we can ignore the term with $a_{11}a_{22}$ because as we move a_{11} to the left, no term will start with a_{12}, which is the first variable of $\mathbf{a}_w$. We get

$$[\mathbf{a}_{23156487}]a_{45}a_{56}a_{64}a_{88}a_{33}a_{77}a_{22}a_{11} = R[\mathbf{a}_{23156487}]a_{45}a_{56}a_{64}a_{88}a_{33}a_{77}a_{12}a_{21} =$$

$$= R[\mathbf{a}_{23156487}]a_{45}a_{56}a_{64}a_{88}a_{33}a_{12}a_{77}a_{21} = R[\mathbf{a}_{23156487}]a_{12}a_{45}a_{56}a_{64}a_{88}a_{33}a_{77}a_{21} =$$

$$= R[\mathbf{a}_{23156487}]a_{12}a_{45}a_{56}a_{64}a_{88}a_{33}a_{21}a_{77} = R^2[\mathbf{a}_{23156487}]a_{12}a_{45}a_{56}a_{64}a_{88}a_{23}a_{31}a_{77} =$$

$$= R^2[\mathbf{a}_{23156487}]a_{12}a_{23}a_{45}a_{56}a_{64}a_{88}a_{31}a_{77} = R^2[\mathbf{a}_{23156487}]a_{12}a_{23}a_{31}a_{45}a_{56}a_{64}a_{88}a_{77} =$$

$$= R^3[\mathbf{a}_{23156487}]a_{12}a_{23}a_{31}a_{45}a_{56}a_{64}a_{78}a_{87} = R^3.$$

Let us figure out how this works in general. As we move a_{1r} to the left, we can either use the relation $a_{jk}a_{1l} = a_{1l}a_{jk}$ for $k < l$ (in which case the variable with the first index 1 remains the same) or the relation $a_{jl}a_{1k} = a_{1k}a_{jl} + (q - q^{-1})a_{1l}a_{jk}$ for $k < l$ (in which case the variable with the first index 1 remains the same in one term and is *increased* in the other).

All in all, the second index of the variable with the first index 1 will stay the same or increase, and the second index of variables with the first index 2 or more will stay the same or decrease. In particular, $w(1) \geq r$ for every w for which the coefficient of $\mathbf{a}_w$ in $a_{\sigma,\pi}$ is non-zero. Since $w(1) \leq 2$ for $w = \gamma_\mu$, we conclude that $r \leq 2$. Moreover, in the process of moving the variable a_{11} or a_{12} to the left, the second index can never increase above 2.

Now take the variable $a_{2r'}$ and assume that $r' > 3$. That means that as we move a_{1r} to the left, we do not use the relation $a_{2l}a_{1k} = a_{1k}a_{2l} + (q - q^{-1})a_{1l}a_{2k}$, and we still have the variable $a_{2r'}$. As we move this variable to the left, the second index can only increase, and we cannot end up with $\mathbf{a}_w$ for $w(2) \leq 3$. Therefore, if $[\mathbf{a}_w]a_{\sigma,\pi} \neq 0$, then we have $r' \leq 3$.

The conclusion we can make in general is that if $[\mathbf{a}_w]a_{\sigma,\pi} \neq 0$, then $\sigma_i \leq \pi_i + 1$. Now assume that $w(i) = i+1$ and that a_{ir} with $r \leq i$ appears in $a_{\sigma,\pi}$. After we move $a_{1r_1}, \ldots, a_{i-1,r_{i-1}}$ to the left, the second index of the variable with the first index i can only decrease, so it is $\leq i$. Furthermore, the variable with the second index $i+1$ is still on the same side of a_{is} as in $a_{\sigma,\pi}$. That means that in order for the coefficient at $\mathbf{a}_w$ to be non-zero, the variable with the second index $i+1$ must be weakly to the left of the variable with the first index i in $a_{\sigma,\pi}$. In other words, either we have $a_{i,i+1}$ in $a_{\sigma,\pi}$, or the slot of $i+1$ is strictly to the left of the slot of i.

On the other hand, assume that $a_{i,i+1}$ appears as a variable in $a_{\sigma,\pi}$. When we move $a_{1r_1}, \ldots, a_{i-1,r_{i-1}}$ to the left, we can ignore the terms in which we replace $a_{i,i+1}$ by a_{ir}

for $r \leq i$, since that would create a variable $a_{j,i+1}$ with $j < i$. And as we move $a_{i,i+1}$ to the left, we can only increase the second index. That means that $w(i) = i+1$.

Now take an r-tuple $\mathcal{J} = (J_1, \ldots, J_r)$ of multisets J_k for which $|J_k| = \mu_k$ and for which $\cup J_k$ contains λ_j copies of j, and let us find all π that φ maps into $\mathcal{J}$ and for which $[\mathbf{a}_w]a_{\sigma,\pi} \neq 0$. For example, for $\lambda = (4,2,1,1)$, $\mu = (3,3,2)$ (i.e. $w = 23156487$) and $\mathcal{J} = (234, 111, 12)$, the first 4 entries of π with $\varphi(\pi) = \mathcal{J}$ are, by definition of φ, $4, 5, 6$ and either 7 or 8 (in some order). However, for $[\mathbf{a}_w]a_{\sigma,\pi}$ to be non-zero, it cannot be 7, as we would have $w(7) = 8$, but the slot of 7 would be strictly to the left of the slot of 8. Similar reasoning shows that the next 2 entries of π must be 3 (it cannot be 1, because that would place 2 strictly to the right of 1 even though $w(1) = 2$, and it cannot be 2, because that would place 3 strictly to the right of 2 even though $w(2) = 3$) and 7 (as 8 already appears among the first 4 entries of π). We get that the 7th entry of π must be 2 and the 8th 1. Also, $w(4) = 5$ and $w(5) = 6$, and we proved that this implies that a_{45} and a_{56} appear in $a_{\sigma,\pi}$. In other words, there is only one option for π, and that is $\pi = 56483721$. Recall that indeed $[\mathbf{a}_{23156487}]a_{45}a_{56}a_{64}a_{88}a_{33}a_{77}a_{22}a_{11} = R^3 \neq 0$.

In general, our previous reasoning implies that for each $\mathcal{J}$, there is exactly one π with $[\mathbf{a}_w]a_{\sigma,\pi} \neq 0$ and $\varphi(\pi) = \mathcal{J}$, and it is constructed as follows. Rearrange each of the r multisets $J_1, \ldots, J_r$ into a weakly decreasing word, and concatenate them into one word α of length m, but write bars in between – in our example, we get $432|111|21$. Then for each k from 1 to s do the following:

1. write the positions of k in α – we get a word g_k of length λ_k (in our example, we get $g_1 = 4568$, $g_2 = 37$, $g_3 = 2$ and $g_4 = 1$)

2. rotate cyclically to the right the positions in g_k that belong to the same multiset J_j; this is h_k (in our example, we get $h_1 = 5648$, $h_2 = 37$, $h_3 = 2$ and $h_4 = 1$);

3. write $h_1, \ldots, h_s$ together; this is π (56483721 in our example).

As another example, take $\lambda = (4,4,3,2,1)$ and $\mu = (2,3,6,3)$. Note that in this case,

$$\mathbf{a}_w = a_{12}a_{21}a_{34}a_{45}a_{53}a_{67}a_{78}a_{89}a_{9,10}a_{10,11}a_{11,6}a_{12,13}a_{13,14}a_{14,12}.$$

For

$$\mathcal{J} = ((1,3),(1,4,5),(1,1,2,2,2,4),(2,3,3)),$$

we get $\alpha = 31|541|422211|332|$, $g_1 = (2,5,10,11)$, $g_2 = (7,8,9,14)$, $g_3 = (1,12,13)$, $g_4 = (4,6)$, $g_5 = (3)$, $h_1 = (2,5,11,10)$, $h_2 = (8,9,7,14)$, $h_3 = (1,13,12)$ $h_4 = (4,6)$ $h_5 = (3)$ and

$$a_{\sigma,\pi} = a_{22}a_{55}a_{10,11}a_{11,10}a_{78}a_{89}a_{97}a_{14,14}a_{11}a_{12,13}a_{13,12}a_{44}a_{66}a_{33}$$

It remains to see what the (non-zero) coefficient of $\mathbf{a}_w$ in $a_{\sigma,\pi}$ is. For example, take $\mu_3 = 6$ in the last example, and the corresponding multiset $J_3 = (1,1,2,2,2,4)$ and the word 422211. The entry corresponding to 4 is a_{66}, the terms corresponding to 222 are $a_{78}a_{89}a_{97}$, and the terms corresponding to 11 are $a_{10,11}a_{11,10}$. As we are moving the terms to the left, we replace $a_{97}a_{66}$ by $a_{67}a_{96}$ and $a_{11,10}a_{96}$ by $a_{9,10}a_{11,6}$, so the coefficient is R^2. In general, it is easy to use the same reasoning to see that the multiset J_k contributes $R^{N_<(J_k)}$ to the coefficient. On the other hand, it contributes $q^{N_=(J_k)}$ to $q^{\mathrm{inv}(\pi)-\mathrm{inv}(\sigma)}$. Therefore

$$q^{\mathrm{inv}(\pi)-\mathrm{inv}(\sigma)}[\mathbf{a}_w]a_{\sigma,\pi} = q^{\sum_j N_=(J_j)}(q-q^{-1})^{\sum_j N_<(J_j)} = q^{r-m}q^{2\sum_j N_=(J_j)}(q^2-1)^{\sum_j N_<(J_j)}.$$

This completes the proof. □

Proof of Theorem 6.5.2. The basic idea is the same is in the proof of [Sta99, Theorem 7.17.1]. Fix n, let $\delta = (n-1, n-2, \ldots, 0)$, and, for $\alpha \in \mathbb{N}^n$, write $a_\alpha = \det(x_i^{\alpha_j})_{i,j=1}^n$. The classical definition of Schur functions says that $a_{\lambda+\delta}/a_\delta = s_\lambda(x_1, \ldots, x_n)$. It is therefore enough to prove

$$a_{\mu+\delta} \cdot \overline{p}_r = \sum_\lambda (-1)^{\mathrm{ht}(\lambda/\mu)} q^{2\,\mathrm{wt}(\lambda/\mu)}(q^2-1)^{\mathrm{st}(\lambda/\mu)-1} a_{\lambda+\delta},$$

where the sum goes over all partitions $\lambda \supseteq \mu$ for which λ/μ is a broken border strip of size r and n is at least the number of parts of λ, and let $n \to \infty$ in order to prove the lemma. Throughout the proof, all functions depend on $x_1, \ldots, x_n$.

It is easy to see that for a partition ν with n parts

$$a_{\mu+\delta} \cdot m_\nu = \sum a_{\mu+\delta+\sigma(\nu)},$$

where the sum runs over all permutations σ of n (here $\sigma(\nu)$ is the composition we get if we shuffle the entries of ν according to σ, i.e. $\sigma(\nu)_{\sigma(i)} = \nu_i$). For example, for $\mu = 31$, $n = 4$ and $\nu = (2,2,1,0)$, then

$$a_{6310} \cdot m_{221} =$$

$$= a_{8520} + a_{8511} + a_{8430} + a_{8412} + a_{8331} + a_{8322} + a_{7332} + a_{7512} + a_{7530} + a_{6531} + a_{6522} + a_{6432}.$$

Of course, for every composition α, a_α is equal to $\pm a_\mu$ for some partition μ, and if α has a repeated part, then $a_\alpha = 0$. For example,

$$a_{6310}m_{221} = a_{8520} + a_{8430} - a_{8421} - a_{7521} + a_{7530} + a_{6531} + a_{6432}.$$

Let us find the coefficient of $a_{\lambda+\delta}$ in $a_{\mu+\delta} \cdot \overline{p}_r$. Assume without loss of generality that λ and μ have n parts (some of which can be 0). We divide the calculations into two parts.

Assume first that λ/μ is a broken border strip (of size r). As a running example, let us take $\lambda = 5431$, $\mu = 33$, $r = 7$ and $n = 4$. We want to find the coefficient of a_{8641} in $a_{6510} \cdot \overline{p}_7$. Since λ/μ is a broken border strip, we have $\lambda_i \le \mu_{i-1} + 1$ for $i \ge 2$. In other words, $(\lambda+\delta)_i \le (\mu+\delta)_{i-1}$. We want to find all partitions ν (with possible zeros at the end) of r so that $\sigma(\nu) + \mu + \delta$ is a permutation of $\lambda + \delta$ for some σ. Equivalently, we want to find all compositions ν so that $\pi(\nu + \mu + \delta) = \lambda + \delta$ for some permutation $\pi \in S_n$. In our example, $\pi_1((2,1,3,1) + (6,5,1,0)) = \pi_2((0,3,3,1) + (6,5,1,0)) =$

$\pi_3((2,1,0,4)+(6,5,1,0)) = \pi_4((0,3,0,4)+(6,5,1,0)) = (8,6,4,1)$ for $\pi_1 = 1234$, $\pi_2 = 2134$, $\pi_3 = 1243$ and $\pi_4 = 2143$. Note that the signs of these permutations are $1,-1,-1,1$, respectively.

Since $(\nu+\mu+\delta)_{i-1} \geq (\mu+\delta)_{i-1} \geq (\lambda+\delta)_i$, we must have $\pi(i) \leq i+1$ for $i \leq n-1$. Furthermore, if $\lambda_i \leq \nu_{i-1}$, then $\pi(i) \leq i$. Denote by $\mathcal{I} \subseteq \{2,\ldots,n\}$ the set of i with $\lambda_i = \nu_{i-1}+1$. In our example, $\mathcal{I} = \{2,4\}$. Note that the elements in $\mathcal{I}$ correspond to rows that contain cells of the broken border strip λ/μ, but are not the first row of a border strip of λ/μ. Furthermore, denote by $\mathcal{K}$ the set of all i with $\lambda_i = \nu_i$. The elements of $\mathcal{K}$ correspond to empty rows of λ/μ.

To a composition ν with $\pi(\nu+\mu+\delta) = \lambda+\delta$ for some π, assign $\mathcal{I}_\nu \subseteq \mathcal{I}$ by $\mathcal{I}_\nu = \{i \in \mathcal{I} \colon \pi(i) = i+1\}$. In our example, we have $\mathcal{I}_{2131} = \emptyset$, $\mathcal{I}_{0331} = \{2\}$, $\mathcal{I}_{2104} = \{4\}$, $\mathcal{I}_{0304} = \{2,4\}$. It is very easy to see that this assignment is a bijection between compositions ν for which $\pi(\nu+\mu+\delta) = \lambda+\delta$ for some permutation $\pi \in S_m$, and subsets of $\mathcal{I}$. It remains to figure out the appropriate sign and weight, and to sum over all subsets of $\mathcal{I}$. In the running example, the weights of $m_{3211}, m_{3310}, m_{4210}, m_{4300}$ in $\overline{p}_7$ are $q^6(q^2-1)^3, q^8(q^2-1)^2, q^8(q^2-1)^2, q^{10}(q^2-1)$, respectively, so the coefficient of a_{5431} in $a_{33} \cdot \overline{p}_7$ is

$$q^6(q^2-1)^3 - q^8(q^2-1)^2 - q^8(q^2-1)^2 + q^{10}(q^2-1) =$$

$$= q^6(q^2-1)\left((q^2-1)^2 - 2q^2(q^2-1) + q^4\right) = q^6(q^2-1)(q^2-(q^2-1))^2 = q^6(q^2-1).$$

Note that λ/μ is composed of two border strips of widths 2 and 1 and heights 1 and 1, so the result matches with (6.5.2).

For $\mathcal{J} \subseteq \mathcal{I}$, the corresponding π satisfies $\pi(i) = i+1$ for $i \in \mathcal{J}$, and the remaining elements appear in increasing order in π. In other words, the disjoint cycle decomposition of π is of the form $(1,2,\ldots,i_1)(i_1+1,i_1+2,\ldots,i_2)\cdots$, where $i_0 = 1, i_1, i_2, \ldots$ are precisely the elements of $\{1,\ldots,n\} \setminus \mathcal{J}$. Since cycles of odd length are even permutations and cycles of even length are odd permutations, that means that the sign of π is $(-1)^{|\mathcal{J}|}$.

Recall that the weight of m_ν in $\overline{p}_r$ is $q^{2(r-s)}(q^2-1)^{s-1}$, where s is the number of different non-zero parts of ν. That means that for a subset $\mathcal{I}_\nu$ of $\mathcal{I}$, m_ν appears with weight $q^{2(r-n+|\mathcal{I}_\nu|+|\mathcal{K}|)}(q^2-1)^{n-|\mathcal{I}_\nu|-|\mathcal{K}|-1}$. In turn, this implies that the coefficient of $a_{\lambda+\delta}$ in $a_{\mu+\delta} \cdot \overline{p}_r$ is

$$\sum_{\mathcal{J} \subseteq \mathcal{I}} (-1)^{|\mathcal{J}|} q^{2(r-n+|\mathcal{J}|+|\mathcal{K}|)}(q^2-1)^{n-|\mathcal{J}|-|\mathcal{K}|-1} =$$

$$= (-1)^{|\mathcal{I}|} q^{2(r-n+|\mathcal{K}|)}(q^2-1)^{n-|\mathcal{I}|-|\mathcal{K}|-1} \sum_{k=0}^{|\mathcal{I}|} \binom{|\mathcal{I}|}{k} (-1)^{|\mathcal{I}|-k} q^{2k}(q^2-1)^{|\mathcal{I}|-k} =$$

$$= (-1)^{|\mathcal{I}|} q^{2(r-n+|\mathcal{K}|)}(q^2-1)^{n-|\mathcal{I}|-|\mathcal{K}|-1} \left(q^2-(q^2-1)\right)^{|\mathcal{I}|} =$$

$$= (-1)^{\mathrm{ht}(\lambda/\mu)} q^{2\,\mathrm{wt}(\lambda/\mu)}(q^2-1)^{\mathrm{st}(\lambda/\mu)-1}.$$

The second part of the proof deals with the case when λ/μ is *not* a broken border strip. Let us start with an example. Choose $\lambda = (6,5,4,3,2)$, $\mu = (4,2,2,2)$, $r = 10$ and $n = 5$. Then $\lambda + \delta = (10,8,6,4,2)$ and $\mu + \delta = (8,5,4,3,0)$, so we get the following table of compositions ν and permutations π for which $\pi(\nu + \mu + \delta) = \lambda + \delta$ (weight(ν) denotes the coefficient of m_ν in $\overline{p}_r$):

ν	π	sign π	weight(ν)
23212	12345	$+1$	$q^{10}(q^2-1)^4$
23032	12435	-1	$q^{12}(q^2-1)^3$
21412	13245	-1	$q^{10}(q^2-1)^4$
21052	13425	$+1$	$q^{12}(q^2-1)^3$
05212	21345	-1	$q^{12}(q^2-1)^3$
05032	21435	$+1$	$q^{14}(q^2-1)^2$
01612	23145	$+1$	$q^{12}(q^2-1)^3$
01072	23415	-1	$q^{14}(q^2-1)^2$

The involution

$$12345 \overset{\varphi}{\longleftrightarrow} 13245, \quad 12435 \overset{\varphi}{\longleftrightarrow} 13425, \quad 21345 \overset{\varphi}{\longleftrightarrow} 23145, \quad 21435 \overset{\varphi}{\longleftrightarrow} 23415$$

reverses signs and preserves weights, so the total coefficient of a_{65432} in $a_{4222} \cdot \overline{p}_{10}$ is 0.

In general, a sign-reversing weight-preserving involution is constructed as follows. Take the maximal i for which $\mu_{i-1} + 1 < \lambda_i$ (in our example, $i = 3$). Such an i exists because λ/μ is not a broken border strip. Choose a composition ν and permutation π with $\pi(\nu + \mu + \delta) = \lambda + \delta$. Note that the maximality of i implies $(\lambda+\delta)_{\pi(i-1)} = (\nu+\mu+\delta)_{i-1} \geq (\mu+\delta)_{i-1} > (\mu+\delta)_i \geq (\lambda+\delta)_{i+1}$ and so $\pi(i-1) \leq i$. Similarly, for $i \leq j < n$, we have $(\lambda+\delta)_{\pi(j)} = (\nu+\mu+\delta)_j \geq (\mu+\delta)_j \geq (\lambda+\delta)_{j+1}$ and $\pi(j) \leq j+1$. Choose the smallest $k \geq i$ with $\pi(k) \leq k$. Part of π is

$$\begin{pmatrix} \dots & i-1 & i & i+1 & \dots & k-1 & k & \dots \\ \dots & \pi(i-1) & i+1 & i+2 & \dots & k & \pi(k) & \dots \end{pmatrix}.$$

Note that $\pi(i-1), \pi(k) \leq i$. In the example, we have $k = 3,4,3,4,3,4,3,4$.

Define $\varphi(\pi) = \pi \cdot (i-1, k)$. Then $\varphi(\pi)$ has the following form:

$$\begin{pmatrix} \dots & i-1 & i & i+1 & \dots & k-1 & k & \dots \\ \dots & \pi(k) & i+1 & i+2 & \dots & k & \pi(i-1) & \dots \end{pmatrix}.$$

Clearly, φ is a sign-reversing involution. Furthermore,

$$\nu_{i-1} = (\nu+\mu+\delta)_{i-1} - (\mu+\delta)_{i-1} = (\lambda+\delta)_{\pi(i-1)} - (\mu+\delta)_{i-1} \geq (\lambda+\delta)_i - (\mu+\delta)_{i-1} > 0,$$

$$\nu_k = (\nu+\mu+\delta)_k - (\mu+\delta)_k = (\lambda+\delta)_{\pi(k)} - (\mu+\delta)_k > (\lambda+\delta)_i - (\mu+\delta)_{i-1} > 0.$$

These are the only entries that change in ν when we take $\varphi(\pi)$ instead of π, and since they are both strictly positive before and after the change, φ preserves weight. □

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