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CODED, DIVERSIFIED, RAYLEIGH-FADED BINARY
SYMMETRIC THRESHOLD CHANNELS

RUSSELL R. PFEIFFER

TECHNICAL REPORT 384

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
RESEARCH LABORATORY OF ELECTRONICS
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Abstract

In a digitalized communication network, the accuracy of the digital data is vital. Errors in these data are caused by signal fading and noise. In an effort to reduce the number of errors, null-zone detecting, diversity, and coding can be used. This paper examines the combination of these three methods of improving reliability. The expressions for the probabilities associated with a Rayleigh-faded channel with null-zone detecting and diversity used are derived. Quantitative results of machine computation are presented for different orders of diversity. The bounds of decoding error of a coding scheme designed for a null-zone channel are evaluated. Quantitative results are presented for various word lengths and transmission rates. The channel statistics are combined with the coding-error bounds, and an examination of the effective power gains and increases in reliability is carried out. A few hypothetical tropospheric scatter links are set up to investigate the practicality of the combination. A qualitative determination of the situation for which it appears feasible to combine null-zone detecting, diversity, and coding is given.

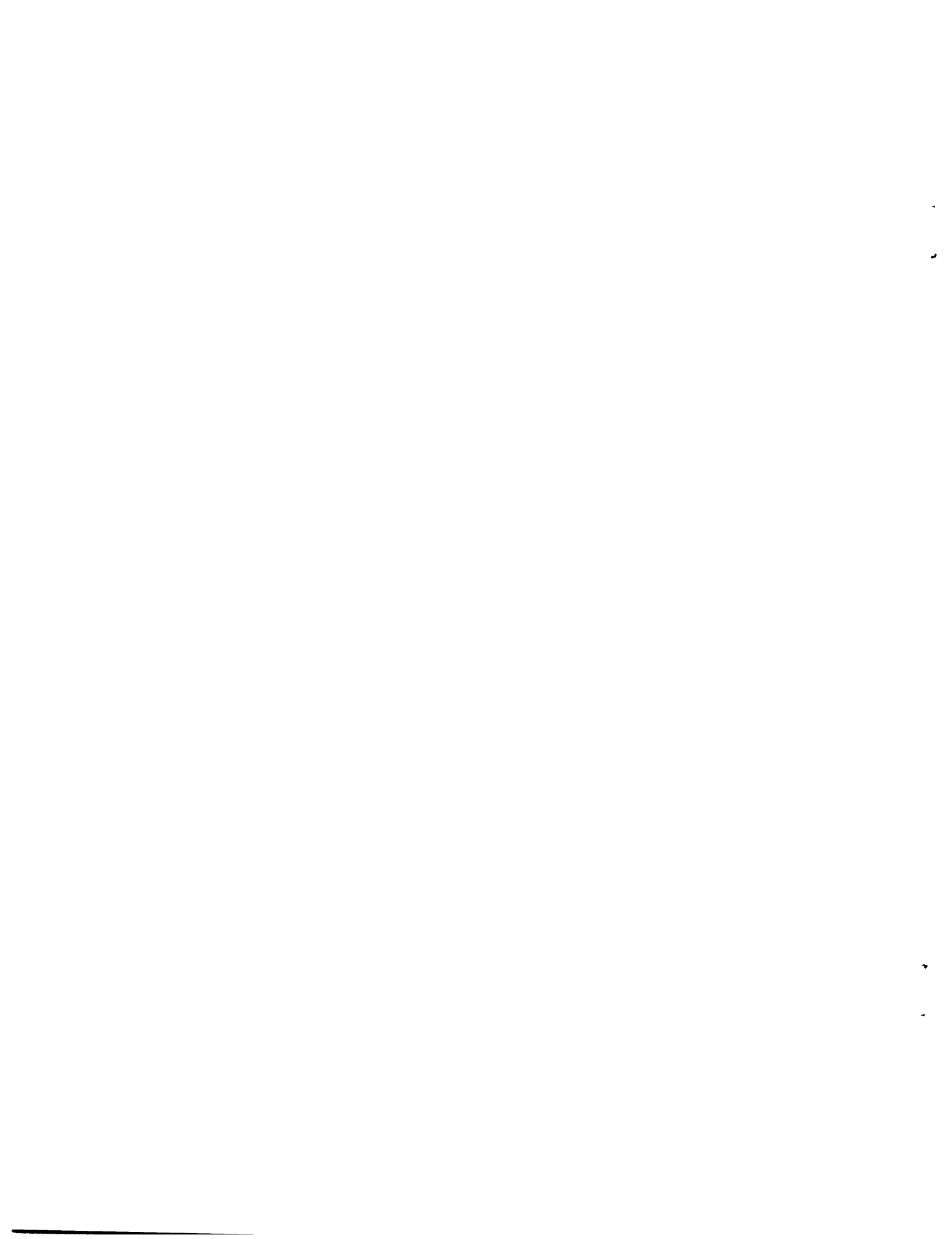


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I. INTRODUCTION

In the past decade, the amount of digital communication equipment that is in use has increased considerably, and there seems to be no question as to its ever increasing importance. The accuracy of digital data is extremely important. However, in many instances, the propagation medium incorporated into a system introduces errors in the received digital sequences.

In long-distance high-frequency communication, the reliability of transmission is often greatly affected by signal fading. Fading is a characteristic multipath problem resulting from the fact that the signal travels over various paths of different physical or electrical lengths. The contributions from the different paths sometimes add and sometimes cancel, and produce a variation in the strength of the received signal. Most errors in binary digital sequences are caused by random noise when the received signal strength is low. A direct solution to this problem would involve increasing the transmitter power or enlarging the receiving antenna, thereby increasing the signal-to-noise ratio at the input of the receiver. But because of the present state of technology and that of economic conditions, the size of these units is limited. However, other means of combating the fading problem do exist.

At the present time, diversity is in use, and has long been known to increase the reliability of reception when fading occurs. Diversity is used to obtain and combine a number of independent samples of the same signal so that over-all reliability is increased.

Diversity may be realized in a number of ways. If, for example, two or more receivers, each of which has a separate antenna, are used and the antennas have sufficient physical separation, each receiver's signal becomes statistically independent of the others. Therefore, the possibility that all of the receptions will fade at the same time is reduced. Signal reliability increases with the number of receivers (higher order of diversity). Other forms of diversity are: frequency diversity, transmitting the same message at two or more different frequencies; time diversity, retransmitting the same message one or more times; polarization diversity, making use of the polarization of the transmitted signals; and so on. Each of these is a method of obtaining independent samples of the same transmitted message.

Obtaining the independent signals is one of the main diversity problems. Another major problem is that of optimally combining these signals (1, 2). Kahn (3) derived an optimum combining technique for dual diversity reception, the "Ratio Squarer" or "Combiner" diversity. Brennan (4, 5) extended this method and compared it with other types of diversity combining techniques. Pierce (6) made a further investigation of diversity, and determined optimum methods of combination when no prior signal information, except fading properties, was known. He derived probabilities of error for binary symmetric channels (see Fig. 2) using frequency shift keying with signals suffering from Rayleigh fading and white noise. Portions of this report extend Pierce's

work, utilizing a different type of binary channel.

Another approach to eliminating fading errors is null-zone detecting. As we have stated, digital errors are likely to occur when the signal strength is low. Null-zone detecting essentially eliminates this possibility by "erasing" all digits whose signal strength is not greater than some preset threshold. Certainly, no information is obtained when a digit is erased, but more importantly, no false information is generated. However, "null-zoning" requires some means of restoring the identity of the erased digits. For example, this could be accomplished by calling for a retransmission of the message or by coding the digital sequences that are transmitted. Bloom, Chang, and Harris (7), investigating a channel subject to additive Gaussian noise, obtained results concerning channel capacity as a function of the null-zone threshold level. They have shown that an increase in channel capacity may be obtained when an erasure zone is generated. It appears that null-zoning for a fading channel is very appropriate because it deals directly with weak faded signals. It will be seen that the reliability of reception can be greatly increased by using null-zone detecting. Further discussion of the erasure zone will follow.

Another approach to increasing signal reliability is coding. Coding can be used to restore the identity of the digits erased by null-zone detecting (8). But in contrast to restoring erased digits, coding can also increase signal reliability by detecting and correcting errors in received binary sequences. Recently, Elias (9) has proposed a coding scheme that considers both errors and erasures (that is, for a Binary Symmetric Threshold Channel, which will be discussed later). He has shown that an appreciable effective power gain is obtained by using this method as compared with using diversity under comparable circumstances. This report is closely related to Elias' results.

Signal reliabilities can be increased by employing diversity, null-zone detecting or coding. Independent samples of the signal are obtained by diversity; weak signals are erased by null-zone detecting; erased digits are restored and incorrect digits are corrected by coding. Each of these methods increases the reliability by reducing the effects of signal fading. An examination of the combination of these three schemes will follow.

This report is concerned with the examination of the statistics of a Rayleigh-faded channel when both null-zone detecting and diversity are employed. The expressions for the associated probabilities are derived. Quantitative results of machine computations of the channel statistics are presented for a variety of situations. (This work was partly done at the Computation Center, M. I. T.) Quantitative results of machine computation of the decoding error bounds, derived by Elias, are presented for a number of word lengths and transmission rates. The channel statistics are then combined with the coding-error bounds, and an examination of the effective power gains that have been realized is carried out. Finally, a few hypothetical tropospheric scatter links are set up to investigate the practicality of the system. A qualitative determination of when it would appear economically feasible to use the system is given.

II. CHANNEL STATISTICS AND CHARACTERISTICS

2.1 The Binary Symmetric Threshold Channel

To simplify the representation of a system, it is convenient to combine the modulator, transmitter, transmitting antenna, coupling medium, receiving antenna, receiver, and the decision-making device and call this the channel. If diversity is employed, the channel includes the complete system, that is, all of the receiving apparatus. The fading phenomenon is introduced in the coupling medium. For digital data transmission, the input to the channel is a sequence of binary digits, and the output may take a variety of forms that depend on the noise and the decision-making device.

The Binary Symmetric Threshold Channel (BSTC) is schematically a combination of a Binary Symmetric Erasure Channel (BSEC) and a Binary Symmetric Channel (BSC). The BSEC and the BSC are shown in Fig. 1.

The BSEC is a form of null-zoner. It has two inputs, mark and space, and three outputs, mark, space, and erasure, where the probability of erasure, p_x , is a function of the threshold level in the decision-making device. In this channel there is no possibility of a mark becoming a space or a space becoming a mark because the noise is considered insufficient to cause an error in the output. On the other hand, the BSC has two inputs, and two outputs, mark and space, but there is a possibility of a mark becoming a space or a space becoming a mark with probability p_c . This transfer event is commonly called a "crossover."

The BSTC, Fig. 2, is formed by including the possibility of adequate noise interference in the BSEC. There are two inputs to the channel, mark and space, and three

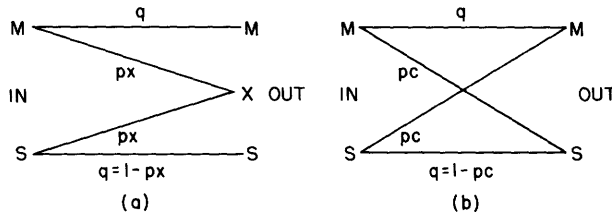


Fig. 1. (a) Binary Symmetric Erasure Channel.
(b) Binary Symmetric Channel.

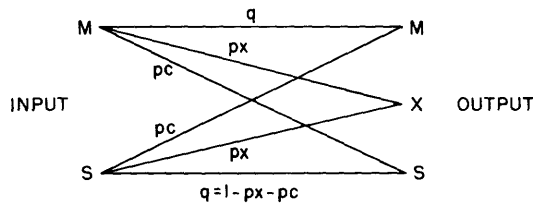


Fig. 2. Binary Symmetric Threshold Channel.

outputs, mark, space, and erasure. Now, in addition to having erasures generated by the decision-making device, there is a possibility that a mark may crossover to a space, or vice versa. As indicated in Figs. 1 and 2, q is the probability of a correct reception, px is the probability of an erasure, and pc is the probability of a crossover. The BSC and the BSEC can be considered special cases of the BSTC. The BSC is obtained by setting the threshold so that no erasures are generated; the BSEC is approximately obtained by setting the threshold so that the probability of crossover, pc , is made very small.

2.2 Channel Capacity of the BSTC

When evaluating the results of coding, we shall have some use for the value of the information capacity of the BSTC. The capacity of this channel is found to be

$$C = pc + q + pc \log_2 \left(\frac{pc}{pc+q} \right) + q \log_2 \left(\frac{q}{pc+q} \right) \quad (1)$$

$$= (1-px) - (1-px) \log_2 (1-px) + q \log_2 q + pc \log_2 pc \text{ bits/symbol} \quad (2)$$

where $[pc + q + px] = 1$.

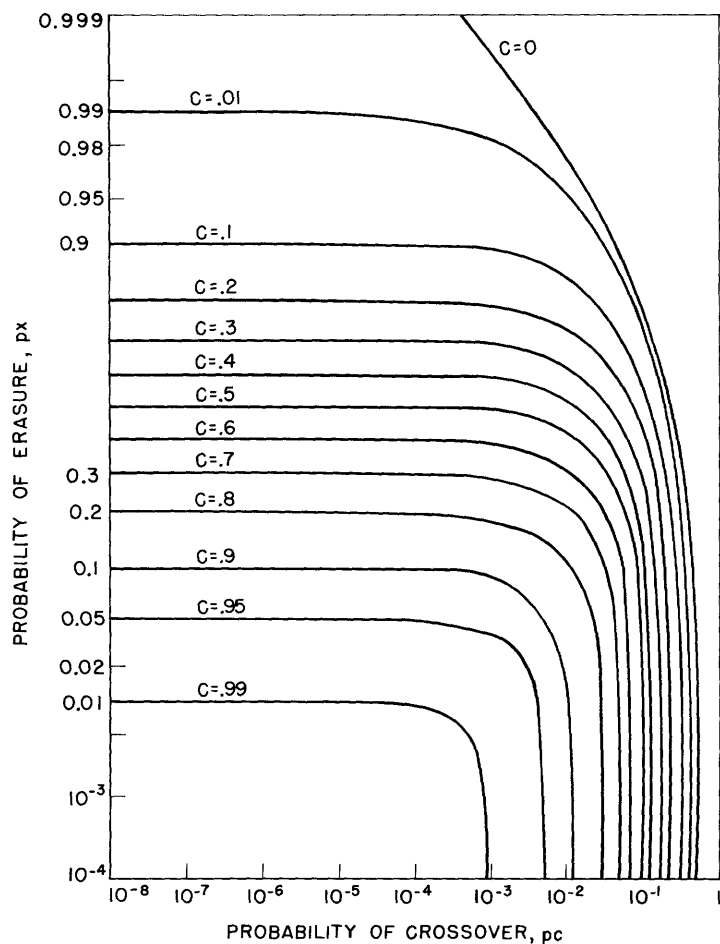


Fig. 3. Constant capacity lines for BSTC.

The iso-capacity lines for the BSTC are plotted in Fig. 3. At low values of p_c , the channel closely represents a BSEC, hence its capacity is approximated by $1 - p_x$. At low values of p_x , the channel closely represents the BSC, hence the capacity values are approximated by $1 + q \log q + p \log pc$. Figure 3 illustrates the transition from one extreme to the other.

2.3 Channel Fading Characteristics

We are aware of the fading that occurs in the propagation medium, and to make an analysis, it is necessary to represent this fading phenomenon in a statistical manner. It appears that the fading encountered in high-frequency transmission (in the absence of aircraft multipath) is closely represented by a Rayleigh probability density (10). This assumption predominates in much of the current literature on high-frequency transmission, and some empirical investigations strengthen the validity of this assumption (11,19). Here, too, Rayleigh fading will be assumed.

The fading is caused by single or multiple reflected waves, waves reflected from different regions of the ionosphere, and so forth, which results in paths of different electrical and physical lengths. Hence the composite signal at the receiving antenna may or may not be attenuated; the choice depends on the relative phases of the signals arriving from the various paths.

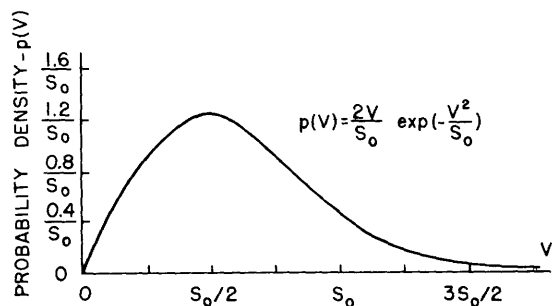


Fig. 4. Rayleigh probability density function.

The Rayleigh probability density function, pdf, of the signal amplitudes, and the probability density function of the received power are, respectively,

$$\begin{aligned}
 p(V) dV &= \frac{2V}{S_0} \exp\left(-\frac{V^2}{S_0}\right) dV & V \geq 0 & & p(S) = \frac{1}{S_0} \exp\left(-\frac{S}{S_0}\right) dS & S \geq 0 \\
 &= 0 & V \leq 0 & & & = 0 & S \leq 0
 \end{aligned} \tag{3}$$

where V is the signal amplitude, S is the signal power, and S_0 is the average (over a long time) signal power. A plot of the Rayleigh density function is given in Fig. 4.

2.4 Characteristics of the Received Signals

Evaluation of the channel statistics requires a knowledge of the probability density functions of the received signals. In addition, a number of assumptions must be made

to make the problem workable.

It will be assumed that the signal amplitude and phase are constant during each baud (The signal that comprises one digit of the transmitted sequence). This does not preclude the possibility of a fairly rapid fade, but does eliminate fading times in the order of baud lengths. Independent white noise at the receivers, with each receiver noise being independent of the other, will also be assumed. This assumption appears to be closely met in practice. When considering diversity, we shall assume that the fading and noise seen by a receiver is independent of that seen by the other receivers. This condition can be obtained by proper separation of the antennas in space diversity, proper time lags in time diversity, and proper polarization of antennas for polarization diversity. The detailed requirements have been discussed by Baghdady (12) and by Bailey, Bateman, and Kirby (13). Staras (14) has made an investigation of the effect of correlation between two Rayleigh-faded channels, and has shown that most of the advantage of diversity is realized when the correlation coefficient between channels has dropped to 0.6. In other words, low correlation between channels is almost as effective as complete independence. An antenna separation of 10-15 wavelengths will often yield coefficients less than 0.6. Antenna separations of 30-50 wavelengths will yield coefficients less than 0.3 most of the time. It is important to remember that independence of signals is the keynote to diversity, for if the signals were identical, that is, having the same fading and noise, there would be no advantage in using diversity. To continue with the assumptions, Frequency Shift Keying (FSK) will be utilized with matched filter (crosscorrelation) detection at the receivers. (Each receiver has a mark filter and a space filter, the signal is crosscorrelated in each channel and the outputs compared.)

Pierce, using the assumptions stated above, carried out an investigation for the BSC, which will be closely followed to a certain point. Pierce's results can be obtained from the expressions derived by using the proper threshold level.

Given a signal power S , baud length T , and a receiver noise n_o watts per cycle of bandwidth, the probability density functions for the envelope samples in the mark and space channels, given that a mark was sent, are:

$$p(w) dw = \frac{wdw}{n_o} \exp\left(-\frac{w^2+2ST}{2n_o}\right) I_0\left(\frac{w\sqrt{2ST}}{n_o}\right) \quad (4)$$

$$p(z) dz = \frac{zdz}{n_o} \exp\left(-\frac{z^2}{2n_o}\right) \quad (5)$$

where I_0 is the modified Bessel function, w is the mark filter sample, and z is the space filter sample from the crosscorrelators in each receiver. The derivation of $p(w)$ has been obtained in detail by Rice (15).

2.5 The Decision Process

It is necessary to develop a decision process that will determine the most probable transmitted signal by examining the mark and space filter samples. This can be realized

by forming a likelihood ratio – a sufficient statistic that utilizes all of the information that can be extracted from the samples. For a discussion of the likelihood function, the reader is referred to Woodward (16).

We shall assume that the mark and the space signals are equally likely to be transmitted, that is, the sequence of binary digits put into the channel has approximately equal numbers of ones and zeros. We form the joint probability density function of the samples from the mark and space filters of all of the receivers under the assumption of transmitted mark and transmitted space. The set of received samples will be called G . For M^{th} order diversity, the joint probability densities become $2M$ dimensional functions. Then the conditional probability density functions, $p(G|\text{mark})$ and $p(G|\text{space})$ are formed.

This manipulation yields a decision process which compares likelihood functions.

$$p(G|\text{mark}) \underset{\text{mark}}{\overset{\text{space}}{>}} p(G|\text{space}) \quad (6)$$

In FSK, the mark and space channels are symmetric. Therefore, assuming that a mark is transmitted yields the same results as assuming that a space is transmitted.

Since the amplitudes of the incoming signals are Rayleigh-faded, it is necessary to integrate the mark envelope distribution (Eq. 4), over all the possible values of signal power, S . From Eqs. 3 and 4 (assuming that a mark is sent), we obtain

$$p(w) = \int_0^\infty p(w|S) p(S) dS = \int_0^\infty \frac{w}{n_o} \exp\left(-\frac{w^2+2ST}{2n_o}\right) I_0\left(\frac{w\sqrt{2ST}}{n_o}\right) \exp\left(-\frac{S}{S_o}\right) \frac{dS}{S_o} \quad (7)$$

$$= \frac{w}{n_o + S_o T} \exp\left(-\frac{w^2}{2n_o + 2S_o T}\right) \quad (8)$$

$$p(z) = \frac{z}{n_o} \exp\left(-\frac{z^2}{2n_o}\right) \quad (9)$$

The space sample remains the same as Eq. 5 because it is independent of S . As the noise power goes to zero, $p(w)$ goes to a Rayleigh density, and $p(z)$ goes to zero. If we have M^{th} order diversity, we must obtain the joint probability density functions for all M receivers. Since the various channels are assumed to be statistically independent, we need merely obtain the product of the probability density functions of each channel. When m is the m^{th} channel, the following expressions result:

$$p(G|\text{mark}) = \prod_{m=1}^M \frac{w_m}{n_o + S_o T} \exp\left(-\frac{w_m^2}{2n_o + 2S_o T} - \frac{z_m^2}{2n_o}\right) \frac{z_m}{n_o} \quad (10)$$

$$p(G|\text{space}) = \prod_{m=1}^M \frac{z_m}{n_o + S_o T} \exp\left(-\frac{z_m^2}{2n_o + 2S_o T} - \frac{w_m^2}{2n_o}\right) \frac{w_m}{n_o} \quad (11)$$

Now if we use Eqs. 10 and 11, Eq. 6 can be reduced by a straightforward algebra to

$$\sum_{m=1}^M \left[\frac{w_m^2}{2n_o + 2S_o T} - \frac{z_m^2}{2n_o} \right] \begin{array}{l} \text{space} \\ < \\ \text{mark} \end{array} > \sum_{m=1}^M \left[\frac{z_m^2}{2n_o + 2S_o T} - \frac{w_m^2}{2n_o} \right] \quad (12)$$

or simply,

$$\sum_{m=1}^M w_m^2 \begin{array}{l} \text{space} \\ < \\ \text{mark} \end{array} > \sum_{m=1}^M z_m^2 \quad (13)$$

Thus, a square-law combination of each of the filter outputs is made; these sums are compared to determine whether a mark or space has been transmitted.

The comparisons above apply to the BSC. Therefore, no erasure zone is generated. At this point, the decision process will be slightly altered to allow for the more general BSTC.

When considering the BSC, the w^2, z^2 space is divided as shown in Fig. 5 (assuming that a mark was sent). The dividing line (decision line) corresponds to expression 19. We wish to eliminate the cases for situations such as point A in Fig. 5. Here, it would be decided that a mark had been sent, but obviously with a small change in noise level, it would be decided to be a space. If there was no noise present, all of the points in the sample space would fall on the w^2 axis, and no errors could result (see Eqs. 8 and 9). However, the presence of noise shifts these points from the w^2 axis. A mark sent at one frequency can become a signal at the space frequency because of the presence of the white noise. If the signal is badly faded, the frequency component of the noise corresponding to the mark frequency may be sufficient in amplitude and phase to eliminate the mark signal, and the noise component at the space frequency may be large enough to cause a crossover.

The effects of the random noise can be reduced by generating an erasure zone in the w^2, z^2 space as shown in Fig. 6. By generating this zone, we increase the reliability by eliminating the most doubtful decisions such as point A. In essence, we generate a buffer zone for the noise characteristics. Points such as B and C, in Fig. 6, can be

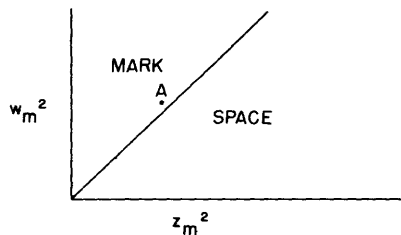


Fig. 5. Sample space and decision line for BSC.

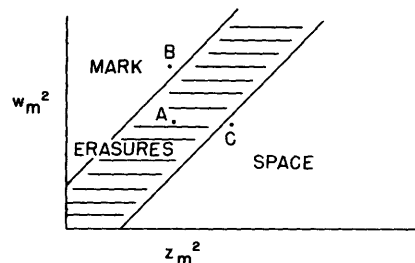


Fig. 6. Sample space and erasure zone for BSTC.

designated mark and space, respectively, with some degree of assurance. The channel statistics, which will be derived, will determine the probabilities that a designated mark is correct or in error.

Now that erasures are going to be generated, it is necessary to determine a criterion that will put a given decision in the erasure zone or out of it. The criterion appears to be arbitrary as far as the threshold level is concerned. For lack of any better, the decision will be as described below (assuming that a mark was sent). A mark will be decided upon if

$$p(G|\text{mark}) > Kp(G|\text{space}) \quad (14)$$

and a space will be decided upon if

$$p(G|\text{space}) > Kp(G|\text{mark}) \quad (15)$$

The boundaries of the erasure zone in Fig. 5 are seen to correspond to lines, where the probability of being correct is $K/(K+1)$ and the probability of being in error is $1/(K+1)$. However, these are only boundary lines, and after integrating over the proper areas, the probability of error will be much less than $1/(K+1)$, and the probability of being correct (out of the decisions made) will be greater than $K/(K+1)$. Hereafter, when a threshold value is mentioned, it will refer to the value of K . The actual equations of the erasure-zone boundary lines are obtained by equating both sides of inequality 14 and both sides of inequality 15. This yields (after some reduction),

$$\sum \frac{w_m^2}{2n_o} - \sum \frac{z_m^2}{2n_o} > \frac{n_o + S_o T}{S_o T} \ln K \quad (16)$$

Thus, whenever the absolute value of the difference is greater than the constant at the right (which will be called the threshold level) of inequality 16, a decision is made — mark if $w_m^2/2n_o > z_m^2/2n_o$, and space if $z_m^2/2n_o > w_m^2/2n_o$. However, an erasure is generated if the absolute value of the difference is not greater than the threshold level. The value of the threshold level defines the intercepts for the boundary lines of the erasure zone. It can be seen that as n_o decreases, or as $S_o T$ increases, the erasure zone becomes narrower. This indicates that a decision with requirement K can be made with a smaller difference between the sums of the mark and space samples.

It is helpful at this point to get a better physical picture of the probability space under consideration. For convenience, let $x = z^2/2n_o$, $y = w^2/2n_o$, and call the threshold level σ . Then $p(x)$ and $p(y)$ are obtained.

$$p(x) = \exp(-x) \quad (17)$$

$$p(y) = \frac{1}{1 + S_o T/n_o} \exp - \frac{y}{1+S_o T/n_o} \quad (18)$$

The three-dimensional volume for a single channel (first-order diversity) is illustrated in Fig. 7 for two values of $S_o T/n_o$. As $S_o T/n_o$ increases, the volume decreases in its

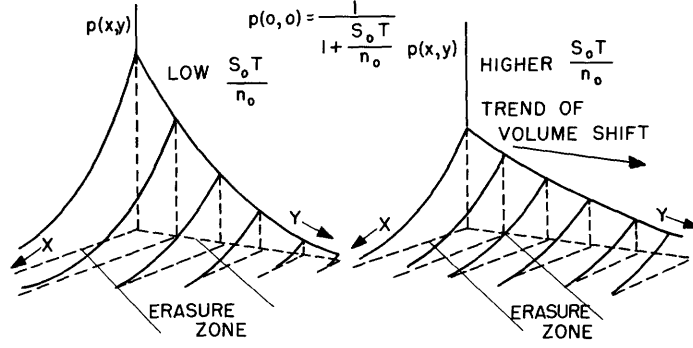


Fig. 7. Probability volume for first-order diversity.

maximum height and shifts closer to the y axis. By imagining the volume at the left of the erasure zone to be that which causes errors, and the volume at the right of the erasure zone to be that which yields correct results, we can see that as $S_o T/n_o$ increases we have a greater percentage of the volume in the desired region. By great increase in $S_o T/n_o$, only a very small portion of the volume will lie at the left of the erasure zone, and the erasure zone will have its width reduced until its intercepts are approximately $\ln K$. Thus the combination of volume shift and erasure zone narrowing gives a physical picture of the probability changes as signal-to-noise ratio increases.

2.6 Channel Probabilities

Now we wish to derive the expressions for the probabilities of error, erasure, and correct decision. Using $p(x)$ and $p(y)$ as defined above, we let $X = \sum_{m=1}^M x_m$ and $Y = \sum_{m=1}^M y_m$. The density functions of X and Y are obtained by taking the M -fold convolution over the identical component distributions, which yields

$$p(X) dX = \frac{X^{M-1} dX}{(M-1)!} \exp(-X) \quad (19)$$

$$p(Y) dY = \frac{Y^{M-1} dY}{(1+S_o T/n_o)^M (M-1)!} \exp\left(-\frac{Y}{1+S_o T/n_o}\right) \quad (20)$$

Remembering the decision process, we decide upon a mark when $Y > \sigma + X$. To obtain the probability of being correct, we must integrate over the X and Y densities in the following manner,

$$q = \int_0^{\infty} \int_{\sigma+X}^{\infty} p(X)p(Y) dXdY \quad (21)$$

For the probability of error (crossover), the following integration is carried out:

$$p_c = \int_0^{\infty} \int_{\sigma+Y}^{\infty} p(X)p(Y) dYdX \quad (22)$$

Referring to Fig. 6, Eq. 21 integrates all of the area to the left of the erasure zone, and indicates what percentage, on the average, of all the marks sent will fall in this area. Likewise, Eq. 22 indicates the average per cent of the marks sent which will fall in the area at the right of the erasure zone, and will be incorrectly determined as a space. The probability of erasure, p_x , is obtained from the relation $p_x + p_c + q = 1$.

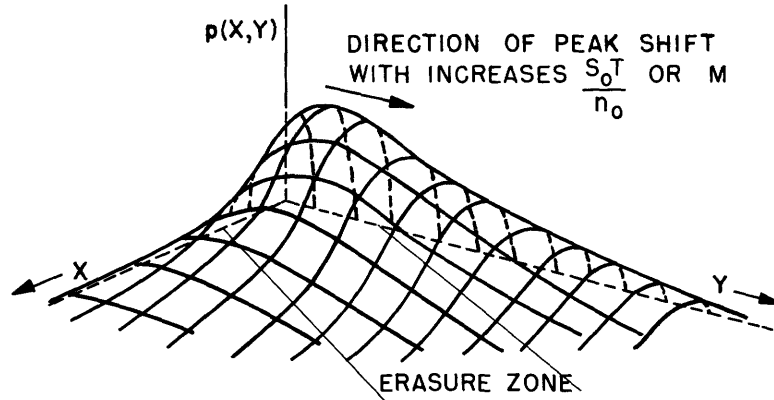


Fig. 8. Probability volume for dual diversity ($M=2$).

A general picture of the joint probability density volume for dual diversity ($M=2$) is shown in Fig. 8. Comparing this with Fig. 7, we can see the trend of volume shift as diversity is employed. Higher orders of diversity produce the same general shape, except that the bulk of the volume stretches out and lies closer to the Y axis. As the volume shifts, the amount remaining in the erasure zone and in the crossover region (for any constant $S_o T/n_o$) decreases. By maximizing $p(X)$ and $p(Y)$, the position of the peak of the volume can be obtained. Solving $\frac{d[p(X)]}{dX} = 0$ and $\frac{d[p(Y)]}{dY} = 0$ yields

$$X_{\max} = (M-1); \quad Y_{\max} = (M-1)(1+S_o T/n_o) \quad (23)$$

From Eq. 23 we can see how the order of diversity and the signal-to-noise ratio determine where the peak of the probability volume is situated. If the $S_o T/n_o$ is high, the peak lies close to the Y axis. On the other hand, if the $S_o T/n_o$ is low, the peak lies close to (or in) the erasure zone, unless the order of diversity is very high. This means that if the $S_o T/n_o$ is low, the percentage of erasures will not be reduced appreciably by increasing M . As a result, as $S_o T/n_o$ decreases, the advantages of diversity also decrease.

The integrations expressed by Eqs. 21 and 22 yield (see Appendix A):

$$q = \frac{K^{-\left(\frac{n_o}{S_o T}\right)}}{(M-1)!(2+S_o T/n_o)^M} \sum_{m=0}^{M-1} \frac{1}{m!(1+S_o T/n_o)^{m-M}} \sum_{\ell=0}^m \binom{m}{\ell} \left(\frac{n_o+S_o T}{S_o T} \ln K\right)^{m-\ell} \left[\frac{\left(1+\frac{S_o T}{n_o}\right)^\ell}{\left(2+\frac{S_o T}{n_o}\right)^\ell} (M-1+\ell)! \right] \quad (24)$$

$$pc = \frac{K \left(\frac{1+S_o T/n_o}{S_o T/n_o} \right)^{-M}}{(2+S_o T/n_o)^M (M-1)!} \sum_{m=0}^{M-1} \frac{1}{m!} \sum_{\ell=0}^m \binom{m}{\ell} \left(\frac{n_o+S_o T}{S_o T} \ln K \right)^{m-\ell} \left(\frac{1+S_o T/n_o}{2+S_o T/n_o} \right)^\ell (M-1+\ell)! \quad (25)$$

and $px = 1 - q - pc$. Pierce's results may be extracted by letting K approach 1; this reduces inequality 16 to expression 13.

For convenience, the expressions for single and dual diversity are given below.

Single Diversity: (that is, one transmitter and one receiver):

$$pc = \frac{1}{2 + S_o T/n_o} K \left(\frac{1+S_o T/n_o}{S_o T/n_o} \right)^{-1} \quad (26)$$

$$q = \left(1 + \frac{1}{2+S_o T/n_o} \right) K \left(\frac{n_o}{S_o T} \right)^{-1} \quad (27)$$

Dual Diversity:

$$pc = K \left(\frac{1+S_o T/n_o}{S_o T/n_o} \right)^{-1} \left[\frac{4+3S_o T/n_o + \ln K \{3+2n_o/S_o T + S_o T/n_o\}}{(2+S_o T/n_o)^3} \right] \quad (28)$$

$$q = K \left(\frac{n_o}{S_o T} \right)^{-1} \left[\frac{(2+S_o T/n_o)^3 - 3S_o T/n_o - 4 + \ln K \{(S_o T/n_o)^2 + 4S_o T/n_o + 5+2n_o/S_o T\}}{(2+S_o T/n_o)^3} \right] \quad (29)$$

The general expressions (24 and 25) were evaluated by machine for single, dual, quadruple, and fifteenth-order diversity. The results are presented in Figs. 9-12. The assumption of channel independence for fifteenth-order diversity is questionable, but the direction of various trends will be more easily recognized by examining this special case. The values of px are plotted along the ordinate; the values of pc are plotted along the abscissa. The signal-to-noise ratio at the receiver input is indicated by the family of curves labeled in db. The threshold value is indicated by the family of vertical curves. The complete channel statistics are obtainable from these figures for the desired order of diversity.

The flexibility obtained by changing the value of K can easily be seen. At a given $S_o T/n_o$, the range of K (which is theoretically not limited) offers a variety of pc , px combinations.

The channel capacity at any K and $S_o T/n_o$ can be determined by using Fig. 3 and Figs. 9-12 in an overlay fashion. It will be found that the channel capacity is maximized, for a given $S_o T/n_o$, by a low value of K (between 1 and 20). However, it will be seen

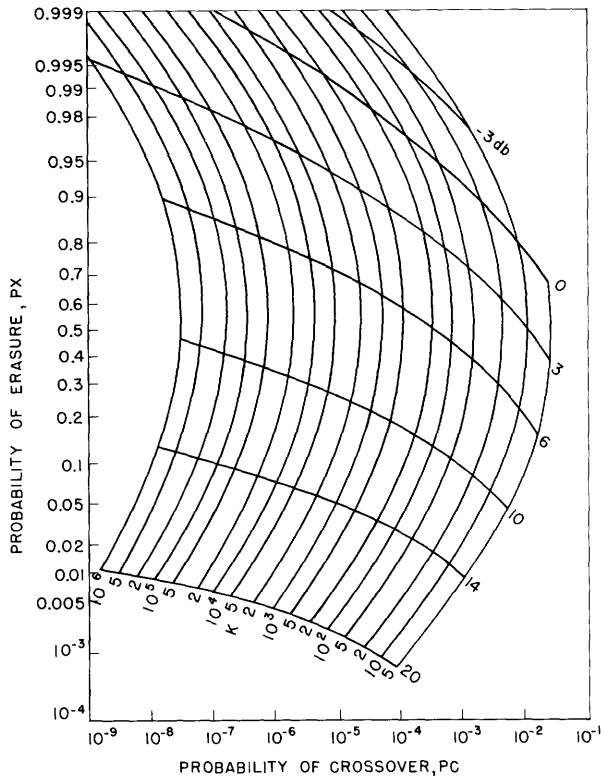


Fig. 9. Channel characteristics, first-order diversity.

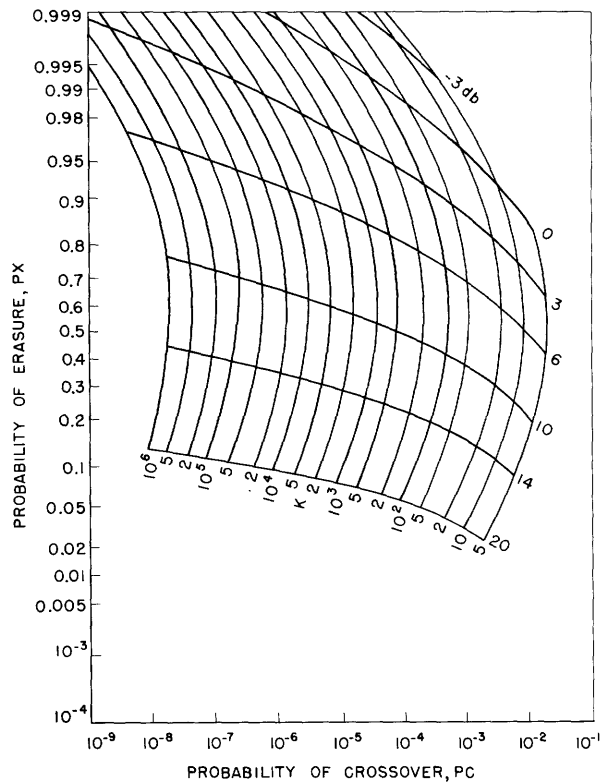


Fig. 10. Channel characteristics, second-order diversity.

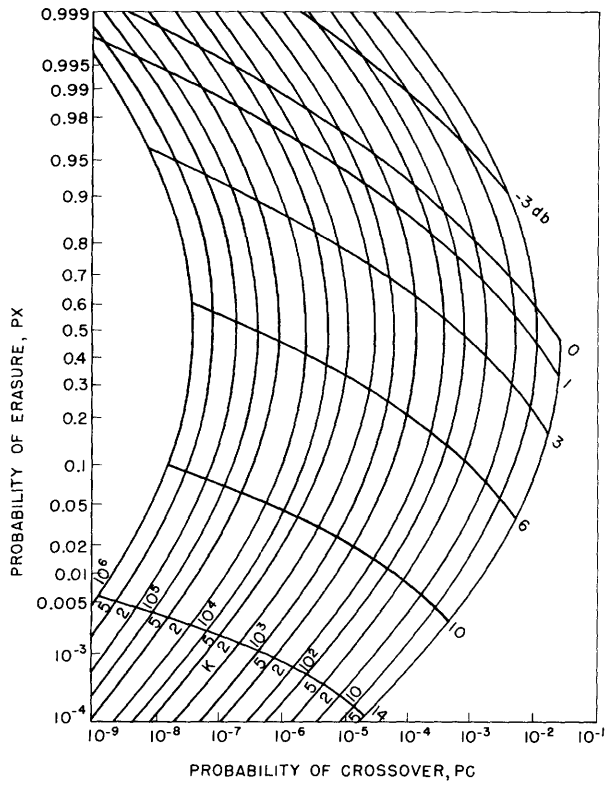


Fig. 11. Channel characteristics, fourth-order diversity.

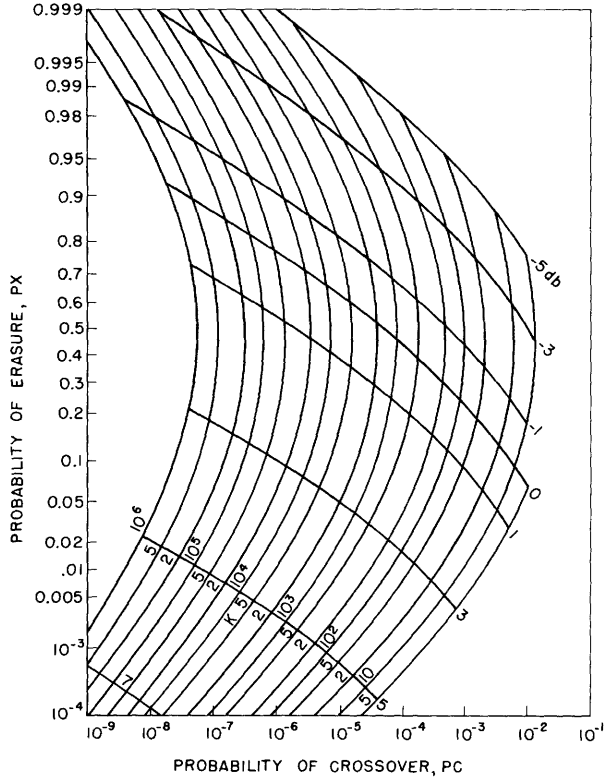


Fig. 12. Channel characteristics, fifteenth-order diversity.

that it is better to use large values of K to avoid high values of p_c ; the reduction in channel capacity is not significant. The maximum capacity obtained by using Figs. 3 and 9-12 refers only to the single erasure zone that has been discussed. It has been shown (7) that the capacity of the channel can be increased further by quantizing the erasure zone.

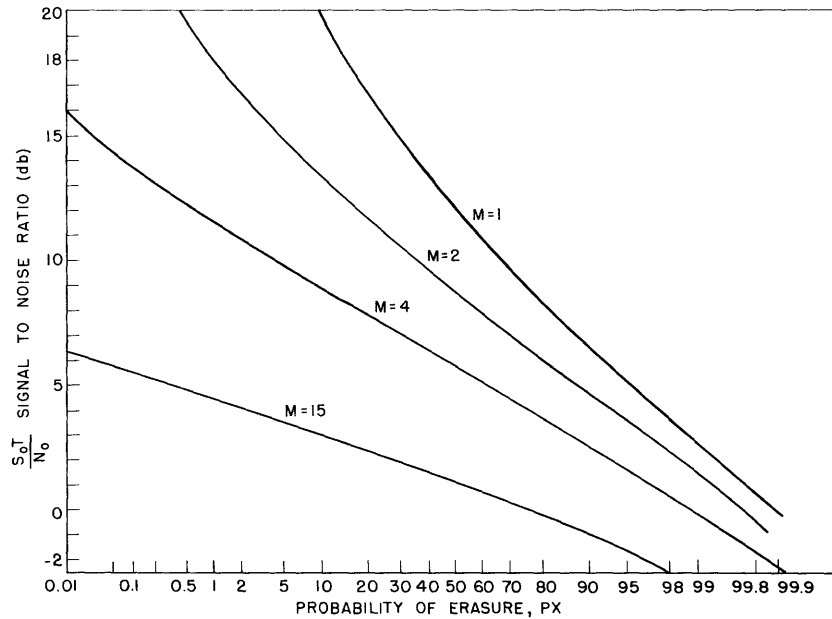


Fig. 13. Requirements for $p_c = 10^{-6}$.

In many situations the value of $S_O T/n_O$ is fixed, and the maximum value of p_c that can be utilized is critical. Thus it is useful to examine the data in a different fashion. In Fig. 13, the data for a crossover probability of 10^{-6} are given. This form of representation requires a separate graph for each value of p_c . From the figure, we can easily see the diminishing gains realized by increasing M and the relative diminishing gains realized by increasing M as p_x is increased. We can determine what percentage of erasures we must restore to obtain the p_c with a given $S_O T/n_O$ and M ; or we can determine what M we need for a given $S_O T/n_O$ and p_x . Similar plots for other values of p_c can be constructed from the data represented by Figs. 9-12.

The number of significant points that can be obtained from Figs. 9-12 will not be discussed here; these points will be seen more readily in specific examples that will be given in Sections III and IV.

III. RESTORING MESSAGE IDENTITY

We have obtained probabilities of the events of a crossover, erasure, and correct reception as functions of $S_o T/n_o$, K , and M . It was observed that for a given M and $S_o T/n_o$ the probability of crossover, p_c , could be varied through a wide range of values by changing the value of K . And, as p_c was made smaller, the probability of erasure, p_x , increased. To restore the identity of the received messages, a means of restoring the identity of the individual erased digits is necessary. In addition, to increase the message reliability beyond $(ID)p_c$ (where ID is the number of digits in a message), crossovers must be corrected.

Restoration of erased digits could be carried out by retransmitting any message that contained one or more erasures. However, if a crossover occurred, it would not be detected unless at least one erasure occurred in the same message. If a crossover did originally occur in a message that was retransmitted, a discrepancy would be noted if the original and the retransmitted sequences were compared. But it would not be known whether the crossover occurred in the original or the retransmitted message; one or more additional transmissions would be necessary to be relatively sure of the value of the digit in question. A more powerful technique of restoring the complete identity of any message is by proper coding at the transmitter and decoding at the receiver.

It has long been known that coding of messages can be used to advantage for increasing the reliability of transmission. It is a fundamental result of Information Theory that if one transmits at a rate that is less than channel capacity, the probability of a message error can be made arbitrarily small at the expense of longer and longer code word lengths (17). Many ingenious coding schemes have been developed for communication channels. However, until recently, most of the coding methods were concerned with binary symmetric channels or binary erasure channels. Elias (9) has proposed a coding technique specifically designed for use with the binary symmetric threshold channel. This coding scheme corrects any specific number of crossovers in a message in addition to restoring large numbers of erasures. It is important, when coding for the BSTC, to be able to restore large numbers of erased digits, but the number of crossovers that must be corrected can be held to a minimum by regulating p_c . Coding is not an absolute solution to the reliability problem; that is, errors can be made. Hence, probabilities are associated with any coding scheme, and the probability that a decoding error will be made is obviously of prime interest to the information user. We shall now evaluate the probabilities of decoding error associated with Elias' scheme.

3.1 Bounds for the Probability of Decoding Error

We start here with the expressions for the probabilities of decoding error derived by Elias. For a detailed explanation of this code, the reader is referred to Elias (9).

For the particular coding scheme that is being examined, the expression for the

probability of decoding error depends upon the number of crossovers one desires to correct. Since the probability of crossover can be controlled quite easily by adjusting the threshold level, it is not necessary to restore large numbers of crossovers. This is fortunate because the amount of computation required by the decoder increases rapidly as more and more crossovers are to be corrected. For present speeds, it seems that correcting for two or three crossovers is about the limit of practicality of this code. Correcting any more than this would require large time delays at the receiver, or extremely fast machines.

The expressions for the bound of decoding error for correcting for 0, 1, 2, and 3 crossovers are as follows:

$$Q_0(k) \leq 2^{-N(1-R)+k} \begin{cases} k \leq k_0 = N(1-R) \\ k \geq k_0 + 1 \end{cases} \leq 1 \quad (30)$$

$$Q_1(k) \leq (N-k-1) 2^{-N(1-R)+k+2} \begin{cases} k \leq k_1 \\ k \geq k_1 + 1 \end{cases} \leq 1 \quad (31)$$

$$Q_2(k) \leq \frac{(N-k-2)(N-k-3)}{2!} 2^{-N(1-R)+k+4} \begin{cases} k \leq k_2 \\ k \geq k_2 + 1 \end{cases} \leq 1 \quad (32)$$

$$Q_3(k) \leq \frac{(N-k-3)(N-k-4)(N-k-5)}{3!} 2^{-N(1-R)+k+6} \begin{cases} k \leq k_3 \\ k \geq k_3 + 1 \end{cases} \leq 1 \quad (33)$$

The subscript on Q indicates how many crossovers are to be corrected in a message of word length N ; R is the rate of transmission of information digits; N is composed of ID information digits and CD check digits, where $ID + CD = N$ and $R = ID/N$; k is the number of erasures occurring in N ; and k_0 , k_1 , k_2 , and k_3 indicate the value of k at which the first bound becomes larger than the second in each expression. This transition point closely corresponds to the situation in which more unknowns than equations exist in the decoder. Since Q is a probability, the second bound in each case is made equal to 1.

The bounds on the probability of decoding error may be used in conjunction with the probabilities of erasure and the probabilities of crossover derived in the previous section, or with corresponding probabilities of some other channel of interest. The results will yield the over-all reliability of the assumed channel when coding is employed. From the data obtained in Section I, the value of signal-to-noise ratio necessary for realizing any desired reliability can be found.

3.2 Message Reliability

To make the probabilities independent of where the erasures or crossovers occur in the message, we shall assume that successive crossovers and erasures are statistically independent. This may seem a contradiction of our previous assumption (Section I) of constant amplitude during a baud, or slow fading relative to baud length. However, we could, in principle, without a great increase in complexity, scramble the digits that are being transmitted so that the digits corresponding to a given message are strung out over a time greater than NT . Then the digits could be placed in their proper position by an unscramble in the receiver. If a few adjacent bauds are faded, the dependence will be lost in the unscramble. We also decrease any correlation that may exist between crossed-over digits that may have been originated by correlated fading.

First, examining the case in which we restore erasures only, we shall obtain a message error if one or more crossovers occurs in the message. This event is bounded by pcN . (If one or more digits is in error, the complete message is in error.) The probability of a decoding error is bounded by taking a binomially weighted average of $Q_0(k)$. The total probability of error is bounded by the sum of these quantities. This yields

$$Q_0 \leq Npc + \left[\sum_{k=1}^{k_0} \binom{N}{k} px^k(1-px)^{N-k} 2^{k-N(1-R)} + \sum_{k=N(1-R)+1}^N \binom{N}{k} px^k(1-px)^{N-k} \right] \quad (34)$$

It can be seen that the results can never be made better than pcN ; thus as word length increases, the probability of crossover must be decreased (by adjusting the threshold) to keep the bounds low. By adjusting the values of pc and px , expression 34 can be dominated by either the first or the second term. It will be shown that the best operating point (that is, pair of pc , px values) occurs when each term of expression 34 contributes an equal amount to the total expression.

In a similar manner, the probability of decoding error for correcting one crossover is obtained. Here, if two or more crossovers occur, a decoding failure results. This event is bounded by $[N(N-1)/2]pc^2$. All of the erased digits must still be restored, which constitutes an event identical to the second term in expression 34; this will be designated β_0 . Now, similarly, we weight $Q_1(k)$ over the values of k and multiply by the probability of one crossover in the remaining $(N-k)$ digits. The final result is the sum of these three separate events, since they are mutually exclusive. This procedure yields

$$Q_1 \leq \frac{N(N-1)pc^2}{2!} + \left[\beta_0 + \sum_{k=0}^{k_1} (N-k)pc \binom{N}{k} px^k(1-px)^{N-k} 2^{-N(1-R)+k+2} \right. \\ \left. + \sum_{k=k_1+1}^N pc(N-k) \binom{N}{k} px^k(1-px)^{N-k} \right] \quad (35)$$

and similarly, for Q_2 and Q_3 ,

$$Q_2 \leq \frac{N(N-1)(N-2)pc^3}{3!} + \left[\beta_1 + \sum_{k=0}^{k_2} \frac{(N-k)(N-k-1)pc^2}{2} \binom{N}{k} px^k (1-px)^{N-k} \frac{(N-k-2)(N-k-3)}{2!} 2^{-N(1-R)+k+4} \right. \\ \left. + \sum_{k=k_2+1}^N \frac{(N-k)(N-k-1)pc^2}{2!} \binom{N}{k} px^k (1-px)^{N-k} \right] \quad (36)$$

$$Q_3 \leq \frac{N(N-1)(N-2)(N-3)pc^4}{4!} + \left[\beta_2 + \sum_{k=0}^{k_3} \frac{(N-k)(N-k-1)(N-k-2)pc^3}{3!} \binom{N}{k} px^k (1-px)^{N-k} \frac{(N-k-3)(N-k-4)(N-k-5)}{3!} 2^{-N(1-R)+k+6} \right. \\ \left. + \sum_{k=k_3+1}^N \frac{(N-k)(N-k-1)(N-k-2)pc^3}{3!} \binom{N}{k} px^k (1-px)^{N-k} \right] \quad (37)$$

The first term in each expression is the probability of 3 and 4 crossovers in word length N , respectively. The other terms are obtained by a direct extension of the case for Q_1 , where β_1 equals the second term of expression 35 and β_2 equals the second term of expression 36.

Expressions 34-37 were evaluated by machine for word lengths of 100, 200, 500, and 1000 digits, and transmission rates of $1/4$, $1/2$, and $1/15$. The results of the computation are shown in Figs. 14-17. In each case, the solid straight lines labeled Q_0 , Q_1 , Q_2 , and Q_3 indicate the lower bounds of the code when correcting 0, 1, 2, 3 crossovers, respectively. The pc scale is used in conjunction with these solid lines. For a desired probability of word in error, we find the maximum allowable channel crossover probability by projecting the intersection of the word error probability and the number of the crossover correction line downward to the pc scale. Any value of pc smaller than that indicated will be satisfactory. The dashed curved lines are used in conjunction with the px scale. Similarly, we project the intersection of the desired word error probability and the line corresponding to the rate of transmission downward to determine what maximum value of px can be used. Thus, we determine the best values of pc and px for a given desired Q , N , and R . The values obtained by the downward projections of the two intersections are clearly the best because they give the maximum pc and the maximum px allowable, which in turn determines the lowest allowable $S_o T/n_o$. Using Figs. 9-12, we can determine what values of $S_o T/n_o$ and K are necessary for a given M and for the conditions given above.

It can be seen that as rate, R , approaches channel capacity (see Fig. 3) Q becomes large, that is, it approaches 1. However, when the word length, N ,

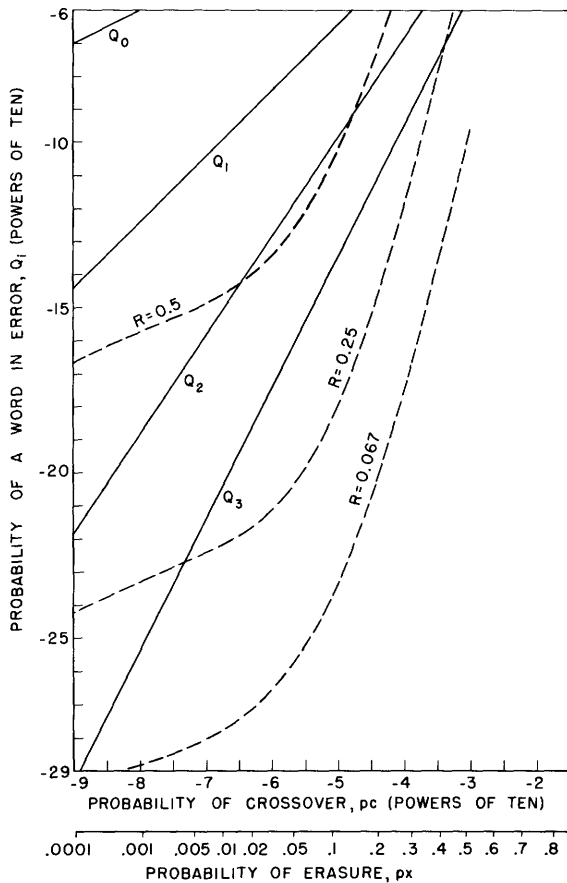


Fig. 14. Probability of decoding error (bounds). $N = 100$.

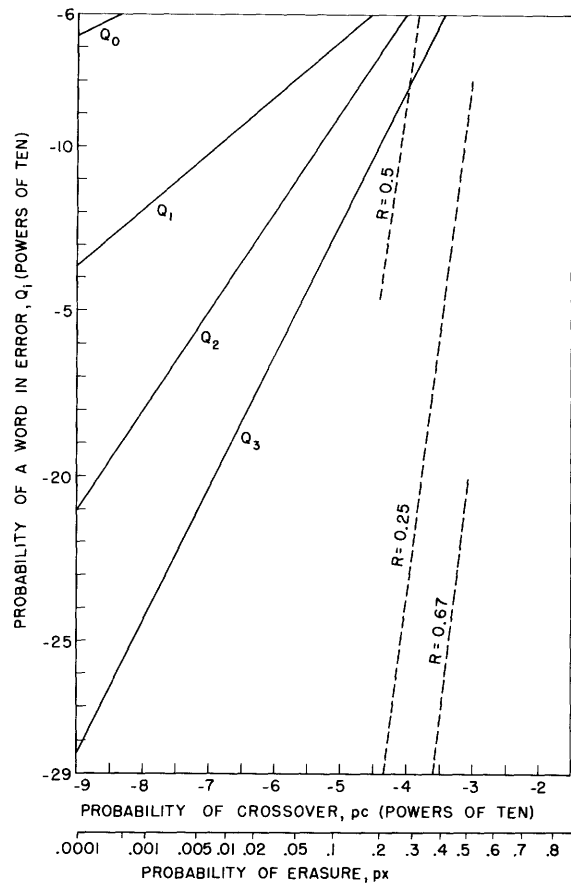


Fig. 15. Probability of decoding error (bounds). $N = 200$.

is increased, we can transmit at a rate closer to channel capacity and still maintain a low Q . We can observe this by holding the rate constant and seeing how channel capacity approaches rate. For example, when transmitting at a rate $R = 0.25$, we need a channel capacity of at least 0.25 to employ the code profitably. Assume that $Q = 10^{-20}$, and examine the channel capacities as N becomes larger; these capacities are approximately equal to $(1 - p_x)$ when p_c is held small. We have $N = 100$, $C = .93$; $N = 200$, $C = .67$; $N = 500$, $C = .48$; and $N = 1000$, $C = .39$; at some larger N , C will be very close to 0.25 (a fundamental result of Information Theory). Conversely, if we transmit at a rate greater than C , the probability of error cannot be made arbitrarily small.

The values of p_x and p_c listed in Figs. 14-17 are general; if a different channel is assumed, they will still apply if the assumptions of statistical independence between crossovers and between erasures are still tolerable.

Let us take a specific example to see what effective power gains can be realized for various combinations of N , R , and M . Assume that we desire a reliability of $(1 - 10^{-12})$ and are willing to decode for the correction of two

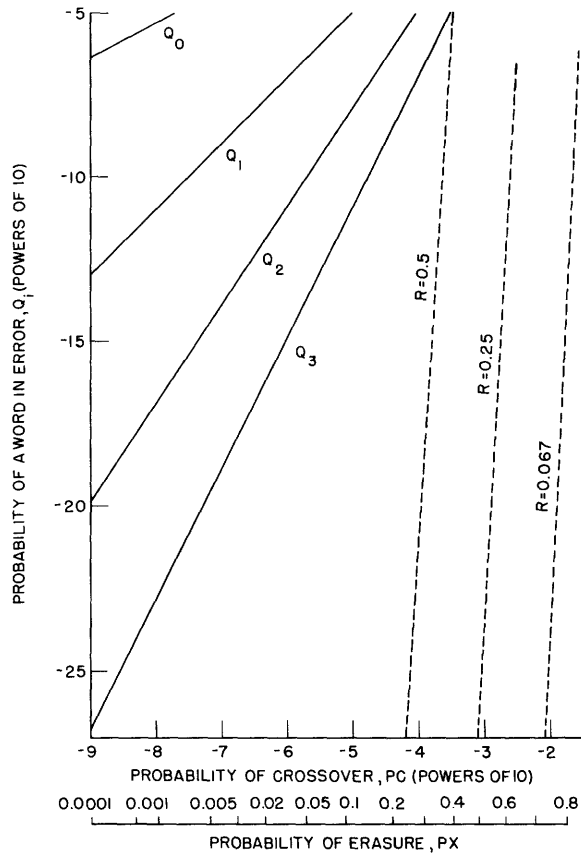


Fig. 16. Probability of decoding error (bounds). $N = 500$.

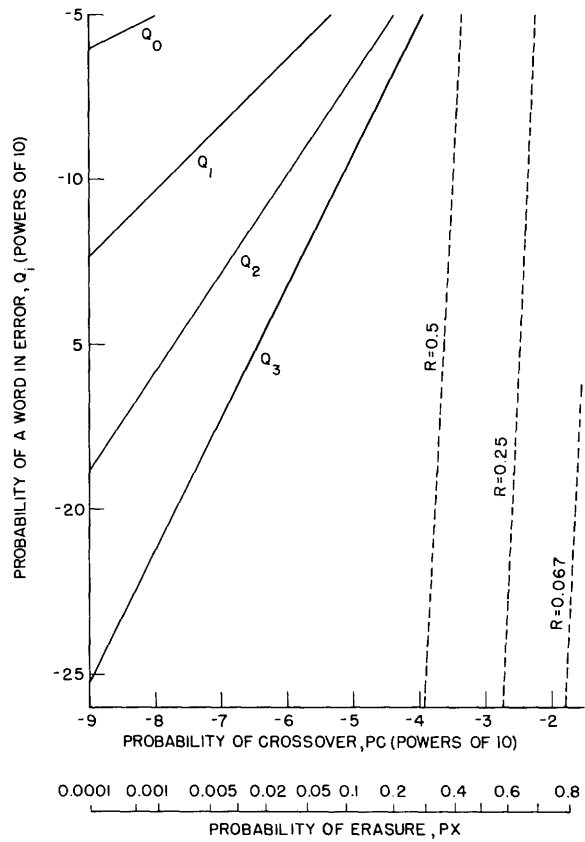


Fig. 17. Probability of decoding error. $N = 1000$.

crossovers. The p_c and p_x values required for each combination of R and N are given in Table I. From the channel statistics of Section I, we can find the $S_0 T/n_0$ necessary at each receiver from the cases stated in Table I.

Table I. Probability requirements for message reliability of $1 - 10^{-12}$.

N \ R	100	200	500	1000
0.5	$p_x = .07$ $p_c = 1.9 \times 10^{-6}$	$p_x = .23$ $p_c = 1 \times 10^{-6}$	$p_x = .35$ $p_c = 4.5 \times 10^{-7}$	$p_x = .38$ $p_c = 1.9 \times 10^{-7}$
0.25	$p_x = .28$ $p_c = 1.9 \times 10^{-6}$	$p_x = .47$ $p_c = 1 \times 10^{-6}$	$p_x = .59$ $p_c = 4.5 \times 10^{-7}$	$p_x = .66$ $p_c = 1.9 \times 10^{-7}$
0.067	$p_x = .45$ $p_c = 1.9 \times 10^{-6}$	$p_x = .6$ $p_c = 1 \times 10^{-6}$	$p_x = .81$ $p_c = 4.5 \times 10^{-7}$	$p_x = .85$ $p_c = 1.9 \times 10^{-7}$

Table II shows what gains in power can be obtained by using more diversity, slower rate, longer word length or combinations of these. It can be seen that increasing word length yields diminishing gains: the gains realized by diversity decrease as rate decreases; and the gains realized by increasing the order of diversity decrease as the orders become high. In Section IV it will be shown that the relative gains over simple transmission increase as the reliability requirements are increased. Also, it will be shown that increasing word length greatly increases reliability when the $S_o T/n_o$ is held constant.

Table II. Necessary $S_o T/n_o$ for reliability of $(1-10^{-12})$.

M	no coding	R = .5				R = .25				R = .067			
		N = 100	200	500	1000	100	200	500	1000	100	200	500	1000
1	120	21	16.4	14.6	14	15.5	12.6	11	10.8	12.8	10.9	8.5	8
2	62.5	14	11.5	10.3	10.1	10.5	9.3	8.3	8	9	8	6.1	6
4	34	9.3	7.7	7	6.9	7.1	6	5.4	5.3	6	5.2	4	3.9
15	13.5	3.2	2.3	2.0	2.0	1.9	1.4	1.1	1	1.2	1	0.1	0.0

A comparison can be made with an existing coding scheme, namely the Reed (18) code. The Reed code will correct seven crossovers in a message when using a word length of 128 and a rate of 0.5 (it is for a BSC, hence erasures are not considered). To obtain a reliability of $(1-10^{-12})$, it is necessary to have a signal-to-noise ratio of 32.5 db at the receiver. Comparing this with erasure coding of $N = 100$ and $R = 0.5$, we find an effective power gain of 11.5 db.

To get a more specific idea of what can be realized by these power gains, an investigation of hypothetical tropospheric scatter links will be carried out in Section IV.

IV. PRACTICAL EMPLOYMENT OF CODING AND DIVERSITY

We have seen what effective power gains can be realized by the employment of coding, diversity, and null-zoning. However, it is to be seen whether it is profitable to utilize these methods when considering a communication link. In most cases, the economic situation determines whether one system should be used in preference to another, but in some applications other requirements such as mobility are the determining factors.

When designating a long-distance communication circuit, it is essential to have available specific details of the expected (desired) performance, inasmuch as these factors will influence the design of the entire link. Thus we might first specify a certain message reliability requirement. We have seen that this reliability may be expressed as a function of the signal-to-noise ratio at the input of the receiver. Therefore, an examination of the signal attenuations and signal gains over the communication channel is necessary. Equipment reliability factors will not be considered. (An equipment down time of approximately 8 hours per year would cut the reliability to less than 10^{-3} .) The reliabilities that are considered here are those that concern the accuracy of the received messages.

4.1 Factors Affecting Tropospheric Communication Links

The hypothetical communication links that will be examined here are commonly called "tropospheric scatter" links. This type of radio transmission utilizes tropospheric paths to communicate over distances greater than line-of-sight ranges. A large amount of experimental data concerning tropospheric transmission is available (19, 20, 21) from which the power gains and losses occurring in the circuit can be calculated.

Some factors affecting signal-to-noise ratios in tropospheric scatter links are transmitter power, antenna gain, receiver noise figure, methods of modulation, and path attenuation. In addition, seasonal variations, changes in bandwidth, climate, rate of transmission, and baud length affect the average $S_o T/n_o$ at the receiver.

If one desires to use only a simple system consisting of one transmitter and one receiver, a limitation exists on the maximum signal-to-noise ratio that can be obtained at the receiver. One factor affecting the signal-to-noise ratio is the maximum power available at frequencies employed by tropospheric scatter. At present, many tens of kilowatts are available at 300 mc, but the maximum power that can be obtained drops off to a few kilowatts at 8000 mc. Also, the power gain obtained from the antennas has its limitations, mainly due to the fact that large accurate antennas are difficult to build, but also because the full theoretical gain of large antennas is not always realized. Another factor limiting the available signal power is the distance between the transmitter and the receiver. As the range is increased, the free-path signal attenuation becomes very large.

Since these restrictions exist, the high signal-to-noise ratios necessary for high reliability cannot normally be obtained in a simple link. Thus, at some point, one must

abandon the simple link and use diversity. If extremely high reliabilities are required, diversity may also be insufficient to obtain the necessary signal power at the receiver. At this point it seems feasible to employ coding. The effective power gains obtained by utilizing diversity and coding can be very useful. The gain could be used to increase the reliability of existing systems or it could be used to decrease the necessary transmitter and antenna sizes, and to increase range or bandwidth.

In some instances, a simple link could be designed to obtain a desired reliability, but it appears that even for relatively short ranges (100-150 miles) reliability or mobility requirements could motivate the employment of threshold coding, or diversity, or both. A few examples will be given to illustrate what may be achieved by employment of these power-saving techniques.

4.2 Antenna Costs

In the examples to be given, the size of the antenna is the main factor that will control the signal level in the receiver. Antennas used in the frequency range of tropospheric

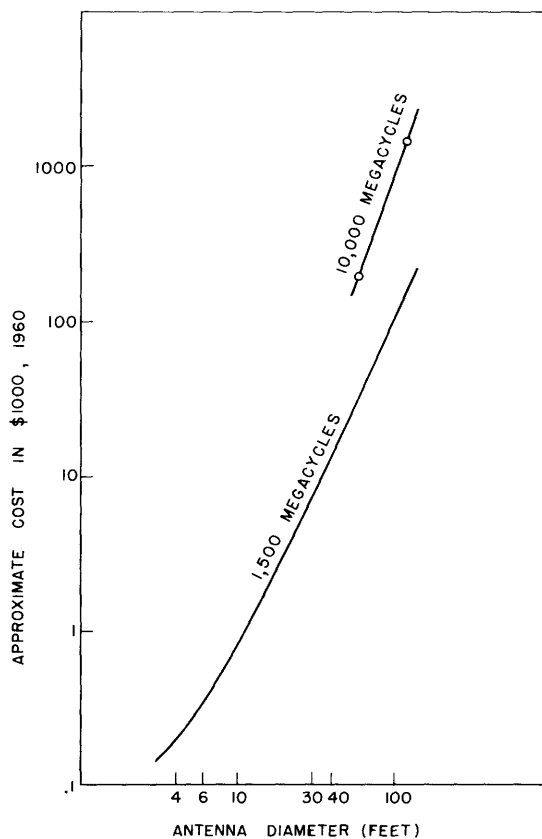


Fig. 18. Approximate antenna costs.

transmission require very accurate surfaces because of the short wavelengths of these high frequencies. As the size of an antenna increases, greater mechanical rigidity is necessary to keep the surfaces accurate. As a result, the cost of antenna construction

increases rapidly with size and frequency.

It is desirable now to discuss the relative costs of antennas, so that this factor may be kept in mind when examining the hypothetical links that will be set up. The antenna costs are indicated in Fig. 18. These figures are, necessarily, approximate because of the many factors affecting the cost estimate. (The estimates shown were obtained from Lincoln Laboratory, M. I. T. through personal communications.) The prices quoted are for antennas operating at approximately 1500 mc with surfaces accurate within approximately 1/2 inch; and for antennas operating at 10,000 mc with surface accuracies of approximately 1/8 inch. It must be remembered that these values are approximate because other factors such as the height above the ground, mobility, etc. also affect costs. It is clear from Fig. 18 that if we can reduce the required size of the antenna, we can, in some instances, save considerable sums of money.

4.3 Hypothetical Communication Circuits

The following links were set up in accordance with a procedure suggested by Altman (21). We shall examine three different cases; in each case a 50-kw transmitter at 1000 mc and a 2-kc bandwidth per 1000 bits per second of transmission are used. The ranges that will be examined are 100, 300, and 600 miles. In the examples, the number of information digits transmitted per second will be held constant. For example, if we send 1000 digits per second in simple transmission, we shall send 2000 per second when using a code at a rate of 1/2, 4000 per second at a rate of 1/4, and so forth. After making these assumptions, the only variables left for controlling the signal-to-noise ratio of the circuit are the antenna size and the signal bandwidth.

When referring to space diversity, we shall assume that only one antenna is being used at the transmitter and one antenna of equal diameter at each receiver. To obtain the total number of antennas that are necessary for two-way communication, we must multiply the order of diversity by two (assuming that one antenna at each end can be used as both a receiving and transmitting antenna). We could reduce the required number of antennas by a factor of two by employing dual feeds that allow polarization and space diversity to be used simultaneously.

The particulars for each link are listed in Table III, and the antenna diameter required for any desired $S_o T/n_o$ is obtained from Fig. 19. These antenna diameters (Fig. 19) are calculated with the assumption of a 2-kc bandwidth; an additional 3 db loss must be added each time the bandwidth is doubled (since the noise is doubled).

Referring to Table III, the total loss consists of the following quantities: free space loss, medium beyond-the-horizon loss, and terminal loss. The power gain is the number of decibels above 1 watt at the transmitter. The difference between the total loss and the sum of the antenna and power gains yields the median carrier level below 1 watt at the receiver. The total antenna gain, which was used in calculating Fig. 19, is for two antennas of equal diameter (one at each end of the link). We also assumed that receiver noise was 9 db. These values, in conjunction with the bandwidth and noise constant,

Table III. Communication circuit data.

Range (miles)	100	300	600
Total loss	217 db	251	283
Power gain [50 kw]	47 db	47	47
Number of antennas	2	2	2
Antenna coupling loss	2 db	2	2
Noise in db below 1 watt	-162 db	-162	-162
Frequency	1000 mc	1000 mc	1000 mc
(Noise constant figured at 293°K)			

yield the median noise level below one watt at the receiver. We only need to determine, from the results of Sections I and II, what $S_o T/n_o$ is necessary for a given reliability. Use of this value with Fig. 19 specifies the necessary size of the antenna for the range in question. It must be remembered that 3 db must be added to the required $S_o T/n_o$ every time the bandwidth is doubled. Throughout this report, we have used a normalized signal-to-noise ratio. Therefore, under some circumstances, a correction factor must

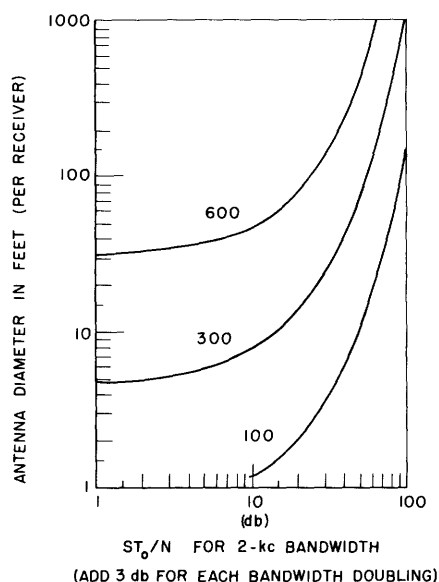


Fig. 19. Required antenna diameter for a given $S_o T/n_o$.

be used when the baud length is varied. In these examples, the correction has already been made, but in other calculations care must be taken to avoid errors.

A number of different situations for reliabilities of $(1-10^{-9})$ and $(1-10^{-18})$ are illustrated in Table IV. Listed are $S_o T/n_o$, antenna size, antenna number, and bandwidth for each range under consideration.

A number of qualitative conclusions can be drawn from these examples. It is apparent that, under the given assumptions, coding and diversity allows one to obtain higher

Table IV. Antenna requirements for tropospheric links.

Situation	Probability of Error	Required $S_o T/n_o$ (db)	Range (miles)	Antenna Diameter (ft)	No. of Antennas	Bandwidth (kc)
Simple Transmission	10^{-9}	90	100	120	2	2
			300	790	2	2
			600	1000+	2	2
Dual Space Diversity No Coding	10^{-9}	47.5	100	9.5	4	2
			300	65	4	2
			600	400	4	2
Erasure Coding N = 100 R = .25, Q_2 , No Diversity	10^{-9}	12.6	100	1.9	2	8
			300	13	2	8
			600	75	2	8
Erasure Coding N = 100 R = .25, Q_2 , Quad. Diversity	10^{-9}	5.6	100	1.3	8	8
			300	8.5	8	8
			600	50	8	8
Erasure Coding N = 500 R = .5, Q_2 , No Diversity	10^{-9}	13.5	100	1.7	2	4
			300	12	2	4
			600	66	2	4
Erasure Coding N = 500 R = .5, Q_1 , No Diversity	10^{-9}	14.9	100	1.9	2	4
			300	12.8	2	4
			600	74	2	4
Erasure Coding N = 1000 R = .25, Q_2 , Dual Diversity	10^{-9}	6.2	100	1.35	4	8
			300	9	4	8
			600	52	4	8
REED Coding N = 128 R = .5, No Diversity	10^{-9}	27.2	100	3.1	2	4
			300	21	2	4
			600	132	2	4
Erasure Coding N = 100 R = .5, Q_2 , No Diversity	10^{-9}	17	100	2.1	2	4
			300	14.1	2	4
			600	84	2	4
Simple Transmission	10^{-18}	180	100	1000+	2	2
			300	1000+	2	2
			600	1000+	2	2
Dual Space Diversity No Coding	10^{-18}	92.5	100	130	4	2
			300	900	4	2
			600	1000+	4	2
Erasure Coding N = 100 R = .25, Q_2 , No Diversity	10^{-18}	20.5	100	2.9	2	8
			300	15	2	8
			600	94	2	8
Erasure Coding N = 100 R = .25, Q_2 , Quad. Diversity	10^{-18}	10	100	1.7	8	8
			300	11.5	8	8
			600	64	8	8
Erasure Coding N = 500 R = .5, Q_2 , No Diversity	10^{-18}	17	100	2.1	2	4
			300	14.1	2	4
			600	64	2	4
Erasure Coding N = 1000 R = .25, Q_2 , Dual Diversity	10^{-18}	9.6	100	1.6	4	8
			300	11	4	8
			600	62	4	8
REED Coding N = 128 R = .5, No Diversity	10^{-18}	37.5	100	5.1	2	4
			300	47	2	4
			600	210	2	4

reliabilities than could be obtained by simple transmission of diversity without coding. A reliability of $(1-10^{-9})$ can easily be obtained at short ranges with dual diversity. However, when the reliability requirement or the range is increased, diversity without coding becomes inadequate. It can be seen that at constant reliability, increasing code-word length does not decrease the antenna requirements (see Table II), except at the 600-mile range. On the other hand, with constant signal level, increasing word length greatly increases the reliability, for example, increasing word length from 100 to 500 digits at rate $1/2$ decreases the probability of error from 10^{-9} to 10^{-18} , with a constant $S_o T/n_o$ of 17 db. Also, it can be seen that correcting for two crossovers instead of for one in the decoding process does not appreciably change the requirements. However, if greater reliabilities are required, the difference may become significant. Another interesting point is that employing diversity and coding simultaneously only seems profitable at very long ranges. Finally, the gain realized over that of the Reed code also becomes significant at long ranges.

From the information in Tables III and IV, we can obtain an approximate idea of what relative savings could be made by employing the power-saving techniques. In many instances, it appears that the reduction in antenna costs could be great enough to warrant the use of coding and decoding equipment. Each specific example can be examined to see if, under the given assumptions, it would be profitable to use a coded system.

Obviously, this is not the complete story. But we hope that these special examples and the examples given in Section II show that a combination of null-zone detecting, coding, and diversity is powerful and practical. The examples illustrate the general conclusions; specific conclusions must be drawn from specific cases of interest.

Little has been done in comparing erasure coding for the BSTC with other existing coding schemes. As an extension to this investigation, it may be interesting to examine other existing coding schemes to see if they could be adapted for use with the BSTC, and to see what improvements in reliability can be made through the generation of erasures.

APPENDIX

DERIVATION OF CHANNEL PROBABILITIES

To obtain the probability of a correct reception, q , the integration indicated by Eq. 21 must be carried out. This is done with the aid of the following relation.

$$\int x^m \exp(ax) dx = \frac{x^m \exp(ax)}{a} - \frac{m}{a} \int x^{m-1} \exp(ax) dx \quad (A-1)$$

Thus substituting Eqs. 19 and 20 in Eq. 21 and using Eq. A-1, we obtain

$$\begin{aligned} q &= \int_0^\infty \frac{X^{M-1}}{(M-1)!} \exp(-X) dX \int_{X+\sigma}^\infty \frac{Y^{M-1}}{(1+S_o T/n_o)^M (M-1)!} \exp\left(-\frac{Y}{1+S_o T/n_o}\right) dY \\ &= \int_0^\infty \frac{X^{M-1}}{(M-1)!} \exp(-X) dX \exp\left(-\frac{X+\sigma}{1+S_o T/n_o}\right) \sum_{m=0}^{M-1} \frac{(X+\sigma)^m (1+S_o T/n_o)^{M-m}}{(1+S_o T/n_o)^M m!} \\ &= \exp\left(-\frac{\sigma}{1+S_o T/n_o}\right) \sum_{m=0}^{M-1} \int_0^\infty \frac{X^{M-1} (X+\sigma)^m}{(M-1)! m! (1+S_o T/n_o)^m} \exp\left(-X \frac{2+S_o T/n_o}{1+S_o T/n_o}\right) dX \\ &= \frac{\exp\left(-\frac{\sigma}{1+S_o T/n_o}\right)}{(M-1)!} \sum_{m=0}^{M-1} \int_0^\infty \frac{X^{M-1}}{m (1+S_o T/n_o)^m} \sum_{\ell=0}^m \binom{m}{\ell} X^\ell \sigma^{m-\ell} \exp\left(-X \frac{2+S_o T/n_o}{1+S_o T/n_o}\right) dX \\ &= \frac{\exp\left(-\frac{\sigma}{1+S_o T/n_o}\right)}{(M-1)!} \sum_{m=0}^{M-1} \frac{1}{m! (1+S_o T/n_o)^m} \sum_{\ell=0}^m \binom{m}{\ell} \sigma^{m-\ell} \int_0^\infty X^{M-1+\ell} \exp\left(-X \frac{2+S_o T/n_o}{1+S_o T/n_o}\right) dX \\ &= \frac{\exp\left(\frac{-\sigma}{1+S_o T/n_o}\right)}{(M-1)!} \sum_{m=0}^{M-1} \frac{1}{m! (1+S_o T/n_o)^m} \sum_{\ell=0}^m \binom{m}{\ell} \sigma^{m-\ell} \left[\left(\frac{1+S_o T/n_o}{2+S_o T/n_o} \right)^{M+\ell} (M-1+\ell)! \right] \\ &= \frac{\exp\left(\frac{-\sigma}{1+S_o T/n_o}\right)}{(M-1)!} \sum_{m=0}^{M-1} \frac{1}{m! (1+S_o T/n_o)^{m-M}} \sum_{\ell=0}^m \binom{m}{\ell} \sigma^{m-\ell} \frac{(1+S_o T/n_o)^\ell}{(2+S_o T/n_o)^{M+\ell}} (M-1+\ell)! \end{aligned}$$

Substituting $[(n_o + S_o T)/S_o T] \ln K$ for σ yields Eq. 24.

Similarly for pc, we have

$$\begin{aligned}
pc &= \int_0^\infty \frac{Y^{M-1} dY}{(1+S_o T/n_o)^M (M-1)!} \exp\left(-\frac{Y}{1+\frac{S_o T}{n_o}}\right) \int_{Y+\sigma}^\infty \frac{X^{M-1}}{(M-1)!} \exp(-X) dX \\
&= \int_0^\infty \frac{Y^{M-1}}{(1+S_o T/n_o)^M (M-1)!} \exp\left(-\frac{Y}{1+\frac{S_o T}{n_o}}\right) dY \exp(-Y-\sigma) \sum_{m=0}^{M-1} \frac{(Y+\sigma)^m}{m!} \\
&= \exp(-\sigma) \frac{1}{(1+S_o T/n_o)^M (M-1)!} \sum_{m=0}^{M-1} \int_0^\infty \frac{X^{M-1}}{m!} \sum_{\ell=0}^m Y^\ell \sigma^{m-\ell} \binom{m}{\ell} \exp\left[-Y \frac{2+\frac{S_o T}{n_o}}{1+\frac{S_o T}{n_o}}\right] dY \\
&= \exp(-\sigma) \frac{1}{(1+S_o T/n_o)^M (M-1)!} \sum_{m=1}^{M-1} \frac{1}{m!} \sum_{\ell=0}^m \binom{m}{\ell} \int_0^\infty \sigma^{m-\ell} Y^{M-1+\ell} \exp\left[-Y \frac{2+\frac{S_o T}{n_o}}{1+\frac{S_o T}{n_o}}\right] dY \\
&= \exp(-\sigma) \frac{1}{(1+S_o T/n_o)^M (M-1)!} \sum_{m=0}^{M-1} \frac{1}{m!} \sum_{\ell=0}^m \binom{m}{\ell} \sigma^{m-\ell} \left(\frac{1+\frac{S_o T}{n_o}}{2+\frac{S_o T}{n_o}}\right)^{M+\ell} (M-1+\ell)! \\
&= \exp(-\sigma) \frac{1}{(2+S_o T/n_o)^M (M-1)!} \sum_{m=0}^{M-1} \frac{1}{m!} \sum_{\ell=0}^m \binom{m}{\ell} \sigma^{m-\ell} \left(\frac{1+S_o T/n_o}{2+S_o T/n_o}\right)^\ell (M-1+\ell)!
\end{aligned}$$

Substituting $[(n_o + S_o T)/S_o T] \ln K$ for σ yields Eq. 25.

Px is obtained from the relation $px + pc + q = 1$.

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