ELECTROMAGNETIC INDUCTION ON AN EXPANDING CONDUCTING SPHERE

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Abstract

One of the explanations that has been proposed for the origin of the radio signals emitted by nuclear explosions in the atmosphere is that such signals are generated by the interaction of the hot, expanding, electrically conducting gases, which result from the explosion, with the earth's magnetic field. The model adopted here for the study of this process consists of a conducting sphere expanding radially in a uniform, static magnetic field. The velocity is prescribed, as a function of space and time, for all points in the interior of the sphere. The problem consists in the determination of the electromagnetic fields and the electromagnetic force acting on the sphere as a result of its expansion. By prescribing the velocity, the hydrodynamic problem, which is difficult to solve, is avoided and the problem is restricted to the domain of classical Electromagnetic Theory.

The major part of the work reported here is concerned with the case of a perfectly conducting expanding sphere. This problem is solved rigorously with the help of an integral equation relating the current density to the magnetic vector potential. The fields are determined in integral form for any arbitrary velocity of expansion, and calculated explicitly for the particular case of constant velocity of expansion. The physical interpretation of these solutions is discussed. The electromagnetic force acting on the sphere is found, and the energy-power balance at the surface of the sphere is investigated with the help of Poynting's theorem.

The case of the expanding sphere with finite conductivity is formulated, but exact solutions are not given. A high-conductivity approximation is obtained under the assumption that the external magnetic field at the surface of the sphere is essentially the same as that found for the case of infinite conductivity. The approximate solution is found by deriving and solving the differential equation satisfied by the magnetic vector potential inside the sphere. Constant velocity of expansion is assumed. The solution is reduced to a particularly simple form for points not too far from the surface.

A rough calculation of the expected electric field 1000 km from a typical nuclear explosion shows that the field is measurable, and its value is of the same order of magnitude as that predicted by another proposed theory.

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I. INTRODUCTION

1.1 STATEMENT OF THE PROBLEM

This report concerns a theoretical investigation of the electromagnetic fields associated with the radial expansion of a spherical conductor immersed in an originally uniform, static, magnetic field. Radial expansion, as used here, implies an increase in size without change of geometrical shape. We have a conducting sphere of initial radius a_o in a static magnetic field which, in the absence of the sphere, is uniform. At a certain time (t=0) the sphere begins to expand. The radius of the sphere is a prescribed arbitrary function of time, a(t). Since the sphere is electrically conducting, its interaction with the magnetic field as it expands results in the induction of currents that, to a greater or lesser extent, shield the magnetic field from the interior of the sphere. The determination of the fields generated by these currents is the objective of this research. In general, it is necessary to prescribe not only the time dependence of the radius but also the velocity as a function of space and time for all points inside the sphere. The forces that are necessary to cause the expansion are assumed to exist, but are not prescribed. Once the fields and currents are found, these forces can be determined. The cases of perfect and imperfect conductivity will be investigated.

1.2 MOTIVATION OF THE PROBLEM

Since the early days of electromagnetic theory, problems involving the interaction of moving conductors and magnetic fields have been of interest to scientists and engineers. In particular, the experience gained in the study of the motion of rigid conductors in magnetic fields led to the present techniques of electromechanical energy conversion in motors and generators. Presumably, because of their importance for such application, rigid conductors in motion have received more attention than nonrigid or deformable conductors.

Problems involving moving deformable conductors were first restricted to the domain of cosmic physics. Since the turn of the century, physicists have been aware of the existence of magnetic fields associated with the solar sunspots and with the sun as a whole. More recently, the existence of strong magnetic fields in the stars has been proved. The sun and the stars are partly formed of masses of ionized gases possessing, to a greater or lesser extent, electrical conductivity. The motion of these masses in the presence of the magnetic field is the source of some of the important characteristics of the sun and stars.[1] Attempts to describe these phenomena quantitatively led to the study of the motion of conducting fluids in magnetic fields and eventually to the rapidly growing field of magnetohydrodynamics.[1,2] Magnetohydrodynamic processes have recently created considerable interest among engineers, who are looking for the possibilities of using these processes as a basis for new energy-conversion devices,[3,4] as well as other engineering applications.[5] The formulation of magnetohydrodynamics will not be discussed here, and the interested reader is referred to Alfvén and Cowling.[1,2] Suffice it to say that such formulation, as it involves the coupling between the moving conducting fluid and the external magnetic field, must include the equations of fluid motion in addition to the electromagnetic equations of Maxwell. As a result of the greater complexity of the equations, their solutions are in general more difficult to obtain than those of problems in either fluid dynamics or electromagnetic theory alone.

In some cases, the difficulties involved in the magnetohydrodynamic formulation can be avoided. In fact, if the velocity of the moving conductor can be specified as a function of space and time for all points inside the conductor, the problem may be handled by classical electromagnetic theory. Whether or not the results obtained by this method are significant will depend largely on the particular physical process to which the problem is related. The foregoing remarks apply to the problem whose solution is reported here. Although it involves the motion of a deformable conductor, the motion of the conductor is prescribed, so that the hydrodynamic treatment, with its inherent difficulties, is avoided. As a result, exact solutions may be obtained for the case of the expanding sphere with infinite conductivity which would be impossible otherwise. These solutions are meaningful if only for the insight that they provide into the basic features of the fields generated by the motion of deformable conductors in magnetic fields. Moreover, a practical problem of current interest for which the expanding sphere can be used as a model justifies serious investigation.

We refer to the problem of detecting explosions (and in particular nuclear explosions) in the atmosphere. Such detection may be carried out, among other ways, by measuring the electromagnetic signals generated by the burst. In fact, nuclear explosions are followed by electromagnetic signals that are detectable up to distances of thousands of kilometers.[6] Some theoretical work has been done to investigate the origin of such signals. They have been attributed to the emission of gamma rays[7] and to plasma oscillations in the ionized medium surrounding the explosion.[7,8] A different mechanism, conceptually simpler than those mentioned above, was suggested by O. I. Leipunskii.[9] It may be explained as follows. Within a few millionths of a second of the detonation of a nuclear bomb, the intensely hot gases at extremely high pressure resulting from the explosion appear as a roughly spherical highly luminous mass.[10] This mass is referred to as "the ball of fire." Immediately after its formation, the ball of fire begins to grow in size. Within seven-tenths of a millisecond from the detonation, the ball of fire from a 1-megaton bomb reaches a radius of approximately 220 feet, and this increases to a maximum of 3,600 feet in 10 seconds. Because of the high pressures and temperatures involved, the ball of fire is electrically conducting. As it expands against the earth's magnetic field, circulating currents must be induced in the ball of fire to shield the field from its interior. The time-variant fields resulting from this interaction are the ones suggested by Leipunskii as the cause of the detected signals. The attempts to embody the features of the process described above in a model resulted in the problem of the expanding sphere which is reported here.

This concludes our exposition of the background and the practical relevance of the problem. Before closing, it is appropriate to recall that the so-called electrodynamics of moving bodies, to which this problem is related, has always been a source of controversy among scientists and engineers.

We do not intend to discuss the relative merits of existing theories. It is enough to say that inconsistencies may be found in the treatment of polarizable and magnetizable matter. Recently, a new macroscopic formulation of the problem has been developed by L. J. Chu.[11] This formulation is not only esthetically satisfying, because of the symmetry that it brings into Maxwell's equations, but also free from the inconsistencies mentioned above. Although such difficulties do not arise in the solution of the problem reported here, Chu's formulation is implicitly adopted because of its inherent simplicity and symmetry.

1.3 OUTLINE AND SCOPE OF THE PROBLEM

The problem consists of the determination of the fields induced by the expansion of a conducting sphere in a static magnetic field. The motion of the sphere is prescribed, so that the problem is restricted to the domain of classical electromagnetic theory. Those interested in the hydrodynamic side of the problem are referred to the work of Jarem.[12] He analyzed the stability of an interface between dissimilar ionized gases undergoing variable acceleration. Among the cases investigated is included a spherical plasma configuration confined by a magnetic field. The sphere is driven radially by a plasma mass source located at the origin. If no magnetic field exists, the resulting expansion is radial, as one would expect from the symmetry of the source. The magnetic field is introduced as a perturbation, and it is found that the sphere elongates into an axisymmetric spheroid. The analysis is valid only for small departures from the original spherical shape. These results underline the difficulties associated with the hydrodynamic treatment.

This report may be divided into two parts. The first part is concerned with the expansion of a perfectly conducting sphere ($\sigma = \infty$), and it constitutes the major part of this research. Within the assumptions stated in Section II, this part of the problem is solved without further approximations, so that all results obtained are exact and therefore valid for relativistic velocities of expansion. The second part of this investigation examines the case of finite conductivity. An approximate solution is obtained for very high conductivities, and reduced to a particularly simple form for points close to the surface of the sphere. This part is treated in Section V.

The problem of the expanding sphere with infinite conductivity is stated in Section II, where the basic equations pertinent to the problem are listed, as well as the appropriate initial and boundary conditions. Section III is concerned with the formulation and solution of the problem. The conventional boundary value approach is discussed, but not followed. Instead, after showing that under the prescribed initial conditions the fields vanish inside the expanding sphere, an integral equation relating the induced current density to the magnetic vector potential is derived. This integral is interpreted as a convolution integral, on the basis of which an impulse response is defined as the potential measured, as a function of space and time, because of an excitation consisting of a spherical shell of current impulsive in time. The impulse response is then evaluated, and introduced in the integral equation. Integral expressions for the induced fields outside the sphere are derived. These expressions hold for any arbitrary rate of expansion of the sphere. In Section IV the general results of Section III are applied to the particular case of the sphere expanding at constant velocity. The induced current density is determined explicitly, as well as the induced fields outside the sphere. It is shown that these fields are those of a time-variant magnetic dipole. From the fields, the electromagnetic force acting on the sphere is found. Poynting's theorem is then applied, and shown to hold at the surface of the sphere. The various terms in the expression of Poynting's theorem are interpreted, and it is shown that the radially directed power density originates from two contributions. The first is the power density converted from mechanical into electromagnetic form. The second is the energy per unit time pushed out by the surface of the sphere as it moves. This is called "convected power" for brevity. It is also shown that, although for very low velocities these contributions are approximately equal, as the velocity increases, more power comes from the convected energy than from mechanical work.

The problem of the expanding sphere with finite conductivity is treated in Section V. For very high conductivities, it is assumed that the magnetic field at the surface of the sphere is substantially the same as if the conductivity were infinite. A differential equation is derived for the vector potential inside the sphere. No general solutions of this equation could be found in the time domain. The Laplace transformation is used to obtain solutions in the frequency domain. The magnetic field is then required to match a prescribed value at the surface. This value is the one obtained before, for the case of infinite conductivity, which is assumed not to change. The matching of fields at a moving surface in the frequency domain presents difficulties, which are overcome by using a convolution method. The solution is obtained first for an impulse in magnetic field, and superposition is used in the time domain to construct the final solution. The solutions obtained by this process include a term in integral form. It is shown that the integral term may be neglected for points near the surface, whereupon the fields may be reduced to a particularly simple form.

Finally, Section VI summarizes the results obtained and discusses the conclusions that may be drawn. The appendices contain the derivation of some of the expressions used throughout the work.

II. BASIC EQUATIONS

2.1 CONVENTIONS

The problem that we propose to solve involves a perfectly conducting sphere ($\sigma = \infty$, $\epsilon = \epsilon_o$, $\mu = \mu_o$) of radius a_o, situated in free space, where (in the absence of the sphere) a static uniform magnetic field of intensity H_o exists. At time t = 0, a uniform expansion of the sphere begins, so that the radius a is given as a function of time by

$$a(t) = a_o + a_1(t). \tag{1}$$

Here, $a_1(t)$ is an arbitrary prescribed function of time. The solution that we seek includes the determination of the electric and magnetic fields as functions of space and time, of the induced charges and currents, and of the forces of electromagnetic origin acting on the sphere.

Before we discuss the basic equations, as well as the initial and boundary conditions pertinent to the problem, the following conventions regarding the choice of a frame of reference must be stated. The fields, the velocity u(t), as well as any other physical quantities of interest are referred to a frame whose origin coincides with the center of the sphere. The observer is stationary with respect to this frame. The primed symbols $\bar{r}'(x', y', z')$ or $\bar{r}'(r', \theta', \phi')$ stand for source (charges and currents) coordinates, whereas the unprimed symbols $\bar{r}(x, y, z)$ or $\bar{r}(r, \theta, \phi)$ stand for the coordinates of the point of observation. Both $\bar{r}$ and $\bar{r}'$ are independent of time. This last statement is rather obvious when applied to the field coordinates $\bar{r}$. Thus, for instance, $\bar{r} = \bar{r}_o$ defines a point of coordinates (x_o, y_o, z_o). The question, what is the electric field at $\bar{r} = \bar{r}_o$ as a function of time, is perfectly unambiguous. The source coordinates, however, describe the location of charges and currents. In a problem involving moving matter, charges and currents are generally in motion, and it must be clearly understood that a vector $\bar{r}' = \bar{r}_o'$ is not associated with a given element of moving charge or current, but with a fixed point in space. This convention is the one used in the well-known Eulerian formulation of fluid dynamics, as opposed to the Lagrangian formulation, in which the independent variables are the coordinates of a given fluid element as it moves with respect to the fixed coordinate system.[13]

2.2 MAXWELL'S EQUATIONS FOR MOVING CONDUCTORS

The electromagnetic fields in the presence of moving matter are related through Maxwell's equations, suitably modified to include the effects of motion upon the electric and magnetic properties of matter. A thorough treatment of the subject may be found in Fano, Chu, and Adler.[14] We are concerned here with a moving conductor, which we assumed to have the constituent parameters of free space ($\mu = \mu_o$, $\epsilon = \epsilon_o$). Hence, polarization and magnetization of matter do not occur, and the corresponding terms drop out of Maxwell's equations, which reduce to

$$
\begin{aligned}
&\text{curl } \overline{H} - \epsilon_o \frac{\partial \overline{E}}{\partial t} = \overline{J} && \text{(a)}\\
&\text{curl } \overline{E} + \mu_o \frac{\partial \overline{H}}{\partial t} = 0 && \text{(b)}\\
&\text{div } \mu_o \overline{H} = 0 && \text{(c)}\\
&\text{div } \epsilon_o \overline{E} = \rho && \text{(d)}
\end{aligned}
\qquad (2)
$$

Here, $\overline{E}$ and $\overline{H}$ are the electric and magnetic field, ρ the electric charge density, and $\overline{J}$ the electric current density. Both conduction and convection currents are included in J. Since polarization effects are excluded, ρ and J refer to free (or true) charges and currents. From Eqs. 2a and 2d, it follows that ρ and J are related by the law of conservation of charge,

$$\text{div } J + \frac{\partial \rho}{\partial t} = 0.$$

One relation is necessary to complete the set of basic equations. This is Ohm's law for a perfect moving conductor

$$\overline{E} + \overline{u} \times \mu_o \overline{H} = 0. \qquad (3)$$

Here, $\overline{u}$ is the velocity of a given grain (macroscopic element of volume) of the moving conductor. Equation 3 must hold everywhere within the conductor. It stems from the requirement that the total force exerted on a charge by the macroscopic fields must vanish inside a perfect conductor.

These are all the equations that we need for the determination of the fields. The appropriate initial and boundary conditions will be examined presently.

We have not specified the velocity $\overline{u}(\overline{r}, t)$ everywhere, as required for the application of Ohm's law (3). It will be shown later that under the assumed initial conditions, both $\overline{E}$ and $\overline{H}$ vanish for all time inside the expanding sphere, in which case it is sufficient to specify the velocity of its surface. This velocity is given implicitly by Eq. 1.

2.3 BOUNDARY CONDITIONS AT A MOVING INTERFACE

The solutions of Maxwell's equations inside and outside the expanding sphere have to be matched across a moving surface. In the absence of polarizable and magnetizable matter, the following relations may be shown to hold (see Appendix A) between the fields on the two sides of a moving surface:

$$
\begin{aligned}
&\overline{n} \times [(\overline{E}_1 + \overline{v} \times \mu_o \overline{H}_1) - (\overline{E}_2 + \overline{v} \times \mu_o \overline{H}_2)] = 0\\
&\overline{n} \times [(\overline{H}_1 - \overline{v} \times \epsilon_o \overline{E}_1) - (\overline{H}_2 - \overline{v} \times \epsilon_o \overline{E}_2)] = \overline{K} - \sigma \overline{v}_T\\
&\overline{n} \cdot \epsilon_o (\overline{E}_1 - \overline{E}_2) = \sigma, \qquad \overline{n} \cdot \mu_o (\overline{H}_1 - \overline{H}_2) = 0.
\end{aligned}
\qquad (4)
$$

Here, $\overline{v}$ is the velocity of the moving surface, $\overline{v}_T$ its tangential component, the normal vector $\overline{n}$ points in the direction of medium 1, and the symbols σ and $\overline{K}$ refer to surface charge and surface current density, respectively.

2.4 CONDITIONS AT INFINITY

The currents and/or charges induced on the expanding sphere are the sources of the secondary or induced fields. These fields must satisfy boundary conditions at infinity. Let ψ be any component of the induced electric or magnetic field. The following conditions are imposed on ψ:

(a) ψ must vanish in such a way that $\lim_{r\to\infty} (r\psi)$ is bounded. This is the condition known as regularity at infinity.

(b) At large distances from the sources, ψ must represent an outward-traveling wave. This is the so-called radiation condition.

2.5 INITIAL CONDITIONS

According to Eq. 1, at $t = 0$ we have a sphere of radius a_o and infinite conductivity, immersed in a uniform static magnetic field H_o. The solutions of this static problem are the initial conditions that we seek. Hence the initial electric field is zero everywhere. As far as the magnetic field is concerned, however, we have no unique choice, unless additional assumptions are made. In fact, inside a stationary perfect conductor, Maxwell's equations require only that the magnetic field be time-invariant, since

$$\mu_o \frac{\partial \overline{H}}{\partial t} = - \text{curl}\, \overline{E} = 0.$$

The choice of $\overline{H}$ inside the conductor depends on the past history of the static problem. If, at $t = -\infty$, the field vanished inside the conductor, then

$$\overline{H}(t) = 0, \qquad t \leqslant 0$$

and the initial magnetic field for the expanding sphere is zero inside the sphere. Surface currents must exist to provide for the necessary discontinuity of $\overline{H}$ at the surface. These currents produce a secondary field outside, which can be easily shown to be the field of a static magnetic dipole located at the center of the sphere. The initial $\overline{H}$ field outside consists therefore of the superposition of this dipole field on the applied uniform field.

This we shall assume to be the case, so that the initial conditions are given by

$$\left.\begin{aligned} \overline{E}(\overline{r}, 0) &= 0 \\ \overline{H}(\overline{r}, 0) &= 0 \end{aligned}\right\} \qquad r < a_o \tag{5}$$

and

$$\left.\begin{aligned} \overline{E}(\overline{r}, 0) &= 0 \\ \overline{H}(\overline{r}, 0) &= \overline{H}_o + \overline{H}(\overline{r}) \end{aligned}\right\} \qquad r > a_o \tag{6}$$

where

$$\overline{H}_o = -H_o i_z = -H_o(\cos\theta\, i_r - \sin\theta\, i_\theta)$$

and

$$\overline{H}(\overline{r}) = \frac{1}{2} H_o \frac{a_o^3}{r^3}(2\cos\theta\, \tau_r + \sin\theta\, \tau_\theta).$$

We have collected all of the information necessary to proceed with the solution of the problem. The detailed formulation leading to the solution is the object of Section III.

III. THE SPHERE WITH ARBITRARY RADIAL EXPANSION

We shall now obtain in integral form the solution of the problem of the sphere whose velocity of expansion is an arbitrary function of time. The method of solution is based on the determination of the induced current density, which will be related to the magnetic vector potential through an integral equation. For this purpose, the induced fields are shown to be completely determined by the magnetic vector potential. Next, the differential equation satisfied by the vector potential is solved through the use of Green's function. The solution is obtained in integral form, and is interpreted as a convolution integral, which expresses the vector potential resulting from an arbitrary time-dependent current density as a superposition of elementary responses caused by impulsive excitations. The concepts of convolution integral and impulse response adopted here are slightly different from the conventional ones used extensively in linear circuit theory.[15] The impulse response that is due to a spherical shell of current is evaluated and interpreted. Finally, the integral equation for the current density is obtained, and the induced fields outside the sphere are expressed in integral form.

3.1 FORMULATION OF THE PROBLEM

A formal approach to the solution of the problem would consist of taking the basic equations established in Section II and, by eliminating one of the field vectors, arriving at the differential equation satisfied by the other field. The procedure must be applied to the two regions of interest, $r < a(t)$ (inside the sphere) and $r > a(t)$ (outside the sphere). Thus, for $r < a(t)$, Eqs. 2 and 3 yield

$$\text{curl } (\overline{u} \times \overline{H}) - \frac{\partial \overline{H}}{\partial t} = 0, \tag{7}$$

whereas for $r > a(t)$, Eqs. 2 and 3 (with $\overline{J} = 0$, $\rho = 0$ as required for free space) yield

$$\text{curl curl } \overline{H} + \frac{1}{c^2} \frac{\partial^2 \overline{H}}{\partial t^2} = 0, \tag{8}$$

where $c^2 = \frac{1}{\mu_o \epsilon_o}$ is the velocity of light in free space.

Solutions of (7) and (8) would then be sought, by taking into consideration the initial and boundary conditions set forth in sections 2.3, 2.4, and 2.5.

We shall not, however, follow the conventional approach, but rather formulate the problem in terms of an integral equation involving the induced current density. As a first step in the formulation, it will be shown that for the given initial conditions, the fields inside the sphere vanish for all time. That is,

$$\left.\begin{aligned} \overline{E}(r,t) &= 0 \\ \overline{H}(r,t) &= 0 \end{aligned}\right\} \qquad r < a(t). \tag{9}$$

In order to prove the foregoing statement, we take Eq. 7 and integrate it over a surface A bounded by a contour C that moves with the conductor. Each point of C is rigidly attached to a given grain of the moving conductor. Both C and A are therefore functions of time.

$$\int_A \frac{\partial \overline{H}}{\partial t} \cdot d\overline{a} = \int_A \text{curl } (\overline{u} \times \overline{H}) \cdot d\overline{a}.$$

Applying Stoke's theorem to the surface integral on the right-hand side, we obtain

$$\int_A \frac{\partial \overline{H}}{\partial t} \cdot d\overline{a} = \oint_C \overline{u} \times \overline{H} \cdot d\overline{s},$$

or

$$\int_A \frac{\partial \overline{H}}{\partial t} \cdot d\overline{a} + \oint_C \overline{H} \times \overline{u} \cdot d\overline{s} = 0. \tag{10}$$

But it may be shown that (see Appendix B)

$$\int_A \frac{\partial \overline{H}}{\partial t} \cdot d\overline{a} + \oint_C \overline{H} \times \overline{u} \cdot d\overline{s} = \frac{d}{dt} \int_A \overline{H} \cdot d\overline{a}. \tag{11}$$

Hence, (10) states that the flux of $\overline{H}$ across an arbitrary surface bounded by a contour moving with the conductor cannot change with time,

$$\frac{d\phi}{dt} = \frac{d}{dt} \int_A \overline{H} \cdot d\overline{a} = 0.$$

This is a well-known property of moving perfect conductors, which was first pointed out by H. Alfvén. He described it picturesquely by stating that the lines of force are "frozen" in the conductor.[16]

From the assumed initial conditions, $\overline{H}(\overline{r}, 0) = 0$, so that the flux is initially zero. Hence it must remain equal to zero for all time. This must be true for any arbitrary contour moving with the conductor. Therefore $\overline{H}(\overline{r}, t)$ must vanish for all time inside the conductor. Since $\overline{E}$ and $\overline{H}$ are related by Eq. 3, $\overline{E}(\overline{r}, t)$ must also vanish for all time, and (9) follows. This completes the proof.

The next step is to express the total fields as superpositions of primary fields (the fields that would exist in the absence of the sphere) and secondary (induced) fields:

$$\overline{H}_T = \overline{H}_p + \overline{H}$$

$$\overline{E}_T = \overline{E}_p + \overline{E} = \overline{E}.$$

Here, the subscript T stands for total, the subscript P for primary, and $\overline{E}_T = \overline{E}$ because there is no primary electric field. Since we have just shown that the total fields vanish inside the sphere, the secondary fields are specified by

$$\left.\begin{aligned} \overline{E} &= 0 \\ \overline{H} &= -\overline{H}_p \end{aligned}\right\} \quad r < a(t) \tag{12}$$

Outside the sphere, $\overline{E}$ and $\overline{H}$ are unspecified.

The sources of the secondary fields are the currents and/or charges induced on the surface of the sphere [r = a(t)]. It can be shown that, because of the symmetry of the problem, there are no induced charges. In fact, according to (12), the secondary $\overline{H}$ field must cancel the primary $\overline{H}_p$ field inside the sphere. Since the primary $\overline{H}_p$ field is uniform and z-directed, the secondary $\overline{H}$ field is also uniform and z-directed. The surface current distribution necessary to create such a field must be ϕ-directed, and independent of ϕ. Hence,

$$\text{div } \overline{K} = \frac{\partial K_\phi}{\partial \phi} = 0.$$

But conservation of charge requires that

$$\text{div } \overline{K} + \frac{\partial \sigma}{\partial t} = 0,$$

where σ is the surface charge density. It follows that

$$\frac{\partial \sigma}{\partial t} = 0,$$

and since at $t = 0$, $\sigma = 0$, then $\sigma = 0$ for all time.

The problem may now be restated, in terms of the secondary fields and the induced surface current density as follows: Given the fields $\overline{E}$ and $\overline{H}$ inside a spherical region of free space, the radius of which varies with time in a prescribed way; determine the surface currents (located over the boundary of the region) which generate the given fields. From the surface currents determine also the fields outside the spherical region.

Rather than relating the fields directly to the surface current density, we shall make use of the magnetic vector potential $\overline{A}$. In the general case,[17] an additional scalar potential, ϕ, is necessary, and the fields are expressed as

$$\mu_o \overline{H} = \text{curl } \overline{A}$$

$$\overline{E} = -\frac{\partial \overline{A}}{\partial t} - \text{grad } \phi.$$

The divergence of $\overline{A}$ may be specified by the additional equation

$$\text{div } \overline{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0.$$

It then follows that $\overline{A}$ and ϕ must satisfy the inhomogeneous wave equations

$$\nabla^2 \overline{A} - \frac{1}{c^2}\frac{\partial^2 \overline{A}}{\partial t^2} = -\mu_o \overline{J}$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_o}.$$

For the particular situation with which we are dealing, since we have shown that there are no charges, the scalar potential must satisfy the homogeneous wave equation,

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0.$$

From the prescribed initial conditions,

$$\phi(\overline{r}, 0) = \frac{\partial\phi}{\partial t}(\overline{r}, 0) = 0.$$

There are no boundary conditions to be satisfied, except regularity at infinity. Hence the scalar potential must vanish. The problem is entirely determined by the magnetic vector potential $\overline{A}$, which is related to the fields by

$$\begin{cases} \overline{H} = \frac{1}{\mu_o}\,\mathrm{curl}\,\overline{A} \\ \overline{E} = -\frac{\partial\overline{A}}{\partial t} \end{cases} \tag{13}$$

and is the solution of

$$\nabla^2\overline{A} - \frac{1}{c^2}\frac{\partial^2\overline{A}}{\partial t^2} = -\mu_o\overline{J}, \tag{14}$$

subject to the appropriate initial and boundary conditions.

3.2 AN INTEGRAL EQUATION FOR THE INDUCED CURRENT DENSITY

The next step to be taken is that of finding the solution of the inhomogeneous vector wave equation (14). The vector equation actually implies that the Cartesian components of $\overline{A}$ satisfy the scalar wave equation with the Cartesian components of $\overline{J}$ as driving terms. Let $\psi(\overline{r}, t)$ be any Cartesian component of $\overline{A}$, and $g(\overline{r}, t)$ the corresponding source density. We seek for solutions of the scalar wave equation in the general form

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -g(\overline{r}, t).$$

We shall express the solution in terms of a Green's function, $G(\overline{r}, t/\overline{r}', t')$.

This function is interpreted as the potential measured at the observation point $\overline{r}$ at time t because of an element of current at the source point $\overline{r}'$ at time t'. This element of current may be represented by

$$g(\overline{r}, t) = \delta(\overline{r} - \overline{r}')\,\delta(t - t'),$$

where the notation $\delta(x)$ is used for the impulse function (or delta function) located at

$x = 0$. It follows that the Green's function is a solution of

$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -\delta(\bar{r}-r')\,\delta(t-t'), \tag{15}$$

subject to the appropriate initial and boundary conditions. The determination of ψ in terms of the Green's function G has been carried out:

$$\psi(\bar{r},t) = \int_0^t dt' \int_V G\, g(\bar{r}',t')\, dv' + \int_0^t dt' \oint_S (G\nabla'\psi - \psi\nabla' G)\cdot d\bar{a}'$$

$$-\frac{1}{c^2}\int_V \left[\left[\frac{\partial G}{\partial t'}\right]_{t'=0} \psi_{t'=0} - G_{t'=0}\left[\frac{\partial \psi}{\partial t'}\right]_{t'=0}\right] dv'.$$

The surface S and the volume V enclosed by it are arbitrary. We shall let the surface S recede to infinity, and express the solution in terms of the volume integrals only. This is possible if the surface integral vanishes when S goes to infinity. By requiring that both ψ and G satisfy the conditions at infinity, the surface integral may be made to vanish. The volume integrals become integrals over all space:

$$\psi(\bar{r},t) = \int_0^t dt' \int_{\text{space}} G\, g(\bar{r}',t')\, dv' - \frac{1}{c^2}\int_{\text{space}} \left[\left[\frac{\partial G}{\partial t'}\right]_{t'=0} \psi_{t'=0} - G_{t'=0}\left[\frac{\partial \psi}{\partial t'}\right]_{t'=0}\right] dv'. \tag{16}$$

The Green's function for the problem may be obtained by solving Eq. 15. It is found to be

$$G(\bar{r},t/\bar{r}',t') = \frac{\delta(t-t'-R/c)}{4\pi R} \qquad \text{for} \begin{cases} R > 0 \\ t > t' \end{cases}$$

where $R \equiv |\bar{r}-\bar{r}'|$.

In Eq. 16, the second volume integral on the right-hand side accounts for the effect of the initial conditions. Henceforth, we shall be concerned with the situation for which

$$\psi_{t'=0} = \left[\frac{\partial \psi}{\partial t'}\right]_{t'=0} = 0.$$

This holds for the case of a sphere whose radius is initially zero, where $a_o = 0$ in Eq. 1. Under these conditions, Eq. 16 becomes

$$\psi(\bar{r},t) = \frac{1}{4\pi}\int_0^t dt' \int_{\text{space}} g(\bar{r}',t')\frac{\delta(t-t'-R/c)}{R}\, dv'. \tag{17}$$

We shall express the source density as

$$g(\bar{r}',t') = f(t')\, F(\bar{r}',t'),$$

where the function $F(\bar{r}',t')$ accounts for the time variation of $g(\bar{r}',t')$ that is due to the motion of the source, for which in general we cannot expect to be able to separate the space and time dependences. The function $f(t')$ accounts for a possible time variation of $g(\bar{r}', t')$, independently of its motion.

Since the integration over the volume in (17) does not involve time, $f(t')$ may be taken out, and (17) becomes

$$\psi(\bar{r}, t) = \int_0^t dt' f(t') \oint dv' \frac{F(\bar{r}', t')}{4\pi R} \delta\left(t - t' - \frac{R}{c}\right). \tag{18}$$

The volume integral represents the potential at point $\bar{r}$ that is due to an impulse in time, occurring at time $t = t'$ simultaneously over the entire source distribution. If this latter integral is called $h(\bar{r}, t, t')$,

$$h(\bar{r}, t, t') = \oint dv' \frac{F(\bar{r}', t')}{4\pi R} \delta\left(t - t' - \frac{R}{c}\right) \tag{19}$$

and the potential $\psi(\bar{r}, t)$ is given by

$$\psi(\bar{r}, t) = \int_0^t h(\bar{r}, t, t') f(t') dt'. \tag{20}$$

This integral expresses the potential as a superposition of partial responses caused by impulses emitted by the sources as they move. Because of the obvious similarity of this treatment to the method of treatment of linear systems, the integral (20) will be referred to as the convolution integral expression of the potential. Similarly, the function $h(\bar{r}, t, t')$, given by the integral (19) will be called the impulse response for the problem.

It must be stressed that the impulse response used here depends not only on the difference $t - t'$ but also on t', the particular instant when the impulse occurs. This is due to the fact that the source is moving and/or deforming. The impulse response resulting from an impulse at $t = t'$ is necessarily dependent on the position and shape of the source at $t = t'$.

This is not the only way to express the integral solution (18). In fact, by interchanging the order of integration, and integrating over t', we obtain

$$\psi(\bar{r}, t) = \frac{1}{4\pi} \oint_V g \frac{(\bar{r}', t-R/c)}{R} dv'. \tag{21}$$

This is the classical retarded-potential solution.

We are now ready to establish the integral equation for the current density induced on the expanding sphere. Let the primary magnetic field be given by

$$\bar{H}_p = -H_o \bar{i}_z = -H_o(\bar{i}_r \cos\theta - \bar{i}_\theta \sin\theta).$$

The vector potential from which this field can be derived is

$$\bar{A}_p = -\bar{i}_\phi \frac{1}{2} \mu_o H_o \, r \sin\theta,$$

as can be verified by taking

$$\mu_o \overline{H}_p = \text{curl } \overline{A}_p.$$

According to the considerations leading to Eq. 12, the secondary $\overline{H}$ field must cancel the primary $\overline{H}$ field inside the sphere. Furthermore, according to (13) the magnetic vector potential is sufficient to determine the fields. Hence, inside the sphere, we must have

$$\overline{A} = -\overline{A}_p \qquad r < a(t),$$

where $\overline{A}$ is the secondary vector potential. Equation 21 (or the equivalent form Eq. 18) applies to each Cartesian component of $\overline{A}$:

$$A_x(\overline{r}, t) = \frac{\mu_o}{4\pi} \oint \frac{J_x(\overline{r}', t-R/c)}{R} dv'$$

$$A_y(\overline{r}, t) = \frac{\mu_o}{4\pi} \oint \frac{J_y(\overline{r}', t-R/c)}{R} dv',$$

where

$$A_x(\overline{r}, t) = -A_x^p(\overline{r}, t) = -\frac{1}{2} \mu_o H_o \, r \sin\theta \sin\phi$$

$$A_y(\overline{r}, t) = -A_y^p(\overline{r}, t) = \frac{1}{2} \mu_o H_o \, r \sin\theta \cos\phi.$$

The two integral equations given above are not independent. In fact, from symmetry considerations we have shown that the current density must be ϕ-directed. Since

$$J_y = J_\phi \cos\phi',$$

we can use the second integral alone to obtain

$$\frac{\mu_o}{4\pi} \oint_v \frac{J_\phi(\overline{r}', t-R/c)}{R} \cos\phi' \, dv' = \frac{1}{2} \mu_o H_o r \sin\theta \cos\phi,$$

or

$$\oint_v \frac{J_\phi(\overline{r}', t-R/c)}{R} \cos\phi' \, dv' = \frac{1}{2} H_o r \sin\theta \cos\phi,$$

which is the desired integral equation for the current density. Alternatively, taking (18), and letting

$$g(\overline{r}', t') = J_\phi(\overline{r}', t') \cos\phi' = f(t') \, F(\overline{r}', t') \cos\phi', \tag{22}$$

we obtain

$$\int_0^t dt' \, f(t') \oint_v dv' \, \frac{F(\overline{r}', t')}{4\pi R} \, \delta\left(t - t' - \frac{R}{c}\right) \cos\phi' = \frac{1}{2} H_o r \sin\theta \cos\phi \tag{23}$$

which is the integral equation expressed in convolution form.

3.3 THE IMPULSE RESPONSE FOR THE EXPANDING SPHERE

The induced current density was related to the secondary magnetic vector potential inside the sphere through the integral equation (23). In order to solve this equation, the impulse response, as defined by Eq. 19, will be evaluated for the expanding sphere. Strictly speaking, the function $F(\bar{r}', t')$ in the integrand of (19) is part of the solution we seek. The characteristics of the problem, however, are such that $F(\bar{r}', t')$ may be obtained by inspection as follows. From the axial symmetry of the problem, there should be no ϕ'-dependence in $F(\bar{r}', t')$. The θ'-dependence is suggested by the θ-dependence of the secondary $\bar{H}$ field inside the sphere. The tangential component of $\bar{H}$ has a $\sin\theta$ dependence, and since this component is related to the surface current density through the boundary conditions, we expect $F(\bar{r}', t')$ to have a $\sin\theta'$-dependence. Finally, since the current density must be a surface current, we express $F(\bar{r}', t')$ as

$$F(\bar{r}', t') = \delta[r' - a(t')] \sin\theta', \tag{24}$$

where the impulse function implies that the current density is located over the moving surface $r' = a(t')$. (Henceforth we shall write simply a for $a(t')$, the time dependence being understood.)

Under these conditions, the impulse response (19) may be rewritten as

$$h(\bar{r}, t, t') = \frac{1}{4\pi} \oint \frac{\delta(r' - a)\ \delta(t - t' - R/c)}{R} \sin\theta' \cos\phi'\, dv'. \tag{25}$$

In spite of the apparent complexity of its integrand, Eq. 25 may be easily interpreted. The function $h(\bar{r}, t, t')$ is the potential at a point $P(r, \theta, \phi)$ produced by an impulse in surface current at $t = t'$, occurring simultaneously over all the spherical surface $r = a$. (Actually, $h(\bar{r}, t, t')$ does not have the dimensions of magnetic vector potential because the common factor μ_o is omitted in Eq. 22. Keeping this in mind, we shall go on using $h(\bar{r}, t, t')$ as in (24), for the sake of convenience.) It is apparent that $h(\bar{r}, t, t')$ must also be a function of t', since the size of the sphere increases with time. Thus, for instance, the duration of the pulse measured at point P, outside the sphere, because of an impulse at time t', increases as t' increases. This is illustrated in Fig. 1.

We now turn to the evaluation of (25). Only the major steps will be outlined here, the detailed derivation being given in Appendix C. By using spherical coordinates, (25) may be expressed as

$$h(\bar{r}, t, t') = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{\delta(r' - a)\ \delta\left(t - t' - \dfrac{|\bar{r} - \bar{r}'|}{c}\right)}{|\bar{r} - \bar{r}'|} r'^2 \sin^2\theta' \cos\phi'\, dr'd\theta'd\phi',$$

where $R = |\bar{r} - \bar{r}'| = (r^2 + r'^2 - 2rr'\cos\gamma)^{1/2}$ and γ is the angle between $\bar{r}$ and $\bar{r}'$.

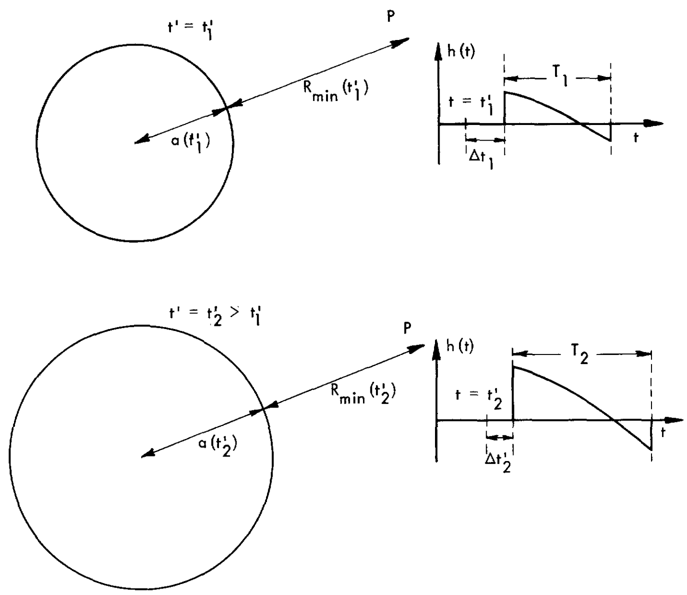

Fig. 1. Illustrating the duration and time delay of the pulse $h(\bar{r}, t, t')$ for two different values of t' and $r > a$. (Not drawn to scale.)

The Fourier representation of the δ-function,

$$\delta(x-x_o) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{j\omega x} e^{-j\omega x_o},$$

is used to express

$$\delta\left(t - t' - \frac{|\bar{r}-\bar{r}'|}{c}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{j\omega t} e^{-j\omega t'} e^{-j\frac{\omega}{c}|\bar{r}-\bar{r}'|}.$$

When this expression is introduced above, we obtain, after interchanging the order of integration,

$$h(\bar{r}, t, t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{j\omega(t-t')} H(\omega), \tag{26}$$

where

$$H(\omega) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \delta(r'-a) \frac{e^{-j\frac{\omega}{c}|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} r'^2 \sin^2\theta \cos\phi' \, dr'd\theta'd\phi'.$$

This integral is evaluated by expanding the factor

$$\frac{e^{-j\frac{\omega}{c}|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}$$

in a series of spherical harmonics, and using the corresponding orthogonality conditions. After a long but straightforward algebraic manipulation, (25) leads to two distinct solutions, for the regions inside and outside the sphere. The functions $h_i(\bar{r}, t, t')$ and $h_o(\bar{r}, t, t')$ will be referred to as the impulse response inside and outside the sphere, respectively. They are

$$h_i(\bar{r}, t, t') = \begin{cases} \frac{1}{2} \sin\theta \cos\phi \left[-\frac{1}{2} c^3 \frac{(t-t_1)^2}{r^2} - (a-r)\, c^2 \frac{(t-t_1)}{r^2} + \frac{ac}{r} \right], & \text{for } t_1 < t < t_2 \\ 0, & \text{otherwise} \end{cases} \tag{27a}$$

for $0 > r > a$, where

$$t_1 \equiv t' + \frac{a-r}{c}, \qquad t_2 \equiv t' + \frac{a+r}{c} \tag{27b}$$

and

$$h_o(\bar{r}, t, t') = \begin{cases} \frac{1}{2} \sin\theta \cos\phi \left[-\frac{1}{2} c^3 \frac{(t-t_1)^2}{r^2} - c^2 (r-a) \frac{(t-t_1)}{r^2} + \frac{ac}{r} \right], & \text{for } t_1 < t < t_2 \\ 0, & \text{otherwise} \end{cases} \tag{28a}$$

for $r > a$, where

$$t_1 = t' + \frac{r-a}{c}, \qquad t_2 = t' + \frac{r+a}{c}. \tag{28b}$$

We recall that, when applying (19) to obtain (23), the source density $g(\bar{r}, t)$ was identified with the y component of J, and the right-hand side of (23) is the y component of the magnetic vector potential.

Hence, the function $h(\bar{r}, t, t')$, given by (25), which led to (27a) and (28a) is the y component of the magnetic vector potential (except for a factor of μ_o) owing to an impulse in surface current. For the x component, the calculation is analogous, leading to functions identical to (27a) and (28a), except that they have a sin ϕ-dependence rather than the cos ϕ-dependence as above. The total (ϕ-directed) impulse response may be obtained by combining the x and y components, and is independent of ϕ. For instance, the total impulse response inside the sphere is

$$h_{ti}(\bar{r}, t, t') = \frac{1}{2} \sin\theta \left[-\frac{1}{2} c^3 \frac{(t-t_1)^2}{r^2} - c^2 (r-a) \frac{(t-t_1)}{r^2} + \frac{ac}{r} \right], \qquad t_1 > t > t_2.$$

The same applies to $h_{to}(\bar{r}, t, t')$. It is worth while to spend some time with the physical interpretation of these functions. We shall look first at the impulse response inside the sphere, as given above by $h_{ti}(\bar{r}, t, t')$. From the detailed derivation in Appendix C, we learn that h_{ti} is obtained as a superposition of two traveling waves, namely:

$$h_{ti}^{-} = \frac{1}{2}\sin\theta\left[-\frac{1}{2}c^3\frac{(t-t_1)^2}{r^2} - c^2(r-a)\frac{(t-t_1)}{r^2} + \frac{ac}{r}\right]u(t-t_1),$$

$$h_{ti}^{+} = \frac{1}{2}\sin\theta\left[\frac{1}{2}c^3\frac{(t-t_2)^2}{r^2} + c^2(a+r)\frac{(t-t_2)}{r^2} + \frac{ac}{r}\right]u(t-t_2),$$

where u(x) is the step function,

$$u(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

and t_1 and t_2 are defined by (27b). These waves change their shape as they travel. In order to present a clear picture of their behavior, they have been sketched in Fig. 2, in the range $0 < r < a$, for several representative values of t. For simplicity, we let $t' = 0$ and $\sin\theta = 1$. For other values of these quantities only the magnitudes change; the physical characteristics remain the same.

Figure 2a, 2b, and 2c apply for $0 < t < a/c$. During this interval, only the h_{ti}^{-} wave exists. Notice that it steepens as it moves toward the center of the sphere, but is positive everywhere. At $t = a/c$, the wave front of h_{ti}^{-} reaches the center of the sphere, where, at this same time, h_{ti}^{+} appears. At $t = a/c$ the two waves cancel each other at the origin (Fig. 2d). For $t > a/c$, h_{ti}^{+} moves out toward the surface of the sphere, cancelling h_{ti}^{-} everywhere as it moves (Fig. 2e and 2f). Note that the h_{ti}^{-} wave now becomes partly negative, the negative portion extending farther toward the surface as t increases. Eventually, it becomes all negative, as in Fig. 2g. Finally, at $t = \frac{2a}{c}$ there is no more field inside the sphere. The potential at any given point inside the sphere is seen to be given by a pulse whose duration decreases as r decreases. This is shown more clearly in Fig. 3, where h_{ti} is plotted against time for three different values of r. At the surface (r=a) the pulse lasts for the full $\Delta t = \frac{2a}{c}$, which corresponds to the time necessary to travel along a diameter of the sphere, between two diametrically opposed points on the surface with the velocity of light. For $r > a$, the duration decreases, and the amplitude of the pulse increases. At the center, (r=0), the duration is zero, that is, the potential is zero for all time.

A similar investigation of the impulse response outside the sphere,

$$h_{to}(r,t,t') = \frac{1}{2}\sin\theta\left[-\frac{1}{2}c^3\frac{(t-t_1)^2}{r^2} - c^2(r-a)\frac{(t-t_1)}{r^2} + \frac{ac}{r}\right], \qquad t_1 < t < t_2$$

reveals that again the impulse response is the superposition of two waves. Both waves, however, are outward-traveling waves. They are

$$h_{to}^{+1} = \frac{1}{2}\sin\theta\left[-\frac{1}{2}c^3\frac{(t-t_1)^2}{r^2} - c^2(r-a)\frac{(t-t_1)}{r^2} + \frac{ac}{r}\right]u(t-t_1)$$

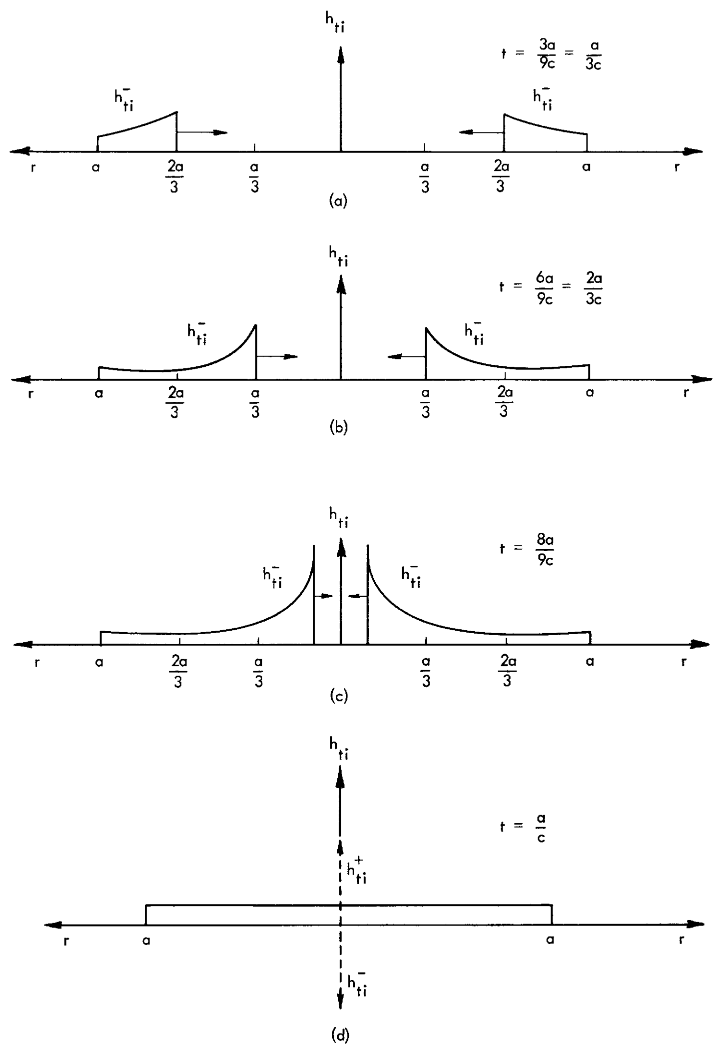

Fig. 2. (Continued on next page.)

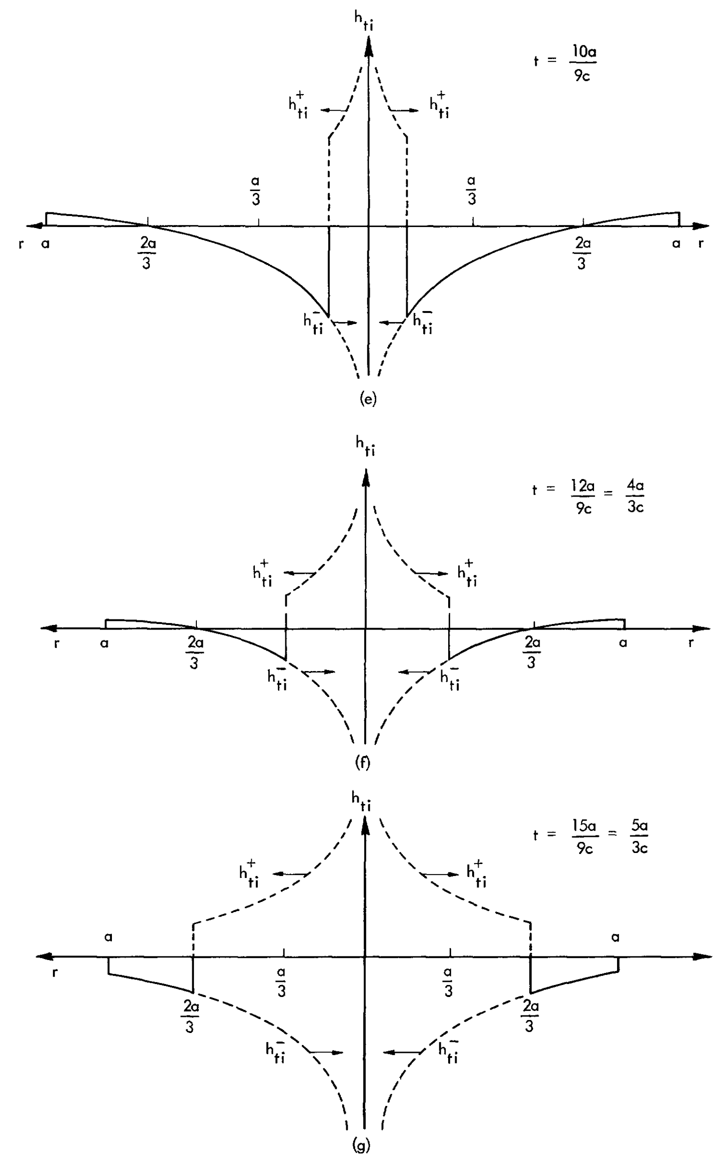

Fig. 2. Behavior of $h^{-}_{ti}(r)$ and $h^{+}_{ti}(r)$ for several values of t. (Not drawn to scale.)

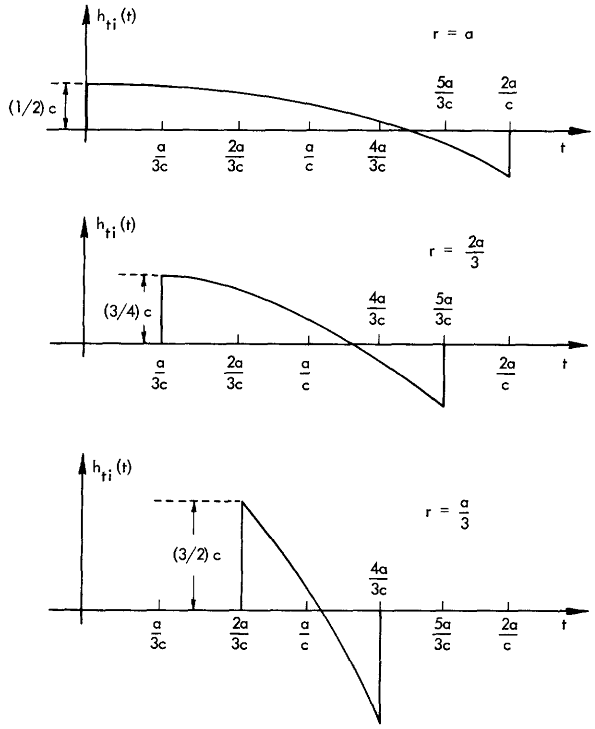

Fig. 3. Behavior of $h_{ti}(t)$ for three values of r. (Not drawn to scale.)

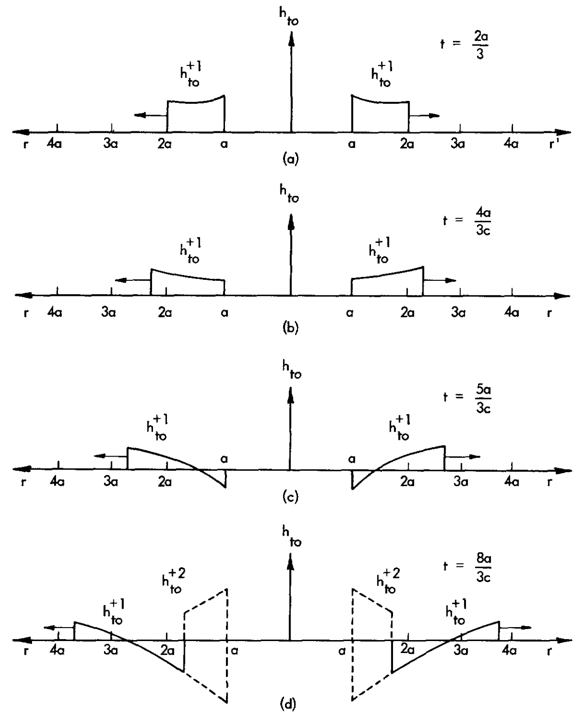

Fig. 4. Behavior of $h_{to}^{+1}(r)$ and $h_{to}^{+2}(r)$ for several values of t. (Not drawn to scale.)

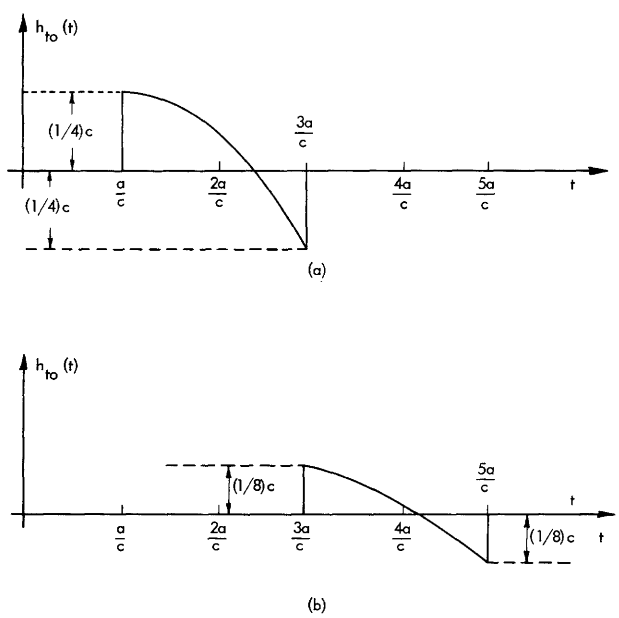

Fig. 5. Behavior of $h_{to}(t)$ for two values of r. (Not drawn to scale.) (a) r = 2a. (b) r = 4a.

and

$$h_{to}^{+2} = \frac{1}{2} \sin\theta \left[\frac{1}{2} c^3 \frac{(t-t_2)^2}{r^2} + c^2(r-a) \frac{(t-t_2)}{r^2} + \frac{ac}{r} \right] u(t-t_2),$$

where t_1 and t_2 are given by (28b). Figure 4 shows sketches of these waves, as a function of r, for several values of t. Again, the waves change their shape as they travel. In Fig. 4 the h_{to}^{+1} wave is shown as it progresses outward and eventually becomes partly negative. At $t = \frac{2a}{c}$, the h_{to}^{+2} wave appears at $r = a$. For all $t > \frac{2a}{c}$, the two waves cancel each other for all $r < ct - a$. Figure 5 shows sketches of h_{to} as a function of t, for $r = 2a$ and $r = 4a$. The duration of the pulse now remains constant ($T = 2a/c$), while its amplitude decreases as r increases.

The behavior of the functions $h_{ti}(\bar{r}, t, t')$ and $h_{to}(\bar{r}, t, t')$, as shown in Figs. 2-5, agrees with what one would expect from physical reasoning, if the potential at a given point is looked upon as the superposition of the elementary contributions from all points of the surface, taken with their respective retardations.

We note that in expressions (27a) and (28a), a is to be considered as the instantaneous value of the radius at $t = t'$. When applying the impulse response for the evaluation of the convolution integral (20), the time-dependence of a must be brought out explicitly. At this point, however, the expressions derived for the impulse response are perfectly general and hold for any time dependence.

3.4 THE CONVOLUTION INTEGRAL FOR ARBITRARY RADIAL EXPANSION

Having obtained the impulse response, we are now ready to take Eq. 23, whose solution yields the desired time-dependence of the current density, and cast it into final form. Let the radius a be an arbitrary function of time, $a = a(t')$. The impulse response inside the sphere, (27a), may be rewritten as

$$h_i(\bar{r}, t, t') = \frac{1}{2} \sin\theta \cos\phi \left[-\frac{1}{2} c^3 \frac{(t-t_1)^2}{r^2} + c^2 \frac{(t-t_1)}{r} - \left(\frac{c^2(t-t_1)}{r^2} - \frac{c}{r} \right) a(t') \right], \qquad t_1 < t < t_2 \tag{29}$$

where

$$\left. \begin{aligned} t_1 &= t' + \frac{a(t') - r}{c} \\ t_2 &= t' + \frac{a(t') + r}{c} \end{aligned} \right\} . \tag{30}$$

In order to determine the limits on t' between which $h_i(\bar{r}, t, t')$ does not vanish, we must solve Eqs. 30 for t'. Let t_1' and t_2' be the solutions of (30) for $t = t_1$ and $t = t_2$; that is,

$$a(t_1') + c\, t_1' = ct + r$$

$a(t_2') + c\, t_2' = ct - r.$

The function $h_i(\bar{r}, t, t')$ vanishes outside the interval

$$t_2' < t' < t_1'. \tag{31}$$

Hence, Eq. 23, by using (29) and (31), may be cast into the form

$$\int_{t_2'}^{t_1'} dt'\, f(t') \left\{ -\frac{1}{2} c^3 \frac{(t-t_1)^2}{r^2} + c^2 \frac{(t-t_1)}{r} - \left[\frac{c^2(t-t_1)}{r^2} - \frac{c}{r} \right] a(t') \right\} = H_o r,$$

$$t_1 = t' + \frac{a(t') - r}{c}. \tag{32}$$

This is the general form of the convolution integral, which holds for an arbitrary time-dependence of the radius. The solution yields $f(t')$, the desired time-dependence of the current density, which, from (22) and (24), is then completely determined as

$$J_\phi(r', t') = f(t')\, \delta(r'-a) \sin \theta'.$$

Once $f(t')$ is obtained, the convolution integral may be used again, with the impulse response $h_o(\bar{r}, t, t')$, to find the vector potential outside the sphere,

$$A_\phi \cos \phi = \mu_o \int_{t_2'}^{t_1'} dt'\, f(t')\, h_o(\bar{r}, t, t'), \tag{33}$$

where $h_o(\bar{r}, t, t')$ is given by (28a) and t_1' and t_2' are the corresponding limits, found from (28b) for $t = t_1$ and $t = t_2$ when the appropriate $a(t')$ is inserted. From the potential (33), the secondary fields may be obtained through (13). By using (28a), (33) and the expressions for the fields outside the sphere become

$$\left.\begin{aligned}
A_\phi &= \frac{1}{2} \mu_o \sin \theta \int_{t_2'}^{t_1'} dt'\, f(t') \left[-\frac{1}{2} c^3 \frac{(t-t_1)^2}{r^2} - c^2[r-a(t')] \frac{(t-t_1)}{r^2} + \frac{ca(t')}{r} \right] \\
E_\phi &= \frac{1}{2} \mu_o \sin \theta \frac{\partial}{\partial t} \int_{t_2'}^{t_1'} dt'\, f(t') \left[-\frac{1}{2} c^3 \frac{(t-t_1)^2}{r^2} - c^2[r-a(t')] \frac{(t-t_1)}{r^2} + \frac{ca(t')}{r} \right] \\
H_\theta &= -\frac{1}{2r} \sin \theta \frac{\partial}{\partial r} r \int_{t_2'}^{t_1'} dt'\, f(t') \left[-\frac{1}{2} c^3 \frac{(t-t_1)^2}{r^2} - c^2[r-a(t')] \frac{(t-t_1)}{r^2} + \frac{ca(t')}{r} \right] \\
H_r &= \frac{1}{r} \cos \theta \int_{t_2'}^{t_1'} dt'\, f(t') \left[-\frac{1}{2} c^3 \frac{(t-t_1)^2}{r^2} - c^2[r-a(t')] \frac{(t-t_1)}{r^2} + \frac{ca(t')}{r} \right]
\end{aligned}\right\} \tag{34}$$

with $t' \equiv t' + \dfrac{r - a(t')}{c}$.

This is our final result. In Section IV the problem will be solved explicitly for the special case of the sphere with constant radial expansion, in which case $a(t') = vt'$.

IV. THE SPHERE WITH CONSTANT RADIAL EXPANSION

The results of Section III will now be applied to the special case for which the radius is given as a function of time by the expression $a = vt$. We shall refer to this situation as the case of constant radial expansion. The current density is found by solving the integral equation (32). Next, the secondary fields are determined, and identified as the fields of a time-variant magnetic dipole. The appropriate boundary conditions are shown to be satisfied by these fields. Finally, the electromagnetic force acting on the surface of the sphere is determined, and the energy-power balance is investigated through the use of Poynting's theorem.

4.1 CURRENT DENSITY FOR CONSTANT RADIAL EXPANSION

The integral equation for the current density has been established (Section III) in terms of a convolution integral (32). We shall now proceed to solve this equation, for the particular time dependence

$$a(t') = vt'.$$

In expression (27a), if we let $a(t') = vt'$, we find

$$t - t_1 = t - t' - \frac{a-r}{c} = t - t' - \frac{vt'-r}{c} = t + \frac{r}{c} - t'(1+v/c).$$

The limits of integration, from (27b), are

$$\left.\begin{aligned} &\text{For } t = t_1 = \frac{vt'-r}{c} + t' \longrightarrow t' = \frac{1}{1+v/c}(t+r/c) \equiv t_1' \\ &\text{For } t = t_2 = \frac{vt'+r}{c} + t' \longrightarrow t' = \frac{1}{1+v/c}(t-r/c) \equiv t_2' \end{aligned}\right\} \qquad (35)$$

Hence, the impulse response inside the sphere is

$$h_i(\bar{r}, t, t') = \frac{1}{2}\sin\theta\cos\phi\left[-\frac{1}{2}c^3\frac{(t-t_1)^2}{r^2} - (vt'-r)c^2\frac{(t-t_1)}{r^2} + \frac{cvt'}{r}\right]$$

$$t_2' < t' < t_1'. \qquad (36)$$

The function h_i vanishes outside the given interval.

Following the same procedure, for expression (28a) we find

$$t - t_1 = t - t' + \frac{r-vt'}{c} = t + \frac{r}{c} - t'\left(1+\frac{v}{c}\right),$$

and the limits of integration, from (28b), are

$$\left.\begin{aligned} &\text{For } t = t_1 = t' + \frac{r - vt'}{c} \longrightarrow t' = \frac{1}{1 - v/c}(t - r/c) \equiv t_1' \\ &\text{For } t = t_2 = t' + \frac{r + vt'}{c} \longrightarrow t' = \frac{1}{1 + v/c}(t - r/c) \equiv t_2' \end{aligned}\right\} \tag{37}$$

Therefore, the impulse response outside the sphere is

$$h_o(\bar{r}, t, t') = \frac{1}{2} \sin\theta \cos\phi \left[-\frac{1}{2} c^3 \frac{(t-t_1)^2}{r^2} - c^2(r - vt') \frac{(t-t_1)}{r^2} + \frac{cvt'}{r} \right]$$

$$t' < t' < t_1'. \tag{38}$$

$h_o(\bar{r}, t, t')$ vanishes outside the interval.

In order to gain further insight into the superposition method involved in this solution, the regions of existence of $h_i(\bar{r}, t, t')$ and $h_o(\bar{r}, t, t')$ are plotted on a t' versus t plane (Fig. 6).

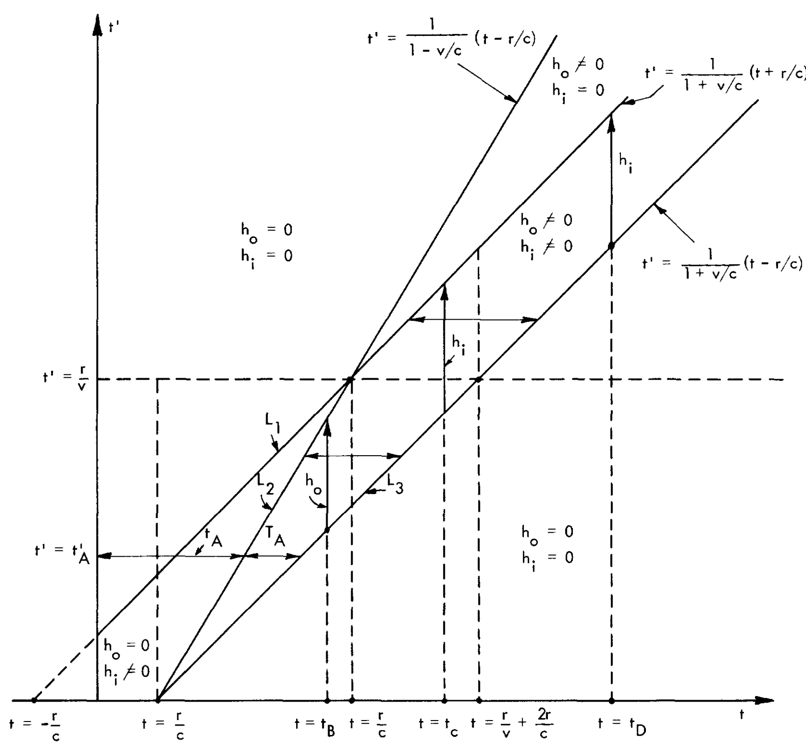

Fig. 6. The functions $h_i(\bar{r}, t, t')$ and $h_o(\bar{r}, t, t')$ in the t' versus t plane.

The relations (35) and (37) are represented by straight lines. There are only three distinct lines, because the second relation of Eqs. 35 and the second of Eqs. 37 are identical. With respect to these lines we define several different regions in the t-t' plane. In some regions, both $h_i(\bar{r}, t, t')$ and $h_o(\bar{r}, t, t')$ are finite (nonzero), whereas in others one or both of these functions vanish. The diagram applies to a fixed observation point r.

If we move along a horizontal line, for instance, for $t' = t'_A$ we conclude that the pulse caused by the impulse at $t' = t'_A$ arrives at the observation point r at time $t = t_A$, and has a duration T_A, measured between lines L_2 and L_3. If the impulse occurs at $t' < t'_A$, the sphere has a smaller radius and the duration of the pulse is less than T_A. For $t' > t'_A$, the opposite occurs.

It is apparent that, for all $t' < r/v$, the observation point is outside the sphere, and the fact that the pulse duration increases as the sphere expands agrees with our previous conclusions drawn from inspection of Fig. 1. For $t' = r/v$, the surface of the sphere reaches the point of observation, and for all $t' > r/v$, the point of observation is inside the sphere. For a point inside the sphere, the duration of the pulse is independent of the size of the sphere, as will become clear from inspection of Fig. 7. Accordingly,

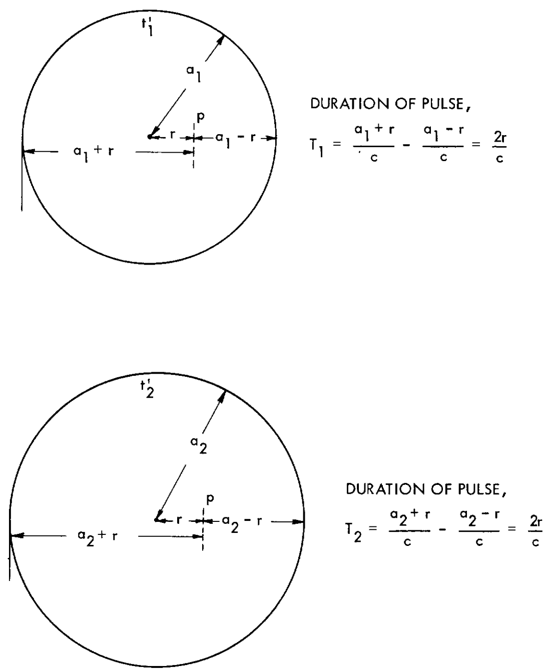

Fig. 7. Illustrating the duration of the pulse $h_i(\bar{r}, t, t')$ for two different values of t and $r < a$.

in the t-t' plane the duration of the pulse for $t' > r/v$ is shown as the horizontal distance between lines L_1 and L_3, which remains constant henceforth.

If we move along a vertical line we are able to gather information on the contributions from the impulses at successive values of t', which superpose to add up to the value of the potential at point r and time t. For instance, at time $t = t_B$, the point r is outside the sphere. The potential is the superposition of pulses emitted between $t' = \frac{1}{1 + \frac{v}{c}}\left(t_B - \frac{r}{c}\right)$ and $t' = \frac{1}{1 - \frac{v}{c}}\left(t_B - \frac{r}{c}\right)$. This operation is indicated by the vertical path between lines L_3 and L_2, where the function h_o is to be integrated between the limits given above.

For $t > r/v$, the point is inside the sphere and the function h_i must be used. It is interesting to note that, for values of t such that

$$\frac{r}{v} < t < \frac{r}{v} + \frac{2r}{c},$$

of which $t = t_c$ is representative, part of the contribution is emitted (the vertical segment between L_3 and the dashed line $t' = r/v$), while the point P is still outside the sphere. When retardation is accounted for, these contributions actually arrive at P after P is already inside the sphere.

We now recognize that the solution we seek, as given by the convolution integral (32) is represented in the t-t' plane by the integration of $f(t')\, h_i(\bar{r}, t, t')$ for all $t > r/c$ between the lines L_3 and L_1, that is, between the limits

$$\frac{1}{1+\frac{v}{c}}\left(t-\frac{r}{c}\right) < t' < \frac{1}{1+\frac{v}{c}}\left(t+\frac{r}{c}\right).$$

Equation 32 becomes

$$\int_{\frac{t-r/c}{1+v/c}}^{\frac{t+r/c}{1+v/c}} dt'\, f(t') \left[-\frac{1}{2}c^3 \frac{(t-t_1)^2}{r^2} + c^2 \frac{(t-t_1)}{r} - vt'\left[c^2 \frac{(t-t_1)}{r^2} - \frac{c}{r}\right]\right] = H_o r, \qquad (39)$$

where $t_1 = t'\left(1+\frac{v}{c}\right) - \frac{r}{c}$.

The solution is carried out in Appendix D. It is found that the function $f(t')$ reduces to a constant K_o, given by

$$K_o = \frac{3}{2} H_o \frac{\left(1+\frac{v}{c}\right)^2}{1+\frac{2v}{c}}. \qquad (40)$$

Hence, the induced surface current density

$$J\phi(\bar{r}', t') = \frac{3H_o}{2} \frac{\left(1+\frac{v}{c}\right)^2}{1+\frac{2v}{c}} \delta(r'-vt') \sin\theta' \qquad (41)$$

is the exact solution of the problem when the radius increases linearly with time.

4.2 EXACT SOLUTION FOR CONSTANT RADIAL EXPANSION

Having obtained the induced surface current density, it is now possible to determine the secondary fields outside the sphere. Rather than applying the convolution integral (33), we shall take the more direct route of Eq. 21. With $J\phi$ as given by Eq. 41, and K_o as in (40), expression (22) becomes

$$g(\bar{r}', t') = J\phi(\bar{r}', t') \cos\phi' = K_o \delta(r'-vt') \sin\theta' \cos\phi'$$

$$\psi = A_\phi \cos\phi = \frac{\mu_o}{4\pi} \oint \frac{K_o \delta\left[r' - c\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right)\right]}{|\bar{r}-\bar{r}'|} \sin\theta' \cos\phi' \, dv'$$

or

$$A_\phi = \frac{\mu_o}{4\pi\cos\phi} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{K_o \delta\left[r' - v\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right)\right]}{|\bar{r}-\bar{r}'|} r'^2 \sin^2\theta' \cos\phi' \, dr' \, d\theta' \, d\phi'. \tag{42}$$

Since the impulse function is an even function of its argument, we can write

$$\delta\left[r' - vt + v\frac{|\bar{r}-\bar{r}'|}{c}\right] = \delta\left(vt - r' - v\frac{|\bar{r}-\bar{r}'|}{c}\right) = \delta\left(a - r' - v\frac{\bar{r}-\bar{r}'}{c}\right).$$

By using, for convenience, the variable $a = vt$ rather than t, the δ-function may be expressed as

$$\delta\left(a - r' - v\frac{|\bar{r}-\bar{r}'|}{c}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)\, e^{jka}\, dk,$$

where

$$f(k) = \int_{-\infty}^{\infty} \delta\left(a - r' - v\frac{|\bar{r}-\bar{r}'|}{c}\right) e^{-jka}\, da$$

or $f(k) = \exp\left[-jk\left(r' + \frac{v}{c}|\bar{r}-\bar{r}'|\right)\right]$.
Hence,

$$\delta\left(a - r' - v\frac{\bar{r}-\bar{r}'}{c}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[-jk\left(r' + \frac{v}{c}|\bar{r}-\bar{r}'|\right)\right] e^{jka}\, dk.$$

Introduction of this expression into (42) leads to

$$A_\phi = \frac{\mu_o K_o}{8\pi^2 \cos\phi} \int_{-\infty}^{\infty} e^{jka}\, dk \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{-jkr'} \frac{\exp\left(-jk\frac{v}{c}|\bar{r}-\bar{r}'|\right)}{|\bar{r}-\bar{r}'|}$$
$$\times r'^2 \sin^2\theta' \cos\phi' \, dr' d\theta' d\phi'. \tag{43}$$

The evaluation of this integral is carried out in Appendix E. The final result is

$$A_\phi(\bar{r}, t-r/c) = \frac{3\mu_o H_o v^3}{2(1-v/c)^2 (1+2v/c)} \left[\frac{(t-r/c)^2}{rc} + \frac{(t-r/c)^3}{3r^2}\right] \sin\theta, \quad t > r/c \tag{44}$$

$$A_\phi(\bar{r}, t-r/c) = 0, \quad t < r/c.$$

Note that, since r is measured with respect to the origin, as far as the fields outside are concerned, the source may be represented by a singularity at the origin. In view of the sin θ-dependence, we recognize this as the vector potential of a time-variant magnetic dipole. Thus suggests a more elegant and compact treatment of the problem, to which we shall return presently. From the vector potential, the secondary fields outside the sphere upon application of (13) are found to be

$$\left.\begin{aligned}
H_\theta &= \frac{3H_o v^3}{(1-v/c)^2\,(1+2v/c)}\left[\frac{\tau}{rc^2}+\frac{\tau^2}{2r^2c}+\frac{\tau^3}{6r^3}\right]\sin\theta \\
H_r &= \frac{3H_o v^3}{(1-v/c)^2\,(1+2v/c)}\left[\frac{\tau^2}{r^2c}+\frac{\tau^3}{3r^3}\right]\cos\theta \\
E_\phi &= \frac{-3H_o v^3}{(1-v/c)^2\,(1+2v/c)}\sqrt{\frac{\mu_o}{\epsilon_o}}\left[\frac{\tau}{rc^2}+\frac{\tau^2}{2r^2c}\right]\sin\theta
\end{aligned}\right\} \qquad (45)$$

for $\tau > 0$, where $\tau \equiv t - r/c$.

Figure 8 is a sketch of the secondary $\overline{H}$ field at time $t = t_o$. The field is zero beyond the wavefront defined by the surface $r = ct_o$. Within the region $r < ct_o$, it is the field of a time-variant magnetic dipole.

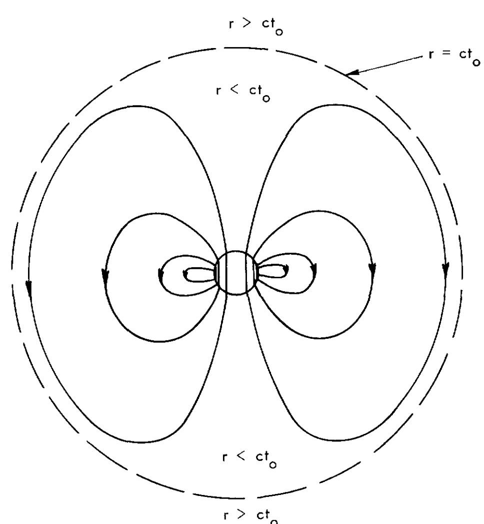

Fig. 8. Secondary H field at $t = t_o$. (Not drawn to scale.)

The more compact treatment of the problem mentioned above involves the use of the dual potentials ϕ^* and $\overline{A}^*$.[19] The fields given by Eqs. 45 might have been obtained from an electric vector potential $\overline{A}^*$ and a magnetic scalar potential ϕ^*, as follows:

$$\left.\begin{aligned} \overline{E} &= -\frac{1}{\epsilon_o}\,\text{curl}\,\overline{A}^* \\ \overline{H} &= -\frac{\partial \overline{A}^*}{\partial t} - \text{grad}\,\phi^* \end{aligned}\right\} \tag{46}$$

Here, as in the case of the potentials ϕ and $\overline{A}$, the divergence of $\overline{A}^*$ is as yet unspecified. We shall stipulate that ϕ^* and $\overline{A}^*$ be related as

$$\text{div}\,\overline{A}^* + \frac{1}{c^2}\frac{\partial \phi^*}{\partial t} = 0. \tag{47}$$

Hence, once $\overline{A}^*$ is determined, ϕ^* may be found by (47).

The electric vector potential $\overline{A}^*$ that generates the fields (45) is

$$\overline{A}^* = \frac{3H_o v^3}{2\left(1-\frac{v}{c}\right)^2\left(1+\frac{2v}{c}\right)}\,\frac{\tau^2}{rc^2}\,\overline{i}_z. \tag{48}$$

Since

$$\overline{A}^* = \frac{\epsilon_o}{4\pi r}\frac{\partial}{\partial t}\mu_o \overline{m}(\tau),$$

(48) corresponds to a magnetic moment,

$$\overline{m} = \frac{2\pi H_o v^3 t^3}{\left(1-\frac{v}{c}\right)^2\left(1+\frac{2v}{c}\right)}\,\overline{i}_z = \frac{2\pi H_o a^3}{\left(1-\frac{v}{c}\right)^2\left(1+\frac{2v}{c}\right)}\,\overline{i}_z. \tag{49}$$

This is a simple but very meaningful result. In fact, the magnetic moment of a stationary sphere having the same surface current distribution is

$$\overline{m}_s = 2\pi a^3 H_o\,\overline{i}_z,$$

to which (49) reduces in the limit of very low velocities ($v \ll c$).

If retardation inside the sphere is neglected, the use of $\overline{m}_s$ as a first approximation leads to fields that are exactly those of Eqs. 45, except for the factors involving v/c in the denominators.

To the induced fields (45) we now add the primary field,

$$\overline{H}_p = -H_o(i_r \cos\theta - i_\theta \sin\theta)$$

to find the complete solution:

$$H_\theta^T = H_o\left[1 + \frac{3v^3}{\left(1-\frac{v}{c}\right)^2\left(1+\frac{2v}{c}\right)}\left[\frac{\tau}{rc^2} + \frac{\tau^2}{2r^2c} + \frac{\tau^3}{6r^3}\right]\right]\sin\theta$$

$$H_r^T = H_o\left[-1 + \frac{3v^3}{\left(1-\frac{v}{c}\right)^2\left(1+\frac{2v}{c}\right)}\left[\frac{\tau^2}{r^2c} + \frac{\tau^3}{3r^3}\right]\right]\cos\theta$$

$$E_\phi^T = -\frac{3H_o v^3}{\left(1-\frac{v}{c}\right)^2\left(1+\frac{2v}{c}\right)}\sqrt{\frac{\mu_o}{\epsilon_o}}\left[\frac{\tau}{rc^2} + \frac{\tau^2}{2r^2c}\right]\sin\theta \qquad \tau > 0\,. \tag{50}$$

For $r < a$, $\overline{E}^T = \overline{H}^T = 0$.

The boundary conditions (1-3) reduce to

$$\left.\begin{aligned} &\overline{i}_r \times \left(\overline{E}^T + \overline{v}\times\mu_o\overline{H}^T\right)_{r=a} = 0 \\ &\overline{i}_r \times \left(\overline{H}^T - \overline{v}\times\epsilon_o\overline{E}^T\right)_{r=a} = \overline{K} \\ &\overline{i}_r \cdot \overline{E}^T(a) = \overline{i}_r \cdot \overline{H}^T(a) = 0 \end{aligned}\right\} \tag{51}$$

Here, $\overline{E}^T$ and $\overline{H}^T$ are the fields outside, given by Eqs. 50, evaluated at $r = a$.

Substitution of $r = a$ in Eqs. 50 yields

$$H_r^T(a) = 0$$

$$H_\theta^T(a) = \frac{3H_o\sin\theta}{2\left(1+\frac{2v}{c}\right)}\frac{\left(1+\frac{v}{c}\right)}{\left(1-\frac{v}{c}\right)} \tag{52}$$

$$E_\phi^T(a) = -\frac{3\mu_o v H_o\sin\theta}{2\left(1+\frac{2v}{c}\right)}\frac{\left(1+\frac{v}{c}\right)}{\left(1-\frac{v}{c}\right)}.$$

It is easy to verify that these values fulfill all of the boundary conditions.

This proves conclusively that the fields given by Eqs. 50 furnish the exact solution of the problem of the linearly expanding sphere.

Before closing, we observe that the ratio between $E_\phi^T(a)$ and $H_\theta^T(a)$, which we might call the "impedance" at the surface,

$$Z(a) = -\frac{v}{c}\sqrt{\frac{\mu_o}{\epsilon_o}},$$

is a measure of the effectiveness of the expanding process in generating an electric field. We see that the induced electric field increases as v/c. The relation between $\overline{E}_\phi^T(a)$ and $\overline{H}_\phi^T(a)$ approaches the characteristic impedance of free space as $v \to c$. The total fields at a given $r = r_o$ are sketched in Fig. 9 as a function of time. Time $t = r_o/v$ is the instant when the surface of the sphere reaches the point of observation. Hence, for $t > r_o/v$,

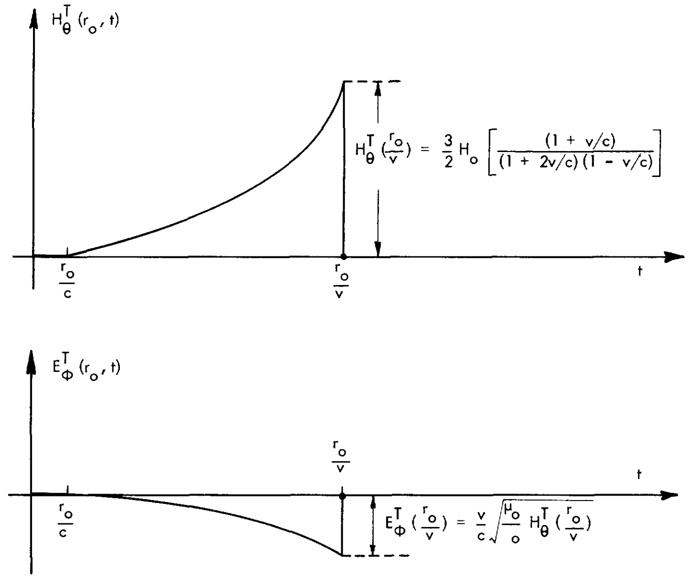

Fig. 9. Total fields H_θ^T and E_ϕ^T at $r = r_o$ and $\theta = \pi/2$ as a function of time. (Not drawn to scale.)

the point is inside the sphere and the fields vanish.

4.3 ENERGY BALANCE FOR THE LINEARLY EXPANDING SPHERE

Having obtained the exact solutions (50), we shall now investigate the electromagnetic energy and forces involved in the expanding process.

The force per unit area exerted by the macroscopic field on the surface current density $\overline{K}$ is

$$f = \frac{1}{2}\mu_o \overline{K} \times \overline{H}(a),$$

where $\overline{H}(a)$ is the outside field evaluated at $r = a$. Using (40) and (50), we find this to be

$$f_r = -\frac{1}{2} \times \frac{3H_o}{2} \frac{\left(1 + \frac{v}{c}\right)^2}{\left(1 + \frac{2v}{c}\right)} \sin\theta \times \frac{3H_o}{2} \mu_o \frac{\left(1 + \frac{v}{c}\right)}{\left(1 + \frac{2v}{c}\right)\left(1 - \frac{v}{c}\right)} \sin\theta$$

$$f_r = \frac{9\mu_o H_o^2}{8} \left[\frac{\left(1 + \frac{v}{c}\right)^3}{\left(1 + \frac{2v}{c}\right)^2 \left(1 - \frac{v}{c}\right)}\right] \sin^2\theta. \qquad (53)$$

The mechanical power per unit area converted into electromagnetic form is found to be

$$p_m = -\bar{f}\cdot\bar{v} = \frac{9}{8}\mu_o v\, H_o^2 \sin^2\theta \left[\frac{\left(1+\frac{v}{c}\right)^3}{\left(1+\frac{2v}{c}\right)^2\left(1-\frac{v}{c}\right)}\right]. \tag{54}$$

The total mechanical power converted is then obtained by integration over the surface:

$$P_m = \frac{9}{8}\mu_o\, v\, H_o^2 \frac{\left(1+\frac{v}{c}\right)^3}{\left(1+\frac{2v}{c}\right)^2\left(1-\frac{v}{c}\right)} \int_0^{2\pi}\int_0^{\pi} a^2 \sin^3\theta\, d\theta d\phi$$

$$P_m = 3\pi\mu_o\, v\, H_o a^2 \left[\frac{\left(1+\frac{v}{c}\right)^3}{\left(1+\frac{2v}{c}\right)^2\left(1-\frac{v}{c}\right)}\right]$$

or

$$P_m = 3\pi\mu_o\, H_o^2 \left[\frac{\left(1+\frac{v}{c}\right)^3}{\left(1+\frac{2v}{c}\right)^2\left(1-\frac{v}{c}\right)}\right] v^3 t^2. \tag{55}$$

We now turn to the investigation of the balance of power on the surface of the sphere. For this purpose, the limiting form of Poynting's theorem for a moving surface shall be used. This expression is derived in Appendix A, and, in this particular case, reduces to

$$\bar{n}\cdot\bar{S}_1 = (\bar{v}\cdot\bar{n})\, w_1 - \frac{1}{2}K\cdot\bar{E}_1, \qquad r = a \tag{56}$$

or

$$i_r\cdot(\bar{E}\times\bar{H})_a = v\left[\frac{1}{2}\epsilon_o|\bar{E}(a)|^2 + \frac{1}{2}\mu_o|H(a)|^2\right] - \frac{1}{2}\bar{K}\cdot E(a),$$

and finally

$$S_r = E_\phi(a)\, H_\theta(a) = v\left[\frac{1}{2}\epsilon_o E_\phi^2(a) + \frac{1}{2}\mu_o H_\theta^2(a)\right] - \frac{1}{2}K_\phi E_\phi(a),$$

where $E_\phi(a)$ and $H_\theta(a)$ are the outside fields, evaluated at $r = a$, as given by Eqs. 52. Evaluation of the term on the left-hand side yields

$$S_r = E_\phi(a)\, H_\theta(a) = \frac{9}{4}\mu_o v H_o^2 \left[\frac{\left(1+\frac{v}{c}\right)}{\left(1+\frac{2v}{c}\right)\left(1-\frac{v}{c}\right)}\right]^2 \sin^2\theta. \tag{57}$$

This is the radial component of the Poynting vector, interpreted here (as in the more conventional case of stationary sources) as the power per unit area crossing a <u>stationary</u>

surface, coinciding with the surface of the sphere at a given time t.

For the first term on the right-hand side, we find

$$v\,w(a) = v\,w_e(a) + v\,w_m(a)$$

$$v\,w_e(a) = \frac{1}{2} v\,\epsilon_o E_\phi^2(a) = \frac{9}{8}\epsilon_o \mu_o^2 v^3 H_o^2 \left[\frac{\left(1+\frac{v}{c}\right)}{\left(1+\frac{2v}{c}\right)\left(1-\frac{v}{c}\right)}\right]^2 \sin^2\theta$$

$$v\,w_m(a) = \frac{1}{2} v\,\mu_o H_\theta^2(a) = \frac{9}{8}\mu_o H_o^2 \left[\frac{\left(1+\frac{v}{c}\right)}{\left(1+\frac{2v}{c}\right)\left(1-\frac{v}{c}\right)}\right]^2 \sin^2\theta$$

so that

$$p_c = v\,w(a) = \frac{9}{8}\mu_o\,v\,H_o^2\left[1+\frac{v^2}{c^2}\right]\left[\frac{\left(1+\frac{v}{c}\right)}{\left(1+\frac{2v}{c}\right)\left(1-\frac{v}{c}\right)}\right]^2 \sin^2\theta. \qquad (58)$$

This fraction of power is related to the energy stored, and may be visualized as being bodily carried (or pushed) by the surface of the sphere as it moves. For compactness, we shall refer to this contribution as the power convected per unit area, p_c.

The second term on the right-hand side is found to be

$$-\frac{1}{2}K_\phi E_\phi(a) = \frac{9}{8}\mu_o\,v\,H_o^2\sin^2\theta\,\frac{\left(1+\frac{v}{c}\right)^2\left(1+\frac{v}{c}\right)}{\left(1+\frac{2v}{c}\right)\left(1+\frac{2v}{c}\right)\left(1-\frac{v}{c}\right)}$$

$$-\frac{1}{2}K_\phi E_\phi(a) = \frac{9}{8}\mu_o\,v\,H_o^2\left[\frac{\left(1+\frac{v}{c}\right)^3}{\left(1+\frac{2v}{c}\right)^2\left(1-\frac{v}{c}\right)}\right]\sin^2\theta.$$

Inspection of Eq. 54 reveals that this is equal to the mechanical power converted per unit area, p_m. This expression may be rearranged to read

$$p_m = \frac{9}{8}\mu_o\,v\,H_o^2\left[1-\frac{v^2}{c^2}\right]\left[\frac{\left(1+\frac{v}{c}\right)}{\left(1+\frac{2v}{c}\right)\left(1-\frac{v}{c}\right)}\right]^2\sin^2\theta. \qquad (59)$$

From the results given by Eqs. 57, 58, and 59, it can be verified that Eq. 56 is indeed satisfied; that is,

$$S_r = \frac{9}{4}\mu_o \, v \, H_o^2 \left[\frac{\left(1+\frac{v}{c}\right)}{\left(1+\frac{2v}{c}\right)\left(1-\frac{v}{c}\right)} \right]^2 \sin^2\theta = p_c + p_m$$

$$= \frac{9}{8}\mu_o \, v \, H_o^2 \left[1 + \frac{v^2}{c^2}\right] \left[\frac{\left(1+\frac{v}{c}\right)}{\left(1+\frac{2v}{c}\right)\left(1-\frac{v}{c}\right)} \right]^2 \sin^2\theta$$

$$+ \frac{9}{8}\mu_o \, v \, H_o^2 \left[1 - \frac{v^2}{c^2}\right] \left[\frac{\left(1+\frac{v}{c}\right)}{\left(1+\frac{2v}{c}\right)\left(1-\frac{v}{c}\right)} \right]^2 \sin^2\theta .$$

It is interesting to note that the relative contributions of p_c and p_m change as the velocity increases. For very low velocities, if we neglect the terms in v^2/c^2, the two contributions are equal. As the velocity increases, more power comes from the convected energy than from mechanical work.

V. THE EXPANDING SPHERE WITH FINITE CONDUCTIVITY

5.1 INTRODUCTION

We shall be concerned here with the problem of a finitely conducting sphere ($\sigma \neq \infty$, $\epsilon = \epsilon_o$, $\mu = \mu_o$) expanding in a uniform magnetic field. Except for finite conductivity, the basic assumptions are those stated in section 2.1. For the present situation, Eq. 3 must be replaced by

$$\overline{J}_c = \sigma(\overline{E} + \overline{u} \times \mu_o \overline{H}), \tag{60}$$

where $\overline{J}_c$ is the conduction current density, σ is the conductivity, and $\overline{u}$ is the velocity of the given grain of matter. All measurements are referred to a stationary frame of reference.

In order to specify the problem completely, the velocity $\overline{u}(\overline{r}, t)$ must be prescribed. The conductivity σ in Eq. 60 is a function of the velocity.[20] If an arbitrary velocity field is specified, the problem becomes exceedingly complicated. In order to make the mathematics more tractable, it will be assumed that the velocity at all points inside the sphere is given by

$$\overline{u}(\overline{r}, t) = \overline{i}_r u(a) = \overline{i}_r v, \qquad 0 < r < a, \tag{61}$$

where v is the velocity of expansion of the surface, assumed to be a constant. For this particular case, the conductivity is a constant.

Even after this simplifying assumption is introduced it is not possible to solve the problem rigorously. Therefore a high-conductivity approximation, which is valid for points not too far from the surface of the sphere, will be presented. In section 5.2, the problem is formulated, and the approximations that will be introduced are explained. In section 5.3, a differential equation for the vector potential is derived for the region inside the sphere ($r < a$). Methods of solution are discussed, with particular emphasis on the time dependence of the proposed solutions. In section 5.4, the approximations discussed in section 5.2, together with the solutions of section 5.3, are applied to the problem; this procedure leads to a high-conductivity approximation. Further simplification is achieved if the approximation is restricted to points not too far from the surface.

5.2 FORMULATION OF THE PROBLEM

The basic equations involved in the problem are now presented for the two regions of interest: inside the sphere ($r < a$), and outside the sphere ($r > a$). Inside the sphere, Maxwell's equations (2) and Ohm's law (60) may be combined to give

$$\left.\begin{aligned}
&\operatorname{curl} \overline{E} = -\mu_o \frac{\partial \overline{H}}{\partial t} \\
&\operatorname{curl} \overline{H} = \sigma \overline{E} + \sigma \overline{u} \times \mu_o \overline{H} + \epsilon_o \frac{\partial \overline{E}}{\partial t} \\
&\operatorname{div} \epsilon_o \overline{E} = \rho \\
&\operatorname{div} \mu_o \overline{H} = 0
\end{aligned}\right\} \qquad (62)$$

for $r < a$, where $\overline{u} = \overline{i}_r v$.

Outside the sphere, Maxwell's equations (2) apply, with $\rho = 0$ and $\overline{J} = 0$, as is appropriate for free space. At $r = a$, the boundary conditions set forth in section 2.3 must be applied in order to ensure proper connection between the fields inside and outside the sphere. The solutions outside the sphere must satisfy the conditions at infinity that were stated in section 2.4. As initial conditions we shall assume that both fields $\overline{E}$ and $\overline{H}$ inside the sphere vanish at $t = 0$, which corresponds to the case of the sphere whose initial radius is zero.

This is a much more difficult problem than the one that we solved for $\sigma = \infty$. We shall not attempt to proceed with the solution of the boundary-value problem in its whole generality. Instead, we shall try to obtain a high-conductivity approximation, which is based on the following argument. If the conductivity is very high, we do not expect the fields outside the sphere to be too different from those for the case $\sigma = \infty$. We have obtained solutions for $\sigma = \infty$; they lead to the following results for the total fields (primary plus induced) at the boundary $r = a$:

$$\left.\begin{aligned}
&E_\phi(a) = -\frac{3}{2} \frac{\mu_o v(1+v/c)}{(1+2v/c)(1-v/c)} H_o \sin\theta \\
&H_\theta(a) = \frac{3}{2} \frac{(1+v/c)}{(1+2v/c)(1-v/c)} H_o \sin\theta \\
&H_r(a) = 0
\end{aligned}\right\} \qquad (63)$$

The effective fields,[21]

$$\overline{E}_e = \overline{E} + \overline{v} \times \mu_o \overline{H}$$

$$\overline{H}_e = \overline{H} - \overline{v} \times \epsilon_o \overline{E},$$

for $r = a$, are

$$\left.\begin{aligned}
&\overline{E}_e(a) = 0 \\
&\overline{H}_e(a) = i_\theta \frac{3}{2} \frac{(1+v/c)^2}{(1-2v/c)^2} H_o \sin\theta
\end{aligned}\right\}. \qquad (64)$$

The vanishing of $\overline{E}_e$ at the boundary is a requirement stemming from the assumption

$\sigma = \infty$. In fact, the derivation of the boundary conditions for a moving interface (Appendix A) shows that the effective electric field $\overline{E}_e$ must be continuous across the interface. Since, under the assumed initial conditions, the $\overline{E}_e$ field inside the perfectly conducting sphere vanishes, the result $\overline{E}_e(a) = 0$ follows. For the present problem, with the fields expected to be finite (nonzero) inside the sphere, $\overline{E}_e(a)$ is not zero, although if σ is very high, we expect $\overline{E}_e(a)$ to be very small. Since $\overline{E}_e(a)$ changes from zero (when $\sigma = \infty$) to a finite value (when $\sigma \neq \infty$), even when in this case $\overline{E}_e(a)$ is very small, the relative change in $\overline{E}_e(a)$ is large. The same reasoning applies to the radial component of the effective magnetic field, $\overline{n} \cdot \overline{H}_e(a)$, which is zero for $\sigma = \infty$, but should be finite for $\sigma \neq \infty$. The relative change in the tangential component of the effective magnetic field, however, should be small if σ is very high. If $\overline{H}_{ie}(\overline{r}, t)$ is the effective magnetic field inside the sphere, for $\sigma \neq \infty$ (which is part of the solution we are seeking), it is reasonable to make the approximation that, for very high σ,

$$\overline{n} \times \overline{H}_{ie}(a) = \overline{n} \times \overline{H}_e(a), \qquad r = a. \tag{65}$$

The time dependence in $\overline{H}_{ie}$ has been dropped, since (from Eqs. 64) $\overline{H}_e(a)$ is independent of time. If Eq. 65 is accepted, we have reduced our problem to a simpler one: <u>To find the solutions of Eqs. 62 inside a spherical region of radius a(t), given as a function of time by a(t) = vt, which reduce to zero at t = 0, and have the property that Eq. 65 is satisfied at r = a for all t > 0.</u> The approximation involved is a consequence of the acceptance of (65), which is not exact.

The problem now involves only the region $r < a$, for which solutions that reduce to prescribed values at the surface $r = a$ must be found. Difficulties are still expected, since the boundary $r = a$ is moving. These difficulties will be more clearly appreciated when we attempt to solve Eqs. 62. Accordingly, we shall postpone the discussion of such difficulties and the approach that will be suggested to overcome them.

5.3 THE SOLUTIONS OF MAXWELL'S EQUATIONS INSIDE THE FINITELY CONDUCTING EXPANDING SPHERE

We shall derive a differential equation for the vector potential in the region $r < a$, which follows from Eqs. 62 and the velocity prescription (61). In (62) we set $\rho = 0$, using the symmetry arguments discussed in Section III (that is, we expect the current density to be ϕ-directed and independent of ϕ). This current density is divergence-free, and we conclude that there are no charges, since $\frac{\partial \rho}{\partial t} = 0$ and $\rho = 0$ at $t = 0$. From the symmetry of the problem, we expect to find only three components of the fields, which we express in the form

$$\begin{aligned} E_\phi(r, \theta, t) &= E_\phi(r, t) \sin \theta \\ H_\theta(r, \theta, t) &= H_\theta(r, t) \sin \theta \\ H_r(r, \theta, t) &= H_r(r, t) \cos \theta. \end{aligned} \tag{66}$$

Introduction of these expressions into Eqs. 62 (with $\rho = 0$) eliminates the θ-dependence, and leads to the following set of equations:

$$\text{(a)}\ \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) + H_r = \epsilon_o \frac{\partial E_\phi}{\partial t} + \sigma E_\phi + \sigma v \mu_o H_\theta$$

$$\text{(b)}\ E_\phi = -\frac{1}{2}\mu_o r \frac{\partial H_r}{\partial t}$$

$$\text{(c)}\ \frac{\partial}{\partial r}(rE_\phi) = \mu_o r \frac{\partial H_\theta}{\partial t}$$

$$\text{(d)}\ \frac{1}{r}\frac{\partial}{\partial r}(r^2 H_r) + 2H_\theta = 0. \tag{67}$$

Among the three field components, only H_r can be isolated in a second-order partial differential equation. We chose to relate H_r to A_ϕ, which is the only necessary component of the vector potential, and obtain the differential equation in terms of A_ϕ. Thus, since $\mu_o \overline{H} = \text{curl}\ \overline{A}$, we find

$$\mu_o H_r = \frac{2A_\phi}{r}. \tag{68}$$

Here, as in (66), $A_\phi(r, \theta, t) = A_\phi(r, t) \sin\theta$. If we substitute (68) in Eqs. 67, and solve Eqs. 67a, b, and d for rA_ϕ, we obtain

$$\frac{\partial^2}{\partial r^2}(rA_\phi) - 2q\frac{\partial}{\partial r}(rA_\phi) - \frac{2}{r^2}(rA_\phi) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(rA_\phi) + \frac{2\alpha}{c^2}\frac{\partial}{\partial t}(rA_\phi), \tag{69}$$

where

$$q \equiv \frac{1}{2}\mu_o \sigma v$$

$$\alpha = \frac{1}{2}\mu_o \sigma c^2$$

$$c^2 = \frac{1}{\mu_o \epsilon_o}.$$

From A_ϕ the field components may then be obtained as follows:

$$E_\phi = -\frac{\partial A_\phi}{\partial t}, \tag{70a}$$

$$\mu_o H_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi), \tag{70b}$$

$$\mu_o H_r = \frac{2A_\phi}{r}. \tag{70c}$$

No general solutions of the differential equation (69) could be found in the time domain. The only other choice is to make use of the classical method of separation of variables. Applying to Eq. 69 the usual Laplace transformation[22] with respect to t, we obtain an ordinary differential equation involving r alone, which is then solved by conventional techniques. The application of the Laplace transformation in this case leads to some difficulties, particularly when matching the fields at a moving boundary. We shall proceed with the solution, and return to these difficulties later. Let

$$A_\phi(r, s) = \int_0^\infty A_\phi(r, t)\, e^{-st}\, dt$$

and $\psi \equiv rA_\phi(r, s)$.

For the prescribed initial conditions, application of the Laplace transformation to (69) yields

$$\frac{d^2\psi}{dr^2} - 2q\frac{d\psi}{dr} - \left[p^2 + \frac{2}{r^2}\right]\psi = 0, \tag{71}$$

where

$$p^2 \equiv \frac{1}{c^2}(s^2+2\alpha s). \tag{72}$$

A series solution of (71) leads to

$$\psi = \psi_o\sqrt{r}\, e^{qr}\, Z_{3/2}(j\beta r),$$

in which $Z_{3/2}$ is the appropriate form of the Bessel function of order 3/2, and

$$\beta^2 = \frac{1}{c^2}(s^2+2\alpha s+q^2c^2). \tag{73}$$

We have no reason to expect solutions that are singular at r = 0. Hence, in order to obtain a finite potential at the center, we choose

$$Z_{3/2}(j\beta r) = J_{3/2}(j\beta r) = -jI_{3/2}(\beta r)$$

and choose β to be the square root of β^2 which is positive when β^2 is real and positive. The vector potential $A_\phi(r, s)$ is then given by

$$A_\phi(r, s) = A(s)\frac{e^{qr}}{r} I_{3/2}(\beta r). \tag{74}$$

From (74) the fields may be found at once by using expressions (70).

At this point the difficulties involved in matching a field across a moving boundary can be fully appreciated. Assume, for instance, that some solution F(r, s), of the same form as (74), that is, F(r, s) = A(s) f(r), must be matched to the prescribed value $F_o(s)$ at the moving surface r = vt. Then, A(s) would be determined by

$$F(vt, s) = A(s)\, f(vt) = F_o(s)$$

or

$$A(s) = \frac{F_o(s)}{f(vt)}.$$

That is, A(s) would be also a function of t. But this is inconsistent with the original assumption, implicit in the Laplace transformation, that the time dependence of the solutions is of the form e^{st}. It is, at best, not clear how the correct solution could be obtained in this way.

This difficulty may be avoided by matching at a fixed surface and then using superposition in the time domain to obtain the final result. The proposed method is essentially an extension of the convolution approach that was used previously for the case of infinite conductivity. It may be described as follows: Let another variable t' be introduced, and related only to the motion of the boundary surface, that is, $a(t') = vt'$. For a particular value of t', let the value to be matched at the surface be $F_o(t,t') = F_1(t)\,\delta(t-t')$. Here, $F_1(t)$ is the time dependence of $F_o(t, t')$ which is not related to the motion of the boundary, and the impulse function $\delta(t-t')$ ensures that only the value of $F_1(t)$ at $t = t'$ is taken into consideration, since $F_o(t, t')$ is zero for $t \neq t'$. Now, if we solve the problem in the frequency domain, we must match the solution at the fixed (instantaneous) value $r = a$, to the function

$$F_o(s, t') = F_1(t')\, e^{-st'} \tag{75}$$

which is the Laplace transform of $F_o(t, t')$. Then we find for the function F(r, s) used in the example above,

$$F(a, s) = A(s, t')\, f(a) = F_1(t')\, e^{-st'}.$$

Hence

$$A(s, t') = \frac{F_1(t')}{f(a)}\, e^{-st'}$$

is now a function of s and t' only. Since t' is not involved in the Laplace transformation, it behaves as a parameter throughout this operation. We can apply the inverse transformation unambiguously to the function F(r, s, t') (obtained by using A(s, t') as determined above) and find a function F(r, t, t') that has the nature of an impulse response, since it is the result of the application of an impulse at the boundary, at time $t = t'$. Now in fact, this process may be applied to impulse excitations at all successive instants of time, from $t' = 0$ to $t' = t$, and the respective contributions superposed to yield the desired result. Of course, as we integrate to sum over all the elementary contributions, we must let a in F(r, t, t') be a function of t', so that the displacement of the boundary is taken into consideration. The final result is given by

$$F(r,t) = \int_0^t dt' \; F(r,t,t').$$

In view of the above-mentioned conclusions, we once more reformulate our problem. Since we must match the $\overline{H}$ field at a fixed boundary, uniqueness requires that the tangential component of $\overline{H}$ be prescribed at r = a. In section 5.2 we showed that $\overline{H}_e(a)$ should change very little, so that this value was prescribed at the boundary (Eq. 65). If $\overline{H}_e(a)$ does not change much then $H_\theta(a)$ (Eqs. 63) cannot change much either, since the term $\overline{v} \times \epsilon_o \overline{E}$ in $\overline{H} = \overline{H}_e + \overline{v} \times \epsilon_o \overline{E}$ is of the order of v^2/c^2 with respect to both $\overline{H}$ and $\overline{H}_e$. If we restrict ourselves to nonrelativistic velocities, the change in $E_\phi(a)$, because of the finite conductivity, has a very small effect on $H_\theta(a)$.

Hence we replace the assumption given by (65) with the following requirement. At any (instantaneous) fixed boundary r = a, if $\overline{H}_i(r,s)$ is the magnetic field inside the sphere, we must have

$$\overline{n} \times \overline{H}_i(a,s) = \overline{n} \times \overline{H}(a), \qquad r = a \tag{76}$$

where $\overline{H}(a)$, from (63), is given by

$$\overline{H}(a) = \overline{i}_\theta \frac{3}{2} \frac{(1+v/c)}{(1+2v/c)(1-v/c)} H_o \sin\theta \equiv H_a \overline{i}_\theta.$$

Here, the symbol H_a has been introduced for convenience.

We must find the solutions of (62) in the frequency domain, which fulfill (76) for all instantaneous values of a. Once the H field is found, it is transformed back to the time domain, and superposition applied as described before. That is, if $H_\theta(r,t,t')$ is the impulse response, the desired solution is given by

$$H_\theta(r,t) = \int_0^t dt' \; H_\theta(r,t,t'). \tag{77}$$

5.4 A HIGH-CONDUCTIVITY APPROXIMATION

We have finally formulated the problem in such a way that it can be solved unambiguously. But one difficulty still remains. Solutions of the type (74) (and still more complicated ones for $\overline{E}$ and $\overline{H}$) are difficult to transform back into the time domain, particularly after matching boundary conditions, since the coefficient A(s) will complicate the solution even further. Since we are in fact looking for solutions that are valid for very high conductivities, it is reasonable to try to approximate the solution (74), if such approximation is possible for very large values of σ. The solution can be simplified if we assume that $\beta r \gg 1$. From expression (73),

$$\beta r = \frac{r}{c} \sqrt{s^2 + 2as + q^2c^2}.$$

The expression under the radical has two zeros, corresponding to the values

$$s_{1,2} = -\alpha\left(1 \pm \sqrt{1 - v^2/c^2}\right).$$

These values of s are both real and negative. If we choose the path of integration for the inverse Laplace transformation along the imaginary axis of the s-plane, βr will never be zero along the path. It will go through a minimum for s = 0, where

$$\beta r(s{=}0) = qr = \frac{1}{2}\mu_o\sigma\, vr.$$

If we assume $\frac{1}{2}\mu_o\sigma\, vr \gg 1$, the condition $\beta r \gg 1$ will be satisfied everywhere. It must be stressed that the condition

$$\frac{1}{2}\mu_o\sigma\, vr \gg 1 \tag{78}$$

implies not only high conductivity, but excludes the region near the center of the sphere (r=0). It also implies that the results obtained from this approximation are not valid in the limit of vanishingly small velocities ($v \to 0$). Thus we cannot use the problem of a stationary sphere in a uniform field (which is trivial) to check our results as $v \to 0$.

If these restrictions are accepted, the Bessel function may be replaced by its asymptotic approximation,

$$A_\phi(r, s) = A(s)\frac{e^{qr}}{r} I_{3/2}(\beta r) \approx A(s)\frac{e^{(q+\beta)r}}{r}. \tag{79}$$

The electric and magnetic fields are found to be

$$\begin{aligned} E_\phi(r, s) &= sA(s)\frac{e^{(q+\beta)r}}{r} \\ \mu_o H_\theta(r, s) &= -(q+\beta)\, A(s)\frac{e^{(q+\beta)r}}{r} \\ \mu_o H_r(r, s) &= 2A(s)\frac{e^{(q+\beta)r}}{r^2}. \end{aligned} \tag{80}$$

We are now in a position to match the tangential H field at the fixed (instantaneous) boundary r = a. From (75), (76), and (80), we find

$$\mu_o H_\theta(a, s) = -(q+\beta)\, A(s)\frac{e^{(q+\beta)a}}{a} = \mu_o H_a\, e^{-st'},$$

so that

$$A(s) = -\mu_o \frac{aH_a}{(q+\beta)}\, e^{-st'}\, e^{-(q+\beta)a},$$

and

$$H_\theta(r, s, t') = H_a \frac{a}{r} e^{-(q+\beta)(a-r)}\, e^{-st'}, \tag{81a}$$

$$E_\phi(r,s,t') = -H_a \frac{sa}{(q+\beta)r} e^{-(q+\beta)(a-r)} e^{-st'}, \tag{81b}$$

$$H_r(r,s,t') = -\frac{2H_a}{(q+\beta)} \frac{a}{r} e^{-(q+\beta)(a-r)} e^{-st'}. \tag{81c}$$

For $H_\theta(r,t,t')$ we then have

$$H_\theta(r,t,t') = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} H_\theta(r,s,t')\, e^{st}\, ds = \mathscr{L}^{-1}[H_\theta(r,s,t')],$$

where $\mathscr{L}^{-1}$ indicates the inverse Laplace transform of $H_\theta(r,s,t')$. If $H_\theta(r,s,t')$ is put into the form

$$H_\theta(r,s,t') = H_a \frac{a}{r} e^{-q(a-r)} \left[\exp\left[-\frac{(a-r)}{c}\sqrt{(s+\alpha)^2-(\alpha^2-q^2c^2)}\right] - \exp\left[-\frac{(a-r)}{c}(s+\alpha)\right] + \exp\left[-\frac{a-r}{c}(s+\alpha)\right]\right] e^{-st'}.$$

Here, the term $\exp\left[-\frac{a-r}{c}(s+\alpha)\right]$ has been added and subtracted; the inverse transformation can be found in published works.[23] Omitting a few intermediate steps, we find

$$H_\theta(r,t,t') = H_a \frac{a}{r} e^{-q(a-r)} e^{-\alpha(t-t')} \left\{ \delta\left[(t-t') - \frac{(a-r)}{c}\right] - \sqrt{\alpha^2-q^2c^2}\,\frac{(a-r)}{c}\, \frac{I_1\left[\sqrt{\alpha^2-q^2c^2}\sqrt{(t-t')^2-\frac{(a-r)^2}{c^2}}\right]}{\sqrt{(t-t')^2-\frac{(a-r)^2}{c^2}}} \right\} \tag{82}$$

for $(t-t') > (a-r)/c$, where I_1 is the Bessel function of order 1, and $H_\theta(r,t,t') = 0$, otherwise.

We may now find $H_\theta(r,t)$, by letting $a = vt'$ and integrating over t',

$$H_\theta(r,t) = \int_0^{\frac{t+r/c}{1+v/c}} H_\theta(r,t,t')\, dt', \qquad r > vt.$$

The first term inside the braces in (82) integrates very easily. The second seems to lead to a very difficult integral. For the first term in (82), if we let $a = vt'$, we find

$$\int_0^{\frac{t+r/c}{1+v/c}} dt'\, vt'\, e^{-\alpha(t-t')}\, e^{-qvt'}\, \delta\left[\left(t+\frac{r}{c}\right) - t'\left(1+\frac{v}{c}\right)\right]$$

$$= \frac{v(t+r/c)}{(1+v/c)}\, e^{-\alpha t} \exp\left(\alpha\, \frac{t+r/c}{1+v/c}\right) \exp\left(-qv\, \frac{t+r/c}{1+v/c}\right).$$

If we let $H_\theta(r,t) = H_{\theta 1}(r,t) + H_{\theta 2}(r,t)$, where $H_{\theta 1}$ and $H_{\theta 2}$ correspond to the two terms in (82), we find, by using the results of the integration above and a few algebraic steps,

$$H_\theta(r,t) = H_a \frac{v(t+r/c)}{r(1+v/c)} \exp\left(-\alpha\, \frac{vt-r}{c}\right) + H_{\theta 2}(r,t), \tag{83}$$

where

$$H_{\theta 2}(r,t) = H_a \sqrt{\alpha^2 - q^2c^2}\, \frac{v}{c}\, \frac{e^{qr}}{r}\, e^{-\alpha t} \int_0^{\frac{t+r/c}{1+v/c}} dt'\, t'(vt'-r)$$

$$\times\, e^{(\alpha - qv)t'}\, \frac{I_1\left[\sqrt{\alpha^2 - q^2c^2}\, \sqrt{(t-t')^2 - \frac{(vt'-r)^2}{c^2}}\right]}{\sqrt{(t-t')^2 - \frac{(vt'-r)^2}{c^2}}}. \tag{84}$$

We note that, at $r = a$, $H_{\theta 2}(a,t) = 0$ (because the integrand is zero) and $H_\theta(a,t) = H_{\theta 1}(a,t) = H_a$ as expected, since H_a was the prescribed tangential field at $r = a$.

The expressions for $E_\phi(r,s,t')$ and $H_r(r,s,t')$, given by (81b) and (81c) are very difficult to transform back to the time domain. At this point, very little can be done to extend our results, unless further approximations are introduced. A possible one seems to be for points near the surface, $r \approx a$.

If we neglect the second term in (83), on the grounds that its contribution near the surface should be small compared with that of the first term, then

$$H_\theta(r,t) = H_a \frac{v(t+r/c)}{r(1+v/c)} \exp\left(-\alpha\, \frac{(vt-r)}{c}\right). \tag{85}$$

Since

$$\mu_o \overline{H} = \text{curl}\, \overline{A}$$

$$\overline{E} = -\frac{\partial \overline{A}}{\partial t},$$

$E_\phi(r,t)$ may be found from $H_\theta(r,t)$. The intermediate steps are straightforward and will be omitted here. We find

$$E_\phi(r,t) = -\sqrt{\frac{\mu_o}{\epsilon_o}}\frac{v}{c}H_a\left[\frac{v(t+v/c)}{r(1+v/c)} - \frac{2}{\mu_o\sigma c r}\right]\exp\left(-\alpha\frac{(vt-r)}{c}\right). \tag{86}$$

The second term in brackets is responsible for the small change in $E_\phi(a,t)$, at the surface, with respect to the value of $E_\phi(a,t)$ for $\sigma = \infty$. In fact, at $r = a$,

$$E_\phi(a,t) = -\sqrt{\frac{\mu_o}{\epsilon_o}}\frac{v}{c}H_a + 2\sqrt{\frac{\mu_o}{\epsilon_o}}\frac{v}{c}\frac{H_a}{\mu_o\sigma c v t}$$

and if the second term is neglected,

$$E_\phi(a,t) = -\sqrt{\frac{\mu_o}{\epsilon_o}}\frac{v}{c}H_\theta(a,t)$$

which is the relation obtained in Section IV for the case of perfect conductivity. From (85) and (86), since

$$\overline{J} = \sigma(\overline{E} + \overline{v}\times\mu_o\overline{H}),$$

we obtain the expression for the current density,

$$J_\phi(r,t) = 2H_a\frac{v}{c}\frac{\exp\left(-\alpha\frac{(vt-r)}{c}\right)}{r}. \tag{87}$$

This is as far as this approximation has led us. Perhaps the integral in (84) can be evaluated, or at least well approximated. We note that neglecting this term is equivalent to neglecting the second term in braces in expression (82). Thus, it implies that the response to an impulse at $r = a$,

$$H_\theta(a,t,t') = H_a\,\delta(t-t'),$$

is given by

$$H_\theta(r,t,t') = H_a\frac{a}{r}e^{-q(a-r)}\,e^{-\alpha(t-t')}\,\delta\left[(t-t') - \frac{(a-r)}{c}\right]$$

for $t - t' > (a-r)/c$.

Thus, the response to an impulse at the boundary is an impulse traveling toward the center, attenuating as it travels. This suggests, by analogy with the problem of propagation of a pulse in a stationary conducting medium,[24] the interpretation of the integral term (82) as a residual field, or "wake."

VI. CONCLUSION

The general results obtained for the case of the expanding sphere with infinite conductivity are summarized in the integral equation (32) and expressions (34).

The integral equation (32) holds for any arbitrary time dependence of the radius, for a sphere whose initial radius is zero. For a finite initial radius, the equation must be modified by the inclusion of the term in (16) which accounts for the initial conditions. This term may be evaluated once the initial radius a_o and the initial conditions on the vector potential and its time derivative are prescribed. Hence, within the scope of the problem, the integral equation (32) is perfectly general, and from it the induced current density may be found for any particular prescribed time dependence. Once the induced current is found, expressions (34) may be used to determine the induced fields. These expressions are likewise general. If it is preferred, the vector potential may be determined directly from the retarded potential integral (21). This was the method used in Section IV.

In Section IV, the general expressions of Section III were used to obtain the current density and the fields for the particular case of constant velocity of expansion. Apart from their usefulness as exact solutions of the problem of constant radial expansion, they serve the purpose of confirming the applicability of the general formulas derived in Section III.

The application of the boundary conditions, as carried out at the end of section 4. 2, proves the correctness of these solutions. Furthermore, they do confirm what would be expected from physical reasoning. The current dènsity given by (41) leads, in the limit of very low velocities $\left(\frac{v}{c} \ll 1\right)$, to

$$J_\phi(\overline{r'}, t') = \frac{3H_o}{2} \delta(r' - vt') \sin \theta'$$

which corresponds to a surface current density

$$K_\phi(r', t') = \frac{3}{2} H_o \sin \theta'$$

located at the surface of the sphere, r = a. This surface current can be shown to be the solution of the problem of a stationary sphere immersed in a uniform static magnetic field, for which the fields are those given by (5) and (6). It is seen, then, that for low velocities all that we have is the static solution, which is carried along by the surface as it moves. This is what would be expected from physical intuition. This surface current density may be shown to produce a magnetic moment,

$$\overline{m}_s = 2\pi a^3 H_o \overline{i}_z,$$

which is the value to which the exact magnetic moment (49) reduces if $\frac{v}{c} \ll 1$. This suggests an easy way of obtaining low-velocity approximations, mentioned in section 4. 2, which is to neglect retardation inside the sphere in the evaluation of the integral equation (21). This is equivalent to determining the current density for a stationary sphere. Once the current density is found, the magnetic moment is obtained easily. Since $a = vt$ in the expression for $\overline{m}_s$ given above, we can interpret the problem immediately as that of a time-variant magnetic dipole. The solution leads to fields identical to those of expressions (45), except for the absence of the terms containing v/c.

This approach may be extended immediately to problems involving other geometries with rotational symmetry, such as an infinite cylinder with radial expansion.

The energy-power balance at the surface of the sphere, as expressed by the application of Poynting's theorem in section 4.3, yields the result

$$S_r = p_c + p_m,$$

where S_r is the radial component of the Poynting vector, p_c is the power convected per unit area (as defined in section 4.3), and p_m is the power per unit area converted from mechanical to electromagnetic form. It is interesting to note how the values of S_r, p_c, and p_m vary with the velocity. In particular,

$$\frac{S_r}{p_m} = \frac{2}{(1-v^2/c^2)},$$

and

$$\frac{S_r}{p_c} = \frac{2}{(1+v^2/c^2)},$$

which shows that, for very low velocities,

$$S_r \approx 2p_m \approx 2p_c,$$

and

$$\frac{p_m}{p_c} \approx 1.$$

The contributions of p_m and p_c to S_r are approximately equal. For very high velocities, however,

$$\frac{p_m}{p_c} \rightarrow 0,$$

that is, most of the power density S_r comes from the convected power density.

The results obtained for the problem of the sphere with finite conductivity are of restricted validity, in view of the approximations that we had to introduce. We wish to point out that the exponential dependence of the fields near the surface, given by expressions (85) and (86) does suggest a steep decay of the fields as we move away from the surface. This seems to indicate that the behavior of the fields when σ is very high is analogous to that of the classical skin-effect solutions. This impression is certainly confirmed by the fact that when $\sigma \to \infty$, the fields are completely expelled from the interior of the sphere. Since we did not in fact find solutions except for points near the surface, the results are not conclusive.

Finally, since this problem was partly motivated by the desire to predict the expected values of the radio signals created by explosions in the magnetic field of the earth (section 1.2), it is reasonable to ask whether or not the results of this analysis lead to acceptable values for these signals. For this purpose, we shall make use of Eq. 50 for the electric field, and take for the constant velocity of expansion the average velocity of expansion of a 20-kiloton nuclear bomb. This average velocity, during the first 15 msec, is approximately 4.40×10^3 meter/sec.[10] At the end of this interval the so-called breakaway occurs, after which both the velocity and the temperature decrease rapidly with time. Hence the first 15 msec is taken as the significant interval for the purpose of our calculations. The earth's magnetic field is $H_o = \frac{1}{8\pi} \times 10^3$ amp/meter. The details of the calculation are omitted. At a distance of 1000 km the expected electric field is of the order of 2×10^{-9} volt/meter. For these same conditions, the predictions of Johnson and Lippmann,[7] who postulated a different mechanism for the generation of the signals (see section 1.2), lead to the value 1.2×10^{-8} volt/meter.

APPENDIX A

Boundary Conditions at a Moving Interface

A.1 Maxwell's Equations

The derivation of boundary conditions for fields in free space[25] must be modified if the surface of discontinuity is moving. We assume the situation where there are no magnetic or dielectric materials, and the velocity is continuous across the boundary. The notation used here is that of Fano, Chu, and Adler.[25] The surface Σ moves with velocity $\overline{v}$ with respect to the observer's frame. The contour is stationary with respect to

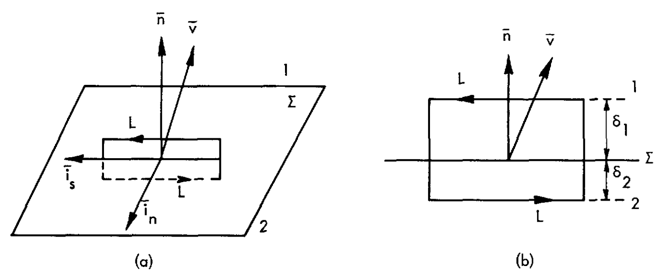

Fig. 10. (a) Stationary contour of integration, L, and moving surface, Σ. (b) Detail of the contour of integration.

this frame and the surface Σ moves past it (Fig. 10a). The contour is shown in more detail in Fig. 10b, where the width δ of the contour has been split into δ_1 and δ_2. Since Σ moves, δ_1 and δ_2 are functions of time (and only of time). Hence,

$$\frac{d\delta_1}{dt} = \frac{\partial \delta_1}{\partial t} = -\overline{v} \cdot \overline{n}$$

$$\frac{d\delta_2}{dt} = \frac{\partial \delta_2}{\partial t} = \overline{v} \cdot \overline{n}.$$

If we take the first of Maxwell's equations in integral form,

$$\oint \overline{E} \cdot d\overline{s} = -\frac{\partial}{\partial t} \int \mu_o \overline{H} \cdot d\overline{a},$$

and apply it to the contour, we get

$$(\overline{E}_1 L - \overline{E}_2 L) \cdot \overline{i}_s = -\frac{\partial}{\partial t} [\mu_o \overline{H}_1 \delta_1 L + \mu_o \overline{H}_2 \delta_2 L] \cdot \overline{i}_n$$

$$(\overline{E}_1 L - \overline{E}_2 L) \cdot \overline{i}_s = -\mu_o L \left[\delta_1 \frac{\partial \overline{H}_1}{\partial t} + \delta_2 \frac{\partial \overline{H}_2}{\partial t} + \overline{H}_2 (\overline{v} \cdot \overline{n}) - \overline{H}_1 (\overline{v} \cdot \overline{n}) \right] \cdot \overline{i}_n.$$

The partial derivatives $\frac{\partial \overline{H}_1}{\partial t}$ and $\frac{\partial \overline{H}_2}{\partial t}$ are assumed to be bounded, as in the stationary case. Hence, as we take the limit $L \to 0$, $\delta_1, \delta_2 \to 0$, the expression above reduces to

$$\overline{i}_s \cdot (\overline{E}_1 - \overline{E}_2) = \mu_o(\overline{v} \cdot \overline{n})(\overline{H}_1 - \overline{H}_2) \cdot \overline{i}_n.$$

Since $\overline{i}_s = \overline{i}_n \times \overline{n}$,

$$\overline{i}_n \times \overline{n} \cdot (\overline{E}_1 - \overline{E}_2) = \mu_o(\overline{v} \cdot \overline{n})\, \overline{i}_n \cdot (\overline{H}_1 - \overline{H}_2)$$

and

$$\overline{i}_n \cdot \overline{n} \times (\overline{E}_1 - \overline{E}_2) = \mu_o \overline{i}_n \cdot [(\overline{v} \cdot \overline{n})(\overline{H}_1 - \overline{H}_2)].$$

Using the vector identity

$$\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \cdot \overline{c})\overline{b} - (\overline{a} \cdot \overline{b})\overline{c}$$

or

$$(\overline{a} \cdot \overline{b})\overline{c} = (\overline{a} \cdot \overline{c})\overline{b} - \overline{a} \times (\overline{b} \times \overline{c})$$

we find

$$(\overline{n} \cdot \overline{v})(\overline{H}_1 - \overline{H}_2) = [\overline{n}(\overline{H}_1 - \overline{H}_2)]\, \overline{v} - \overline{n} \times [\overline{v} \times (\overline{H}_1 - \overline{H}_2)].$$

The boundary condition $\overline{n} \cdot (\overline{H}_1 - \overline{H}_2) = 0$ still holds. Hence,

$$\overline{i}_n \cdot \overline{n} \times (\overline{E}_1 - \overline{E}_2) = -\mu_o \overline{i}_n \cdot \overline{n} \times \overline{v} \times (\overline{H}_1 - \overline{H}_2)$$

$$\overline{i}_n \cdot [\overline{n} \times (\overline{E}_1 - \overline{E}_2) + \overline{n} \times \overline{v} \times \mu_o (\overline{H}_1 - \overline{H}_2)] = 0.$$

The vector $\overline{i}_n$ is an arbitrary tangential vector. Since both quantities inside the brackets are tangential, $\overline{i}_n$ may be dropped to yield

$$\overline{n} \times [(\overline{E}_1 + \overline{v} \times \mu_o \overline{H}_1) - (\overline{E}_2 + \overline{v} \times \mu_o \overline{H}_2)] = 0.$$

In other words, the <u>effective field</u> $\overline{E}_{eff} = \overline{E} + \overline{v} \times \mu_o \overline{H}$ is continuous across the boundary.

If we take the second of Maxwell's equations in integral form

$$\oint \overline{H} \cdot d\overline{s} = \frac{\partial}{\partial t} \int \epsilon_o \overline{E} \cdot d\overline{a} + \int J \cdot d\overline{a}$$

the derivation proceeds along the same lines. Two extra terms appear.

(i) Since $\epsilon_o \overline{n} \cdot (\overline{E}_1 - \overline{E}_2) = \sigma$ (surface charge), we have a surface convection current $\sigma \overline{v}_T$, where $\overline{v}_T = \frac{\overline{v} \cdot \overline{K}}{|K|}$, and $\overline{K}$ is the surface conduction current.

(ii) As the contour shrinks to zero, the contribution of any finite current density goes to zero. If $\overline{J}$ is infinite but concentrated in a layer of vanishing thickness d at the interface, so that $\lim_{\substack{d \to 0 \\ J \to 0}} (\overline{J}d) = \overline{K}$, the surface current density $\overline{K}$ must be included as in the stationary case.

Taking these terms into account, we get

$$\bar{n} \times [(\bar{H}_1 - \bar{v} \times \epsilon_o \bar{E}_1) - (\bar{H}_2 - \bar{v} \times \epsilon_o \bar{E}_2)] = \bar{K} - \sigma \bar{v}_T.$$

We now summarize the conditions that hold for a moving interface, under the assumed restrictions (continuous velocity, no magnetic or dielectric materials):

$$\bar{n} \times [(\bar{E}_1 + \bar{v} \times \mu_o \bar{H}_1) - (\bar{E}_2 + \bar{v} \times \mu_o \bar{H}_2)] = 0$$

$$\bar{n} \times [(\bar{H}_1 - \bar{v} \times \epsilon_o \bar{E}_1) - (\bar{H}_2 - \bar{v} \times \epsilon_o \bar{E}_2)] = \bar{K} - \sigma \bar{v}_T$$

$$\bar{n} \cdot (\bar{H}_1 - \bar{H}_2) = 0, \qquad \bar{n} \cdot (\bar{E}_1 - \bar{E}_2) = \sigma/\epsilon_o.$$

A. 2 Poynting's Theorem

The form of Poynting's theorem to be applied in the case of a moving discontinuity will be derived, subject to the same restrictions as in A. 1. Poynting's theorem in integral form is given by

$$\oint \bar{S} \cdot \bar{n}\, da + \frac{\partial}{\partial t} \int_V w\, dv + \int_V (\bar{E} \cdot \bar{J} + \bar{H} \cdot \bar{J}^*)\, dv = 0,$$

where

$$w = w_e + w_m = \frac{1}{2} \epsilon_o |E|^2 + \frac{1}{2} \mu_o |H|^2.$$

A cylindrical volume element of height δ ($\delta = \delta_1 + \delta_2$) is taken stationary with respect to the observer. The surface Σ sweeps past it with velocity v. The area of the upper and

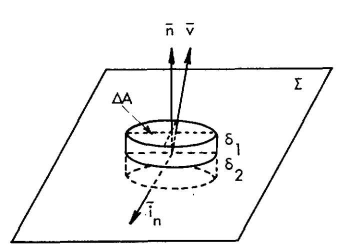

Fig. 11. Stationary cylindrical volume element and moving surface, Σ.

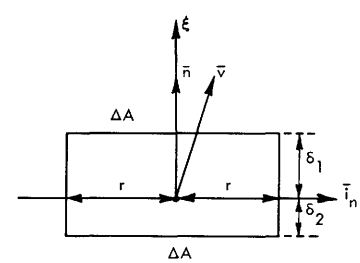

Fig. 12. Cross section of the volume element.

lower faces of the element is ΔA (Fig. 11). A cross section of the volume element is shown in Fig. 12. For the surface integral in Poynting's theorem, we get

$$\oint \bar{S} \cdot \bar{n}\, da = (\bar{S}_1 - \bar{S}_2) \cdot \bar{n} \Delta A + \int_{\text{sides}} \bar{S} \cdot \bar{i}_n\, da = (\bar{S}_1 - \bar{S}_2) \cdot \bar{n} \Delta A + 2\pi r[\delta_1 (\bar{S}_1 + \bar{S}_2) \cdot \bar{i}_n].$$

For the volume integral of the stored energy density, we find:

$$\frac{\partial}{\partial t} \int w\, dv = \frac{\partial}{\partial t} \Delta A[w_2\delta_2 + w_1\delta_1] = \Delta A\left[\left(\frac{\partial w_2}{\partial t}\delta_2 + \frac{\partial w_1}{\partial t}\delta_1\right) + (v \cdot n)(w_2 - w_1)\right].$$

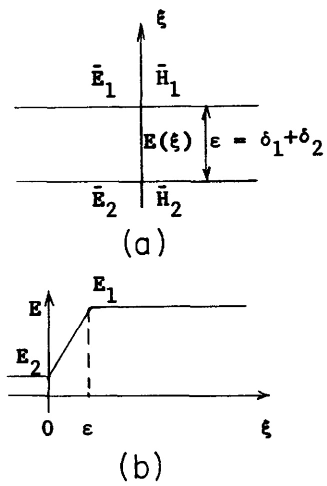

Fig. 13. (a) Region of discontinuity of the fields. (b) Continuous linear transition of the field, E, that replaces the discontinuity across the surface, Σ.

The evaluation of the volume integral involving the fields and current densities must be carried out with more care. Since we are going to take the limits as δ_1 and δ_2 go to zero, the current densities may tend to surface currents in the limit. But the fields are discontinuous at the surface and it is not clear what the limiting form of the integral will be (for stationary systems, E_{Tang} is always continuous, and the difficulty does not arise for $\int E \cdot \bar{J}\, dv$; it does arise for $\int \bar{H} \cdot J^*\, dv$, but this integral is seldom evaluated).

We shall replace the discontinuity of the fields by a rapid but continuous linear transition (Fig. 13).

$$\bar{E}(\xi) = \bar{E}_2 + \frac{(\bar{E}_1 - \bar{E}_2)}{\epsilon}\xi$$

$$\bar{H}(\xi) = \bar{H}_2 + \frac{(\bar{H}_1 - \bar{H}_2)}{\epsilon}\xi$$

$$\int \bar{J} \cdot \bar{E}\, dv = \Delta A \int_0^{\epsilon} \bar{J} \cdot \left[\bar{E}_2 + \frac{(\bar{E}_1 - \bar{E}_2)}{\epsilon}\xi\right] d\xi = \Delta A\left[\bar{J} \cdot \bar{E}_2\epsilon + \frac{1}{2}\epsilon\bar{J} \cdot (\bar{E}_1 - \bar{E}_2)\right]$$

$$= \Delta A \times \frac{1}{2}(\bar{E}_1 + \bar{E}_2) \cdot \bar{J}\epsilon \quad \lim_{\epsilon \to 0} \int \bar{J} \cdot \bar{E}\, dv = \frac{1}{2}\bar{K} \cdot (\bar{E}_1 + \bar{E}_2)\, \Delta A,$$

where $\bar{K} = \lim_{\substack{\epsilon \to 0 \\ \bar{J} \to \infty}} (\bar{J}\epsilon)$. Similarly,

$$\lim_{\epsilon \to 0} \int \bar{J}^* \cdot \bar{H}\, dv = \frac{1}{2}\bar{K}^* \cdot (\bar{H}_1 + \bar{H}_2)\, \Delta A.$$

When we take the limit as δ_1, δ_2 and ΔA go to zero, we obtain the following form for Poynting's theorem:

$$(\bar{S}_1 - \bar{S}_2) \cdot \bar{n} - (\bar{v} \cdot \bar{n})(w_1 - w_2) + \frac{1}{2}\bar{K} \cdot (\bar{E}_1 + \bar{E}_2) + \frac{1}{2}\bar{K}^* \cdot (\bar{H}_1 + \bar{H}_2) = 0.$$

APPENDIX B

Proof of Expression (11)

The purpose now is to prove the identity

$$\frac{d}{dt}\int_A \overline{H}\cdot d\overline{a} = \int_A \frac{\partial \overline{H}}{\partial t}\cdot d\overline{a} + \oint_C \overline{H}\times\overline{u}\cdot d\overline{s}, \tag{B-1}$$

where $\overline{H}$ is an arbitrary, continuous function of position and A is an arbitrary surface enclosed by the contour C. Each point of the contour C is rigidly attached to a moving grain of the medium, whose motion is specified by the velocity $\overline{u}$.

In order to prove the validity of expression (B-1) one starts with the definition of the derivative:

$$\frac{df}{dt} = \lim_{\Delta t\to 0}\frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

For the integral on the left-hand side of (B-1), we can write

$$\frac{d}{dt}\int_A \overline{H}\cdot d\overline{a} = \lim_{\Delta t\to 0}\frac{\int_{A(t+\Delta t)} \overline{H}(t+\Delta t)\cdot d\overline{a} - \int_{A(t)} \overline{H}(t)\cdot d\overline{a}}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{\Delta A} \overline{H}(t+\Delta t)\cdot d\overline{a} + \int_{A(t)} \overline{H}(t+\Delta t)\cdot d\overline{a} - \int_{A(t)} \overline{H}(t)\cdot d\overline{a}\right].$$

Here, ΔA is the increment in A arising from the motion of the contour during the interval Δt. If we let

$$\overline{H}(t+\Delta t) = \overline{H}(t) + \frac{\partial \overline{H}}{\partial t}\Delta t, \tag{B-2}$$

the last two integrals on the right-hand side can be combined to give

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{A(t)}\left(\overline{H}(t) + \frac{\partial \overline{H}}{\partial t}\Delta t\right)\cdot d\overline{a} - \int_{A(t)} \overline{H}(t)\cdot d\overline{a}\right]$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\times\Delta t\int_A \frac{\partial \overline{H}}{\partial t}\cdot d\overline{a} = \int_A \frac{\partial \overline{H}}{\partial t}\cdot d\overline{a}. \tag{B-3}$$

It remains to evaluate the first integral on the right-hand side:

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{\Delta A} \overline{H}(t+\Delta t)\cdot d\overline{a}. \tag{B-4}$$

For this purpose, we first express $d\overline{a}$ as

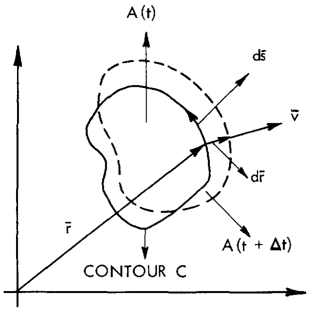

Fig. 14. Illustrating the position of moving contour, C, at times t and t + Δt.

$$d\bar{a} = d\bar{r} \times d\bar{s} = \bar{u}\Delta t \times d\bar{s},$$

where $d\bar{r} = \bar{u}\Delta t$ is the displacement in the position of the contour during the interval Δt, and $d\bar{s}$ is the element of arc (Fig. 14). The integral (B-4) can then be expressed as

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\Delta A} \bar{H}(t+\Delta t) \cdot d\bar{a}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\Delta A} \bar{H}(t+\Delta t) \cdot \bar{u}\Delta t \times d\bar{s}$$

$$= \oint_C \bar{H} \cdot \bar{u} \times d\bar{s} = \oint_C \bar{H} \times \bar{u} \cdot d\bar{s}, \quad \text{(B-5)}$$

where the contour C is shown in Fig. 14. From (B-3) and (B-5) we obtain (B-1).

APPENDIX C

Impulse Response of the Expanding Sphere

In order to find the impulse response for the problem of the expanding sphere, we assume a current density of the form

$$J_i = \delta(r-a) \sin \theta,$$

where $\delta(r-a)$ is the impulse function at $r = a$. Equation 25 yields

$$h(\bar{r}, t, t') = \frac{1}{4\pi} \oint \frac{\delta(r'-a)\, \delta\left(t - t' - \frac{R}{c}\right)}{R} \sin \theta' \cos \theta' \, dv'.$$

In spherical coordinates,

$$R = |\bar{r}-\bar{r}|' = \sqrt{r^2 + r'^2 - 2rr' \cos \gamma}\ ,$$

where γ is the angle between $\bar{r}$ and $\bar{r}'$, and $dv' = r'^2 \sin \theta' \, dr'd\theta'dy'$.

Equation 25 can be rewritten

$$h(\bar{r}, t, t') = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{\delta(r'-a)\, \delta\left(t - t' - \frac{|\bar{r}-\bar{r}|'}{c}\right)}{|\bar{r}-\bar{r}'|} r'^2 \sin^2 \theta' \cos \theta' \, dr'd\theta'd\phi'. \tag{C-1}$$

Note that the radius $\underline{a}$ involved in the δ-function is the value $a(t')$ at time t', when the impulse occurs all over the surface. As far as this integration is concerned, $a(t')$ is not a function of time t. The only function of time in the integrand is the δ-function that contains the retardation factor, $\delta\left(t - t' - \frac{|\bar{r}-\bar{r}'|}{c}\right)$. The time dependence may be transformed away by the use of the Fourier representation of the δ-function,

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{j\omega t}$$

which, for this case, becomes

$$\delta\left(t - t' - \frac{|\bar{r}-\bar{r}'|}{c}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{j\omega(t-t')} \, e^{-jk|\bar{r}-\bar{r}'|},$$

where

$$k \equiv \frac{\omega}{c}.$$

Since nothing else depends on t, the volume and frequency integrations may be interchanged, and Eq. C-1 becomes

$$h(\bar{r}, t, t') = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} d\omega \, e^{j\omega(t-t')} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \delta(r'-a) \frac{e^{-jk|\bar{r}-\bar{r}'|}}{\bar{r} - \bar{r}'}$$

$$\times r'^2 \sin^2 \phi' \cos \phi' \, dr' d\phi' d\phi'. \qquad \text{(C-2)}$$

We now make use of the following series expansions:

$$\frac{e^{-jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} = \begin{cases} -jk \sum_0^\infty (2n+1) h_n^{(2)}(kr') j_n(kr) P_n(\cos\gamma) & r < r' \\ -jk \sum_0^\infty (2n+1) h_n^{(2)}(kr) j_n(kr') P_n(\cos\gamma) & r > r' \end{cases}$$

where the spherical Bessel functions $j_n(x)$ and $h_n^{(2)}(x)$ are defined by

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x)$$

$$h_n^{(2)}(x) = \sqrt{\frac{\pi}{2x}} H_{n+1/2}^{(2)}(x).$$

The solution <u>inside</u> the sphere will be evaluated first ($r < r'$). For this case, the first expansion given holds, and (C-2) may be rewritten

$$h(\bar{r}, t, t') = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} d\omega \, e^{j\omega(t-t')} (-jk) \int_0^{\infty} r'^2 \delta(r'-a) \, dr'$$

$$\times \sum_0^\infty (2n+1) h_n^{(2)}(kr') j_n(kr) \int_0^{2\pi} \int_0^{\pi} \sin^2 \theta' \cos \phi' \, P_n(\cos\gamma) \, d\theta' d\phi', \qquad \text{(C-3)}$$

where the summation and integration over the angular variables have been interchanged. Since

$$P_1^1(\cos\theta) = \sin\theta,$$

the double integral over the angular variables may be rewritten

$$\int_0^{2\pi} \int_0^{\pi} \sin^2 \theta' \cos \phi' \, P_n(\cos\gamma) \, d\theta' d\phi' =$$

$$\int_0^{2\pi} \int_0^{\pi} P_n(\cos\gamma) \, P_1^1(\cos\theta) \cos \phi' \sin \theta' \, d\theta' d\phi'.$$

From the addition theorem for Legendre polynomials,[26]

$$\int_0^{2\pi} \int_0^{\pi} P_n(\cos\gamma) \, P_n^m(\cos\theta') \cos m\phi' \sin\theta' \, d\theta' d\phi' = \frac{4\pi}{2n+1} P_n^m(\cos\theta) \cos m\phi.$$

This procedure reduces the integral over the angular variables to

$$\int_0^{2\pi} \int_0^{\pi} P_n(\cos\gamma)\, P_1^1(\cos\theta)\, \cos\phi'\, \sin\theta'\, d\theta' d\phi'$$

$$= \frac{4\pi}{3} P_1^1(\cos\theta) \cos\phi = \frac{4\pi}{3} \sin\theta \cos\phi, \qquad n = 1$$

$$\int_0^{2\pi} \int_0^{\pi} P_n(\cos\gamma)\, P_1^1(\cos\theta)\, \cos\phi'\, \sin\theta'\, d\theta' d\phi' = 0, \qquad n \neq 1.$$

Hence, the summation in Eq. C-3 reduces to its n = 1 term only, and (C-3) becomes

$$h(\overline{r}, t, t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, (-jk)\, e^{+j\omega(t-t')} \int_0^{\infty} \delta(r'-a)\, j_1(kr) h_1^{(2)}(kr')\, r'^2 \sin\theta \cos\phi\, dr'. \tag{C-4}$$

The integral over the radial variable can immediately be evaluated by the use of the definition of the δ-function,

$$\int_0^{\infty} f(x)\, \delta(x-\xi)\, dx = f(\xi), \qquad 0 < \xi < \infty.$$

Hence,

$$h(r, t, t') = \frac{1}{2\pi} a^2 \sin\theta \cos\phi \int_{-\infty}^{\infty} (-jk) j_1(kr)\, h_1^{(2)}(ka)\, e^{j\omega(t-t')}\, d\omega. \tag{C-5}$$

In order to evaluate the remaining integral, we make use of the definition of the spherical Bessel functions:

$$j_1(kr) = \sqrt{\frac{\pi}{2kr}}\, J_{3/2}(kr)$$

$$h_1^{(2)}(ka) = \sqrt{\frac{\pi}{2ka}}\, H_{3/2}^{(2)}(ka).$$

Furthermore, the Bessel functions of order 3/2 can be expressed in closed form:

$$H_{3/2}^{(2)}(ka) = -\sqrt{\frac{2}{\pi ka}} \left[1 + \frac{1}{jka}\right] e^{-jka}$$

$$J_{3/2}(kr) = \frac{1}{2}\left[H_{3/2}^{(2)}(kr) + H_{3/2}^{(2)*}(kr)\right] = -\frac{1}{2} \sqrt{\frac{2}{\pi kr}} \left[\left(1 + \frac{1}{jkr}\right) e^{-jkr} + \left(1 - \frac{1}{jkr}\right) e^{jkr}\right].$$

Hence,

$$h_1^{(2)}(ka) = -\frac{1}{ka}\left[1 + \frac{1}{jka}\right] e^{-jka}$$

$$j_1(kr) = -\frac{1}{2kr}\left[\left(1 + \frac{1}{jkr}\right) e^{-jkr} + \left(1 - \frac{1}{jkr}\right) e^{jkr}\right]$$

and

$$j_1(kr)\, h_1^{(2)}(ka) = \frac{1}{2k^2 ar}\left\{\left[\left(1+\frac{1}{jka}\right)\left(1-\frac{1}{jkr}\right) e^{-jk(a-r)}\right]+\left[\left(1+\frac{1}{jka}\right)\left(1+\frac{1}{jkr}\right) e^{-jk(a+r)}\right]\right\}.$$

We may now rewrite (C-5) as

$$h(\overline{r}, t, t') = \frac{1}{4\pi r} a \sin\theta \cos\phi\, f(t), \tag{C-6}$$

where

$$f(t) = \int_{-\infty}^{\infty} \frac{c}{jk}\left[\left(1+\frac{1}{jka}\right)\left(1-\frac{1}{jkr}\right) e^{-jk(a-r)}\right] + \left[\left(1+\frac{1}{jka}\right)\left(1+\frac{1}{jkr}\right) e^{-jk(a+r)}\right] e^{jkc(t-t')}\, dk. \tag{C-7}$$

The integrand, f(k), can be rearranged as follows:

$$f(k) = \frac{c}{jar}\left[\frac{+k^2 ar + jk(a-r) + 1}{k^3} e^{jk[c(t-t')-(a-r)]} + \frac{+k^2 ar - jk(a+r) - 1}{k^3} e^{jk[c(t-t')-(a+r)]}\right].$$

The integral can be evaluated by integration in the complex plane and by the method of residues. The integrand has a third-order pole at the origin, so that the path of integration must avoid it. Accordingly, taking into account the fact that the response must

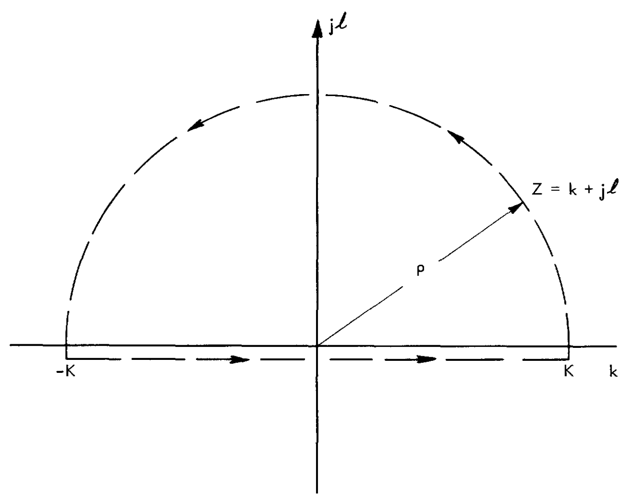

Fig. 15. Contour of integration for expression (C-7).

vanish for all times less than the time of arrival of the wave front, we choose the path shown in Fig. 15. The contour is closed by the semicircle of radius ρ, whose contribution may be easily seen to vanish as $\rho \to \infty$. It encloses the pole at the origin, and yields the correct value if

$$ct > ct' + (a-r) \quad \text{and} \quad ct > ct' + (a+r),$$

respectively, for the first and second terms of the integrand. For the inverse inequalities,

$$ct < ct' + (a-r) \qquad \text{and} \qquad ct < ct' + (a+r),$$

the contour is closed by a semicircle in the lower half-plane, and the integration yields

$$f(t) \equiv 0.$$

For the finite contribution of the pole at the origin, Cauchy's residue theorem may be applied to yield

$$f(t) = 2\pi j[\text{Res } f(z)]_{\ell=0}, \tag{C-8}$$

where z is the complex variable $z = k + j\ell$, and

$$\text{Res } f(z)_{z=0} = \frac{1}{2}\frac{d^2}{dz^2}[z^3 f(z)]_{z=0}.$$

For the first term in the integrand, the residue is found to be

$$\text{Res } f_1(z)_{z=0} = \frac{c}{jar}\left\{-\frac{1}{2}[c(t-t')-(a-r)]^2 - (a-r)[c(t-t')-(a-r)] + ar\right\},$$

whereas for the second term,

$$\text{Res } f_2(z)_{z=0} = \frac{c}{jar}\left\{+\frac{1}{2}[c(t-t')-(a+r)]^2 + (a+r)[c(t-t')-(a+r)] + ar\right\}.$$

Hence, using (C-8), we obtain

$$f(t) = 0, \qquad t < t' + \frac{a-r}{c}$$

$$f(t) = \frac{2\pi c}{ar}\left\{-\frac{1}{2}c^2\left[t - t' - \frac{(a-r)}{c}\right]^2 - (a-r)c\left[t - t' - \frac{(a-r)}{c}\right] + ar\right\},$$

$$t' + \frac{a-r}{c} < t < t' + \frac{a+r}{c}$$

and

$$f(t) = \frac{2\pi c}{ar}\left\{-\frac{1}{2}c^2\left(t - t' - \frac{a-r}{c}\right)^2 - (a-r)c\left(t - t' - \frac{a-r}{c}\right) + \frac{1}{2}c^2\left(t - t' - \frac{a+r}{c}\right)^2\right.$$

$$\left. + (a+r)c\left(t - t' - \frac{a+r}{c}\right) + 2ar\right\} \equiv 0, \qquad t > t' + \frac{a+r}{c}.$$

This function may be expressed in more compact form through the use of step functions. Let

$$t_1 \equiv t' + \frac{a-r}{c}$$

$$t_2 \equiv t' + \frac{a+r}{c}.$$

Then,

$$f(t) = \frac{2\pi c}{ar}\left[-\frac{1}{2}c^2(t-t_1)^2 - (a-r)c(t-t_1) + ar\right] u(t-t_1)\; u(t_2-t). \tag{C-9}$$

The impulse response inside the sphere, through use of (C-6), is finally found to be

$$h_i(\bar{r}, t, t') = \frac{1}{2} \sin\theta \cos\phi \left[-\frac{1}{2}\frac{c^3(t-t_1)^2}{r^2} - (a-r)\frac{c^2(t-t_1)}{r^2} + \frac{ac}{r}\right]$$
$$\times\, u(t-t_1)\; u(t_2-t), \qquad (r < r'). \tag{C-10}$$

The impulse response for $r > r'$ follows along the same lines. When the appropriate series expansion is used, the expression analogous to (C-3) is found to be

$$h_o(r, t, t') = \frac{1}{8\pi^2}\int_{-\infty}^{\infty} d\omega\; e^{j\omega(t-t')}\,(-jk)\int_0^{\infty} r'^2\delta(r'-a)\; dr'$$
$$\times \sum_0^{\infty}(2n+1) j_n(kr')\, h_n^{(2)}(kr)\int_0^{2\pi}\int_0^{\pi}\sin^2\theta'\cos\phi'\; P_n(\cos\gamma)\; d\theta' d\phi'. \tag{C-11}$$

After application of the addition theorem, (C-11) is reduced to

$$h_o(\bar{r}, t, t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\;(-jk)\; e^{j\omega(t-t')}$$
$$\times \int_0^{\infty}\delta(r'-a)\; j_1(kr')\; h_n^{(2)}kr\; r'^2 \sin\theta\cos\phi\; dr'. \tag{C-12}$$

Again, straightforward algebraic manipulation leads to the expression

$$h_o(\bar{r}, t, t') = \frac{1}{4\pi r}\, a \sin\theta\cos\phi\; f_o(t), \tag{C-13}$$

where

$$f_o(t) = \int_{-\infty}^{\infty}\frac{c}{jk}\left[\left(1+\frac{1}{jkr}\right)\left(1-\frac{1}{jka}\right)e^{-jk(r-a)}\right.$$
$$\left. + \left(1+\frac{1}{jkr}\right)\left(1+\frac{1}{jka}\right)e^{-jk(a+r)}\right] e^{jkc(t-t')}\; dk. \tag{C-14}$$

Evaluation of this integral leads to the impulse response

$$h_o(\bar{r}, t, t') = \frac{1}{2}\sin\theta\cos\phi\left[-\frac{1}{2}c^3\frac{(t-t_1)^2}{r^2} - c^2\frac{(r-a)}{r^2}(t-t_1) + \frac{ac}{r}\right], \qquad t_1 < t < t_2$$
$$h_o(\bar{r}, t, t') = 0, \qquad \text{otherwise.} \tag{C-15}$$

Here, $t_1 = t' + \frac{r-a}{c}$ and $t_2 = t' + \frac{r+a}{c}$.

This expression can be rewritten more compactly in terms of the step functions:

$$h_o(\overline{r}, t, t') = \frac{1}{2} \sin\theta \cos\phi \left[-\frac{1}{2} c^3 \frac{(t-t_1)^2}{r^2} - c^2 \frac{(r-a)}{r^2} (t-t_1) + \frac{ac}{r} \right] u(t-t_1)\, u(t_2-t). \tag{C-16}$$

This completes the determination of the impulse response.

APPENDIX D

Solution of the Convolution Integral for Constant Velocity of Expansion

The desired surface current density is the solution of the integral equation (39):

$$\int_{\frac{t-r/c}{1+v/c}}^{\frac{t+r/c}{1+v/c}} dt' \, f(t') \left\{ -\frac{1}{2} c^3 \frac{(t-t_1)^2}{r^2} + c^2 \frac{(t-t_1)}{r} - vt' \left[c^2 \frac{(t-t_1)}{r^2} - \frac{c}{r} \right] \right\} = H_o r$$

$$t_1 = t'(1+v/c) - r/c$$

which we now proceed to determine. Let

$$\alpha \equiv 1 + \frac{v}{c}, \qquad t_1 = \alpha t' - \frac{r}{c}$$

Make the following change of variables. Let

$$\eta \equiv t + \frac{r}{c} - \alpha t' = t - t_1$$

$$d\eta = -\alpha dt'.$$

The impulse response $h_i(\bar{r}, t, t')$ may be rewritten

$$h_i(\bar{r}, t, t') = -\frac{1}{2}\frac{c^3}{r^2}(t-t_1)^2 + \frac{c^2}{r}(t-t_1) + \frac{c^2 v}{\alpha r^2}(-\alpha t')(t-t_1)$$

$$+ \frac{c^2 v}{\alpha r^2}\left(t + \frac{r}{c}\right)(t-t_1) - \frac{c^2 v}{\alpha r^2}\left(t + \frac{r}{c}\right)(t-t_1) - \frac{cv}{\alpha r}(-\alpha t') - \frac{cv}{\alpha r}\left(t + \frac{r}{c}\right) + \frac{cv}{\alpha r}\left(t + \frac{r}{c}\right)$$

or

$$h_i(\bar{r}, t, t') = -\frac{1}{2}\frac{c^3}{r^2}(t-t_1)^2 + \frac{c^2}{r}(t-t_1) + \frac{c^2 v}{\alpha r^2}(t-t_1)^2$$

$$- \frac{c^2 v}{\alpha r^2}\left(t + \frac{r}{c}\right)(t-t_1) - \frac{cv}{\alpha r}(t-t_1) + \frac{cv}{\alpha r}\left(t + \frac{r}{c}\right).$$

Letting $t - t_1 \equiv \eta$,

$$h_i(\bar{r}, t, \eta) = -\left(\frac{1}{2}\frac{c^3}{r^2} - \frac{c^2 v}{\alpha r^2}\right)\eta^2 - \left[\frac{c^2 v}{\alpha r^2}\left(t + \frac{r}{c}\right) - \frac{c^2}{r} + \frac{cv}{\alpha r}\right]\eta + \frac{cv}{\alpha r}\left(t + \frac{r}{c}\right).$$

The limits of integration become

$$t' = \frac{t - (r/c)}{1 + \frac{v}{c}} = \frac{t - (r/c)}{\alpha}, \qquad \eta = t - \alpha t' + \frac{r}{c} = \frac{2r}{c}$$

$$t' = \frac{t + (r/c)}{1 + v/c} = \frac{t + (r/c)}{a}, \qquad \eta = t - at' + \frac{r}{c} = 0.$$

In terms of the new variable, the integral equation becomes

$$+ \int_0^{\frac{2r}{c}} \frac{d\eta}{a} f(\eta) \left[\frac{c^3}{2r^2} - \frac{c^2 v}{ar^2} \right] \eta^2 + \left[\frac{c^2 v}{ar^2} \left(t + \frac{r}{c}\right) - \frac{c^2}{r} + \frac{cv}{ar} \right] \eta - \frac{cv}{ar} \left(t + \frac{r}{c}\right) = -H_o r.$$

It is now easy to show that $f(\eta) = K_o$, a constant, satisfies this equation. Straightforward integration yields

$$\frac{K_o}{a} \left\{ \left(\frac{c^3}{2r^2} - \frac{c^2 v}{ar^2} \right) \frac{8r^3}{3c^3} + \left[\frac{c^2 v}{ar^2} \left(t + \frac{r}{c}\right) - \frac{c^2}{r} + \frac{cv}{ar} \right] \frac{2r^2}{c^2} - \frac{cv}{ar} \left(t + \frac{r}{c}\right) \frac{2r}{c} \right\} = -H_o r$$

$$\frac{K_o}{a} \left\{ \left[\frac{4}{3} r - \frac{8vr}{3ac} \right] + \left[\frac{2v}{a} \left(t + \frac{r}{c}\right) + \frac{2vr}{ac} - 2r - \frac{2v}{a} \left(t + \frac{r}{c}\right) \right] \right\} = -H_o r$$

$$\frac{K_o}{a} \left[\frac{4}{3} r \left(1 - \frac{2v}{ac}\right) - 2r \left(1 - \frac{v}{ac}\right) \right] = -H_o r.$$

But

$$1 - \frac{2v}{ac} = 1 - \frac{2v}{c(1+v/c)} = \frac{c - v}{c + v} = \frac{1 - v/c}{1 + v/c}$$

$$1 - \frac{v}{ac} = \frac{c + v - v}{c + v} = \frac{c}{c + v} = \frac{1}{1 + v/c}$$

and

$$\frac{K_o}{(1+v/c)^2} \left[\frac{4}{3} r \left(1 - \frac{v}{c}\right) - 2r \right] = -H_o r$$

$$K_o \left(-\frac{2}{3} r - \frac{4}{3} r \frac{v}{c} \right) = -H_o (1+v/c)^2 \, r \; - \frac{2}{3} K_o \left(1 + \frac{2v}{c}\right) = -H_o (1+v/c)^2.$$

Finally,

$$K_o = \frac{3}{2} H_o \frac{(1+v/c)^2}{(1+2v/c)}.$$

This completes the solution.

APPENDIX E

Determination of the Fields outside the Sphere for Constant Velocity of Expansion

The vector potential outside the sphere may be found by integration of Eq. 43.

$$A_\phi = \frac{\mu_o K_o}{8\pi^2 \cos\phi} \int_{-\infty}^{\infty} e^{jka}\, dk$$

$$\times \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} e^{-jkr'} \frac{\exp\left(-jk\frac{v}{c}|r-r'|\right)}{|\bar{r}-\bar{r}'|} r'^2 \sin^2\theta' \cos\phi'\, dr' d\theta' d\phi'.$$

If we let $a \equiv \frac{v}{c}$, and use the expansion

$$\frac{e^{-jka|r-r'|}}{|r-r'|} = -jka \sum_{n=0}^{\infty} (2n+1) j_n(kar') h_n^{(2)}(kar) P_n(\cos\gamma),$$

Eq. 43 becomes

$$A_\phi = \frac{\mu_o K_o}{8\pi^2 \cos\phi} \int_{-\infty}^{\infty} (-jka)\, e^{jka}\, dk \int_0^{\infty} e^{-jkr'} r'^2\, dr' \sum_{n=0}^{\infty} (2n+1) h_n^{(2)}(kar) j_n(kar')$$

$$\times \int_0^{\pi} \int_0^{2\pi} P_n(\cos\gamma) \sin^2\theta' \cos\phi'\, d\theta' d\phi'. \qquad \text{(E-1)}$$

The integration over the angular coordinates may be evaluated through the properties of the Legendre polynomials. Thus,

$$\int_0^{\pi} \int_0^{2\pi} P_n(\cos\gamma) P_1^1(\cos\theta') \sin\theta' \cos\phi'\, d\theta' d\phi' = \frac{4\pi}{3} \sin\theta \cos\phi \qquad n = 1$$

$$= 0 \qquad n \neq 1$$

Hence, (E-1) becomes

$$A_\phi = \frac{\mu_o K_o \sin\theta}{2\pi} \int_{-\infty}^{\infty} (-jka)\, e^{jka}\, dk \int_0^{\infty} e^{-jkr'} r'^2 h_1^{(2)}(kar) j_1(kar')\, dr' \qquad \text{(E-2)}$$

The spherical Bessel functions may be expressed in closed form:

$$h_1^{(2)}(kar) = -\frac{1}{kar}\left[1 + \frac{1}{jkar}\right] e^{-jkar}$$

$$j_1(kar') = -\frac{1}{2kar'}\left[\left(1 + \frac{1}{jkar'}\right) e^{-jkar'} + \left(1 - \frac{1}{jkar'}\right) e^{jkar'}\right]$$

The integration over r' becomes

$$\int_0^\infty e^{-jkr'}\, r'^2 (k\alpha r')\, dr' = -\frac{1}{2}\int_0^\infty \frac{r'}{k\alpha}\left[\left(1+\frac{1}{jk\alpha r'}\right)e^{-jk\alpha r'} + \left(1-\frac{1}{jk\alpha r'}\right)e^{-jkr'}\right] e^{-jkr'}\, dr'$$

$$= -\frac{1}{2}\int_0^\infty \left\{ \frac{r'}{k\alpha}\left[e^{-jk(1+\alpha)r'} + e^{-jk(1-\alpha)r'}\right] + \frac{1}{j(k\alpha)^2} \times \left[e^{-jk(1+\alpha)r'} - e^{-jk(1-\alpha)r'}\right]\right\} dr'$$

which yields

$$\int_0^\infty e^{-jkr'}\, r'^2 j_1(k\alpha r')\, dr' = \frac{1}{2k^3}\left[\frac{1}{\alpha(1+\alpha)^2} + \frac{1}{\alpha(1-\alpha)^2} + \frac{1}{\alpha^2(1+\alpha)} - \frac{1}{\alpha^2(1-\alpha)}\right] = \frac{2\alpha}{k^3(1-\alpha^2)^2}.$$

We may now rewrite (E-2) as

$$A_\phi = \frac{\mu_o K_o \sin\theta}{2\pi r}\int_{-\infty}^{\infty} j\left[1+\frac{1}{jk\alpha r}\right]\frac{2\alpha}{k^3(1-\alpha^2)^2}\, e^{jk[a-\alpha r]}\, dk, \tag{E-3}$$

or

$$A_\phi = \frac{2\mu_o K_o \sin\theta}{r}\ \frac{\alpha}{(1-\alpha^2)^2}\, f(t)$$

in which

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(1+\frac{1}{jk\alpha r}\right)\frac{1}{(jk)^3}\, e^{jk(a-\alpha r)}\, dk.$$

For the evaluation of f(t) we let $s = jk$, carry out the contour integral

$$f(t) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(\frac{1}{s^3} + \frac{1}{\alpha r s^4}\right) e^{s(a-\alpha r)}\, ds,$$

and obtain, for $a > \alpha r$ or $t > \frac{r}{c}$,

$$f(a-\alpha r) = \frac{(a-\alpha r)^2}{2} + \frac{(a-\alpha r)^3}{6\alpha r},$$

or, for $a = vt$,

$$f(t-r/c) = \frac{v^2(t-r/c)^2}{2} + \frac{v^3(t-r/c)^3}{6\alpha r}.$$

Hence,

$$A_\phi(r,t) = \frac{\mu_o K_o v^3}{(1-v^2/c^2)^2}\left[\frac{(t-r/c)^2}{rc} + \frac{(t-r/c)^3}{3r^2}\right]\sin\theta, \qquad t > r/c.$$

Letting $t - r/c \equiv \tau$, and

$$K_o = \frac{3H_o(1+v/c)^2}{2(1+2v/c)},$$

we find

$$A_\phi(\overline{r}, t) = \frac{3\mu_o H_o v^3}{(1-v/c)^2(1+2v/c)} \left[\frac{\tau^2}{2rc} + \frac{\tau^3}{6r^2}\right] \sin\theta, \qquad \tau > 0.$$

For $\tau < 0$, $t < r/c$, and $A_\phi = 0$, the fields are found to be

$$\mu_o H_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)$$

$$\mu_o H_\theta = -\frac{3\mu_o H_o v^3 \sin\theta}{(1-v/c)^2(1+2v/c)\, r} \frac{\partial}{\partial r}\left[\frac{\tau^2}{2c} + \frac{\tau^3}{6r}\right]$$

$$H_\theta = \frac{3H_o v^3}{(1-v/c)^2(1+2v/c)} \left[\frac{\tau}{rc^2} + \frac{\tau^2}{2r^2c} + \frac{\tau^3}{6r^3}\right] \sin\theta$$

$$\mu_o H_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A\phi \sin\theta)$$

$$\mu_o H_r = \frac{3\mu_o H_o v^3}{(1-v/c)^2(1+2v/c)} \frac{1}{r\sin\theta} \left[\frac{\tau^2}{2rc} + \frac{\tau^3}{6r^2}\right] \frac{\partial}{\partial\theta}(\sin^2\theta)$$

$$H_r = \frac{3H_o v^3}{(1-v/c)^2(1+2v/c)} \left[\frac{\tau^2}{r^2c} + \frac{\tau^3}{3r^3}\right] \cos\theta$$

$$E_\phi = -\frac{\partial A_\phi}{\partial t} = -\frac{3\mu_o H_o v^3}{(1-v/c)^2(1+2v/c)} \left[\frac{\tau}{rc} + \frac{\tau^2}{2r^2}\right] \sin\theta$$

$$E_\phi = -\frac{3H_o v^3}{(1-v/c)^2(1+2v/c)} \sqrt{\frac{\mu_o}{\epsilon_o}} \left[\frac{\tau}{rc^2} + \frac{\tau^2}{2r^2c}\right] \sin\theta.$$

We summarize the results as follows:

$$A_\phi = \frac{3\mu_o H_o v^3}{(1-v/c)^2(1+2v/c)} \left[\frac{\tau^2}{2rc} + \frac{\tau^3}{6r^2}\right] \sin\theta$$

$$H_\theta = \frac{3H_o v^3}{(1-v/c)^2(1+2v/c)} \left[\frac{\tau}{rc^2} + \frac{\tau^2}{2r^2c} + \frac{\tau^3}{6r^3}\right] \sin\theta$$

$$H_r = \frac{3H_o v^3}{(1-v/c)^2(1+2v/c)} \left[\frac{\tau^2}{r^2c} + \frac{\tau^3}{3r^3}\right] \cos\theta$$

$$E_\phi = -\frac{3H_o v^3}{(1-v/c)^2(1+2v/c)} \sqrt{\frac{\mu_o}{\epsilon_o}} \left[\frac{\tau}{rc^2} + \frac{\tau^2}{2r^2c}\right] \sin\theta.$$

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25. R. M. Fano, L. J. Chu, and R. B. Adler, op. cit., p. 86, Par. 3. 3.
26. J. A. Stratton, op. cit., p. 407.