NONLINEAR WAVES ON THE SURFACE OF A RADIALLY FLOWING FILM

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ABSTRACT

The behavior of waves on the surface of an axisymmetric film produced by the impingement of a vertical jet has been investigated. The theoretical study is composed of analytical work and numerical experiments. The flow was modeled by the inviscid Euler equations with the appropriate nonlinear boundary conditions. The first of three nonlinear wave analyses is a numerical study of the shallow water wave equations. Initial data tested showed the outgoing characteristic decays rapidly, while the ingoing breaks. Secondly, a Stokes wave expansion is assumed as a perturbation on an arbitrary fluid depth to include nonlinear dispersive effects. An average Lagrangian is calculated, and the variational principle of water waves leads directly to modulation equations. A stability criteria is summarized as a kH vs. λk^2 map. It is found that long waves become unstable. Lastly, a possible mechanism for splattering is considered. A multiple scale analysis applied to the water wave equations leads directly to a variable coefficient KP equation. For axisymmetric waves, the equation reduces to a variable coefficient K-dV equation. It is found that for slow radial variations the equation allows cnoidal wave solutions for the region close to the jet. As the film thins, the amplitude to width ratio increases turning them into a train of solitary waves. Dispersion ceases to be a major effect; hence, the propagation becomes hyperbolic. Numerical solution shows that solitary waves of small amplitude propagate without breaking, but any of large amplitude breaks.

Thesis Supervisor: Dr. John H. Lienhard V

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Nomenclature

Roman Letters

$A = \frac{E}{c_p}$	wave momentum
A_n	Fourier coefficients of ϕ expansion
a_n	Fourier coefficients of h expansion
$c_\pm = u \pm \sqrt{Mh}$	characteristic shallow water speeds
$\bar{c}_g = \frac{\partial \bar{\omega}_0}{\partial k}$	group velocity evaluated at $\bar{H}$
$c = \sqrt{1 + \frac{4\sigma}{\rho U_0^2 d}}$	nondimensional Bernoulli constant
c	characteristic speed of the system of modulation equations
$\bar{c}_p = \frac{\bar{\omega}_0}{k}$	phase velocity evaluated at $\bar{H}$
$E = \frac{1}{2} g a^2 (1 + \lambda k^2)$	amplitude proportional to the wave energy density
$F(\lambda k^2, kH)$	stability function
g	gravitational body force
h(r,t)	local thickness of liquid film
ho	characteristic film thickness
$\bar{H}$	total mean film thickness
H(r)	steady state film thickness
J	variational principle
k	radial wave number

K(p), E(p)	complete elliptic integrals of the first and second kind
L	Lagrangian
$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} L d\theta$	average Lagrangian
l	radial length scale
$M = \frac{gh_0}{U_0^2} = \frac{1}{Fr}$	inverse Froude number
$m = \frac{3^{3/2}\, Md^2}{c^3 l^2}\frac{1}{r}$	nondimensional radial group
p (x,τ)	slow varying modulus of the Jacobian elliptic function
p_o	initial value of p
$Q = \frac{d^2}{l}$	nondimensional flow rate group
r	radius measured from point of jet impact
$Re_d = \frac{U_0 d}{\nu}$	Reynold's number based on the jet diameter
$R_{\pm} = 2\sqrt{Mh} \pm u$	Riemann invariant
$\overline{T} \equiv \tanh k\overline{H}$	hyperbolic tangent evaluated at $\overline{H}$
t	time
u	depth average radial velocity
U(r)	steady state film speed
U_o	characteristic velocity of impinging jet
v	depth average azimuthal velocity
$\frac{1}{We} = \frac{\sigma h_0^2}{\rho \lambda_r^2 U_0^2 h_0}$	jet Weber number

$\frac{1}{We_{h_0}} = \frac{\sigma}{\rho U_0^2 h_0}$	jet Weber number based on ho
x	phase shifted time
x_o	initial value of x
z	distance normal to the wall

Greek Letters

$\alpha = \frac{lh_0}{d^2}$	nondimensional thickness group
$\alpha\,(x,\tau)$	slow wave amplitude
α_o	initial value of α
$\beta = \frac{\lambda_\tau}{\lambda_{\tilde{\theta}}}$	azimuthal wave parameter
$\tilde{\beta}$	pseudo wave number (mean flow)
$\Delta = \varepsilon^{2n}\mu^{2m+2}$	small expansion parameter
$d\Omega = drd\tilde{\theta}$	differential section
ε	small amplitude parameter
Φ	flow potential
ϕ	z independent potential
$\gamma\,(x,\tau)$	slow wave width
γ	pseudo wave frequency
$\Gamma(x,\tau)$	mean disturbance height
γ_o	initial value of γ

$\eta(r,t)$	mean wave disturbance (not periodic)
$\lambda_{\tilde{\theta}}$	characteristic azimuthal wavelength
λ_r	characteristic radial wavelength
$\lambda = \sigma/\rho g$	capillary wavelength
$\lambda(\tau)$	dispersion coefficient of the variable coefficient K-dV
λ_o	initial value of λ
$\mu = \frac{h_0}{\lambda_r}$	radial shallow water parameter
ν	kinematic viscosity
$\nu(\tau)$	nonlinearity coefficient of the variable coefficient K-dV
ν_o	initial value of ν
Θ	phase function
θ	fast phase coordinate
$\tilde{\theta}$	azimuthal coordinate
$\theta_0(x,\tau)$	first correction to the fast varying phase
ρ	fluid density
σ	surface tension coefficient
$\tau^*(\tau_0)$	location of first crossing of characteristics
τ_o	initial value of τ
τ	slow radial coordinate
ω	wave frequency
ω_o	linear dispersion relation evaluated at H

$\overline{\omega}_0^2 = gk\overline{T}(1+\lambda k^2)$	linear dispersion relation evaluated at $\overline{H}$
ξ	scaled azimuthal coordinate
ψ	pseudo phase

1. Introduction

Jet impingement is a very common physical phenomenon that occurs in everyday life. Take for example an open fuaset discharching a jet of fluid onto a horizontal sink. The impingement of this vertical circular jet of fluid creates a very thin radial film, whose depth suddenly increases at a certain location to an almost constant thickness. This sudden increase is known as a circular hydraulic jump or a standing circular wave. Most of this phenomenon is affected by viscosity except for a region near the stagnation zone (see figure 1a) where the flow is essentially inviscid. This region extends to the radial position r_0 (see figure 1a) where the growing viscous boundary layer on the surface of the plate reaches the free surface of the film. By increasing the Reynolds number of the incoming flow, it is found experimentally that the region can be enlarged.

Figure 1a shows the hydrodynamic evolution of the spreading film flow and figure 1b is a photograph of the actual physical phenomenon. As we can see from figure 1a, the flow is extremely complicated with a series of regions governed by different types of fluid flows. When the incoming flow is perturbed by creating disturbances on the surface of the jet we encounter another extremely interesting phenomenon known as splattering. The physical mechanism is still not well understood, but observations indicate that when disturbances are created on the jet surface, they will be transported onto the thin film. Once on the film, the disturbances are observed to propagate as wave trains that develop into either unstable large amplitude cylindrical waves that eventually breakup into droplets and splatter into the air or else they simply propagate without breaking as finite amplitude waves.

This phenomenon is depicted on figure 2a. Photographs showing two types of actual physical processes are presented in figures 2b and 2c. Since these waves are transported through the inviscid region of the flow, we attempt to study the wave motion in this region. The fluid is modelled as inviscid, irrotational, isothermal, nonconducting, and constant property.

The methods of analysis used herein are the following:

1. Nondispersive nonlinear waves

2. Nonlinear dispersive of Stokes waves with surface tension.

3. Nonlinear long-wave evolution.

The first case is essentially a numerical study of hyperbolic waves on a nonuniform axisymmetric flow. Here we find that solitary wave type disturbances break in the typical hyperbolic fashion.

The second analysis uses Whitham's averaged Lagrangian theory (Whitham's *Linear and Nonlinear Waves*, 1974) to analyze a modulated wave train of Stokes waves on a steady nonuniform flow characterized by the steady film thickness, H(r), and the steady radial velocity U(r). We derive the modulation equations that govern the transport properties of the wave train. From the analysis, we obtain a nonlinear dispersion relation that includes surface tension effects; we also derive a stability criterion that includes the effects of gravity, surface tension, and arbitrary film thickness. The stability corresponds to the hyperbolic stability of the modulation equations. We also look at specific cases by specifying the density and surface tension of the fluid, and thus obtaining a map of H(r) versus the wave number k.

In the last analysis, we derive an evolution equation for the development of a wave disturbance on the surface of a nonuniform axisymmetric flow, again characterized by H(r) and U(r). For certain values of H and U, this equation reduces to the nearly concentric Korteweg-deVries equation (ncK-dV). We devote the rest of the investigation to studying this general evolution equation. The three cases studied include: slowly varying periodic cnoidal waves, slowly varying solitary waves, and nondispersive waves. Ignoring dispersive effects in small enough amplitude disturbances leads to hyperbolic propagation without breaking, while for large enough amplitudes it leads to breaking.

Also, we find that periodic wave trains are possible only in regions close to the jet and that as r increases the waves turn into solitary waves. Finally, we show that slowly varying solitary waves are also solutions of the evolution equation. We again note that all the analyses are based on a long wave approximation, since the film thickness to disturbance wavelength ratio is very small.

These analyses do not predict splattering per se; rather, they attempt to identify the mechanism by which waves propagate, and the conditions which can result in sharpening of initially smooth disturbances. This mechanism can be summarized as a simple nonlinear filtering process that takes periodic wave trains and turns them into solitary waves propagating down the surface of an inviscid axisymmetric thin film.

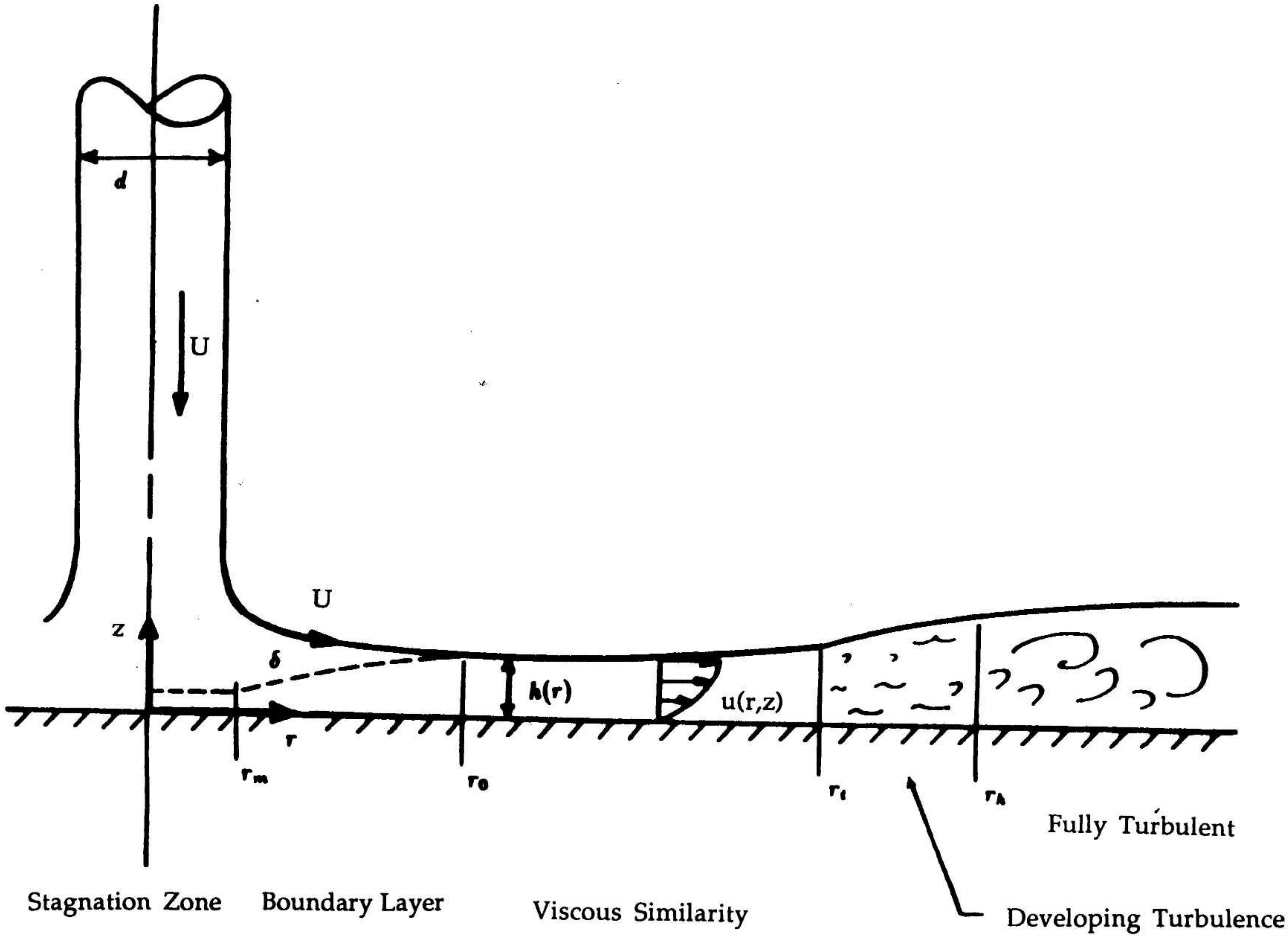

Figure 1a: The jet and film flow showing hydrodynamic evolution (not to scale).

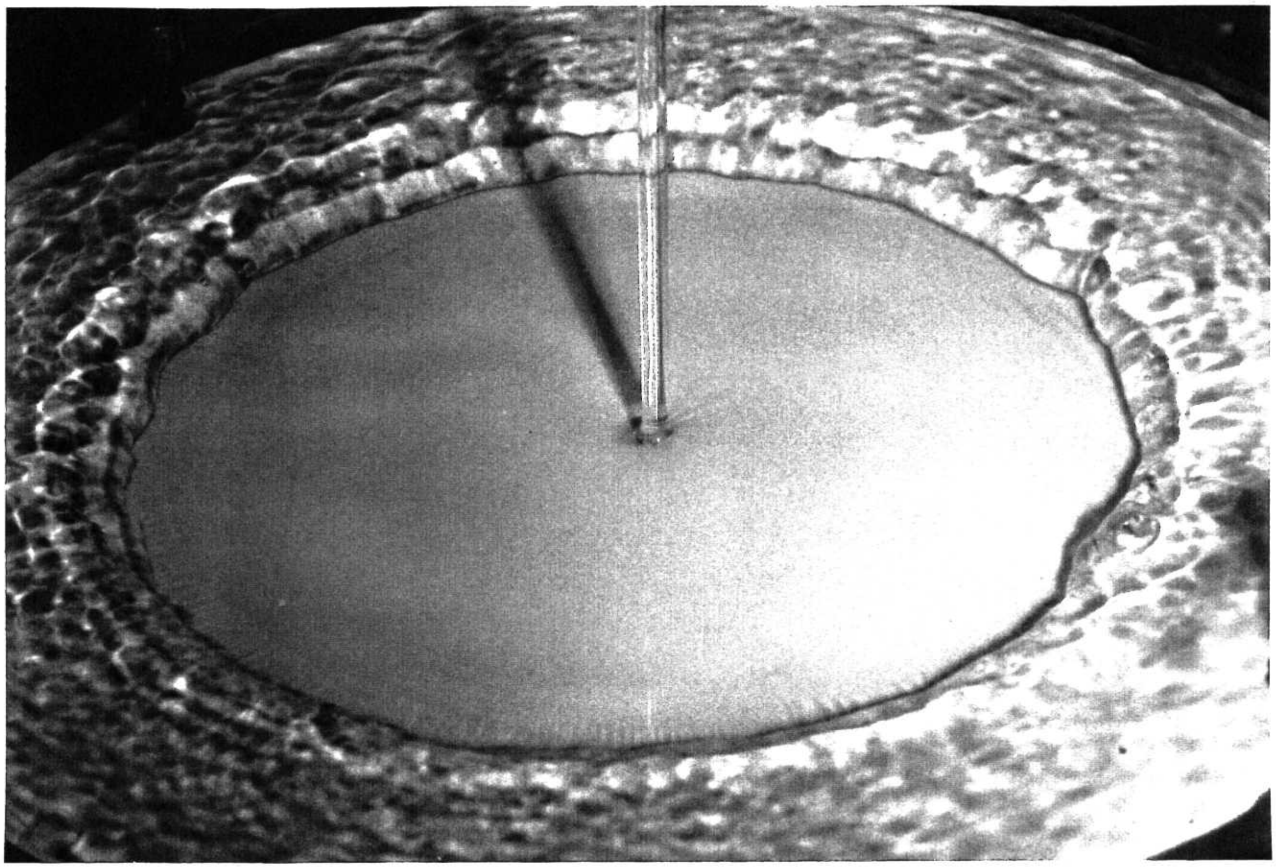

Figure 1b: Photograph of a laminar jet impinging on a horizontal surface, showing the circular hydraulic jump.

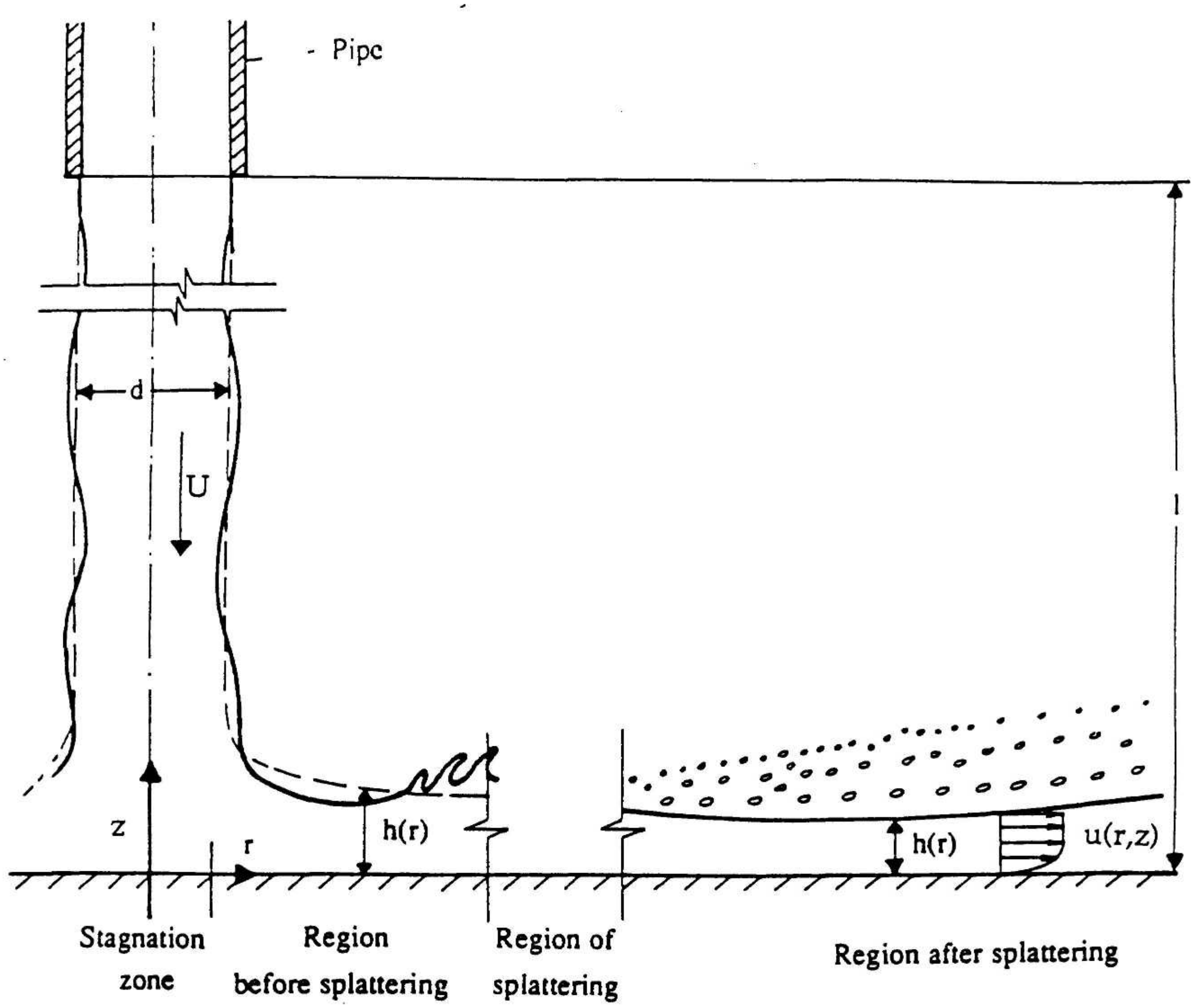

Figure 2a: Perturbed jet and splattering mechanism.

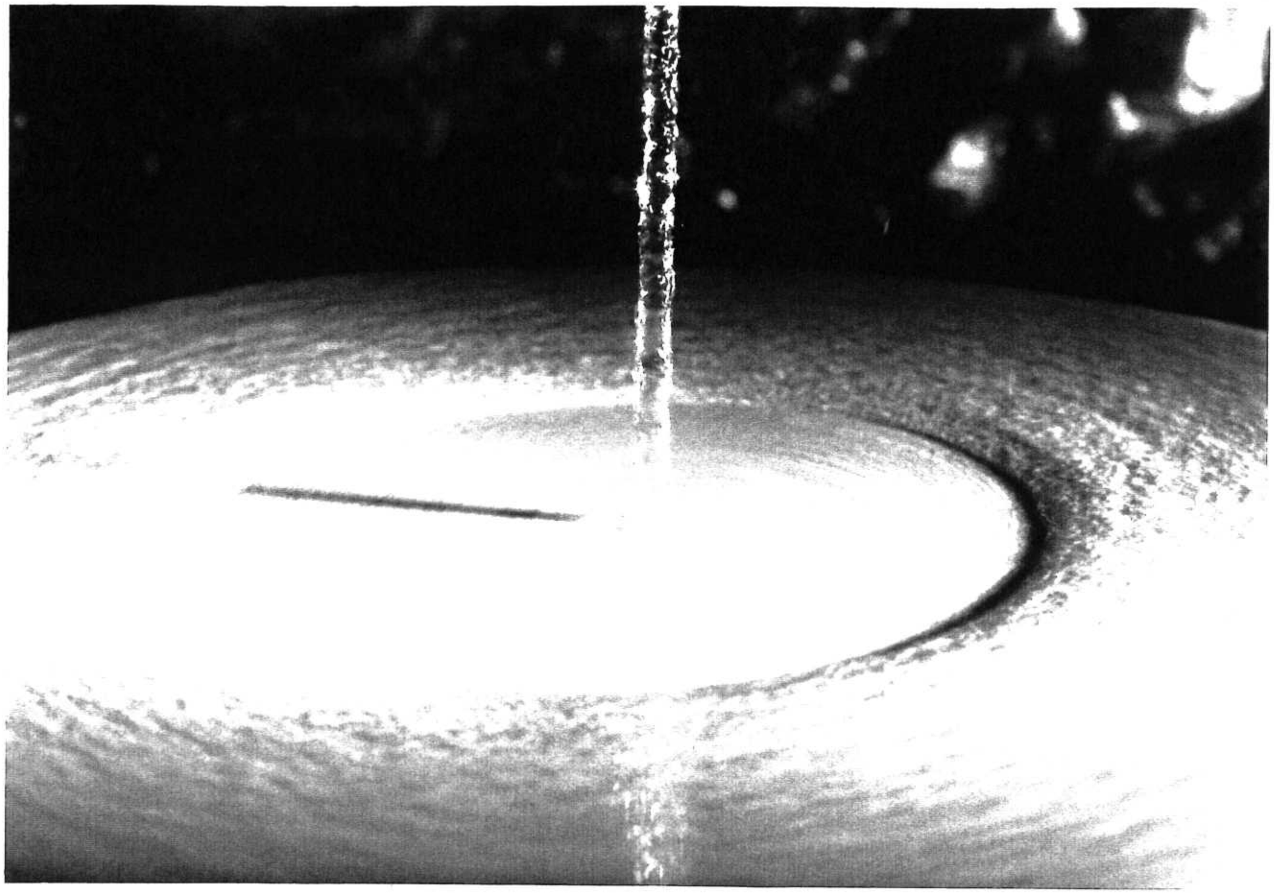

Figure 2b: Photograph of a turbulent jet showing the wave motion of non-breaking waves on the thin film.

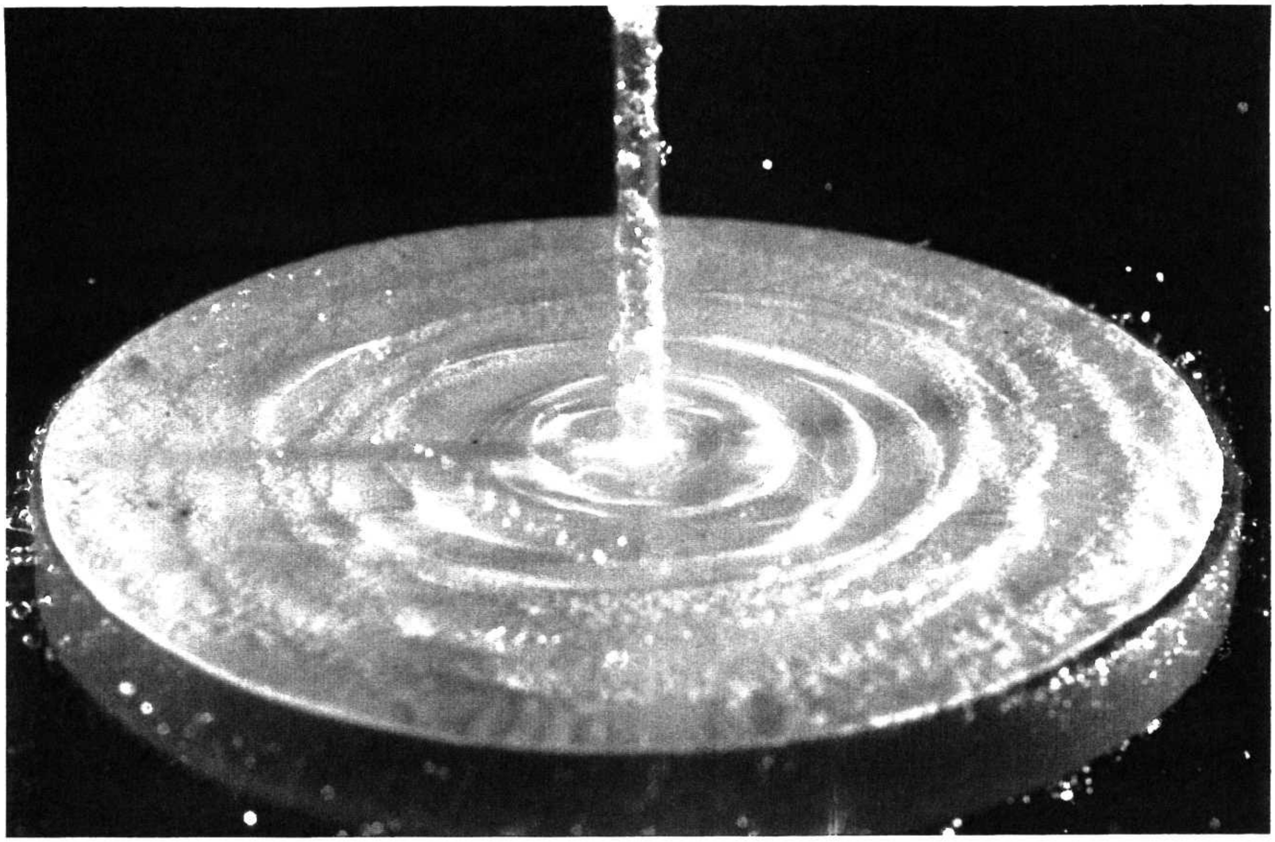

Figure 2c: Photograph of a turbulent jet showing the development of large amplitude splattering waves.

2. Derivation of Equations of Motion

It is convenient to derive the inviscid equation of motion from a variational principle, so that its Lagrangian can be used later for analyzing nonlinear water waves in a cylindrical geometry. We define the Lagrangian as

$$L = \int_0^h \left[\Phi_t + \frac{1}{2}\left(\Phi_r^2 + \Phi_z^2 + \frac{1}{r^2}\Phi_{\tilde{\theta}}^2\right) + gz\right] dz + \frac{\sigma}{\rho}\sqrt{1 + h_r^2 + \frac{1}{r^2}h_\theta^2}$$

(1)

where $\Phi(r,z,\tilde{\theta},t)$ is the potential, $h(r,\tilde{\theta},t)$ is the free surface position ρ, g, σ are the density, gravitational body force, and surface tension respectively. And we introduce the integral

$$J = \int_{t_1}^{t_2} \int_{\tilde{\theta}_1}^{\tilde{\theta}_2} \int_{r_1}^{r_2} L\, r\, dr\, d\tilde{\theta}\, dt \qquad (2)$$

and its first variation

$$\delta J = 0 \qquad (3)$$

We note that this variational principle leads to the inviscid equations of motion as follows

$$\delta \int \int_\Omega Lr\, d\Omega dt \qquad (4)$$

where $d\Omega = drd\tilde{\theta}$.

After integrating by parts, rearranging, and collecting like terms, we have

$$\int\int_{\Omega} d\Omega dt \left[\frac{\partial}{\partial t}\int_0^h r\,\delta\phi dz\right] + \int\int_{\Omega} d\Omega dt \left[\frac{\partial}{\partial r}\int_0^h r\Phi_r\delta\Phi dz\right] + \int\int_{\Omega} d\Omega dt \left[\frac{\partial}{\partial \theta}\int_0^h \frac{1}{r}\Phi_\theta\delta\Phi dz\right]$$

$$- \int\int_{\Omega} d\Omega dt\left[r\Phi_z(0)\delta\Phi(0)\right] - \int\int_{\Omega} d\Omega dt\left[r\delta\Phi(h)\left[h_t + \Phi_r h_r + \frac{1}{r^2}\Phi_\theta h_\theta - \Phi_z\right]_{z=h}\right]$$

$$- \int\int_{\Omega} d\Omega dt\left[\int_0^h r\left[\frac{(r\Phi_r)_r}{r} + \frac{1}{r^2}\Phi_{\theta\theta} + \Phi_{zz}\right]\delta\Phi dz\right]$$

$$+ \int\int_{\Omega} d\Omega dt\left[r\left[\Phi_t + \frac{1}{2}(\nabla\Phi)^2 + gz\right]_{z=h} - \frac{\sigma}{\rho}\left\{\frac{\partial}{\partial r}\left[\frac{rh_r}{\sqrt{1+(\nabla h)^2}}\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{h_\theta}{\sqrt{1+(\nabla h)^2}}\right]\right\}\right] = 0 \qquad (5)$$

where ∇ is the gradient operator in cylindrical coordinates.

The first three integrals integrate out to the boundaries of Ω and vanish if $\delta\Phi$ is chosen to vanish at the boundary. Now requiring that (4) vanish for appropriate $\delta\Phi$ and δh, and applying the usual variational argument (see for example Whitham's *Linear and Nonlinear Waves,* 1974) with the following choices:

- $d\Phi = 0$ on $z = h$ and $z = 0$: Laplace's eq.

- $\delta h \neq 0$ on $z = h$: Dynamic Boundary Condition @ $z = h$

- $\delta\Phi > 0$ on $z = h$: Kinematic Boundary Condition @ $z = h$

  $\delta\Phi = 0$ on $z = 0$

- $\delta\Phi = 0$ on $z = h$: Boundary Condition at $z = 0$

  $\delta\Phi > 0$ on $z = 0$

Leads directly to :

$$\nabla^2\Phi = 0 \qquad \forall(\mathbf{x},t) \tag{6}$$

along with the boundary conditions at z = h,

$$\Phi_t + \frac{1}{2}(\nabla\Phi)^2 + gh - \frac{\sigma}{\rho}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{rh_r}{\sqrt{1+(\nabla h)^2}}\right] + \frac{1}{r^2}\frac{\partial}{\partial\tilde{\theta}}\left[\frac{h_{\tilde{\theta}}}{\sqrt{1+(\nabla h)^2}}\right]\right\} = 0 \tag{7}$$

$$h_t + \nabla\Phi\cdot\nabla h = \Phi_z \tag{8}$$

and the boundary condition z = 0,

$$\Phi_z = 0 \tag{9}$$

These are the inviscid water wave equations. We note that the variational principle is one of stationary total pressure, where the surface pressure due to the curvature of the free surface has also been accounted for.

The objective of this investigation is to attempt to solve the above equations in some region of space and time where long waves can be analyzed. Before continuing with the analysis, we non-dimensionalize the variables as follows:

$$[t] \to \frac{\lambda_r}{U_0} \qquad [r] \to \lambda_r \qquad [z] \to h_0$$

$$[\Phi] \to \lambda_r U_0 \qquad [r\tilde{\theta}] \to \lambda_{\tilde{\theta}}$$

where h_o is the characteristic film thickness, U_0 is a characteristic speed of the incoming jet and λ_r, $\lambda_{\tilde{\theta}}$ are the characteristic wavelengths in the r and $\tilde{\theta}$ directions respectively. We also define the following non-dimensional parameters: inverse Froude number; inverse Weber number; and two parameters measuring the dispersive effects of waves in the radial and the azimuthal directions.

$$M = \frac{gh_0}{U_0^2} = \frac{1}{Fr} \qquad \frac{1}{We} = \frac{\sigma h_0^2}{\rho \lambda_r^{\,2} U_0^2\, h_0}$$

$$\mu = \frac{h_0}{\lambda_r} \qquad \beta = \frac{\lambda_r}{\lambda_{\tilde{\theta}}} \qquad \frac{1}{We} = \frac{\mu^2}{We_{h_0}}$$

so that $$\frac{1}{We_{h_0}} = \frac{\sigma}{\rho U_0^2\, h_0} \tag{10}$$

The long wave analysis requires that the two shallow water parameters μ and β be much smaller than O(1).

Typical values of the above parameters can be found for a water jet at 20 m/s and having a diameter d=6.3 mm. For r/d ~ 3, we have $h_0 \sim \frac{d^2}{8r}$ = 0.26 mm. A typical wavelength is $\lambda_r \sim d$

This implies that $M = 6.4 \times 10^{-6}$

$$\frac{1}{We_{h_0}} = 6.7\times 10^{-4}$$

$$\mu = 0.04 = o(1)$$

Another case of interest is U_o=1 m/s with d=1 mm which leads to h_o=4.2x10^{-5} m. In this case $M = 4 \times 10^{-4}$

$$\frac{1}{We_{h_0}} = 1.7$$

$$\mu = 0.04 = o(1)$$

We take β to be of the same order as μ.

Observation of the disturbances present on real jets (which are $\lambda_r \sim d$) show that μ much smaller than one is a reasonable assumption. Observations of natural waves on laminar sheets (with a strobe light) also suggest $\mu = o(1)$ is viable.

With the above definitions, we can write

$$(\nabla h)^2 = \mu^2\left[h_r^2 + \frac{\beta^2}{r^2}h_\theta^2\right] \tag{11}$$

The different streachings in r, z and θ introduce the crucial steps in the expansions of the equations. In the non-dimensional variables, the problem is reformulated as

$$\Phi_{zz} + \mu^2\left[\Phi_{rr} + \frac{1}{r}\Phi_r + \frac{\beta^2}{r^2}\Phi_{\tilde{\theta}\tilde{\theta}}\right] = 0 \tag{12}$$

in the domain, $r > 0$, $0 < z < h$, $0 < \tilde{\theta} < 2p$, and for all t, with the boundary conditions

$$\Phi_t + \frac{1}{2}\left[\Phi_r^2 + \frac{\beta^2}{r^2}\Phi_\theta^2 + \frac{1}{\mu^2}\Phi_z^2\right] + Mh - \frac{1}{We}\left[h_{rr} + \frac{1}{r}h_r + \frac{\beta^2}{r^2}h_{\tilde{\theta}\tilde{\theta}} + O(\mu^2)\right] = 0\,; \quad @\ z = h \tag{13}$$

$$\mu^2\left[h_t + \Phi_r h_r + \frac{\beta^2}{r^2}\Phi_{\tilde{\theta}} h_{\tilde{\theta}}\right] = \Phi_z\,; \quad @\ z = h \tag{14}$$

$$\Phi_z = 0 \qquad @\ z = 0 \tag{15}$$

We pose the solution to Laplace's equation as a formal expansion in z and upon application of the boundary condition at z=0, we have

$$\Phi = \left[\cosh \mu z \bar{\delta}\right] \phi\left(r, \tilde{\theta}, t\right) = \left[1 + \frac{1}{2}\mu^2 z^2 \bar{\delta}^2 + \frac{1}{4!}\mu^4 z^4 \bar{\delta}^4 + \cdots\right]\phi \quad ; \mu \to 0 \quad (16)$$

where

$$\bar{\delta}^2 = -\left(\partial_{rr} + \frac{\partial_r}{r}\right) - \frac{\beta^2}{r^2}\partial_{\tilde{\theta}\tilde{\theta}} \equiv \bar{\delta}^2 = \delta^2 - \frac{\beta^2}{r^2}\partial_{\tilde{\theta}\tilde{\theta}} \quad (17)$$

and the operator $\cosh \mu z \bar{\delta}$ is defined by its Taylor expansion when μ is assumed small compared to one. Now the derivatives of Φ become

$$\Phi_t = \phi_t + \frac{1}{2}\mu^2 z^2\left(\bar{\delta}^2 \phi_t\right) + \cdots \quad (18)$$

$$\Phi_r = \phi_r + \frac{1}{2}\mu^2 z^2\left(\bar{\delta}^2 \phi\right)_r + \cdots \quad (19)$$

$$\Phi_z = \mu^2 z\left(\bar{\delta}^2 \phi\right) + \frac{1}{6}\mu^4 z^3\left(\bar{\delta}^4 \phi\right) + \cdots \quad (20)$$

$$\frac{1}{r}\Phi_{\tilde{\theta}} = \frac{1}{r}\phi_{\tilde{\theta}} + \frac{1}{2}\frac{\mu^2 z^2}{r}\left(\bar{\delta}^2 \phi\right)_{\tilde{\theta}} + \cdots \quad (21)$$

with $\mu \to 0$. This implies that the flow is independent of z to O(1).

Substituting the above expressions into the free surface boundary conditions and collecting terms up to order μ^2, β^2, $\beta^2\mu^2$, reduces the problem to

$$\phi_t + \frac{1}{2}\left[\phi_r^2 + \frac{\beta^2}{r^2}\phi_{\tilde{\theta}}^2\right] + \frac{1}{2}\mu^2 h^2\left[\phi_r\left(\delta^2\phi\right)_r + \frac{\beta^2}{r^2}\phi_{\tilde{\theta}}\left(\delta^2\phi_{\tilde{\theta}}\right) + \left(\delta^2\phi\right)^2 + \left(\delta^2\phi_t\right)\right] + Mh + \frac{1}{We}\left(\delta^2 h\right) \sim 0$$

(22)

and

$$h_t + \phi_r h_r + \frac{\beta^2}{r^2}\phi_{\tilde{\theta}} h_{\tilde{\theta}} - h\left(\delta^2\phi\right) + \mu^2\left[\frac{1}{2}h^2 h_r\left(\delta^2\phi\right)_r - \frac{1}{6}h^3\left(\delta^4\phi\right) + \frac{1}{2}\frac{h^2 h_{\tilde{\theta}}\beta^2}{r^2}\left(\delta^2\phi_{\tilde{\theta}}\right)\right] \sim 0 \qquad (23)$$

with $\mu \to 0$.

Since Φ is the potential we define the radial and azimuthal velocities as Φ_r and $\frac{1}{r}\Phi_\theta$ respectively.

We differentiate the first equation, (22), with respect to r and introduce the depth-averaged radial and azimuthal velocities defined by

$$hu = \int_0^h \frac{\partial}{\partial r}[\cosh \mu z\bar{\delta}]\phi \, dz \sim h\phi_r + \frac{1}{3!}\mu^2 h^3\left(\bar{\delta}^2\phi\right)_r \quad , \quad \mu \to 0 \qquad (24)$$

$$hv = \frac{\beta}{r}\int_0^h \frac{\partial}{\partial\tilde{\theta}}[\cosh \mu z\bar{\delta}]\phi \, dz \sim \frac{\beta}{r}\left[h\phi_{\tilde{\theta}} + \frac{1}{3!}\mu^2 h^3\left(\bar{\delta}^2\phi_{\tilde{\theta}}\right)\right] \quad , \quad \mu \to 0 \qquad (25)$$

and find their inverse, ϕ_r and $\phi_{\tilde{\theta}}$, in terms of u and v to be

$$\phi_r \sim u + \frac{1}{3!}\mu^2 h^2\left(u_r + \frac{u}{r}\right)_r \quad , \quad \mu \to 0 \qquad (26)$$

$$\frac{\beta\phi_{\tilde{\theta}}}{r} \sim v + \frac{1}{3!}\mu^2 h^2\left(v_{rr} + \frac{v_r}{r}\right) \quad , \quad \mu \to 0 \qquad (27)$$

A consistency relation is obtained from equations (26) and (27) as

$$(rv)_r = \beta u_{\tilde{\theta}} \quad , \quad \mu \to 0 \qquad (28)$$

Upon substitution of (24), (25), (26), and (27) into (22) and (23) we obtain the depth-averaged radial momentum equation

$$u_t + \frac{\partial}{\partial r}\left[\frac{1}{2}\left(u^2 + v^2\right) + Mh + \frac{1}{We}\left(\delta^2 h\right)\right] + \frac{1}{3}\frac{\mu^2}{h}\frac{\partial}{\partial r}\left[2h^3\left\{u_r^2 + \frac{uu_r}{r} + \frac{u^2}{r^2}\right\} - Mh^3\left(\delta^2 h\right)\right] + \cdots = 0 \tag{29}$$

and the continuity equation

$$(rh)_t + (rhu)_r + \frac{\beta}{r}(rhv)_{\tilde{\theta}} = 0 \tag{30}$$

To make this a consistent system, we require the relation already derived in equation (28)

$$\frac{(rv)_r}{r} - \frac{\beta}{r}u_{\tilde{\theta}} = 0 \tag{31}$$

Equations (29), (30), and (31) are the unsteady, cylindrical, nonlinear, long wavelength, dispersive water wave equations. This is the system that we choose to work with in determining the propagation of wave disturbances on the film. Clearly, this system is valid for a region where $\mu = o(1)$ and we note that no other restrictions have been imposed on the other non-dimensional parameters. In applying this system, we assume that the mean flow is obtained from these equations by setting $\mu = o(1)$ where $\mu = h_0/l$ and l is a characteristic length of the flow field, such as the radial position where the viscous boundary layer reaches the free surface. Such a mean flow is essentially unrestricted, in the sence of the normal potential flow of an impinging jet (see next section). Different types of wave motion on this mean flow can be analyzed by considering different regions of space and time, and we look at some of these in the subsequent sections.

3. Steady Axisymmetric Flow

We begin by analyzing the steady impingement of an axisymmetric jet of fluid onto a flat surface located at z = 0. The evolution of the free surface of the thin film formed will be studied in the region where the film is still governed by the inviscid flow equations. A boundary layer forms growing from the stagnation point at r=0 to a location downstream where it reaches the free surface of the film. This implies that any wave analysis will only apply in the region away from r=0, since the flow would be singular there, and up to the point where the boundary layer reaches the free surface (see figure 3).

In the case of steady axisymmetric flow the water wave equations provide a steady state that can be subsequently perturbed. The system of equations (29-31) up to O(1) reduces to

$$\frac{1}{2}u^2 + Mh + \frac{1}{We}\left(h_{rr} + \frac{h_{rr}}{r}\right) = \frac{1}{2}c^2 \qquad (32)$$

$$rhu = Q \qquad (33)$$

where
$$c = \sqrt{1 + \frac{4\sigma}{\rho U_0^2 d}}$$

$$Q = \frac{d^2}{l}$$

Here l is the characteristic radial length, which we can take as the location where the viscous boundary layer reaches the free surface, c is the nondimensional head of the incoming flow and Q is the incoming dimensionless mass flow rate. Note that for large U_0 surface tension effects are negligible; hence c~1. By ignoring surface tension effects in equation (32) (since 1/We is very small compared to one $O(10^{-4})$) the flow becomes essentially a momentum driven flow, which can be solved in closed form to this order of approximation.

The above system results in the solution of a cubic algebraic equation for h and u.

$$\frac{1}{2}\frac{Q^2}{r^2h^2} + Mh - \frac{1}{2}c^2 = 0 \tag{34}$$

Whose roots are

$$u_i = \frac{2c}{\sqrt{3}}\cos\left[\frac{1}{3}\cos^{-1}(-m) + \begin{Bmatrix} 0 \\ 2\pi/3 \\ 4\pi/3 \end{Bmatrix}\right] \tag{35}$$

$$h_i = \frac{\sqrt{3}}{2}\frac{1}{\alpha r\cos\left[\frac{1}{3}\cos^{-1}(-m) + (0,\, 2\pi/3,\, 4\pi/3)\right]} \tag{36}$$

where

$$m = \frac{3^{3/2}\, Md^2}{c^3 l^2}\frac{1}{r} \tag{37}$$

$$\alpha = \frac{l h_0}{d^2} \tag{38}$$

The index i=1,2,3 represents the three roots of the cubic. The i=1 solution represents the spreading radial film, the i=2 is not a physical solution and i=3 gives an major increase in the film thickness. The i=3 root resembles what we can call an inviscid hydraulic jump. This discontinuity does not predict the location of the physical jump since the viscous shear is not accounted for in the momentum balance. For a thin film, the radial length scale, l, is larger than the jet diameter and the inverse Froude number is O(10^{-3} to 10^{-6}), hence m<<1 seems like a good approximation. We therefore have the limiting solution (relevant root, i=1)

$$h \sim \frac{1}{\alpha r} \qquad u \sim 1 \quad , \mu \to 0 \tag{39}$$

This is the classical approximation for an axisymmetric jet hitting a flat surface under negligible gravitational effects. However, when gravitational effects are still negligible, but viscosity becomes important as in the case of very thin films, we can use Watson's similarity solution which gives the surface speed and position of a fully viscous film as

$$u(r) = \frac{27n^2}{8\pi^2} \frac{Q}{\nu(r^3 + l^3)} \tag{40}$$

$$h(r) = \frac{2\pi^2}{3\sqrt{3}} \frac{\nu}{Q}\left(\frac{l^3}{r} + r^2\right) \tag{41}$$

where

$$n = 1.40218$$

$$l = (0.3243)dRe_d^{1/3}$$

And $Re_d = \frac{U_0 d}{\nu}$, is the Reynold's number based on the jet diameter

In terms of d and r, h becomes:

$$h(r) = 0.1713\left(\frac{d^2}{r}\right) + \frac{5.147}{Re_d}\left(\frac{r^2}{d}\right) \tag{42}$$

According to Watson's analysis, the location at which the viscous boundary layer reaches the free surface is

$$r_0 = 0.1833dRe_d^{1/3}$$

We therefore take $l = r_0$, so that the indeterminacy of l is finally resolved.

Thus any wave analysis will be limited to the region where $r < r_o$. The present analyses are best for very high Reynolds number jets. We note that laminar jets with $Re_d > 10^5$ are easily produced in the laboratory.

4. Unsteady Nondispersive Axisymmetric Flow

When considering the equations to be functions of r and t only and with negligible dispersion and negligible surface tension effects, we obtain the radial shallow water wave equations for axisymmetric flow

$$u_t + uu_r + Mh_r = 0 \tag{43}$$

$$(hr)_t + (rhu)_r = 0 \tag{44}$$

The equivalent conservation equations are

momentum:

$$(rhu)_t + \left(rhu^2 + \frac{1}{2}Mrh^2\right)_r = \frac{1}{2}Mh^2 \tag{45}$$

continuity:

$$(rh)_t + (rhu)_r = 0 \tag{46}$$

In order to analyze this nonlinear system, we apply the method of characteristics and find the system's preferred directions by combining (45) and (46) into its Riemann invariant form. The system becomes

$$\frac{d}{dt}\left(2\sqrt{Mh} \pm u\right) = -\frac{u\sqrt{Mh}}{r} \tag{47}$$

$$\frac{dr}{dt} = u \pm \sqrt{Mh} \tag{48}$$

where $\quad \frac{d}{dt} = \frac{\partial}{\partial t} + \left(u \pm \sqrt{Mh}\right)\frac{\partial}{\partial r}$

The Riemann "invariants" of the system are

$$R_{\pm} = 2\sqrt{Mh} \pm u \tag{49}$$

These are not really invariants since the right hand side of equation (47) is non-zero. The characteristic speeds are

$$c_{\pm} = u \pm \sqrt{Mh} \tag{50}$$

Here the $\pm$ refers to outgoing and incoming waves respectively.

To analyze this system we notice that the right hand side of the Riemann invariants will depend on the integral

$$\int_{c_{\pm}} \frac{u\sqrt{Mh}}{r} dt \tag{51}$$

taken along the characteristics. It turns out from numerical evidence (see r-t plane on figure 9) that when the integral is taken along c_-, near the head of the wave the range of integration diminishes, thus making this contribution relatively small compared to that of the c_+. This suggests that R_- is almost constant along the c_- characteristic.

Hence,

$$2\sqrt{Mh} - u = 2\sqrt{Mh_0} - u_0 \equiv F_0 \tag{52}$$

Eliminating $\sqrt{Mh}$ from above and substituting into the R_+ equation we obtain a single first order equation for u which requires integration along the c_+ characteristic.

The equation for u becomes

$$\frac{\partial u}{\partial t} + \left(\frac{3}{2} + \frac{1}{2}F_0\right)\frac{\partial u}{\partial r} + (F_0 + u)\frac{u}{2r} = 0 \tag{53}$$

and whose exact solution is

$$u(F_0 + u)^2 = \frac{G_0(\tau)}{r} \tag{54}$$

where $G_0(\tau)$ is a function of integration and $\tau = \tau\ (r,t)$ is the characteristic variable (τ is the " time " along the r-t characteristic) to be determined from

$$\frac{dt}{dr} = \frac{2}{(3u + F_0)} \sim \frac{2}{F_0} - \frac{6u}{F_0^2} \quad , \quad \frac{u}{F_0} \to 0 \tag{55}$$

A uniformly valid approximation to O(1) is

$$\frac{u}{F_0^2} = \frac{G_0(\tau)}{rF_0^4}$$

with

$$t = \frac{2r}{F_0} - \frac{6G_0(\tau)}{F_0^4} \ln r + G_1(\tau) \tag{56}$$

where $G_1(\tau)$ is a second function of integration. From (52), we have

$$2\sqrt{Mh} = F_0 + \frac{G(\tau)}{F_0^2 r} \tag{57}$$

The functions $G_0(\tau)$ and $G_1(\tau)$ can be determined from the initial conditions. However, we do not calculate these since that requires a higher order of approximation. We merely note that these are only the geometrical acoustics form of h and u.

We now look at the full hyperbolic problem defined by equations (45)-(48) and solve the initial value problem numerically using a second order ENO scheme ("essentially non-oscillatory," see Harten and Osher 1987). In the numerical calculations, we first look at steady solutions of the equations, and we find these lead to a hydraulic jump downstream of the stagnation point. Once this steady shock wave is located, we analyze the behavior of prescribed initial data on the spreading film. The initial conditions that were tested correspond to a sech-squared disturbance whose amplitude and phase obey the solitary wave solution of the Korteweg and de Vries equation.

$$3a^2 \operatorname{sech}^2 \frac{a}{2}(r - r_0)$$

where a is the amplitude of the disturbance and r_0 is the radial location of the peak.

First, we look at the time evolution of a steady shock starting from a given initial profile. Figure 3a shows the formation of a hydraulic jump from an arbitrary initial film thickness. The surface is a mesh plot showing the film thickness as a function of time and radial position. In figure 3b we show the numerical convergence of the solution towards a specific jump location. This plot corresponds to the projection of figure 3a on to the height versus position plane for different values of time. The jump is essentially a standing hyperbolic wave that forms a typical hyperbolic discontinuity at the shock location. The downstream data that we specified were the flowrate and the final film thickness (typical values were O(1)). We do not pay too much attention this solution of the hydraulic jump, since it is predicted from a model that neglects shear stresses.

Although this hydraulic jump does not include the effect of viscosity, we can still see the large change of height from a radially decreasing thin film to an almost constant height region. Figures 4a and 4b are the corresponding plots for the velocity field. From these we can see the acceleration of the flow until a critical condition is reached at the jump, at which point the flow slows from supercritical to subcritical. We again note that this prediction of the hydraulic jump is not physically significant since we have neglected the effect of viscosity which is of utmost importance in the case of thin films. However, our interest focuses only on the dynamics of the thin film formed upstream of jump. Consequently, we confine our numerical domain to the region ahead of the discontinuity.

Once an exact solution is reached, in other words a steady state solution, we can perturb the unsteady equations by looking at the dynamics of localized disturbances such as an initial solitary wave profile with compact support. We begin with the dynamics of a solitary wave initial condition. Figure 5a shows a height versus position versus time mesh, from which we can see the splitting of the "soliton" into two waves of smaller amplitude. The diagonal disturbance in the figure is a numerical pressure wave that stabilizes the flow. These waves travel along the c_+ and c_- characteristics, and we can see that as in the approximate analytic solution the c_- characteristic is the important one. The c_+ characteristic runs along the flow direction and thus radial spreading takes over and diffuses the disturbance into the film. However the c_- characteristic runs into the flow and steepens until a shock is formed. Hence breaking in the typical hyperbolic fashion, occurs in finite time. Figure 5b shows the time development of the initial disturbance. From this plot we see that after the "soliton" splits the c_+ characteristic decays to the right and the c_- forms a shock whose amplitude develops into a sharp peak that is eventually washed down by the flow.

Figure 6a shows the interaction of a two solitary wave initial conditions. The dynamics of each wave are similar to those of a single one for initial times. They each split into two waves, c_+ and c_-. However, the c_+ from the left wave eventually collides with the c_- from the right wave, thus giving birth to a new localized wave of smaller amplitude. This is due to the fact that the c_+ was already decaying via the radial spreading of the film. This new wave will again split into two new c_+^* and c_-^* and its dynamics are similar to the ones of a single wave disturbance. This is the case because the c_-^* will never cross the path of the c_- from the left wave disturbance and both c_+^* and c_+ from the right wave will decay into the flow. Both c_- from the left wave disturbance and c_-^* will break in finite time, but the c_-^* forms a weaker shock since its amplitude is smaller than that of the c_-. This is easier to visualize by looking at figure 6b , which is a position versus time plot, or an r-t plane of the wave interactions. From here we can see how c_- from the left wave disturbance forms an envelope of characteristics, in other words a shock. The c_-^* forms a weaker shock that diffuses back into the flow. Again the diagonal disturbance that we see is a numerical pressure wave that is artificially introduced for stabilizing the flow. Thus by studying this hyperbolic approximation we can predict the breaking of finite amplitude disturbances, but of course this breaking mechanism refers to the typical hyperbolic shock formation. In order to better understand this process, we need to include dispersive effects along with surface tension effects.

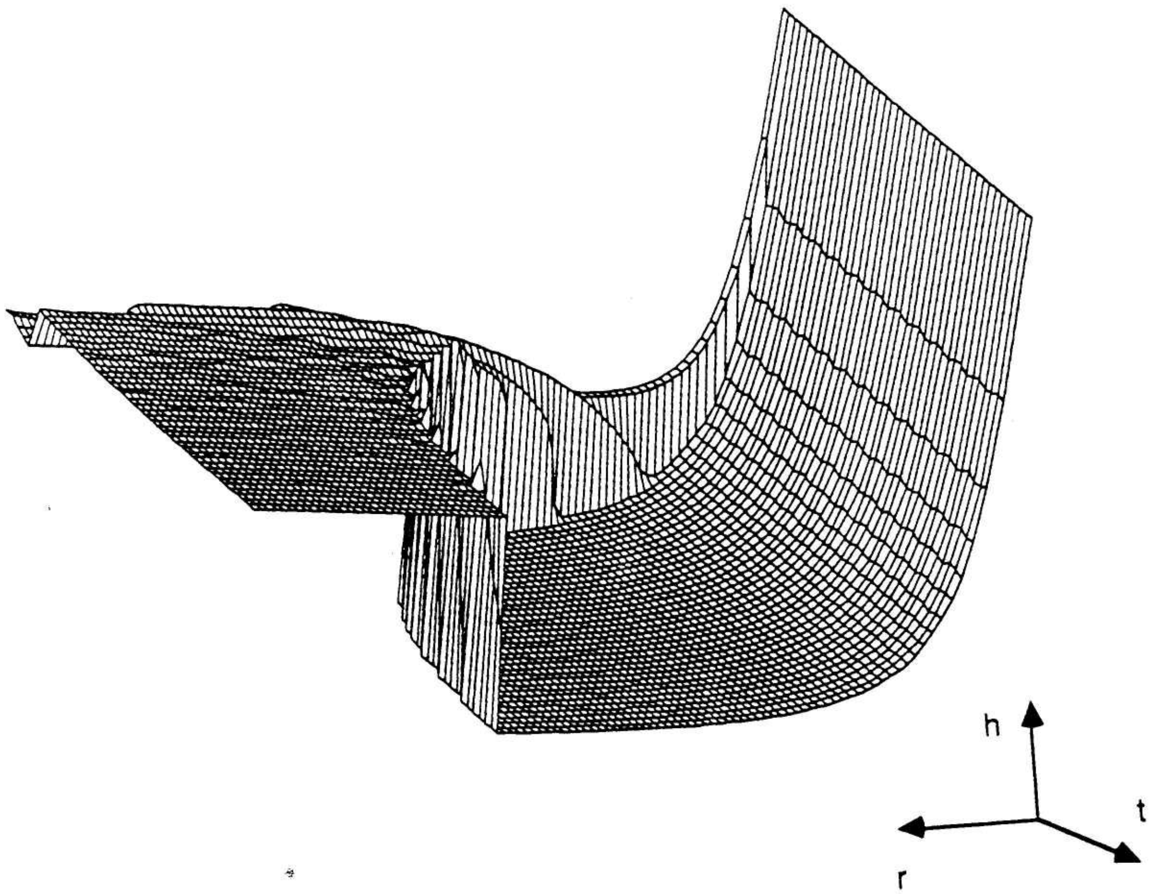

Figure 3a: Steady shock formation height vs. radial position vs. time.

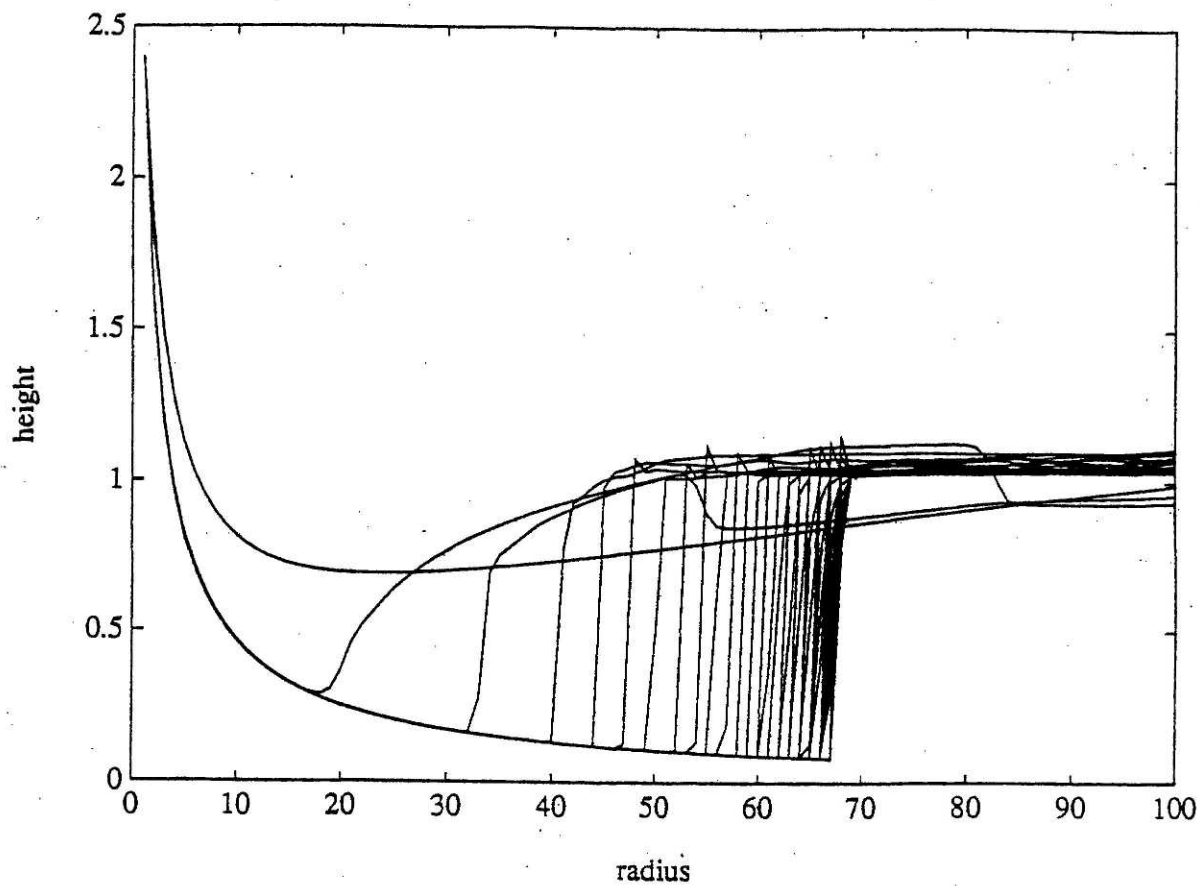

Figure 3b: Steady shock formation height vs. radial position, showing numerical convergence as time progresses.

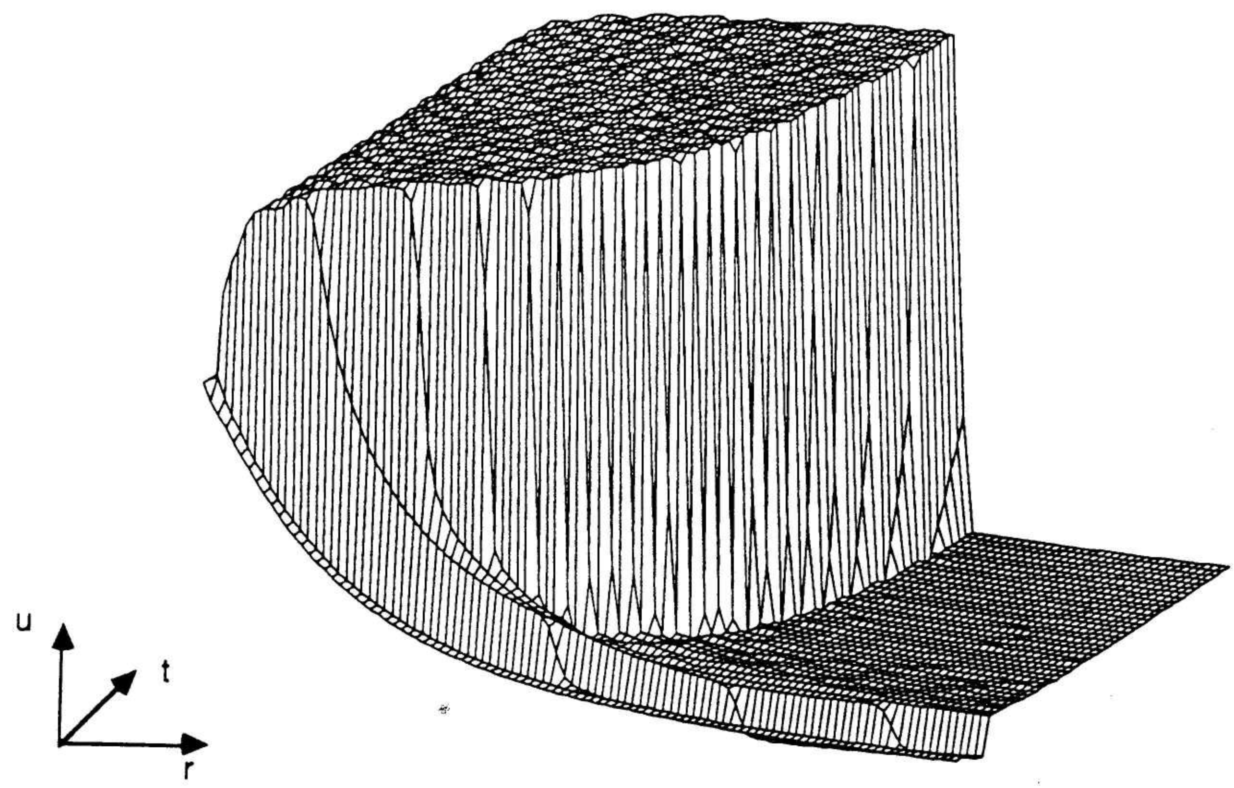

Figure 4a: Steady shock formation velocity vs. radial position vs. time.

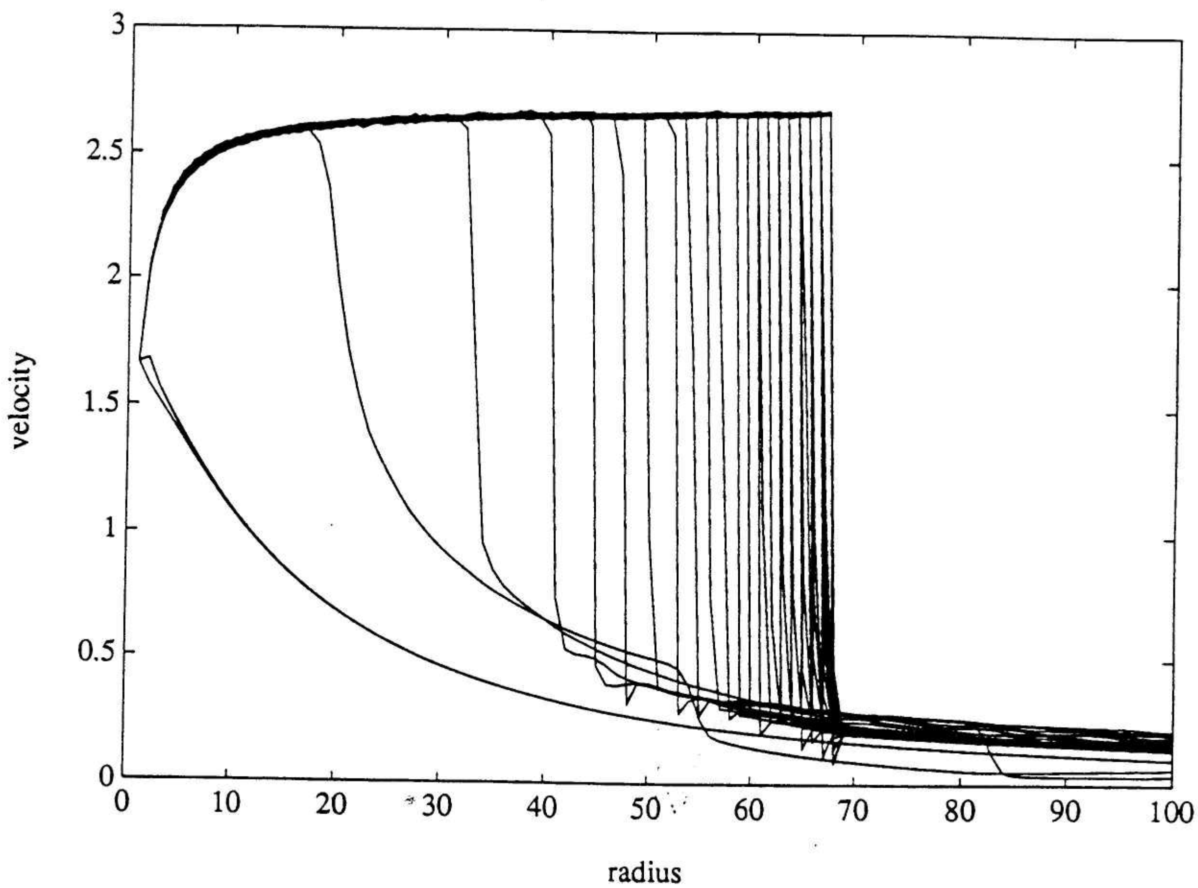

Figure 4b: Steady shock formation velocity vs. radial position, showing numerical convergence as time progresses.

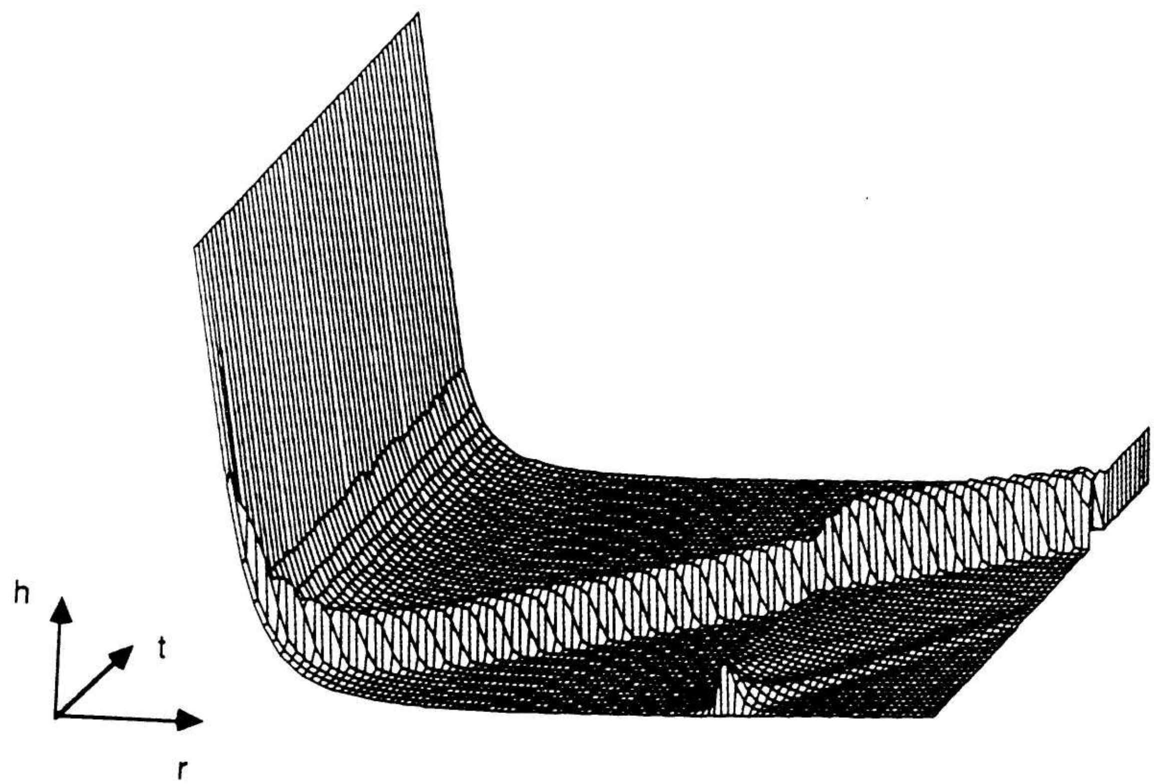

Figure 5a: Height vs. radial position vs. time, showing the evolution of a localized initial disturbance.

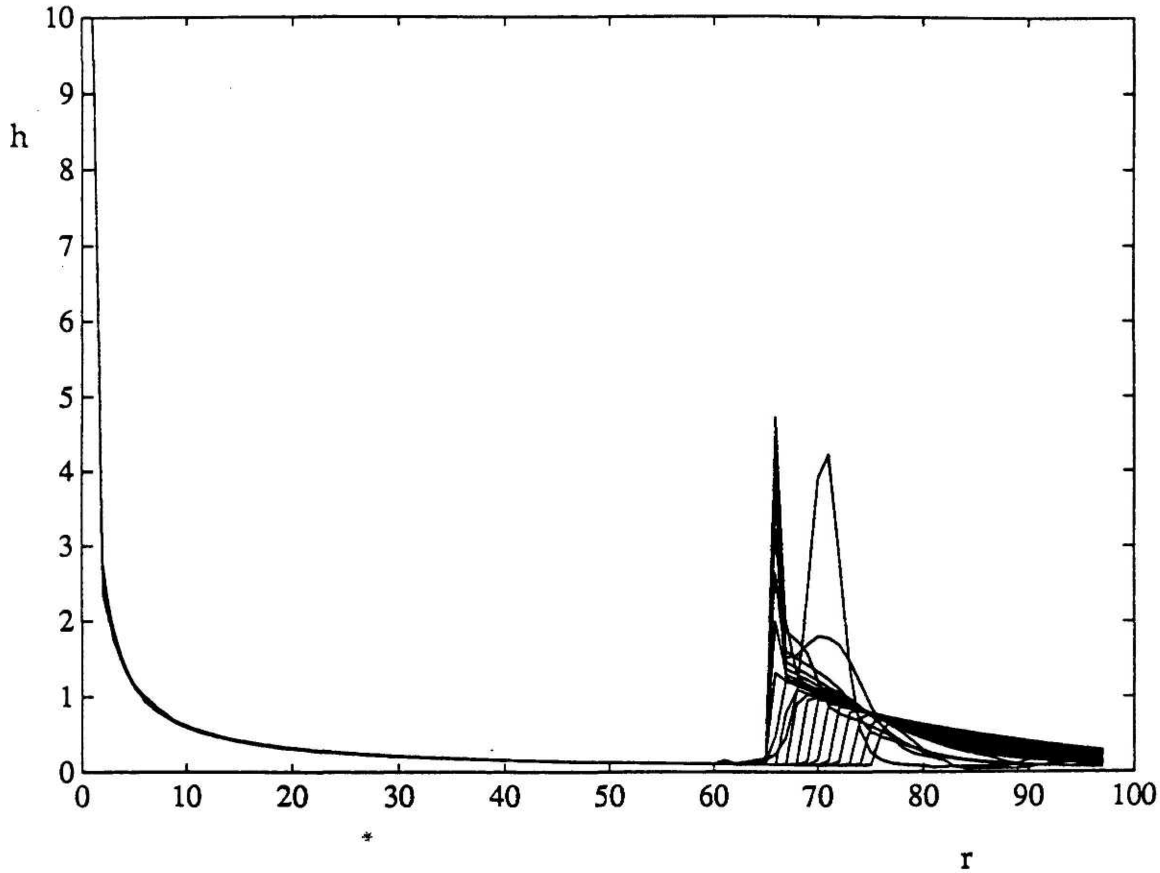

Figure 5b: Time evolution of localized disturbance in an height vs. radius plane, showing the splitting and steepening of a localized disturbance as it evolves in time.

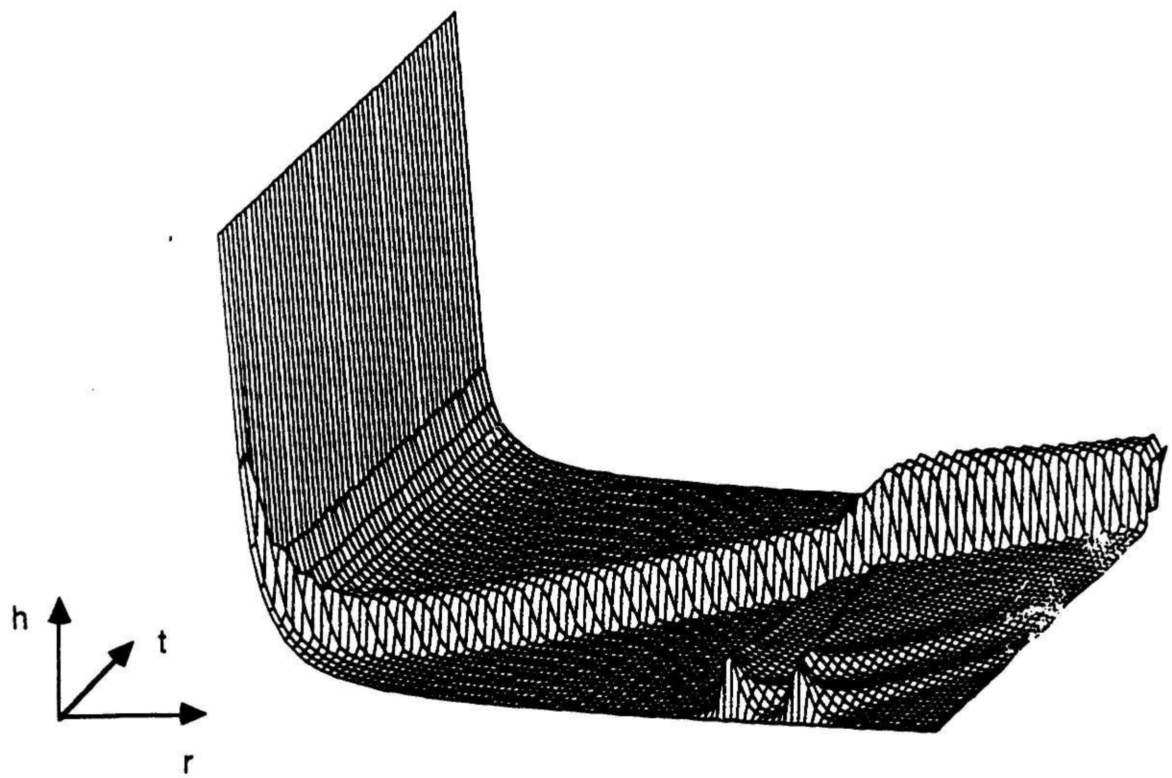

Figure 6a: Height vs. radial position vs. time, showing the dynamics of two localized initial disturbances.

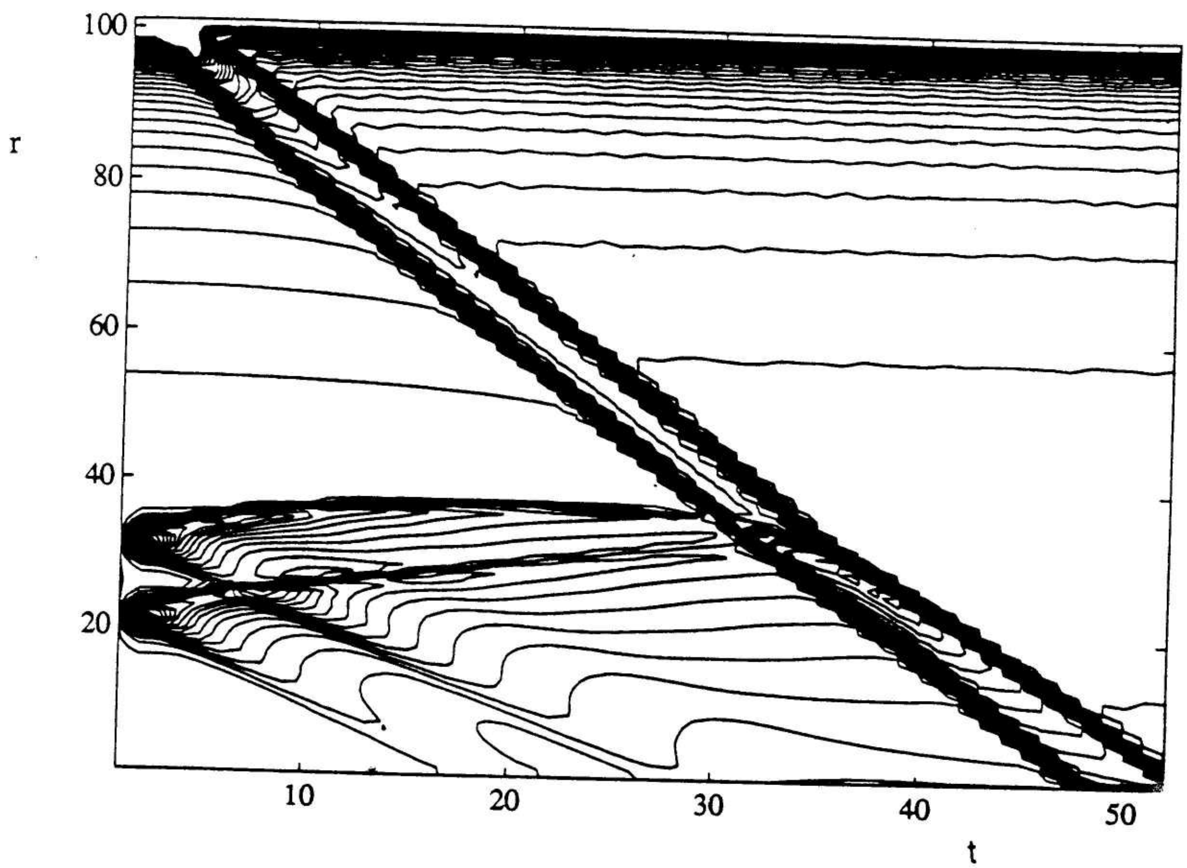

Figure 6b: r-t plane showing the interacting dynamics of the two localized disturbances.

5. Nonlinear Dispersion of Axisymmetric Water Waves, Including Effects of Surface Tension.

In order to analyze the effect of dispersion on radial water waves, we look at periodic modulations on the surface of the film and study the equations governing the modulated wave train. The analysis applies Whitham's general theory of averaged Lagrangians (see *Linear and Nonlinear Waves*, 1974) to the propagation of Stokes waves on a nonuniform radially spreading axisymmetric film. Thus instead of working with the water wave equations, we work directly with the variational principle (in dimensional form) proposed by equations (1) and (2).

Following from Whitham's (1967) paper on nonlinear dispersion of water waves without surface tension in a planar geometry, we generalize his problem by including surface tension effects and applying his technique to a cylindrical geometry.

A uniform periodic wave train is specified by certain parameters such as amplitude, wave-number, etc. The theory treats non-uniform wave trains in which these parameters vary slowly in space and time, in the sense that the changes in one wavelength and in one period are relatively small.

In this section, the modulation equations for the slow variations of amplitude, wave-number, etc. will be established for arbitrary depth. The wave train is calculated by a Stokes expansion in powers of the amplitude; however, the expansion breaks down for long waves, and these are covered separately in the next sections.

For finite depth, variations in mean height and mean fluid velocity occur, and they are coupled nonlinearly with variations of the amplitude and wave-number. These four quantities are the fundamental parameters of the non-uniform wave train analysis.

The uniform periodic solution of the water wave equations (in rectangular coordinates x,t) takes the form

$$h = h(\theta) \qquad \theta = kx - \omega t$$
$$\Phi = \tilde{\beta}x - \gamma t + \Phi(\theta, z) \tag{58}$$

where $k, \omega, \tilde{\beta}, \gamma$ are constant parameters. Here k, ω, θ are wave-number, frequency, and the phase function respectively. The term $\tilde{\beta}x - \gamma t$ must be allowed in Φ, since it is only the derivatives of Φ that represent periodic physical quantities. Physically, $\tilde{\beta}$ is the mean velocity, and γ corresponds to absorbing the Bernoulli constant into the potential. Mathematically $\tilde{\beta}, \gamma$ act like a pseudo wave-number and frequency in Φ corresponding to the real wave-number and frequency (k,ω) in θ.

We study the propagation of non-uniform temporally periodic (sinusoidal) wave trains characterized by an amplitude a.

For non-uniform wave trains in a cylindrical geometry, the generalized form of the solution is proposed as

$$h = h(\theta, r, t) \qquad \Phi = \psi(r,t) + \Phi(\theta, r, t, z)$$
$$k = \theta_r \qquad \omega = -\theta_t$$
$$\tilde{\beta} = \psi_r \qquad \gamma = -\psi_t \tag{59}$$

The pair (k, ω) are the wave-number and frequency. The term ψ acts like a pseudo-phase, since in the Lagrangian in (1) only the derivatives of Φ represent periodic physical quantities. The parameter $\tilde{\beta}$ is the mean of the radial velocity and γ is related to the mean height of the waves. Now we take all quantities to be slowly varying functions of space and time. The functions h and Φ are 2π periodic in the phase function θ.

We define the average Lagrangian over one period to be

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} L d\theta \tag{60}$$

and since the space and time scales are considered long compared with the wavelength, the averaged variational principle

$$\delta\int\int \mathcal{L} \ rdrdt = 0 \tag{61}$$

provides the required Euler equations. Note that in terms of q,r,t Laplace's equation, equation (6), becomes

$$\Phi_{zz} + k^2\Phi_{\theta\theta} + O\left(\Phi_{rr} + \frac{1}{r}\Phi_r + ...\right) = 0$$

where terms describing the slow variations in r have been neglected.

We expand, to the same order of approximation as Laplace's equation, h and F into their Fourier series in the phase function θ

$$h = \overline{H}(r,t) + h(\theta,r,t) \tag{62}$$

$$\overline{H}(r,t) = H(r) + \eta(r,t)$$

$$\Phi = \psi + \sum_{n=1}^{\infty} \phi^{(n)}(\theta,r,t)\cosh nkz \tag{63}$$

where

$$h(\theta,r,t) = a\cos\theta + a_2\cos 2\theta + \cdots \tag{64}$$

$$\phi^{(n)}(\theta,r,t) = \frac{A_n}{n}\sin n\theta \tag{65}$$

Here H is the steady undisturbed profile, η represents non-periodic variations from the undisturbed depth, and Φ has been chosen to satisfy Laplace's equation to first order along with the zero flux condition at z=0. The two new sets of variables a_n and A_n are amplitude parameters. The main one of these, we define as $a_1= a$. The parameters a and η (both of which are supposed to be small compared to one) are the other two important quantities that describe a modulated wave train.

To the same order of approximation the derivatives of Φ and h become:

$$\Phi_t = -\gamma + \sum_{n=1}^{\infty} -\omega\phi_\theta^{(n)}\cosh nkz + ... \tag{66}$$

$$\Phi_r = \tilde{\beta} + \sum_{n=1}^{\infty} k\,\phi_\theta^{(n)}\cosh nkz + ... \tag{67}$$

$$\Phi_z = \sum_{n=1}^{\infty} nk\phi^{(n)}\sinh nkz + ... \tag{68}$$

$$h_r = kh_\theta + ... \tag{69}$$

where the "+ ..." represents terms neglected due to the slow dependence of the parameters on r and t. After substitution of (62), (63) and (66)-(69) into (1) we obtain the approximate Lagrangian

$$L = \left(\tfrac{1}{2}\tilde{\beta}^2 - \gamma\right)h + \tfrac{1}{2}gh^2 + \left[\left(\tilde{\beta}k-\omega\right)\phi_\theta\right]\frac{\sinh kh}{k} + \left[\left(\tilde{\beta}k-\omega\right)\phi_\theta^{(2)}\right]\frac{\sinh 2kh}{2k}$$

$$+ \tfrac{1}{2}k^2\phi_\theta^2\left[\frac{\sinh 2kh}{2k} + h\right] + \tfrac{1}{4}k^2\phi^2\left[\frac{\sinh 2kh}{2k} - h\right]$$

$$+ \tfrac{1}{2}k^2\phi_\theta\phi_\theta^{(2)}\left[\frac{1}{3}\frac{\sinh 3kh}{k} + \frac{\sinh kh}{k}\right] + \tfrac{1}{2}k^2\phi\phi^{(2)}\left[\frac{1}{3}\frac{\sinh 3kh}{k} - \frac{\sinh kh}{k}\right]$$

$$+ \tfrac{1}{2}k^2\phi_\theta^{(2)}\left[\frac{1}{4}\frac{\sinh 4kh}{k} + h\right] + k^2\left(\phi^{(2)}\right)^2\left[\frac{1}{4}\frac{\sinh 4kh}{k} - h\right]$$

$$+ \frac{\sigma}{\rho}\left[1 + \tfrac{1}{2}k^2h_\theta^2\right] + \ldots \qquad (70)$$

Now, to calculate $\mathcal{L}$, we substitute equations (64) and (65) into (70) and substitute the result into (60). The average of L over one period is

$$\mathcal{L} = \left(\tfrac{1}{2}\tilde{\beta}^2 - \gamma\right)\overline{H} + \tfrac{1}{2}g\overline{H}^2 + \tfrac{1}{4}ga^2 + \tfrac{1}{4}ga_2^2 - \frac{\left(\omega-\tilde{\beta}k\right)}{k}\left[e_1A_1 + e_2A_2\right]$$

$$+ \tfrac{1}{2}k\left[e_{11}A_1^2 + 2e_{12}A_1A_2 + e_{22}A_2^2\right] + \frac{\sigma}{\rho}\left[1 + \tfrac{1}{4}k^2a^2 + k^2a_2^2\right]\cdots \qquad (71)$$

where

$$e_1 = \tfrac{1}{2}ka\cosh k\overline{H} + \tfrac{1}{4}k^2aa_2\sinh k\overline{H} + \tfrac{1}{10}k^3a^3\cosh k\overline{H}$$

$$e_2 = \tfrac{1}{2}ka_2\cosh 2k\overline{H} + \tfrac{1}{4}k^2a^2\sinh 2k\overline{H}$$

$$e_{11} = \tfrac{1}{4}ka\sinh 2k\overline{H} + \tfrac{1}{4}k^2a^2\sinh k\overline{H} + \tfrac{1}{4}ka_2$$

$$e_{12} = \tfrac{1}{4}ka\cosh 3k\overline{H}$$

$$e_{22} = \tfrac{1}{8}\sinh 4k\overline{H} \qquad (72)$$

The Stokes expansion is in powers of ka, and it is not uniformly valid as $kH \to 0$. It also requires that ka be small compared with $(kH)^3$. The fundamental parameters in the problem are (ω,k,a) and (γ, $\tilde{\beta}$, η). All the other quantities a_n and A_n can be eliminated by taking the variations with respect to them, since they are independent of each other.

The Euler equations from the averaged variational principle (61) are

$$\delta a_n: \qquad \mathcal{L}_{a_n} = 0 \tag{73}$$

$$\delta \overline{H} \qquad \mathcal{L}_{\overline{H}} = 0 \tag{74}$$

$$\delta A_n: \qquad \mathcal{L}_{A_n} = 0 \tag{75}$$

$$\delta\theta: \qquad \frac{\partial}{\partial t}\left(r\mathcal{L}_{\omega}\right) - \frac{\partial}{\partial r}\left(r\mathcal{L}_{k}\right) = 0 \tag{76}$$

$$\delta\psi: \qquad \frac{\partial}{\partial t}\left(r\mathcal{L}_{\gamma}\right) - \frac{\partial}{\partial r}\left(r\mathcal{L}_{\beta}\right) = 0 \tag{77}$$

To complete the system we introduce the consistency relations to eliminate θ and ψ

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial r} = 0 \tag{78}$$

$$\frac{\partial \tilde{\beta}}{\partial t} + \frac{\partial \gamma}{\partial r} = 0 \tag{79}$$

The variations with respect to A_1 and A_2 lead to

$$\mathcal{L}_{A_1} = 0 \quad : \quad e_{11}A_1 + e_{12}A_2 - \frac{\omega - \tilde{\beta} k}{k^2} e_1 = 0 \tag{80}$$

and

$$\mathcal{L}_{A_2} = 0 \quad : \quad e_{12}A_1 + e_{22}A_2 - \frac{\omega-\tilde{\beta}k}{k^2}e_2 = 0 \tag{81}$$

Solving (80) and (81) for A_1 and A_2, we have

$$\Rightarrow \qquad A_1 = \frac{\omega-\tilde{\beta}k}{k^2}\left[\frac{e_1e_{22} - e_2e_{12}}{e_{22}e_{11} - e_{12}^2}\right] \tag{82}$$

$$\Rightarrow \qquad A_2 = \frac{\omega-\tilde{\beta}k}{k^2}\left[\frac{e_2e_{11} - e_1e_{12}}{e_{22}e_{11} - e_{12}^2}\right] \tag{83}$$

Substituting (82) and (83), back into (71) and combining the A_1 and A_2 terms we obtain

$$\mathcal{L} = \left(\frac{1}{2}\tilde{\beta}^2 - \gamma\right)\overline{H} + \frac{1}{2}g\overline{H}^2 + \frac{1}{4}ga^2 + \frac{\sigma}{\rho}\left[1 + \frac{1}{4}k^2a^2 + k^2a_2^2\right]$$
$$- \frac{1}{4}\frac{\left(\omega-\tilde{\beta}k\right)^2}{k\overline{T}}\left[a^2 - \frac{2\overline{T}^2 - 1}{4\overline{T}^2}k^2a^4 - \frac{3-\overline{T}^2}{2\overline{T}}ka^2a_2 + \left(1+\overline{T}^2\right)a_2^2\right] + \cdots \tag{84}$$

where

$$\overline{T} \equiv \tanh k\overline{H}$$

To eliminate a_2 we simply take the variation with respect to a_2

$$\mathcal{L}_{a_2} = 0$$

and solving for a_2 we obtain the following relation

$$\frac{a_2}{a^2} = \frac{\left(3-\overline{T}^2\right)\left(\omega-\tilde{\beta}k\right)^2k}{4\overline{T}\left[\left(\omega-\tilde{\beta}k\right)^2\left(1-\overline{T}^2\right) - k\overline{T}\left(g+4\frac{\sigma}{\rho}k^2\right)\right]} \tag{85}$$

Substitution of (85) into (84) leads to

$$\mathcal{L} = \sigma + \rho\left(\frac{1}{2}\widetilde{\beta}^2 - \gamma\right)\overline{H} + \frac{1}{2}\rho g \overline{H}^2 + \frac{1}{4}\rho a^2\left[g\left(1+\lambda k^2\right) - \frac{\left(\omega - \widetilde{\beta}k\right)^2}{k\overline{T}}\right]$$

$$+ \frac{1}{4}\rho a^2\left[g\left(1+\lambda k^2\right) - \frac{\left(\omega - \widetilde{\beta}k\right)^2}{k\overline{T}}\right]$$

$$+ \frac{1}{4}\rho a^2\left\{\frac{gk^2}{4^2\overline{T}^2}\frac{\left[9\overline{T}^4 - 10\overline{T}^2\left(\frac{1+\frac{17}{5}\lambda k^2}{1+\lambda k^2}\right) + 9\left(\frac{1+\frac{7}{3}\lambda k^2}{1+\lambda k^2}\right)\right]}{\left[1 - \left(\frac{3\lambda k^2}{1+\lambda k^2}\right)\frac{1}{\overline{T}^2}\right]}\left(1+\lambda k^2\right)\right\} + \ldots \quad (86)$$

where $\lambda = \sigma/\rho g$. This is a useful version of the Lagrangian for calculating modulation quantities required in the Euler equations (76) - (79).

In order to obtain the modulation equations, we first need to calculate the following partial derivatives

$$\mathcal{L}_\omega = -\frac{1}{2}ga^2\frac{\left(1+\lambda k^2\right)}{\overline{\omega}_o} + O\left(a^4\right) \quad (87)$$

$$\mathcal{L}_k = \frac{1}{2}ga^2\left[\lambda k + \frac{1+\lambda k^2}{\overline{\omega}_0}\left\{\widetilde{\beta} + \overline{c}_g\left[1 - \frac{\lambda k^2}{1+\lambda k^2}\left(\frac{\overline{c}_p}{\overline{c}_g}\right)\right]\right\}\right] + O\left(a^4\right) \quad (88)$$

$$\mathcal{L}_\gamma = -\overline{H} \quad (89)$$

$$\mathcal{L}_{\widetilde{\beta}} = \widetilde{\beta}\overline{H} + \frac{1}{2}\frac{\left(1+\lambda k^2\right)}{\overline{c}_p}ga^2 + O\left(a^4\right) \quad (90)$$

along with the variation with respect to $\overline{H}$

$$\mathcal{L}_{\overline{H}} = 0$$

from these

$$\gamma = \frac{1}{2}\widetilde{\beta}^2 + g\overline{H} + \frac{1}{4}\frac{ga^2}{\overline{H}}\left[2\left(\frac{\overline{c}_g}{\overline{c}_p}\right)\left(1+\lambda k^2\right) - \left(1+3\lambda k^2\right)\right] \quad (91)$$

where the following definitions have been made

$$\overline{\omega_0^2} = gk\overline{T}\left(1+\lambda k^2\right) \quad , \quad \bar{c}_p = \frac{\overline{\omega_0}}{k} \qquad \text{and} \qquad \bar{c}_g = \frac{\partial\overline{\omega_0}}{\partial k} \tag{92}$$

$$\bar{c}_p^2 = \frac{g\overline{T}}{k}\left(1+\lambda k^2\right) \tag{93}$$

$$\frac{2\bar{c}_g}{\bar{c}_p} = \frac{2k\overline{H}}{\sinh 2k\overline{H}} + \left(\frac{1+3\lambda k^2}{1+\lambda k^2}\right) \tag{94}$$

Note that (91) is a Bernoulli type equation for the pseudo phase as coupled to the mean height.

To obtain the dispersion relation we take the variation with respect to a

$$\mathcal{L}_a = 0$$

$$\Rightarrow \qquad \frac{\left(\omega-\tilde{\beta}k\right)^2}{\overline{\omega_0^2}} = 1 + \frac{2\overline{D}_0 k^3 E}{\overline{\omega_0^2}} + O\left(E^2\right) \tag{95}$$

where

$$E = \frac{1}{2}ga^2\left(1+\lambda k^2\right) \tag{96}$$

and

$$\overline{D}_0\left(\overline{H},k\right) = \frac{9\overline{T}^4 - 10\overline{T}^2\left(\frac{1+\frac{17}{5}\lambda k^2}{1+\lambda k^2}\right) + 9\left(\frac{1+\frac{7}{3}\lambda k^2}{1+\lambda k^2}\right)}{8\overline{T}^3\left(1 - \left(\frac{3\lambda k^2}{1+\lambda k^2}\right)\right)} \tag{97}$$

This provides the nonlinear dispersion relation and we note the dependence of frequency on amplitude, a, as well as on $\tilde{\beta}$ and $\overline{H}$. The parameter E, which is proportional to the energy density, is introduced as a more convenient small amplitude parameter to work with.

Therefore with the above definitions the modulation equations can be written as

$$\frac{\partial}{\partial t}\left[\frac{rE}{\bar{\omega}_0}\right]+\frac{\partial}{\partial r}\left[\left(\tilde{\beta}+\bar{c}_g\right)\frac{rE}{\bar{\omega}_0}\right]=0 \tag{98}$$

$$\frac{\partial k}{\partial t}+\frac{\partial \omega}{\partial r}=0 \tag{99}$$

$$\frac{\partial}{\partial t}\left[r\bar{H}\right]+\frac{\partial}{\partial r}\left[r\tilde{\beta}\bar{H}+\frac{rE}{\bar{c}_p}\right]=0 \tag{100}$$

$$\frac{\partial \tilde{\beta}}{\partial t}+\frac{\partial \gamma}{\partial r}=0 \tag{101}$$

At this point, it is convenient to separate $\bar{H}$ and $\tilde{\beta}$ into their steady and unsteady components. Hence, we expand all quantities containing $\bar{H}$ about H(r) and retain only linear terms in h and expand $\tilde{\beta}$ about U(r) ,the unperturbed steady flow (i.e. $\tilde{\beta} \rightarrow U(r)+\tilde{\beta}$).

We have

$$T \equiv \tanh kH(r)$$

$$\omega_0^2 = gkT\left(1+\lambda k^2\right) \quad , \quad c_p = \frac{\omega_0}{k} \qquad \text{and} \qquad c_g = \frac{\partial \omega_0}{\partial k}$$

$$\bar{\omega}_0 = \omega_0\left[1+\frac{1}{2}k\eta\left(\frac{1-T^2}{T}\right)\right]+\ldots$$

$$\bar{D}_0 = D_0 + O(\eta)$$

$$B_0 \equiv c_g - \frac{1}{2}c_p\left(\frac{1+3\lambda k^2}{1+\lambda k^2}\right) \tag{102}$$

When expanding any function to O(1), we merely drop all over bars and evaluate quantities at H.

Making use of the new definitions and keeping terms to linear order in η, $\tilde{\beta}$, and E, the dispersion relation becomes

$$\omega = \omega_0 + \tilde{\beta} k + \frac{D_0 k^2}{c_p} E + \frac{k B_0}{H} \eta + \cdots \tag{103}$$

In order to analyze the fourth order system (98)-(101), we make one final definition purely for algebraic purposes

$$A = \frac{E}{c_p} \equiv \text{wave momentum} \tag{104}$$

Substituting (91), and (102)-(104) into the system (98)-(101), we obtain the following modulation equations

$$\frac{\partial}{\partial t}\left(\frac{rA}{k}\right) + \frac{\partial}{\partial r}\left[\left(\tilde{\beta} + c_g\right)\frac{rA}{k}\right] = 0 \tag{105}$$

$$\frac{\partial}{\partial t}[r\eta] + \frac{\partial}{\partial r}\left[r\left\{\tilde{\beta} H + U\eta + \tilde{\beta}\eta + A\right\}\right] = 0 \tag{106}$$

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial r}\left[\left(U + \tilde{\beta}\right)k + \omega_0 + \frac{k B_0}{H}\eta + D_0 k^2 A\right] = 0 \tag{107}$$

$$\frac{\partial \tilde{\beta}}{\partial t} + \frac{\partial}{\partial r}\left[U\tilde{\beta} + g\eta + \frac{B_0}{H} A\right] = 0 \tag{108}$$

These equations determine to first order in A, η, and $\tilde{\beta}$ the evolution of the wave momentum, A, mean wave elevation, η , wave-number, k, and mean radial velocity, $\tilde{\beta}$.

In order to study the stability of a Stokes wave train, we need to examine the characteristic forms of (105-108). The stability is decided by the equation type of the full set (105-108) which is fourth order in $(A,k,\eta,\tilde{\beta})$.

The system can be rewritten as

$$\eta_t + U\eta_r + H\tilde{\beta}_r + A_r + \frac{1}{r}\left[(rH)_r\tilde{\beta} + (rU)_r\eta + A\right] = 0 \tag{109}$$

$$A_t + (U+c_g)A_r + \left(\frac{\partial c_g}{\partial k}\right)k_r + \left[2U_r + \left(\frac{\partial c_g}{\partial k}\right)H_r + \frac{(U+c_g)}{r} + \frac{\partial \omega_0}{\partial H}\frac{H_r}{k}\right]A = 0 \tag{110}$$

$$\tilde{\beta}_t + U\tilde{\beta}_r + g\eta_r + \left(\frac{B_0}{H}\right)A_r + \left(\frac{B_0}{H}\right)_k Ak_r + \left(\frac{B_0}{H}\right)_H AH_r = 0 \tag{111}$$

$$\begin{aligned} &k_t + \left[U + \tilde{\beta} + c_g + \eta\left(\frac{kB_0}{H}\right)_k + A(D_0k^2)_k\right]k_r \\ &+ k\tilde{\beta}_r + \frac{kB_0}{H}\eta_r + D_0k^2A_r + kU_r + \left[\frac{\partial \omega_0}{\partial H} + (D_0k^2)_H A + \left(\frac{(kB_0)}{H}\right)_H \eta\right]H_r = 0 \end{aligned} \tag{112}$$

where nonlinear terms like AA_r and so on have been neglected for the calculation of the characteristic speeds.

To find the characteristic forms of the system (109-112), we compose a linear combination: L_1 times (109), L_2 times (110), and L_3 times (111) added to (112). The composition is

$$\begin{aligned} &k_t + k_r\left[U + c_g + v + L_2\frac{\partial C_g}{\partial k}A + L_3\left(\frac{B_0}{H}\right)A\right] + L_1\left[\eta_t + \eta_r\left(U + \frac{kB_0}{HL_1} + \frac{L_3 g}{L_1}\right)\right] \\ &+ L_2\left[A_t + A_r\left(U + c_g + \frac{D_0k^2}{L_2} + \frac{L_1}{L_2} + \frac{L_3B_0}{HL_2}\right)\right] + L_3\left[\beta_t + \beta_r\left(U + \frac{L_1H}{L_3} + \frac{k}{L_3}\right)\right] = RHS \end{aligned} \tag{113}$$

where

$$v = \tilde{\beta} + \left(\frac{kB_0}{H}\right)_k \eta + (D_0k^2)_k A$$

The combinations can be written in terms of a characteristic speed c

$$k_t + ck_r + L_1(\eta_t + c\eta_r) + L_2(A_t + cA_r) + L_3(\beta_t + c\beta_r) = RHS \tag{114}$$

which requires that L_1, L_2, L_3, and c satisfy the system of algebraic equations

$$c = U + c_g + v + L_2\frac{\partial c_g}{\partial k}A + L_3\frac{B_0 A}{H} \tag{115}$$

$$\begin{bmatrix} 1 & c_0\text{-}c & \dfrac{\frac{\partial B_0}{\partial k}}{H} \\ U\text{-}c & 0 & g \\ H & 0 & U\text{-}c \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = - \begin{bmatrix} D_0 k^2 \\ \dfrac{kB_0}{H} \\ k \end{bmatrix} \tag{116}$$

where

$$c_0 = U + c_g$$

Solving this system of equations, we have the condition that c must satisfy

$$(c_0\text{-}c)^2\left[(c\text{-}U)^2\text{-}gH\right] + (c_0\text{-}c)\left[\frac{\frac{\partial B_0}{\partial k}}{H}(c\text{-}U+B_0)A + v\left((c\text{-}U)^2\text{-}gH\right)\right]$$

$$+\frac{\partial c_g}{\partial k}Ak\left[\text{-}D_0 k\left\{\text{-}(c\text{-}U)^2 + gH\right\} + g + 2\frac{B_0}{H}(c\text{-}U) + \frac{B_0^2}{H}\right] = 0 \tag{117}$$

The resulting multipliers are

$$L_1 = \frac{k[(c\text{-}U)B_0 + gH]}{(c\text{-}U)^2 - gH}$$

$$L_2 = \frac{c\text{-}c_0 + v}{A\frac{\partial c_g}{\partial k}} + \frac{k\frac{\partial B_0}{\partial k}}{H\frac{\partial c_g}{\partial k}}\left[\frac{(c\text{-}U) + B_0}{gH - (c\text{-}U)^2}\right]$$

$$L_3 = \frac{k[(c\text{-}U) + B_0]}{(c\text{-}U)^2 - gH} \tag{118}$$

Now to solve for c we look at the limits as $A \to 0$ and find that $c \to c_0$ is a double root which corresponds to linear wave theory where the double characteristics collapse onto one. Note that the errors in (114) are $O(A^2)$. Therefore, for the roots when $A \to 0$, we get c_o plus a correction. These are given approximately by

$$c = c_0 \pm \sqrt{\frac{\partial c_g}{\partial k}\frac{kA}{H}\left[-kHD_0 + \frac{gH + 2B_0(c_0-U) + B_0^2}{gH - (c_0-U)^2}\right]} \tag{119}$$

$$c = c_0 \pm \sqrt{\frac{\omega_0 A}{4}F(\lambda k^2, kH)}$$

where

$$\frac{\partial c_g}{\partial k} = -\frac{1}{4}\frac{\sqrt{gkT(1+\lambda k^2)}}{k^2}\left[\frac{1-6\lambda k^2-3\lambda^2 k^4}{(1+\lambda k^2)^2} - 2kH\left(\frac{1-T^2}{T}\right)\left(\frac{1+3\lambda k^2}{1+\lambda k^2} - \frac{1}{2}\frac{kH}{T}(1+3T^2)\right)\right] \tag{120}$$

and D_o was defined in (97) and B_o in (102) and $F(\lambda k^2, kH)$ is defined below in equation (122).

Due to surface tension, $\frac{\partial c_g}{\partial k}$ changes sign and so does the bracketed term in (119); hence the radicand is negative for certain regions of kH and λk^2 space.

The terms inside the bracket arise from the nonlinear coupling of frequency with amplitude and from the coupling of mean height η with mean velocity β. Also note that the correction to c_o is $O(\sqrt{A})$ and not $O(A)$.

The other roots that we find as we let $A \to 0$ are

$$c = U \pm \sqrt{gh} + O(A) \quad , A \to 0 \tag{121}$$

These correspond to the simple nonlinear shallow water wave theory (corresponding to equation (48) linearized about a steady state). The correction to these roots is of O(A) and does not produce any kind of instability since the roots are real. The equations derived so far are first order in A, η, $\tilde{\beta}$, since the waves are of very small amplitude.

To answer the question of the stability of the Stokes nonuniform wave train, we simply determine when the characteristic speeds become imaginary. This implies that the radicand should be negative. If the radicand is positive the waves would propagate in a typical hyperbolic fashion; otherwise, the equations are elliptic and modulation tends to grow. This does not necessarily imply that the motion will be unbounded. We conjecture based on results for rectangular geometries (Whitham's *Linear and Nonlinear Waves*, 1974) , that the next stage would be the development of modulations where the envelope of the wave develops into a sequence of solitary waves.

In order to look at the critical regions of stability we set to zero the radicand appearing in the equation that determines the characteristic speeds and obtain a stability map of kH versus λk^2. We note that in the absence of capillary effects (λ=0) we recover Whitham's stability result (Whitham 1967) of kH=1.36 as the critical threshold for purely gravitational waves.

The neutral stability of the modulation is determined from the zeros of the function

$$F(\xi, \Omega) = \left[\frac{1-6\xi-3\xi^2}{(1+\xi)^2} - 2\Omega\left(\frac{1-T^2}{T}\right)\left(\frac{1+3\xi}{1+\xi} - \frac{1}{2}\frac{\Omega}{T}(1+3T^2)\right)\right] .$$

$$\left\{\left[\frac{9T^4-10T^2\left(\frac{1+\frac{17}{5}\xi}{1+\xi}\right)+9\left(\frac{1+\frac{7}{3}\xi}{1+\xi}\right)}{8T^3\left(1-\frac{3\xi}{T^2(1+\xi)}\right)}\right]-\frac{\left[1+\frac{1}{2}(1+\xi)(1-T^2)\left[\Omega\left(\frac{1-T^2}{T}\right)+\frac{1+3\xi}{1+\xi}\right]+\frac{1}{4}\frac{\Omega}{T}(1+\xi)(1-T^2)^2\right]}{\left[\Omega-\frac{T}{4}(1+\xi)\left\{\Omega\left(\frac{1-T^2}{T}\right)+\frac{1+3\xi}{1+\xi}\right\}^2\right]}\right\} \tag{122}$$

and

$$F(\xi,\Omega) = 0$$

where, for simplicity, we have defined $(\xi,\Omega) \equiv (\lambda k^2, kH)$

The regions where $F(\xi,\Omega)$ turns from negative to positive correspond to a change of type of the modulation equations. The system equations turns from elliptic to hyperbolic. Thus, the zeros of $F(\xi,\Omega)$ define the neutral stability curves and the wave train propagates in a typical hyperbolic fashion for $F(\xi,\Omega)$ greater than zero.

Figure 7 is a plot of the mentioned curve $F(\xi,\Omega)=0$. Notice that the critical value is $kH=1.36$ in the absence of surface tension. Region I represents the typical deep water instability. However in the case of finite surface tension effects, we find that for large kH and moderate λk^2, greater than about 0.4, we move into a stable hyperbolic region (region II). This region again turns elliptic as λk^2 increases so that region III is entered. Region IV is again stable for any λk^2 but becomes unstable as kH is increased or decreased enough so that we return to region III . As kH is decreased, we reentered the unstable region III but we quickly enter region V which is again stable for all λk^2 except in the bubble defined by region VI which again turns elliptic for small enough λk^2 . In other words, for long enough wavelengths the wave train becomes unstable; we tend to think that this mechanism leads to solitary wave modulations.

Region VII is again stable and corresponds to the classic limit of kH tending towards zero. The intersection of regions I, II, III, and VII is an unstable saddle point where dynamics are very intricate, depending on the region from which one is traversing it.

Since we are interested in the case of water waves, for which $\lambda=7.54\times10^{-6}$ meters squared, we look at a map where the l dependence is extracted and the contour becomes one of H versus k.

Figure 8a shows the stability regions for water waves in a graph of the critical dependence of H on k.

One of the main features to observe is that as k tends to zero H tends to a finite value of about 0.0048 meters which was not obvious from figure 7. Again the regions of stability are marked on the figure with an S for stable and a U for unstable regimes. Our interest is in the limit as H approaches zero, and we notice that in this limit waves are unstable for waves having wavelengths in the order of 2.8 cm or longer. However as k increases and H decreases we see that waves of small amplitude could also become unstable. Therefore we conclude that small amplitude waves or "ripples" can become unstable depending on the film thickness. However, we should be reminded that this instability could imply that a small periodic modulated wave train could develop into a train of solitary waves.

Figures 8b, 8c, and 8d show stability maps for different values of λ. We conclude that by changing the value of surface tension to density ratio changes, we are able to change the cut-off wave number for region VII. Note that λ can be changed by adding surfactants, changing the density or the temperature of the fluid, or by using a different fluid (or by lowering g, as in micro-gravity applications.)

By decreasing λ, the cut-off wave number increases. This implies that the critical wavelengths are smaller for thinner films. As λ increases to order 10^{-5} m^2, the cut-off wave number decreases, making the instability become important for thicker films on the order of 0.8 cm, but stable for very thin films except again in the limit of extremely large wave numbers (see figures 8b, 8c, and 8d).

In order to understand what happens when the modulations become unstable we need to include higher order dispersive effects in the expansions as well as in the calculation of the averaged Lagrangian. However, instead of doing this, we choose to look at a long wave analysis applied to the reduced inviscid equations (29)-(31).

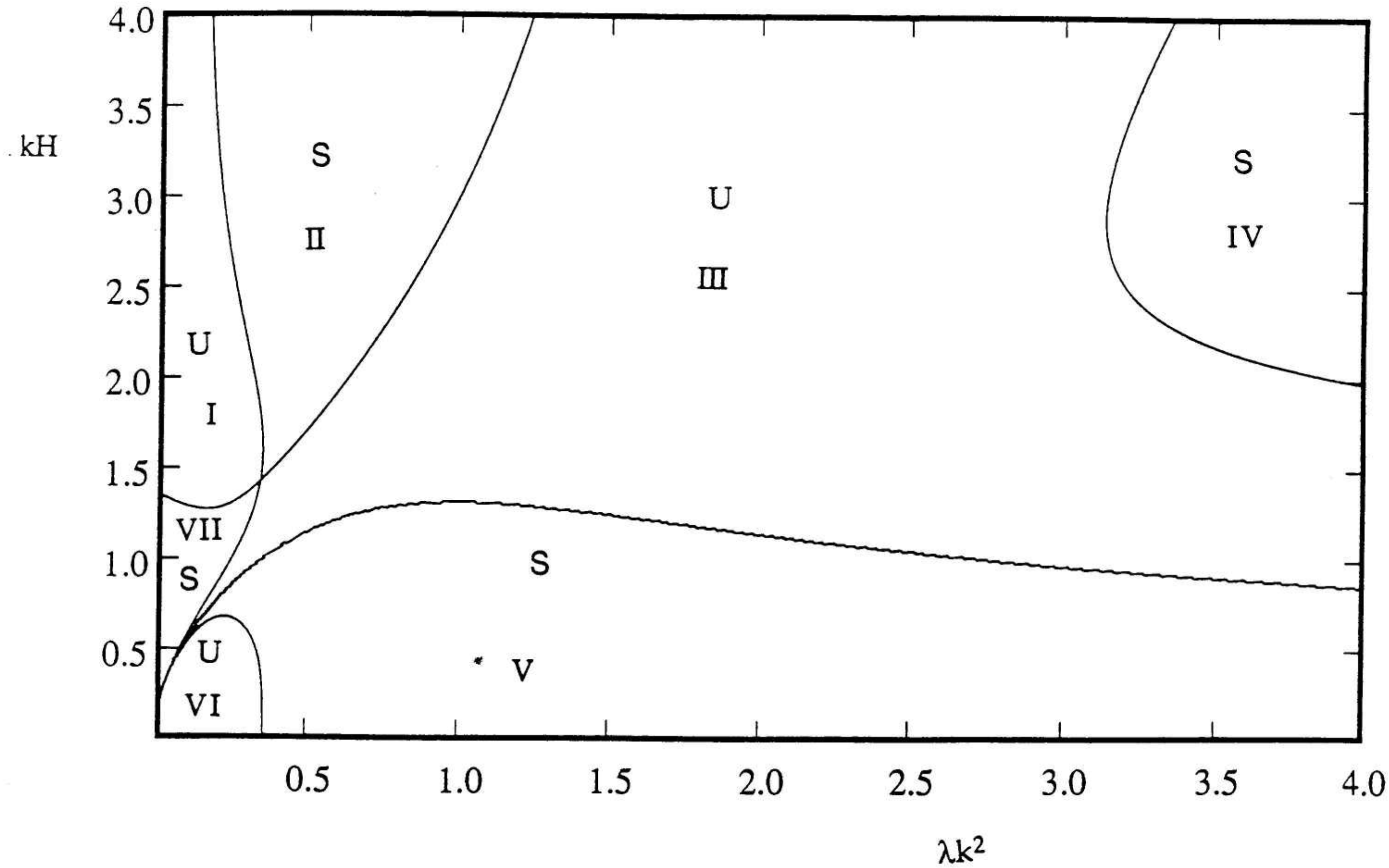

Figure 7: Stability map of kH vs. λk^2 for Stokes wavetrains on a non-uniform radial film.

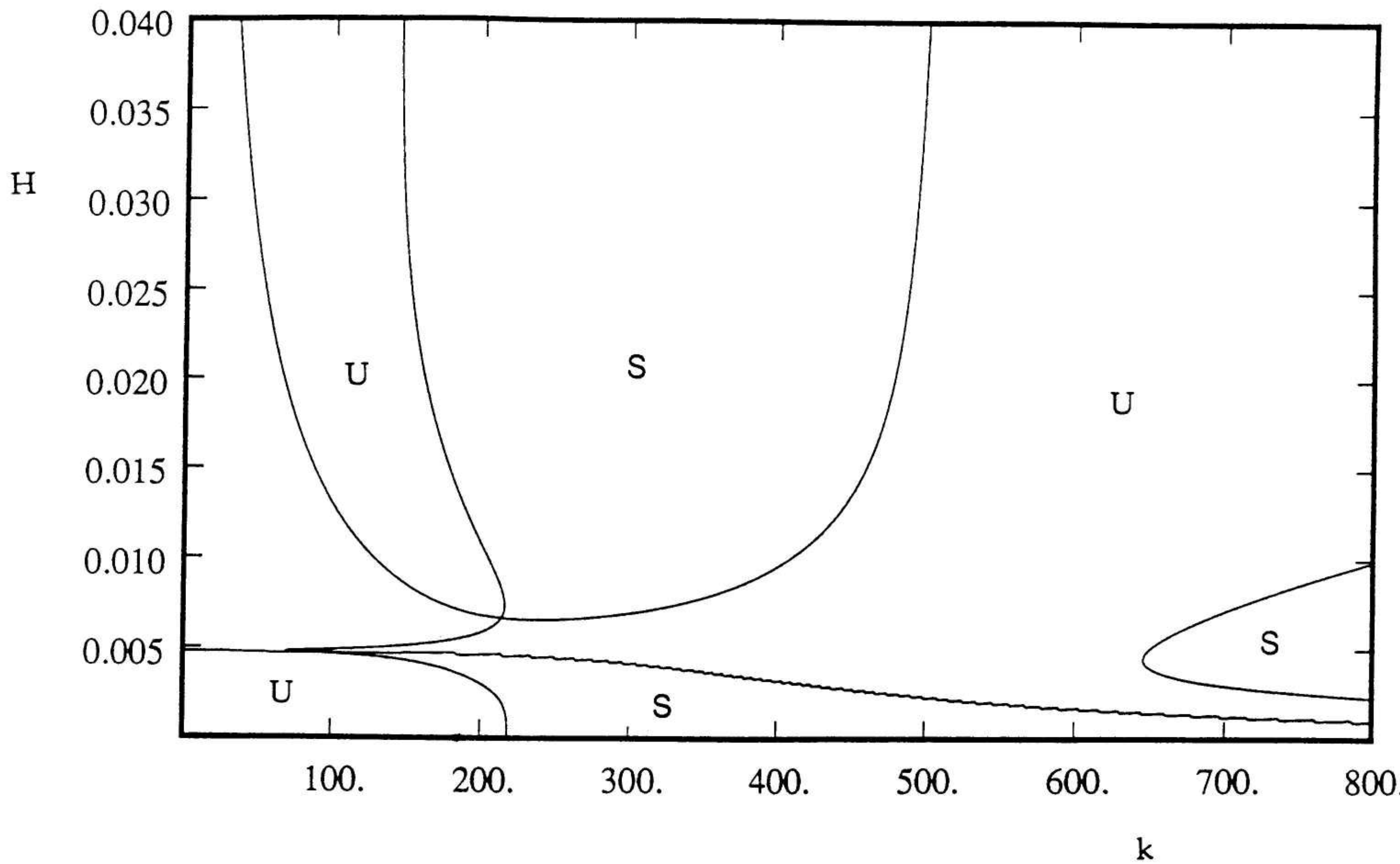

Figure 8a: Stability map, H vs. k, for water waves (λ= 7.54e-6).

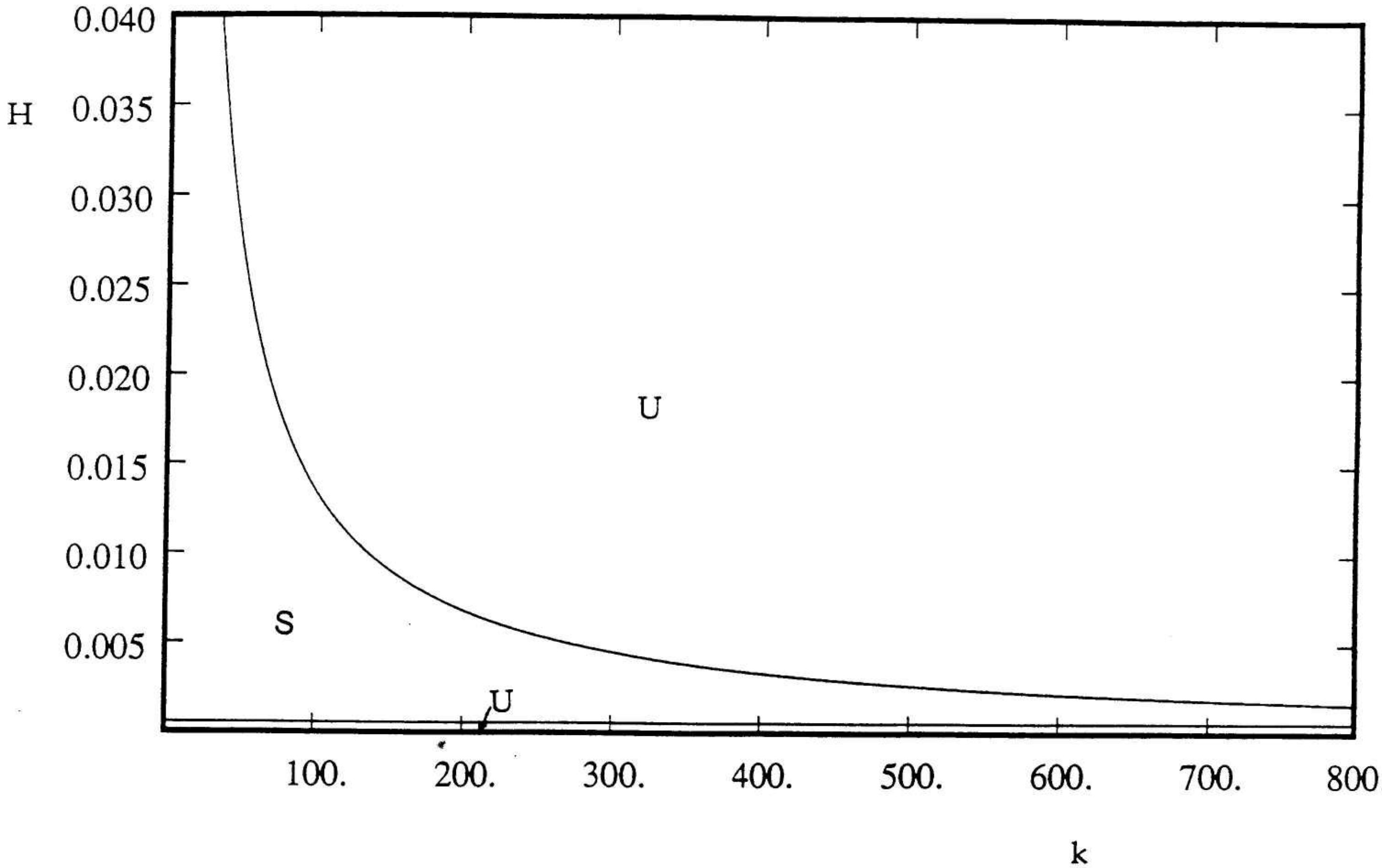

Figure 8b: Stability map, H vs. k (λ= 1e-7).

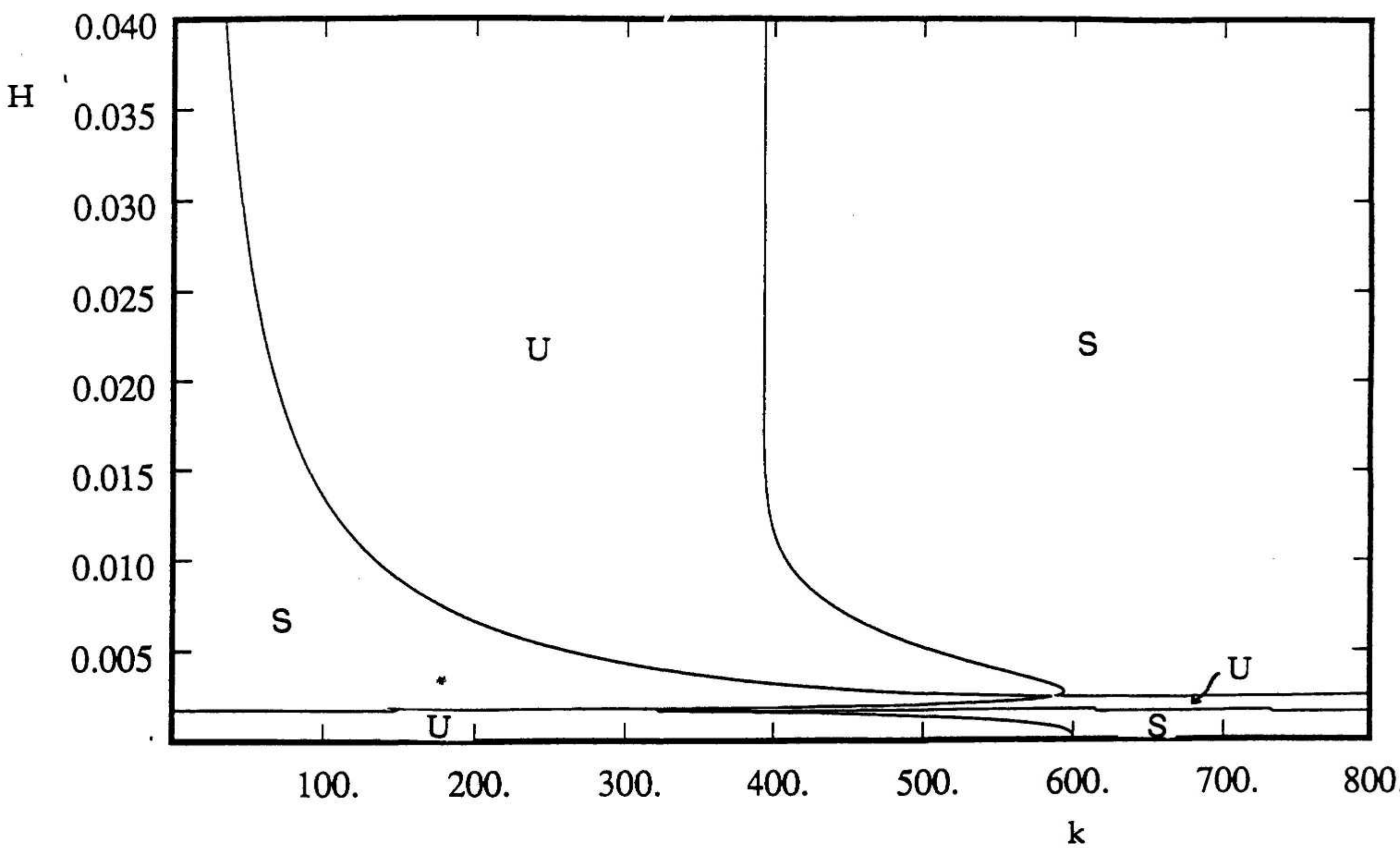

Figure 8c: Stability map, H vs. k (λ= 1e-6).

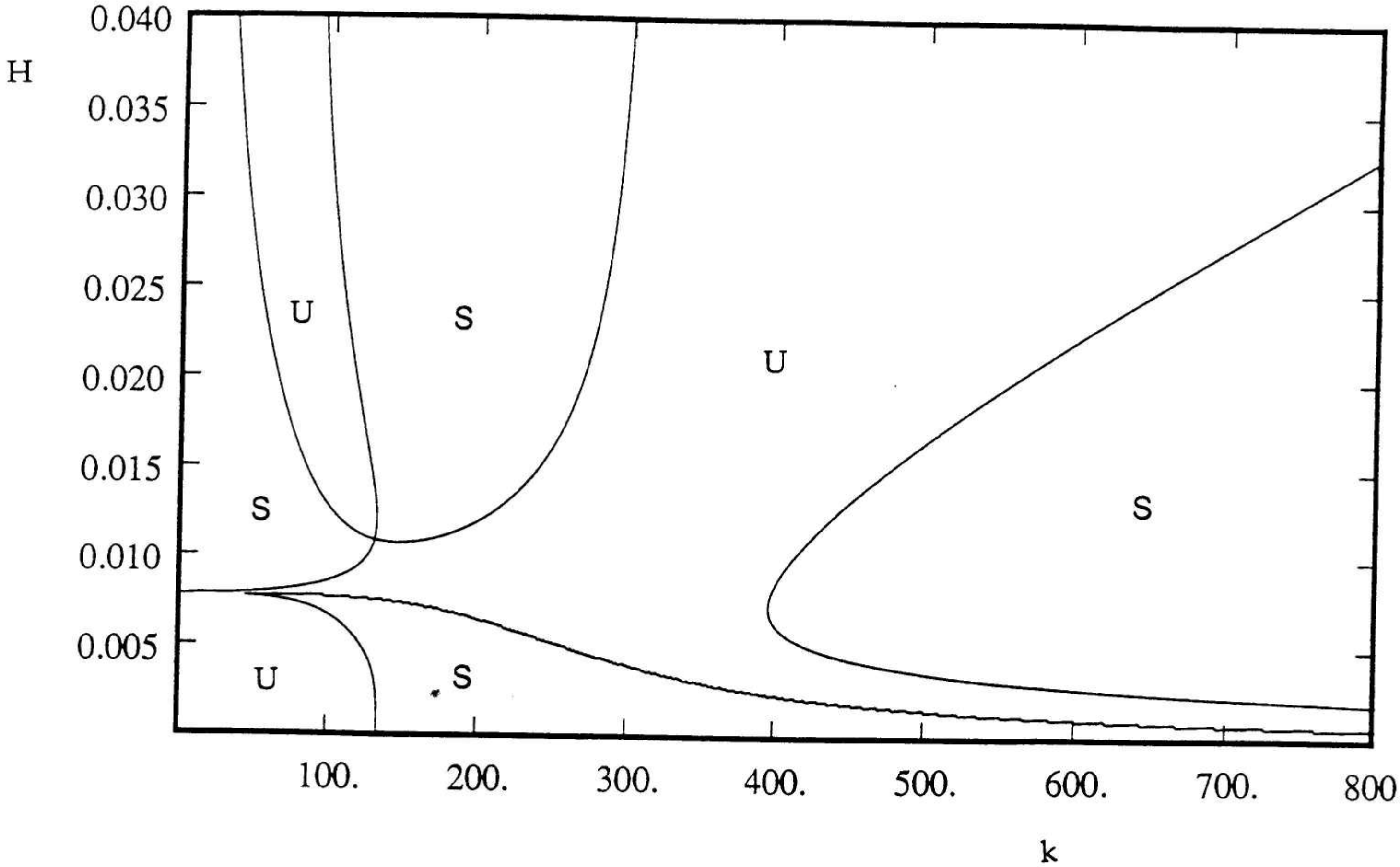

Figure 8d: Stability map, H vs. k (λ= 2e-5).

6. Derivation of an Evolution Equation

We analyze the evolution of a disturbance on the surface of a radially spreading film whose thickness is considered to be a slowly varying function of the spatial dimension, in the sense that changes over one wavelength and one period are relatively small. However, the steady state solution is considered to be of order one in amplitude when compared with the disturbance. By assuming long, nearly one dimensional wave disturbances, we have the two small parameters already introduced in equation (10)

$$\mu = \frac{h_0}{\lambda_r} \qquad \beta = \frac{\lambda_r}{\lambda_{\tilde{\theta}}} \tag{123}$$

which account for dispersive effects in the radial and azimuthal directions respectively. Expanding the inviscid water wave equations, (12)-(15) in these parameters, we derive the simplified version of the equations for the velocity field $(u(r,\theta,t), v(r,\theta,t))$ and the film thickness $h(r,\theta,t)$. The momentum equation in the radial direction is

$$u_t + \frac{\partial}{\partial r}\left[\frac{1}{2}\left(u^2 + v^2\right) + Mh + \frac{1}{We}\left(\delta^2 h\right)\right] + \frac{1}{3}\frac{\mu^2}{h}\frac{\partial}{\partial r}\left[2h^3\left\{u_r^2 + \frac{uu_r}{r} + \frac{u^2}{r^2}\right\} - Mh^3\left(\delta^2 h\right)\right] + \cdots = 0 \tag{124}$$

to order 1 in μ. The continuity equation

$$(rh)_t + (rhu)_r + \frac{\beta}{r}(rhv)_{\tilde{\theta}} = 0 \tag{125}$$

is an exact equation with u and v defined in equations (24) and (25). Along with the consistency relation

$$\frac{(rv)_r}{r} - \frac{\beta}{r}u_{\tilde{\theta}} = 0 \qquad (126)$$

equations (124-125) determine the system to $O\left(\mu^4, \mu^2\beta^2, \beta^4\right)$. These equations are identical to (29)-(31).

To study disturbances on a steady film characterized by H(r) and U(r), where H(r) is the steady film profile and U(r) is the steady radial velocity distribution as obtained from equations (35) and (36), we perturb these quantities with small but finite amplitude disturbances. However, since the film spreads radially, we need to scale the amplitude of the disturbance as well as the time and space coordinates so that we can achieve a balance of all effects. By considering waves traveling mainly in the radial direction we can introduce a phase-shifted time coordinate, x, that takes into account the steady fluid velocity. We also introduce a slow length scale, τ, to account for the slow flow dependence on position. In order to find a solution that would take into account radial spread, nonlinearity, and dispersion, we must look for non-trivial scalings that combine all three effects. The method is similar to that of singular perturbation theory, in this case we have an amplitude and a shallow water parameter that must be combined in some non-trivial fashion. Hence, we propose the scaled field variables:

$$h \Rightarrow H + \Delta h \quad ; \quad \Delta = \varepsilon^c \mu^d \to 0$$

$$u \to U + \Delta u$$

$$v \to \varepsilon^a \mu^b v \qquad (127)$$

as well as the scaled independent variables

$$x = \varepsilon^n \mu^m (G(r) - t)$$

$$\tau = \varepsilon^k \mu^j r$$

$$\xi = \varepsilon^p \mu^q \tilde{\theta} \tag{128}$$

where x is the phase-shifted time coordinate that acts as a length scale, and t is a slow radial coordinate that acts as a time scale and G(r) is an unknown function to be determined from the analysis. This scaling happens to be valid for our purpose since we are interested in analyzing disturbances on the film away from the stagnation point. The ξ coordinate takes into account slow azimuthal variations of the disturbances. We further specialize to the case where

$$\beta = O(\mu)$$

or equivalently

$$\beta = \beta_0 \mu \tag{129}$$

where β_0 is an O(1) constant.

In equations (127-129), ε represents an amplitude parameter; however, we have not yet specified its relative magnitude. For now, we assume that both m and e are much smaller than one.

These scalings are equivalent to a multiple scale analysis in the variables x and τ. Here, any function of r becomes a slowly varying function of τ.

With these new independent variables, derivatives transform as

$$\frac{\partial}{\partial t} = -\varepsilon^n \mu^m \frac{\partial}{\partial x} \tag{130}$$

$$\frac{\partial}{\partial r} = \varepsilon^n \mu^m G' \frac{\partial}{\partial x} + \varepsilon^k \mu^j \frac{\partial}{\partial \tau} \tag{131}$$

(Note that primes refer to differentiation with respect to r or equivalently with respect to τ.)

All exponents need to be determined from the equations of motion. G(r) is a function that takes into account the fact that the steady state is non-uniform. We further impose the condition that G(0) = 0 so that we can map the value of x =0 to the point (r,t) = (0,0).

Substitution of (128), (129), (130), (132), and (133) into (125), (126), and (127) leads to the momentum equation:

$$(uG' - 1)u_x + MG'h_x + \varepsilon^{k-n}\mu^{j-m}[(Uu)_\tau + Mh_\tau] + \Delta G' uu_x + \varepsilon^{2n}\mu^{2m+2}\left(\frac{1}{3}MH^2 - \frac{1}{We_{h_0}}\right)(G')^3 h_{xxx}$$
$$+ \frac{2}{3}H^2 G'\left[\varepsilon^{k+n}\mu^{j+m+2}\left(2G'U'u_x + G'Uu_x\right) + \varepsilon^{2k}\mu^{2j+2}2\left(U'u_\tau + \frac{Uu}{\tau^2} + (Uu)_\tau\right)\right]_x$$
$$+ \frac{\varepsilon^{2a}\mu^{2b}}{\Delta} vv_x + O\left(\Delta^2\right) = 0 \quad , \quad \Delta \to 0 \tag{132}$$

Balancing the nonlinear term with the dispersive term requires that

$$\varepsilon^{2n}\mu^{2m+2} \equiv \Delta \tag{133}$$

The continuity equation becomes

$$\left(G'U - 1\right)h_x + G'Hu_x + \Delta G'(hu)_x$$
$$+ \varepsilon^{k-n}\mu^{j-m}\left[(Hu)_\tau + (Uh)_\tau + \frac{Hu + Uh}{\tau} + \frac{\varepsilon^{a+p}\mu^{1+b+q}}{\Delta}\beta_0 \frac{Hv_\xi}{\tau}\right] + O\left(\Delta^2\right) = 0 , \quad \Delta \to 0 \tag{134}$$

which yields the requirements

$$\varepsilon^{a+p}\mu^{1+b+q} \equiv \Delta$$

$$\varepsilon^{k-n}\mu^{j-m} \equiv \Delta \tag{135}$$

The consistency relation becomes

$$G'v_x - \beta_0(\Delta\varepsilon^{p-a-n+k}\mu^{q-m-b+1+j})\frac{1}{\tau}u_\xi + \varepsilon^{k-n}\mu^{j-m}\left[v_\tau + \frac{v}{\tau}\right] = 0 \tag{136}$$

with the condition

$$\varepsilon^{n+a-p-k}\mu^{m+b-q-1-j} \equiv \Delta \tag{137}$$

Solving for the exponents from the above requirements , we have

$$\Delta = \varepsilon^{2n}\mu^{2m+2}$$

and

$$k = 3n \qquad p = -n \qquad a = 3n$$

$$j = 3m + 2 \qquad q = -m - 2 \qquad b = 3(m+1) \tag{138}$$

for arbitrary n and m as long as $\Delta \rightarrow 0$. This result does not impose any additional requirements on the size of ε.

From the above relations, we see that Δ can be used as an expansion parameter once the non-trivial scalings of the amplitudes have been taken into account. The final form of the scaled variables is

$$h \rightarrow H + \Delta h(x,\tau,\xi)$$

$$u \rightarrow U + \Delta u(x,\tau,\xi)$$

$$v \rightarrow \Delta^{3/2} v(x,\tau,\xi) \tag{139}$$

where h,u, and v are now O(1) quantities with the independent variables scaled as

$$x = \frac{\Delta^{1/2}}{\mu}(G(r) - t)$$

$$\tau = \frac{\Delta^{3/2}}{\mu} r$$

$$\xi = \frac{\theta}{\mu \Delta^{1/2}} \tag{140}$$

The governing equations are now:

the momentum equation,

$$\left(UG' - 1\right)u_x + MG'h_x + \Delta\left[(Uu + Mh)_\tau + G'uu_x + \left(\frac{MH^2}{3} - \frac{1}{We_{h_0}}\right)(G')^3 h_{xxx}\right] + O\left(\Delta^2\right) = 0 \quad , \ \Delta \to 0 \tag{141}$$

the continuity equation,

$$\left(uG'-1\right)h_x + G'Hu_x + \Delta\left[(Hu+Uh)_\tau + \frac{(Hu+Uh)}{\tau} + \beta_0 H\frac{v_\xi}{\tau} + G'(hu)_x\right] + O\left(\Delta^2\right) = 0 \ , \quad \Delta \to 0 \tag{142}$$

and the consistency relation,

$$G'v_x - \beta_0\frac{u_\xi}{\tau} + \Delta\left[v_\tau + \frac{v}{\tau}\right] + O\left(\Delta^2\right) = 0 \quad , \Delta \to 0 \tag{143}$$

We now look for solutions of the above equations by expanding the order one quantities u, h, and v into their Taylor series expansions about $\Delta = 0$. For any function

$$f(\mathbf{x};\Delta) = f(\mathbf{x};0) + \Delta f_\Delta(\mathbf{x};0) + O\left(\Delta^2\right) \tag{144}$$

Therefore, let

$$u = u + \Delta u^{(1)} + \cdots$$

$$h = h + \Delta h^{(1)} + \cdots$$

$$v = v + \Delta v^{(1)} + \cdots \tag{145}$$

Substituting the above expansions into the governing equations, collecting terms in powers of Δ, and requiring that the coefficients of all powers of Δ vanish exactly furnishes a sequence of problems for $u^{(n)}$, $h^{(n)}$, and $v^{(n)}$.

We now solve the first two problems in the sequence.

The O(1) Problem:

$$(UG'-1)h_x + HG'u_x = 0 \tag{146}$$

$$(UG'-1)u_x + MG'h_x = 0 \tag{147}$$

$$G'v_x - \beta_0 \frac{u_\xi}{\tau} = 0 \tag{148}$$

The solution to this set of equations can be expressed in terms of the

unknown function h so that

$$-u = \pm \frac{Mh}{\sqrt{MH}} \tag{149}$$

$$-G'v_x = \pm \left(\frac{M\beta_0}{\sqrt{MH}\,\tau}\right) h_\xi \tag{150}$$

and

$$G(r) = \int_0^r \frac{dr}{U(r) -\pm \sqrt{MH(r)}} \tag{151}$$

where "- $\underline{+}$" stands for minus or plus.

To this order we have completely determined the phase function G(r) with the condition G(0) = 0. We note that $U - \underline{+}\sqrt{MH}$ are the linearized versions of the characteristic speeds of the hyperbolic shallow water wave equations, as found previously in equation (48). We also notice that, if M is negligible, the characteristic speeds collapse onto a single speed equal to the mean flow speed. Hence, in a purely momentum driven flow, waves will propagate predominantly in one direction. In that case, the mean flow speed is much greater than the gravity wave speed (found here) and disturbances of this type are swept downstream rapidly.

The order one problem is not sufficient to fully determine the three unknown functions, hence we look at the next order problem to evaluate h.

The O(Δ) - problem:

$$HG'u_x^{(1)} + (UG'-1)h_x^{(1)} + \left[\left(U \pm \sqrt{MH}\right)'h + \left(U \pm \sqrt{MH}\right)\left(h_\tau + \frac{h}{\tau}\right) + \beta_0 H \frac{v_\xi}{\tau} \pm \frac{2MG'}{\sqrt{MH}} h h_x\right] = 0 \tag{152}$$

$$(UG'-1)u_x^{(1)} + MG'h_x^{(1)} + \left[\left(Mh \pm \frac{MUh}{\sqrt{MH}}\right)_\tau + \frac{G'M^2}{MH} h h_x + \left(\frac{1}{3} \frac{MH^2}{We_{h_0}}\right)(G')^3 h_{xxx}\right] = 0 \tag{153}$$

$$G'v_x^{(1)} - \beta_0 \frac{u_\xi^{(1)}}{\tau} + \left[v_\tau + \frac{v}{\tau}\right] = 0 \tag{154}$$

Since we are interested mainly in determining h, we eliminate $u^{(1)}$ and $h^{(1)}$ from the first two equations. For the two equations to be consistent the following relation between h and v must be satisfied.

$$2h_\tau + \frac{h}{\tau} + h\left[\frac{2c'}{c} - \frac{H'}{2H}\right] \pm \left[\frac{3M}{c^2\sqrt{MH}}\right]hh_x \pm \left[\frac{\sqrt{MH}}{c^4}\left(\frac{H^2}{3} - \frac{1}{MWe_{h_0}}\right)\right]h_{xxx} + \frac{\beta_0 H v_\xi}{c\tau} = 0 \tag{155}$$

where

$$c_\pm = U - \pm \sqrt{MH}$$

This condition together with the results from (149-151) provides a single evolution equation for $h(x,\tau,\xi)$:

$$\left(2h_\tau + \frac{h}{\tau} + h\left\{\frac{2c'}{c} - \frac{H'}{2H}\right\} \pm \left\{\frac{3\sqrt{MH}}{Hc^2}\right\}hh_x \pm \left\{\frac{\sqrt{MH}}{c^4}\left(\frac{1}{3}H^2 - \frac{1}{MWe_{h_0}}\right)\right\}h_{xxx}\right)_x \pm \beta^2\sqrt{MH}\frac{h_{\xi\xi}}{\tau^2} = 0 \tag{156}$$

Equation (156) is valid for arbitrary ε and μ as long as $\Delta \to 0$.

This evolution equation is the desired condition that completes the O(1) problem. In other words, we have found an equation that provides the evolution of nonlinear (i.e. finite amplitude), dispersive, long wavelength disturbances in nonuniform axisymmetric fluids.

Equation (156) is an extremely rich equation, that has the nearly concentric Korteweg-deVries equation (ncK-dV) as well as the cylindrical Korteweg-deVries equation (CK-dV) as special cases. In general, equation

(156) is called a variable coefficient Kadomtsev-Petviashvili (KP) equation (Santini 1981).

For example, consider the case of nearly concentric ripples propagating on a quiescent constant depth fluid. This implies U=0 and we can normalize $c = \pm 1$ and $G(r) = \pm r$. Equation (156) reduces to

$$\left(2h_\tau + \frac{h}{\tau} \pm 3hh_x \pm \left(\frac{1}{3} - \frac{1}{MWe_{h_0}}\right)h_{xxx}\right)_x \pm \frac{1}{\tau^2}h_{\xi\xi} = 0 \tag{157}$$

which is the ncK-dV equation, first derived by Johnson (1980).

If we impose axisymmetry in the above case, we have

$$2h_\tau + \frac{h}{\tau} \pm 3hh_x \pm \left(\frac{1}{3} - \frac{1}{MWe_{h_0}}\right)h_{xxx} \tag{158}$$

which is the CK-dV equation (Johnson, 1980). This equation describes concentric ripples on still shallow water.

Another special case is the one in which the flow is momentum driven.

$$\sqrt{MH} \ll U \qquad \Rightarrow \qquad c \sim U, \ \frac{\sqrt{MH}}{U} \to 0$$

$$G(r) = \int \frac{dr}{U} + O(\sqrt{MH}) \ , \quad \sqrt{MH} \to 0 \tag{159}$$

The evolution equation becomes

$$\left(2h_\tau - \left(\frac{5}{2}\frac{H'}{H} + \frac{1}{\tau}\right)h \pm \left\{\frac{3\sqrt{MH}}{HU^2}\right\}hh_x \pm \left\{\frac{\sqrt{MH}}{U^4}\left(\frac{1}{3}H^2 - \frac{1}{3}\frac{1}{MWe_{h0}}\right)\right\}h_{xxx}\right)_x \pm \frac{\beta_0^2\sqrt{MH}}{\tau^2}h_{\xi\xi} = 0$$

(160)

In the case of the inviscid far field (from equation 39) U is almost a constant and H is inversely proportional to τ. We normalize U to one, H to $1/\alpha_0\tau$ ($\alpha_0 = \alpha\mu/\Delta^{3/2}$); G(r) becomes

$$G(r) \sim r \pm 2\sqrt{\frac{M}{\alpha_0}}r^{1/2}, \quad \sqrt{\frac{M}{\alpha_0}} \to 0$$

The equation for h is

$$\left(2h_\tau + \frac{3}{2}\frac{h}{\tau} \pm \left\{3\sqrt{\alpha_0 M}\,\tau^{1/2}\right\}hh_x \pm \left\{\sqrt{\frac{M}{\alpha_0}}\left(\frac{1}{3}\frac{1}{\tau^{5/2}} - \frac{1}{\tau^{1/2}MWe_{h0}}\right)\right\}h_{xxx}\right)_x \pm \beta_0^2\sqrt{\frac{M}{\alpha_0}}\frac{h_{\xi\xi}}{\tau^{5/2}} = 0 \qquad (161)$$

We can rescale the variables as follows:

$$h \to \frac{1}{\alpha_0^{2/3}M^{4/3}We_{h0}}h$$

$$x \to \frac{1}{\alpha_0^{1/6}M^{1/12}We_{h0}^{1/4}}x$$

$$\tau \to (MWe_{h0})^{1/2}\,\tau$$

$$\xi \to \frac{\beta_0}{\alpha_0^{1/3}M^{1/6}We_{h0}^{1/2}}\xi \qquad (162)$$

and we have

$$\left(2h_\tau + \frac{3}{2}\frac{h}{\tau} \pm 3\tau^{1/2}hh_x \pm \left(\frac{1}{3}\frac{1}{\tau^{5/2}} - \frac{1}{\tau^{1/2}}\right)h_{xxx}\right)_x \pm \frac{1}{\tau^{5/2}}h_{\xi\xi} = 0 \qquad (163)$$

which is the same as (161) but rescaled so that it is parameter independent.

The rest of the analysis is devoted to the study of certain limits of the general evolution equation (156).

6.1. Variational Principle for the Variable Coefficient Kadomtsev-Petviashvili Equation

Before continuing with the analysis of the general equation (156), it is interesting to note that our evolution equation belongs to the class of variable coefficient K-P equations (Santini, 1981). Note that the variable coefficient K-P equation can be derived from the variational principle

$$\delta L = 0 \tag{164}$$

where the Lagrangian L is defined as

$$L = -\frac{1}{2} P_x P_\tau - \frac{1}{6}\nu P_x^3 + \frac{1}{2}\lambda P_{xx}^2 - \frac{1}{2}Q_x^2 \pm i\Lambda^{1/2} P_x Q_\xi \tag{165}$$

and P and Q are potential functions; ν, λ and Λ are functions of τ, and i is the imaginary number.

The variational equations of the above Lagrangian are:

δP:

$$P_{x\tau} + \nu P_x P_{xx} - \pm \Lambda^{1/2} Q_{\xi x} + \lambda P_{xxxx} = 0 \tag{166}$$

δQ:

$$-Q_{xx} \pm i\Lambda^{1/2} P_{x\xi} = 0 \tag{167}$$

This system can be reduced to the single equation

$$\left(\zeta_\tau + \nu\zeta\zeta_x + \lambda\zeta_{xxx}\right)_x + \Lambda\zeta_{\xi\xi} = 0 \tag{168}$$

where $\zeta = P_x$ (169)

and these relate to the original variables as

$$\zeta = \frac{c\tau^{1/2}}{H^{1/4}} h \qquad \nu = \pm \left\{ \frac{3}{2} \frac{\sqrt{MH}}{c^3 H^{3/4} \tau^{1/2}} \right\}$$

$$\lambda = \pm \left\{ \frac{\sqrt{MH}}{c^4} \left(\frac{1}{6} H^2 - \frac{1}{2MWe_{h_0}} \right) \right\} \qquad \Lambda = \pm \frac{\beta^2 \sqrt{MH}}{2\tau^2} \tag{170}$$

ζ satisfies the variable coefficient K-P equation. The variable coefficient K-dV equation can be derived by specializing to exponentially decaying one dimensional waves (Ablowitz and Segur, 1979).

Even though knowledge of a Lagrangian is extremely useful (since we can then apply all the powerful tools of the average Lagrangian theory for modulated wave trains), we choose to study the equations in a more direct way. Instead of looking at the exact modulated equations that can be derived from (165 and 168), we use perturbation techniques to study certain asymptotic limits of (168).

6.2 Axisymmetric Dispersive Disturbances

For the case of axisymmetric disturbances, we examine the full evolution equation (156) and look for solutions that decay for large values of x. The equation becomes

$$\left(2h_\tau + \frac{h}{\tau} + h\left\{\frac{2c'}{c} - \frac{H'}{2H}\right\} \pm \left\{\frac{3\sqrt{MH}}{Hc^2}\right\}hh_x \pm \left\{\frac{\sqrt{MH}}{c^4}\left(\frac{1}{3}H^2 - \frac{1}{MWe_{h_0}}\right)\right\}h_{xxx}\right)_x = 0 \qquad (171)$$

and for solutions that decay exponentially as $x \to \pm\infty$, we have

$$2h_\tau + \frac{h}{\tau} + h\left\{\frac{2c'}{c} - \frac{H'}{2H}\right\} \pm \left\{\frac{3\sqrt{MH}}{Hc^2}\right\}hh_x \pm \left\{\frac{\sqrt{MH}}{c^4}\left(\frac{1}{3}H^2 - \frac{1}{MWe_{h_0}}\right)\right\}h_{xxx} = 0 \qquad (172)$$

This is the equation that we choose to study in sections 6.2 and 6.3. We can transform (172) to a more compact equation by the following amplitude scaling:

$$\eta = \frac{c\tau^{1/2}}{H^{1/4}}h \qquad (173)$$

The corresponding evolution equation for h is the variable-coefficient Korteweg and de Vries equation

$$\eta_\tau + \nu\eta\eta_x + \lambda\eta_{xxx} = 0 \qquad (174)$$

where

$$\nu = \pm\frac{3}{2}\frac{\sqrt{MH}}{c^3H^{3/4}\tau^{1/2}} \qquad (175)$$

$$\lambda = \pm\frac{\sqrt{MH}}{c^4}\left(\frac{1}{6}H^2 - \frac{1}{2}\frac{1}{MWe_{h_0}}\right) \qquad (176)$$

for general $H(\tau)$ and $c(\tau)$.

In order to analyze the variable-coefficient K-dV, we note that ν and λ are slowly-varying functions of τ . With this in mind, we look at slowly-varying modulated wave trains by introducing slow scales to describe the modulated quantities and a fast scale to describe a rapidly-varying phase. It should be noted that the K-dV equation is itself a model equation that describes the evolution of a weakly nonlinear, weakly dispersive wave.

We study both slowly varying periodic solutions and slowly varying solitary wave-like solutions. Slow modulation problems are based on a small parameter that is intrinsic to the problem. In the case of the thin film, we can choose the parameter appearing in the steady state solution $H(\alpha r)$ or equivalently $H(\mu\alpha\Delta^{-3/2}\tau)$. We therefore let the small parameter be

$$\varepsilon \equiv \mu\alpha\Delta^{-3/2} << 1 \tag{177}$$

6.2.1. Slow Periodic Modulations of the Variable Coefficient Korteweg and de Vries Equation

The governing equation reduces to the variable coefficient K-dV equation (174)

$$\eta_y + \nu\eta\eta_\chi + \lambda\eta_{\chi\chi\chi} = 0$$

where ν and λ are functions of τ, and are defined in equations (175) and (176). Note that we have changed the notation in the K-dV equation (174) from τ to y, and from x to χ in order to introduce the symbols τ and x as the slow variables.

We choose the fast and slow time scales

$$\theta = \frac{\Theta}{\varepsilon}$$

$$\tau = \varepsilon y = \alpha r$$

$$x = \varepsilon \chi \qquad (178)$$

where Θ is the phase function, and ε is a small parameter defined in (177). With the new independent variables the derivatives transform as follows:

$$\frac{\partial}{\partial y} = -\omega\frac{\partial}{\partial \theta} + \varepsilon\frac{\partial}{\partial \tau}$$

$$\frac{\partial}{\partial \chi} = k\frac{\partial}{\partial \theta} + \varepsilon\frac{\partial}{\partial x} \qquad (179)$$

where

$$\omega = -\Theta_\tau \qquad \text{and} \qquad k = \Theta_x$$

Equation (176) becomes

$$-\omega\eta_\theta + kv\eta\eta_\theta + \lambda k^3\eta_{\theta\theta\theta} + \varepsilon\left[\eta_\tau + v\eta\eta_x + 3\lambda\left(k^2\eta_{\theta\theta x} + kk_x\eta_{\theta\theta}\right)\right]$$

$$+ \lambda\varepsilon^2\left[3k\eta_{\theta xx} + 3k_x\eta_{\theta x} + k_{xx}\eta_\theta\right] + \lambda\varepsilon^3\eta_{xxx} = 0 \qquad (180)$$

The method of solution follows by expanding η into an infinite series in powers of ε,

$$\eta = \sum_{n=0}^{\infty} \varepsilon^n \eta^{(n)}(\theta, x, \tau) \quad , \quad \varepsilon << 1 \qquad (181)$$

In order to analyze slow periodic modulations, we impose the condition that all the $\eta^{(n)}$ in the expansion for η be periodic in θ. Requiring that all $\eta^{(n)}$ have the same period as $\eta^{(0)} \equiv \eta$ ensures that the expansion be uniformly valid.

Substituting (181) into (180) and equating the coefficients of like powers of ε to zero, we obtain a sequence of problems for $\eta^{(n)}$.

The O(1) - problem is

$$-\omega\eta_\theta + kv\eta\eta_\theta + \lambda k^3\eta_{\theta\theta\theta} = 0 \tag{182}$$

This is a local steady state K-dV equation, whose general periodic solutions are cnoidal waves. We therefore write down its general solution as

$$\eta = \Gamma(x,\tau) + \alpha(x,\tau)\mathrm{cn}^2\left[\gamma(x,\tau)\theta;p(x,\tau)\right] \tag{183}$$

where p is the modulus of the Jacobian elliptic function, and $0 \leq p \leq 1$. The case p = 1 gives the solitary wave solution, and p = 0 the zero period and zero amplitude cnoidal wave. The modulation parameters β, α, γ, and p are related to ω, k, v, and λ. The goal now is to determine these relations. We begin by setting $\varphi = \gamma\theta$ and by calculating the derivatives of η.

$$\eta_\theta = -\,2\gamma\alpha\ \mathrm{cn}\varphi\mathrm{sn}\varphi\mathrm{dn}\varphi$$

$$\eta_{\theta\theta} = -\,2\gamma^2\alpha\left[-\mathrm{sn}^2\varphi + p^2\mathrm{sn}^4\varphi + \mathrm{cn}^2\varphi - 2p^2\mathrm{cn}^2\varphi\mathrm{sn}^4\varphi\right]$$

$$\eta_{\theta\theta\theta} = -\,3\gamma^3\alpha\left[-\mathrm{sn}\varphi\mathrm{cn}\varphi\mathrm{dn}\varphi + 2p^2\mathrm{sn}^3\varphi\mathrm{cn}\varphi\mathrm{dn}\varphi - p^2\mathrm{cn}^3\varphi\mathrm{sn}\varphi\mathrm{dn}\varphi\right] \tag{184}$$

where $\mathrm{sn}\varphi$, $\mathrm{cn}\varphi$, and $\mathrm{dn}\varphi$ are all Jacobian elliptic functions. After substituting (184) into (182) and if η, as in equation (183), is to be a solution of (182) for all θ, then

$$\frac{\alpha}{p^2\gamma^2} = \frac{12\lambda k^2}{v} \tag{185}$$

and

$$\omega = kv\Gamma + 4\gamma^2 k^3(-1+2p^2) \tag{186}$$

To these we add the consistency relation that eliminates Θ

$$k_t + \omega_x = 0 \tag{187}$$

To close the system, three more conditions can be determined from the $O(\varepsilon)$ problem by ensuring that $\eta^{(1)}$ be periodic with the same period as η.

The $O(\varepsilon)$ - problem is

$$-\omega\eta_\theta^{(1)} + kv(\eta\eta^{(1)})_\theta + \lambda k^3\eta_{\theta\theta\theta}^{(1)} = -\left[\eta_\tau + v\eta\eta_x + 3\lambda(k^2\eta_{\theta x} + kk_x\eta_\theta)_\theta\right] \tag{188}$$

and if we require that the wave profile decay to zero as it travels then β must take a particular form. This implies that

$$\eta \to \Gamma(x,\tau) \tag{189}$$

$$\eta^{(1)} \to 0$$

and from (188) we see that the right hand side (RHS) gives

$$\left.\eta_\tau + v\eta\eta_x\right]_{\eta\to\Gamma} = 0 \tag{190}$$

or

$$\Gamma_\tau + v\Gamma\Gamma_x = 0 \tag{191}$$

This equation decouples from the rest and hence determines $\Gamma(x,t)$ from any initial value of Γ. The implicit solution of the Cauchy problem is

$$\Gamma = \Gamma(x_0,\tau_0) \tag{192}$$

where $x_0(x,t)$ and $\tau_0(x,t)$ are determined from the family of characteristic curves

$$x = x_0 + \Gamma(x_0,\tau_0)\int_{\tau_0}^{\tau} v\,d\tau \qquad (193)$$

Hence if $\Gamma(x_0, \tau_0)$ is zero, $\Gamma(x, \tau)$ would be zero for all time.

Another conservation equation can be deduced by integrating (188) over a complete period. This implies

$$\int_0^{2\pi} d\theta\,[\eta_\tau + v\eta\eta_x] = 0 \qquad (194)$$

which is equivalent to

$$\frac{\partial}{\partial\tau}\int_0^{2\pi} \eta\,d\theta + \frac{v}{2}\frac{\partial}{\partial x}\int_0^{2\pi} \eta^2 d\theta = 0 \qquad (195)$$

where the symbol "2π " represents any complete cycle rather than the actual value of the period. We now define the following quantities:

$$I_2 \equiv \int_0^K cn^2\,\varphi d\varphi = \frac{1}{p^2}[E - (1-p^2)K] \qquad (196)$$

$$I_4 \equiv \int_0^K cn^4\,\varphi d\varphi = \frac{1}{3p^4}[(2-3p^2)(1-p^2)K + 2(2p^2 - 1)E] \qquad (197)$$

$$I_6 \equiv \int_0^K cn^6\,\varphi d\varphi = \frac{4(2p^2 - 1)I_4 + 3I_2}{5p^2} \qquad (198)$$

where K(p) and E(p) are the complete elliptic integrals of the first and second kind, respectively. With these definitions and combining (191) with (195) we write

$$\frac{\partial}{\partial t}\left[\frac{\alpha}{\gamma}I_2\right] + v\frac{\partial}{\partial x}\left[\frac{\Gamma\alpha}{\gamma}I_2 + \frac{1}{2}\frac{\alpha^2}{\gamma}I_4\right] = 0 \tag{199}$$

To obtain the final modulation equation we let $\eta^{(1)} = \eta_\theta F(\theta,x,\tau)$ and integrate

$$\int_0^\theta d\theta\left[-\omega\eta_\theta^{(1)} + kv\left(\eta\eta^{(1)}\right)_\theta + \lambda k^3\eta_{\theta\theta\theta}^{(1)}\right] \tag{200}$$

The O(ε) equation becomes

$$\frac{\partial}{\partial\theta}\left[-\omega\eta_\theta F + kv\eta\eta_\theta F + \lambda k^3\left[\eta_{\theta\theta\theta}F + 2\eta_{\theta\theta}F_\theta + \eta_\theta F_{\theta\theta}\right]\right]$$

$$= -\left[\eta_\tau + v\eta\eta_x + 3\lambda\left(k^2\eta_{\theta x} + kk_x\eta_\theta\right)_\theta\right] \tag{201}$$

and integrated over θ and making use of the O(1) equation we have

$$\lambda k^3[2\eta_{\theta\theta}F_\theta + \eta_\theta F_{\theta\theta}] = -\int_0^\theta (\eta_\tau + v\eta\eta_x)d\theta - 3\lambda\left(k^2\eta_{\theta x} + kk_x\eta_\theta\right) \tag{202}$$

If we now multiply by η_θ and integrate over one period we have

$$\lambda k^3\int_0^{2\pi}\frac{\partial}{\partial\theta}\left(\eta_\theta^2 F_\theta\right)d\theta = -\int_0^{2\pi}\eta_\theta\int_0^{\theta}(\eta_\tau + v\eta\eta_x)d\theta' d\theta - 3\lambda\int_0^{2\pi}\eta_\theta\int_0^{\theta'}\left(k^2\eta_{\theta x} + kk_x\eta_\theta\right)_\theta d\theta' d\theta$$

(203)

This equation reduces to

$$0 = \frac{\partial}{\partial \tau}\int_0^{2\pi} \tfrac{1}{2}\eta^2 d\theta + v\frac{\partial}{\partial x}\int_0^{2\pi} \tfrac{1}{3}\eta^3 d\theta - \frac{3\lambda}{2}\frac{\partial}{\partial x}k^2\int_0^{2\pi} \eta_\theta^2 d\theta \tag{204}$$

This is the required secular condition that closes the system of modulation equations. Combining the integrals $I_n(p)$ with the mean flow equation (191) and making use of the conservation equation (199), we reduce (204) to

$$\frac{\partial}{\partial \tau}\left[\frac{1}{2}\frac{\alpha^2}{\gamma}I_4\right] + v\frac{\partial}{\partial x}\left[\frac{1}{2}\frac{v\Gamma\alpha^2}{\gamma}I_4 + \frac{1}{3}\frac{\alpha^3 v}{\gamma}I_6 - \frac{3}{2}\lambda\gamma k^2\alpha^2 I_5\right] + \frac{1}{2}\frac{v\alpha^2}{\gamma}I_4\Gamma_x = 0 \tag{205}$$

The above equations describe the slow modulations of a periodic wave train of cnoidal waves in terms of the wave-number, k, the modulus of the Jacobian elliptic function, p, and the amplitude of the wave α. If we now specialize to the case in which the parameters are all functions of τ only, this implies that we look for solutions to the parameters that are solely dependent on the radial position, and not on the time.

$$k_\tau = 0 \qquad \Gamma_\tau = 0 \tag{206}$$

and

$$\frac{\partial}{\partial \tau}\left[\frac{\alpha}{\gamma}I_2\right] = 0 \Rightarrow \frac{\alpha}{\gamma}I_2 = c_0 \tag{207}$$

$$\frac{\partial}{\partial \tau}\left[\frac{1}{2}\frac{\alpha^2}{\gamma}I_4\right] = 0 \Rightarrow \frac{\alpha^2}{\gamma}I_4 = c_1 \tag{208}$$

or combining the above with the dispersion relation and the amplitude-width relation, we get

$$\frac{\alpha}{\alpha_0} = \frac{I_2(p)}{I_2(p_0)} \cdot \frac{I_4(p_0)}{I_4(p)} \tag{209}$$

$$\frac{\gamma}{\gamma_0} = \frac{p_0}{p} \sqrt{\frac{\nu}{\nu_0} \cdot \frac{\lambda_0}{\lambda} \cdot \frac{I_2(p)}{I_2(p_0)} \cdot \frac{I_4(p_0)}{I_4(p)}} \tag{210}$$

$$\omega = k\nu\left[\Gamma + \tfrac{1}{3}\alpha\left(\frac{2p^2 - 1}{p^2}\right)\right] \tag{211}$$

and

$$\frac{\lambda}{\nu} = \frac{\lambda_0}{\nu_0} \cdot \frac{p_0^2}{p^2} \cdot \frac{I_2^3(p)}{I_2^3(p_0)} \cdot \frac{I_4(p_0)}{I_4(p)} \tag{212}$$

where the subscript refers to quantities evaluated at some initial time.

The last equation provides an implicit relation between the modulus of the Jacobian elliptic and the ratio λ/ν. Hence, we know $p(\lambda/\nu)$ or equally $p(\tau)$. Since Γ is constant we may set its value to zero because we are considering η to be a disturbance on a steady profile. Therefore,

$$\Gamma=0$$

From the equation for ω we can determine the phase function Θ by integration, so that

$$\Theta = x - \frac{1}{3}\int_{\tau_0}^{\tau} \nu\alpha\left(\frac{2p^2 - 1}{p^2}\right)d\tau \tag{213}$$

with p(τ) given implicitly by

$$\frac{\left(\frac{\lambda}{\nu}\right)\left[E - \left(1-p^2\right)K\right]^3}{\left[\left(2-3p^2\right)\left(1-p^2\right)K + 2\left(2p^2-1\right)E\right]} = \text{constant} \tag{214}$$

From (214) we infer that as p varies from one to zero the $I_4/p^2I_2^3$ is a monotonically increasing function that quickly approaches infinity as p tends to zero. This implies that λ/ν increases as $p\to 0$, and for the case of an inviscid film λ/ν is decreasing function of τ. Thus for small values of τ, p tends to zero which implies that periodic wave trains with modulated wavelengths would be observed in this region of space. However, as τ increases, p tends to one and the wave trains degenerate onto solitary waves. In other words, the nonuniform film acts as a filtering mechanism that takes propagating ripple-like disturbances and turns them into modulated solitary waves running down the film.

We can conclude that oscillatory solutions are only possible for small values of τ. Thus, for regions far from the stagnation point, we look at solitary wave-like solutions. In the next section, we take a closer look at cylindrical solitary wave disturbances.

6.2.2. Solitary Wave Modulations of the Variable Coefficient Korteweg de Vries Equation

The question of finding slowly varying solitary wave solutions to the variable coefficient K-dV has been considered by many authors such as Ko and Kuel (1978), and Grimshaw (1979) among others. Thus, we merely outline the method of solution and quote the main results from the references. From here, we proceed to apply them to the case of a radially spreading film. The method used is very similar to that employed for periodic waves, except that we now consider a solitary wave which is no longer periodic.

Again expanding h in powers of ε, as in equation (181), we obtain an O(1)- problem:

$$-\omega\eta_\theta + \frac{1}{2}k\nu\left(\eta^2\right)_\theta + \lambda k^3\eta_{\theta\theta\theta} = 0 \tag{215}$$

whose solitary wave solution is

$$\eta = \Gamma + \alpha\,\mathrm{sech}^2\,\gamma\left(\theta+\theta_0\right) \tag{216}$$

where θ_0 is the first correction to the phase.

Direct substitution of (216) into (215) leads to the dispersion relation and to the amplitude-width relation

$$\omega = \nu k\left(\Gamma + \frac{1}{3}\alpha\right) \tag{217}$$

$$\frac{\alpha}{\gamma^2} = 12\left(\frac{\lambda}{\nu}\right)k^2 \tag{218}$$

The third modulation equation is the consistency relation for eliminating $\boldsymbol{\Theta}$ between k and ω, namely,

$$\mathbf{k}_\tau + \omega_\mathbf{x} = 0 \tag{219}$$

In order to determine the rest of the modulation equations, we continue the expansion to the next order.

The O(ε)- problem is

$$-\omega\eta_{\theta}^{(1)} + kv\left(\eta\eta^{(1)}\right)_{\theta} + \lambda k^3\eta_{\theta\theta\theta}^{(1)} = -\left[\eta_{\tau} + v\eta\eta_x + 3\lambda k(k\eta_{\theta})_{x\theta}\right] \qquad (220)$$

We seek a solution for $\eta^{(1)}$ such that $\eta^{(1)}$ tends to a constant as $\theta \rightarrow \pm\infty$. It follows at once from equation (220) that

$$\Gamma_{\tau} + v\Gamma\Gamma_x = 0 \qquad (221)$$

Thus, as expected, the mean quantity Γ satisfies the 'shallow water' approximation to the K-dV equation; which again decouples Γ from the modulation equations and provides $\Gamma(x, \tau)$ based on its initial value at (x_0, τ_0).

In order to determine the last relation, we follow an approach similar to the previous section, which is to let

$$\eta^{(1)} = \eta_{\theta}F(\theta,x,\tau) \qquad (222)$$

which, after making use of (215), leads to

$$\lambda k^3\frac{\partial}{\partial\theta}\left[2\eta_{\theta\theta}F_{\theta} + \eta_{\theta}F_{\theta\theta}\right] = -\left[\eta_{\tau} + v\eta\eta_x + 3\lambda k(k\eta_{\theta})_{\theta x}\right] \qquad (223)$$

Upon integration from zero to θ, multiplication by η_{θ}, and integration from $-\infty$ to $+\infty$ leads to the last secular condition. Explicitly, this is

$$\int_{-\infty}^{\infty} d\theta\eta_{\theta}\int_0^{\theta} \lambda k^3\frac{\partial}{\partial\theta}\left[2\eta_{\theta\theta}F_{\theta} + \eta_{\theta}F_{\theta\theta}\right] d\theta = \int_{-\infty}^{\infty} d\theta\eta_{\theta}\int_0^{\theta} -\left[\eta_{\tau} + v\eta\eta_x + 3\lambda k(k\eta_{\theta})_{\theta x}\right]d\theta \qquad (224)$$

By noticing that the first integration of the LHS leads to a perfect differential, which vanishes at the boundaries the LHS equals zero. After evaluating the double integrals of the RHS, we have

$$\frac{\partial}{\partial\tau}\left[\frac{\alpha^2}{\gamma}\right] + v\frac{\partial}{\partial x}\left[\Gamma\frac{\alpha^2}{\gamma} + \frac{1}{3}\frac{\alpha^3}{\gamma}\right] + \frac{v\alpha^2}{\gamma}\Gamma_x = 0 \qquad (225)$$

Combination of equations (217) and (219) with (225) can be rearranged in the following form:

$$\frac{\partial}{\partial\tau}\left[\frac{\alpha^2}{k\gamma}\right] + \nu\left(\Gamma + \tfrac{1}{3}\alpha\right)\frac{\partial}{\partial x}\left[\frac{\alpha^2}{k\gamma}\right] + \nu\Gamma_x\left[\frac{\alpha^2}{k\gamma}\right] = 0 \tag{226}$$

This is the required modulation equation. Notice that to this level in the expansion, we have determined α, Γ, γ, k, ω, but not θ_0. Examining the form of (226) and combining it with the amplitude-width relation we see that the grouping

$$\frac{\alpha^2}{k\gamma} = \sqrt{12}\left(\frac{\lambda}{\nu}\right)^{1/2}\alpha^{3/2} \tag{227}$$

is a function only of α and τ. Thus equation (226) is an evolution equation for α since Γ is a given function of x and τ.

We summarize the modulation equations as

$$\frac{\partial}{\partial\tau}\left[\left(\frac{\lambda}{\nu}\right)^{1/2}A^3\right] + \nu\left(\Gamma + \tfrac{1}{3}A^2\right)\frac{\partial}{\partial x}\left[\left(\frac{\lambda}{\nu}\right)^{1/2}A^3\right] + \nu\Gamma_x\left(\frac{\lambda}{\nu}\right)^{1/2}A^3 = 0 \tag{228}$$

$$\frac{\partial k}{\partial\tau} + \nu\left(\Gamma + \tfrac{1}{3}A^2\right)k_x + \nu\left(\Gamma + \tfrac{1}{3}A^2\right)_x k = 0 \tag{229}$$

$$\Gamma = \Gamma(x_0,\tau_0) \qquad \text{along} \qquad x = x_0 + \Gamma(x_0,\tau_0)\int_{\tau_0}^{\tau}\nu(\xi)d\xi \tag{230}$$

$$\omega = \nu k\left(\Gamma + \tfrac{1}{3}A^2\right) \tag{231}$$

where $A = \alpha^{1/2}$.

Our interest again is on the case in which these quantities are functions of τ only, since the undisturbed film is changing only with radial position and not with time. This implies

$$\frac{\partial}{\partial\tau}\left[\left(\frac{\lambda}{\nu}\right)^{1/2} A^3\right] = 0$$

$$\frac{\partial k}{\partial \tau} = 0 \qquad\qquad \Gamma = \Gamma(\tau_0) \tag{232}$$

Again we can set k=1 and β=0, since η is a disturbance on a steady film with β=0 initially. With these values we have

$$\Theta = x - \frac{1}{3}\frac{\alpha_0\lambda_0^{1/3}}{\nu_0^{1/3}}\int_{\tau_0}^{\tau}\frac{\nu^{4/3}}{\lambda^{1/3}}d\tau \tag{233}$$

$$\alpha = \alpha_0\left(\frac{\nu}{\nu_0}\right)^{1/3}\left(\frac{\lambda_0}{\lambda}\right)^{1/3} \tag{234}$$

$$\gamma = \sqrt{\frac{\alpha_0}{12}}\left(\frac{\lambda_0}{\nu_0}\right)^{1/6}\left(\frac{\nu}{\lambda}\right)^{2/3} \tag{235}$$

the phase function, the amplitude, and the width as functions of τ.

In order to calculate θ_0 we need to study the next order problem, but it suffices to look for solutions that are only functions of θ and τ. Hence, the $O(\varepsilon^2)$ problem is

$$-\omega\eta_\theta^{(2)} + k\nu\left(\eta\eta^{(2)}\right)_\theta + \lambda k^3\eta_{\theta\theta\theta}^{(2)} = -\left[\eta_\tau^{(1)} + \nu k\eta^{(1)}\eta_\theta^{(1)}\right] \tag{236}$$

To solve for θ_0, we first must solve the $O(\varepsilon)$-problem completely for $\eta^{(1)}(\theta,\tau)$ and then apply the same technique used to determine the last secular condition, or equation (225). Namely

$$\int_{-\infty}^{\infty} d\theta\eta_\theta \int_0^\theta \left[-\omega\eta_\theta^{(2)} + k\nu(\eta\eta^{(2)})_\theta + \lambda k^3\eta_{\theta\theta\theta}^{(2)}\right]d\theta = \int_{-\infty}^{\infty} d\theta\eta_\theta \int_0^\theta -\left[\eta_\tau^{(1)} + \nu k\eta^{(1)}\eta_\theta^{(1)}\right]d\theta$$

(237)

Rather than to carry out the lengthy calculations, we simply quote the results from Ko and Kuehl (1978), in which they find that

$$\gamma\theta_0 = \frac{1}{2}\left[1 - \frac{\gamma}{\gamma_0}\right] + \frac{1}{2}\gamma\int_{\tau_0}^{\tau} d\xi \int_{\xi_0}^{\xi} \frac{\gamma_\xi^2}{\gamma^3} d\xi \tag{238}$$

where $\gamma_0 = \sqrt{\frac{\alpha_0\nu_0}{12\lambda_0}}$ and the first order correction is to η is

$$\eta^{(1)} = \eta\left[\frac{\theta_{0\tau}}{\omega}(\varphi\tanh - 1) - \frac{1}{4}\frac{\gamma_\tau}{\omega\gamma^2}\left\{4(1+\varphi) - \left[4\varphi^2 + 6\varphi - e^{-2\varphi} + D\right]\tanh\ \varphi\right\}\right] \tag{239}$$

where we let $\varphi = \gamma(\theta + \theta_\theta)$, and D is an arbitrary function of integration that can be calculated from the next order problem.

Summarizing the above results, the disturbance to the mean flow is

$$\eta = \alpha \operatorname{sech}^2\gamma(\theta + \theta_\theta) + \varepsilon\eta^{(1)} \tag{240}$$

where the argument of the sech-squared is

$$\gamma(\theta + \theta_0) = \sqrt{\frac{\alpha_0}{12}}\left(\frac{\lambda_0}{\nu_0}\right)^{1/6}\left(\frac{\nu}{\lambda}\right)^{2/3}\left[x - \frac{1}{3}\frac{\alpha_0\lambda_0^{1/3}}{\nu_0^{1/3}}\int_{\tau_0}^{\tau}\frac{\nu^{4/3}}{\lambda^{1/3}}d\tau\right]$$

$$+\frac{1}{2}\left[1-\left(\frac{\nu}{\nu_0}\right)^{2/3}\left(\frac{\lambda_0}{\lambda}\right)^{2/3}\right]+\frac{1}{2}\gamma\int_{\tau_0}^{\tau}d\xi\int_{\xi_0}^{\xi}\frac{\gamma_\xi^2}{\gamma^3}d\xi \tag{241}$$

The amplitude and the width are respectively

$$\alpha = \alpha_0\left(\frac{\nu}{\nu_0}\right)^{1/3}\left(\frac{\lambda_0}{\lambda}\right)^{1/3} \tag{242}$$

$$\gamma = \sqrt{\frac{\alpha_0}{12}}\left(\frac{\lambda_0}{\nu_0}\right)^{1/6}\left(\frac{\nu}{\lambda}\right)^{2/3} \tag{243}$$

The O(ε) correction is given by equation (239), and the slowly varying functions ν and λ are given

$$\frac{\nu}{\lambda} = \frac{3c}{H^{3/2}\tau^{1/2}\left(\frac{1}{3}H^2 - \frac{1}{MWe_{h_0}}\right)} \tag{244}$$

$$\nu = \pm\frac{\frac{3}{2}\sqrt{MH}}{c^3H^{3/4}\tau^{1/2}} \tag{245}$$

Note that the only place in the solution of equations (240) - (243) where there is M dependence is in the phase function, through the following

integral

$$\int_{\tau_0}^{\tau}\frac{\nu^{4/3}}{\lambda^{1/3}}d\tau \tag{246}$$

Since M is very small, the solution is practically independent of the Froude number. However, M becomes important in regions of weak dispersive effects ($\lambda \to 0$) .

Finally, the transformations to physical variables are given from equation (140) as

$$x = \frac{\Delta^{1/2}}{\mu}[G(r) - t] \qquad \tau = \frac{\Delta^{3/2}}{\mu} r \quad , \Delta \to 0$$

$$G(r) = \int_0^r \frac{dr}{U - \pm\sqrt{MH}}$$

where we have eliminated ε in favor of the original nondimensional parameters. Here M, We_{h_0} , and μ are defined by equation (10), U and H are given by equations (35) and (36) or by any pair (U,H) satisfying the steady equations, and c is given by equation (155) (U, H are the same as little u, h in equations (35) and (36)).

This is the required solution to $O(\varepsilon^2)$. We now study the dynamics represented by the analytic solution on the following figures. First, we analyze the motion of a solitary wave travelling in the direction of flow on a monotonically decreasing film.

Figures 9a shows the spatial and temporal evolution of a modulated solitary wave on the surface of a monotonically decreasing radial film (initial conditions were taken at an arbitrary radial location.) We also observe, from figures 9b and 9c, that as time and radial distance increase, the amplitude to width ratio increases showing how the wave develops into a narrow larger amplitude wave. In the limit as r approaches the jet , we see that the amplitude decreases considerably and the motion seems to resemble that of infinitesimal waves. This observation is in full agreement with the periodic analysis, since in the limit as r tends towards the origin, the modulus of the cnoidal wave tends to zero making the Jacobian elliptic function degenerate into sinusoidal waves. Figure 9d contains an r-t plane showing the surface contours of the modulated solitary wave on the same monotonically decreasing film. We again see, from figure 9d, that the nonuniform film induces a filtering mechanism that turns small amplitude modulated waves into narrower and steeper amplitude solitary waves. This can be seen by noticing that the crest contours converge rapidly to a narrow larger amplitude soliton that travels down the film. As was mentioned in equation (246), the Froude number does not play an important role in the dynamics of the wave disturbance (as long as λ is non-zero). In other words, the characteristic speeds collapse onto one, namely, the mean flow speed. This makes the disturbances travel only in the direction of the flow.

In order to understand the effect of a non-monotonic film, we substitute into the analytic solution a profile similar to the one derived by Watson for the outer inviscid flow. Thus, we look at a the influence of a downstream film with a quadratic radial dependence. Figures 10a and 10b show the spatial and temporal evolution of a solitary wave on this type of film. We observe, that as the wave travels down on the decreasing part of the film, it evolves in the same way it did in figure 9a. However, as the wave passes through the minimum film thickness, the amplitude to width ratio begins to decrease back to the original ratio. This is again in full agreement with the predicted behavior from the periodic analysis, which specifies that as the film thickness increases the modulations become periodic. For this particular case, it just happens that as r increases past a certain point the film thickness ceases to decrease and begins to increase, hence making the solitary waves turn back into periodic waves of smaller amplitude. This effect can also be observed in Figure 10c, which is an r-t plane for the corresponding nonuniform film. Figure 11 shows a snapshot of the slowly varying solitary wave on the surface of a radially thinning film superimposed on the undisturbed film.

The analysis breaks down in a region where λ, the coefficient of h_{xxx}, tends to zero. At that point, the equation can be approximated by a hyperbolic version of the variable-coefficient K-dV. This case is treated in some detail in the next section. This condition corresponds to minimizing dispersion, namely when the film thickness satisfies

$$\frac{1}{3}H^2 - \frac{1}{MWe_{h_0}} \qquad (247)$$

The dispersion coefficient is proportional to that in equation (247), this implies that its magnitude could become negative at which point disturbances would become depression waves. We do not analyze this type of motion in the present investigation.

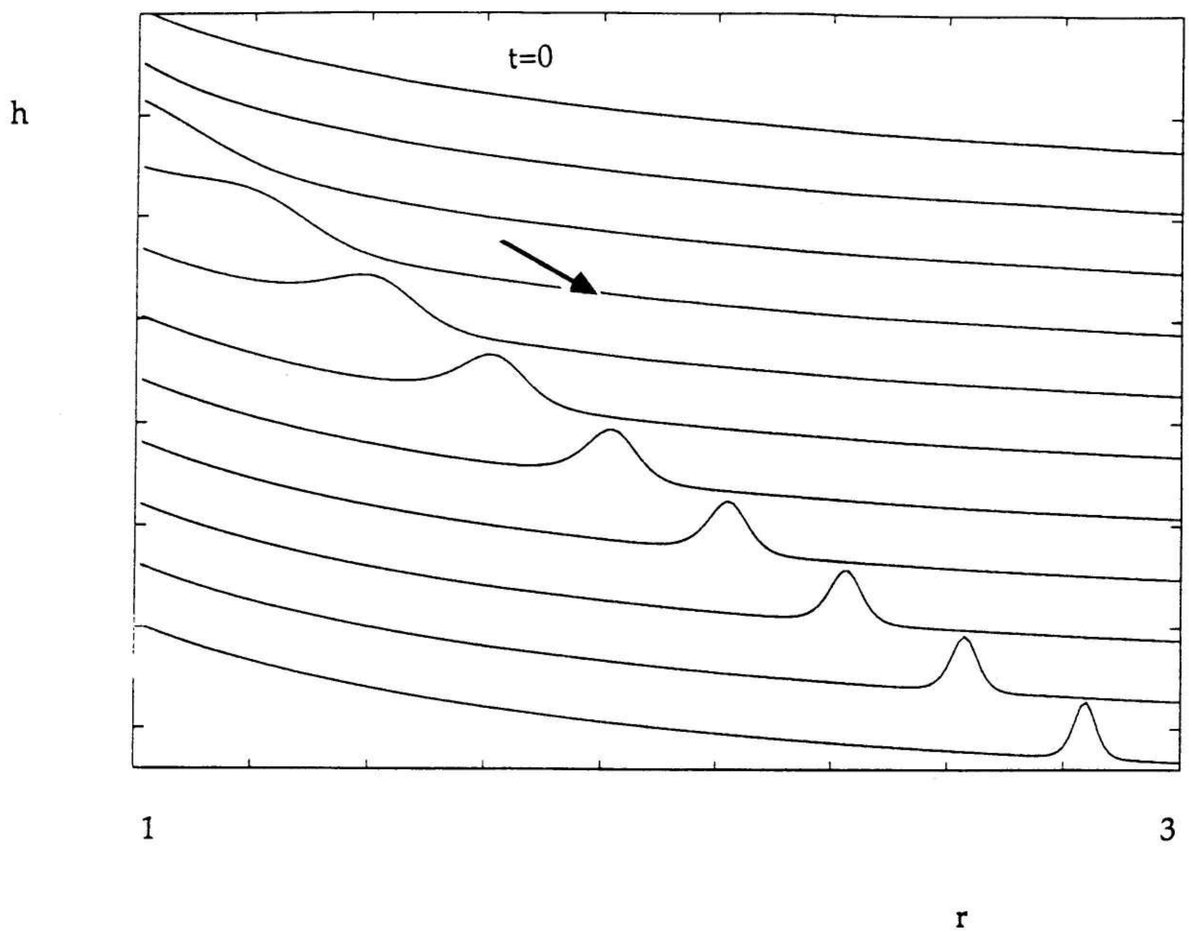

Figure 9a: Spatial and temporal evolution of a modulated solitary wave on the surface of a monotonically decreasing radial film.

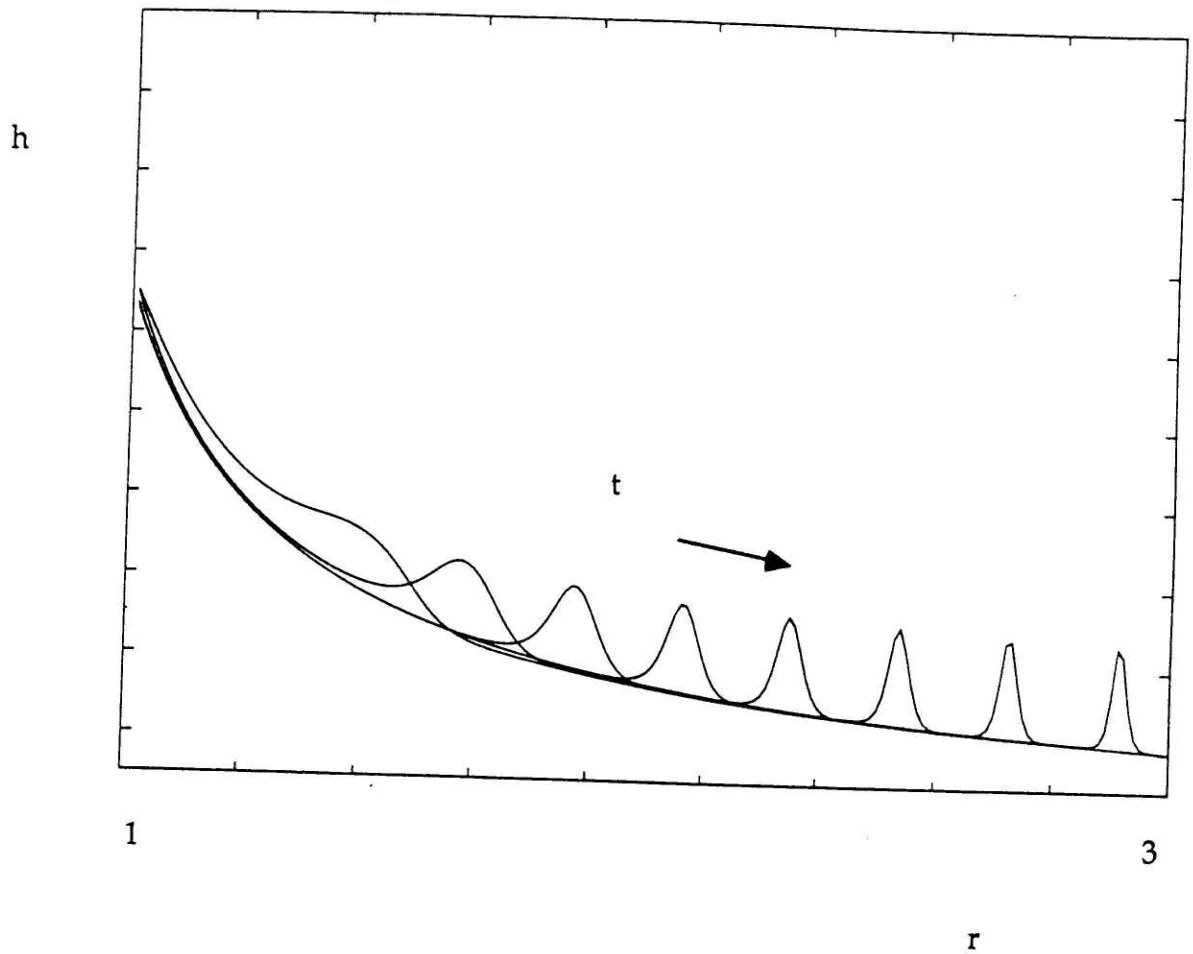

Figure 9b: Time evolution of the same modulated solitary wave as in figure 9a, plotted on the same axis (h vs. r).

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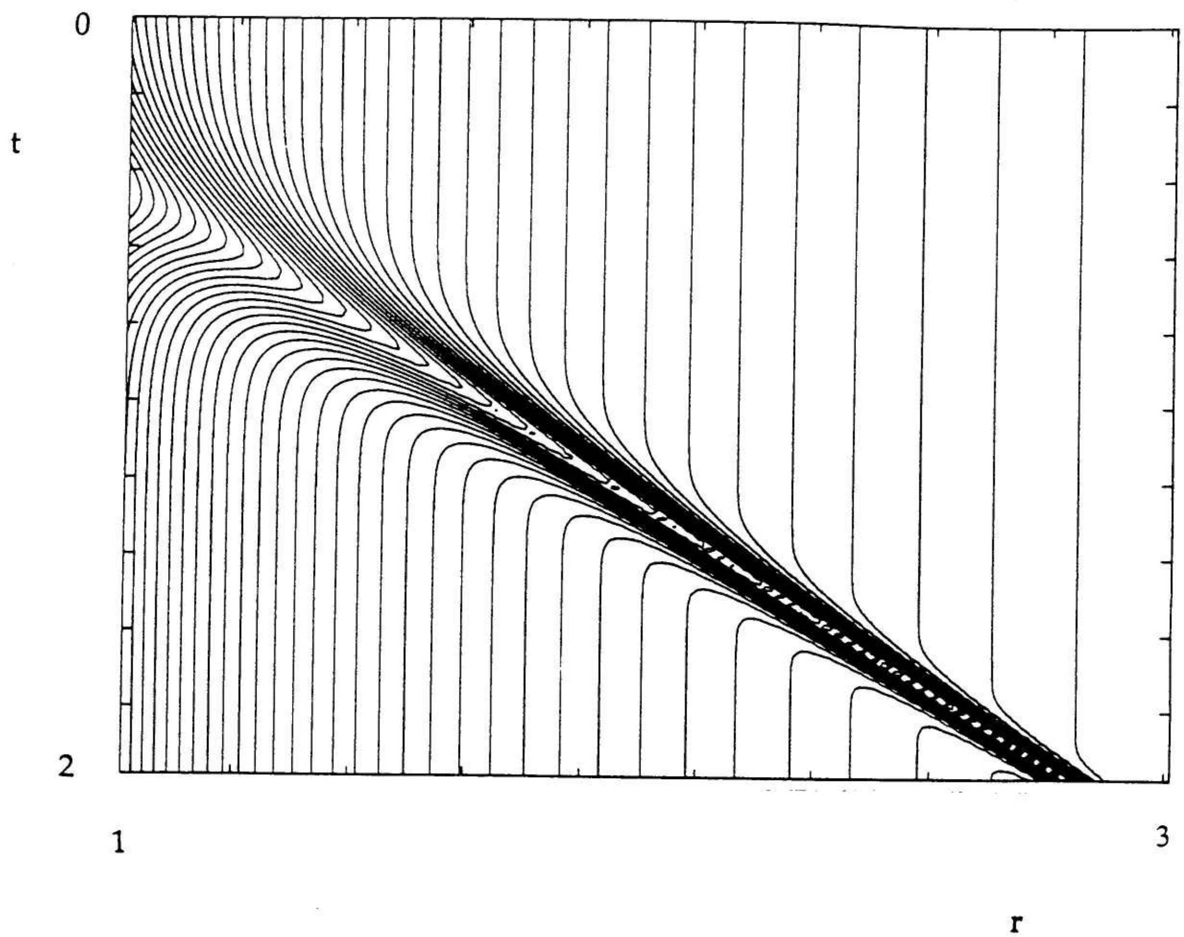

Figure 9d: r-t plane showing the surface contours for modulated solitary wave on a monotonically decreasing film.

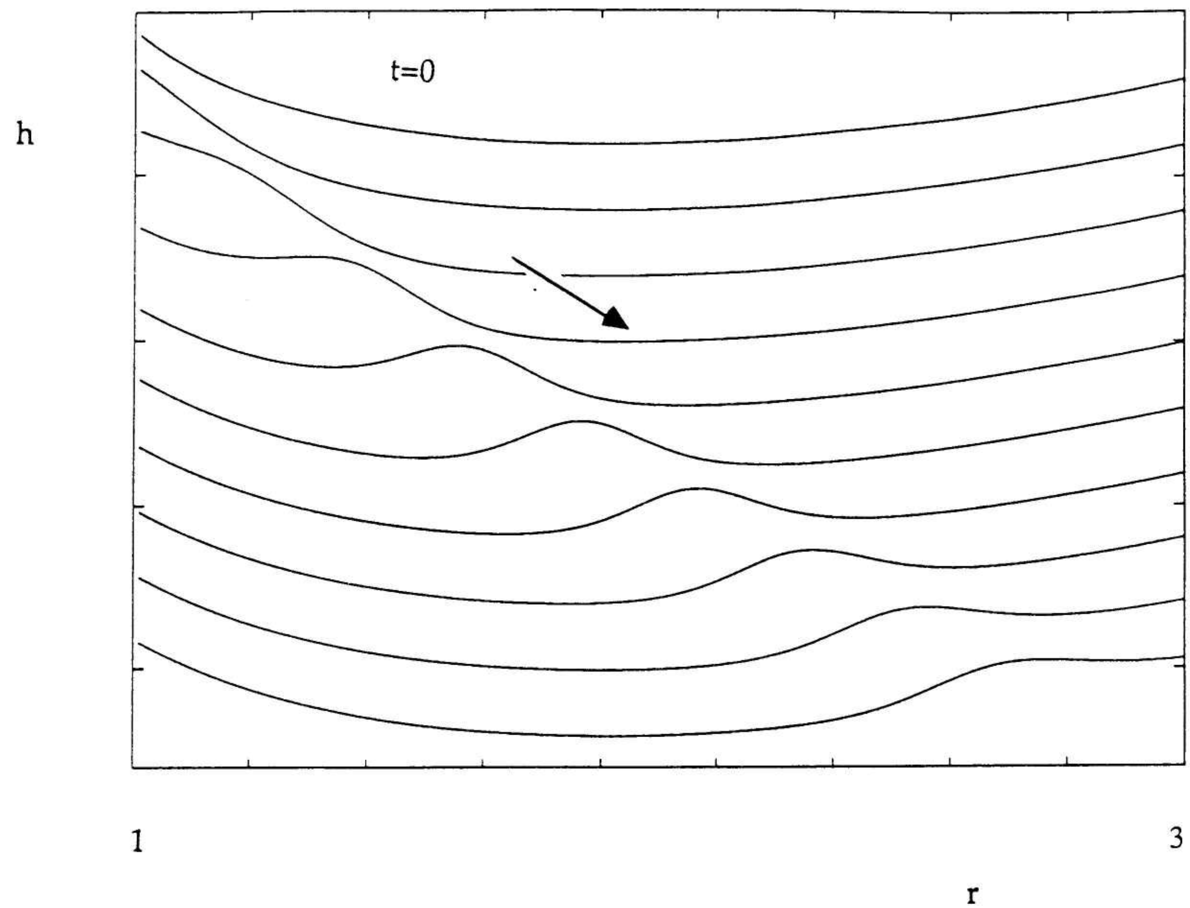

Figure 10a: Spatial and temporal evolution of a modulated solitary wave on the surface of a strictly non-monotonically decreasing radial film (specifically, we look at the influence of a film with a downstream quadratic r dependence).

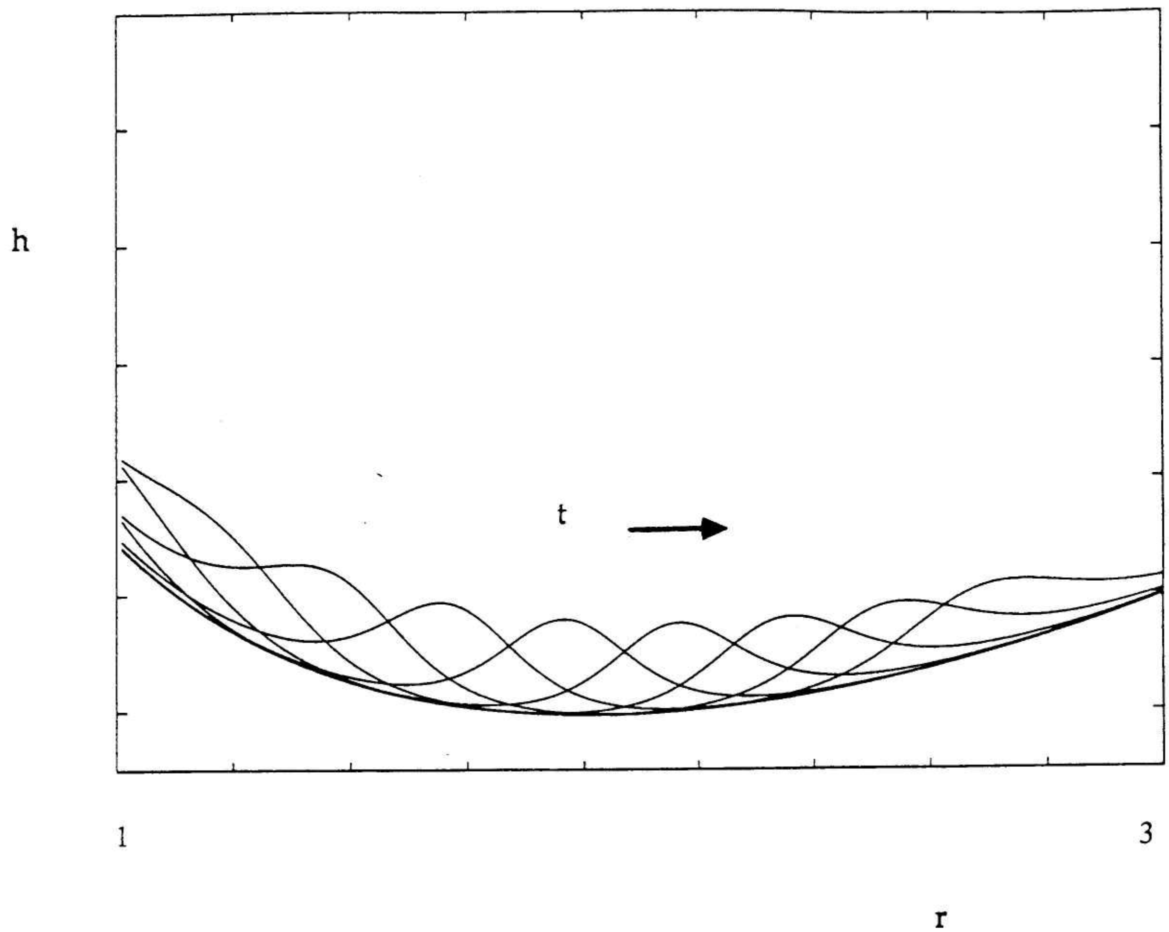

Figure 10b: Time evolution of the same modulated solitary wave as in figure 10a, plotted on the same axis (h vs. r).

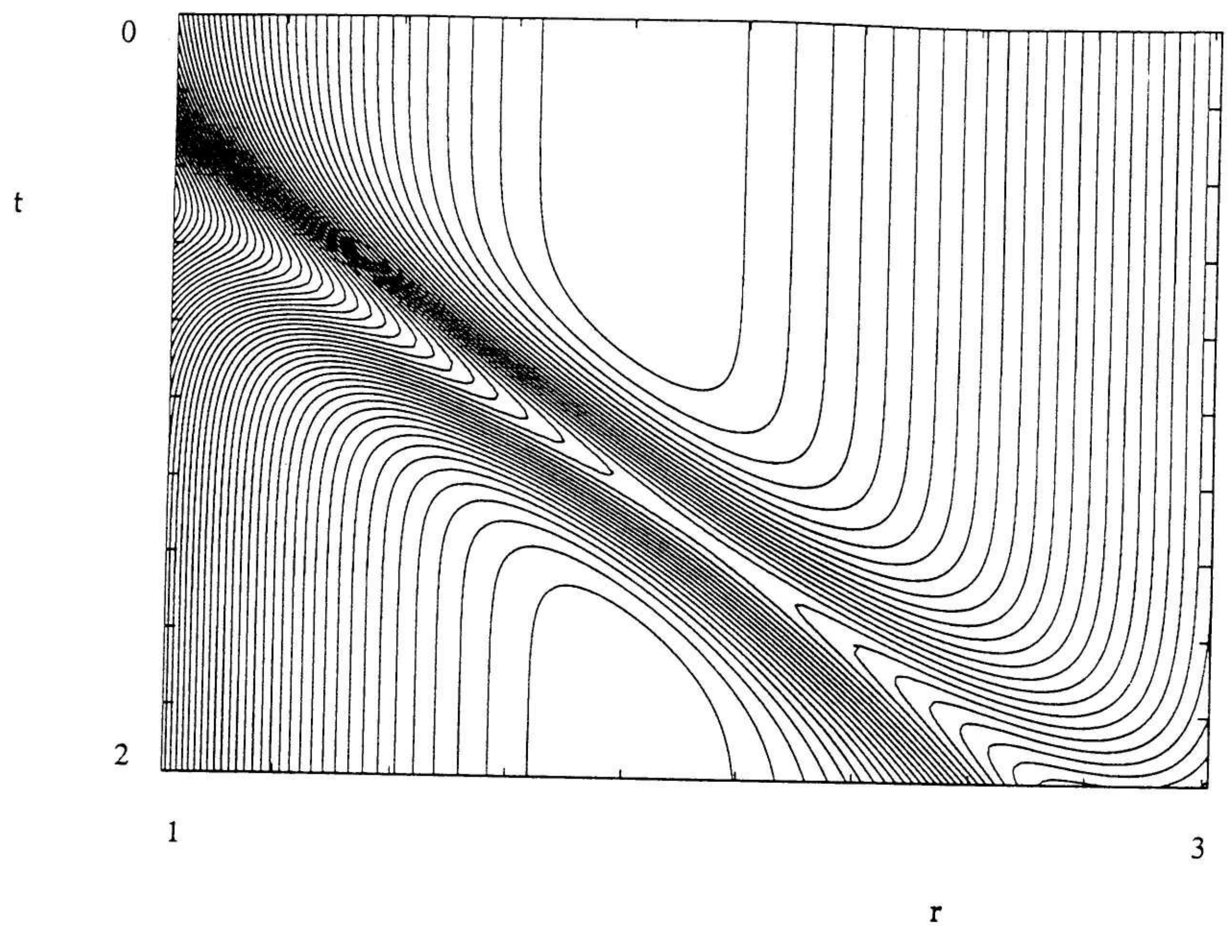

Figure 10c: r-t plane showing the surface contours for a modulated solitary wave on the surface of a strictly non-monotonically decreasing radial film (specifically, we look at the influence of a film with a downstream quadratic r dependence).

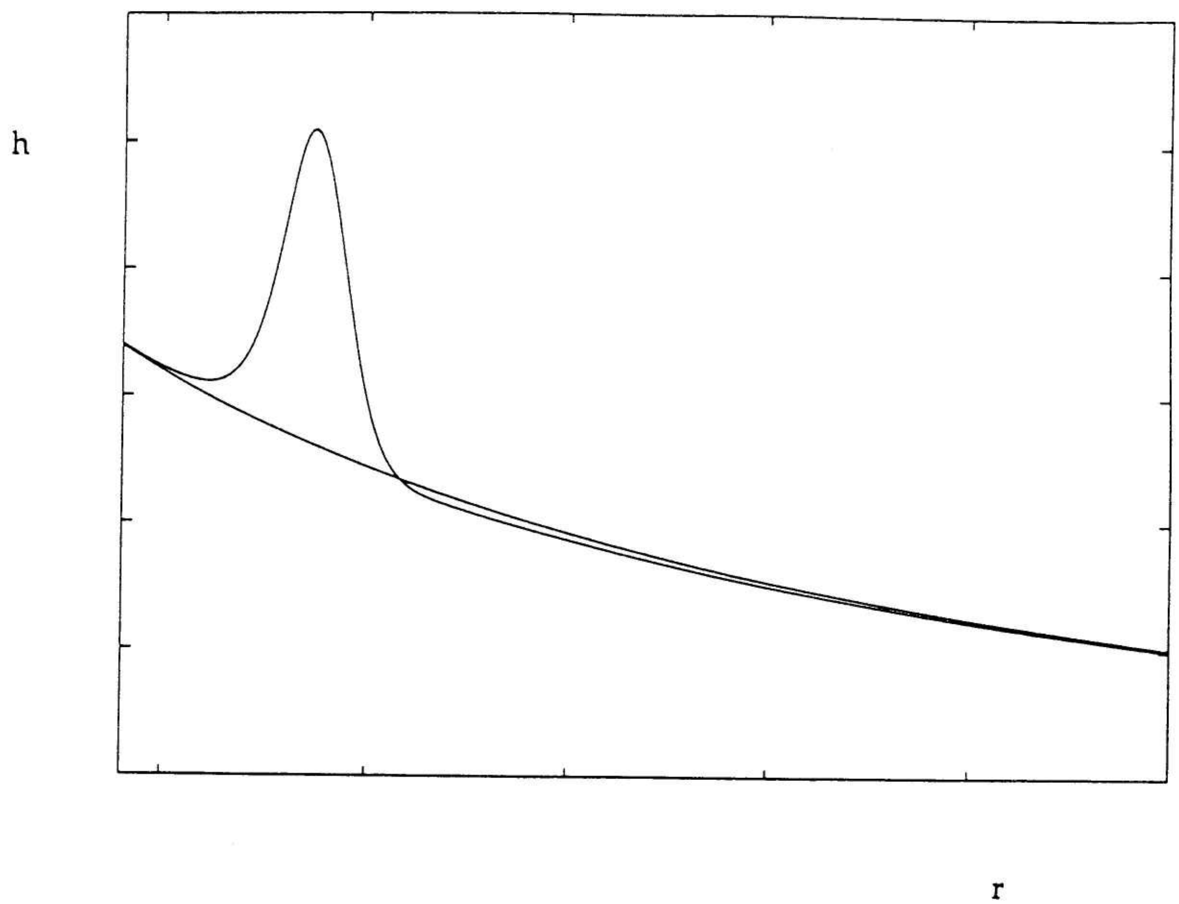

Figure 11: Modulated solitary wave on the surface of a radially thinning film (the figure shows the wave profile at a fixed time superimposed on the undisturbed film).

6.3. Axisymmetric Nondispersive Disturbances

The full equation reduces to a simpler problem in the presence of negligible dispersion on the (x,ξ) plane. This is the case for a region of space where the coefficient of h_{xxx} is much smaller than one, hence producing minimal dispersion which leads to axisymmetric disturbances (assume $\beta=0$.) The problem becomes one of a hyperbolic type and the full evolution equation leads to

$$\frac{\partial}{\partial x}\left[2h_\tau + \left\{\frac{1}{\tau} + \frac{2c'}{c} - \frac{H'}{2H}\right\}h \pm \left\{\frac{3\sqrt{MH}}{c^2H}\right\}hh_x\right] \sim 0 \quad , \ \Delta \rightarrow 0 \tag{247}$$

and for vanishing h and bounded h_τ, h_x, c', H' as $x \rightarrow \infty$, we have

$$h_\tau \pm \left[\frac{3}{2}\frac{\sqrt{MH}}{c^2H}\right]hh_x + \left[\frac{1}{2\tau} + \frac{c'}{c} - \frac{1}{4}\frac{H'}{H}\right]h = 0 \tag{248}$$

Now, multiplying h by the amplification factor, we get the modified amplitude

$$\eta = \frac{c\tau^{1/2}}{H^{1/4}}h \tag{249}$$

the problem reduces to the simpler equation

$$\eta_\tau \pm \left\{\frac{3}{2}\frac{\sqrt{MH}}{c^3H^{3/4}\tau^{1/2}}\right\}\eta\eta_x = 0 \tag{250}$$

Implicit solutions may be found by the method of characteristics as

$$\frac{d\eta}{d\tau} = 0 \tag{251}$$

along the family of characteristic curves described by

$$\frac{dx}{d\tau} = \pm\frac{3}{2}\frac{\sqrt{MH}}{c^3H^{3/4}\tau^{1/2}}\eta \tag{252}$$

Thus, the Riemann invariant $\eta=\eta(x_o,\tau_o)$ is an implicit function of τ and x given by

$$x = x_0 \pm \frac{3}{2}\eta(x_0,\tau_0)\int_{\tau_0}^{\tau}\frac{\sqrt{MH}}{c^3H^{3/4}\tau^{1/2}}d\tau \tag{253}$$

here (x_0,τ_0) are the initial values of (x,τ), but x is a function of r and t, from equation (140); hence at t=0, x_0 is a function of τ_0.

We note that the amplification factor for the general hyperbolic wave is

$$h = h_0\left(\frac{H}{H_0}\right)^{1/4}\left(\frac{c_0}{c}\right)\left(\frac{\tau_0}{\tau}\right)^{1/2} \tag{254}$$

This is a generalized Green's Law for axisymmetric disturbances. If c and H are both constants we obtain the typical $\tau^{-1/2}$ decay law for cylindrical waves. Also if U = 0 then c is proportional to $\sqrt{H}$ and we obtain

$$h = h_0\left(\frac{H}{H_0}\right)^{1/4}\left(\frac{\tau_0}{\tau}\right)^{1/2} \tag{255}$$

which resembles the normal Green's law for infinitesimal waves (Lamb, 1932, §185).

Wave breaking occurs on any characteristic x_o, t_o, such that the Jacobian of the transformation vanishes, which is equivalent to the crossing of characteristics as in the case of shock waves.

In other words,

$$\frac{\partial x}{\partial x_0} = 0 \tag{256}$$

Then $\tau = \tau*(\tau_0)$ is the first crossing of characteristics and it corresponds to

$$\frac{2}{3}\eta_{x_0}^{-1}(x_0,\tau_0) = -\pm \int_{\tau_0}^{\tau*(\tau_0)} \frac{\sqrt{MH}}{c^3 H^{3/4} \tau^{1/2}} d\tau \tag{257}$$

The wave disturbance shocks in the typical hyperbolic fashion.

The film thickness is given from equations (254) and (255)

$$h = \left(\frac{H}{H_0}\right)^{1/4} \left(\frac{c_0}{c}\right)\left(\frac{\tau_0}{\tau}\right)^{1/2} h(x_0,\tau_0)$$

along

$$x = x_0 \pm \frac{3}{2}\eta(x_0,\tau_0) \int_{\tau_0}^{\tau} \frac{\sqrt{MH}}{c^3 H^{3/4} \tau^{1/2}} d\tau$$

The breaking condition thus corresponds to

$$\frac{2}{3} \frac{H_0^{1/4}}{c_0 \tau_0^{1/2} h_{x_0}(x_0,\tau_0)} = -\pm \int_{\tau_0}^{\tau*} \frac{\sqrt{MH}}{c^3 H^{3/4} \tau^{1/2}} d\tau \tag{258}$$

In the case of concentric nondispersive waves on still shallow water the evolution equation reduces to the nondispersive CK-dV with $c = -\pm 1$, $\sqrt{MH} = 1$ and $H = 1$ as was proposed on equation (157). We have

$$h = \left(\frac{\tau_0}{\tau}\right)^{1/2} h(x_0,\tau_0) \tag{259}$$

which is the typical decay law of cylindrical waves, and the characteristic family of curves is given by

$$x = x_0 \pm 3\tau_0^{1/2} h(x_0,\tau_0)\left[\tau^{1/2} - \tau_0^{1/2}\right] \quad (260)$$

Breaking corresponds to

$$\tau^*(\tau_0) = \tau_0\left[1 \pm \frac{1}{3}\frac{1}{\tau_0 h_{x_0}(\tau_0,x_0)}\right]^2 \quad (261)$$

Similarly, for the momentum driven flow, equation (161), with

$c \sim U \equiv 1$ and $H = 1/\tau$,

$$h = \left(\frac{\tau_0}{\tau}\right)^{3/4} h(\tau_0,x_0) \quad (262)$$

along

$$x = x_0 \pm 2\left(\tau^{3/4} - \tau_0^{3/4}\right)\tau_0^{3/4} h(\tau_0,x_0) \quad (263)$$

With the above characteristics, the breaking criterion leads to

$$\tau^*(\tau_0) = \tau_0\left[1 \pm \frac{1}{2}\frac{1}{\tau^{3/2} h_{x_0}(\tau_0,x_0)}\right]^{4/3} \quad (264)$$

In order to visualize these results, we numerically integrate the initial value problem of equation (250), by using an ENO scheme (Harten and Osher, 1987).

The integration is carried out for a region in the inviscid far field where a modulated solitary wave has already developed, as from section 6.2 and figure 9, and where dispersion is negligible compared to nonlinearity.

We study the amplitude dependence of the initial prescribed data for a fixed Froude number. Figure 12a shows the time evolution of an initial solitary wave profile of very small amplitude (0.1). We note that the profile steepens without breaking. Thus, waves of very small amplitude can be sustained by the film.

Figure 12b is an r-t plane for the modulated wave of figure 12a. We can see from the contours ahead of the wave that the profile does not break.

Figure 13a and 13b are time evolution plots of disturbances with amplitudes 0.25 and 0.5, respectively. We see from figure 13a that the profile has steepened to almost its maximum slope before breaking, and from figure 13b, we see that the profile has already formed a shock, and its amplitude has begun to diminish.

For even larger amplitude, figure 14a presents the evolution of a solitary wave profile of unit amplitude. Again, we see the steepening, breaking and diminishing of the profile. Figure 14b is an r-t plane of the evolution of the wave profile shown in figure 14a. A three dimensional plot of the spatial and temporal evolution of the wave disturbance is presented in figures 15a and 15b for the same unit amplitude wave.

It is interesting to note that for the small amplitudes, 0.1 and 0.25, the peak amplitude seems to be preserved. This could be the case since the waves do not shock at any location and hence do not loose any energy as they propagate. On the other hand, for larger amplitudes (0.5 and 1.0), the waves steepen and shock. Thus dissipating their potential energy, and this is why the amplitude diminishes as they travel.

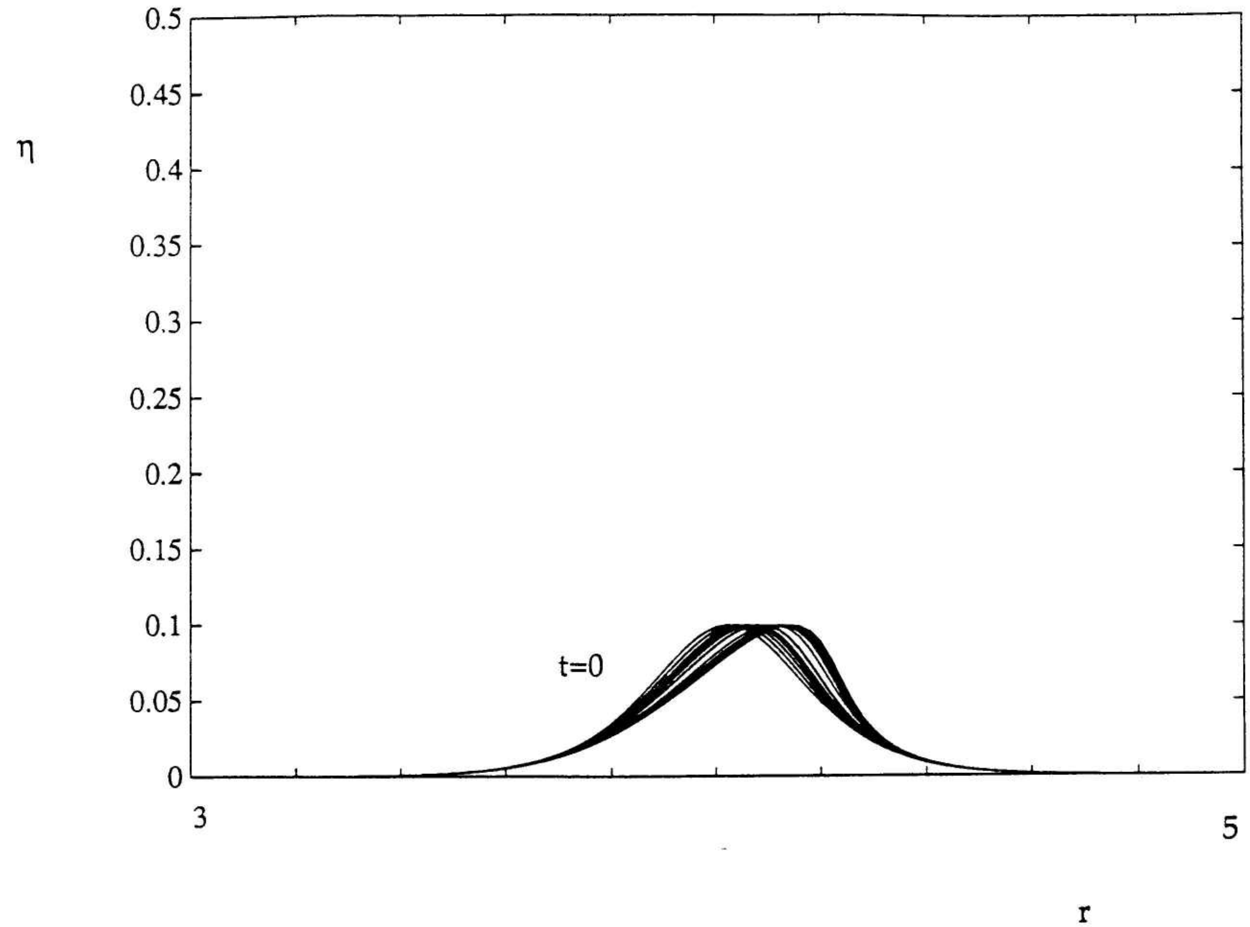

Figure 12a: Time evolution in the h vs. r plane of an initial modulated solitary wave profile. For this small amplitude (0.1), notice that the profile steepens but does not break (note that the profile is not superimposed on the undisturbed film).

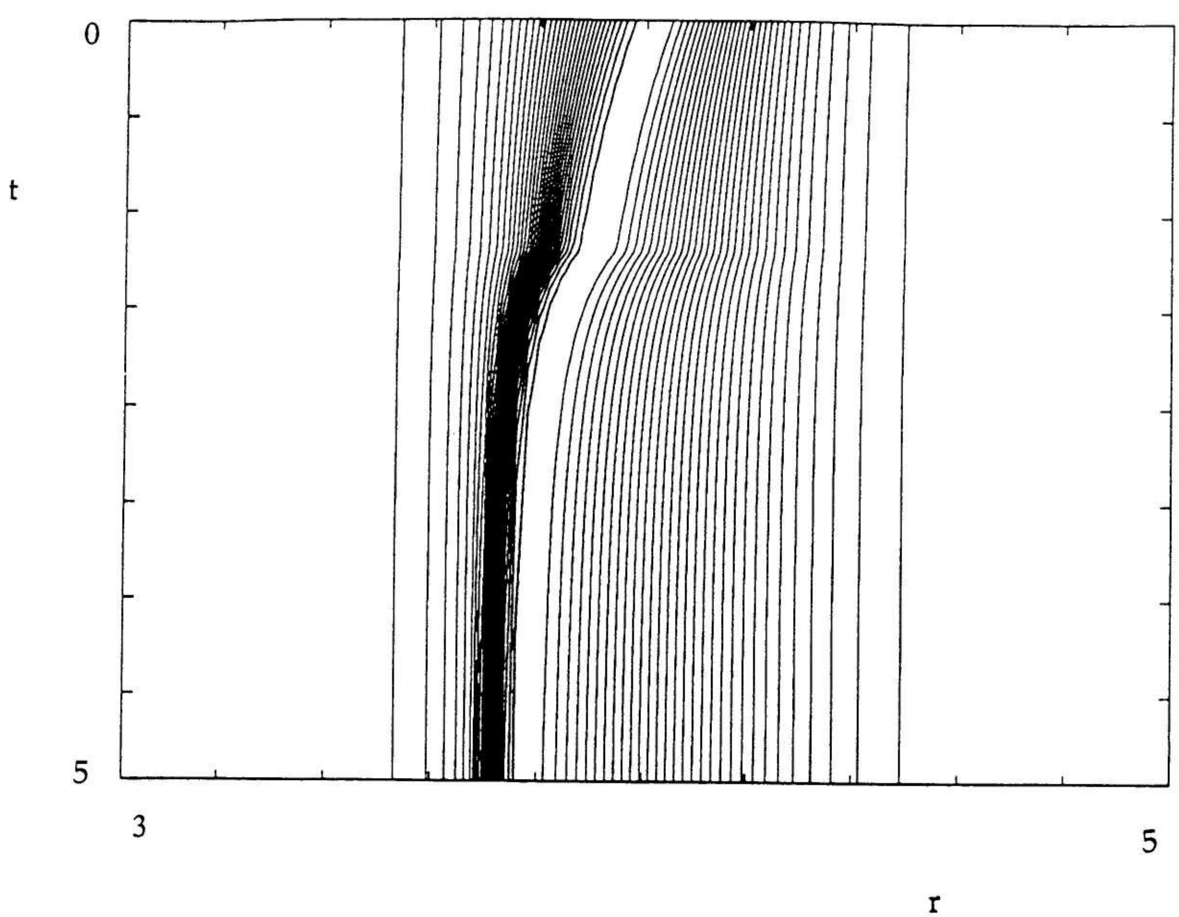

Figure 12b: r-t plane for the same modulated solitary wave as in figure 12a.

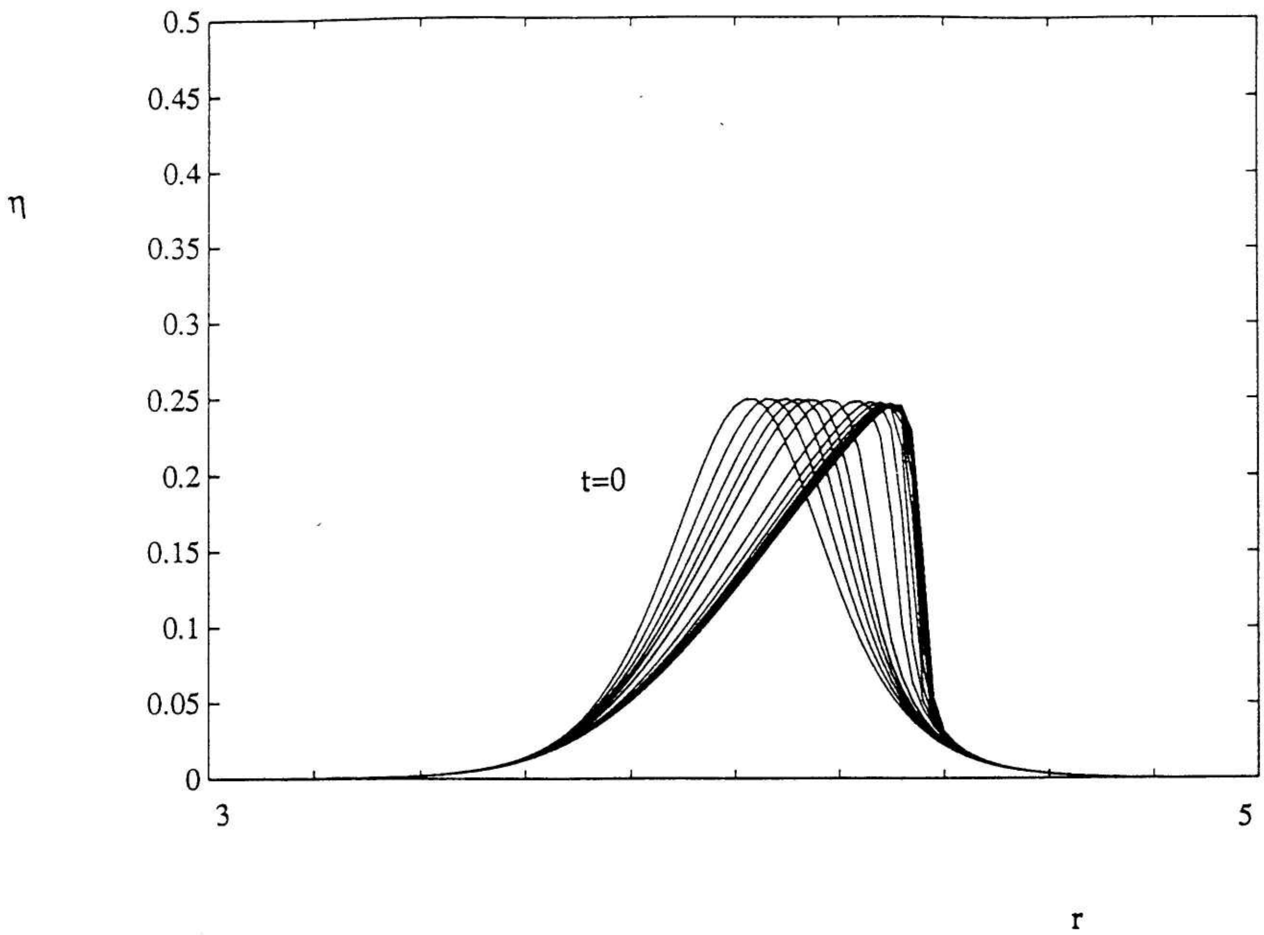

Figure 13a: Time evolution in the h vs. r plane of an initial modulated solitary wave profile. For increasing amplitude (=0.25), notice that the profile steepens almost to a maximum slope, but still no breaking occurs (note that the profile is not superimposed on the undisturbed film).

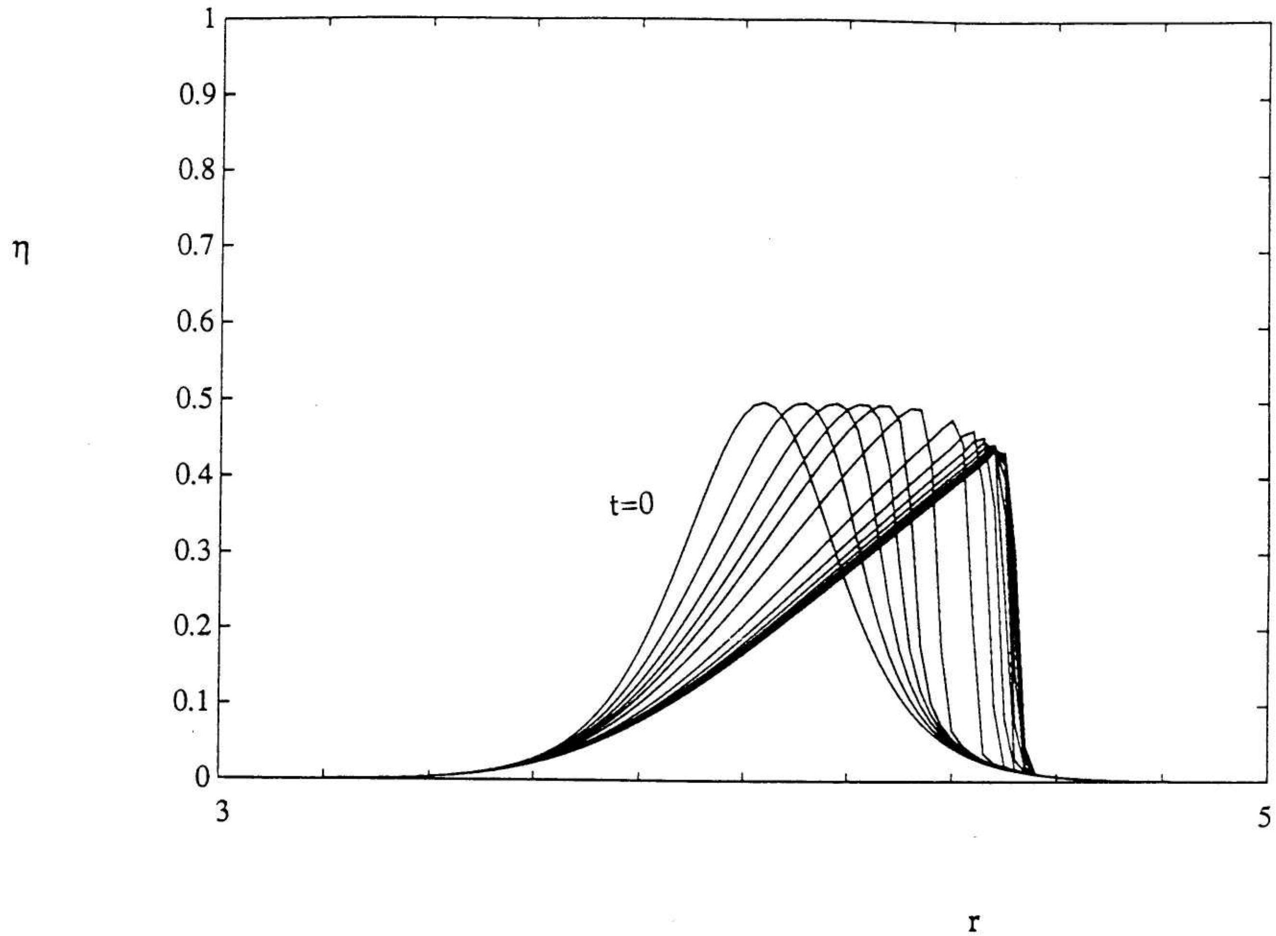

Figure 13b: Time evolution in the h vs. r plane of an initial modulated solitary wave profile. For increasing amplitude (=0.5), notice that the profile steepens and breaks in the typical hyperbolic fashion (note that the profile is not superimposed on the undisturbed film).

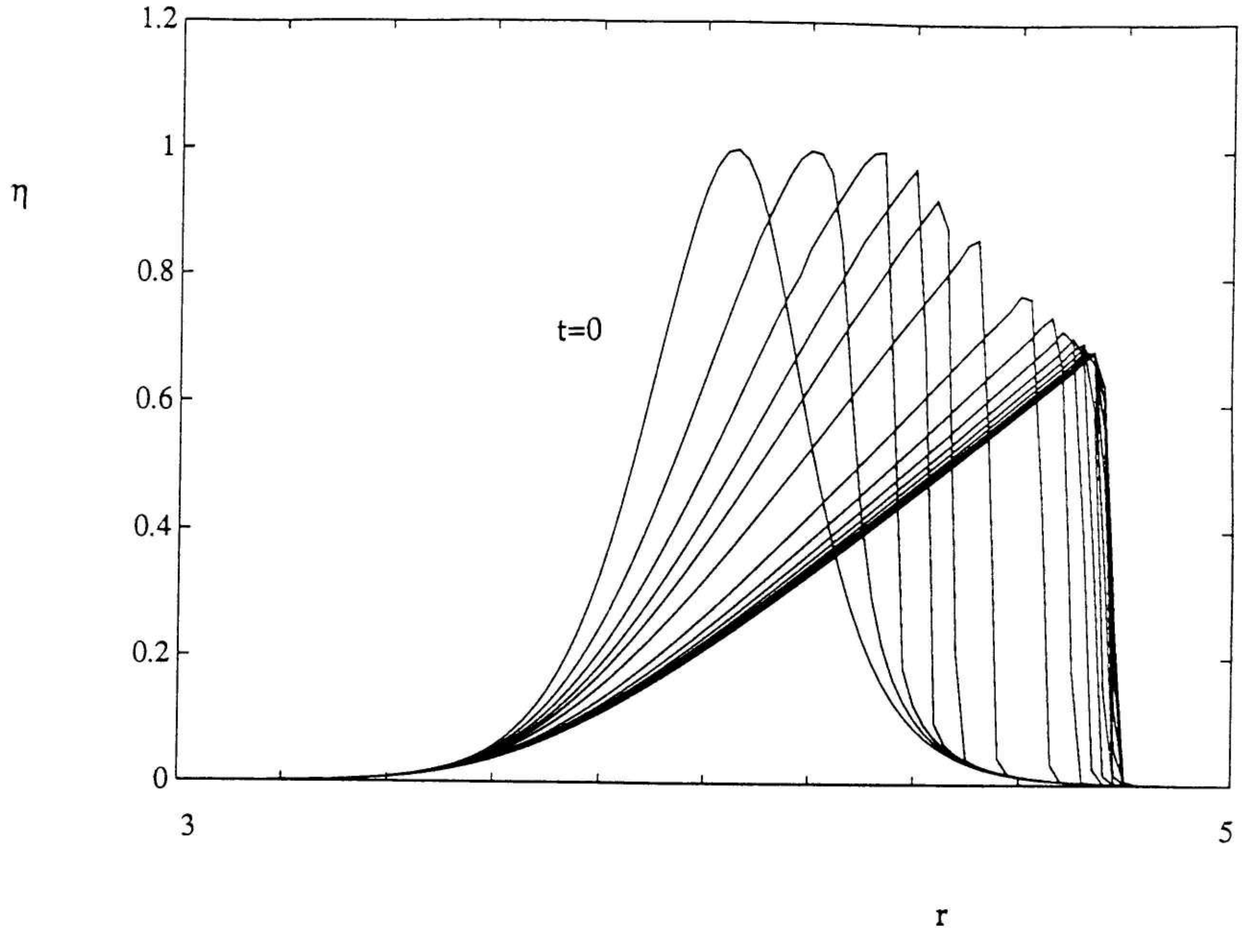

Figure 14a: Time evolution in the h vs. r plane of an initial modulated solitary wave profile. For even higher amplitudes (=1), the wave profile breaks in the typical hyperbolic fashion , and it is dissipated by the flow (note that the profile is not superimposed on the undisturbed film).

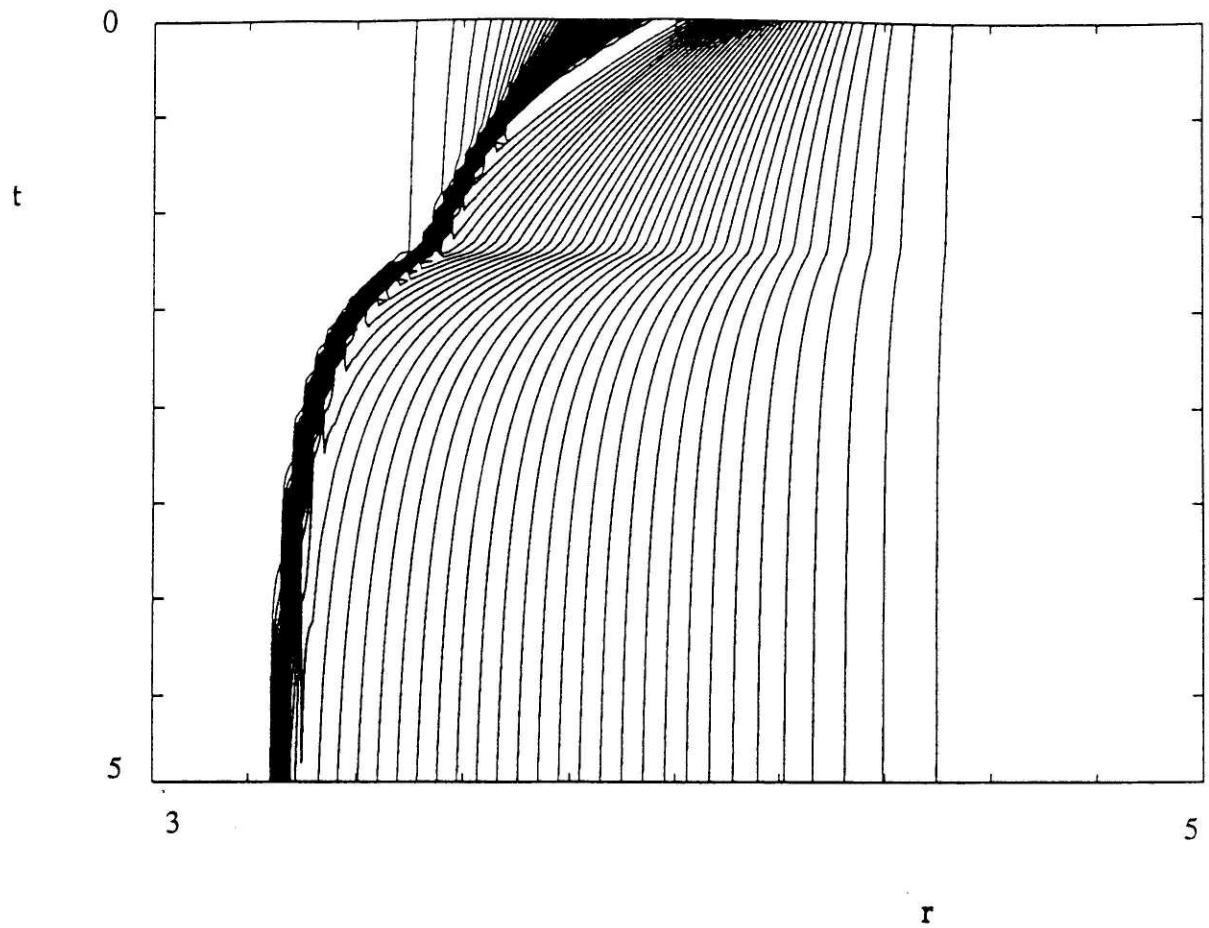

Figure 14b: r-t plane for the same modulated solitary wave as in figure 14a.

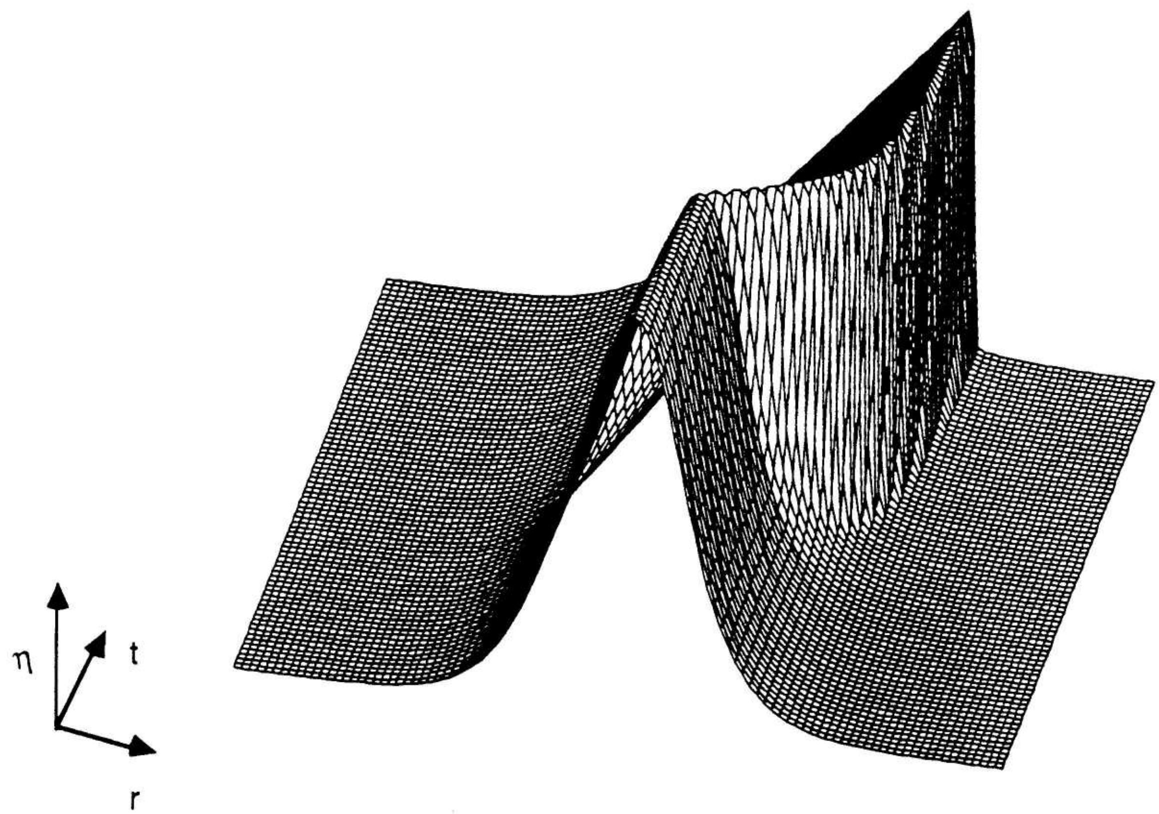

Figure 15a: Mesh plot of the wave disturbance (η) vs. radial position vs. time, showing the evolution of the modulated solitary wave (viewed from the t=0 plane).

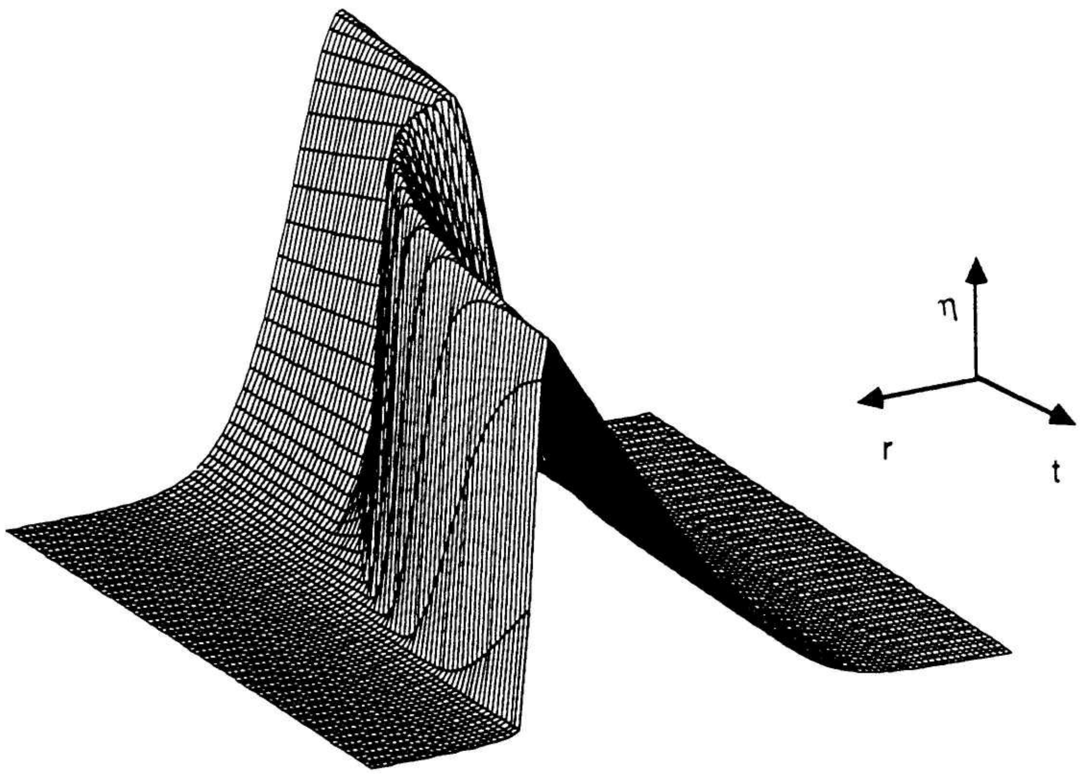

Figure 15b: Mesh plot of the wave disturbance (η) vs. radial position vs. time, showing the shock profile of the modulated solitary wave (viewed from the t=5 plane).

7. Conclusion

The behavior of wave disturbances on the surface of an axisymmetric nonuniform film produced by the impingement of a vertical round jet against a flat surface has been investigated. The research is a theoretical study composed mainly of analytical work along with numerical experiments to support the analysis.

In the analysis, we first find a steady state solution to the inviscid Euler equations, and this solution is then perturbed in three different ways:

1. The hyperbolic initial value problem for the axisymmetric shallow water wave equations is solved numerically.

2. A Stokes wave expansion, for small amplitude, is assumed as a perturbation on an arbitrary fluid depth in order to include nonlinear dispersive effects.

3. A multiple scale analysis, based on the slow changes of the liquid film relative to the disturbance wavelength, is performed on the full dispersive water wave equations in order to derive an evolution equation for the wave disturbance.

From the first of these analyses, we find that solitary wave initial disturbances split along the characteristic directions and lose energy as they propagate on the nonuniform film. We also find that this type of initial data breaks in finite time. Of course, the breaking corresponds to the typical hyperbolic shock formation at the crossing of characteristics. The numerics also show that collisions between cylindrical "solitons" are not amplitude preserving as in the case of planar geometry.

The Stokes wave expansion is used to calculate an averaged Lagrangian from the variational principle for water waves. This average Lagrangian provides a system of equations that govern the slow modulation of the Stokes wave train. By finding the characteristic forms of the system, we determine a stability map, kH versus λk^2. The instability corresponds to a change of type of the equations from hyperbolic to elliptic when the characteristic speeds become imaginary. We conclude that long waves can become unstable, in the sense that modulations will grow large in finite time. However, the analysis does not predict the subsequent development of the wave envelope for larger times.

By using multiple scale analysis, we derive an evolution equation for the development of nearly concentric wave disturbances on the surface of a radially flowing film. The equation belongs to the class of variable coefficient KP equations, and it reduces to the variable coefficient K-dV equation for axisymmetric disturbances. We analyze slow periodic modulations of the variable coefficient K-dV and find that these types of waves are confined to regions of increasing depth. The analysis predicts that small amplitude periodic cnoidal waves degenerate onto solitary wave trains as they propagate down the film. By analyzing solitary wave modulations, we find that they indeed belong in the region of decreasing depth and that as they propagate down the film, the amplitude to width ratio increases considerably. In the case when dispersive effects stop affecting the wave propagation, the equation becomes the hyperbolic version of the variable coefficient K-dV. This happens when the film thickness balances with the capillary length. We find that, for fixed Froude numbers, waves of amplitude smaller than a certain critical value will propagate on the film without breaking, but any profile with amplitude larger than the critical (for the case under study the critical value was found to be 0.25) will steepen until it breaks.

These analyses do not predict splattering per se; rather, they attempt to identify the mechanism by which waves propagate, and the conditions which can result in sharpening of initially smooth disturbances. This mechanism can be summarized as a simple nonlinear filtering process that takes periodic wave trains and turns them into solitary waves propagating down the surface of an inviscid axisymmetric thin film.

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