

SIGNAL RECONSTRUCTION FROM PHASE OR MAGNITUDE

by

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ABSTRACT

This thesis addresses two issues related to the problem of reconstructing a one-dimensional or a multidimensional sequence from either the phase or the magnitude of its Fourier transform. The first concerns the development of conditions under which a sequence is uniquely defined in terms of only phase or magnitude information. For example, it is shown that a one-dimensional sequence is, in most cases, uniquely specified by the phase of its Fourier transform if the sequence is finite in length. In the case of magnitude, a condition for uniqueness is presented which is a generalization of the minimum and maximum phase constraints and includes them as a special case. For multidimensional sequences, on the other hand, it is shown that a finite support constraint is sufficient, in most cases, for the sequence to be uniquely defined by either the phase or magnitude of its Fourier transform.

The second issue which is addressed concerns the development of algorithms for reconstructing a sequence from either the phase or magnitude of its Fourier transform. In particular, several algorithms are presented for reconstructing a sequence from its phase. These algorithms, which include iterative as well as non-iterative approaches, always lead to the correct solution provided the appropriate uniqueness constraints are fulfilled. An iterative procedure is also described for reconstructing a multidimensional sequence from the magnitude of its Fourier transform. However, the convergence of this algorithm to the desired sequence appears to depend upon the availability of an initial estimate which is sufficiently close to the correct solution. Finally, a number of examples are presented which illustrate the use of these algorithms.

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*To my lovely wife Sandy and my precious son Michael for the sacrifices
they endured in the many hours spent away from them.
Without their loving patience and continuous support,
this thesis would not have been possible.*

and

*To my parents for their ever present love, encouragement
and enthusiastic support.*

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TABLE OF SYMBOLS

| | | |
|-----------------------------------|--|----------|
| $B(-)$ | Frequency domain mapping in the phase-only iteration | 111 |
| $d(\underline{x}, \underline{y})$ | The standard Euclidean metric on R^n | 111, 142 |
| E_p | Normalized mean square error after p iterations | 106 |
| $F(n)$ | The set of all m -Dimensional sequences with finite support | 56 |
| $\mathcal{F}(z)$ | Set of all polynomials in $\underline{z}=(z_1, \dots, z_m)$ over the field \mathcal{F} | 66 |
| $\text{Im}(x)$ | The imaginary part of the complex number x | 42 |
| $L(\Omega)$ | Lattice of points in the m -dimensional Fourier plane | 69 |
| $L(\underline{A})$ | A lattice of points over the sets $\underline{A} = \{A_1, \dots, A_m\}$ | 68 |
| $P(n, m)$ | Set of all polynomials of degree n in m variables | 136 |
| $Q(\underline{n}, m)$ | Set of all polynomials in m variables of degree $\underline{n}=(n_1, \dots, n_m)$ in $\underline{z}=(z_1, \dots, z_m)$ | 140 |
| $Q_x(z)$ | The polynomial in \underline{z}^{-1} which is equal to $\underline{z}^{-N}R_x(z)$ | 81 |
| $r_x(n)$ | Autocorrelation of $x(n)$ | 50, 81 |
| $R_x(z)$ | z -transform of $r_x(n)$ | 50, 81 |
| R^n | n -dimensional Euclidean space | 110 |
| $R(N)$ | The set of all points in the region defined by $0 \leq n_x < N_x$ | 56 |
| $S_r(x)$ | The sphere of radius r about \underline{x} | 144 |
| $T(-)$ | Time-domain mapping in phase-only iteration | 110 |
| $W(-)$ | DFT mapping | 111 |
| $W^{-1}(-)$ | Inverse DFT mapping | 111 |
| $X(\underline{k})_M$ | The M -point Discrete Fourier Transform of the sequence $x(n)$ | 33, 56 |
| $X(\omega)$ | The Fourier transform of the sequence $x(n)$ | 32, 56 |

| | | |
|------------------|---|--------|
| $X(z)$ | The z-transform of the sequence $x(n)$ | 32, 55 |
| $\tilde{X}(z)$ | The z-transform of the "time-reversed" sequence $\tilde{x}(n)=x(N-n)$ | 72 |
| $ X(\omega) $ | The magnitude of the Fourier transform $X(\omega)$ | 33, 56 |
| $ X(k) _M$ | The magnitude of the M -point Discrete Fourier Transform of $x(n)$ | 56 |
| $x_1(n)$ | Convolutional inverse of the sequence $x(n)$ | 47, 90 |
| $x_i(n)$ | The imaginary part of $x(n)$ | 33, 46 |
| $x_r(n)$ | The real part of $x(n)$ | 33, 46 |
| $\alpha(n,m)$ | The number of coefficients in a polynomial $p_n(x)$ in $P(n,m)$ | 136 |
| $\beta(n,m)$ | The number of coefficients in a polynomial $q_n(x)$ in $Q(n,m)$ | 140 |
| $\delta(n)$ | The unit sample function | |
| $\phi_x(\omega)$ | The phase of the Fourier transform of $x(n)$ | 33, 56 |
| $\phi_x(k)_M$ | The phase of the M -point Discrete Fourier Transform of $x(n)$ | 56 |
| $\Phi(-)$ | Mapping which replaces the DFT phase of a sequence with $\phi_x(k)_M$ | 110 |
| \sim | An equivalence relation of the set of multidimensional sequences | 82 |
| \times | Cartesian cross product | 68 |
| \in | "is an element of" | |
| \subset | "is a subset of" | |

CHAPTER I: INTRODUCTION

I.1: Introduction

The Fourier transform of a signal is a complex-valued function of a continuous variable which, when written in polar form, is expressed in terms of its magnitude and phase. Without any additional information about the signal, the magnitude and phase are independent functions in the sense that knowledge of one is not sufficient to deduce the other. In other words, both magnitude and phase information are generally required in order to uniquely define a signal. Therefore, each of these components contains a certain piece of "information" about the signal. Although it is not easy to characterize the type of information which is encoded in the phase, it has been observed in many applications that phase is an important component in the representation of signals. For example, it has been shown that for those signals which typically arise in such applications as x-ray crystallography, speech processing, image processing, and optics, many of the important features and characteristics are preserved if the signal is reconstructed on the basis of only phase information [34]. Such a phase-only synthesis may, for example, be accomplished by combining the correct phase with a unity magnitude function or with a magnitude which is in some way representative of the class of signals of interest. If, on the other hand, the reconstruction of a signal is based only on Fourier transform magnitude information, the result will not, in general, contain any of the important features of the original signal.

Although Fourier transform phase or magnitude information alone is not, in general, sufficient to uniquely specify a signal, the ability to reconstruct a signal from only phase or magnitude information would be useful in a number of important applications. For example, in many problems which arise in x-ray crystallography [42],

electron microscopy [44], coherence theory [31], and optics [28], only the magnitude of the Fourier transform of an electromagnetic wave may be recorded or is available for measurement. Therefore, the specification of the electromagnetic wave depends upon the retrieval of the Fourier transform phase of the wave from only spectral magnitude information. In other applications, either the spectral magnitude or phase of a signal may be severely distorted so that the restoration of the signal must rely only on the undistorted component. For example, in the class of problems referred to as blind deconvolution, a desired signal is to be recovered from an observation which is the convolution of the desired signal with some unknown signal [46]. Since little is usually known about either the desired signal or the distorting signal, deconvolution of the two signals is generally a very difficult problem. However, in the special case in which the distorting signal is known to have a phase which is identically zero, the phase of the signal is undistorted. Such a situation occurs, at least approximately, in long-term exposure to atmospheric turbulence or when images are blurred by severely defocused lenses with circular aperture stops [1]. In this case, except for phase reversals, the phase of the observed signal is approximately the same as the phase of the desired signal and, therefore, it is of interest to consider signal reconstruction from phase information alone.

1.2: Scope of thesis

Due to its theoretical as well as practical importance, this thesis considers the problem of reconstructing a discrete-time signal, or sequence, from only the phase or magnitude of its Fourier transform. Specifically, there are two issues related to phase-only and magnitude-only reconstruction which are addressed. The first concerns the issue of uniqueness. In particular, although a sequence may not generally be

uniquely defined in terms of only its Fourier transform phase or magnitude, there are certain classes of sequences for which such a unique specification is possible. It is well known, for example, that the log magnitude and phase of the Fourier transform of a minimum phase sequence are related by a Hilbert transform [36]. For minimum phase sequences, therefore, magnitude information alone is sufficient to uniquely recover the phase and, hence the sequence, whereas phase information alone is sufficient to recover the magnitude to within a scale factor which, in turn, specifies the sequence to within a scale factor. However, since the minimum phase requirement is fairly restrictive, many of the signals which arise in practice do not satisfy this constraint. Therefore, the first issue which is to be addressed concerns the development of some alternative conditions under which a sequence is uniquely defined by either the phase or magnitude of its Fourier transform. Although it is possible to derive many different sets of conditions for which such a unique specification is possible, it will be of interest to find conditions which are satisfied by many of the signals which are found in practical applications.

The second issue which is addressed concerns the numerical reconstruction of a sequence from only Fourier transform phase or magnitude information. In particular, for those sequences which are determined to be uniquely defined in terms of the phase or magnitude of their Fourier transform, the development of practical algorithms for phase-only or magnitude-only reconstruction are investigated.

1.3: Outline of thesis

The organization of this thesis is as follows. In Chapter II is a review of some of the results which have appeared in the literature and are related to the problem of reconstructing a signal from either the phase or magnitude of its Fourier transform.

Due to its importance in applications such as x-ray crystallography, spatial and temporal coherence theory, electron microscopy, and optics, the recovery of phase from only magnitude information has received considerable attention in the literature. The results which are briefly reviewed include a discussion of the question of uniqueness in the phase retrieval problem as well as a brief survey of some of the algorithms which have been proposed for recovering phase from magnitude. Although there has not been a similar treatment of the problem of recovering a signal from only phase information, it has been noted in a number of different contexts and applications that phase is an important component in the representation of a signal. Therefore, a survey of some of the literature which address the importance of phase in signals is also presented.

In Chapter III, the uniqueness of a one-dimensional discrete-time signal in terms of the phase or magnitude of its Fourier transform is considered. In particular, it is shown that any finite length sequence which contains no zero phase components is uniquely defined to within a scale factor by the phase of its Fourier transform. Furthermore, it is shown that for a sequence of length N , the phase need only be known over a set of $N-1$ distinct frequencies in the open interval $(0, \pi)$. Unfortunately, however, a finite length constraint does not lead to a similar result for the uniqueness of a sequence in terms of only the magnitude of its Fourier transform. Nevertheless, a condition for uniqueness in terms of magnitude is presented which is an extension of the minimum and maximum phase constraint and which includes these constraints as a special case.

In Chapter IV, the uniqueness constraints presented in Chapter III are extended to the case of multidimensional sequences. This extension is first performed by mapping a multidimensional sequence into a one-dimensional sequence by means of an invertible transformation. The one-dimensional uniqueness constraints in Chapter III are

then applied to this one-dimensional sequence. Although this approach is straightforward, it has the limitation that the uniqueness constraints are expressed in terms of a one-dimensional function of a multidimensional sequence rather than directly in terms of the multidimensional sequence. Therefore, a different approach is then presented which relies on some basic results in the algebra of multivariable polynomials. The uniqueness constraints which follow consist of restrictions on the types of factors which are allowed in the multidimensional z-transform of a multidimensional sequence. In particular, as is the case for a one-dimensional sequence, it is shown that a finite extent constraint is sufficient, in most cases, to insure the unique specification of a multidimensional sequence in terms of the phase of its Fourier transform. Furthermore, unlike the case for one-dimensional sequences, it is shown that for most multidimensional sequences, a finite extent constraint is also sufficient for a unique solution in terms of Fourier transform magnitude. Finally, it is shown that the phase or magnitude only needs to be known over a finite set of frequency values when these samples are obtained from a Discrete Fourier Transform (DFT) of the appropriate size.

Chapters III and IV present a number of conditions which are sufficient for a sequence to be uniquely specified in terms of its Fourier transform phase or magnitude. Chapter V addresses the problem of numerically reconstructing a sequence from only Fourier transform phase or magnitude information. In particular, for the problem of reconstructing a sequence from only phase information, several practical algorithms are described which always lead to the correct result provided the appropriate uniqueness conditions are fulfilled. The algorithms presented include iterative as well as non-iterative procedures, and a number of examples of reconstruction from phase are presented to illustrate these algorithms. In reconstructing a sequence from the magni-

tude of its Fourier transform, however, it appears that there is, as yet, no practical algorithm which will always yield the correct solution even when it is known that the solution is unique. Nevertheless an iterative algorithm is described which appears to yield the correct solution when an initial estimate of the unknown sequence may be found which is sufficiently close to the correct sequence.

Finally, Chapter VI provides a brief summary of the results presented in the previous chapters. In addition, some unanswered questions and some areas for future research are described.

Although this thesis addresses only the problem of reconstructing a discrete-time signal from either the phase or magnitude of its Fourier transform, it should be pointed out that all of the results which are presented may be easily applied to the dual problem of reconstructing a periodic (complex-valued) continuous-time signal from its phase or magnitude. Specifically, since a periodic continuous-time signal may be written in terms of a Fourier series, the coefficients in this series may be considered to be a discrete-time signal or sequence. Consequently, if the Fourier series coefficients satisfy the appropriate set of constraints, it may be possible to recover these coefficients from either the phase or magnitude of the continuous-time signal. From the reconstructed coefficients, the continuous-time signal may then be recovered.

CHAPTER II: BACKGROUND

II.1: Introduction

When the Fourier transform of a signal is written in polar form, it is expressed in terms of its magnitude and phase. Although the Fourier transform magnitude and phase are generally independent functions, with some additional information about the signal it may be possible to derive one from the other. It is well known, for example, that if a signal satisfies the minimum phase constraint, then the log magnitude and phase of its Fourier transform are related by a Hilbert transform [36]. Therefore, phase may be uniquely recovered from magnitude information and magnitude may be uniquely recovered to within a scale factor from phase information. Signals which arise in practice, however, may not generally satisfy the minimum phase constraint. Nevertheless, in many applications it is desirable to be able to reconstruct a signal from only phase or magnitude information.

II.2: The phase retrieval problem

The recovery of phase from only magnitude information, generally referred to as the *phase retrieval problem*, arises in a variety of different contexts and applications. In x-ray crystallography, for example, the molecular structure of crystals is to be inferred from the observed diffraction pattern of x-rays [42]. Although the diffraction pattern is related to the scattering density of the crystal by a Fourier transformation, only the intensity (squared magnitude) of the diffraction pattern may be measured. Since knowledge of the phase of the diffracted wave is indispensable for the determination of the crystal structure, the phase retrieval problem is particularly important. A

similar problem arises in optical and electron microscopy when, for example, the index of refraction of a thin object or the height distribution of a surface is to be determined from the intensity of the wave distribution in the image plane or some other plane in the microscope. Again, in order to determine the object structure, phase information of the wave is required. In some applications, however, it is only the wave intensity in the image plane of an optical system which is of interest. Nevertheless, the retrieval of phase information may still be an important problem. Imaging through a turbulent atmosphere, for example, may reduce the resolution of objects well below the diffraction limits of the telescope. With interferometric techniques, however, it is possible to obtain diffraction limited information about the Fourier transform magnitude of the object [13]. Therefore, if it were possible to recover the phase of the Fourier transform of an object from only its magnitude, it may then be possible to obtain a diffraction limited image through a turbulent atmosphere.

Due to the importance of the phase retrieval problem in a number of practical applications, a considerable effort has been devoted to the development of conditions under which the phase of a complex function may be uniquely recovered from its magnitude. For one-dimensional functions, the uniqueness of the phase retrieval problem is now fairly well understood. In order to briefly review some of these results, let $f(x)$ be a complex-valued function of the continuous variable x , and let $F(\omega)$ be its Fourier transform. In an optical imaging system, the functions $f(x)$ and $F(\omega)$ may, for example, represent the wave distributions in the image plane and Fraunhofer plane (exit pupil), respectively. On the other hand, $f(x)$ may represent the electron density distribution in a crystal with $F(\omega)$ the x-ray diffraction pattern. Since both $f(x)$ and $F(\omega)$ are, in general, complex-valued functions, they may be written in terms of their magnitude and phase as

$$f(x) = |f(x)| \exp[j\phi_f(x)] \quad (2-1 a)$$

$$F(\omega) = |F(\omega)| \exp[j\phi_F(\omega)] \quad (2-1 b)$$

Without any additional information or constraints, it should be clear that $F(\omega)$ may not be recovered from only its magnitude, $|F(\omega)|$, or from any function of $|F(\omega)|$ since it is always possible to combine the magnitude of $F(\omega)$ with the phase of some other function, $G(\omega)$, to obtain another function with the same magnitude. In many situations of interest, however, physical constraints restrict the set of admissible solutions. In optical imaging systems, for example, a finite aperture in the exit pupil imposes a band-limited constraint on the field in the image plane, $f(x)$. Similarly, crystals which are analyzed by x-ray diffraction methods are of finite size and, therefore, the diffracted field must be the Fourier transform of a space-limited function, $f(x)$. It is natural to ask, therefore, whether $|F(\omega)|$ uniquely defines the phase of $F(\omega)$ under the constraint that $f(x)$ is zero outside an interval $[a,b]$ or, equivalently, whether $|f(x)|$ uniquely defines the phase of $f(x)$ under the constraint that $F(\omega)$ is band-limited to an interval $[\omega_1, \omega_2]$. These problems were addressed independently by Hofstetter [20] and Walther [49,32] who were able to prove that space-limitation (or band-limitation) is not, in general, sufficient to insure a unique solution to the phase retrieval problem. The reason for this lack of uniqueness follows from the possibility of "zero-flipping". In particular, suppose that $f(x)$ is zero outside the interval $[a,b]$. In this case, the Laplace transform of $f(x)$, $F(s)$, is an entire function and is uniquely defined to within a factor by the distribution of its zeros in the s -plane. Therefore, with s_k a zero of $F(s)$, define $H_k(s)$ as

$$H_k(s) = \frac{s + s_k^*}{s - s_k} \quad (2-2)$$

and consider the product $G_k(s) = F(s)H_k(s)$. Since $|H_k(\omega)| = 1$ for all ω , $H_k(s)$ is the transfer function of an all-pass filter [in many applications $H_k(s)$ is referred to as a Blaschke factor]. Therefore, the output of this filter, $g_k(x)$, has a Fourier transform with the same magnitude as $F(\omega)$. In addition, it is straightforward to show that if $f(x)$ is zero outside the interval $[a,b]$, then $g_k(x)$ is also zero outside $[a,b]$. Since the effect of the filter $H_k(s)$ on $F(s)$ is to flip the zero at s_k about the $j\omega$ -axis, it follows that the flipping of a zero about the $j\omega$ -axis preserves the magnitude of the Fourier transform of the signal as well as the duration of the signal. It may also be shown that if an arbitrary (i.e. possibly infinite) number of zeros are flipped about the $j\omega$ -axis, the result will be another signal which is zero outside $[a,b]$ with a Fourier transform with the same magnitude [20]. Therefore, zero flipping allows for a possibly infinite number of different signals to be found which have the same duration and Fourier transform magnitude as $f(x)$. However, if all of the zeros of $F(s)$ are imaginary, then there will be a unique solution to the phase retrieval problem. For example, since $F(s) = 2\sinh(s)/s$ has only imaginary zeros, the unit pulse is the only signal which is zero outside the interval $[-1,1]$ and has a Fourier transform with magnitude $[2\sin\omega/\omega]^2$.

Although these uniqueness results have been stated in terms of signals which are functions of a continuous variable, they are easily extended to the case of discrete time signals. For example, let $f(n)$ be a discrete time signal which is zero outside the interval $[N_A, N_B]$. In this case, the z -transform of $f(n)$, $F(z)$, is analytic for all finite and non-zero values of z and is uniquely defined to within a factor by its $(N_A - N_B)$ zeros in the region $0 < |z| < \infty$. With z_k a zero of $F(z)$ and $H_k(z)$ defined as

$$H_k(z) = \frac{z^{-1} - z_k^*}{z - z_k} \quad (2-3)$$

it is straightforward to show that the product $G_k(z)=F(z)H_k(z)$ represents the z-transform of a signal which is zero outside the interval $[N_A, N_B]$ and has a Fourier transform with the same magnitude as $f(n)$. Since the effect of $H_k(z)$ on $F(z)$ is to flip the zero at z_k about the unit circle, it follows that if an arbitrary number of zeros of $F(z)$ are flipped about the unit circle, the result will be a signal of the same duration with a Fourier transform with the same magnitude as $f(n)$.

The ambiguity which appears as the result of zero-flipping has led to a search for solutions to the phase retrieval problem which are based on the availability of additional information. In electron microscopy, for example, the field in the object plane of the microscope is of finite extent. Therefore, the field in the exit pupil is known to be an entire function. In addition, due to the finite size of the aperture in the exit pupil, the field in the image plane is also an entire function. In this case, Hoenders [19] has shown that these constraints are sufficient to reduce the phase ambiguity to a single field, $f(x)$, or its "twin", $f^*(-x)$. In a similar vein, it has been shown that the presence of any interval $[c,d]$ over which the field in the object plane is known to be zero is sufficient to insure a unique solution [16].

The uniqueness question has also been considered when the field intensity in two planes is known. This may be the case, for example, in electron microscopy when the field may be measured both in the image plane as well as in the exit pupil plane, thus providing information about $|F(\omega)|$ as well as $|f(x)|$. In this case, it was demonstrated by Huiser et. al. that the solution to the phase retrieval problem is unique to within a constant phase factor assuming the analyticity of the functions involved [21]. As another example, in an optical imaging system it may be possible to measure the field intensity in two slightly defocused planes. For this case it has also been shown that there is a unique solution to the phase retrieval problem [9,19].

Solutions to the phase retrieval problem have also been considered for those cases in which a known reference signal is added to the unknown signal prior to the observation or measurement of the wave intensities [3]. Such a procedure is used in holography, for example, to encode phase information into the intensity of the wavefront which is then recorded on a photographic plate. Knowledge of the reference signal may then allow for the phase information to be retrieved and the original complex wavefront to be reconstructed. Related procedures are used in x-ray crystallography by incorporating, for example, heavy atoms into the unknown crystal structure [42]. With the appropriate reference signal (such as a point source of sufficient magnitude) it may also be possible to insure that the signal intensity which is observed corresponds to a minimum phase signal. In this case, the Hilbert transform may then be used to retrieve the unknown phase.

Most of the results which have been reported concerning the uniqueness of the solution to the phase retrieval problem have considered only one-dimensional functions. However, in many of the applications in which the phase retrieval problem is important, the signals of interest are functions of two or more variables. The difficulty encountered in considering the uniqueness of the solution to the phase retrieval problem for functions in more than one variable lies in the fact that entire functions in two or more variables may not generally be characterized by a countable collection of zeros. Nevertheless, some results have recently appeared which address the ambiguity of the solution to the phase retrieval problem for two dimensional fields. Huiser and van Toorn, for example, address the two-dimensional version of the problem considered by Hofstetter and Walther and show that a phase ambiguity is not ruled out for a two-dimensional field, $f(x,y)$, which is known to be zero outside some closed convex set S in R^2 [22]. In particular, if $f(x,y)$ is zero outside S , and if $f(x,y)$ can be written

as a product of the form $f(x,y)=p(x,y)g(x,y)$ where $p(x,y)$ is a polynomial and $g(x,y)$ is an entire function, then $\tilde{f}(x,y)=p^*(x^*,y^*)g(x,y)$ is also zero outside S and has a Fourier transform with the same magnitude as $f(x,y)$.

A slightly different approach was taken by Bruck and Sodin who considered the case of a real-valued discrete two-dimensional field, $f(n_1,n_2)$ which is finite in extent [5]. Since the z -transform of this two-dimensional field is a polynomial in two variables, $F(z_1,z_2)$, the authors assert that if this polynomial is irreducible, i.e. if it cannot be factored, then the ambiguity of the phase retrieval problem is reduced to only one of two fields, the field $f(n_1,n_2)$ or the field which is obtained by rotating $f(n_1,n_2)$ by 180 degrees.

In addition to a discussion of the uniqueness of the solution to the phase retrieval problem, a number of algorithms have been proposed for recovering phase from magnitude information. Perhaps the most familiar algorithm is the iterative approach proposed by Gerchberg and Saxton for recovering phase information from intensity measurements in both the image plane and diffraction plane (exit pupil) of an imaging system [15]. Specifically, given $|f(x)|$ and $|F(\omega)|$, their algorithm is an iterative procedure which is characterized by the repeated Fourier transformation between the image and diffraction planes where in each plane, the known intensities are incorporated into the current estimate of the unknown field. Similar iterative procedures have also been proposed for solving the phase retrieval problem under different sets of constraints. Fienup, for example, modified the Gerchberg-Saxton algorithm in order to retrieve the phase from the magnitude of the Fourier transform of an image under the constraint that the desired solution is non-negative [12]. Misell also proposed an iterative procedure similar to the Gerchberg-Saxton algorithm for the case in which the intensity distributions in two slightly defocused planes are known [28]. For each of these iterative

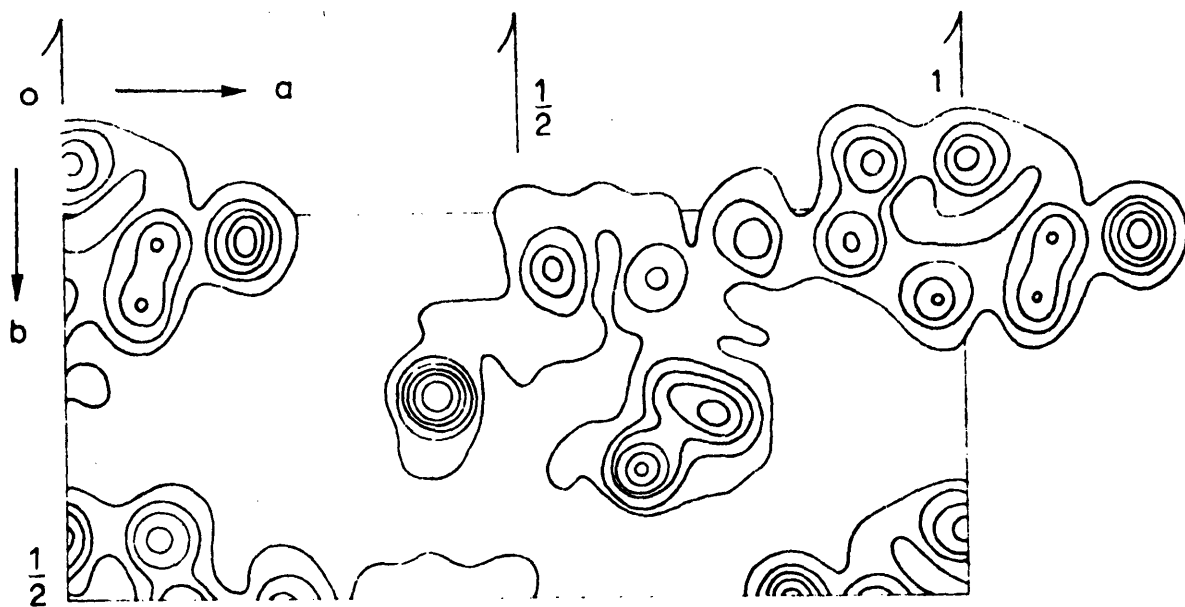
procedures, however, it appears that the convergence of the algorithm to the correct solution depends upon the properties of the signal to be recovered as well as on the initial estimate which is used to begin the iteration.

Other algorithms which have been proposed for retrieving phase from intensity measurements in the image and exit pupil planes include the algebraic approaches of Gerchberg and Saxton [14], Dallas [8], and Bates [2], and a recursive algorithm proposed by Quatieri [40]. Although this list is not intended to be exhaustive, references to some additional algorithms may be found in [11].

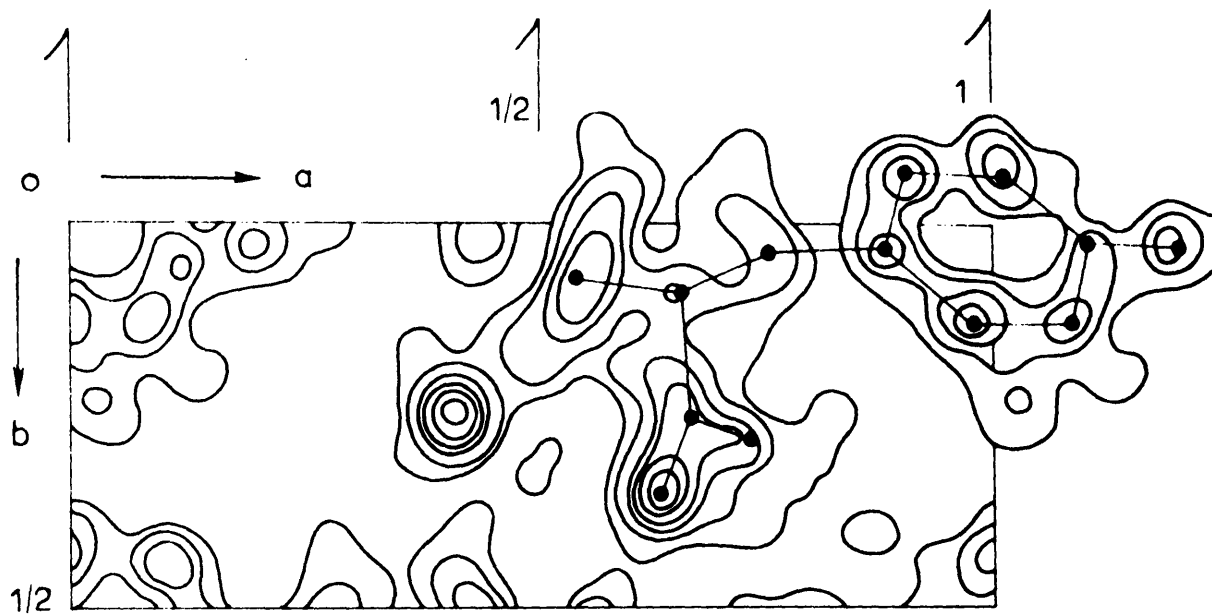
II.2: The importance of phase.

Although there has been a large amount of research directed towards the development of an understanding of the uniqueness questions related to the phase retrieval problem, it appears that similar research has not been undertaken to investigate the question of uniqueness for the case in which only phase information is available. Nevertheless, it has been noted in a number of different contexts and applications that many of the important features of a signal are contained within the phase of its Fourier transform. Specifically, it has been observed that a phase-only synthesis of a signal, formed by combining the phase of the Fourier transform of the signal with a constant or ensemble average magnitude, contains a number of similarities to the true signal. It appears that the first context in which the similarity between a signal and its phase-only synthesis was noted is in the field of x-ray crystallography [45]. Since the x-ray diffraction pattern of a crystal is related to the electron density distribution in the crystal by a Fourier transform, if both the magnitude and phase of the diffracted wave could be measured, then an inverse Fourier transform of the diffraction pattern would yield the desired structure. However, since only the intensity of the diffraction

pattern may be measured, the only available information which is obtained by measurement is that which is contained in the autocorrelation function (Patterson diagram) of the electron density distribution. Therefore, although the autocorrelation function contains peaks at positions corresponding to the interatomic vectors between the atoms in the crystal, phase information is necessary to uniquely recover the structure. In order to determine the importance of phase information in crystal structure determination, Srinivasan performed a number of experiments in the Fourier synthesis of crystal structures by combining the correct phase with various other magnitude functions [45]. For example, shown in Figure 2.1a is a contour diagram of the electron density distribution of L-tyrosine HCL which was obtained by combining the correct phase and magnitude information. The contours in this plot correspond to regions of constant electron density with the peaks representing the locations of the atomic positions. Shown in Figure 2.1b is the contour plot which is obtained by combining the correct phase with a magnitude function which is inversely proportional to frequency (a property which is known to be characteristic of the diffracted wave). Therefore, although the magnitude information has been essentially discarded, the resulting synthesis still contains many of the important features and properties of the correct structure (the correct atomic structure is overlaid in solid lines). Shown in Figure 2.1c is the synthesis which is obtained by randomly permuting the correct magnitudes and again many of the important properties are preserved. Finally, in Figure 2.1d is the result which is obtained by combining the correct phase with the magnitude of a totally different structure which has atomic locations at the positions indicated by the crosses. It is apparent that the resulting structure most closely resembles the structure corresponding to the phase used in the synthesis. The conclusion which may be drawn from this series of experiments is that, at least for those signals which typically arise in x-ray crystallography, most of the essential information of interest is contained in the phase.

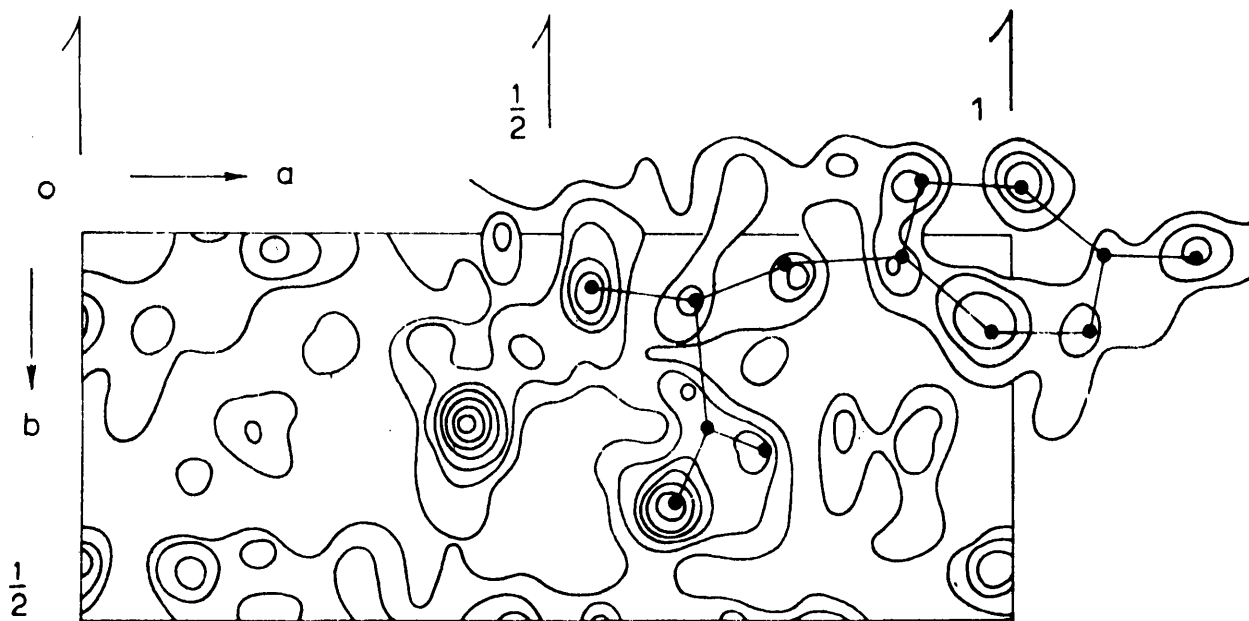


(a)

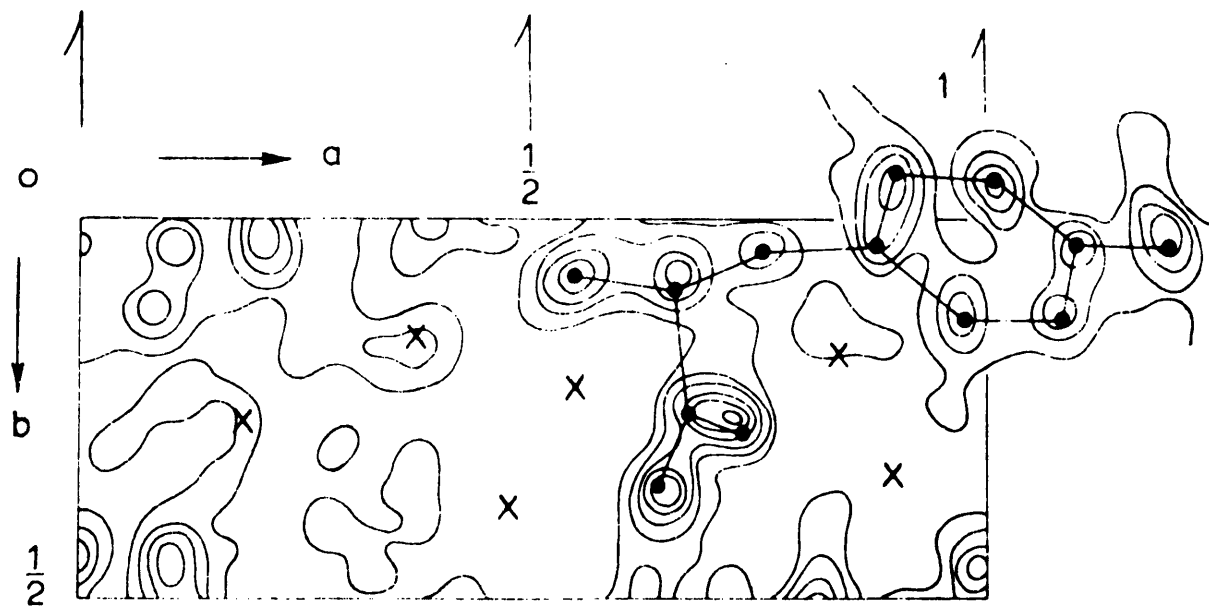


(b)

Figure 2.1: The Fourier synthesis of crystallographic structures. (a) Contour diagram of L-tyrosine HCL synthesized from the correct phase and magnitude. The contours correspond to constant electron density with peaks occurring at the atomic positions. (b) Fourier synthesis from correct phase and a magnitude which is inversely proportional to frequency.



(c)

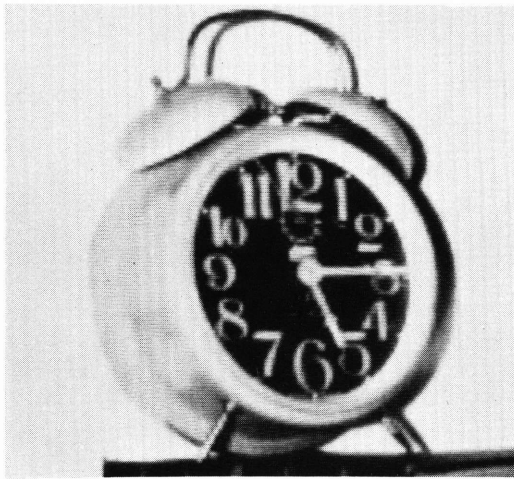


(d)

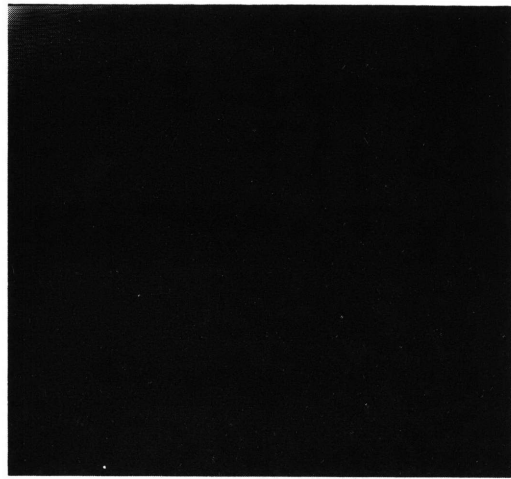
Figure 2.1 (cont.): (c) Fourier synthesis from correct phase and a random permutation of correct magnitude. (d) Fourier synthesis from correct phase and a magnitude associated with a totally different structure whose atoms are located at the positions indicated by the crosses (after Ramachandran and Srinivasan [42]).

The similarity between a signal and its phase-only synthesis has also been noted in the context of image processing by performing a similar set of experiments [34,35]. In particular, it has been observed that when only phase information is used to synthesize an image, the result contains many of the important features of the original image. An image synthesized on the basis of magnitude information alone, however, will not, in general, bear any similarity to the original. Shown in Figure 2.2a, for example, is an original image and in Figure 2.2b is the magnitude-only image which is formed by combining the magnitude of the Fourier transform of the image in (a) with zero phase. While this magnitude-only image contains no recognizable features, the phase-only image which is formed by combining the phase of the Fourier transform of the image in (a) with a constant magnitude contains many of the features of the original image as shown in Figure 2.2c. An even better phase-only Fourier synthesis is possible if the correct phase is combined with a magnitude which is representative of the class of images of interest. Shown in Figure 2.2d, for example, is the result which is obtained by combining the phase of the Fourier transform of the image in (a) with an ensemble average magnitude which was computed by averaging the Fourier transform magnitudes of several completely different images.

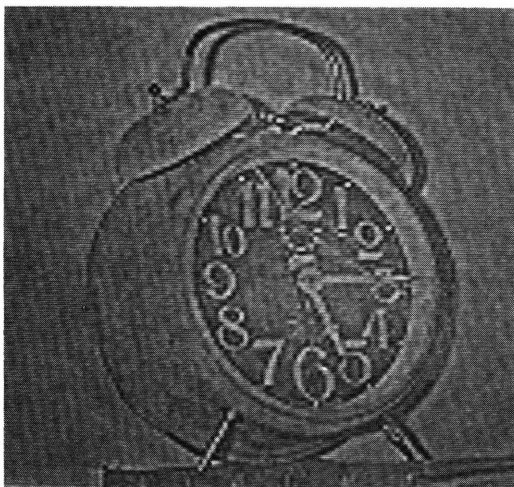
Finally, Figure 2.3 illustrates the effect of combining the phase of the Fourier transform of one image with the magnitude of another. Specifically, in Figure 2.3a and b are two original images, and in Figure 2.3c is the result which is obtained by combining the phase of the Fourier transform of image (a) with the magnitude of the Fourier transform of image (b). Similarly, in Figure 2.3d is the result which is obtained by combining the phase of the Fourier transform of image (b) with the magnitude of the Fourier of image (a). As is apparent from this figure, the images with the same phase are the ones which most closely resemble each other.



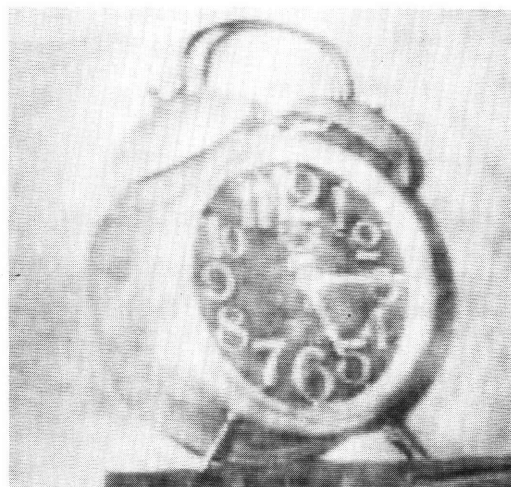
(a)



(b)

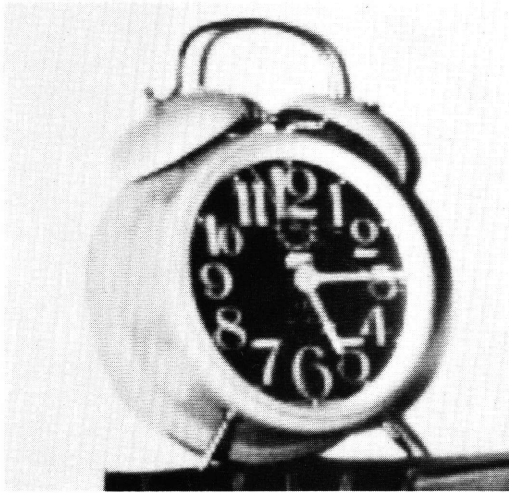


(c)



(d)

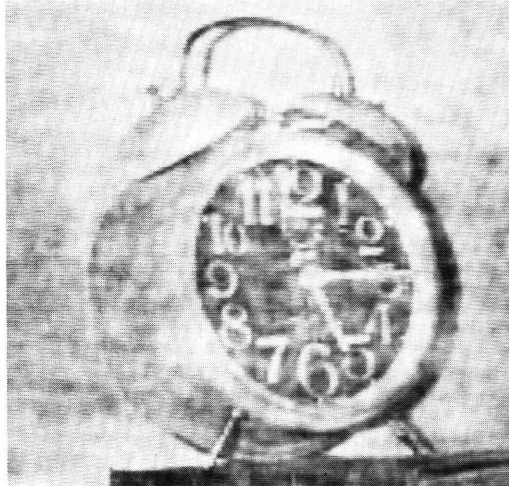
Figure 2.2: Phase-only and magnitude-only synthesis of images. (a) Original image. (b) Image formed by combining the magnitude of the Fourier transform of image (a) with zero phase. (c) Image formed by combining the phase of the Fourier transform of image (a) with a constant magnitude. (d) Image formed by combining the phase of the Fourier transform of image (a) with an ensemble average magnitude.



(a)



(b)



(c)



(d)

Figure 2.3: Images synthesized by combining the phase of the Fourier transform of one image with the magnitude of another. (a) Original image A. (b) Original image B. (c) Image formed by combining the phase of the Fourier transform of image A with the magnitude of the Fourier transform of image B. (d) Image formed by combining the phase of the Fourier transform of image B with the magnitude of the Fourier transform of image A.

The importance of phase in the representation of signals has also been noted in a number of other applications such as speech processing [35], and acoustical and optical holography [24,25,29,39]. In speech processing, for example, a set of experiments similar to those described above have been performed with similar results [35]. Specifically, if the Fourier transform of a sentence of speech is computed and the phase is set equal to zero, the intelligibility is lost in the signal which is obtained by an inverse Fourier transform. However, if the phase is preserved and the magnitude is set equal to a constant, an inverse Fourier transform leads to an intelligible synthesis of the original sentence of speech.

Although the importance of phase in the representation of signals has been noted in a number of applications, it should be pointed out that the importance of phase depends, in part, on the class of signals of interest. In particular, although it may be appropriate to conclude that phase is an important component in the representation of those signals which typically arise in speech processing, image processing, and x-ray crystallography, this statement is certainly not true for all signals. For example, for any signal which may be written in the form $f(x)=g(x)*g(-x)$ where $g(x)$ is an arbitrary real-valued function, the phase of the Fourier Transform of $f(x)$ is identically zero. Therefore, while the magnitude-only synthesis of such a signal will reproduce the signal $f(x)$ exactly, a phase-only synthesis with unit magnitude will yield only an impulse at the origin.

Finally, it should be noted that a number of different approaches have been taken in order to explain or quantify the relative importance of phase in the representation of signals. Tescher, for example, considered the rms error introduced when the phase or magnitude of the Fourier transform is quantized [47]. For random signals, it was concluded that for the same rms error, approximately two more bits are required

in quantizing the phase than in quantizing the magnitude. Pearlman and Grey reached a similar conclusion using rate distortion theory in their investigations into Fourier transform coding [38]. Specifically, it was concluded that phase must be encoded with approximately 1.37 more bits than magnitude to achieve an equivalent distortion. From another approach, Kermisch provided a quantitative analysis of the effect of reconstructing an image from a phase-only hologram (kinoform) [23]. Specifically, it was shown that in the phase-only reconstructed image, approximately 78% of the total image irradiance exactly reconstructs the original image. The remaining 22% of the image irradiance represents the image degradation and consists of convolutions of the original image with itself.

CHAPTER III: UNIQUENESS CONSTRAINTS (1-D)

III.1: Introduction

This chapter is concerned with the uniqueness of a one-dimensional sequence in terms of either the phase or magnitude of its Fourier transform. In general, phase or magnitude information alone is not sufficient to uniquely specify a sequence since convolution with a zero phase sequence produces another sequence with the same phase whereas convolution with an all-pass sequence produces another sequence with the same magnitude. Therefore, phase or magnitude information alone may, at best, uniquely define a sequence to within an arbitrary zero-phase or all-pass convolutional factor, respectively. In spite of this ambiguity, it is possible to include some additional information or constraints on a sequence so that it is uniquely defined by the phase or magnitude of its Fourier transform. For example, a minimum phase sequence may be uniquely recovered from the magnitude of its Fourier transform and may be recovered to within a scale factor from the phase of its Fourier transform [36]. Although this result is important and has been exploited in a number of applications, the minimum phase requirement is very restrictive since it is unlikely that an arbitrary sequence will satisfy this constraint. Therefore, it is the purpose of this chapter to develop some other classes of sequences which are uniquely specified by the phase or magnitude of their Fourier transform.

The organization of this chapter is as follows. After a brief review of some notation and terminology in Section III.2, the uniqueness of a sequence in terms of the phase of its Fourier transform is considered in Section III.3. Then, in Section III.4, uniqueness in terms of Fourier transform magnitude is addressed.

III.2: Notation and framework

A one-dimensional sequence is a function of an integer-valued variable, n , and will be denoted by $x(n)$. Unless otherwise specified, all sequences will be assumed to be real-valued. The z -transform of $x(n)$, denoted by $X(z)$, is defined by

$$X(z) = \sum_n x(n) z^{-n} \quad (3-1)$$

When $x(n)$ is real, $X(z)$ has the following property:

$$X(z) = X^*(z^*) \quad (3-2)$$

Although the sum in (3-1) will not, in general, converge for all values of z , it will always be assumed that $X(z)$ is a rational function of z with a region of convergence which includes the unit circle, $|z|=1$. In this case, the Fourier transform exists and is given by

$$X(\omega) = X(z)|_{z=e^{j\omega}} = \sum_n x(n) e^{-jn\omega} \quad (3-3)$$

Furthermore, when $x(n)$ is real then $X(\omega)$ is a conjugate symmetric function of ω , i.e.,

$$X(\omega) = X^*(-\omega) \quad (3-4)$$

Since $X(\omega)$ is, in general, a complex-valued function of ω , it may be expressed in terms of its real and imaginary parts as

$$X(\omega) = X_R(\omega) + jX_I(\omega) \quad (3-5)$$

or, in terms of its magnitude and phase as

$$X(\omega) = |X(\omega)| \exp[j\phi_x(\omega)] \quad (3-6)$$

where

$$|X(\omega)|^2 = [X_R(\omega)]^2 + [X_I(\omega)]^2 \quad (3-7 \text{ a})$$

and
$$\tan[\phi_x(\omega)] = X_I(\omega) / X_R(\omega) \quad (3-7 \text{ b})$$

Finally, if $X(\omega)$ is sampled at M uniformly spaced frequencies between zero and 2π , then these samples correspond to the M -point Discrete Fourier Transform (DFT), $X(k)_M$, of $x(n)$. Specifically,

$$X(k)_M = X(\omega) \Big|_{\omega=2\pi k/M} = \sum_n x(n) e^{-j2\pi nk/M} \quad (3-8)$$

Furthermore, if $x(n)$ is zero outside an interval of length N where $N \leq M$ then $X(k)_M$ is sufficient to uniquely recover $x(n)$.

Thus, it is the purpose of this chapter to develop some conditions under which $x(n)$, or equivalently $X(\omega)$, is uniquely specified in terms of either $\phi_x(\omega)$ or $|X(\omega)|$. In the case of phase, these conditions include the case in which $\phi_x(\omega)$ is known for all ω as well as the case in which $\phi_x(\omega)$ is known for only a finite number of frequencies and, in particular, when only the phase of $X(k)_M$ is known.

III.3: Uniqueness in terms of phase

In this section, some conditions are developed under which a one-dimensional sequence is uniquely defined by the phase of its Fourier transform. Specifically, it is shown that almost all finite duration sequences are uniquely defined (to within a scale factor) by the phase of their Fourier transform. This result is established first for the case in which the phase of the Fourier transform of a real sequence is known for all frequencies and then is extended to the case in which only a finite number of phase

values are known. Similar uniqueness constraints are then developed for complex sequences and for sequences whose convolutional inverses are finite in length (i.e. all-pole sequences). Finally, the dual problem of uniquely defining a periodic continuous-time signal in terms of its phase or magnitude is briefly addressed.

III.3.1: Uniqueness in terms of continuous phase

Recall from Section III.1 that for any sequence, $x(n)$, it is always possible to find another sequence which has a Fourier transform with the same phase by simply convolving $x(n)$ with a zero phase sequence $g(n)$,

$$y(n) = x(n) * g(n) \quad : \quad \phi_g(\omega) = 0 \quad (3-9)$$

It is also true, however, that if two sequences, $x(n)$ and $y(n)$, have a Fourier transform with the same phase then they are related by (3-9). This result follows simply by noting that

$$Y(\omega) = X(\omega) [Y(\omega)/X(\omega)] = X(\omega) G(\omega) \quad (3-10)$$

Therefore, since $\phi_y(\omega) = \phi_x(\omega) + \phi_g(\omega)$, if $\phi_x(\omega) = \phi_y(\omega)$, then $\phi_g(\omega) = 0$ for all ω and $g(n)$ is a zero phase sequence.

In order to gain some insight into the uniqueness question, it will be useful to characterize zero phase sequences. Therefore, note that if $g(n)$ is a zero-phase sequence, then its Fourier transform, $G(\omega)$, must be real and non-negative. However, if $G(\omega)$ is real, then $g(n)$ must be conjugate symmetric,

$$g(n) = g^*(-n) \quad (3-11)$$

and therefore

$$G(z) = G^*(1/z^*) \quad (3-12)$$

Furthermore, if $g(n)$ is real then (3-11) implies that $g(n)$ is even

$$g(n) = g(-n) \quad (3-13)$$

and therefore

$$G(z) = G(1/z) \quad (3-14)$$

From (3-12) it follows that the singularities of $G(z)$ (i.e. its poles and zeros) occur in conjugate reciprocal pairs. In other words, if $G(z)$ has a zero (pole) at $z=z_0$, then $G(z)$ must also have a zero (pole) at $z=1/z_0^*$ which is the mirror image of z_0 about the unit circle. Note, however, that (3-12) places no restrictions or constraints on the zeros of $G(z)$ which lie on the unit circle.

Since (3-12) is a necessary and sufficient condition that $G(\omega)$ be real, it is equivalent to the constraint

$$\tan[\phi_g(\omega)] = 0 \quad \text{for all } \omega \quad (3-15)$$

i.e., for each ω , $\phi_g(\omega)$ is either equal to zero or π . However, in order for $g(n)$ to be a zero phase sequence, in addition to being real, $G(\omega)$ must also be non-negative. Although (3-12) places no restrictions on the zeros of $G(z)$ which lie on the unit circle, in order for $G(\omega)$ to be non-negative, any unit circle zeros of $G(z)$ must be of even multiplicity (Spectral Factorization Theorem). Therefore, if $g(n)$ is a zero-phase sequence, its z -transform may always be written in the form:

$$G(z) = \sigma^2 A(z) \cdot A^*(1/z^*) \quad (3-16)$$

where σ is a real number and $A(z)$ is a rational function of z^\dagger .

[†] Recall that it is assumed that all sequences have rational z -transforms.

Since convolution with a zero-phase sequence produces another sequence with the same phase, it will be useful to consider the effect of convolution with a zero-phase sequence. Therefore, let $x(n)$ be a finite duration sequence which has a z -transform with no zeros on the unit circle or in conjugate reciprocal pairs, and let $g(n)$ be an arbitrary zero-phase sequence. With $Y(z)=X(z)G(z)$, at least one of the following statements about $Y(z)$ must be true:

- (1) $Y(z)$ contains conjugate reciprocal zeros or poles.
- (2) $Y(z)$ contains zeros of even multiplicity on the unit circle.
- (3) The zeros of $X(z)$ are replaced with poles in $Y(z)$ at the conjugate reciprocal locations.

Furthermore, if $g(n)$ is only constrained to be an even sequence, i.e. $\tan[\phi_g(\omega)]=0$, then again at least one of these statements must apply to $Y(z)$ if the zeros on the unit circle in (2) are not constrained to be of even multiplicity. In either case, in order for $y(n)$ to be finite in length, $Y(z)$ must either have zeros on the unit circle or in conjugate reciprocal pairs. Therefore, this leads to the following:

Theorem 3.1: Let $x(n)$ and $y(n)$ be real finite length sequences with z -transforms which have no zeros on the unit circle or in conjugate reciprocal pairs. If $\phi_x(\omega)=\phi_y(\omega)$ for all ω , then $x(n)=\beta y(n)$ for some positive real constant β . If $\tan[\phi_x(\omega)]=\tan[\phi_y(\omega)]$ for all ω , then $x(n)=\beta y(n)$ for some real constant β .

Proof: Let $x(n)$ and $y(n)$ satisfy the constraints of the theorem and consider the sequence $g(n)$ defined by:

$$g(n) = x(n) * y(-n) \quad (3-17)$$

which has a z -transform given by

$$G(z) = X(z) Y(z^{-1}) \quad (3-18)$$

By noting that the phase of the Fourier transform of $g(n)$ is

$$\phi_g(\omega) = \phi_x(\omega) - \phi_y(\omega) \quad (3-19)$$

it follows that if $\phi_x(\omega) = \phi_y(\omega)$ then $\phi_g(\omega) = 0$. Also, if $\tan[\phi_x(\omega)] = \tan[\phi_y(\omega)]$ then $\phi_x(\omega)$ and $\phi_y(\omega)$ are equal to within a factor of π and, therefore, $\tan[\phi_g(\omega)] = 0$. In either case, $G(\omega)$ is real and

$$G(z) = G(z^{-1}) \quad (3-20)$$

Therefore, from (3-18) and (3-20)

$$X(z) Y(z^{-1}) = X(z^{-1}) Y(z) \quad (3-21)$$

Now, suppose that $X(z)$ has a k th-order zero at $z = z_0$ where $0 < |z_0| < \infty$ (since $X(z)$ has no zeros on the unit circle then, in addition, $|z_0| < 1$). Since $y(n)$ is finite in length then $Y(1/z_0)$ must be finite and

$$G(z^{-1}) = X(z^{-1}) Y(z) \quad (3-22)$$

must also have a k th-order zero at $z = z_0$. However, since $X(z)$ has no zeros in conjugate reciprocal pairs, it follows that $X(1/z_0)$ is non-zero. Therefore, $Y(z)$ must also have a k th-order zero at $z = z_0$. Repeating the argument for the zeros of $Y(z)$, it follows that $X(z)$ and $Y(z)$ have the same zero set for all z such that $0 < |z_0| < \infty$. Consequently,

$$X(z) = \beta z^k Y(z) \quad (3-23)$$

where β is a real number and k is an integer. However, if either the phase or the tangent of the phase of the Fourier transforms of $x(n)$ and $y(n)$ are equal, then k must be equal to zero. Finally, if $\phi_x(\omega) = \phi_y(\omega)$, then β must be a positive real number and the theorem follows. /////

Although the conditions in Theorem 3.1 exclude the possibility of zeros on the unit circle, the theorem may be generalized to allow first-order zeros on the unit circle for the case in which $\phi_x(\omega) = \phi_y(\omega)$. In particular, if $\phi_x(\omega) = \phi_y(\omega)$, then $g(n)$ in (3-17) is a zero phase sequence. Therefore, if $|z_0| = 1$ and if $G(z_0) = 0$, then $G(z)$ must have at least a second-order zero at z_0 . However, since

$$G(z) = X(z) Y^*(1/z^*) \quad (3-24)$$

and since $X(z)$ and $Y(z)$ have at most one zero at z_0 , then the zero of $G(z)$ at z_0 must be second-order. Consequently, since neither $X(z)$ nor $Y(z)$ may have a second-order zero at z_0 , then both $X(z)$ and $Y(z)$ must have a simple zero at z_0 and, thus, the zeros of $X(z)$ and $Y(z)$ on the unit circle are identical.

These results may be summarized in a slightly different form as follows:

- (1) If $x(n)$ is finite in length and cannot be written as the convolution of two finite-length sequences, $x(n) = x_0(n) * g(n)$, where $g(n)$ is a zero-phase sequence, then $\phi_x(\omega)$ uniquely specifies $x(n)$ to within a positive scale factor.
- (2) If $x(n)$ is finite in length and cannot be written as the convolution of two finite-length sequences, $x(n) = x_0(n) * g(n)$, where $\tan[\phi_g(\omega)] = 0$ [i.e. $g(n) = g(-n)$], then $\tan[\phi_x(\omega)]$ uniquely specifies $x(n)$ to within a scale factor.

In Chapter V, several algorithms are presented for reconstructing a sequence which satisfies the constraints of Theorem 3.1 from the phase of its Fourier transform. At this point, however, it will be instructive to consider the following algorithm which may, in theory, be used to reconstruct a sequence from the phase of its Fourier transform. In addition, this algorithm may be viewed as an alternative proof of Theorem 3.1.

* * * * *

Let $x(n)$ be a sequence which satisfies the constraints of Theorem 3.1 and let $\phi_x(\omega)$ be the phase of its Fourier transform. With $\hat{\phi}_x(\omega)$ the associated unwrapped phase [36], it follows that the z -transform of $x(n)$ is restricted to be of the form:

$$X(z) = C z^{n_0} \prod_{k=1}^{N_1} (1 - a_k z^{-1}) \prod_{k=1}^{N_2} (1 - b_k z) \quad (3-25)$$

where C is a real number, n_0 is an integer, $|a_k| < 1$, $|b_k| < 1$ for all k , and $a_k \neq b_i^*$ for any k and i .

Step 1: From $\phi_x(\omega)$, the algebraic sign of C may be determined by using the fact that $\phi_x(0)$ is zero if and only if C is positive [36]. In addition, the value of n_0 may be obtained from the unwrapped phase by

$$n_0 = 1/\pi [\hat{\phi}_x(\pi) - \hat{\phi}_x(0)] \quad (3-26)$$

Step 2: From the unwrapped phase and the value of n_0 obtained in Step 1, a new phase function may be defined as

$$\psi_x(\omega) = \hat{\phi}_x(\omega) - n_0 \omega - \hat{\phi}_x(0) \quad (3-27)$$

Using the Hilbert transform, a minimum phase sequence $x_{\min}(n)$ may then be found which has a Fourier transform with a phase equal to $\psi_x(\omega)$ [36]. The z-transform of $x_{\min}(n)$, $X_{\min}(z)$, is given by [36]:

$$X_{\min}(z) = \frac{\prod_{k=1}^{N_1} (1 - a_k z^{-1})}{\prod_{k=1}^{N_1} (1 - b_k^* z^{-1})} \quad (3-28)$$

where the coefficients a_k and b_k are identical to those in (3-25).

Step 3: Since pole/zero cancellations cannot occur in (3-28) by virtue of the fact that $a_k \neq b_i^*$ for any k and i , the coefficients a_k and b_k in (3-25) may be obtained from the zeros and poles of $X_{\min}(z)$.

Step 4: From the sign of C , the value of n_0 , and the coefficients a_k and b_k which are obtained in Steps 1 through 3, $X(z)$ in (3-25) is uniquely specified to within a positive scale factor. Therefore, $x(n)$ may then be recovered to within a positive scale factor by an inverse z-transform.

* * * * *

The condition in Theorem 3.1 that there are no zeros in reciprocal pairs insures that there are no pole/zero cancellations in (3-28). If the original sequence has reciprocal zeros, then this algorithm may still be used. However, due to pole/zero cancellations in (3-28), only those zeros which are not in reciprocal pairs may be recovered.

Although the algorithm outlined above may be used, in theory, to reconstruct a sequence from the phase of its Fourier transform, due to the difficulties involved in obtaining the unwrapped phase in (3-27) and in determining the locations of the poles

and zeros in (3-28), this algorithm would probably not be used in practice. More practical algorithms for recovering a sequence from the phase of its Fourier transform will be described in Chapter V.

III.3.2: Uniqueness in terms of phase samples

Theorem 3.1 provides a set of conditions under which a sequence is uniquely defined to within a scale factor by the phase of its Fourier transform. Although the theorem assumes that the phase is known for all frequencies, due to the finite length constraint, it is possible to extend this result to the case in which the phase is known only for a finite number of frequency values. Specifically,

Theorem 3.2: Let $x(n)$ and $y(n)$ be real sequences which are zero outside the interval[†] $[0, N-1]$ with z -transforms which have no zeros in conjugate reciprocal pairs or on the unit circle. If $\phi_x(\omega) = \phi_y(\omega)$ at $(N-1)$ distinct frequencies in the open interval $(0, \pi)$, then $x(n) = \beta y(n)$ for some positive real number β . If $\tan[\phi_x(\omega)] = \tan[\phi_y(\omega)]$ at $(N-1)$ distinct frequencies in the interval $(0, \pi)$, then $x(n) = \beta y(n)$ for some real number β .

Proof: Let $x(n)$ and $y(n)$ satisfy the constraints of the theorem. As in the proof of Theorem 3.1, consider the sequence $g(n)$ defined by

$$g(n) = x(n) * y(-n) \quad (3-29)$$

[†] More generally, $x(n)$ and $y(n)$ may be taken to be zero outside any interval of length N .

If $\{\omega_1, \omega_2, \dots, \omega_{N-1}\}$ are $(N-1)$ distinct frequencies in the interval $(0, \pi)$ and if $\phi_x(\omega_k) = \phi_y(\omega_k)$ or $\tan[\phi_x(\omega_k)] = \tan[\phi_y(\omega_k)]$ for each k , then

$$\tan[\phi_g(\omega_k)] = 0 \quad \text{for } k=1, 2, \dots, (N-1) \quad (3-30)$$

Therefore, $G(\omega_k)$ is real for each k and, since $g(n)$ is zero outside the interval $[-N+1, N-1]$, then

$$\text{Im}[G(\omega_k)] = \sum_{n=-N+1}^{N-1} g(n) \sin(n\omega_k) = 0 \quad (3-31)$$

Thus, (3-31) may be expressed as

$$\sum_{n=1}^{N-1} [g(n) - g(-n)] \sin(n\omega_k) = 0 \quad (3-32)$$

for $k=1, 2, \dots, (N-1)$. Since the functions $\{\sin\omega, \sin 2\omega, \dots, \sin(n\omega)\}$ form a Chebyshev set, it follows that $[g(n) - g(-n)] = 0$, i.e., $g(n)$ is an even sequence [43]. As a result, $\tan[\phi_g(\omega)] = 0$ for all ω which implies that $\tan[\phi_x(\omega)] = \tan[\phi_y(\omega)]$ for all ω . Therefore, from Theorem 3.1, $x(n) = \beta y(n)$ where β is a real number. Finally, if $\phi_x(\omega) = \phi_y(\omega)$ for at least one $\omega \in (0, \pi)$, then β must, in addition, be positive. /////

As a special case of this theorem, note that if $x(n)$ is zero outside the interval $[0, N-1]$, then for any $M \geq 2N-1$, the phase of its M -point DFT uniquely specifies $x(n)$ to within a scale factor.

Although it was possible to extend Theorem 3.1 to allow first-order zeros on the unit circle for the case in which $\phi_x(\omega) = \phi_y(\omega)$, this extension is not possible in the case of Theorem 3.2 as illustrated by the following example:

Example: Let $x(n)$ and $y(n)$ be two sequences which are zero outside the interval $[0,2]$ and which are defined by

$$x(n) = \delta(n-1) \quad \text{and} \quad y(n) = \delta(n) + \delta(n-2) \quad (3-33)$$

Since $Y(\omega) = 2\cos\omega \cdot X(\omega)$, then $\phi_x(\omega) = \phi_y(\omega)$ for all $\omega \in (-\pi/2, \pi/2)$. Therefore, although $x(n)$ and $y(n)$ have Fourier transform with the same phase over any set frequency values in the interval $(0, \pi/2)$, they are not related to one another by a scale factor.

Theorems 3.1 and 3.2 assert that, within the set of all finite length sequences which have z -transforms with no zeros on the unit circle or in conjugate reciprocal pairs, a sequence is uniquely defined to within a scale factor by its phase or, if the interval over which the sequence is non-zero is known, by a finite number of phase samples. In Chapter V, a number of algorithms are developed for reconstructing a sequence which satisfies the constraints of Theorem 3.2 from samples of the phase of its Fourier transform. Although the sequences obtained from these algorithms have the correct phase samples and are zero outside a given interval, some additional information about the sequence is required in order to guarantee that the solution has a z -transform with no zeros on the unit circle or in conjugate reciprocal pairs. For example, suppose that a sequence is known to be zero outside the interval $[0,2]$ with phase $\phi_x(\omega) = -\omega$. Although scaled versions of $x(n) = \delta(n-1)$ are the only sequences consistent with this information and which satisfy the constraints of Theorem 3.2, there are many other sequences which do not satisfy the constraints of Theorem 3.2 but which are zero outside the interval $[0,2]$ and have Fourier transforms with phase $\phi_y(\omega) = -\omega$, e.g., any sequence of the form $y(n) = \delta(n) + \lambda\delta(n-1) + \delta(n-2)$ for any $\lambda \geq 2$. However, if it is known that the first non-zero value of $x(n)$ is at $n=1$, then scaled versions of $x(n)$ are

the only sequences consistent with the known information. This result is true in general as the following theorem states:

Theorem 3.3: Let $x(n)$ be a real sequence which is zero outside the interval $[0, N-1]$ with $x(0) \neq 0$ and which has a z -transform with no zeros in conjugate reciprocal pairs or on the unit circle. Let $y(n)$ be any real sequence which is also zero outside the interval $[0, N-1]$. If $\phi_x(\omega) = \phi_y(\omega)$ at $(N-1)$ distinct frequencies in the interval $(0, \pi)$, then $y(n) = \beta x(n)$ for some positive constant β . If $\tan[\phi_x(\omega)] = \tan[\phi_y(\omega)]$ for $(N-1)$ distinct frequencies in the interval $(0, \pi)$, then $y(n) = \beta x(n)$ for some real constant β .

Note that, in contrast to Theorem 3.2, there are no constraints on the z -transform of $y(n)$. In particular, $y(n)$ may be any finite duration sequence which is zero outside the interval $[0, N-1]$. It should also be pointed out that although Theorem 3.3 assumes that the first non-zero point of $x(n)$ is at $n=0$, the theorem may easily be extended to the case in which the first non-zero point of $x(n)$ is at an arbitrary $n=n_0$. Furthermore, knowledge of the location of the first non-zero point of $x(n)$ may be replaced with the knowledge of the location of the last non-zero point of $x(n)$.

Proof: Let $x(n)$ and $y(n)$ satisfy the constraints of the theorem and consider the sequence $g(n)$ defined in (3-29). As in the proof of Theorem 3.2, $g(n)$ is an even sequence which is zero outside the interval $[-N+1, N-1]$. Now let N_1-1 represent the location of the last non-zero point of $x(n)$, i.e., $x(n)=0$ for $n \geq N_1$ and $x(N_1-1) \neq 0$. Then,

$$G(z) = X(z) Y(z^{-1}) = \prod_{n=0}^{N_1-1} x(n) z^{-n} \prod_{n=0}^{N-1} y(n) z^n \quad (3-34)$$

Since $g(n)$ is even and $x(0) \neq 0$, then $y(n) = 0$ for $n \geq N_1$. Therefore, the number of zeros in $Y(z)$ is less than or equal to the number of zeros in $X(z)$. For reasons identical to those in the proof of Theorem 3.1, since $g(n)$ is an even sequence and $x(n)$ has no zeros in reciprocal pairs or on the unit circle, if $X(z)$ has a k th-order zero at $z = z_0$, then $Y(z)$ must also have a k th-order zero at $z = z_0$. Therefore, since $Y(z)$ cannot have more zeros than $X(z)$, it follows that $y(n) = \beta x(n)$ for some constant β . Furthermore, if $\phi_x(\omega) = \phi_y(\omega)$ for at least one ω , then β must, in addition, be positive. /////

III.3.3: Extensions

Theorems 3.1–3.3 provide some conditions under which a sequence is uniquely defined to within a scale factor by the phase of its Fourier transform. Although these constraints include the restriction that the sequence be real-valued, it is possible to develop similar theorems which apply to complex sequences. For example, the statement of Theorem 3.1 is unchanged when the sequences are allowed to be complex. Furthermore, the only significant change required in the proof is to redefine $g(n)$ in (3-17) by

$$g(n) = x(n) * y^*(-n) \quad (3-35)$$

The statements of Theorems 3.2 and 3.3, on the other hand, are slightly different for complex sequences. Specifically, for a sequence of length N , instead of $(N-1)$ phase samples in the interval $(0, \pi)$ for a real sequence, $(2N-1)$ phase samples in the half open interval $[0, 2\pi)$ are necessary for the unique specification of a complex sequence. Since the proofs of these results are slightly different from those in Theorems 3.2 and 3.3, the statement and proof of the extension of Theorem 3.2 to complex sequences is

presented below. The proof of the extension of Theorem 3.3 to complex sequences follows in a similar manner.

Theorem 3.4: Let $x(n)$ and $y(n)$ be complex sequences which are zero outside the the interval[†] $[0, N-1]$ with z -transforms which have no zeros in conjugate reciprocal pairs or on the unit circle. If $M \geq 2N-1$ and $\phi_x(\omega) = \phi_y(\omega)$ at M distinct frequencies in the the interval $[0, 2\pi)$, then $x(n) = \beta y(n)$ for some positive real number β . If $M \geq 2N-1$ and $\tan[\phi_x(\omega)] = \tan[\phi_y(\omega)]$ at M distinct frequencies in the interval $[0, 2\pi)$, then $x(n) = \beta y(n)$ for some real number β .

Proof: Let $x(n)$ and $y(n)$ satisfy the constraints of the theorem and consider the sequence $g(n)$ defined by

$$g(n) = x(n) * y^*(-n) = g_R(n) + jg_I(n) \quad (3-36)$$

where $g_R(n)$ and $g_I(n)$ are the real and imaginary parts of $g(n)$ respectively. If $\{\omega_1, \omega_2, \dots, \omega_M\}$ are M distinct frequencies in the interval $[0, 2\pi)$ and if $\phi_x(\omega) = \phi_y(\omega)$ or $\tan[\phi_x(\omega)] = \tan[\phi_y(\omega)]$ for each k , then

$$\tan[\phi_g(\omega_k)] = 0 \quad \text{for } k=1, 2, \dots, M \quad (3-37)$$

Therefore, $G(\omega_k)$ is real and, since $g(n)$ is zero outside the interval $[-N+1, N-1]$,

$$\text{Im}[G(\omega_k)] = \sum_{n=-N+1}^{N-1} [g_I(n) \cos n\omega_k - g_R(n) \sin n\omega_k] = 0 \quad (3-38)$$

[†] As before, $x(n)$ and $y(n)$ may be taken to be zero outside any interval of length N .

Thus, with

$$\mu(n) = g_R(n) - g_R(-n) \quad (3-39)$$

and

$$\nu(n) = g_I(n) + g_I(-n) \quad (3-40)$$

then (3-38) becomes

$$\sum_{n=0}^{N-1} [\nu(n) \cos n\omega_k] - \sum_{n=1}^{N-1} [\mu(n) \sin n\omega_k] = 0 \quad (3-41)$$

Since the functions $\{1, \cos\omega, \sin\omega, \dots, \cos(n\omega), \sin(n\omega)\}$ form a Chebyshev set, it follows that if $M \geq 2N-1$ then $\mu(n)=0$ and $\nu(n)=0$ for $n=0, 1, \dots, M$ [43]. Consequently,

$$g_R(n) = g_R(-n) \quad \text{and} \quad g_I(n) = -g_I(-n) \quad (3-42)$$

so $g(n)$ is conjugate symmetric. As a result, $\tan[\phi_g(\omega)]=0$ for all ω which implies that $\tan[\phi_x(\omega)]=\tan[\phi_y(\omega)]$ for all ω . Therefore, it follows in a style similar to that used in the proof of Theorem 3.1 that $x(n)=\beta y(n)$ where β is a real number. Finally, if $\phi_x(\omega)=\phi_y(\omega)$ for at least one $\omega \in [0, 2\pi)$, then β must, in addition, be positive. /////

Although the results which have been presented thus far have been confined to finite length sequences, an extension is easily made to those sequences whose convolutional inverses are finite in length. Specifically, let $x_1(n)$ denote the convolutional inverse of a sequence $x(n)$, i.e.,

$$x(n) * x_1(n) = \delta(n) \quad (3-43)$$

where $\delta(n)$ is the unit sample function. Now suppose that $x(n)$ is a stable all-pole sequence so that the z-transform of $x(n)$ is given by

$$X(z) = 1 / P(z) \quad (3-44)$$

where $P(z)$ is a polynomial in z and z^{-1} . In this case, the convolutional inverse of $x(n)$ is finite in length with a z -transform given by

$$X_1(z) = P(z) \quad (3-45)$$

Furthermore, the phase of $x_1(n)$ is uniquely specified by the phase of $x(n)$:

$$\phi_{x_1}(\omega) = -\phi_x(\omega) \quad (3-46)$$

Therefore, a stable all-pole sequence is uniquely specified to within a scale factor by the phase of its Fourier transform if its convolutional inverse satisfies the constraints of Theorem 3.1. A similar statement applies if the phase is known for only a finite number of frequency values or if $x(n)$ is complex.

Finally, note that the results of this section may be applied to the dual problem of uniquely defining a periodic continuous-time signal in terms of its phase. Specifically, let $x(t)$ be a complex-valued function of the continuous variable t with phase $\phi_x(t)$, i.e.,

$$x(t) = |x(t)| \exp[j\phi_x(t)] \quad (3-47)$$

If $x(t)$ is periodic with period T , then $x(t)$ has a Fourier series expansion of the form

$$x(t) = \sum_n x_n e^{j2\pi nt/T} \quad (3-48)$$

where x_n are the Fourier series coefficients of $x(t)$ which may be viewed as a sequence of complex-valued numbers. Therefore, the results of this section provide conditions under which the Fourier series coefficients x_n , and thus $x(t)$, is uniquely defined to within a scale factor by $\phi_x(t)$. For example, if $x(t)$ is a band-limited periodic signal,

then the Fourier series coefficients are a finite length sequence of complex numbers. Therefore, using Theorem 3.4 it follows that $x(t)$ is uniquely defined to within a scale factor by its phase, $\phi_x(t)$, if $x(t)$ cannot be factored into a product of two band-limited signals, $x(t) = x_0(t)g(t)$, where $g(t)$ is a positive real function of t , i.e., $g(t)$ has zero phase.

III.4: Uniqueness in terms of magnitude

This section considers the uniqueness of a one-dimensional sequence in terms of the magnitude of its Fourier transform. As is the case for phase, without any additional information or constraints, a sequence is not uniquely defined by the magnitude of its Fourier transform. Specifically, for any sequence $x(n)$, another sequence $y(n)$ may always be found which has a Fourier transform with the same magnitude by simply convolving $x(n)$ with an all-pass sequence, $g(n)$:

$$y(n) = x(n) * g(n) \quad : \quad |G(\omega)| = 1 \quad (3-49)$$

It is also true, however, that if two sequences $x(n)$ and $y(n)$ have the same magnitude, then they are related by (3-49). This result follows simply by noting that

$$Y(\omega) = X(\omega) [Y(\omega)/X(\omega)] = X(\omega) G(\omega) \quad (3-50)$$

Therefore, since $|Y(\omega)| = |X(\omega)| |G(\omega)|$, if $x(n)$ and $y(n)$ have Fourier transforms with the same magnitude, then $|G(\omega)| = 1$ for all ω .

In order to gain some insight into the uniqueness question, it will be useful to characterize all-pass sequences. Therefore, let $g(n)$ be a sequence which has a Fourier transform with unit magnitude. In this case,

$$|G(\omega)|^2 = G(\omega) G^*(\omega) = 1 \quad (3-51)$$

which implies that the autocorrelation of $g(n)$, $r_g(n)$, equals the unit sample function

$$r_g(n) = g(n) * g^*(-n) = \delta(n) \quad (3-52)$$

and therefore

$$R_g(z) = G(z) G^*(1/z^*) = 1 \quad (3-53)$$

Furthermore, if $g(n)$ is real then (3-52) and (3-53) become

$$r_g(n) = g(n) * g(-n) \quad (3-54)$$

and

$$R_g(z) = G(z) G(z^{-1}) \quad (3-55)$$

From (3-53), however, it follows that

$$G^{-1}(z) = G^*(1/z^*) \quad (3-56)$$

Therefore, $G(z)$ consists of conjugate reciprocal pole/zero pairs. In other words, if $G(z)$ has a zero (pole) at $z=z_0$ then $G(z)$ must also have a pole (zero) at $z=1/z_0^*$. Note, in addition, that $G(z)$ may have no singularities on the unit circle.

Since convolution with an all-pass sequence produces another sequence with the same magnitude, it will be useful to consider the effect of convolution with an all-pass sequence. Therefore, let $x(n)$ be a sequence which has a z -transform with no conjugate reciprocal pole/zero pairs and let $g(n)$ be an arbitrary all-pass sequence, other than a delayed unit sample function. With $Y(z)=X(z)G(z)$, at least one of the following statements about $Y(z)$ must be true:

- (1) $Y(z)$ contains conjugate reciprocal pole/zero pairs.
- (2) Poles or zeros of $X(z)$ are reflected about the unit circle.

Therefore, this lead to the following:

Theorem 3.5: Let $x(n)$ and $y(n)$ be real sequences with z -transforms which have no conjugate reciprocal pole/zero pairs and, in addition,

- a) all the zeros of $X(z)$ and $Y(z)$ (except at $z=0$ or $z^{-1}=0$) are either inside or outside the unit circle.
- b) all the poles of $X(z)$ and $Y(z)$ (except at $z=0$ or $z^{-1}=0$) are either inside or outside the unit circle.

If $|X(\omega)|=|Y(\omega)|$, then $x(n)=\pm y(n+k)$ for some integer k .

Note that minimum phase sequences as well as maximum phase sequences satisfy the constraints of this theorem. However, since minimum (maximum) phase sequences have no singularities at $z^{-1}=0$ ($z=0$), the magnitude of the Fourier transform uniquely specifies a minimum phase or maximum phase sequence to within a multiplicative sign factor.

Proof: Let $x(n)$ and $y(n)$ satisfy the constraints of the theorem. If the Fourier transform magnitudes of $x(n)$ and $y(n)$ are equal, then their autocorrelations are equal and, therefore,

$$X(z) X(z^{-1}) = Y(z) Y(z^{-1}) \quad (3-57)$$

Now consider the case in which all of the zeros of $X(z)$ and $Y(z)$ are inside the unit circle, and suppose that $X(z)$ has a k th-order zero at $z=z_0$ where $0 < |z_0| < 1$. Since $X(z)$ has no conjugate reciprocal pole/zero pairs, then $X(z)$ does not have a pole at $z=1/z_0$ and, consequently,

$$R_x(z) = Y(z) Y(z^{-1}) \quad (3-58)$$

must have a k th-order zero at $z=z_0$. However, since $|z_0^{-1}| > 1$ and since $Y(z)$ has no zeros outside the unit circle, then $Y(1/z_0) \neq 0$ and $Y(z)$ must have at least k zeros at $z=z_0$. Finally, since $Y(z)$ has no conjugate reciprocal pole/zero pairs, then $Y(z)$ must have exactly k zeros at $z=z_0$. Reversing the roles of $X(z)$ and $Y(z)$, it follows that $X(z)$ and $Y(z)$ have the same zero set for $0 < |z| < \infty$. By a similar argument, the same result holds for the case in which the zeros of $X(z)$ and $Y(z)$ are outside the unit circle.

Finally, repeating the argument for poles, it follows that the poles of $X(z)$ and $Y(z)$ are identical for $0 < |z| < \infty$. Thus,

$$Y(z) = \beta z^k X(z) \quad (3-59)$$

where β is a complex number and k is an integer. However, since $|X(\omega)| = |Y(\omega)|$, it follows that $|\beta| = 1$ which, since $x(n)$ and $y(n)$ are real, implies that $\beta = \pm 1$. Therefore, $y(n) = \pm x(n+k)$ as desired. /////

It should be pointed out that this theorem is valid, as well, for complex-valued sequences. Furthermore, the only change required in the proof in this case is to use (3-52) for the autocorrelation of a complex-valued sequence.

Finally, it may be noted that there are other classes of sequences which are uniquely defined by the magnitude of their Fourier transforms. For example, suppose that $x(n)$ and $y(n)$ are even sequences with $|X(\omega)| = |Y(\omega)|$. From (3-57) and the fact that $X(z) = X(z^{-1})$ and $Y(z) = Y(z^{-1})$, it follows that

$$X^2(z) = Y^2(z) \quad (3-60)$$

Therefore, $X(z) = \pm Y(z)$ and, consequently, $x(n) = \pm y(n)$. In other words, an even sequence is uniquely defined to within a sign by the magnitude of its Fourier transform. As another example, suppose that $x(n)$ is a real-valued finite length sequence which is

equal to zero outside the interval $[0, N-1]$ with $x(0) \neq 0$. In this case, since $X(z)$ is a polynomial in z^{-1} over the real numbers, it may be shown (See Section IV.5.2) that if $X(z)$ is irreducible and if $y(n)$ is any finite length sequence with $|Y(\omega)| = |X(\omega)|$, then either $y(n) = \pm x(n)$ or $y(n) = \pm x(-n)$. However, due to the Fundamental Theorem of Algebra [30], no polynomial of degree greater than two is irreducible over the real numbers. Therefore, this result constrains $x(n)$ to be of length three or less.

Although these constraints do not encompass a very large or useful class of one-dimensional sequences, they are special cases of a set of constraints under which a multidimensional sequence is uniquely defined by the magnitude of its Fourier transform (Theorem 4.9). Furthermore, unlike the one-dimensional case, many multidimensional sequences of practical interest satisfy these constraints.

CHAPTER IV: UNIQUENESS CONSTRAINTS (m-D)

IV.1: Introduction

In Chapter III, some conditions were presented under which a one-dimensional sequence is uniquely defined in terms of either the phase or magnitude of its Fourier transform. In this chapter, the uniqueness question is considered for sequences in two or more dimensions. First, however, some notation and terminology related to multidimensional sequences is presented in Section IV.2. Then, in Section IV.3 the results of Chapter III are used to develop some conditions under which a multidimensional sequence is uniquely defined in terms of either the phase or magnitude of its Fourier transform. Specifically, it is shown that a multidimensional sequence may be mapped into a one-dimensional sequence by means of an invertible transformation. Consequently, the one-dimensional uniqueness constraints may then be applied to this one-dimensional sequence. Although this approach is straightforward, it is limited by the fact that the uniqueness constraints are expressed in terms of a one-dimensional function of a multidimensional sequence rather than directly in terms of the properties of the multidimensional sequence. Therefore, another approach is presented in Section IV.5 which leads to uniqueness constraints which involve restrictions of the types of factors which are allowed in the multidimensional z-transform of a multidimensional sequence. However, since this approach requires some results from the algebra of polynomials in two or more variables, Section IV.4 is intended to provide the necessary background. Finally, Section IV.6 presents some further extensions of the theory developed in Section IV.5.

IV.2: Notation and framework

An m -dimensional sequence is a function of m integer-valued variables, n_1 through n_m , which will be denoted by $x(n_1, \dots, n_m)$. As in Chapter III, all sequences are assumed to be real-valued with rational z -transforms. Therefore, with $X(z_1, \dots, z_m)$ the m -dimensional z -transform of $x(n_1, \dots, n_m)$, then

$$X(z_1, \dots, z_m) = \sum_{n_1} \dots \sum_{n_m} x(n_1, \dots, n_m) z_1^{-n_1} \dots z_m^{-n_m} = \frac{A(z_1, \dots, z_m)}{B(z_1, \dots, z_m)} \quad (4-1)$$

where $A(z_1, \dots, z_m)$ and $B(z_1, \dots, z_m)$ are polynomials.

In order to express (4-1) as well as some later results more succinctly, vector notation will be used whenever possible. For example, an m -dimensional sequence and its z -transform will be written as

$$x(\underline{n}) = x(n_1, \dots, n_m) \quad (4-2 \text{ a})$$

$$X(\underline{z}) = X(z_1, \dots, z_m) \quad (4-2 \text{ b})$$

In addition, with $\underline{n}=(n_1, \dots, n_m)$ an integer-valued vector, $\underline{z}^{\underline{n}}$ will be defined by

$$\underline{z}^{\underline{n}} = z_1^{n_1} z_2^{n_2} \dots z_m^{n_m} \quad (4-3)$$

Thus, (4-1) may be written more succinctly as

$$X(\underline{z}) = \sum_{\underline{n}} x(\underline{n}) \underline{z}^{-\underline{n}} = \frac{A(\underline{z})}{B(\underline{z})} \quad (4-4)$$

It will always be assumed that the region of convergence of $X(\underline{z})$ includes the unit polydisc: $|z_k|=1$ for $k=1, \dots, m$. In this case, the Fourier transform of $x(\underline{n})$ exists and is given by

$$X(\omega) = X(z)|_{z=e^{j\omega}} = \sum_n x(n) \exp[-jn\omega] \quad (4-5)$$

Since $X(\omega)$ is, in general, a complex-valued function of ω , it may be written in polar form in terms of its magnitude and phase as:

$$X(\omega) = |X(\omega)| \exp[j\phi_x(\omega)] \quad (4-6)$$

Finally, with $\underline{M}=(M_1, \dots, M_m)$, the \underline{M} -point Discrete Fourier Transform (DFT) of $x(\underline{n})$ will be denoted by $X(\underline{k})_{\underline{M}}$. The magnitude and phase of $X(\underline{k})_{\underline{M}}$ will be denoted by $|X(\underline{k})_{\underline{M}}|$ and $\phi_x(\underline{k})_{\underline{M}}$, respectively.

As in the one-dimensional case, many of the sequences considered in this chapter are assumed to have finite support, i.e. $x(\underline{n})$ is non-zero only for finitely many values of \underline{n} . For convenience, any sequence with finite support is assumed, without any loss in generality, to be non-zero only when $0 \leq \underline{n}^\dagger$. In the general case, any sequence may simply be shifted in order to satisfy this assumption. If $x(\underline{n})$ is zero outside the region $0 \leq \underline{n} < \underline{N}$, i.e. $x(n_1, \dots, n_m) = 0$ whenever $n_k < 0$ or $n_k \geq N_k$ for $k=1, \dots, m$, then the region of support will be denoted by $R(\underline{N})=R(N_1, \dots, N_m)$. Furthermore, $F(\underline{n})$ will be used to denote the set of all m -D sequences which have, for some \underline{N} , support $R(\underline{N})$. Thus, $x \in F(\underline{n})$ will be taken to mean that $x(\underline{n})$ is an m -D sequence with finite support which is non-zero only when $0 \leq \underline{n}$.

[†] If \underline{k} and \underline{n} are two vectors, then $\underline{k} < \underline{n}$ means that $k_i < n_i$ for $i=1, \dots, m$.

IV.3: Uniqueness via projections

In this section, the uniqueness constraints in Chapter III are used to provide some conditions under which a multidimensional sequence is uniquely defined in terms of the phase or magnitude of its Fourier transform. Specifically, a multidimensional sequence is first mapped into a 1-D (one-dimensional) sequence by means of an invertible transformation. This transformation has the property that the Fourier transform phase or magnitude of this one-dimensional sequence may be easily determined from the phase or magnitude of the multidimensional Fourier transform of the original sequence. Therefore, the 1-D uniqueness constraints in Chapter III provide conditions under which this one-dimensional sequence, and hence the multidimensional sequence, is uniquely specified in terms of the phase or magnitude of its Fourier transform. Since the 2-D (two-dimensional) results are easily extended to sequences of higher dimension, the following discussion will focus only on the 2-D case.

Consider first the case of a 2-D sequence with finite support. Specifically, suppose that $x(n_1, n_2)$ has support $R(N_1, N_2)$, i.e., $x(n_1, n_2)$ is zero outside the region $0 \leq n_1 < N_1$ and $0 \leq n_2 < N_2$, and let $\hat{x}_1(n)$ be the 1-D sequence which is defined in terms of $x(n_1, n_2)$ by the following mapping:

$$\hat{x}_1(Nn_1 + n_2) = x(n_1, n_2) \quad \text{with } N \geq N_2 \text{ and } 0 \leq n_2 < N \quad (4-7)$$

When $N = N_2$, the sequence $\hat{x}_1(n)$ in (4-7) is defined simply as the concatenation of the columns of the array $x(n_1, n_2)$. If, on the other hand, $N > N_2$, then each column of $x(n_1, n_2)$ is padded with $(N - N_2)$ zeros before concatenation. Clearly, the mapping (4-7) is invertible for any $N \geq N_2$.

With $X(z_1, z_2)$ the two-dimensional z-transform of $x(n_1, n_2)$ and with $\hat{X}_1(z)$ the z-transform of $\hat{x}_1(n)$, then

$$\begin{aligned}
\hat{X}_1(z) &= \sum_{\mathbf{k}} \hat{x}_1(\mathbf{k}) z^{-\mathbf{k}} = \sum_{n_1} \sum_{n_2} \hat{x}_1(Nn_1+n_2) z^{-(Nn_1+n_2)} \\
&= \sum_{n_1} \sum_{n_2} x(n_1, n_2) z^{-Nn_1} z^{-n_2} = X(z^N, z)
\end{aligned} \tag{4-8}$$

Therefore, $\hat{X}_1(z)$ equals the z -transform of $x(n_1, n_2)$ along the contour $z_1=z_2^N$. The sequence $\hat{x}_1(n)$ is referred to as a *projection* of $x(n_1, n_2)$ and $\hat{X}_1(z)$ is referred to as a *slice* of $X(z_1, z_2)$ [27].

With $z=\exp(j\omega)$, the Fourier transform of $\hat{x}_1(n)$ is

$$\hat{X}_1(\omega) = X(N\omega, \omega) \tag{4-9}$$

Therefore, $X_1(\omega)$ is equal to $X(\omega_1, \omega_2)$ along the line $\omega_1=N\omega_2$. However, since $X(\omega_1, \omega_2)$ is periodic with period 2π in both ω_1 and ω_2 , $\hat{X}_1(\omega)$ is also equal to $X(\omega_1, \omega_2)$ along a series of N parallel lines in the ω_1, ω_2 -plane, each of which makes an angle $\theta=\tan^{-1}(1/N)$ with the ω_1 -axis as shown in Figure 4.1.

Note that (4-7) is not the only invertible transformation which maps a 2-D sequence with finite support into a 1-D sequence. For example, the mapping

$$\hat{x}_2(n_1+Nn_2) = x(n_1, n_2) \quad \text{with } N \geq N_1 \text{ and } 0 \leq n_1 < N \tag{4-10}$$

is also invertible and represents the concatenation of the rows of $x(n_1, n_2)$ which have been padded with $(N-N_1)$ zeros. The Fourier and z -transforms of $\hat{x}_2(n)$ are given by

$$\hat{X}_2(\omega) = X(\omega, N\omega) \quad : \quad \hat{X}_2(z) = X(z, z^N) \tag{4-11}$$

Since $x(n_1, n_2)$ has finite support, both $\hat{x}_1(n)$ and $\hat{x}_2(n)$ are finite length sequences. For example, if $N=N_2$ in (4-7) or if $N=N_1$ in (4-10), then $\hat{x}_1(n)$ and $\hat{x}_2(n)$ are equal to zero outside the interval $[0, M-1]$ where $M=N_1N_2$. In this case, $\hat{x}_1(n)$ and

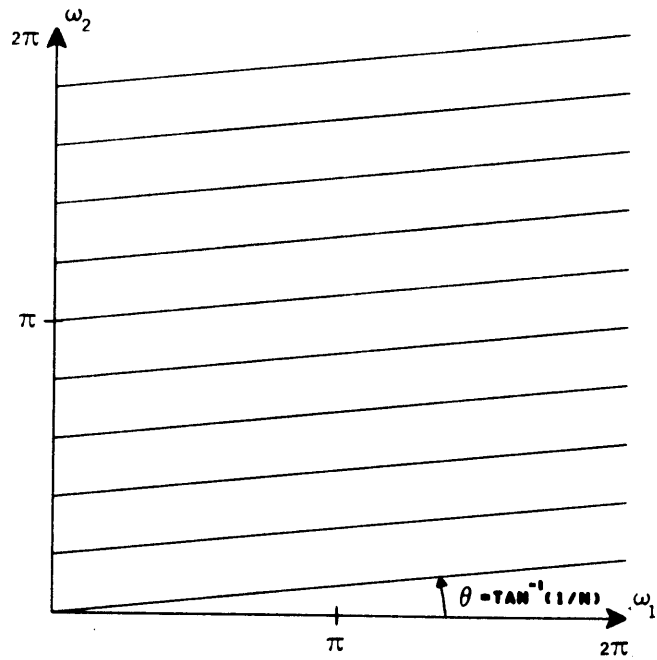


Figure 4.1: Lines in the ω_1, ω_2 -plane along which $X(\omega_1, \omega_2)$ and $\hat{X}_1(\omega)$ are equal.

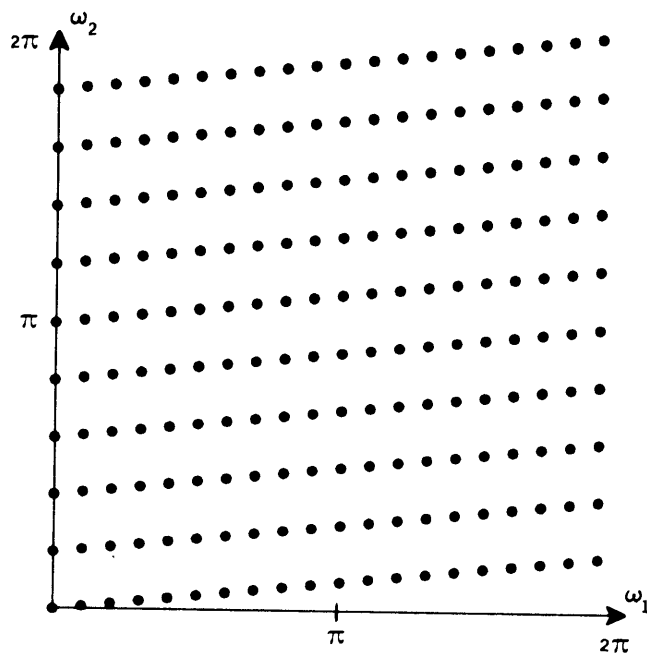


Figure 4.2: Samples in the ω_1, ω_2 -plane which correspond to the DFT of $\hat{x}_1(n)$.

$\hat{x}_2(n)$ are uniquely defined in terms of their one-dimensional M -point DFT's which consist of M equally spaced samples of $\hat{X}_1(\omega)$ and $\hat{X}_2(\omega)$ between 0 and 2π , respectively. In the 2-D Fourier plane, these samples correspond to equally spaced samples along the lines $\omega_1 = N\omega_2$ and $\omega_2 = N\omega_1$, respectively, as shown in Figure 4.2.

Consider now the more general case of a 2-D sequence with a rational z -transform. Specifically, let

$$X(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)} \quad (4-12)$$

where $A(z_1, z_2)$ is a 2-D polynomial of degree N_1 in z_1 and N_2 in z_2 and where $B(z_1, z_2)$ is a 2-D polynomial of degree M_1 in z_1 and M_2 in z_2 . Consider the sequence $\hat{x}_1(n)$ which is defined to have the following z -transform

$$\hat{X}_1(z) = X(z^N, z) \quad (4-13)$$

Since

$$\begin{aligned} X(z^N, z) &= \sum_i \sum_k x(k, i) z^{-Nk-i} \\ &= \sum_n \sum_k x(k, n-Nk) z^{-n} \end{aligned} \quad (4-14)$$

then $\hat{x}_1(n)$ is defined in terms of $x(n_1, n_2)$ by

$$\hat{x}_1(n) = \sum_k x(k, n-Nk) \quad (4-15)$$

Similarly, it follows that if $\hat{x}_2(n)$ is defined to have a z -transform given by

$$\hat{X}_2(z) = X(z, z^M) \quad (4-16)$$

then $\hat{x}_2(n)$ is defined in terms of $x(n_1, n_2)$ by

$$\hat{x}_2(n) = \sum_k x(n-Mk, k) \quad (4-17)$$

Note that if $x(n_1, n_2)$ has support $R(N_1, N_2)$ then (4-15) is equivalent to (4-7) provided $N \geq N_2$ and (4-17) is equivalent to (4-10) provided $M \geq N_1$.

Finally, it may be shown [27] that for $X(z_1, z_2)$ a rational function of the form (4-12), the transformation (4-15) is invertible for any $N \geq \max(M_2, N_2)$. Specifically, note that $\hat{X}_1(z)$ is a rational function of z , i.e.,

$$\hat{X}_1(z) = \frac{A(z^N, z)}{B(z^N, z)} \quad (4-18)$$

Therefore, since the poles and zeros of $\hat{X}_1(z)$ are uniquely defined by $\hat{x}_1(n)$, the numerator and denominator polynomials may (theoretically) be obtained from $\hat{x}_1(n)$. Therefore, since $N \geq M_2$ then $A(z_1, z_2)$ may be recovered from $A(z^N, z)$ and since $N \geq N_2$ then $B(z_1, z_2)$ may be recovered from $B(z^N, z)$ and the result follows. It may similarly be shown that the transformation (4-17) is invertible for any $M \geq \max(M_1, N_1)$.

From (4-9) and (4-11), it is clear that the phase (magnitude) of $\hat{X}_1(\omega)$ and $\hat{X}_2(\omega)$ are specified by the phase (magnitude) of $X(\omega_1, \omega_2)$. Therefore, each of the theorems in Chapter III imply conditions under which a multidimensional sequence is uniquely defined by the phase or magnitude of its Fourier transform. Specifically, a multidimensional sequence may be mapped into a one-dimensional sequence by (4-7) or (4-10) and the 1-D uniqueness theorems in Chapter III may then be applied to the resulting 1-D sequence. From a slightly different point of view, note that the trans-

formations (4-7) and (4-10) may be viewed as invertible mappings of one-dimensional sequences into multidimensional sequences. Note, in addition, that the theorems in Chapter III define certain classes of one-dimensional sequences which are uniquely specified by their Fourier transform phase or magnitude. From these one-dimensional sequences, therefore, the mappings (4-7) and (4-10) may be used to define certain classes of multidimensional sequences which are uniquely specified in terms of a slice of the phase or a slice of the magnitude of their multidimensional Fourier transforms. From either point of view, however, Theorem 3.1 implies the following

Theorem 4.1: Let $x(n_1, n_2)$ and $y(n_1, n_2)$ have finite support and suppose that, for some N , $\hat{x}_1(n)$ and $\hat{y}_1(n)$ as defined in (4-7) have z -transforms with no zeros on the unit circle or in conjugate reciprocal pairs. If $\phi_x(\omega_1, \omega_2) = \phi_y(\omega_1, \omega_2)$ for all ω_1 and ω_2 along the line $\omega_1 = N\omega_2$, then $x(n_1, n_2) = \beta y(n_1, n_2)$ for some positive constant β . If, on the other hand, $\tan[\phi_x(\omega_1, \omega_2)] = \tan[\phi_y(\omega_1, \omega_2)]$ for all ω_1 and ω_2 along the line $\omega_1 = N\omega_2$, then $x(n_1, n_2) = \beta y(n_1, n_2)$ for some real constant β .

Note that this theorem asserts that a 2-D sequence with finite support is uniquely defined to within a scale factor by the phase of its Fourier transform if there exists a projection which satisfies the constraints of Theorem 3.1. Unfortunately, however, although one particular projection may not satisfy these constraints, this does not preclude the possibility that they will be satisfied by another projection.

Example: Consider the 2-D sequence $x(n_1, n_2)$ which, written as an array, is defined by

$$x = \begin{bmatrix} 2 & 2 \\ 5 & 0 \end{bmatrix} \quad (4-19)$$

i.e., $x(0,0)=2$, $x(0,1)=5$, $x(1,0)=2$, and $x(n_1,n_2)=0$ otherwise. Since $\hat{X}_1(z)$ with $N=2$ is given by

$$\hat{X}_1(z) = 2 + 5z^{-1} + 2z^{-2} = (z^{-1}+2)(2z^{-1}+1) \quad (4-20)$$

and, therefore, has a pair of reciprocal zeros, $\hat{x}_1(n)$ is not uniquely defined by the phase of its Fourier transform for this particular projection. In other words, knowledge of $\phi_x(\omega_1, \omega_2)$ along the line $\omega_1=2\omega_2$ is not sufficient to uniquely define $x(n_1, n_2)$. In fact, if $y(n_1, n_2)$ is any sequence of the form

$$y = \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_2 & 0 \end{bmatrix} \quad (4-21)$$

where $\alpha_2 \geq \alpha_1$, then the phase of the Fourier transform of $y(n_1, n_2)$ is equal to the phase of the Fourier transform of $x(n_1, n_2)$ along the line $\omega_1=2\omega_2$, i.e., $\phi_y(2\omega, \omega) = \phi_x(2\omega, \omega)$. However, if $N > 2$ then the z -transform of $\hat{x}_1(n)$ is equal to

$$\hat{X}_1(z) = 2 + 5z^{-1} + 2z^{-N} \quad (4-22)$$

and has no zeros on the unit circle or in reciprocal pairs. Therefore, $x(n_1, n_2)$ is uniquely defined to within a scale factor by the phase of its Fourier transform along the line $\omega_1=N\omega_2$ for any $N > 2$. Finally, suppose that $N \geq 2$ and consider the sequence $\hat{x}_2(n)$. Since

$$\hat{X}_2(z) = 2 + 2z^{-1} + 5z^{-N} \quad (4-23)$$

has no zeros on the unit circle or in reciprocal pairs, then $\phi_x(\omega_1, \omega_2)$ along the line $\omega_2=N\omega_1$ is also sufficient to uniquely specify $x(n_1, n_2)$ to within a scale factor provided $N \geq 2$.

This approach of transforming multidimensional sequences into their 1-D projections provides only a partial answer to the uniqueness question. Specifically, with this approach it is difficult to determine which multidimensional sequences are uniquely specified by their Fourier transform phase or magnitude. Therefore, it is the goal of the remainder of this chapter to develop some constraints which are expressed directly in terms of the multidimensional sequence or its multidimensional z-transform.

IV.4: Polynomials in two or more variables

In order to consider the uniqueness of a multidimensional sequence in terms of its multidimensional Fourier transform phase or magnitude, some theory from the algebra of polynomials in two or more variables is required. This section, therefore, is intended to provide the necessary background. Although only those results which are needed in the following sections are presented, a more detailed treatment may be found in [30].

IV.4.1: Definitions

A *polynomial* in the m variables $\underline{z}=(z_1, z_2, \dots, z_m)$ is a function of the form:

$$p(\underline{z}) = p(z_1, \dots, z_m) = \sum_{k_1 + \dots + k_m \leq N} \dots \sum c(k_1, \dots, k_m) z_1^{k_1} \dots z_m^{k_m} \quad (4-24)$$

where k_1, k_2, \dots, k_m are non-negative integers, and where $c(k_1, \dots, k_m)$ are arbitrary numbers which are referred to as the *coefficients* of the polynomial. Each term in the sum in (4-24) is called a *monomial*. Thus, monomials are functions of the form:

$$f(\underline{z}) = f(z_1, \dots, z_m) = c(k_1, \dots, k_m) z_1^{k_1} \cdots z_m^{k_m} \quad (4-25)$$

The *degree* of the monomial in (4-25) is defined to be

$$d(f) = k_1 + k_2 + \cdots + k_m \quad (4-26)$$

The degree, $d(p)$, of the polynomial (4-24) is therefore defined to be equal to the degree of the monomial which has the largest degree and a non-zero coefficient. Although not standard terminology, polynomials which consist of a sum of two or more monomials will be referred to as *non-trivial* polynomials. Monomials are therefore defined to be *trivial* polynomials.

It is often useful to consider a polynomial in m variables as a function of one variable, say z_k , which has coefficients which are polynomials in the remaining $(m-1)$ variables. For example, $p(\underline{z})$ in (4-24) may be written as

$$p(\underline{z}) = \sum_{n=0}^N \xi_k(n) z_k^n \quad (4-27)$$

where $\xi_k(n)$ for $n=0,1,\dots,N$ are polynomials in the $(m-1)$ variables z_i for $i \neq k$. In this form, the largest value of n for which $\xi_k(n)$ is non-zero is referred to as the degree of $p(\underline{z})$ with respect to the variable z_k . Therefore, $p(\underline{z})$ will be defined to be of degree $\underline{N}=(N_1, \dots, N_m)$ in $\underline{z}=(z_1, \dots, z_m)$ if $p(\underline{z})$ is a polynomial of degree N_k with respect to the variable z_k .

Example: Consider the following polynomial in two variables

$$p(z_1, z_2) = 1 + 2z_1 + z_1 z_2 + 5z_1^2 z_2 \quad (4-28)$$

which has degree $N=(2,1)$ in $z=(z_1, z_2)$. Since $(5z_1^2z_2)$ is the monomial with the largest degree, then $d(p)=3$. Written as a polynomial in z_1 with coefficients which are polynomials in z_2 , then

$$p(z_1, z_2) = \xi_1(0) + \xi_1(1)z_1 + \xi_1(2)z_1^2 \quad (4-29)$$

where $\xi_1(0)=1$, $\xi_1(1)=(2+z_2)$, and $\xi_1(2)=5z_2$.

If all of the coefficients of a polynomial $p(z)$ belong to a particular number field, \mathcal{F} , then $p(z)$ is called a polynomial over \mathcal{F} . The set of all polynomials in m variables over \mathcal{F} will be denoted by $\mathcal{F}(z)$. If two polynomials $p_1(z)$ and $p_2(z)$ in $\mathcal{F}(z)$ are equal to within a factor of zero degree, i.e., $p_1(z)=cp_2(z)$ where $c \in \mathcal{F}$ and $c \neq 0$, then $p_1(z)$ and $p_2(z)$ are called *associated* polynomials. A polynomial $p \in \mathcal{F}(z)$ with $d(p) > 0$ is called a *reducible* polynomial over \mathcal{F} if there are polynomials $p_1, p_2 \in \mathcal{F}(z)$ with $d(p_1) > 0$ and $d(p_2) > 0$ such that $p(z)=p_1(z)p_2(z)$. If no such decomposition is possible, then $p(z)$ is called an *irreducible* polynomial. It may be noted that a polynomial which is irreducible over one field may not be irreducible over another field. For example, although the polynomial $p(z_1, z_2)=z_1^2+z_2^2$ is irreducible over the field of real numbers, over the field of complex numbers $p(z_1, z_2)$ is reducible.

IV.4.2: The factorization of polynomials

The result of interest in this section is the fact that any polynomial of non-zero degree may always be uniquely decomposed to within factors of zero degree into a product of irreducible polynomials. Specifically,

Theorem 4.2 [30, p.335]: Any polynomial $p \in \mathcal{F}(z)$ having non-zero degree can be expressed as a product of factors irreducible in \mathcal{F} . Furthermore, if $p(z)$ has two different factorizations:

$$p(z) = f_1(z)f_2(z) \cdots f_m(z) = g_1(z)g_2(z) \cdots g_n(z) \quad (4-30)$$

then $m=n$ and the factors $f_i(z)$ and $g_i(z)$ can be ordered in such a way that the factors are associated.

It may be noted that the Fundamental Theorem of Algebra states that a polynomial in one variable of degree two or more is always reducible over the field of complex numbers and, therefore, may always be written as a product of first order polynomials. This is not the case, however, for polynomials in two or more variables. In particular, a polynomial in two or more variables of arbitrarily large degree may be irreducible.

Example: If $q(x)$ is a polynomial in $m > 0$ variables over the field of complex numbers, then the polynomial $p(x,y)$ in $m+1$ variables defined by

$$p(x,y) = q(x) + y^k \quad (4-31)$$

is always irreducible for any $k > 0$.

Since polynomials in two or more variables of arbitrarily large degree may be irreducible, it is of interest to determine the probability with which a multivariable polynomial is irreducible. More specifically, given an arbitrary polynomial in two or more variables, is it more likely that it is reducible or that it is irreducible? An answer to this question may be found in Appendix I where it is shown that "almost all" polynomials in two or more variables are irreducible. In a probabilistic setting, this result asserts that a multidimensional polynomial is irreducible with probability one.

IV.4.3: Uniqueness of Polynomials Over a Lattice

It is well known that a polynomial $p(z)$ in one variable of degree N is uniquely defined in terms of its values over a set $A = \{a_1, \dots, a_{N+1}\}$ of $N+1$ distinct points and may be reconstructed from these points by, for example, the Lagrange or Newton interpolation formulas [30]. This result may be extended to polynomials in m variables if the set of points A is replaced with an m -dimensional lattice of points, $L(A_1, \dots, A_m)$. More specifically, let A_k be a set of N_k distinct points in the field \mathcal{F} for $k=1, \dots, m$. The m -dimensional lattice $L(A_1, \dots, A_m)$ is then defined as the m -fold Cartesian product of these m sets of points, i.e.,

$$L(A_1, \dots, A_m) = \prod_{k=1}^m A_k = A_1 \times A_2 \times \dots \times A_m \quad (4-32)$$

With $\underline{A} = \{A_1, \dots, A_m\}$ a set of m sets of points, the lattice (4-32) will be denoted by $L(\underline{A})$. However, if all of the sets A_1, \dots, A_m are the same, the lattice will be written as $L(A^m)$. The result of interest is therefore the following:

Theorem 4.3 [30, p.329]: Suppose $p_1, p_2 \in \mathcal{F}(z)$ are polynomials of degree at most $N-1$. If A is a set of N distinct numbers in the field \mathcal{F} and

$$p_1(\underline{z}) = p_2(\underline{z}) \quad \text{for all } \underline{z} \in L(A^m) \quad (4-33)$$

then $p_1(\underline{z}) = p_2(\underline{z})$ for all \underline{z} .

A slightly different form of this result may be derived which will be of interest in the following discussions. Specifically,

Theorem 4.4: Suppose $p_1, p_2 \in \mathcal{F}(z)$ are polynomials of degree at most $N_k - 1$ in the variable z_k for $k=1, \dots, m$. If, for each k , A_k is a set of N_k distinct numbers in the field \mathcal{F} , and if

$$p_1(\underline{z}) = p_2(\underline{z}) \quad \text{for all } \underline{z} \in L(\underline{A}) \quad (4-34)$$

then $p_1(\underline{z}) = p_2(\underline{z})$ for all \underline{z} .

As an application of Theorem 4.4, note that if $x(n)$ is an m -dimensional sequence with support $R(\underline{N})$, then its z -transform is a polynomial in \underline{z}^{-1} . Therefore, Theorem 4.4 may be used to show that $x(n)$ is uniquely specified by the values of its z -transform over a lattice of points. One such statement of this fact is provided in the following

Example: Suppose $x, y \in F(n)$ have support $R(\underline{N})$ and let A_k be a set of N_k distinct complex numbers. If

$$X(\underline{z})|_{L(\underline{A})} = Y(\underline{z})|_{L(\underline{A})} \quad (4-35)$$

then $x(n) = y(n)$ for all n .

Note that if the elements of the sets A_k are complex numbers with unit magnitude, then $X(\underline{z})|_{L(\underline{A})}$ in (4-35) is equal to the Fourier transform, $X(\underline{\omega})$, evaluated over a lattice in the $\underline{\omega}$ -plane. Specifically, consider the sets:

$$\Omega_k = \{ \beta_{k,i} \} \quad \text{with } 0 \leq \beta_{k,i} < 2\pi \quad \text{for } i=1, 2, \dots, N_k \quad (4-36 \text{ a})$$

and

$$A_k = \{ \exp(j\beta_{k,i}) \} \quad (4-36 \text{ b})$$

where the elements of Ω_k for $k=1,\dots,m$ are assumed to be distinct. Then,

$$X(\underline{z})|_{L(\underline{\Lambda})} = X(\underline{\omega})|_{L(\underline{\Omega})} \quad (4-37)$$

is equal to the Fourier transform of $x(\underline{n})$ over the lattice $L(\underline{\Omega})$ in the $\underline{\omega}$ -plane. Therefore, it also follows from Theorem 4.4 that a multidimensional sequence with support $R(\underline{N})$ is uniquely defined by the values of its Fourier Transform over a lattice in the $\underline{\omega}$ -plane:

Example: Suppose $x, y \in F(\underline{n})$ have support $R(\underline{N})$ and let Ω_k be a set of N_k distinct real numbers in the interval $[0, 2\pi)$ for $k=1, 2, \dots, m$. If

$$X(\underline{\omega})|_{L(\underline{\Omega})} = Y(\underline{\omega})|_{L(\underline{\Omega})} \quad (4-38)$$

then $x(\underline{n})=y(\underline{n})$ for all \underline{n} .

Finally, note that if the numbers $\beta_{k,i}$ in (4-36) are equally spaced between zero and 2π , i.e., $\beta_{k,i}=2\pi i/N_k$, then

$$X(\underline{\omega})|_{L(\underline{\Omega})} = X(\underline{k})_{\underline{M}} \quad (4-39)$$

is the \underline{M} -point DFT of $x(\underline{n})$.

IV.5: Multidimensional uniqueness constraints

In this section, the results presented in Section IV.4 are used to develop conditions under which a multidimensional sequence is uniquely defined by the phase or magnitude of its Fourier transform. In Chapter III, the uniqueness constraints were geometrically motivated in terms of the pole/zero plots of one-dimensional z -transforms. However, since the z -transform of a multidimensional sequence is, in general, a function of two or more independent variables, it is not possible to represent a multidimensional z -transform in terms of a pole/zero plot as in the 1-D case. For example, a 2-D sequence with support $R(N_1, N_2)$ has a z -transform, $X(z_1, z_2)$, which is a polynomial in two complex variables, z_1 and z_2 . Therefore, the zero set of $X(z_1, z_2)$ is given by a continuum of values and represents a contour in the z_1, z_2 -plane. Consequently, a certain degree of abstraction is necessary in order to provide a similar motivation for the multidimensional uniqueness constraints.

The multidimensional equivalent of the poles and zeros of a 1-D z -transform are the zero contours of the irreducible factors of a multidimensional z -transform. Specifically, note that the z -transform of an arbitrary one-dimensional or multidimensional sequence with finite support may always be written as

$$X(\underline{z}) = \alpha \underline{z}^{\underline{n}_0} \prod_{k=1}^p X_k(\underline{z}) \quad (4-40)$$

where α is a real number, \underline{n}_0 is an integer-valued vector, and where $X_k(\underline{z})$ for $k=1, \dots, p$ are non-trivial irreducible polynomials[†]. For one-dimensional sequences, the irreducible polynomials are linear factors of the form $(1-a_k z^{-1})$. Although the irreducible fac-

[†] It should be kept in mind that although $X(\underline{z})$ is used to denote the z -transform of a sequence $x \in F(n)$, $X(\underline{z})$ is in fact a polynomial in \underline{z}^{-1} . Therefore, in the following discussions, whenever $X(\underline{z})$ is said to be a polynomial this should be taken to mean that $X(\underline{z})$ is a polynomial in \underline{z}^{-1} .

tors in (4-40) need not be linear for sequences in two or more dimensions, these factors play a role analogous to that played by the poles and zeros of one-dimensional z-transforms.

IV.5.1: Uniqueness in terms of phase

As in the one-dimensional case, without any additional information or constraints, a multidimensional sequence is not uniquely specified by the phase of its Fourier transform since convolution with a zero phase sequence produces another sequence with the same phase. For one-dimensional sequences, this difficulty was overcome by constraining a sequence to have no zeros on the unit circle or in conjugate reciprocal pairs. By imposing a similar constraint on a multidimensional sequence, the uniqueness theorems in Chapter III may be extended to the multidimensional case. This constraint involves the idea of a symmetric z-transform[†] which is defined as follows. The z-transform of a sequence $x \in F(n)$ is a polynomial in \underline{z}^{-1} and will be defined to be *symmetric* if, for some vector, \underline{k} , of positive integers,

$$X(\underline{z}) = \pm \underline{z}^{\underline{k}} X(\underline{z}^{-1}) \quad (4-41)$$

Note that if $X(\underline{z})$ has no trivial factors and is of degree \underline{N} in \underline{z}^{-1} , then

$$\tilde{X}(\underline{z}) \equiv \underline{z}^{\underline{N}} X(\underline{z}^{-1}) \quad (4-42)$$

is also a polynomial of degree \underline{N} in \underline{z}^{-1} which has no trivial factors. Therefore, it follows that $X(\underline{z})$ will be symmetric if it has no trivial factors and $X(\underline{z}) = \pm \tilde{X}(\underline{z})$. Finally, it should be pointed out that a symmetric z-transform may be reducible, irreducible,

[†] A symmetric z-transform should not be confused with the algebraic definition of symmetric polynomials [30].

trivial, or non-trivial.

Example: The sequence $x(\underline{n})$ defined by

$$\mathbf{x} = \begin{bmatrix} a & b \\ \pm b & \pm a \end{bmatrix} \quad (4-43)$$

has a symmetric z -transform which is irreducible if $a \neq b$. Also, for any polynomial $A(\underline{z})$,

$$\mathbf{X}(\underline{z}) = A(\underline{z}) \cdot \tilde{A}(\underline{z}) \quad (4-44)$$

is a reducible symmetric z -transform. Finally, if $\mathbf{X}(\underline{z})$ is a trivial polynomial, i.e.,

$$\mathbf{X}(\underline{z}) = \beta \underline{z}^{\underline{k}} \quad (4-45)$$

then $\mathbf{X}(\underline{z})$ is a trivial symmetric z -transform.

It should be noted that a 1-D sequence which has all of its zeros on the unit circle or in conjugate reciprocal pairs has a z -transform which is symmetric. Therefore, (4-42) represents an extension of this property to multidimensional sequences. It should also be noted that, except for a linear phase term, the Fourier transform of a sequence which has a symmetric z -transform is either purely real or purely imaginary.

IV.5.1.1: Uniqueness in terms of continuous phase

It was shown in Section III.3 that a one-dimensional sequence $x(n)$ is uniquely specified to within a scale factor by the phase or the tangent of the phase of its Fourier transform if $X(z)$ has no zeros on the unit circle or in reciprocal pairs, i.e., if $X(z)$ contains no non-trivial symmetric factors. This result may be directly extended to the case of multidimensional sequences as follows:

Theorem 4.5: Let $x, y \in F(n)$. If $X(z)$ and $Y(z)$ have no non-trivial symmetric factors and $\phi_x(\omega) = \phi_y(\omega)$ for all ω , then $x(n) = \beta y(n)$ for some positive real number β . If $\tan[\phi_x(\omega)] = \tan[\phi_y(\omega)]$ for all ω , then $x(n) = \beta y(n)$ for some real number β .

Note that the theorem hypothesis excludes only *non-trivial* symmetric factors. Therefore, $X(z)$ and $Y(z)$ may have trivial (linear-phase) factors. It should be emphasized, however, that the non-trivial symmetric factors which are excluded from $X(z)$ and $Y(z)$ need not be irreducible. For example, if $X(z) = P(z)Q(z)$ with $P(z) = A(z)\tilde{A}(z)$, then $x(n)$ does not satisfy the constraints of the theorem since $P(z)$ is a (reducible) symmetric factor of $X(z)$. In effect, the exclusion of symmetric factors from $X(z)$ is equivalent to the constraint that if $A(z)$ is an irreducible factor of $X(z)$ then $\tilde{A}(z)$ is not a factor of $X(z)$. A proof of Theorem 4.5 is as follows:

Proof: Let $x, y \in F(n)$ and let \underline{N} be an integer-valued vector which is large enough so that both $x(n)$ and $y(n)$ are equal to zero outside $R(\underline{N})$. Now consider the sequence $g(n)$ defined by

$$g(n) = x(n) * y(-n) \tag{4-46}$$

which has a z-transform

$$G(z) = X(z) Y(z^{-1}) \quad (4-47)$$

Since the phase of the Fourier transform of $g(n)$ is equal to

$$\phi_g(\omega) = \phi_x(\omega) - \phi_y(\omega) \quad (4-48)$$

it follows that if $\phi_x(\omega) = \phi_y(\omega)$ then $\phi_g(\omega) = 0$ or if $\tan[\phi_x(\omega)] = \tan[\phi_y(\omega)]$ then $\tan[\phi_g(\omega)] = 0$.

In either case, the Fourier transform of $g(n)$ is real which implies that

$$G(z) = G(z^{-1}) \quad (4-49)$$

Therefore, from (4-47) and (4-49)

$$X(z) Y(z^{-1}) = X(z^{-1}) Y(z) \quad (4-50)$$

Multiplying both sides of (4-50) by z^{-n} results in the following polynomial equation in z^{-1} :

$$X(z) \tilde{Y}(z) z^{-m} = \tilde{X}(z) Y(z) z^{-n} \quad (4-51)$$

where \underline{m} and \underline{n} are integer-valued vectors with $\underline{m} \geq \underline{0}$ and $\underline{n} \geq \underline{0}$. Now consider an arbitrary non-trivial irreducible factor $X_k(z)$ of $X(z)$. From Theorem 4.2 in Section IV.4.2, it follows that $X_k(z)$ must be associated either with a factor of $\tilde{X}(z)$ or with a factor of $Y(z)$. However, if $X_k(z)$ is associated with a factor of $\tilde{X}(z)$ then

$$X_k(z) = \alpha \tilde{X}_i(z) \quad (4-52)$$

for some i . If $i=k$, then (4-52) implies that

$$X_k(z) = \alpha^2 X_k(z) \quad (4-53)$$

Therefore, $\alpha = \pm 1$ and $X_k(z)$ is symmetric. If, on the other hand, $i \neq k$, then

$$X_k(z) X_1(z) = \alpha \tilde{X}_1(z) X_1(z) \quad (4-54)$$

and $A(z) = \tilde{X}_1(z) X_1(z)$ is a symmetric factor of $X(z)$. Both cases, however, are excluded by the theorem hypothesis. Consequently, each non-trivial irreducible factor of $X(z)$ must be associated with a factor of $Y(z)$. In addition, however, it follows from (4-51) and Theorem 4.2 that any non-trivial irreducible factor $Y_k(z)$ of $Y(z)$ must be associated either with a factor of $X(z)$ or with a factor of $\tilde{Y}(z)$. Furthermore, since $Y(z)$ contains no non-trivial symmetric factors, it follows that each non-trivial irreducible factor of $Y(z)$ must be associated with a factor of $X(z)$. Therefore, $X(z)$ and $Y(z)$ may differ by at most a trivial factor, i.e.,

$$Y(z) = \beta z^k X(z) \quad (4-55)$$

However, if the phase or the tangent of the phase of $x(n)$ and $y(n)$ are equal, then $k=0$. The theorem then follows by noting that β must be positive if $\phi_x(\omega) = \phi_y(\omega)$. ////

As a special case of this theorem, note that if $x(n)$ has finite support and a z -transform of the form

$$X(z) = z^n \cdot P(z) \quad (4-56)$$

where $P(z)$ is an irreducible non-symmetric polynomial, then $X(z)$ satisfies the constraints of Theorem 4.5. Therefore, since "almost all" polynomials in two or more variables are of the form (4-56) (Appendix I), it follows that "almost all" multidimensional sequences are uniquely defined to within a scale factor by the phase of their Fourier transforms.

IV.5.1.2: Uniqueness in terms of phase samples

Theorem 4.5 provides a set of conditions under which a multidimensional sequence is uniquely defined to within a scale factor by the phase of its Fourier transform. Recall, however, that it was possible to extend the one-dimensional version of this theorem to the case in which the phase is known only over a finite set of points by using the fact that the trigonometric functions form a Chebyshev set. Unfortunately, there are no non-trivial Chebyshev sets of functions in two or more variables [43]. In particular, the trigonometric functions in two or more variables do not form a Chebyshev set. Nevertheless, note that if $X(z)$ and $Y(z)$ have no non-trivial symmetric factors, then the proof of Theorem 4.5 essentially relies on the fact that the Fourier transform of $g(n)$ is real. It is possible, therefore, to show that a sequence with support $R(N)$ is uniquely defined to within a scale factor by the phase of its M -point DFT provided $M \geq 2N-1$. Specifically,

Theorem 4.6: Let $x, y \in F(n)$ with support $R(N)$ and let $M \geq 2N-1$. If $X(z)$ and $Y(z)$ have no non-trivial symmetric factors and $\phi_x(k)_M = \phi_y(k)_M$, then $y(n) = \beta x(n)$ for some positive number β . If $\tan[\phi_x(k)_M] = \tan[\phi_y(k)_M]$, then $y(n) = \beta x(n)$ for some real number β .

Proof: Let $x(n)$ and $y(n)$ satisfy the constraints of the theorem and consider the sequence $g(n) = x(n) * y(-n)$. Since $x(n)$ and $y(n)$ have support $R(N)$, then $g(n)$ is zero outside the region $-N < n < N$. Therefore, if $M \geq 2N-1$ then the M -point DFT of $g(n)$ is equal to the product of the M -point DFT's of $x(n)$ and $y(n)$,

$$G(k)_M = X(k)_M \cdot Y(-k)_M \quad (4-57)$$

Thus, if $\phi_x(\underline{k})_{\underline{M}} = \phi_y(\underline{k})_{\underline{M}}$ or $\tan[\phi_x(\underline{k})_{\underline{M}}] = \tan[\phi_y(\underline{k})_{\underline{M}}]$ then $G(\underline{k})_{\underline{M}}$ must be real and, therefore, the periodic extension of $g(\underline{n})$ must be even. However, since $g(\underline{n})$ is non-zero only for $-\underline{N} < \underline{n} < \underline{N}$, then $g(\underline{n})$ must be even and, consequently, the Fourier transform of $g(\underline{n})$ must be real. Thus, repeating the steps in the proof of Theorem 4.5, the desired result follows. /////

Theorem 4.6 asserts that, within the set of all multidimensional sequences with support $R(\underline{N})$ which have z -transforms with no non-trivial symmetric factors, a multidimensional sequence is uniquely defined to within a scale factor by the phase of its \underline{M} -point DFT when $\underline{M} \geq 2\underline{N} - 1$. In reconstructing a sequence which satisfies the constraints of Theorem 4.6 from $\phi_x(\underline{k})_{\underline{M}}$, however, it is not sufficient to simply find a sequence with support $R(\underline{N})$ and the correct phase since the reconstructed sequence may have non-trivial symmetric factors and, thus, will not represent the correct solution. Therefore, since the factorization of a multidimensional polynomial to check for the presence of non-trivial symmetric factors is, in general, a very difficult problem, it will be useful to include some additional information in order to guarantee that the reconstructed sequence has no non-trivial symmetric factors. For one-dimensional sequences, the additional information which is included to insure the correct solution is the location of the first non-zero point of the sequence. Since the z -transform of any 1-D finite length sequence may always be written as

$$X(z) = C z^{-n_0} \prod_{k=1}^{N-1} (1 - a_k z^{-1}) \quad (4-58)$$

the location of the first non-zero point of $x(n)$ is equal to n_0 . Therefore, Theorem 3.3 asserts that if a sequence of length N has a z -transform with no zeros on the unit circle or in reciprocal pairs (i.e. $X(z)$ has no non-trivial symmetric factors) then it is

uniquely specified to within a scale factor by n_0 and the phase of its M -point DFT provided $M \geq 2N-1$. A similar result is true for multidimensional sequences. Specifically, recall that the z -transform of any multidimensional sequence with support $R(N)$ may always be written in the form given by (4-40). Therefore, suppose that $X(z)$ has no non-trivial symmetric factors and note that if \underline{n}_0 is equal to zero then $X(z)$, in addition, has no trivial symmetric factors. The multidimensional extension of Theorem 3.3 asserts that if $x(\underline{n})$ has support $R(N)$ and if $X(z)$ has no symmetric factors (trivial or non-trivial) then scaled versions of $x(\underline{n})$ are the only sequences with support $R(N)$ and a M -point DFT with phase $\phi_x(\underline{k})_M$ provided $M \geq 2N-1$.

Theorem 4.7: Let $x, y \in F(n)$ with support $R(N)$ and let $M \geq 2N-1$.

If $X(z)$ has no symmetric factors and $\phi_x(\underline{k})_M = \phi_y(\underline{k})_M$ then $y(\underline{n}) = \beta x(\underline{n})$ for some positive number β . If $\tan[\phi_x(\underline{k})_M] = \tan[\phi_y(\underline{k})_M]$, then $y(\underline{n}) = \beta x(\underline{n})$ for some real number β .

Note that, in contrast to Theorem 4.6, there are no constraints on the z -transform of $y(\underline{n})$. Therefore, $y(\underline{n})$ may be any multidimensional sequence with support $R(N)$. In addition, note that although the constraint that $X(z)$ has no (trivial) symmetric factors is equivalent to the condition that $\underline{n}_0 = 0$ in (4-40), this does not necessarily imply that $x(0) \neq 0$. In fact, $\underline{n}_0 = 0$ is equivalent to the constraint that if $\underline{k} \geq 0$, then $x'(\underline{m}) = x(\underline{n}-\underline{k})$ will not be equal to zero for all $\underline{m} < 0$. Finally, it should also be pointed out this theorem may be easily extended to the case in which \underline{n}_0 in (4-40) is non-zero but known. A proof of Theorem 4.7 is as follows:

Proof: If $x(\underline{n})$ and $y(\underline{n})$ satisfy the constraints of the theorem, as in the proof of Theorem 4.6, it follows that $g(\underline{n}) = x(\underline{n}) * y(-\underline{n})$ is an even sequence. Furthermore, as in the proof of Theorem 4.5, since $X(z)$ contains no symmetric factors, then each non-

trivial irreducible factor of $X(\underline{z})$ must be associated with a factor of $Y(\underline{z})$. Therefore, $X(\underline{z})$ and $Y(\underline{z})$ must be related by

$$Y(\underline{z}) = \underline{z}^{\underline{m}} P(\underline{z}) X(\underline{z}) \quad (4-59)$$

where $P(\underline{z})$ is a polynomial and \underline{m} is an integer-valued vector. However, since $Y(\underline{z})$ and $X(\underline{z})$ are polynomials in \underline{z}^{-1} and since $X(\underline{z})$ contains no trivial factors, then $Q(\underline{z}) = \underline{z}^{\underline{m}} P(\underline{z})$ must also be a polynomial in \underline{z}^{-1} . Furthermore, in order for the phase or tangent of the phase of $x(\underline{n})$ and $y(\underline{n})$ to be equal, $q(\underline{n})$ must be an even sequence, i.e., $Q(\underline{z}) = Q(\underline{z}^{-1})$. Therefore, $Q(\underline{z}) = \beta$ and the theorem follows by noting that β must be positive if $\phi_x(\underline{k})_{\underline{M}} = \phi_y(\underline{k})_{\underline{M}}$. /////

IV.5.2: Uniqueness in terms of magnitude

In Section IV.5.1, the uniqueness of a multidimensional sequence in terms of the phase of its Fourier transform was considered. This section addresses the dual problem related to the uniqueness of a multidimensional sequence in terms of its Fourier transform magnitude. It appears that the first treatment of this uniqueness question was provided by Bruck and Sodin [5] who postulated that the uniqueness of a 2-D sequence $x \in F(\underline{n})$ is related to the irreducibility of its z -transform. In this section, a slightly more general result is derived which includes sequences with irreducible z -transforms as a special case. Even more importantly, however, as in Section IV.5.1, the uniqueness of a multidimensional sequence in terms of a finite set (lattice) of samples of its Fourier transform magnitude is considered.

IV.5.2.1: Uniqueness in terms of continuous magnitude

Consider a sequence $x \in F(n)$ for which $|X(\omega)|$ is known for all ω . Since the inverse Fourier transform of $|X(\omega)|^2$ is the autocorrelation, $r_x(n)$, of $x(n)$:

$$r_x(n) = x(n) * x(-n) \quad (4-60)$$

the specification of $|X(\omega)|$ is equivalent to the knowledge of $r_x(n)$ or its z -transform $R_x(z)$:

$$R_x(z) = X(z) X(z^{-1}) \quad (4-61)$$

For any $x \in F(n)$, the most general form for its z -transform, $X(z)$, is given by (4-40). Therefore, substituting (4-40) into (4-61) gives

$$R_x(z) = \alpha^2 \prod_{k=1}^P X_k(z) \cdot X_k(z^{-1}) \quad (4-62)$$

Now suppose that the polynomial

$$P(z) = \prod_{k=1}^P X_k(z) \quad (4-63)$$

is of degree N in z^{-1} . Multiplying $R_x(z)$ by z^{-N} yields a polynomial in z^{-1} which is of degree $2N$ in z^{-1} :

$$Q_x(z) = z^{-N} R_x(z) = \alpha^2 \prod_{k=1}^P X_k(z) \cdot \tilde{X}_k(z) \quad (4-64)$$

It is apparent that $Q_x(z)$ and $|X(\omega)|$ contain exactly the same information about $x(n)$ since one may be uniquely derived from the other. Therefore, the ability to uni-

quely recover $x(n)$ from $|X(\omega)|$ is equivalent to the ability to uniquely recover $X(z)$ from $Q_x(z)$. With this in mind, it follows that $x(n)$ cannot be unambiguously recovered from only the magnitude of its Fourier transform. For example, the sign of a as well as the linear phase term z^n are not recoverable from $Q_x(z)$. Even more important is the observation that, without additional information, it is not possible to determine whether $X_k(z)$ or $\tilde{X}_k(z)$ is a factor of $X(z)$. This ambiguity is not surprising, however, since it represents the multidimensional extension of a result which is familiar for 1-D sequences [36]. Specifically, for any finite duration sequence $x(n)$, another sequence $y(n)$ may be generated which has the same Fourier transform magnitude as $x(n)$ by simply reflecting a zero of $X(z)$ about the unit circle. For m -D sequences, $\tilde{X}_k(z)$ represents the reflection of the zero contour of $X_k(z)$ about the unit polydisc.

It will be useful in the following discussions to define an equivalence relation on the set $F(n)$ as follows:

$$y(n) \sim x(n) \quad \text{if} \quad y(n) = \pm x(\underline{k} \pm n) \quad (4-65)$$

for some integer-valued vector \underline{k} . In other words, the equivalence class generated by a sequence $x \in F(n)$ is defined to be the set of all sequences which may be derived from $x(n)$ by a linear shift, a "time-reversal", or a change in the sign of the sequence. Note that all of the sequences within a given equivalence class have the same Fourier transform magnitude. Thus, it will be convenient to refer to the Fourier transform magnitude of the sequences within an equivalence class as the Fourier transform magnitude of the class.

In general, there will be more than one equivalence class having the same Fourier transform magnitude. More specifically, given a sequence $x \in F(n)$, there may exist another sequence $y \in F(n)$ with the same Fourier transform magnitude as $x(n)$ but

which is not in the same equivalence class as $x(n)$. Therefore, the goal of this section is to develop a set of conditions which guarantee the existence of only one equivalence class with a given Fourier transform magnitude. The first question to be addressed, however, concerns the number of equivalence classes which have a given Fourier transform magnitude. Once this has been established, conditions which guarantee the existence of only one equivalence class may then be easily determined. The answer to the first question is implied by the following theorem:

Theorem 4.8: Let $x \in F(n)$ have a z -transform given by

$$X(z) = \alpha z^n \prod_{k=1}^p X_k(z) \quad (4-66)$$

where $X_k(z)$ are *non-trivial* irreducible polynomials for $k=1, \dots, p$. If $y \in F(n)$ and $|X(\omega)| = |Y(\omega)|$ for all ω , then $Y(z)$ is of the form:

$$Y(z) = \pm \alpha z^m \prod_{k \in I} X_k(z) \prod_{k \notin I} \tilde{X}_k(z) \quad (4-67)$$

where I is a subset of the integers in the interval $[1, p]$.

Proof: Let $x, y \in F(n)$ and let N be large enough so that both $x(n)$ and $y(n)$ are zero outside the region $R(N)$. Since $|X(\omega)| = |Y(\omega)|$, it follows that

$$X(z) X(z^{-1}) = Y(z) Y(z^{-1}) \quad (4-68)$$

Therefore, let the z -transform of $y(n)$ be given by

$$Y(z) = \beta z^m \prod_{k=1}^q Y_k(z) \quad (4-69)$$

where $Y_k(z)$ are non-trivial irreducible factors for $k=1, \dots, q$. Substituting (4-66) and (4-69) into (4-68) and multiplying by z^{-N} yields the following equation in z^{-1} :

$$\alpha^2 z^{-m_1} \prod_{k=1}^p X_k(z) \tilde{X}_k(z) = \beta^2 z^{-m_2} \prod_{k=1}^q Y_k(z) \tilde{Y}_k(z) \quad (4-70)$$

where $m_1 \geq 0$ and $m_2 \geq 0$. From Theorem 4.2 in Section IV.4.2, it follows that $m_1 = m_2$ and $p = q$:

$$\alpha^2 \prod_{k=1}^p X_k(z) \tilde{X}_k(z) = \beta^2 \prod_{k=1}^p Y_k(z) \tilde{Y}_k(z) \quad (4-71)$$

Again from Theorem 4.2 it follows that the factors $Y_k(z)$ may be ordered in such a way that, for each k , $Y_k(z)$ is associated with either $X_k(z)$ or $\tilde{X}_k(z)$. Therefore, from (4-69) and the fact that $|X(\omega)| = |Y(\omega)|$ implies that $\alpha = \pm \beta$, the desired result (4-67) follows. /////

It may be noted that this theorem is simply a statement of the fact that the only way to generate a new sequence, $y(n)$, which has the same Fourier transform magnitude as $x(n)$, is to convolve $x(n)$ with an all-pass sequence, $g(n)$, i.e., a sequence for which $|G(\omega)| = 1$ for all ω . However, if $|G(\omega)| = 1$ then

$$G(z) G(z^{-1}) = 1 \quad (4-72)$$

Therefore, with $G(z)$ a rational function of z , let

$$G(z) = \beta z^m \frac{\prod_{k=1}^p A_k(z)}{\prod_{k=1}^q B_k(z)} \quad (4-73)$$

where $A_k(z)$ and $B_k(z)$ are non-trivial irreducible polynomials and assume that for each i and k , $B_i(z)$ is not associated with $A_k(z)$ (i.e., the numerator and denominator have no common factors). Thus, it follows from (4-72) and (4-73) that

$$\beta^2 \underline{z}^{\underline{n}_1} \prod_{k=1}^p A_k(z) \tilde{A}_k(z) = \underline{z}^{\underline{n}_2} \prod_{k=1}^q B_k(z) \tilde{B}_k(z) \quad (4-74)$$

However, from Theorem 4.2, it follows that $p=q$ and $\underline{n}_1=\underline{n}_2$. Furthermore, since $A_k(z)$ is not associated with $B_i(z)$ for any i , then the factors $B_k(z)$ may be ordered in such a way that

$$A_k(z) = \eta_k \tilde{B}_k(z) \quad (4-75)$$

for each k . Therefore,

$$G(z) = \mu \underline{z}^{\underline{m}} \prod_{k=1}^p B_k^{-1}(z) \cdot \tilde{B}_k(z) \quad (4-76)$$

Finally, since

$$|B_k^{-1}(\omega) \tilde{B}_k(\omega)| = 1 \quad (4-77)$$

then $\mu=\pm 1$. Therefore, any all-pass sequence with a rational z -transform must always be of the form:

$$G(z) = \pm \underline{z}^{\underline{m}} \prod_{k=1}^p B_k^{-1}(z) \cdot \tilde{B}_k(z) \quad (4-78)$$

where $B_k(z)$ are non-trivial irreducible polynomials in \underline{z}^{-1} . Thus, given a sequence $x \in F(n)$ with a z -transform of the form (4-66), $y(n)=x(n)*g(n)$ has finite support if

and only if for each k , $B_k(z) = X_i(z)$ for some $i \in [1, p]$. Consequently, $Y(z)$ must be of the form given by (4-67).

As a consequence of Theorem 4.8, if $x \in F(n)$ is a sequence with a z -transform given by (4-66), and if $y \in F(n)$ with $|Y(\omega)| = |X(\omega)|$, then $Y(z)$ must have the same number, p , of non-trivial irreducible factors. Furthermore, except for a scale factor of (-1) and linear shifts, the only way to generate another sequence $y \in F(n)$ for which $|Y(\omega)| = |X(\omega)|$ is to replace one or more non-trivial factors $X_k(z)$ of $X(z)$ with $\tilde{X}_k(z)$. However, if $X_k(z)$ is symmetric, then this replacement may only change $X(z)$ by a factor of (-1) . Therefore, it follows that the number of equivalence classes with magnitude $|X(\omega)|$ is at most $2^{(p-1)}$ where p is the number of non-symmetric irreducible factors in $X(z)$. Thus, the following is an immediate consequence of Theorem 4.8:

Theorem 4.9: Let $x \in F(n)$ have a z -transform with at most one irreducible non-symmetric factor, i.e.,

$$X(z) = P(z) \prod_{k=1}^p X_k(z) \quad (4-79)$$

where $P(z)$ is irreducible and where $X_k(z)$ for $k=1, \dots, p$ are irreducible and symmetric. If $y \in F(n)$ with $|X(\omega)| = |Y(\omega)|$ for all ω , then $y(n) \sim x(n)$.

As a final remark, note that if $x(n)$ has a z -transform of the form

$$X(z) = z^A \cdot P(z) \quad (4-80)$$

where $P(z)$ is irreducible polynomial, then $X(z)$ satisfies the constraints of Theorem 4.9. Furthermore, since "almost all" polynomials in two or more variables are of the form (4-80) (Appendix I), it follows that "almost all" multidimensional sequences with

finite support are uniquely specified to within a sign, a linear shift, and a "time reversal" by the magnitude of their Fourier transforms.

IV.5.2.2: Uniqueness in terms of magnitude samples

As in Section IV.5.1.2, it may be apparent that the assumption in Theorem 4.9 that $|X(\omega)|=|Y(\omega)|$ holds for all ω is not required if $x(n)$ and $y(n)$ are known to be zero outside some given domain. More specifically, suppose that for some N , $x(n)$ and $y(n)$ are known to be zero outside $R(N)$. Let Ω_k and A_k be sets of M_k distinct points as defined in (4-36) for $k=1,\dots,m$, and let $L(\Omega)$ and $L(A)$ be the lattices generated by these sets. Since

$$|X(\omega)|_{L(\Omega)}^2 = R_x(z)_{L(A)} \quad (4-81)$$

it follows that if

$$|X(\omega)|_{L(\Omega)}^2 = |Y(\omega)|_{L(\Omega)}^2 \quad (4-82)$$

then

$$R_x(z)_{L(A)} = R_y(z)_{L(A)} \quad (4-83)$$

Therefore, (4-82) implies that

$$Q_x(z)_{L(A)} = Q_y(z)_{L(A)} \quad (4-84)$$

where $Q_x(z)$ and $Q_y(z)$ are polynomials of degree at most $2(N-1)$ in z^{-1} . Thus, if $M \geq 2N-1$, it follows from Theorem 4.4 that $Q_x(z)=Q_y(z)$ for all z and, consequently, that $|X(\omega)|=|Y(\omega)|$ for all ω . This leads, therefore, to the following

Theorem 4.10: Let $x, y \in F(N)$ with support $R(N)$ and let Ω_k be a set of M_k distinct real numbers in the interval $[0, 2\pi)$ with $M_k \geq 2N_k - 1$ for $k=1, \dots, m$. If $X(z)$ has at most one irreducible non-symmetric factor and

$$|X(\omega)|_{L(\Omega)} = |Y(\omega)|_{L(\Omega)} \quad (4-85)$$

then $y(n) \sim x(n)$.

A special case of this theorem results when the points in the sets Ω_k are equally spaced between 0 and 2π . In this instance, $X(\omega)|_{L(\Omega)}$ is equal to the \underline{M} -point DFT of $x(n)$, i.e.,

$$X(\omega)|_{L(\Omega)} = X(k)|_{\underline{M}} \quad (4-86)$$

Therefore, (4-85) may be replaced with the constraint that

$$|X(k)|_{\underline{M}} = |Y(k)|_{\underline{M}} \quad (4-87)$$

provided that $\underline{M} \geq 2N - 1$.

IV.6: Extensions

In Section IV.4, conditions are presented under which an m -D sequence is uniquely defined to within a scale factor by the phase of its Fourier transform. A similar set of conditions are presented in Section IV.5 which allow an m -D sequence to be uniquely specified by the magnitude of its Fourier transform to within a delay, a sign, and a "time-reversal". It is of interest to note that these uniqueness constraints are not mutually exclusive. Specifically, suppose that $x \in F(n)$ has a z -transform which, except for trivial factors, is non-symmetric and irreducible, i.e.,

$$X(z) = z^N P(z) \quad (4-88)$$

where $P(z)$ is an irreducible non-symmetric polynomial in z^{-1} . It then follows that $x(n)$ satisfies the constraints of both Theorems 4.5 and 4.9. Furthermore, if $x(n)$ is known to have support $R(N)$, then the constraints of Theorems 4.6 and 4.10 are also satisfied by $x(n)$. Therefore, the following is a direct consequence of these theorems:

Theorem 4.11: If $x \in F(n)$ has a z -transform which, except for trivial factors, is irreducible and non-symmetric, then $x(n)$ is uniquely specified (in the sense of Theorems 4.5 or 4.9) by either the phase or magnitude of its Fourier transform. If, in addition, $x(n)$ is known to have support $R(N)$, then the phase or magnitude of the M -point DFT of $x(n)$ is sufficient for this unique specification provided $M \geq 2N-1$.

In Appendix I, it is shown that almost all polynomials in two or more variables are irreducible. Specifically, it is shown that within the set of all polynomials in $k \geq 1$ variables, the subset of reducible polynomials is a set of measure zero. It may similarly be shown that the set of all symmetric polynomials corresponds to a set of measure

zero. Since the set of reducible polynomials and the set of symmetric polynomials are both sets of measure zero, then so is the union of these sets. However, since the complement of this union corresponds to the set of irreducible and non-symmetric polynomials, it follows that almost all polynomials in two or more variables are irreducible and non-symmetric. Consequently, almost all sequences with finite support satisfy the constraints of Theorem 4.11 and are uniquely defined to within a scale factor by the phase of their Fourier transform or to within a sign, a linear shift, and a time-reversal by the magnitude of their Fourier transform.

Although the results which have been presented thus far have been confined to sequences with finite support, an extension is easily made to those sequences whose convolutional inverses have finite support. Specifically, let $x_1(\underline{n})$ denote the convolutional inverse of an m -D sequence $x(\underline{n})$, i.e.,

$$x(\underline{n}) * x_1(\underline{n}) = \delta(\underline{n}) \quad (4-89)$$

where $\delta(\underline{n})$ is the m -D unit sample function. Now suppose that $x(\underline{n})$ is a stable sequence which has a z -transform of the form

$$X(\underline{z}) = 1 / P(\underline{z}) \quad (4-90)$$

where $P(\underline{z})$ is a polynomial in \underline{z}^{-1} . In this case, the convolutional inverse of $x(\underline{n})$ has a z -transform given by

$$X_1(\underline{z}) = P(\underline{z}) \quad (4-91)$$

so that $x_1 \in F(\underline{n})$. In addition, the phase or magnitude of the Fourier transform of $x_1(\underline{n})$ is uniquely specified by the phase or magnitude, respectively, of the Fourier transform

of $x(n)$, i.e.,

$$|X_1(\omega)| = |X(\omega)|^{-1} \quad (4-92)$$

and

$$\phi_{x_1}(\omega) = -\phi_x(\omega) \quad (4-93)$$

Therefore, if $x(n)$ is a stable sequence with a z-transform given by (4-90), then $x(n)$ is uniquely defined by the phase or magnitude of its Fourier transform (in the sense of Theorems 4.5 or 4.9) if the polynomial $P(z)$ satisfies the appropriate uniqueness constraints.

CHAPTER V: RECONSTRUCTION

V.1: Introduction

In Chapter III, conditions for a 1-D sequence to be uniquely defined in terms of the phase or magnitude of its Fourier transform were presented. These conditions were then extended in Chapter IV to multidimensional sequences. In this chapter, the reconstruction of a sequence from the phase or magnitude of its Fourier transform is considered. Specifically, in Section V.2 several practical algorithms are presented for reconstructing a sequence from the phase of its Fourier transform. These algorithms, which include both iterative as well as non-iterative solutions, always yield the correct sequence provided only that it satisfies the appropriate uniqueness constraints. In addition, several examples of 1-D and 2-D phase-only reconstruction are presented. The problem of reconstructing a sequence from the magnitude of its Fourier transform is then discussed in Section V.3. Unlike reconstruction from phase, however, it appears that a practical algorithm for magnitude-only reconstruction which always produces the correct result is yet to be realized. Nevertheless, an iterative solution which has been proposed [12] is described and several examples are presented.

V.2: Phase-only reconstruction

In this section, the problem of reconstructing a real sequence from the phase of its Fourier transform is addressed. In particular, given a finite length sequence $x(n)$ (or a sequence with a finite length convolutional inverse) which has a Fourier transform with phase $\phi_x(\omega)$, it is the goal of this section to develop some practical algorithms

which may be used to recover $x(n)$ from samples of $\phi_x(\omega)$. The algorithms which are presented may be applied to any phase-only reconstruction problem. However, unless the sequence $x(n)$ satisfies the uniqueness constraints developed in Chapters III and IV, the reconstructed sequence will not, in general, correspond to the desired sequence.

First, two non-iterative solutions are presented in Section V.2.1. An iterative procedure for phase-only reconstruction is then developed in Section V.2.2.

V.2.1: Non-iterative algorithms

This section presents two algorithms for reconstructing a real sequence from the phase of its Fourier transform. These algorithms are non-iterative and only require finding the solution to a set of linear equations. The first is a "time-domain" solution in the sense that the unknowns in the linear equations are the coefficients of the sequence $x(n)$. The second algorithm is a "frequency-domain" solution since the unknowns in the linear equations are the values of the DFT of $x(n)$. Since there are slight differences between the one-dimensional and multidimensional case, the algorithms are first formulated in terms of reconstructing a one-dimensional sequence from the phase of its Fourier transform. The extension of these algorithms to the multidimensional phase-only reconstruction problem is then briefly described.

V.2.1.1: Time-Domain Solution

Let $x(n)$ be a real one-dimensional sequence which is zero outside the interval $[0, N-1]$ with $x(0) = \alpha_0 \neq 0$. In addition, suppose that $X(z)$ has no zeros on the unit circle or in conjugate reciprocal pairs. In this case, $x(n)$ is uniquely defined by α_0 and any $N-1$ samples of its phase in the interval $(0, \pi)$. Therefore, let $\phi_x(\omega)$ be the phase of the

Fourier transform of $x(n)$. Since

$$\tan[\phi_x(\omega)] = X_I(\omega)/X_R(\omega) \quad (5-1)$$

where $X_R(\omega)$ and $X_I(\omega)$ denote the real and imaginary parts of $X(\omega)$, respectively, then

$$\tan[\phi_x(\omega)] = \frac{\sin[\phi_x(\omega)]}{\cos[\phi_x(\omega)]} = - \frac{\sum_{n=0}^{N-1} x(n) \sin(n\omega)}{\sum_{n=0}^{N-1} x(n) \cos(n\omega)} \quad (5-2)$$

Note that if $\phi_x(\omega) \neq \pm\pi/2$, then the denominator in (5-2) is non-zero. Therefore, cross multiplying and using a standard trigonometric identity, (5-2) may be rewritten as

$$\sum_{n=1}^{N-1} x(n) \sin[\phi_x(\omega)+n\omega] = -x(0) \sin[\phi_x(\omega)] \quad (5-3 \text{ a})$$

However, if $\phi_x(\omega) = \pm\pi/2$ then the denominator in (5-2) is equal to zero, i.e.,

$$\sum_{n=1}^{N-1} x(n) \cos(n\omega) = -x(0) \quad (5-3 \text{ b})$$

For each fixed ω , (5-3) is linear in the unknowns $x(n)$. Therefore, let $\{\omega_1, \omega_2, \dots, \omega_{N-1}\}$ be $N-1$ distinct frequency samples in the interval $(0, \pi)$, and let $\phi_x(\omega_k)$ be the corresponding phase samples. Substituting these phase samples into (5-3) then leads to $N-1$ linear equations in $x(n)$. When augmented with the equation $x(0) = \alpha_0$, these equations may be written in matrix form as

$$S \underline{x} = \alpha_0 \underline{b} \quad (5-4)$$

where S is an $N \times N$ matrix and where \underline{x} is the vector representation of the sequence $x(n)$, i.e.,

$$\underline{x}^T = [x(0), x(1), \dots, x(N-1)] \quad (5-5)$$

where a superscript T denotes a vector transpose.

Any solution $\underline{y}^T = [y(0), \dots, y(N-1)]$ to (5-4) corresponds to a finite length sequence of length N with $y(0) = \alpha_0$ and with the tangent of its phase equal to $\tan[\phi_x(\omega_k)]$ for $k=1, 2, \dots, N-1$. Thus, from Theorem 3.3, it follows that $\underline{x} = \underline{y}$ and, therefore, that the solution to (5-4) is unique. Consequently, the inverse of S exists and $x(n)$ may be uniquely reconstructed from its phase by

$$\underline{x} = \alpha_0 S^{-1} \underline{b} \quad (5-6)$$

If $x(0) = \alpha_0$ is not known, then (5-4) only specifies $x(n)$ to within a scale factor, α_0 . Nevertheless, if $\phi_x(\omega)$ is known for at least one $\omega_k \in (0, \pi)$, then the sign of α_0 may be determined. Specifically, with $\underline{y} = S^{-1}\underline{b}$, if $\phi_y(\omega_k) = \phi_x(\omega_k)$, then $\alpha_0 > 0$; otherwise, $\alpha_0 < 0$.

It should be pointed out that if $x(n)$ satisfies the constraints of Theorem 3.2, but the first non-zero point of $x(n)$ is not at $n=0$ but rather at $n=n_0 > 0$, then the matrix S will be singular. This follows from the observation that \underline{x} is a solution to the equation $S\underline{x} = \underline{0}$. However, it is straightforward to show that n_0 may be obtained from

$$n_0 = N - \text{rank}(S) \quad (5-7)$$

Furthermore, by rewriting (5-3) using the fact that $x(n)$ is zero outside the interval $[n_0, N-1]$ with $x(n_0) \neq 0$, a set of linear equations result for which $x(n)$ is the unique solution.

The matrix S will also be singular if X(z) either has zeros on the unit circle or in reciprocal pairs. Specifically, in this case x(n) may be written as the convolution of two finite length sequences

$$x(n) = x_0(n) * g(n) \quad (5-8)$$

where $\tan[\phi_x(\omega)] = 0$ and where $x_0(n)$ satisfies the constraints of Theorem 3.2. Therefore, since the tangent of the phase of $x_0(n)$ is equal to the tangent of the phase of $x(n)$, and since $x_0(0) = 0$, then $x_0(n)$ is a solution to the equation $Sx = 0$ and the singularity of S follows. However, it may easily be verified that the location of the first non-zero point of $x_0(n)$ is given by (5-7). Therefore, as noted above, (5-3) may be rewritten so that $x_0(n)$ may be uniquely reconstructed to within a scale factor.

Example: Let $x(n)$ be zero outside the interval [0,2] with $\phi_x(\omega) = -\omega$.

From (5-3) it follows that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin[\phi_x(\omega_1) + \omega_1] & \sin[\phi_x(\omega_1) + 2\omega_1] \\ 0 & \sin[\phi_x(\omega_2) + \omega_2] & \sin[\phi_x(\omega_2) + 2\omega_2] \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix} = -\alpha_0 \begin{bmatrix} -1 \\ \sin[\phi_x(\omega_1)] \\ \sin[\phi_x(\omega_2)] \end{bmatrix} \quad (5-9)$$

and, since $\phi_x(\omega_1) = -\omega_1$ and $\phi_x(\omega_2) = -\omega_2$, then

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \sin \omega_1 \\ 0 & 0 & \sin \omega_2 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix} = -\alpha_0 \begin{bmatrix} -1 \\ \sin \omega_1 \\ \sin \omega_2 \end{bmatrix} \quad (5-10)$$

Therefore, the matrix S is singular. Furthermore, any solution to (5-10) corresponds to a sequence y(n) of the form

$$y(n) = \alpha_0 \delta(n) + \alpha_1 \delta(n-1) + \alpha_2 \delta(n-2) \quad (5-11)$$

for arbitrary constants α_0 and α_1 . This, however, is not surprising since any sequence which is zero outside $[0,2]$ with phase $\phi_x(\omega)=-\omega$ must be of the form given in (5-11). Finally, since

$$n_0 = N - \text{rank}(\mathbf{S}) = 3 - 2 = 1 \quad (5-12)$$

then, with $x(1)=\alpha_1$, the sequence $x_0(n)$ with phase $\phi_x(\omega)=-\omega$ which satisfies the constraints of Theorem 3.2 is given by the solution to the equations

$$\begin{bmatrix} 1 & 0 \\ 0 & \sin \omega_1 \end{bmatrix} \begin{bmatrix} x_0(1) \\ x_0(2) \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (5-13)$$

Therefore, $x_0(n) = \alpha_1 \delta(n-1)$.

The algorithm defined by (5-6) has been applied to a variety of different examples. Consistent with the theoretical results described above, in all the examples which have been considered, the desired solution was always obtained provided it satisfied the constraints in Theorem 3.2 or 3.3.

Finally, in reconstructing $x(n)$ from $\phi_x(\omega)$, it should be pointed out that there is some flexibility in the choice of the matrix \mathbf{S} . Specifically, since the elements of \mathbf{S} are functions of the samples of $\phi_x(\omega)$, \mathbf{S} may be changed by choosing a different set of frequency values. This control over \mathbf{S} may be useful, for example, in minimizing the reconstruction error when $\phi_x(\omega)$ is not known exactly. In particular, it has been observed that the reconstruction error from noisy phase is generally minimized when the frequency samples are chosen to have the maximum separation [10].

V.2.1.2: Frequency-Domain Solution

The time-domain solution to the phase-only reconstruction problem given by (5-6) is defined by a set of linear equations which are to be solved in terms of the sequence values, $x(n)$. It is also possible to derive a set of linear equations in which the unknowns are the values of the DFT of $x(n)$. As is the case for the time-domain solution, these equations may be solved to uniquely reconstruct $X(k)$, and hence $x(n)$, provided the uniqueness constraints of Theorem 3.3 are satisfied.

As before, let $x(n)$ be a sequence which is zero outside $[0, N-1]$ with $x(0) = \alpha_0 \neq 0$. With $X(k)$ the M -point DFT of $x(n)$, it follows from (5-1) that the real and imaginary parts of $X(k)$ are related by

$$X_I(k) = \tan[\phi_x(k)] X_R(k) \quad (5-14)$$

where $\phi_x(k)$ is the phase of $X(k)$. Therefore, with \underline{X}_R and \underline{X}_I the vectors whose components are $X_R(k)$ and $X_I(k)$ respectively, (5-14) may be written as

$$\underline{X}_I = T_\bullet[\underline{X}_R] \quad (5-15)$$

where T_\bullet is a diagonal matrix with $\{T_\bullet\}_{kk} = \tan[\phi_x(k)]$.

Equation (5-15) provides a relationship between the real and imaginary parts of the DFT of a sequence in terms of its phase. Although (5-15) is not sufficient to reconstruct $X(k)$, it is possible to derive another relationship between \underline{X}_R and \underline{X}_I which, when combined with (5-15), yields a set of linear equations which may be solved to recover $X(k)$. Specifically, note that if $M \geq 2N-1$, then the real and imaginary parts of the M -point DFT of $x(n)$ are related by the Discrete Hilbert Transform (DHT) [7]. Therefore, with $\underline{B}^T = \alpha_0[1, 1, \dots, 1]$ and using the matrix formulation for the DHT, then

\underline{X}_R and \underline{X}_I are related by

$$\underline{X}_R = H_T[\underline{X}_I] + \underline{B} \quad (5-16)$$

and

$$\underline{X}_I = -H_T[\underline{X}_R] \quad (5-17)$$

where H_T is a square circulant $M \times M$ matrix [6]. The desired result may now be obtained by substituting (5-15) into (5-16) which yields, after a rearrangement of terms,

$$(I - H_T T_\bullet) \underline{X}_R = \underline{B} \quad (5-18)$$

where I represents the identity matrix. Note that (5-18) corresponds to a set of M linear equations in the unknowns $X_R(k)$. However, due to the fact that $X_R(k) = X_R(M-k)$, half of these equations may be eliminated.

For reasons identical to those given in justifying the invertibility of the matrix S in the time-domain solution, it follows that the matrix $Q = (I - H_T T_\bullet)$ will be invertible if $x(n)$ satisfies the constraints of Theorem 3.3 and if $M \geq 2N - 1$. In this case, $X(k)$ may be reconstructed as follows

$$\underline{X} = (I - jH_T) \underline{X}_R = (I - jH_T)(I - H_T T_\bullet)^{-1} \underline{B} \quad (5-19)$$

However, if $M < 2N - 1$ or if $x(n)$ does not satisfy the constraints of Theorem 3.3, then Q will be singular since there will exist at least one non-zero solution to the homogeneous equation $Q(\underline{X}_R) = 0$.

The algorithm defined by (5-19) has been applied to a variety of different examples. Consistent with the observations above, in all the examples which have been considered, the desired solution was always obtained provided it satisfied the constraints of Theorem 3.3 and $M \geq 2N - 1$. There are, however, several disadvantages of this algorithm compared with the time-domain solution. Specifically, in contrast to

(5-6), (5-19) requires that the phase samples be uniformly spaced between zero and 2π . In addition, due to the fact that the entries in the matrix T_k correspond to $\tan[\phi_x(k)]$, it is expected that numerical problems will be encountered when $\phi_x(k)$ is close to $\pm \pi/2$ [If $\phi_x(k) = \pm \pi/2$, then $X_R(k) = 0$ and the number of equations in (5-19) may be reduced to remove this singularity].

V.2.1.3: Extension to multidimensional sequences

In the development of the non-iterative solutions to the phase-only reconstruction problem described in Sections V.2.1.1 and V.2.1.2, it was assumed that the sequences to be recovered were one-dimensional. However, by using the projection/slice approach described in Section IV.3, it is possible to use either of these algorithms for reconstructing a multidimensional sequence from its phase. For example, suppose that a 2-D sequence $x(n_1, n_2)$ with support $R(N_1, N_2)$ is to be reconstructed from its phase $\phi_x(\omega_1, \omega_2)$. Let $N_1 \geq N_2$ and assume that $x(0,0) = \alpha_0 \neq 0$. If the projection $\hat{x}_1(n)$ of $x(n_1, n_2)$ defined in (4-7) has a z-transform with no zeros on the unit circle or in conjugate reciprocal pairs, then α_0 along with any $M = N_1 N_2 - 1$ samples of the phase $\phi_x(\omega_1, \omega_2)$ along the line $\omega_1 = N\omega_2$ uniquely specify $\hat{x}_1(n)$ and, hence, $x(n_1, n_2)$. Therefore, if $\{\omega_1, \dots, \omega_M\}$ are M distinct frequencies in the interval $(0, \pi)$, then (5-6) may be used to recover $x_1(n)$ from the phase samples $\phi_x(N\omega_k, \omega_k)$. Consequently, $x(n_1, n_2)$ may be reconstructed from $\hat{x}_1(n)$ by *back-projecting* $\hat{x}_1(n)$ using (4-7).

It is also possible, however, to modify both of the algorithms so that they may be used for multidimensional phase-only reconstruction. Consider, for example, the time-domain solution (5-6). With $x(n_1, n_2)$ a two-dimensional sequence with support $R(N_1, N_2)$, a two-dimensional version of (5-3) may easily be derived from the definition of $\phi_x(\omega_1, \omega_2)$. The incorporation of the phase samples obtained from an

$M_1 \times M_2$ -point DFT leads to $M_1 M_2$ linear equations in the $N_1 N_2$ unknowns of $x(n_1, n_2)$ [by exploiting the fact that the phase is an odd function of ω_1 and ω_2 , approximately half of these equations may be eliminated]. Finally, as in the one-dimensional case, it is straightforward to show that if $M_1 \geq 2N_1 - 1$ and $M_2 \geq 2N_2 - 1$ and if $x(n_1, n_2)$ satisfies the constraints of Theorem 4.10, then these equations may be solved to uniquely recover $x(n_1, n_2)$ using generalized inverses [17].

V.2.2: Iterative algorithms

In Section V.2.1, two algorithms were presented for reconstructing a sequence from samples of the phase of its Fourier transform. Although these algorithms are non-iterative and only require finding the solution to a set of linear equations, for sequences with large support, these algorithms are unfeasible due to the computational difficulties encountered in solving large sets of linear equations. Therefore, it is of interest to consider alternative solutions to the reconstruction problem. One such solution is an iterative procedure which is similar in style to algorithms proposed by Gerchberg and Saxton [15], Fienup [12], and Quatieri and Oppenheim [41]. This algorithm, which is described in detail in Section V.2.2.1, has been used successfully in reconstructing one and two-dimensional sequences from the phase of their Fourier transforms [17,18]. Furthermore, as shown in Section V.2.2.2, it is possible to prove that the iteration will theoretically always converge to the correct solution provided it satisfies the appropriate uniqueness constraints [48]. However, since convergence is generally very slow after the first few iterations, in order to increase the rate of convergence, an "adaptive relaxation" technique has been developed [33] and is described in Section V.2.2.3. Although it has not yet been shown theoretically that the iteration will converge when adaptively relaxed, a significant increase in the convergence rate has been observed.

V.2.2.1: The phase-only iteration

Let $x(\underline{n})$ be a multidimensional sequence with support $R(\underline{N})$ which satisfies the constraints of Theorem 4.6 (or Theorem 3.3 if $x(\underline{n})$ is one-dimensional). For convenience, assume in addition, that $x(\underline{0}) = \alpha_0 \neq 0$. In this case, if $M \geq 2N - 1$ then $x(\underline{n})$ is uniquely specified by the phase of its M -point DFT and the value of α_0 . The mathematical problem to be solved in recovering $x(\underline{n})$ from $\phi_x(\underline{k})$ and α_0 is to find a sequence which is consistent with the known information in both the time and frequency domains. Specifically, in the time domain it is known that $x(\underline{n})$ is non-zero only within the region $R(\underline{N})$ and that $x(\underline{0}) = \alpha_0$ whereas in the frequency domain it is known that the phase of the M -point DFT of $x(\underline{n})$ is $\phi_x(\underline{k})$. Therefore, a heuristic approach for reconstructing $x(\underline{n})$ is the iterative algorithm illustrated in Figure 5.1. As shown, this algorithm is characterized by the repeated transformation between the time and frequency domains where, in each domain, the known information about the desired sequence is imposed on the current estimate. More specifically, the iteration may be described as follows:

* * * * *

Step 1: Beginning with $|X_0(\underline{k})|$, an initial guess of the unknown DFT magnitude, the first estimate, $X_1(\underline{k})$, of $X(\underline{k})$ is formed by combining $|X_0(\underline{k})|$ with the known phase, i.e.,

$$X_1(\underline{k}) = |X_0(\underline{k})| \exp[j\phi_x(\underline{k})] \quad (5-20)$$

Computing the inverse DFT of $X_1(\underline{k})$ provides the first estimate, $x_1(\underline{n})$, of $x(\underline{n})$. Since an M -point DFT is used, $x_1(\underline{n})$ is, in general, non-zero outside $R(\underline{N})$ (In fact, if $x_1(\underline{n})$ is zero outside $R(\underline{N})$, then either Theorem 3.3 or Theorem 4.7 imply that $x_1(\underline{n})$ is equal to $x(\underline{n})$ to within a scale factor).

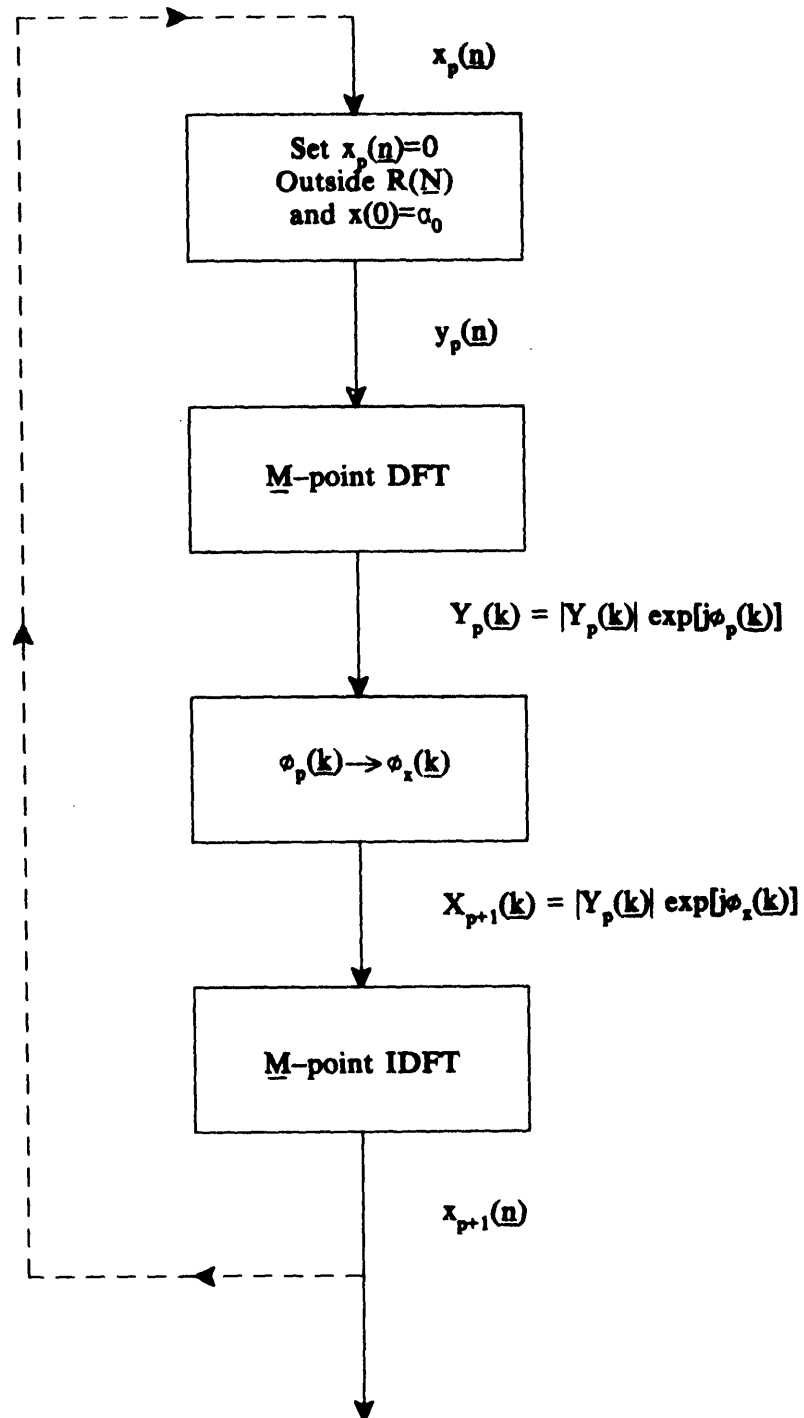


Figure 5.1: Block diagram of the phase-only iteration.

Step 2: From $x_1(\underline{n})$, another sequence, $y_1(\underline{n})$, is formed as follows:

$$y_1(\underline{n}) = \begin{cases} x_1(\underline{n}) & \text{for } \underline{n} < \underline{N} \text{ and } \underline{n} \neq 0 \\ \alpha_0 & \text{for } \underline{n} = 0 \\ 0 & \text{otherwise} \end{cases} \quad (5-21)$$

Step 3: The magnitude $|Y_1(\underline{k})|$ of the \underline{M} -point DFT of $y_1(\underline{n})$ is then used as the new estimate of $|X(\underline{k})|$ and the next estimate of $X(\underline{k})$ is formed by

$$X_2(\underline{k}) = |Y_1(\underline{k})| \exp[j\phi_x(\underline{k})] \quad (5-22)$$

A new estimate, $x_2(\underline{n})$, is then obtained by taking the inverse DFT of $X_2(\underline{k})$. Repeated application of steps two and three defines the iteration.

* * * * *

In Section V.2.2 it is shown that this iterative procedure will always converge to the correct sequence provided it satisfies the requirements of Theorem 3.3 for one-dimensional sequences or Theorem 4.7 for multidimensional sequences. Consistent with this theoretical result, in all of the examples which have been considered, the iteration has always converged to the correct sequence. Two one-dimensional examples are shown in Table 5.1 for a mixed phase sequence of length $N=8$. In the first example, a DFT of length $M=16$ is used whereas in the second example, the DFT length is extended to $M=128$. In both cases, the initial estimate of the unknown magnitude is chosen to be a constant. The results after 10, 100, 500, and 1000 iterations are presented along with the values of the Normalized Mean Square Error (NMSE) which is defined by

Table 5.1:

ONE-DIMENSIONAL ITERATIVE RECONSTRUCTION FROM PHASE

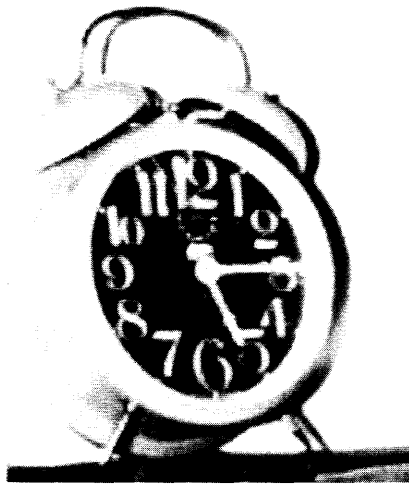
| DFT Length | Number of Iterations | x(0) | x(1) | x(2) | x(3) | x(4) | x(5) | x(6) | x(7) | NMSE |
|-------------------|----------------------|-------|-------|---------|-------|-------|-------|--------|--------|---------------------------|
| 16 | 10 | 4.000 | 0.461 | -3.841 | 0.650 | 2.584 | 3.252 | 5.823 | -2.312 | .14369 |
| | 100 | 4.000 | 1.936 | -8.975 | 1.897 | 4.744 | 6.911 | 11.749 | -4.903 | .52976 · 10 ⁻¹ |
| | 500 | 4.000 | 2.078 | -10.567 | 3.870 | 4.506 | 6.103 | 14.192 | -5.713 | .78196 · 10 ⁻² |
| | 1000 | 4.000 | 2.033 | -10.858 | 4.638 | 4.163 | 5.355 | 14.740 | -5.904 | .76965 · 10 ⁻³ |
| 128 | 10 | 4.000 | 0.765 | -4.178 | 1.727 | 2.108 | 2.632 | 6.139 | -2.222 | .82863 · 10 ⁻¹ |
| | 100 | 4.000 | 1.774 | -9.771 | 4.302 | 3.867 | 4.959 | 13.495 | -5.203 | .15958 · 10 ⁻² |
| | 500 | 4.000 | 1.997 | -10.990 | 4.996 | 4.007 | 5.011 | 14.996 | -5.988 | .96828 · 10 ⁻⁶ |
| | 1000 | 4.000 | 1.999 | -10.998 | 4.999 | 3.999 | 4.999 | 14.997 | -5.999 | .42165 · 10 ⁻⁸ |
| Original Sequence | | 4.0 | 2.0 | -11.0 | 5.0 | 4.0 | 5.0 | 15.0 | -6.0 | |

$$E_p = 1/N \sum_{n=0}^{N-1} [\sigma_x^{-1}x(n) - \sigma_p^{-1}x_p(n)]^2 \quad (5-23)$$

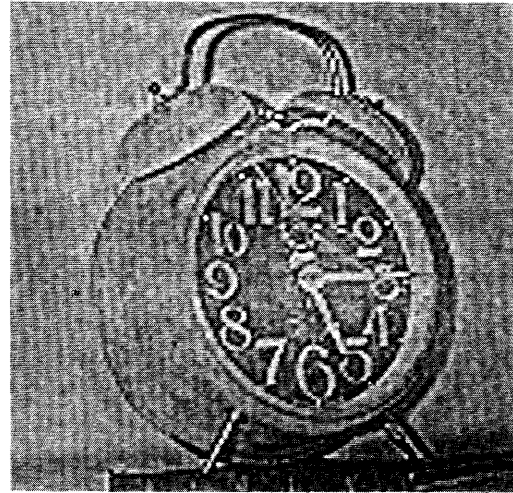
where σ_x and σ_p are the standard deviations of $x(n)$ and $x_p(n)$ respectively. This error criterion was chosen since it is invariant to scaling of either $x(n)$ or $x_p(n)$.

A two-dimensional example of iterative reconstruction from phase is shown in Figures 5.2 and 5.3. In Figure 5.2a is an original image, $x(n_1, n_2)$, which is 128x128 pixels in extent. Using a 256x256-point DFT, the phase-only synthesis of this image, $x_0(n_1, n_2)$, obtained by setting the DFT magnitude equal to a constant, is shown in Figure 5.2b. With the phase-only image as the initial estimate in the iteration, the images obtained after 10, 20, 50, and 100 iterations are shown in Figure 5.3.

As a measure of the error between $x_p(n_1, n_2)$ and $x(n_1, n_2)$, again consider the NMSE given in (5-23) where, in this case, the sum is taken over both indices, n_1 and n_2 . A plot of $\log[E_p]$ versus p is shown in Figure 5.4. Note that the error decreases rapidly over the first few iterations whereas, for large p , the error decreases very slowly. This behavior has been observed to be typical in all of the examples (both 1-D and 2-D) which have been considered. Therefore, since the number of iterations and, consequently, the computation time required to achieve a small error may be quite large, particularly when the support of the unknown sequence is large, it is of interest to consider methods for increasing the rate of convergence of the iteration. One possibility, as indicated in the 1-D example, is to increase the DFT length. Although it has been observed that increasing the DFT length may increase the rate of convergence, such an increase obviously results in an increase in the number of computations required per iteration and increases the memory requirements in the implementation of the algorithm. Another technique which has proved successful is described in Section V.2.2.3. First, however, the issue of convergence is addressed.

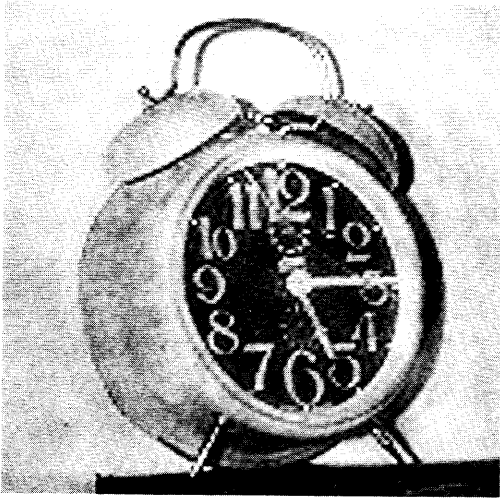


(a)

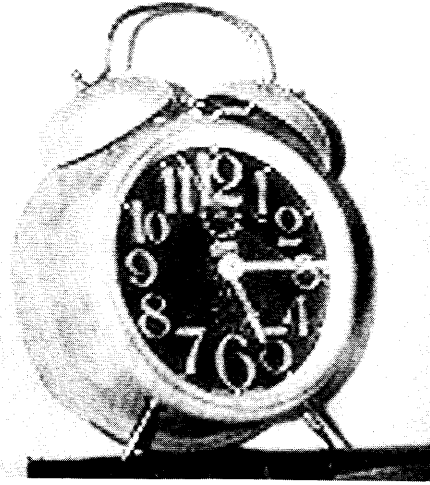


(b)

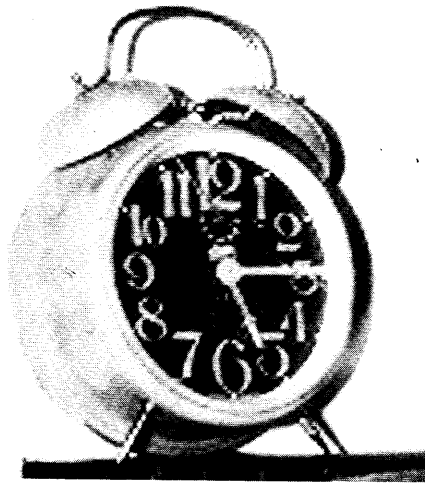
Figure 5.2: Original image and its phase-only synthesis. (a) Original image. (b) Phase-only image formed by combining the phase of the Fourier transform of image (a) with a constant magnitude.



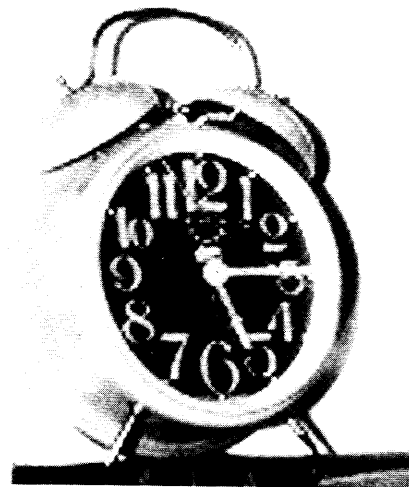
(a)



(b)



(c)



(d)

Figure 5.3: Iterative phase-only image reconstruction. Image reconstructed from phase-only image after (a) 10 iterations, (b) 20 iterations, (c) 50 iterations, and (d) 100 iterations.

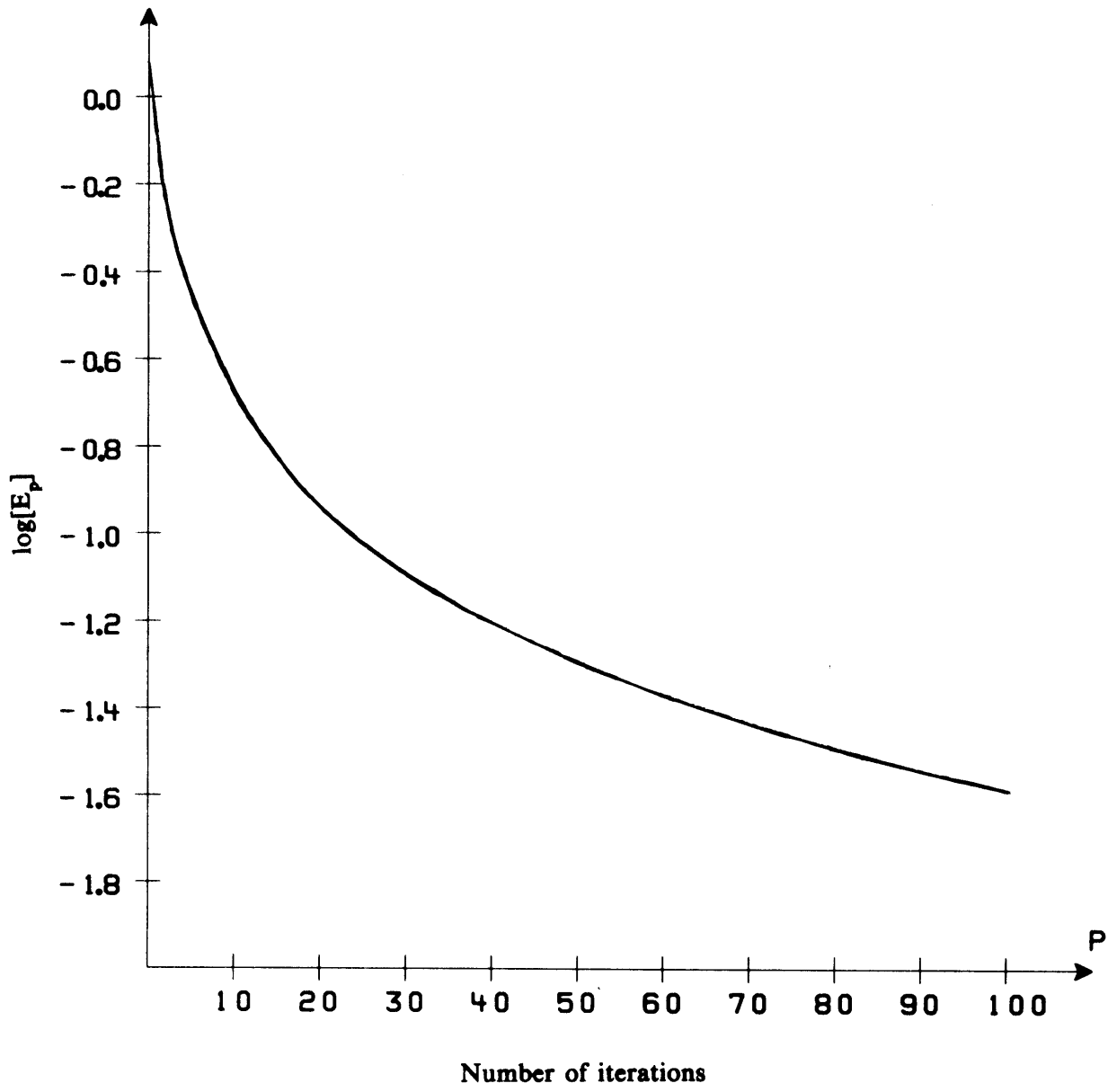


Figure 5.4: Plot of Normalized Mean Square Error

V.2.2.2: Convergence of the phase-only iteration

A fundamental question of considerable practical and theoretical importance concerns the conditions under which the phase-only iteration converges to the correct sequence. Therefore, in this section the issue of convergence is addressed. Specifically, in the one-dimensional case it is shown that the iteration will always converge to the correct sequence provided $x(0)$ and $\phi_x(k)_M$ uniquely specify $x(n)$, i.e., $x(n)$ satisfies the constraints of Theorem 3.3. In the multidimensional case, it may similarly be shown that the iteration will always converge to the correct sequence provided $x(n)$ satisfies the constraints of Theorem 4.7. However, since the proof of convergence for the one-dimensional case may be easily extended to the multidimensional case, the following discussions will focus only on the one-dimensional case.

Mathematical Formulation of the Phase-only iteration

Before the issue of convergence is addressed, it will be necessary to mathematically formalize the phase-only iteration. Therefore, let \underline{x}_p and \underline{y}_p be the vector representations of the sequences $x_p(n)$ and $y_p(n)$, respectively. Since $x_p(n)$ and $y_p(n)$ are sequences of length M , $\underline{x}_p, \underline{y}_p \in R^M$ where R^M is M -dimensional Euclidean space. Note that the functional relationship between \underline{x}_p and \underline{y}_p in the phase-only iteration may be expressed as

$$\underline{y}_p = T [\underline{x}_p] \quad (5-24)$$

where $T(-)$ is the mapping defined by (5-21). In other words, $T(-)$ operates on a sequence by setting it equal to zero outside the interval $[0, N-1]$ and equal to α_0 for $n=0$.

Now consider the mapping $\Phi(-)$ which replaces the phase of the M -point DFT of a sequence with the phase of the M -point DFT of $x(n)$, $\phi_x(k)$. Specifically,

$$\Phi[Y_p(k)] = |Y_p(k)| \exp[j\phi_x(k)] \quad (5-25)$$

With W and W^{-1} the linear mappings which represent the M -point DFT and IDFT, respectively, the functional relationship between \underline{x}_{p+1} and \underline{y}_p may then be expressed as

$$\underline{x}_{p+1} = B [\underline{y}_p] \quad (5-26)$$

where $B=W^{-1} \Phi W$. Therefore, combining (5-24) and (5-26), the phase-only iteration may be written as

$$\underline{x}_{p+1} = G [\underline{x}_p] \quad (5-27)$$

where $G:R^M \rightarrow R^M$ is the nonlinear mapping from R^M into R^M defined as the composition of the mappings T and B , i.e. $G=BT$. Similarly, the iteration may be mathematically formalized in terms of the relationship between \underline{y}_p and \underline{y}_{p+1} as

$$\underline{y}_{p+1} = F \underline{y}_p \quad (5-28)$$

where $F:R^M \rightarrow R^M$ is the mapping $F=TB$. In the following, it is shown that if \underline{x} is the vector representation of the sequence $x(n)$, then

$$\lim_{p \rightarrow \infty} \underline{y}_p = \underline{x} \quad (5-29)$$

provided $x(n)$ satisfies the constraints of Theorem 3.3.

Properties of the mapping F

As a measure of the distance between two vectors $\underline{u}, \underline{v} \in R^M$, let $d(\underline{u}, \underline{v})$ be the standard Euclidean metric which is defined by

$$d^2(\underline{u}, \underline{v}) = \sum_{k=0}^{M-1} [u(k) - v(k)]^2 \quad (5-30)$$

With this metric, it may be shown that F is a *nonexpansive* mapping on \mathbb{R}^M , i.e.,

$$d(F\underline{u}, F\underline{v}) \leq d(\underline{u}, \underline{v}) \quad \text{for all } \underline{u}, \underline{v} \in \mathbb{R}^M \quad (5-31)$$

In order to establish the nonexpansive property of F , consider first the mapping T .

With $I_T = 0 \cup [N, M-1]$ it follows that, for any $\underline{u}, \underline{v} \in \mathbb{R}^M$,

$$\begin{aligned} d^2(\underline{u}, \underline{v}) &= \sum_{k \in I_T} [u(k) - v(k)]^2 + \sum_{k \in I_T} [u(k) - v(k)]^2 \\ &\geq \sum_{k \in I_T} [u(k) - v(k)]^2 = d^2(T\underline{u}, T\underline{v}) \end{aligned} \quad (5-32)$$

where equality holds if and only if $u(k) = v(k)$ for $k \in I_T$. Therefore, T is a nonexpansive map.

It may similarly be shown that the mapping B is nonexpansive. Specifically, it follows from Parseval's Theorem that

$$d^2(\underline{u}, \underline{v}) = 1/M \sum_{k=0}^{M-1} |U(k) - V(k)|^2 \quad (5-33)$$

for any $\underline{u}, \underline{v} \in \mathbb{R}^M$. Using the triangle inequality for vector differences, (5-33) becomes

$$d^2(\underline{u}, \underline{v}) \geq 1/M \sum_{k=0}^{M-1} [|U(k)| - |V(k)|]^2 \quad (5-34)$$

where equality holds if and only if, for each k , $\phi_u(k) = \phi_v(k)$, or $|U(k)| = 0$, or $|V(k)| = 0$.

Therefore,

$$d^2(\underline{u}, \underline{v}) \geq 1/M \sum_{k=0}^{M-1} | U(k) \exp[j\phi_u(k)] - V(k) \exp[j\phi_v(k)] |^2 \quad (5-35)$$

which, again using Parseval's Theorem, becomes

$$d^2(\underline{u}, \underline{v}) \geq d^2(\underline{Bu}, \underline{Bv}) \quad (5-36)$$

as desired.

Finally, since both T and B are nonexpansive mappings, it follows that F is also nonexpansive. Specifically, combining (5-32) and (5-36)

$$d(\underline{Fu}, \underline{Fv}) = d[\underline{T(Bu)}, \underline{T(Bv)}] \leq d(\underline{Bu}, \underline{Bv}) \leq d(\underline{u}, \underline{v}) \quad \text{for all } \underline{u}, \underline{v} \in \mathbb{R}^M \quad (5-37)$$

Note, in addition, that the nonexpansiveness of T and B also implies that the mapping $G=BT$ is nonexpansive.

A consequence of the nonexpansiveness of F is the following property. Let \underline{x} be the vector representation of a sequence, $x(n)$, which is zero outside the interval $[0, N-1]$ with $x(0) = \alpha_0 \neq 0$ and with an M-point DFT with phase $\phi_x(k)$ [$x(n)$ need not necessarily satisfy the additional constraints of Theorem 3.3]. Since $F\underline{x} = \underline{x}$ and $F\underline{y}_p = \underline{y}_{p+1}$, with $\underline{u} = \underline{y}_p$ and $\underline{v} = \underline{x}$ in (5-37), it follows that

$$d(\underline{y}_{p+1}, \underline{x}) \leq d(\underline{y}_p, \underline{x}) \quad (5-38)$$

In other words, the mean square error between \underline{x} and \underline{y}_p is a nonincreasing function of p. However, neither (5-37) nor (5-38) are sufficient to insure that \underline{y}_p will converge to \underline{x} . In fact, if \underline{x} represents a sequence which has a z-transform with zeros on the unit circle or in conjugate reciprocal pairs, then there is more than one sequence which satisfies the time and frequency domain constraints. Therefore, since each of these sequences is a solution to the equation $F\underline{x} = \underline{x}$, even if the iteration (5-28) converges, then it may converge to any one of these solutions.

Finally, it should be pointed out that if (5-28) is modified as

$$\underline{y}_{p+1} = (1-\lambda)\underline{y}_p + \lambda F(\underline{y}_p) \quad (5-39)$$

where λ is a real number with $\lambda \in (0,1)$, then it follows from Theorem 2A.4 in Appendix II that \underline{y}_p will always converge to a solution to the equation $F\underline{x}=\underline{x}$. Therefore, if \underline{x} satisfies the constraints of Theorem 3.3, then the solution to the equation $F\underline{x}=\underline{x}$ is unique and the sequence of vectors \underline{y}_p in (5-39) will always converge to \underline{x} . The next section addresses the convergence of the iteration $F\underline{y}_p=\underline{y}_{p+1}$, i.e., the case in which $\lambda=1$ in (5-39).

Convergence

With the phase-only iteration formalized as a mapping $F:\mathbb{R}^M \rightarrow \mathbb{R}^M$, it is now possible to address the issue of convergence of the iteration $F\underline{y}_p=\underline{y}_{p+1}$. Therefore, let $x(n)$ be a real 1-D sequence which is zero outside the interval $[0,N-1]$ with $x(0)=\alpha_0 \neq 0$ and an M -point DFT with phase $\phi_x(k)$. Since $x(n)$ satisfies both the time and frequency domain constraints of the iteration, then $x(n)$ is a solution to the equation $F\underline{x}=\underline{x}$. Therefore, it follows from Theorem 2A.3 in Appendix II that $\underline{y}_p(n)$ converges to $x(n)$ for any initial estimate $\underline{y}_0(n)$ of $x(n)$ provided F is a *strictly nonexpansive* map on \mathbb{R}^M so that equality holds in (5-37) only if $\underline{u}=\underline{v}$, i.e.,

$$d(F\underline{u},F\underline{v}) < d(\underline{u},\underline{v}) \quad \text{for all } \underline{u},\underline{v} \in \mathbb{R}^M \text{ provided } \underline{u} \neq \underline{v} \quad (5-40)$$

Although F is nonexpansive on \mathbb{R}^M , it is not in general a strictly nonexpansive map. However, if $x(n)$ satisfies the constraints of Theorem 3.3 and if $M \geq 2N-1$, then it may be shown that F is strictly nonexpansive on \mathbb{R}^M . A proof of this only requires showing that if $d(F\underline{u},F\underline{v})=d(\underline{u},\underline{v})$ then $\underline{u}=\underline{v}$. Therefore, suppose that $x(n)$ satisfies the constraints of Theorem 3.3 and that $M \geq 2N-1$ and let \underline{u} and \underline{v} be any two vectors in \mathbb{R}^M for which $d(F\underline{u},F\underline{v})=d(\underline{u},\underline{v})$. In this case it follows that

$$d(T\underline{B}\underline{u},T\underline{B}\underline{v}) = d(\underline{B}\underline{u},\underline{B}\underline{v}) = d(\underline{u},\underline{v}) \quad (5-41)$$

However, recall that the left equality in (5-41) holds if and only if

$$(Bu)(k) = (Bv)(k) \quad \text{for } k \in I_T \quad (5-42)$$

Therefore, consider the vector \underline{z} defined by

$$\underline{z} = \underline{Bu} - \underline{Bv} \quad (5-43)$$

and note that $z(k)=0$ for $k \in I_T$. Since the phase of the M -point DFT's of \underline{Bu} and \underline{Bv} are equal to $\phi_x(k)$, then

$$\tan[\phi_x(k)] = \tan[\phi_x(k)] \quad (5-44)$$

Thus, if $x(n)$ satisfies the constraints of Theorem 3.3 and $M \geq 2N-1$, then $z(n) = \beta x(n)$ for some real number β . However, since $z(0)=0$ and $x(0) = \alpha_y \neq 0$, then $\beta=0$. Therefore, $z(n)=0$ for all n from which it follows that $\underline{Bu} = \underline{Bv}$ and

$$d(\underline{Bu}, \underline{Bv}) = 0 \quad (5-45)$$

Finally, from (5-41), it follows that $d(u,v)=0$ and, consequently, that $\underline{u} = \underline{v}$. Therefore, equality holds in (5-41) only if $\underline{u} = \underline{v}$ and F is strictly nonexpansive. As a result, it follows from Theorem 2A.3 in Appendix II that the iteration $\underline{Fy}_p = \underline{y}_{p+1}$ will converge to \underline{x} for any initial estimate, \underline{y}_0 .

The strict nonexpansiveness of F , and hence the convergence of \underline{y}_p to \underline{x} , depends upon the assumption that $x(n)$ satisfies the constraints of Theorem 3.3 and that $M \geq 2N-1$. Specifically, as previously noted, if $x(n)$ does not satisfy the constraints of Theorem 3.3 or if $M < 2N-1$, then there are two distinct sequences, $u(n)$ and $v(n)$, which satisfy the time and frequency domain constraints of the iteration. Both of these sequences, therefore, are solutions to the equation $F\underline{x} = \underline{x}$. Consequently, F is not strictly nonexpansive on all of R^M since $d(\underline{Fu}, \underline{Fv}) = d(\underline{u}, \underline{v})$.

V.2.2.3: Adaptive relaxation

Although the iteration $F\mathbf{y}_p = \mathbf{y}_{p+1}$ will always converge to the correct sequence provided the solution to $F\mathbf{x} = \mathbf{x}$ is unique, in general the convergence is very slow. Thus, it is of interest to consider methods for increasing the rate of convergence. Therefore, motivated by various relaxation techniques [37], suppose that the phase-only iteration (5-27) is modified as follows:

$$\mathbf{x}_{p+1} = (1-\lambda_p) \mathbf{x}_p + \lambda_p G(\mathbf{x}_p) \quad (5-46)$$

where the relaxation parameter, λ_p , is a scalar which may be allowed to vary as a function of p . With the vector \mathbf{r}_p defined by

$$\mathbf{r}_p = G(\mathbf{x}_p) - \mathbf{x}_p \quad (5-47)$$

(5-46) may be equivalently written as

$$\mathbf{x}_{p+1} = \mathbf{x}_p + \lambda_p \mathbf{r}_p \quad (5-48)$$

Several special cases of (5-48) are immediately apparent. If λ_p is a fixed constant, λ_0 , then (5-48) reduces to the original iteration (5-27) when $\lambda_0=1$, whereas $\lambda_0=0$ produces the trivial result $\mathbf{x}_{p+1} = \mathbf{x}_p$. Intermediate values of λ_0 , i.e., $0 < \lambda_0 < 1$ correspond to what is commonly referred to as the *under-relaxed* form of (5-27) [37].

A common difficulty encountered with iterations of the form (5-48) is the determination of the optimum value of λ_p for which the rate of convergence of the iteration is maximized. However, for the phase-only iteration, it is relatively straightforward to derive an expression for λ_p which is optimum in a certain sense. Specifically, consider partitioning (5-48) as follows:

$$\begin{bmatrix} \underline{x}_{p+1}^{(1)} \\ \underline{x}_{p+1}^{(2)} \end{bmatrix} = \begin{bmatrix} \underline{x}_p^{(1)} \\ \underline{x}_p^{(2)} \end{bmatrix} + \lambda_p \begin{bmatrix} \underline{r}_p^{(1)} \\ \underline{r}_p^{(2)} \end{bmatrix} \quad (5-49)$$

where the lower part of each partitioned vector is a subvector of length $M-N$ which corresponds to the interval over which the desired vector, \underline{x} , is known to be equal to zero. Now, note that the tangent of the phase of the DFT of \underline{x}_{p+1} is equal to the tangent of the phase of the DFT of \underline{x} for any choice of the relaxation parameter λ_p [This follows from (5-46) by observing that the phase of the DFT of \underline{x}_p and $G(\underline{x}_p)$ are equal to the phase of the DFT of \underline{x}]. Consequently, if \underline{x} satisfies the constraints of Theorem 3.3 and $M \geq 2N-1$, then $\underline{x}_{p+1}^{(2)}=0$ implies that $\underline{x}_{p+1}=\beta\underline{x}$ for some scale factor β . Therefore, a reasonable approach for selecting λ_p is to choose that value, $\lambda_p=\lambda_p^*$ which minimizes $\|\underline{x}_{p+1}^{(2)}\|^2$, i.e.,

$$\frac{d}{d\lambda_p} \left[\|\underline{x}_{p+1}^{(2)}\|^2 \right]_{\lambda_p=\lambda_p^*} = 0 \quad (5-50)$$

where $\|\underline{x}_{p+1}^{(2)}\|^2$ is the Euclidean norm of the vector $\underline{x}_{p+1}^{(2)}$. The solution to (5-50) may be easily shown to be

$$\lambda_p = - \frac{\langle \underline{x}_p^{(2)}, \underline{r}_p^{(2)} \rangle}{\|\underline{r}_p^{(2)}\|^2} \quad (5-51)$$

where $\langle \underline{x}_p^{(2)}, \underline{r}_p^{(2)} \rangle$ is the inner product of the vectors $\underline{x}_p^{(2)}$ and $\underline{r}_p^{(2)}$.

Although it has not been shown theoretically that this procedure of "adaptive relaxation" will always lead to a convergent solution, in all of the examples which have been considered, the iteration has always converged to the correct sequence. Furthermore, it has been observed that the number of iterations required to achieve a given normalized mean square error is, in general, substantially reduced when adaptive re-

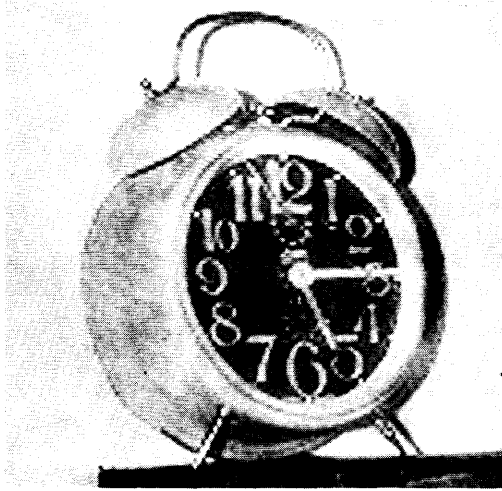
laxation is used. For example, consider the image $x(n_1, n_2)$ shown in Figure 5.2a. Using the phase only image in Figure 5.2b as the initial estimate of $x(n_1, n_2)$, the results which are obtained after 5, 10, 15, and 20 iterations using adaptive relaxation are shown in Figure 5.5. For a quantitative comparison of these results with those obtained with the unaccelerated iteration (5-27), a plot of $\log[E_p]$ versus p is shown in Figure 5.6. As evidenced by this figure, in contrast to the unaccelerated iteration, adaptive relaxation tends to maintain a rapid decrease in the normalized mean square error, even for relatively large values of p .

Assuming that the DFT length used in the iteration is $M=2N$, the number of multiplications required to compute λ_p is M and the number of multiplications required to determine \underline{x}_{p+1} in (5-48) is also M . Therefore, this approach requires an additional $2M$ multiplications per iteration over the unaccelerated iteration (5-27). Since the number of multiplications required for each iteration in (5-27) is on the order of $M \log_2 M$, if $M \gg 1$ this additional computation is negligible. However, an important consideration in the implementation of (5-48) is the requirement for additional memory since two vectors of length M , namely \underline{x}_p and $G(\underline{x}_p)$, need to be stored.

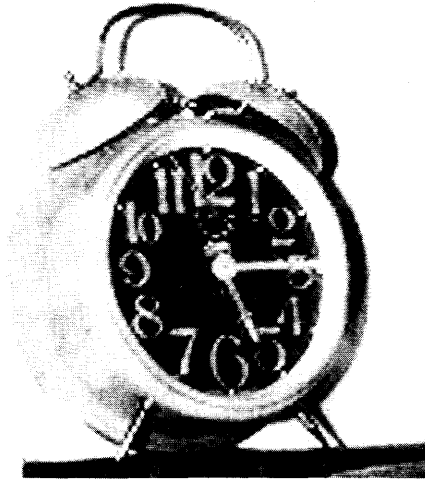
The iteration (5-48) may be considered as a first-order acceleration of the basic iteration (5-27) since it incorporates one previous estimate, \underline{x}_p , of \underline{x} to modify the current estimate $G(\underline{x}_p)$. It is possible, therefore, to consider generalizing (5-48) to include a linear combination of K previous estimates. Specifically, let the set of vectors $\underline{r}_{p,k}$ for $k=0,1,\dots,K-1$ be defined by

$$\underline{r}_{p,k} = G(\underline{x}_p) - \underline{x}_{p-k} \quad (5-52)$$

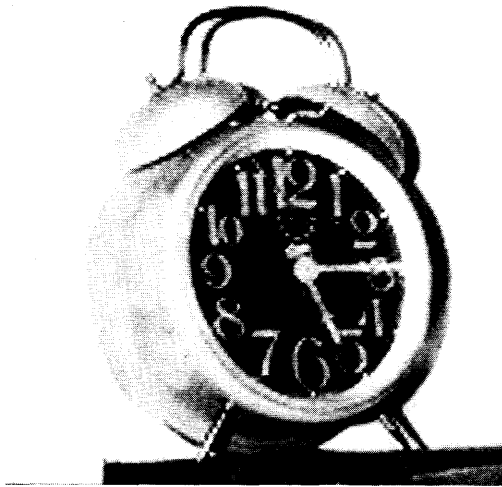
and let $\underline{\lambda}_p^T = [\lambda_{p,0}, \lambda_{p,1}, \dots, \lambda_{p,K-1}]$ be a relaxation vector which may, in general, be a function of p . Then (5-48) may be generalized as follows:



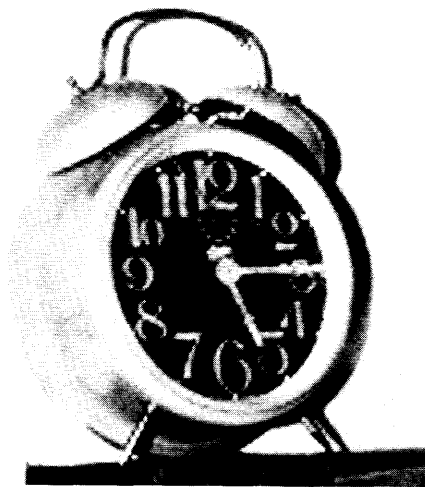
(a)



(b)



(c)



(d)

Figure 5.5: Iterative phase-only image reconstruction with adaptive relaxation. Image reconstructed from phase-only image after (a) 5 iterations, (b) 10 iterations, (c) 15 iterations, and (d) 20 iterations.

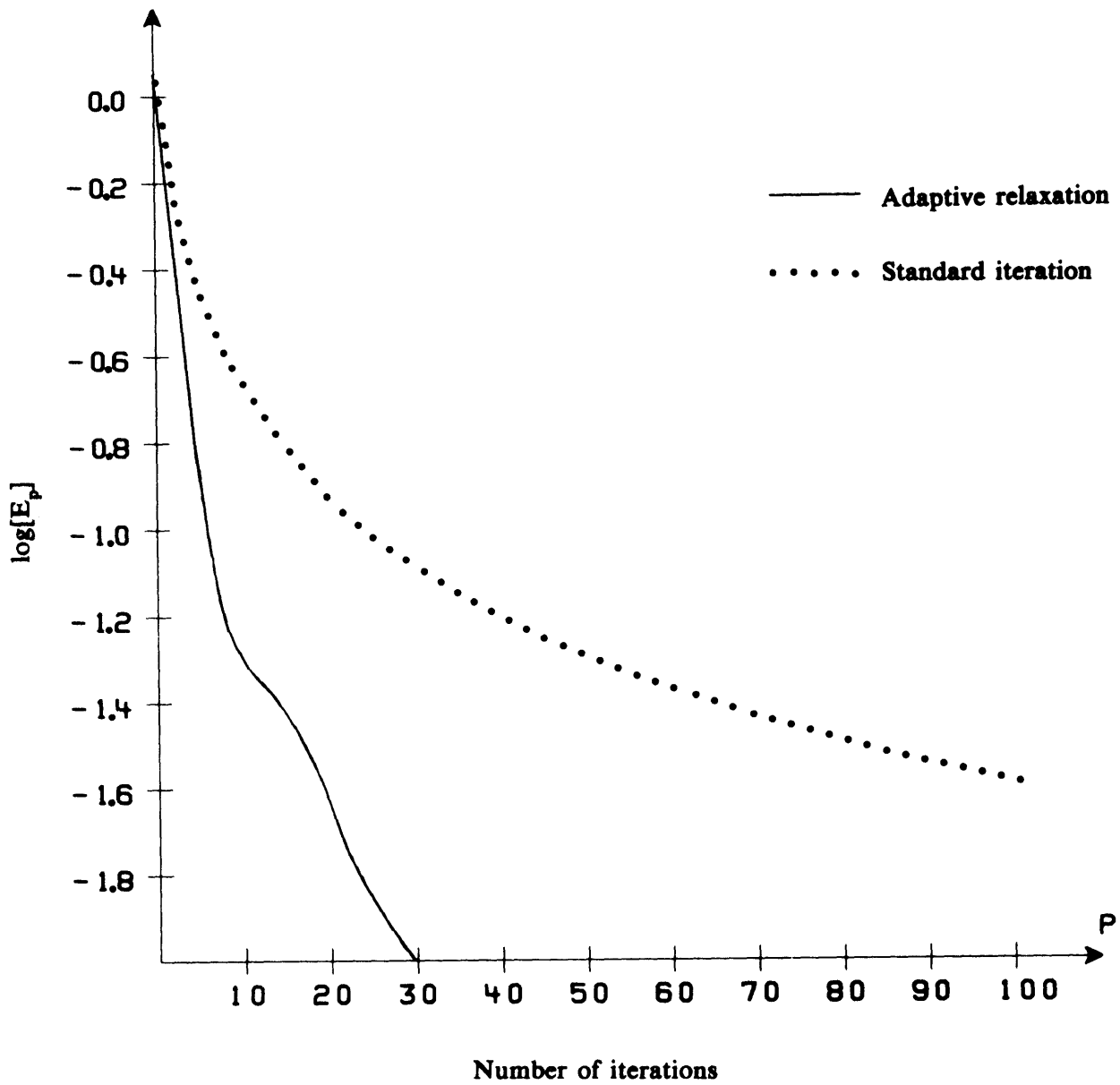


Figure 5.6: Plot of Normalized Mean Square Error with adaptive relaxation.

$$\underline{x}_{p+1} = \underline{x}_p + \mathbf{A} \underline{\lambda}_p \quad (5-53)$$

where

$$\mathbf{A} = [\underline{r}_{p,0}, \underline{r}_{p,1}, \dots, \underline{r}_{p,K-1}] \quad (5-54)$$

is an $M \times K$ matrix. As in (5-49), suppose that (5-53) is partitioned as follows:

$$\begin{bmatrix} \underline{x}_{p+1}^{(1)} \\ \underline{x}_{p+1}^{(2)} \end{bmatrix} = \begin{bmatrix} \underline{x}_p^{(1)} \\ \underline{x}_p^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{G} \\ \mathbf{H} \end{bmatrix} \underline{\lambda}_p \quad (5-55)$$

where \mathbf{G} and \mathbf{H} are $N \times K$ and $(M-N) \times K$ matrices, respectively. The vector $\underline{\lambda}_p$ which minimizes $|\underline{x}_{p+1}^{(2)}|^2$ is then defined by the equation

$$(\mathbf{H}^T \mathbf{H}) \underline{\lambda}_p = -\mathbf{H}^T \underline{x}_p^{(2)} \quad (5-56)$$

If the columns of \mathbf{H} are linearly independent, then $(\mathbf{H}^T \mathbf{H})$ is invertible and

$$\underline{\lambda}_p = -(\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T \underline{x}_p^{(2)} \quad (5-57)$$

Furthermore, if the relaxation vector $\underline{\lambda}_p$ is of length $K=M$ and if the columns of \mathbf{H} are linearly independent, then \mathbf{H} is invertible. In this case,

$$\underline{\lambda}_p = -\mathbf{H}^{-1} \underline{x}_p^{(2)} \quad (5-58)$$

and

$$\underline{x}_{p+1} = \begin{bmatrix} \underline{x}_p^{(1)} - \mathbf{G} \mathbf{H}^{-1} \underline{x}_p^{(2)} \\ 0 \end{bmatrix} \quad (5-59)$$

Therefore, if \underline{x} satisfies the constraints of Theorem 3.3, then \underline{x}_{p+1} must be equal to a scaled version of \underline{x} .

V.3: Magnitude-only Reconstruction

Due to its importance in the phase retrieval problem for wave amplitudes and coherence functions [11], an algorithm for the reconstruction of a multidimensional sequence from the magnitude of its Fourier transform has been the objective of many research efforts and the subject of a considerable number of published papers. Nevertheless, in the absence of any additional knowledge of the desired sequence, there is as yet no practical algorithm which will always recover the correct phase from only magnitude information. It appears that there are at least two problems which have made the development of such an algorithm difficult. The first is the nonlinear relationship between the coefficients of a multidimensional sequence and the magnitude of its Fourier transform. Recall, for example, that the Fourier transform magnitude of a sequence $x(\underline{n})$ may be used to obtain the autocorrelation, $r_x(\underline{n})$, of $x(\underline{n})$. Therefore, with magnitude information alone, it is always possible to define a set of second-order equations which relate the known values of the autocorrelation of $x(\underline{n})$ at various lags with the unknown coefficients of the desired multidimensional sequence, e.g.,

$$r_x(\underline{n}) = \sum_{\underline{k}} x(\underline{n}+\underline{k})x(\underline{k}) \quad (5-60)$$

One possible solution to the phase-retrieval problem may thus consist of solving these nonlinear equations for $x(\underline{n})$. Although there exist techniques for finding a solution to a set of simultaneous nonlinear equations in more than one unknown [37], when the number of equations and the number of unknowns become large, these algorithms are not practical.

The second problem which has contributed to the difficulty in developing a practical phase-retrieval algorithm is that, without any phase information, it is not generally possible to obtain a very accurate estimate of the unknown sequence from

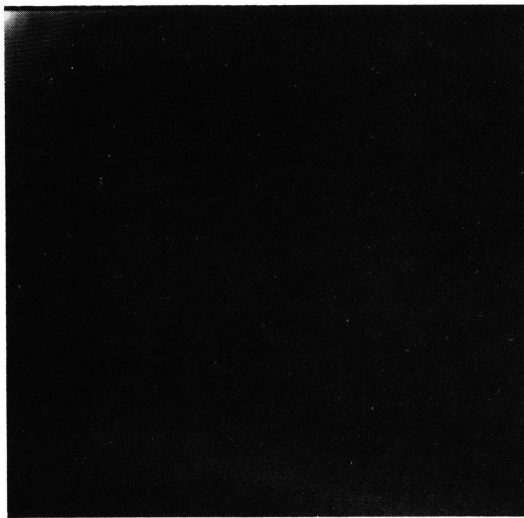
only the magnitude of its Fourier transform. Recall, for example, that when the Fourier transform magnitude of an image is combined with zero phase, the result which is obtained upon an inverse Fourier transformation is an image which generally does not contain any recognizable features (See Figure 2.2b). In contrast, a phase-only image which has a Fourier transform with the correct phase and a constant magnitude contains many of the important characteristics of the original image (See Figure 2.2c).

In spite of these difficulties, an algorithm which has been proposed for 2-D magnitude-only reconstruction [12] is an iterative procedure similar in style to the phase-only algorithm described in Section V.2. Specifically, this algorithm involves the repeated Fourier transformation between the time and frequency domains where, in each domain, the known information about the desired sequence is imposed on the current estimate. In the time domain, for example, a sequence is constrained to have a given region of support whereas in the frequency domain, the sequence is constrained to have a given Fourier transform magnitude. Unlike the phase-only iteration, however, it has been observed that the magnitude-only iteration will not generally converge to the correct solution even if the desired sequence satisfies the uniqueness constraints of Theorem 4.10. There appear to be two factors which determine whether or not the iteration converges to the correct solution. The first pertains to the ability to obtain an initial estimate to begin the iteration which is sufficiently close to the correct solution. It has been observed, for example, that for a 2-D sequence with support $R(N_1, N_2)$, if the initial estimate used in the iteration has a Fourier transform with the correct magnitude and either a zero phase or a random phase, then the iteration will not generally converge to the correct sequence. This conclusion is based on many attempts to reconstruct a 2-D sequence from the magnitude of its Fourier transform and the sequences considered had regions of support which varied from $R(2,2)$ to $R(128,128)$.

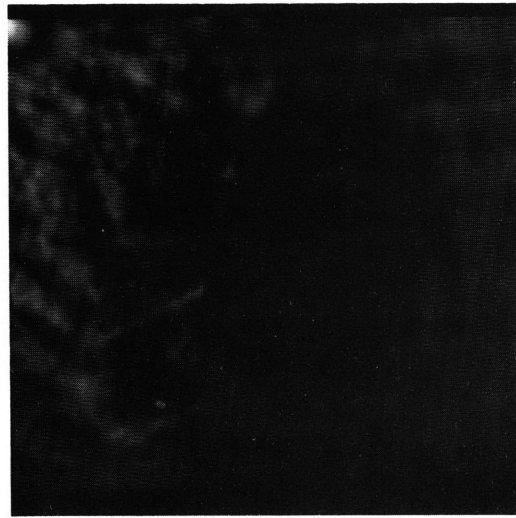
Even in those cases for which the desired sequence was known to have an irreducible z -transform and thus satisfied the constraints of Theorem 4.10, the correct solution was not generally obtained. Two examples are shown in Figure 5.7 for an image which is 128x128 pixels in extent. In Figure 5.7a is the magnitude-only synthesis of the image in Figure 5.2a which was formed by taking a 256x256-point DFT of the original image, setting the phase equal to zero, and taking the inverse DFT. Using this image as the initial estimate in the magnitude-only iteration, the result which is obtained after 30 iterations is shown in Figure 5.7b. If, instead of zero phase, a random phase is used in the magnitude-only synthesis, the result is the image shown in Figure 5.7c. Using this image as the initial estimate in the magnitude-only iteration, the result which is obtained after 30 iterations is shown in Figure 5.7d. Although the results after only 30 iterations are shown, in both cases there is virtually no change in the reconstructed image from one iteration to the next after the first 10 or 20 iterations. Similar results have been observed for 2-D sequences with smaller support, e.g. $R(4,4)$, in which the magnitude-only iteration was run for up to 1000 iterations.

Whereas Figure 5.7 illustrates the effects which typically occur when the magnitude-only iteration is initialized with an estimate which is not a close approximation to the correct solution, Figure 5.8 shows that, with the appropriate initial conditions, the iteration tends to converge to the correct solution. Specifically, shown in Figure 5.8a is an "amplitude-only" image, $x_0(n_1, n_2)$, which was obtained from the Fourier transform of the image in Figure 5.2a by quantizing the phase to one bit so that $X_0(\omega_1, \omega_2)$ has the correct magnitude and a phase which, for each frequency, is equal to either zero or π . In other words,

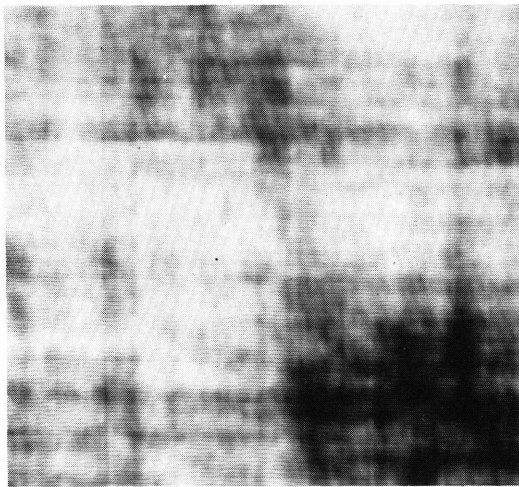
$$X_0(\omega_1, \omega_2) = \begin{cases} |X(\omega_1, \omega_2)| & \text{if } -\pi/2 < \phi_x(\omega_1, \omega_2) \leq \pi/2 \\ -|X(\omega_1, \omega_2)| & \text{otherwise} \end{cases} \quad (5-61)$$



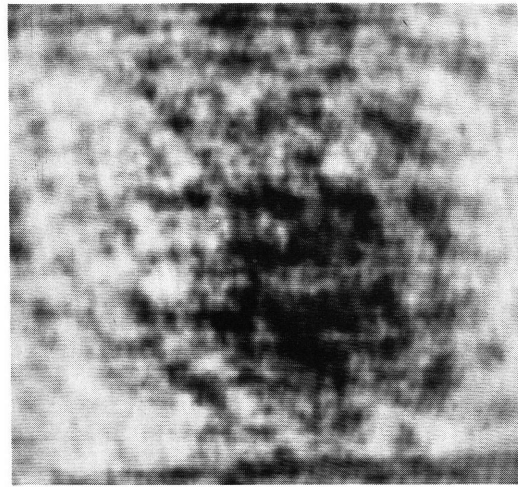
(a)



(b)

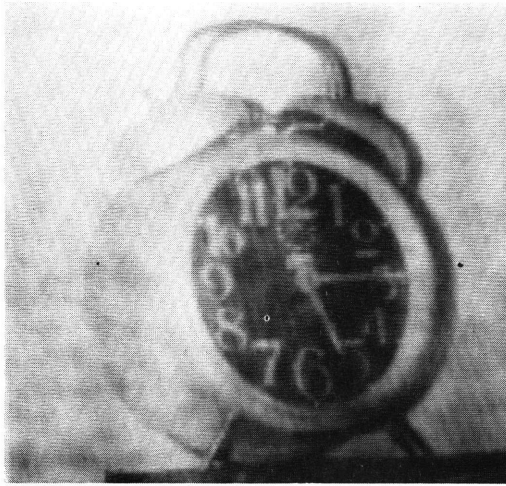


(c)

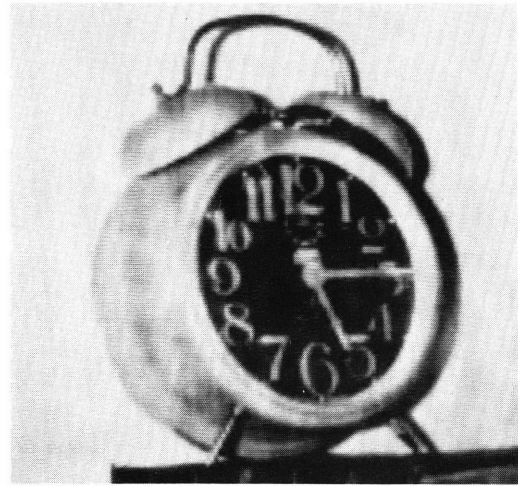


(d)

Figure 5.7: Iterative magnitude-only image reconstruction. (a) Magnitude-only image formed by combining the correct Fourier transform magnitude with zero phase. (b) Image reconstructed after 30 iterations. (c) Magnitude-only image formed by combining the correct Fourier transform magnitude with random phase. (d) Image reconstructed after 30 iterations.



(a)



(b)

Figure 5.8: Iterative amplitude-only image reconstruction. (a) Amplitude-only image formed by combining the correct Fourier transform magnitude with one bit of phase information. (b) Image reconstructed after 20 iterations.

where $X(\omega_1, \omega_2) = |X(\omega_1, \omega_2)| \exp[j\phi_x(\omega_1, \omega_2)]$ is the Fourier transform of the image in Figure 5.2a. With $x_0(n_1, n_2)$ used as the initial estimate in the iteration, the result which is obtained after 20 iterations is shown in Figure 5.8b. Similar results have been observed in other examples in which the initial estimate used to begin the iteration were sufficiently close to the correct solution.

The second factor which appears to have an effect on the convergence of the iteration concerns the shape of the known region of support of the sequence. Specifically, recall from Theorem 4.10 that magnitude information alone may only uniquely specify a multidimensional sequence to within a sign, a linear shift, and a time-reversal. Therefore, if a 2-D sequence $x(n_1, n_2)$ is known to have support $R(N_1, N_2)$, even if information were available to resolve the sign and linear shift ambiguities, there still would be two possible solutions to the magnitude-only reconstruction problem, namely $x(n_1, n_2)$ and $\tilde{x}(n_1, n_2) = x(N_1 - n_1, N_2 - n_2)$. This ambiguity is a result of the fact that $R(N_1, N_2)$ is a "symmetric" region of support so that for any sequence $x(n_1, n_2)$ which has support $R(N_1, N_2)$, then $\tilde{x}(n_1, n_2)$ also has support $R(N_1, N_2)$. This property, however, is not true for "non-symmetric" regions of support. For example, suppose that $x(n_1, n_2)$ is known to have a triangular region of support, $T(N)$, i.e., $x(n_1, n_2) = 0$ whenever $n_1 < 0$, $n_2 < 0$, or $n_1 + n_2 \geq N$. Assuming that no smaller triangular region of support would contain the non-zero values of $x(n_1, n_2)$, the only sequences with support $T(N)$ and a Fourier transform magnitude equal to $|X(\omega_1, \omega_2)|$ are $\pm x(n_1, n_2)$. In this case, therefore, if the triangular support constraint is imposed in the iteration along with a possible scaling by a factor of (-1) to force an arbitrary but fixed non-zero value of $x_p(n_1, n_2)$ to be positive, then there is only one solution consistent with the imposed time and frequency domain constraints. As may be expected, the convergence of the magnitude-only iteration is more likely to occur with a non-symmetric support con-

straint than with a symmetric support constraint. This, in fact, has been observed to be true in many of the examples which were considered. Furthermore, in each case the convergence of the iteration was attributed to the fact that a non-symmetric region of support was imposed in the iteration. Specifically, note that if $x(n_1, n_2)$ has support $T(N)$, then it also has support $R(N, N)$. Therefore, while the iteration usually converges to $x(n_1, n_2)$ when the time domain constraint sets $x_p(n_1, n_2)$ equal to zero outside $T(N)$, when $R(N, N)$ is used as the support constraint, convergence of the iteration is not generally obtained.

CHAPTER VI: SUMMARY AND CONCLUSIONS

This thesis considered the problem of reconstructing either a one-dimensional or a multidimensional sequence from only the phase or magnitude of its Fourier transform. The first issue which was addressed concerned the development of some conditions under which a sequence is uniquely defined in terms of only phase or magnitude information. In particular, it was shown that for a one-dimensional sequence, a finite length constraint is, in most cases, a sufficient condition for the phase to uniquely specify the sequence to within a scale factor. Furthermore, it was shown that if the sequence is of length N , then only $(N-1)$ phase samples are required for this unique specification provided the phase samples correspond to distinct frequency values in the interval $(0, \pi)$. In the case of magnitude, however, due to the possibility of "zero flipping", it was shown that in the absence of additional information or constraints, a finite length constraint is not sufficient for magnitude information alone to uniquely specify a one-dimensional sequence. Nevertheless, a condition for uniqueness in terms of magnitude was presented which is an extension of the minimum and maximum phase requirement and includes them as special cases.

For multidimensional sequences, a set of conditions for a unique solution in terms of phase were developed which are similar to the one-dimensional case. Specifically, it was shown that a finite support constraint is, in most cases, sufficient for a multidimensional sequence to be uniquely defined to within a scale factor by its phase. In addition, however, as opposed to the result for one-dimensional sequences, it was shown that a finite support constraint is also sufficient, in most cases, for a multidimensional sequence to be uniquely defined by the magnitude of its Fourier transform to within a sign, a linear shift, and a time-reversal. Finally, it was shown that if a

multidimensional sequence with finite support is uniquely defined by its phase or magnitude, then only a finite number of phase or magnitude samples, respectively, are required for this unique specification provided these samples are obtained from a Discrete Fourier Transform of the appropriate size.

The second issue which was addressed concerned the development of algorithms for the reconstruction of a one-dimensional or a multidimensional sequence from the phase or magnitude of its Fourier transform. In particular, several practical algorithms were presented for reconstructing a sequence from samples of its phase. These algorithms, which included iterative as well as non-iterative approaches, always lead to the correct solution assuming that the appropriate uniqueness constraints are fulfilled. An iterative procedure was also described for the reconstruction of a sequence from only the magnitude of its Fourier transform. The success of this algorithm, however, appears to depend upon the availability of an initial estimate of the unknown sequence which is sufficiently close to the correct solution or a region of support which is non-symmetric.

Although a number of results have been presented which relate to the reconstruction of a sequence from either the phase or magnitude of its Fourier transform, many important questions and interesting problems still remain to be investigated in future research efforts. First is the effect of noise in the reconstruction of a sequence from only phase information. Since, in any practical setting, there will be limits to the accuracy in which the phase may be measured or computed, it will be important to understand the sensitivity of the various phase-only reconstruction algorithms to errors in the phase samples. Although some experimental results have been obtained on the effect of noisy phase on the reconstruction of a sequence with the non-iterative algorithm in Section V.2.1.1 [10], it will be important to add to these results, a theoretical

analysis of the errors introduced by noisy phase. For the iterative procedure, it will also be important to study the error in the sequence reconstructed from noisy phase as a function of the number of iterations. Although the error in the reconstructed sequence will be the same, in the limit, as the error obtained in the non-iterative algorithm for uniformly spaced phase samples, it may be possible that the error after a finite number of iterations is less than the error of the convergent solution. This may be the case, in particular, if it is possible to obtain an initial estimate which is sufficiently close to the correct solution by, for example, combining the noisy phase with an estimate of the correct magnitude. Such a magnitude estimate may possibly consist of a set of noisy magnitude measurements or simply a magnitude which is in some way 'representative' of the class of sequences of interest.

A related topic is the development of a phase-only reconstruction algorithm which is "robust" in the presence of noise. One approach might involve the incorporation of a noise reduction technique into an algorithm which assumes some statistics on the noise in the phase. Another approach may involve the incorporation of additional information or constraints in the reconstruction algorithm. In the iterative algorithm, for example, an estimate or model of the unknown magnitude may be incorporated as an additional constraint in the iteration. If, on the other hand, the number of phase samples which are known exceeds the minimum number required for a unique solution, it may be possible to reduce the effect of noise by finding the sequence for which the phase is, in some sense, "as close as possible" to the given phase.

Another important topic for future research concerns the development of an efficient algorithm for reconstructing a multidimensional sequence from the magnitude of its Fourier transform. Although a number of algorithms have been proposed, the success of these algorithms appears to be related to the availability of additional in-

formation or, in the case of the iterative algorithm described in Section V.3, on the ability to obtain an initial estimate which is sufficiently close to the correct solution or on a non-symmetric region of support.

There are also a number of theoretical questions related to the uniqueness of a multidimensional sequence in terms of phase or magnitude information which need to be answered. For example, it is of interest to determine whether or not the multidimensional uniqueness constraints in Chapter IV hold for complex-valued sequences. Such a result will be important in the context of the phase retrieval problems described in Chapter II in which an electromagnetic wave, represented in terms of a complex-valued function in two or more variables, is to be reconstructed from the magnitude of its Fourier transform. Another question is concerned with the uniqueness of a multidimensional sequence in terms of its phase. In the one-dimensional case, it was shown that a sequence is uniquely defined by its phase if it is finite in length and contains no zero phase factors. The constraint for multidimensional sequences, however, is slightly different since it restricts sequences with finite support to have no symmetric factors. Although a zero phase factor may always be written as a symmetric factor times a linear phase factor, the converse is not necessarily true due to the plus or minus sign in the definition of a symmetric factor. Therefore, it will be of theoretical interest to determine whether or not it is sufficient to exclude only zero-phase factors in the multidimensional case and, thus, to obtain an equivalence between the one-dimensional and multidimensional uniqueness constraints.

Finally, there are many interesting and important applications to be explored in future research efforts which may potentially benefit from the results presented in this thesis. A number of applications related to the reconstruction of signals from only magnitude information have been outlined in Chapter II and will not be repeated here.

There are in addition, however, a number of applications to be explored in the context of phase-only reconstruction. One such application is the problem of deconvolution described briefly in Chapter I. Specifically, suppose that an observed signal consists of the convolution of a desired signal with the impulse response of some unknown filter. If the frequency response of the filter has a phase which is nearly zero, then the observed signal will have a phase which is approximately equal to the phase of the desired signal. In the absence of noise and assuming that the convolutional model is correct and that the phase of the filter's frequency response is zero, the desired signal is uniquely specified in terms of the phase of the observed signal. For this case, therefore, it is reasonable to consider a solution to the deconvolution problem which consists of a phase-only reconstruction from the phase of the observed signal. In other applications, it may be that the phase of the filter's frequency response is not zero but that the impulse response of the filter is known approximately. In this case, although the desired signal may theoretically be recovered by inverse filtering, this approach is generally very sensitive to noise. It is possible, however, to consider deconvolving the two signals by subtracting the phase of the frequency response of the filter from the phase of the observed signal and then performing a phase-only reconstruction from the resulting phase.

Another application, which may be viewed as a problem of deconvolution, is the multiple arrival or echo removal problem. Specifically, consider the case in which an unknown signal arrives an arbitrary number of times with an unknown attenuation factor at each of two receivers. From these observations, the unknown signal is to be recovered. In the noise-free case, it has been shown that the relative time delays between the signals as well as the relative attenuation factors may be determined exactly by using a phase-only reconstruction [26]. Additional processing will therefore lead to the

recovery of the desired signal. It will be important, however, to determine the effectiveness of this procedure in the presence of noise and to optimize it to minimize any noise effects.

The phase-only results may also be of some potential use in the context of Fourier transform coding. Typically, in a Fourier transform coding system, both the magnitude and the phase are coded and transmitted. However, most signals of interest may be reconstructed from only Fourier transform phase information. It may be possible, therefore, to capitalize on this result by developing a Fourier transform coding system in which some magnitude information is recovered from the coded phase.

APPENDIX I: REDUCIBLE POLYNOMIALS AND MEASURE ZERO

The Fundamental Theorem of Algebra [30] states that any polynomial in one variable of degree two or more is reducible over the field of complex numbers and, therefore, may always be expressed as a product of first order factors. This is not the case, however, for polynomials in two or more variables. Specifically, it is the purpose of this appendix to prove that the set of all reducible polynomials in two or more variables is a set of measure zero. In other words, "almost all" polynomials in two or more variables are irreducible. This result is important in a number of practical applications [4]. For example, it implies that a multidimensional filter may not, in general, be realized in cascade form. It also implies that almost all multidimensional sequences with support $R(\mathbf{N})$ have z -transforms which are irreducible polynomials. Consequently, as discussed in Section IV.6, it follows that almost all multidimensional sequences are uniquely defined to within a scale factor by the phase of their Fourier transform or to within a sign, a linear shift, and a time-reversal by the magnitude of their Fourier transform. Before proving that the reducible polynomials in two or more variables are a set of measure zero, it will be useful to first define what is meant by a set of measure zero and to list a few properties of these sets.

Generally speaking, a subset A of R^n is of measure zero if it has zero volume. More specifically, suppose that \underline{a} and \underline{b} are two vectors in R^n with $a_i < b_i$ for $i=1,2,\dots,n$. The rectangular solid, $S(\underline{a},\underline{b})$, is defined to consist of all those points $\underline{x} \in R^n$ for which $a_i < x_i < b_i$ for $i=1,2,\dots,n$. The volume of $S(\underline{a},\underline{b})$, denoted by $\text{vol}(S)$, is defined by

$$\text{vol}(S) \equiv \prod_{i=1}^n (b_i - a_i) \quad (1A-1)$$

If A is a subset of R^n , then A is said to have measure zero in R^n if, for any $\epsilon > 0$, there exists a countable covering of A by solids S_1, S_2, \dots such that

$$\sum_k \text{vol}(S_k) < \epsilon \quad (1A-2)$$

Two important properties of sets with measure zero follow immediately from this definition. The first is that if $A \subset R^n$ is of measure zero, then any subset of A must also be a set of measure zero. The second property is that a countable union of sets of measure zero is again a set of measure zero.

Finally, an important and useful theorem which will be referred to in the following discussions pertains to the image of a set of measure zero under a continuously differentiable map. Specifically,

Theorem 1A.1 [37, p.131]: If $F: R^m \rightarrow R^n$ is a continuously differentiable map from R^m to R^n , then the image of F , $F(R^m)$, is of measure zero in R^n provided $m < n$.

Having defined what is meant by a set of measure zero and having examined a few basic properties of these sets, it is now possible to prove that within the set of all polynomials in two or more variables with real coefficients, the subset of all reducible polynomials is a set of measure zero. In particular, let $P(n,m)$ denote the set of all polynomials in m variables with real coefficients which are of degree n . A polynomial $p_n(\underline{x})$ in the set $P(n,m)$ is therefore of the form

$$p_n(\underline{x}) = p_n(x_1, \dots, x_m) = \sum_{k_1 + \dots + k_m \leq n} \dots \sum c(k_1, \dots, k_m) x_1^{k_1} \dots x_m^{k_m} \quad (1A-3)$$

Let $\alpha(n,m)$ denote the number of coefficients required to define an arbitrary polynomial $p_n(x)$ in $P(n,m)$. If, for example, $m=2$ then $\alpha(n,m)=n(n+1)/2$. By representing each coefficient of a polynomial $p_n(x)$ as a coordinate of a vector in $R^{\alpha(n,m)}$, it follows that there is a one-to-one correspondence between $P(n,m)$ and $R^{\alpha(n,m)}$. In other words, $P(n,m)$ is *isomorphic* to $R^{\alpha(n,m)}$. The result of interest is the following:

Theorem 1A.2: The set of all polynomials in $P(n,m)$ which are reducible over the field of complex numbers corresponds to a set of measure zero in $R^{\alpha(n,m)}$ provided $m>1$ and $n>1$.

Proof: Consider the subset P_0 of $P(n,m)$ consisting of those polynomials for which the coefficient of the lowest degree term, $c(0)=c(0,0,\dots,0)$, equals zero. Since each polynomial in this subset has $\alpha(n,m)-1$ unconstrained coefficients, the set is isomorphic to $R^{\alpha(n,m)-1}$ and thus is a set of measure zero in $R^{\alpha(n,m)}$. It follows, therefore, that only the set $P'(n,m)$ of polynomials for which $c(0)\neq 0$ need be considered in the proof. If $c(0)=1$ the polynomial will be referred to as being "normalized".

If a polynomial $p_n(x) \in P'(n,m)$ is reducible, then its factors may always be combined in such a way that $p_n(x)$ is either:

- (a) a product of two real factors, or
- (b) a product of two complex factors which are conjugates of each other.

Step 1a: Let $B(k)$ be the subset of $P'(n,m)$ containing those polynomials which can be written as the product of a real polynomial of degree k and a real polynomial of degree $(n-k)$. Therefore, with $p_n(x) \in B(k)$, then

$$p_n(x) = \lambda q_k(x) \cdot r_{n-k}(x) \tag{1A-4}$$

where λ is real, $0 < k < n$, and $q_k(\underline{x})$ and $r_{n-k}(\underline{x})$ are normalized polynomials. Now consider the mapping

$$f: \mathbb{R} \times \mathbb{R}^{\alpha(k,m)-1} \times \mathbb{R}^{\alpha(n-k,m)-1} \rightarrow \mathbb{R}^{\alpha(n,m)} \quad (1A-5)$$

which assigns to the vector consisting of λ and the coefficients of $q_k(\underline{x})$ and $r_{n-k}(\underline{x})$ the vector of coefficients which define the reducible polynomial $p_n(\underline{x})$. It is obvious that the mapping f is continuously differentiable since all third-order partial derivatives exist and are identically zero. Thus, from Theorem 1A.1 it follows that $B(k)$ has measure zero provided

$$\alpha(k,m) + \alpha(n-k,m) - 1 < \alpha(n,m) \quad (1A-6)$$

Step 1b: Let P_C be the subset of $P'(n,m)$ containing those polynomials which can be written as a product of two complex factors. For $p_n(\underline{x}) \in P_C$, it is always possible to write $p_n(\underline{x})$ as

$$p_n(\underline{x}) = \lambda s_k(\underline{x}) \cdot s_k^*(\underline{x}) \quad (1A-7)$$

where λ is real, $k=n/2$, and $s_k(\underline{x})$ is normalized. (Note: n must be even). Since the number of real coefficients in $s_k(\underline{x})$ is $2\alpha(n/2,m)-2$, one may define a continuously differentiable map

$$f_C: \mathbb{R} \times \mathbb{R}^{2\alpha(n/2,m)-2} \rightarrow \mathbb{R}^{\alpha(n,m)} \quad (1A-8)$$

in accordance with (1A-7). Therefore, P_C is a set of measure zero if

$$2\alpha(n/2,m) - 1 < \alpha(n,m) \quad (1A-9)$$

However, this is merely a special case of (1A-6) when $k=n/2$.

Step 2: To establish the validity of (1A-6), note that if $n \geq 1$ and $m \geq 1$ then

$$\alpha(n,m) > \alpha(n-1,m) \quad (1A-10)$$

In other words, the number of coefficients in a polynomial of degree n is greater than the number of coefficients in a polynomial of degree $n-1$. The condition $m \geq 1$ simply requires that the polynomials have at least one variable whereas the condition $n \geq 1$ is required so that the right side of (1A-10) makes sense.

Any polynomial $p_n(x,y)$ of degree n in the $m+1$ variables (x,y) may always be written uniquely as

$$p_n(x,y) = q_n(x) + q_{n-1}(x)y + q_{n-2}(x)y^2 + \dots + q_0(x)y^n \quad (1A-11)$$

where $q_i(x)$ is a polynomial of degree at most i . Therefore,

$$\alpha(n,m+1) = \alpha(n,m) + \alpha(n-1,m) + \dots + \alpha(1,m) + 1 \quad (1A-12)$$

or, alternatively

$$\alpha(n,m+1) = [\alpha(n,m) + \dots + \alpha(k+1,m)] + \alpha(k,m+1) \quad (1A-13)$$

Using (1A-10), it follows from (1A-13) that

$$\alpha(n,m+1) > [\alpha(n-k,m) + \dots + \alpha(1,m)] + \alpha(k,m+1) \quad (1A-14)$$

However, (1A-12) implies that the term in brackets is equal to $\alpha(n-k,m+1)-1$. Therefore,

$$\alpha(n,m+1) > \alpha(n-k,m+1) + \alpha(k,m+1) - 1 \quad (1A-15)$$

which is valid provided $m \geq 1$ and $n \geq 1$. Therefore, (1A-6) is true under the hypothesis of the theorem as desired.

Step 3: Any reducible polynomial in $P(n,k)$ is always contained within the union of the sets P_o , P_c , and $B(k)$ for $k=1,2,\dots,n-1$. Since each of these sets has measure zero in $R^{2n(m)}$, then so does their union and Theorem 1A.2 follows. /////

The result presented in Theorem 1A.2 may be easily extended to other classes of polynomials. For example, it is straightforward to modify the proof of the theorem to show that the set of all reducible polynomials of degree n in m variables with complex coefficients corresponds to a set of measure zero in $R^{2n(m)}$ provided $m > 1$ and $n > 1$.

Another class of polynomials which is often encountered consists of those which have a given degree in each variable. Specifically, let $Q(\underline{n},m)$ be the set of all polynomials which have degree n_i in x_i for $i=1,\dots,m$. A polynomial $q_n(\underline{x})$ in $Q(\underline{n},m)$ is therefore of the form

$$q_n(\underline{x}) = q_n(x_1, \dots, x_m) = \sum_{k_1=1}^{n_1} \cdots \sum_{k_m=1}^{n_m} c(k_1, \dots, k_m) x_1^{k_1} \cdots x_m^{k_m} \quad (1A-16)$$

With $\beta(\underline{n},m)$ the number of coefficients required to specify the polynomial $q_n(\underline{x})$, it is easy to show that

$$\beta(k,m) + \beta(n-k,m) - 1 < \beta(n,m) \quad (1A-17)$$

provided $m > 1$ and $n > 1$. Therefore, it follows in a style similar to that used in the proof of Theorem 1A.2 that the set of all reducible polynomials in $Q(\underline{n},m)$ corresponds to a set of measure zero in $R^{2n(m)}$.

APPENDIX II: FIXED POINTS

It is the purpose of this Appendix to define what is meant by a fixed point of a mapping F and to discuss the convergence issues related to an iterative procedure for finding the fixed points of F . The results in this appendix are used to discuss the convergence properties of the phase-only iteration in Section V.2.2.2.

Consider a mapping F from a subset A of \mathbb{R}^M into \mathbb{R}^M , $F:A \subset \mathbb{R}^M \rightarrow \mathbb{R}^M$. If the image of A under F , $F(A)$, is a subset of A then F is said to map A *into itself*. If a mapping $F:A \subset \mathbb{R}^M \rightarrow \mathbb{R}^M$ has a point $\underline{x}^* \in A$ which is invariant under F , i.e., $F(\underline{x}^*) = \underline{x}^*$, then \underline{x}^* is called a *fixed point* of F . A mapping may have any number of fixed points. For example, if $F:\mathbb{R} \rightarrow \mathbb{R}$ is defined as $F(x) = -x$ then F has a unique fixed point $x^* = 0$ whereas the mapping $F(x) = x + 1$ has no fixed points. If, on the other hand, $F(x) = x$, then every point of \mathbb{R} is a fixed point of F .

Whenever it is desired to numerically determine the value of a fixed point of a mapping, an iterative procedure is often employed. A common iterative approach, based on the method of successive approximation, is defined by

$$\underline{x}_{p+1} = F(\underline{x}_p) \quad (2A-1)$$

where \underline{x}_p is the p th approximation to the fixed point \underline{x}^* . With $F^p(\underline{x}) = F[F^{p-1}(\underline{x})]$ and \underline{x}_0 the initial estimate of \underline{x}^* which is used to begin the iteration, (2A-1) may be written as

$$\underline{x}_p = F^p(\underline{x}_0) \quad (2A-2)$$

Clearly, if the sequence \underline{x}_p converges, then it must converge to a fixed point of F . Unfortunately, however, even if F has a unique fixed point, (2A-1) may not converge.

For example, although the mapping $F: \mathbb{R} \rightarrow \mathbb{R}$ defined by $F(x) = -x$ has a unique fixed point, $x^* = 0$, (2A-1) will not converge unless $x_0 = 0$.

There are many different types of constraints which may be imposed on the mapping F or on the set A to insure the existence or uniqueness of a fixed point of F or to guarantee the convergence of the iteration (2A-1). Perhaps the most familiar result is that which deals with contraction mappings. The definition of a contraction mapping involves the notion of a distance function or metric, d , which is defined on the underlying space. In the following discussion, the metric defined on \mathbb{R}^M will be taken to be the standard Euclidean metric, i.e.,

$$d^2(\underline{x}, \underline{y}) = \sum_{k=0}^{M-1} [x(k) - y(k)]^2 \quad (2A-3)$$

A contraction mapping is thus defined as follows. If $A \subset \mathbb{R}^M$ and if F maps A into itself, then F is called a *contraction mapping* if there is a constant $\mu \in (0, 1)$ such that:

$$d(F\underline{x}, F\underline{y}) \leq \mu d(\underline{x}, \underline{y}) \quad (2A-4)$$

for all $\underline{x}, \underline{y} \in A$. The *contraction mapping theorem* is, therefore, the following [37]

Theorem 2A.1: If F is a contraction mapping on a closed subset A of \mathbb{R}^M , then there is a unique fixed point $\underline{x}^* \in A$. Furthermore, the iteration $\underline{x}_p = F^p(\underline{x}_0)$ converges to \underline{x}^* for any initial estimate $\underline{x}_0 \in A$ and

$$d(\underline{x}_p, \underline{x}^*) \leq \mu^p / (1 - \mu) \cdot d(\underline{x}_1, \underline{x}_0) \quad (2A-5)$$

Although this theorem is useful in many applications, not all iterations which converge to a unique fixed point are characterized by a contraction mapping.

A larger class of mappings are obtained if μ is allowed to be equal to one in (2A-4) and are called *nonexpansive* maps. Specifically, a mapping $F:A \subset \mathbb{R}^M \rightarrow \mathbb{R}^M$ is said to be nonexpansive if

$$d(F\underline{x}, F\underline{y}) \leq d(\underline{x}, \underline{y}) \quad (2A-6)$$

for all $\underline{x}, \underline{y} \in A$. Unlike contractions, nonexpansive mappings may have any number of fixed points. For example, the identity map $F(\underline{x}) = \underline{x}$ is nonexpansive and every point is a fixed point of F whereas the map $F(\underline{x}) = \underline{x} + 1$ is nonexpansive and has no fixed points.

A mapping F is said to be *strictly nonexpansive* if the inequality in (2A-6) is strict whenever $\underline{x} \neq \underline{y}$, i.e.,

$$d(F\underline{x}, F\underline{y}) < d(\underline{x}, \underline{y}) \quad \text{for } \underline{x} \neq \underline{y} \quad (2A-7)$$

Although strictly nonexpansive mappings have at most one fixed point, strict nonexpansiveness is not sufficient to guarantee the existence of a fixed point. For example, the mapping $F:\mathbb{R} \rightarrow \mathbb{R}$ defined by

$$F(x) = \begin{cases} x + \exp(-x/2) & \text{for } x \geq 0 \\ \exp(x/2) & \text{for } x < 0 \end{cases} \quad (2A-8)$$

is strictly nonexpansive but has no fixed point. In order to guarantee the existence of a fixed point, an additional constraint must be imposed on the image of A under F . Specifically,

Theorem 2A.2: Let $F:A \subset \mathbb{R}^M \rightarrow \mathbb{R}^M$ be a strictly nonexpansive mapping which maps a subset A of \mathbb{R}^M into itself. If the image of A under F is compact, then F has a unique fixed point $\underline{x}^* \in A$. Furthermore, the iteration $\underline{x}_p = F^p(\underline{x}_0)$ converges to \underline{x}^* for any $\underline{x}_0 \in A$.

Note that since the nonexpansiveness of F implies that F is continuous, compactness of $F(A)$ may be replaced by the stronger condition that A be compact. A slightly different version of this theorem results if, in addition to being strictly nonexpansive, F is known to have a fixed point. Specifically,

Theorem 2A.3: If $F:R^M \rightarrow R^M$ is a strictly nonexpansive mapping which has a fixed point $\underline{x}^* \in R^M$, then this fixed point is unique and the iteration $\underline{x}_p = F^p[\underline{x}_0]$ converges to \underline{x}^* for any \underline{x}_0 in R^M .

Proof: The uniqueness of the fixed point follows immediately from the strict nonexpansiveness of F . Specifically, suppose there exists two fixed points, \underline{x}^* and \underline{y}^* . Then,

$$d(\underline{x}^*, \underline{y}^*) = d(F\underline{x}^*, F\underline{y}^*) \leq d(\underline{x}^*, \underline{y}^*) \quad (2A-9)$$

where, since F is strictly nonexpansive, equality holds if and only if $\underline{x}^* = \underline{y}^*$. Therefore, since equality must hold in (2A-9), \underline{x}^* must be unique.

To show that the iteration $\underline{x}_{p+1} = F(\underline{x}_p)$ converges to \underline{x}^* for any $\underline{x}_0 \in R^M$, let S_r denote the unit sphere of radius r about \underline{x}^* :

$$S_r = \{ \underline{x} \in R^M : d(\underline{x}, \underline{x}^*) \leq r \} \quad (2A-10)$$

and consider the map $F:S_r \rightarrow R^M$. Since F is strictly nonexpansive and $F(\underline{x}^*) = \underline{x}^*$, it follows that the image of F is a compact subset of S_r . Specifically, for any $\underline{x} \in S_r$,

$$d(F\underline{x}, \underline{x}^*) = d(F\underline{x}, F\underline{x}^*) \leq d(\underline{x}, \underline{x}^*) \leq r \quad (2A-11)$$

Thus, $F(\underline{x}) \in S_r$ and F maps S_r into itself. Finally, since F is continuous and S_r is compact, then the image of F is compact and the result follows. Therefore, it follows

from Theorem 2A.2 that the sequence of vectors $x_{p+1}=Fx_p$ converges to x^* for any $x_0 \in R^M$. //

Although Theorems 2A.2 and 2A.3 do not hold if the nonexpansiveness of F is not strict, the following theorem requires only that F be nonexpansive:

Theorem 2A.4: If F is nonexpansive and maps a convex† compact subset A of R^M into itself, then F has at least one fixed point in A . Furthermore, for any real number $\lambda \in (0,1)$ and for any $x_0 \in A$, the iteration

$$x_{p+1} = (1-\lambda)x_p + \lambda F(x_p) \tag{2A-12}$$

converges to a fixed point of F in A .

For example, consider again the mapping $F:R \rightarrow R$ with $A=[-1,1]$ and $F(x)=-x$. Although the iteration (2A-1) will not converge unless $x_0=x^*=0$, the constraints of Theorem 2A.3 are satisfied and therefore, (2A-12) will converge to the unique fixed point $x^*=0$ for any $x_0 \in [-1,1]$. The iteration (2A-12) is commonly referred to as the *relaxed* form of (2A-1) and λ is referred to as the *relaxation parameter*.

Finally, it should be pointed out that although Theorem 2A.3 guarantees the existence of at least one fixed point in A , this fixed point need not be unique (consider, for example the identity mapping).

† Convexity of A requires that if $x,y \in A$ and $0 < \lambda < 1$, then $z \in A$ where $z = [(1-\lambda)x + \lambda y]$.

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