STRESSES AND DISPLACEMENTS IN VISCOELASTIC BODIES

by

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ABSTRACT

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In most published works on viscoelastic stress analysis the constitutive equations of the materials are expressed in linear differential operator forms. However, due to the mathematical complexity which arises when a realistic number of terms are used to properly characterize the material, these analyses have generally been limited to either short time intervals or unrealistic material representations. To overcome this difficulty, a more general method of representation for the constitutive equations of linear viscoelastic materials is achieved through the use of the hereditary integrals. Use of such constitutive equations permits an easy formulation of the time dependent expressions in the form of integral equations involving multiple convolution integrals which involve all the time dependent variables. The evaluation of these convolution integrals and the numerical solution of the integral equations then provides the response of the materials over broad time intervals.

Two techniques are presented for evaluating the multiple convolution integrals. The first involves numerical integration, while the second is an exact integration which is valid for material functions that can be represented by Dirichlet series. The technique for the numerical solution of the total integral equation is presented and illustrated.

Two examples are presented to illustrate this method of analysis. The first is the deflection of a viscoelastic cantilever beam. The results of this analysis are compared with a certain exact solution. The second example is the analysis of the stresses and displacements in a three-layer viscoelastic half-space.

The elastic solution is derived in an acceptable form, and then the corresponding viscoelastic solution is presented. Numerical results are presented, obtained by both techniques, and are compared.

Certain types of non-linear viscoelasticity are reviewed and considered with respect to the possibility of extending the above techniques to these problems. Ageing effects, thermoviscoelasticity, geometrical non-linearities, and material non-linearities are considered. As an illustration of a technique for solving a certain class of non-linear problem, the deflection of a linear viscoelastic plate on a non-linear viscoelastic foundation is analysed, and numerical results are presented.

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CONTENTS

LIST OF SYMBOLS

σ_{ij}	stress acting in x direction on a plane normal to x direction
F_i	body force acting in the x direction
u_i	displacement component in the x direction
ρ	density
t	time
x_i	cartesion coordinate direction
ε_{jj}	extensional strain
ε_{jk}	shear strain
e_{jk}	components of finite strain
K	bulk or volumetric elastic modulus
G	elastic shear modulus
σ	volumetric stress component
e	volumetric strain component or base of natural logarithms
S_{ij}	deviatoric stress component
δ_{ij}	Kronecker delta function
μ	Poisson's ratio
E	Young's Modulus
$E_r(t)$	relaxation function analogous to Young's modulus
$D_c(t)$	creep compliance function analogous to 1/E
$\mathcal{L}\{f(t)\}$	Laplace transform of f(t)
s	Laplace transform parameter

$E(s)$	transform equivalent of Young's modulus
$G(s)$	transform equivalent of shear modulus
$\mu(s)$	transform equivalent of Poisson's ratio
$K(s)$	transform equivalent of bulk modulus
a_ℓ, b_ℓ	constants
η	dashpot viscosity
τ	relaxation time
$L(\tau)$	retardation spectrum
$H(\tau)$	relaxation spectrum
$\overline{E(\omega)}$	complex modulus
ω	frequency
$E'(\omega)$	real part of complex modulus
$E''(\omega)$	imaginary part of complex modulus
$\overline{D(\omega)}$	complex compliance
$D'(\omega)$	real part of complex compliance
$D''(\omega)$	imaginary part of complex compliance
$\psi(t)$	stress, strain, or displacement
ϕ_i, θ_i	constants with respect to time
α_i, β_i	products of elastic constants
$f_i(t)$	known functions of time
$\nu(t)$	symbol equivalent to $\psi(t)$ or $f_i(t)$
γ_j	elastic constants
$\gamma_j(t)$	relaxation or creep function equivalent to γ_j constant
$\gamma(\tau)$	viscoelastic equivalent to $\prod_{j=1}^{k} \gamma_j$

$\beta_i(t)$, $\alpha_i(t)$	viscoelastic equivalent (multiple convolution integrals) of β_i and α_i terms
$P(t)$	time varying load intensity
$I_j(\ )$	result of j th convolution integration
G_i^j	constant for j th term in Dirichlet series for i th creep or relaxation function
δ^j	inverse of j th relaxation or retardation time
${}_kB_i^j$	constant in multiple convolution integration result for k th integration, i th term in the polynomial multiplying the j th term in the series of exponentials
$\rho(t)$	result of summing m solutions $\phi_i \cdot \beta_i(t)$ of exact multiple convolution integrations
C_ℓ^j	constant in result $\rho(t)$ for ℓ th term in polynomial multiplying j th term in series of exponentials
I	moment of inertia
c_1	depth of cantilever beam
ℓ	length of cantilever beam
h	thickness of the second half-space layer
A, B, C, D	constants in three-layer half-space solution
w	vertical deflection
$J_N(\ )$	Bessel function of Nth order
m	dummy integration variable
$H(t)$	Heaviside step function
r, z, θ	cylindrical coordinates
q	intensity of distributed surface loading
a	radius of circular loaded area

A_i, λ_m^k $B_i, q_{j,k,i}$ C_i $g_j, \phi_{j,k,i}$	constants in three-layer half-space solution
H	thickness of first layer of half-space
$\psi_{k,i}(m,t)$	part of solution for stress or displacement for i th layer for a particular value of m, at time t.
∇^2	Laplacian operator in cylindrical coordinates
$\sigma_r, \sigma_\theta, \sigma_z$ $\tau_{rz}, \tau_{r\theta}, \tau_{\theta z}$	stress components in cylindrical coordinates
u	radial displacement
ϕ	stress function
$\Theta_1(m)$	$J_0(mr)J_1(ma)$
$\Theta_2(m)$	$J_1(mr)\ J_1(ma)$
ξ	reduced time t/a(T)
$a(T)$	experimentally determined shift factor for thermoviscoelasticity
T_o	reference temperature
T	temperature
$\alpha(T)$	temperature dependent coefficient of thermal expansion
$G_j(t-\tau_1, t-\tau_2, \cdots t-t_j)$	kernel functions in multiple integral representation of non-linear viscoelastic constitutive equations
a_{ij}	flexibility coefficient ÷ E for node i with respect to node j
$g_i(t)$	result of two-fold convolution integration of K(t), D(t), and $f(w_i(t))$ in the non-linear problem of Chapter VII

CHAPTER I

INTRODUCTION

An essential part of the rational analysis and design of engineering structures is the analysis of the critical stresses and displacements that the structure is subjected to during its useful life. Except in a few very specialized areas, the totality of such analysis and design is done, in the field of solid mechanics, utilizing the assumption that the materials of concern are linearly elastic. This has resulted in a great amount of literature on such analysis, with "closed" or analytic solutions having been formed for many classical problems.

Although some engineering materials, within a certain range of stress and strain, are indeed governed by constitutive equations which are essentially linear elastic, many new materials (such as polymers) are becoming available having time dependent stress-strain behaviors. In addition, many materials such as Portland cement concrete are now recognized to be decidedly time-dependent. Further examples of materials showing appreciable time-dependency are metals at high temperature, and bituminous concretes. Those materials, where the stress and strain tensors are related through integral or differential relationships with respect to time, are termed viscoelastic, and if these relationships are linear then the materials are termed linear viscoelastic.

The analysis of stresses and deformations in such linear viscoelastic bodies is receiving increased attention. In the past fifteen years this attention has resulted in the solution of some problems of practical significance, but the number of available analyses is very small compared to that of elasticity analyses. However, techniques are now emerging which are applicable to a great variety of problems.

It is the purpose of this work to present and to demonstrate a straight-forward means of analysis for viscoelastic materials which can be applied to a large number of practical problems. The method to be explained and illustrated in the following sections is applicable to analysis using realistic material properties, and is an efficient way to carry out such analysis.

The method employs a formulation of the viscoelastic solution in terms of integral equations involving multiple convolution integrals of the relevant relaxation functions, using the correspondence between elastic and viscoelastic problems. Two different techniques are presented for evaluating the multiple convolution integrals, and then solving the integral equations numerically. Both techniques are illustrated on an arbitrary integral equation of the proper form, and on two example problems.

The first of these examples, the deflection of a viscoelastic cantilever beam, is presented only to illustrate the techniques and their use. The second example, the analysis of a three-layer half-space, is of engineering significance in the analysis of foundations and flexible pavements, and is thus presented in detail.

A discussion on non-linear problems is presented in Chapter VII. Various sources of non-linearity are considered, and potential methods for solving these types of problems (compatible with the method of analysis presented previously) are discussed. A particular form of material non-linearity theorized by several authors in the literature is discussed, and the problem of an infinite linear viscoelastic plate on a non-linear viscoelastic (Winkler) base is solved as an illustration of the correspondence between elastic and viscoelastic problems when this theory is applicable.

CHAPTER II

SURVEY OF LITERATURE ON THE ANALYSIS OF STRESSES AND DISPLACEMENTS IN LINEAR VISCOELASTIC BODIES

In this section, a brief survey of the literature related to the analysis of stresses and displacements in linear viscoelastic bodies is presented, with emphasis on the analysis of viscoelastic half-spaces as is used in Chapter VI as an example.

The difference between elastic and viscoelastic bodies is essentially that an elastic body has a constant ratio between stress and strain, whereas a viscoelastic body has a stress-strain relationship which allows for time effects. Alfrey [5] *, using the fact that some of the equations of elasticity (the equilibrium and strain-displacement equations) are unchanged for a viscoelastic body, formulated the "correspondence principle" for incompressible viscoelastic bodies in 1944. Tsien [131] generalized Alfrey's principle in 1950 to include bodies with the same time characteristics in shear and dilation, and then Lee [73] extended, in 1955, the "correspondence principle" so that it included any linear viscoelastic body. The essence of this principle is that if the

*Numbers in brackets refer to the list of references in the Appendix.

equations of viscoelasticity (equilibrium, stress-strain, strain-displacement and the boundary conditions) are transformed from the time domain to the Laplace domain through the application of the Laplace transform, the partial differential equations with respect to the variable time will be transformed into algebraic equations in the variable s (Laplace parameter) which are in the same form as an associated elastic solution. If this elastic solution can be solved, the inversion of this result through the use of the inverse Laplace transform will yield the time-varying solution. This method is applicable to all problems in which 1) the Laplace transform of all the time-varying equations exists, 2) the associated elastic problem can be solved, and 3) the associated elastic solution can be inverted to the time domain.

Most of the published works on viscoelastic stress and displacement analysis have treated problems which have been handled by the Laplace transform method, and which utilized simple discrete models of springs and dashpots in series and/or parallel to characterize the viscoelastic material behavior. Because of the mathematical complexity which arises when a large number of such spring and dashpot elements are used, only very simple discrete models, composed of from two to five elements, have been used. This type of an approach is able to predict the behavior of real materials accurately only over very short time intervals, and consequently little is known of the responses over long time intervals. However, these analyses do provide some qualitative information on such behavior.

Examples of this type of analyses are numerous: Lee illustrated the basic idea in his paper of 1955 with the solution for a fixed and moving point load on a viscoelastic halfspace which was assumed to behave as a Voigt model in shear, and to behave elastically in hydrostatic tension or compression. In 1961 Pister [98] presented the solution for a viscoelastic plate on a viscoelastic foundation under a uniform circular load where both the plate and the foundation are assumed to behave as incompressible Maxwell materials. In 1962 Pister and Westman [100] used a three-element model to characterize the behavior of a beam on a Winkler foundation, and analysed this for a moving point load. Radok [101] presented a solution in 1957 for a ring of time-varying thickness under an internal pressure in which he assumed that the rings were characterized as an elastic Voigt model. Kraft [61] presented an analysis of the deflection of a two-layer half-space system in 1965 in which the layers were each composed of three-element models, and the volumetric behavior was assumed to be elastic. The applicability of analyses using discrete models has been discussed further by Arnold, Lee, and Panarelli [9] in 1965.

One of the principle problems met when applying the Laplace transform approach is finding the inverse Laplace transform. Schapery [110] has devised and presented some important numerical means that can sometimes be used to facilitate this inversion. Cost and Becker [26] have presented another numerical technique, and compared its accuracy to the Schapery techniques.

An alternative approach to the problem was suggested by Lee and Rogers [72] in 1963, using measured creep or relaxation functions in the form of hereditary integrals for the viscoelastic stress-strain relationships. This method results in integral equations which may be solved numerically. In the paper written by Lee and Rogers, a numerical technique originally suggested by Hopkins and Hamming [54] in 1957 was utilized successfully on their fairly specialized example.

A few results are available using the hereditary form of the stress-strain equations. These have generally covered simple problems, and have been comp...

to the use of the discrete models. Examples of such papers are: Rogers and Lee [107] in 1962 on the finite deflection of a viscoelastic cantilever; Baltrukonis and Vaishnar in 1965 [13] on the creep-bending of a beam column; Huang, Lee, and Rogers [56] in 1965 on the influence of viscoelastic compressibility on a pressurized cylinder; and Anderson [6] in 1965 on the buckling of viscoelastic arches.

In spite of the predominance the discrete models have enjoyed in the literature, the desirability of obtaining solutions over broad time ranges which realistically represent real material properties seems to imply that the more general hereditary forms will have increased use in the future. The alternative to this approach seems to be the use of the spectral representation (an infinite sum of discrete models) for the stress-strain relations. This approach has been summarized nicely by Williams [139] in 1964, and numerical techniques for its application have been discussed by Schapery [110] in 1962.

Several very good survey papers on linear viscoelasticity are available, notably the monograph by Bland [18], and the papers by Williams [139], Hilton [51], and Rogers [106]. In addition, Gurtin and Sternberg [40] have presented a rigorous development of the theory which supplies proof of a large number of theorems normally assumed on a physical basis.

CHAPTER III

STRESS AND DEFORMATION ANALYSIS OF VISCOELASTIC MATERIALS

In the analysis of the stresses, strains, and displacements of a body subject to external forces and displacements, three distinct sets of equations may be formulated in terms of the stresses, strains, and displacements. The solution of these equations which also satisfies the boundary conditions of the problem at hand yields the desired stresses and deformations. The sets of equations necessary are the equilibrium equations, the strain-displacement equations, and the constitutive equations. These will be discussed individually, and then the practical solution of problems formulated with these equations will be discussed.

III-1. Equilibrium Equations

These are dynamical equations, which state the equality of Newton's Second Law f = ma in terms of the stresses and body forces acting on any infinitesimal element of a body. Equations (1) give the equilibrium equations of forces for a body with no couple stresses acting (so that $\sigma_{ij} = \sigma_{ji}$ from the equations of moment equilibrium of an element) in cartesian coordinates, using the conventional indicial notation:

$$\frac{\partial \sigma_{ij}}{\partial X_j} + F_i = \rho \frac{\partial^2 U_i}{\partial t^2} \qquad (1)$$

In these equations σ_{ij} is the stress acting in the X_j direction on a plane, passing through the point, normal to the X_i direction; F_i is the body force acting in the X_i direction; ρ is the density of the material; and U_i is the displacement component in the X_i direction. There are six unknown components of stress and three unknown displacements in these three equations.

III-2. Strain-Displacement Equations

These are kinematic relationships between strains and displacements. They express necessary relationships in order that a set of strains may yield a set of displacements and still preserve the continuity of the body. Letting e_{ij} be the component of finite strain such that the extensional strain in the X_j direction is given as:

$$\varepsilon_{jj} = \sqrt{1 + e_{jj}} - 1 \qquad (2)$$

and the change in angle between the X_j and X_k directions is given as:

$$\varepsilon_{jk} = \sin^{-1}\left(\frac{e_{jk}}{\sqrt{1+e_{jj}}\ \sqrt{1+e_{kk}}}\right) \qquad (3)$$

Then the six strain-displacement equations are given as:

$$e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} \tag{4}$$

This expression represents six equations in six unknown components of strain and three unknown components of displacement. These equations can be simplified somewhat by making certain assumptions such as neglecting the non-linear terms when the strains and rotations are small.

III-3. Constitutive Equations for an Elastic Body

The constitutive equations are the mechanical equations of state for the body. They can be stated in quite general form:

$$\mathcal{E}_{ij} = f\ (\text{stresses, other strains, time, temperature, geometry}) \tag{5}$$

That is, strain is a function of the stresses, the other components of strain, time, temperature, and geometry. In infinitesimal linear elasticity, the contributions to the functional relationship of the other strains, of time, and of temperature variables are disregarded. The assumption of a homogeneous body reduces the relationship to one involving only the stresses, that is:

$$\mathcal{E}_{ij} = f(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{13}, \sigma_{23}, \sigma_{12}) \tag{6}$$

Two further simplifying assumptions are also often made. The first is that the strains are <u>linear</u> functions of the stresses, and the second is that the material is isotropic (i.e., the properties at any point do not depend upon direction). With these two assumptions, the constitutive equations of linear elasticity for an isotropic, homogeneous body can be stated as in equations (7) and (8).

$$\sigma = 3Ke \tag{7}$$

$$S_{ij} = 2G\epsilon_{ij} \tag{8}$$

In these equations, K is the elastic bulk modulus, G is the elastic shear modulus, and σ, e, S_{ij}, and ϵ_{ij} are given by the following relationships:

$$\sigma = \text{volumetric stress} = \sigma_{11} + \sigma_{22} + \sigma_{33} \tag{9}$$

$$e = \text{volumetric strain} = \mathcal{E}_{11} + \mathcal{E}_{22} + \mathcal{E}_{33} \tag{10}$$

$$S_{ij} = \text{deviatoric stress} = \sigma_{ij} - \frac{\sigma}{3}\delta_{ij} \tag{11}$$

$$\epsilon_{ij} = \text{deviatoric strain} = \mathcal{E}_{ij} - \frac{e}{3}\delta_{ij} \tag{12}$$

where

δ_{ij} is the Kronecker delta function:

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \tag{13}$$

The constitutive equations of a linearly elastic body are also often given in terms of Young's modulus E and Poisson's ratio μ . These constants are related to G and K through the relationships given in equations (14) and (15).

$$\mu = \frac{3K - 2G}{2G + 6K} \tag{14}$$

$$E = \frac{9KG}{3K + G} \tag{15}$$

III-4. Constitutive Relations for a Viscoelastic Body

The constitutive equations for a viscoelastic body, in addition to being a function of the variables considered for an elastic body, are also a function of time. There are several ways in which these relationships can be written, which may be shown to be interrelated [39,139] . In the following discussion, only linear viscoelastic constitutive equations will be considered. It should also be pointed out that since temperature does not enter into the constitutive relations, the implicit assumption has been made that there is no variation in properties with temperature, or else isothermal conditions exist. More general constitutive equations will be discussed in Chapter VII.

III-4.1. Hereditary Integral Form

The first form for a viscoelastic constitutive equation to be considered here is the hereditary integral form. Consider a uniaxial relaxation test on a specimen, where $\sigma_{ij}(t)$ is measured for a constant strain $\varepsilon_{ij}(0)$. Then, for this test, a relaxation function can be defined as

$$E_r(t) = \frac{\sigma_{ij}(t)}{\varepsilon_{ij}(0)} \tag{16}$$

Similarly for a creep test, $\varepsilon_{ij}(t)$ could be measured for a constant stress $\sigma_{ij}(0)$, and the creep compliance function is then defined as

$$D_r(t) = \frac{\varepsilon_{ij}(t)}{\sigma_{ij}(0)} \tag{17}$$

Consider now an applied strain which is composed of n pulses at times t_1, t_2, ...t_n of magnitude $\Delta\varepsilon_{ij}(t_K)$, K = 1, 2,...n. If linearity is assumed, then the stress history is the superposition of n discrete histories each following equation (16):

$$\sigma_{ij}(t) = \sum_{K=1}^{n} E_r(t - t_K)\,\Delta\varepsilon_{ij}(t_K) \tag{18}$$

Passing to the limit where $\varepsilon_{ij}(t)$ changes continuously, the hereditary integral form is obtained in terms of the relaxation function $E_r(t)$:

$$\sigma_{ij}(t) = \int_{0^-}^{t} E_r(t-\tau)\,\frac{\partial\varepsilon_{ij}(\tau)}{\partial\tau}\,d\tau \tag{19}$$

In an analogous manner, the hereditary integral form involving the creep compliance function may be written:

$$\varepsilon_{ij}(t) = \int_{0^-}^{t} D_r(t-\tau)\,\frac{\partial\sigma_{ij}(\tau)}{\partial\tau}\,d\tau \tag{20}$$

To avoid the difficulty of dealing with discontinuities at the origin, it is convenient to write (19) and (20) in the following form, where the integration limit t^- together with the initial conditions on $E_r(t)$ or $D_r(t)$ account for such discontinuities:

$$\sigma_{ij}(t) = \left[E_r(0) - \int_{0}^{t^-} (\quad)\,\frac{\partial E_r(t-\tau)}{\partial\tau}\,d\tau \right] \varepsilon_{ij}(t) \tag{21}$$

$$\varepsilon_{ij}(t) = \left[D_r(0) - \int_{0}^{t^-} (\quad)\,\frac{\partial D_r(t-\tau)}{\partial\tau}\,d\tau \right] \sigma_{ij}(t) \tag{22}$$

In equations (21) and (22), the symmetrical properties of the integrals have been utilized so that the initial conditions on the relaxation function and creep compliance function could be written outside of the integral.

The expressions (21) and (22) are written in a form such that the operator within the brackets corresponds to the analogous elastic modulus or elastic compliance.

Consider now the Laplace transforms* of equations (19) and (20):

$$\sigma_{jj}^*(s) = s E_r^*(s) \varepsilon_{jj}^*(s) = E(s) \varepsilon_{jj}^*(s) \tag{23}$$

$$\varepsilon_{jj}^*(s) = s D_r^*(s) \sigma_{jj}^*(s) = \frac{1}{E(s)} \sigma_{jj}^*(s) \tag{24}$$

Equations (23) and (24) are elastic-type relations, where $E(s)$ (analogous to Young's Modulus) $\equiv s E_r^*(s) \equiv 1/sD_r^*(s)$ in the transform plane.

III-4.2. Characterization of Volumetric Behavior

In the above discussion of the hereditary integral form for a viscoelastic constitutive equation, an operator was derived which is useful in equating stress to strain for the case of uniaxial normal stress. For

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*The Laplace transform of f(t) is defined as

$$\mathcal{L}\{f(t)\} = f^*(s) = \int_0^\infty e^{-st} f(t)\, dt$$

three-dimensional analyses, one other material relationship must be given. That is, in the above development an operator equivalent to the elastic modulus was formulated. A constitutive relation giving an equivalent Poisson's ratio, or bulk modulus, or shear modulus, is also needed. The most common assumption for this relationship [139] is that the material behaves in an elastic manner under hydrostatic tension or compression. The second relationship needed is then

$$\sigma(t) = 3K e(t) \tag{25}$$

which has a Laplace transform of

$$\sigma^*(s) = 3K e^*(s) \tag{26}$$

Hence, the equivalent bulk modulus in the transform plane is the elastic bulk modulus. Given two characterizations such as equations (19) and (25), an equivalent shear modulus and Poisson's ratio in the transform plane can be found from the relations

$$G(s) = \frac{3K E(s)}{9K - E(s)} \tag{27}$$

$$\mu(s) = \frac{1}{2} - \frac{E(s)}{6K} \tag{28}$$

Of course, if equation (25) were given in a time-varying form, then K(s) would have to be used in equations (27) and (28). For example, the volumetric behavior might be specified in hereditary integral form as

$$\sigma(t) = \int_{0^-}^{t} K_r(t-\tau)\frac{\partial e(\tau)}{\partial \tau}d\tau \tag{29}$$

where $K_r(t)$ is the bulk relaxation function defined in a fashion analogous to equation (16). Then the Laplace transform of (29) gives the equivalent elastic bulk modulus in the transform plane:

$$\frac{\sigma^*(s)}{e^*(s)} = sK_r^*(s) \equiv 3K(s) \tag{30}$$

However, at the present time very little analysis has been done considering viscoelastic volumetric behavior. This is reasonable because little is known of the actual time variation of the volumetric components of stress and strain. In fact, a further simplification of equation (25) is commonly made by assuming that the bulk modulus is infinite, i.e., the material is incompressible, which also implies, as shown in equation (28), that Poisson's ratio is equal to 1/2.

III-4.3. Differential Operator Form

It is sometimes convenient to express the constitutive equations of linear viscoelasticity in linear differential operator form such as given in equation (31):

$$\sum_{\ell=0}^{n} a_\ell \frac{\partial^\ell \sigma_{jj}(t)}{\partial t^\ell} = \sum_{\ell=0}^{m} b_\ell \cdot \frac{\partial^\ell \varepsilon_{jj}(t)}{\partial t^\ell} \tag{31}$$

This form can conveniently be related to combinations of Hookean springs and Newtonian dashpots which is a helpful aid in visualizing the responses being represented.

The Laplace transform of equation (31) is a polynomial form in s:

$$\sum_{\ell=0}^{n} a_\ell s^\ell \sigma_{jj}^*(s) = \sum_{\ell=0}^{m} b_\ell s^\ell \varepsilon_{jj}^*(s) \tag{32}$$

where the first n-1 derivatives of $\sigma_{jj}(0)$ and the first m-1 derivatives of $\varepsilon_{jj}(0)$ are taken as zero.

This may be rewritten as in equation (33) to give an expression equivalent to the elastic modulus:

$$\sigma_{jj}^*(s) = \frac{\sum_{\ell=0}^{m} b_\ell s^\ell}{\sum_{\ell=0}^{n} a_\ell s^\ell} \varepsilon_{jj}^*(s) \equiv E(s)\varepsilon_{jj}^*(s) \tag{33}$$

As an example of the formulation of a constitutive equation in the differential operator form, consider the three-element model shown in Figure 1. The differential equation describing the force-deformation behavior of this model for uniaxial normal stress is given in equation (34) and is seen to correspond to m = n = 1 in equation (31).

$$\left[\frac{\partial}{\partial t} + \frac{E_2}{\eta_2}\right]\sigma_{ij}(t) = \left[(E_1+E_2)\frac{\partial}{\partial t} + \frac{E_1E_2}{\eta_2}\right]\varepsilon_{ij}(t) \tag{34}$$

For a constant stress $\sigma_{ij}(0)$ (a creep test), the strain is obtained by solving equation (34) [139] to give:

$$\varepsilon_{ij}(t) = \sigma_{ij}(0)\left[\frac{1}{E_1} - \frac{E_2}{E_1(E_1+E_2)}\, e^{-\left[\frac{E_1E_2}{\eta_2(E_1+E_2)}\right]t}\right] \tag{35}$$

where e is the base of the natural logarithm.

To use this characterization one might thus perform a creep test, and then select the constants E_1, E_2, and η_2 in equation (35) so that it would give the best possible fit to the real creep data.

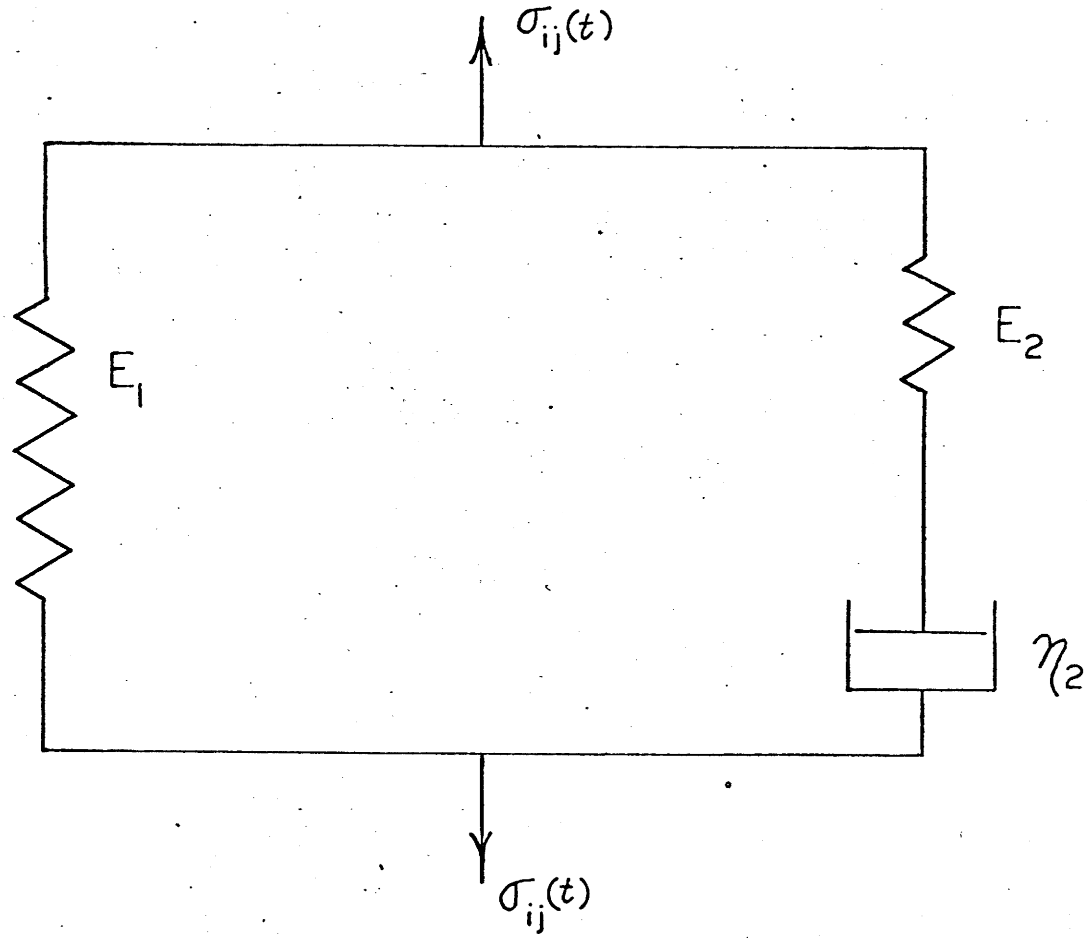

FIGURE 1

THREE - ELEMENT MODEL

Many other combinations of springs and dashpots can be selected that will yield similar differential operator constitutive relations. These have been elaborated on by many writers, and reference [18] gives a comprehensive coverage of the differential equations involved.

The disadvantages related to the use of the differential operator form (which appears so intuitively convenient) arise in trying to fit the actual data (creep, recovery, etc.) to the differential operator equation over long times. Although materials do exist which have viscoelastic characteristics which may be adequately represented by low-order differential operator relations over several decades of time, most materials cannot be accurately represented by such low order expressions [72]. Furthermore, as the order of the equations is increased, additional difficulties arise, among these being a rapid increase in the complexity of analysis when using such relations.

III-4.4. Spectral Representation

One approach to characterization, which follows from the differential operator form, consists in passing from a discrete number of springs and dashpots to an infinite number of such elements. The result can then be expressed as an integral relationship. Figure 2 shows, for example, a repeating combination of springs and dashpots arranged in the so-called Wiechert model. The constitutive equation of this model is [139] :

$$\sigma_{ij}(t) = \left[E_o + \sum_{i=1}^{n} \frac{E_i}{\frac{\partial}{\partial t} + \frac{E_i}{\eta_i}} \frac{\partial}{\partial t} \right] \varepsilon_{ij}(t) \quad (36)$$

The quantity η_i/E_i is the relaxation time for the ith spring and dashpot combination [the time required for the combination to reach 1/e (e being the base of the natural logarithm) of its total stress relaxation in a relaxation test] and is normally denoted τ_i. One can synthesize a function of relaxation times in this model, and substitute this in (36) to express E_i and η_i in terms of only τ_i. Then passing to the limit [$n \rightarrow \infty$ in equation (36)], an integral relationship is obtained.

A convenient form for this function is

$$H(\tau_i) = \frac{\eta_i}{\Delta_i \tau} \quad (37)$$

which gives, after substituting in (36) and passing to

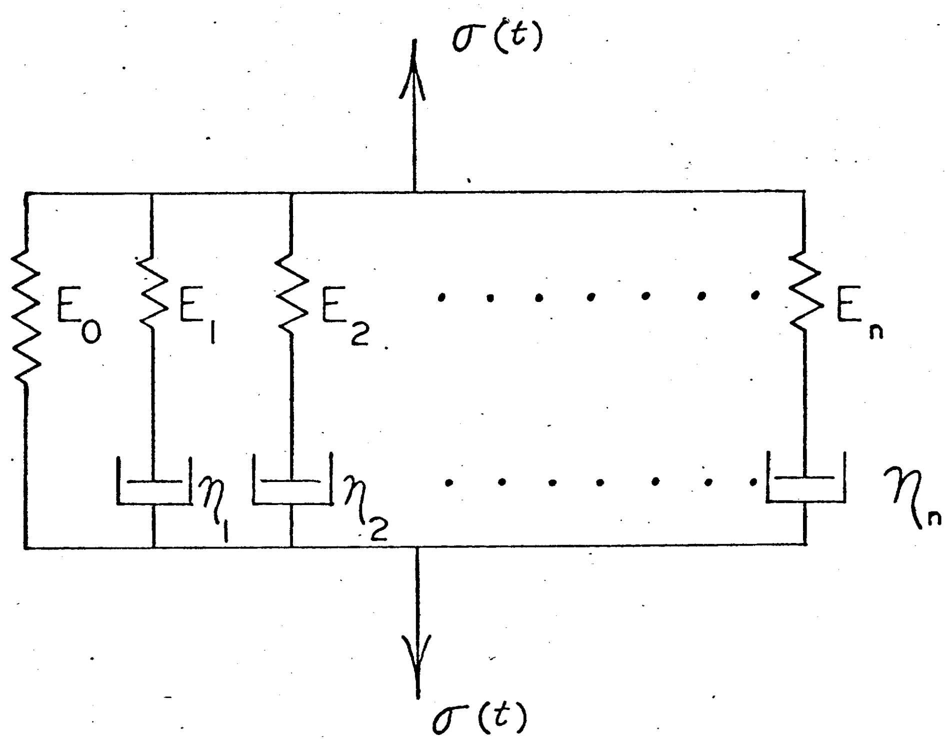

FIGURE 2

WIECHERT MODEL

the limit:

$$\sigma_{jj}(t)=\left[E_o+\int_0^\infty \frac{H(\tau)\,d\tau}{\left[\frac{\partial}{\partial t}+\frac{1}{\tau}\right]\tau}\frac{\partial}{\partial t}\right]\varepsilon_{jj}(t) \qquad (38)$$

which is the spectral representation. $H(\tau)$ is known as the relaxation spectrum, and E_o is the long time elastic modulus.

The use of equation (38) is essentially the same as the use of the discrete models. A known stress-strain history is fitted by finding a suitable form for $H(\tau)$, either by solving the integral equation (38) or by the trial and error procedure of predicting a mathematical form for $H(\tau)$, integrating equation (38), and then comparing this result with the experimental data.

The result expressed in equation (38) for the Wiechert model is most useful when a strain is imposed and the stress history is measured. If the opposite case is used, then another infinite combination, the Kelvin model shown in Figure 3, is more convenient. The response for this model can be developed along the same lines as for the Wiechert model, yielding equation (39) as the constitutive relation in terms of the retardation spectrum $L(\tau)$.

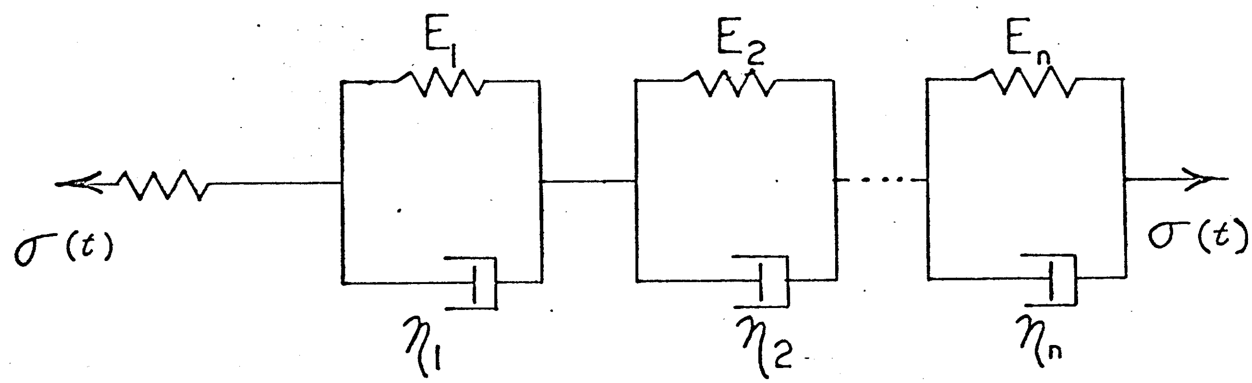

FIGURE 3

KELVIN MODEL

$$\varepsilon_{ij}(t) = \left[\frac{1}{E_o} + \int_0^\infty \frac{L(\tau)\, d\tau}{\left[\frac{\partial}{\partial t} + \frac{1}{\tau}\right] \tau^2} \right] \sigma_{ij}(t) \tag{39}$$

This relation would be fitted to experimental data in a manner similar to that of equation (38).

The Laplace transforms of equations (38) and (39) are given below [139] :

$$\sigma_{ij}^*(s) = \left[E_o + \int_0^\infty \frac{H(\tau)\, d\tau}{\left[s + \frac{1}{\tau}\right] \tau} s \right] \varepsilon_{ij}^*(s) \equiv E(s)\, \varepsilon_{ij}^*(s) \tag{40}$$

$$\varepsilon_{ij}^*(s) = \left[\frac{1}{E_o} + \int_0^\infty \frac{L(\tau)\, d\tau}{\left[s + \frac{1}{\tau}\right] \tau^2} \right] \sigma_{ij}^*(s) \equiv D(s)\, \sigma_{ij}^*(s) \equiv \frac{1}{E(s)} \sigma_{ij}^*(s) \tag{41}$$

III-4.5. Complex Representations

It is often convenient to measure the response of a material to an oscillatory input. Such a technique makes it possible to measure the response at very short times (since no discontinuous changes in stress or strain are required as in a creep or relaxation test) and also gives a fairly direct measurement of the loss characteristics. Use of such dynamic testing methods leads to the definition of a complex modulus or complex compliance, as described below.

Consider a specified strain input $R[\mathcal{E}_o e^{i\omega t}]$ with $\mathcal{E}_o$ the amplitude of the sine wave. The resulting stress can be denoted $\overline{\sigma(\omega)}\, e^{i\omega t}$ where now $\overline{\sigma(\omega)}$ is a complex function of frequency. The complex modulus is then defined to be [39]:

$$\frac{\overline{\sigma(\omega)}}{\mathcal{E}_o} \equiv \overline{E(\omega)} \equiv E'(\omega) + i E''(\omega) \tag{42}$$

and analogously one defines the complex compliance

$$\frac{\overline{\mathcal{E}(\omega)}}{\sigma_o} \equiv \overline{D(\omega)} \equiv D'(\omega) - i D''(\omega) \tag{43}$$

for an input stress of $R[\sigma_o e^{i\omega t}]$.

To show how the complex modulus and compliance are related to the other characterizations, substitute the dynamic input $R[\mathcal{E}_o e^{i\omega t}]$ and output $\overline{\sigma(\omega)}\, e^{i\omega t}$ into the differential operator form of the constitutive equation [equation (31)] :

$$\sum_{j=0}^{n} a_j (i\omega)^j \overline{\sigma(\omega)}\, e^{i\omega t} = \sum_{j=0}^{m} b_j (i\omega)^j \mathcal{E}_o\, e^{i\omega t} \tag{44}$$

or

$$\frac{\overline{\sigma(\omega)}}{\varepsilon_o} = \overline{E(\omega)} = \frac{\sum_{j=0}^{n} q_j (i\omega)^j}{\sum_{j=0}^{m} b_j (i\omega)^j} \equiv E(i\omega) \qquad (45)$$

It is apparent from equations (33) and (45) that the complex modulus is equivalent to the equivalent elastic modulus if s is replaced by $i\omega$.

All of the above methods for measuring and characterizing viscoelastic behavior have been used, and all, as has been briefly shown, can be interrelated. Before proceeding to a consideration of how these constitutive relations can be used in stress and deformation analysis, it is appropriate to point out that the above characterizations often lead to quite complicated constitutive relations, and series expansions and other numerical methods are often necessary in handling these relations. In particular, Schapery [110] has presented methods for developing series representations and approximate numerical methods for performing the inverse Laplace transforms. In addition, Gross [39] has presented a thorough coverage of the interrelationships between these various characterizations.

very simple problems can usually be solved in this manner, and many of these could be handled more easily by the "correspondence principle" to be considered below.

A second approach to solving the equations is to attempt to solve them using numerical methods and high-speed computers. This approach will probably grow in usefulness in the future, but at the present time such solutions seem to be most appropriately used, again, in conjunction with the "correspondence principle".

As has been previously noted, the only differences in the applicable equations of elasticity from those of viscoelasticity are in the constitutive equations, and indeed these constitutive equations are the dividing line between each of the classes of continuum mechanics. It has been noted, furthermore, that the constitutive relations of linear viscoelasticity are similar in form to the constitutive equations of linear elasticity; for example, in the transform plane an algebraic equivalent of E, K, μ, or G exists. Similarly, an operator such as included within the brackets of equation (21) can be considered to be an equivalent to the elastic modulus E in the time domain. These similarities make it possible, in a large number of worthwhile engineering applications, to use the solutions to elastic very simple problems can usually be solved in this manner, and many of these could be handled more easily by the "correspondence principle" to be considered below.

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To further show the correspondence between elastic and viscoelastic problems, consider transforming the equilibrium, strain-displacement, constitutive equations, and the boundary conditions of a viscoelastic problem, using the Laplace transform. The transformed equilibrium equations are still three equations in the six unknown stresses (now the transformed stresses), and the strain displacement equations are essentially unchanged. The constitutive equations have been converted to elastic-type relations. The boundary conditions may or may not have changed form, depending on whether they varied in time originally. In any event, the resulting equations are in the same form as an elastic problem, and, if this problem can be solved, then the time varying solution to the viscoelastic problem can be found by means of the inverse Laplace transform. Of course, if the boundary conditions are unchanged in the transformation and inertia terms can be neglected, then the equivalent elastic problem in the transformed plane will be precisely the same as the original problem in the time plane with the constitutive equations changed to those of an elastic body [73].

In a very similar manner, one can use an operator equivalent of the elastic constants in the original problem, carry out the necessary manipulations to solve the equivalent elastic problem, and then solve the resulting integral or differential equation in the variable time [101].

The "correspondence principle" is thus based on the idea that it is often possible to utilize known elastic solutions to obtain analogous viscoelastic solutions. For the so-called quasi-static problems, where it is assumed that the dependent variables vary sufficiently slowly so that the inertia terms can be neglected in the equilibrium equations, the Laplace transform has usually been used. For this type of problem, and assuming that the Laplace transform of the boundary conditions exists, the correspondence principle may be stated as follows:

Replace the dependent variables and boundary conditions in the elastic solution by their Laplace transforms, and replace the elastic constants by their equivalent forms in the transform plane. Inversion of this result will yield the time-varying viscoelastic solution.

A large number of engineering problems can in principle be solved using this approach. However, its use imposes certain limitations on the type of the problems which can be handled:

1. The assumption that the Laplace transform of the boundary conditions exists, and the assumption of quasi-static behavior, limit the application of the principle [101].

2. It is often difficult to obtain an appropriate analytical expression for the constitutive equations of the material. Experimental data yields curves or a discrete number of points, and the analyst, if he is to obtain realistic answers, needs to select a form which is sufficiently flexible to fit the actual experimental data [72].

3. A major difficulty is in obtaining the inverse Laplace transform of the equivalent elastic solution. Many such inverse transforms are known and have been tabulated [24]. Many complicated forms may be inverted by separating them into simpler forms using the method of partial fractions [24]. In addition, some numerical techniques have been developed for relatively direct inversion [26, 110].

To avoid these difficulties, a method which eliminates the need for analytical expressions for constitutive equations of the material and which can use actual experimental curves or data has been proposed by Lee and Rogers [72]. Furthermore, Radok [101], using a method of functional equations, has shown that some of the restrictions imposed by the use of the Laplace transformation can be removed and that the correspondence principle can be extended to a wider class of problems.

It should be noted that the direct use of the operator approach is completely justified if the boundary conditions do not vary in type (that is, remain of the stress type or remain of the displacement type), but that the procedure is open to some question when this is not true (for instance, a rolling contact problem). [70,71] For the latter type of problems, a check on the significance of the results is necessary. Further research is still necessary to determine the validity of the technique in this case.

This thesis presents a method, based on the combination of the above-mentioned approaches, for the solution of a wide class of viscoelastic stress analysis problems.

The method, to be explained and illustrated below, relies upon the use of the operator equivalents of the elastic constants, using realistic material properties. The problems encountered using this method and the means of handling them are presented and discussed.

The basis of the operator approach relies upon the possibility of using an operator equivalent for each elastic constant occurring in the solution for the elastic body with the same boundary conditions. As has been pointed out previously, two such "equivalent elastic constants" must be known for three-dimensional analysis. With a knowledge of any two of these "equivalent elastic constants", any of the others can be found through the use of equations such as (14) and (15). Also, as has been noted, the assumption that viscoelastic materials are elastic (or sometimes incompressible) in volumetric behavior is usually made due to a lack of detailed knowledge of actual material behavior. This latter assumption is not necessary when using the operator approach, although its use does, of course, simplify the resulting equations somewhat.

To use the operator approach in a straightforward manner, let us assume that the equivalent elastic solution can be arranged, by appropriate algebraic operations, into the following form:

$$\psi(t) = \frac{\sum_{i=1}^{n} \theta_i \alpha_i f_i(t)}{\sum_{i=1}^{m} \phi_i \beta_i} \tag{46}$$

where

$\psi(t)$ = the desired stress or displacement

θ_i, ϕ_i = constants with respect to time

α_i, β_i = products of elastic constants.

For example, $\alpha_j = E^2 K$

$f_i(t)$ = functions of time introduced through time variations in the boundary conditions.

The vast majority of problems to which the correspondence principle is applicable may be arranged in this form. Some solutions, which at first do not appear to be suitable to arrangement in this form, can be modified through series expansions.*

If each of the elastic constants in the α_i and β_i terms can be replaced by its viscoelastic operator equivalents, then equation (46) can be converted to the viscoelastic solution. However, the operators that occur here must be applied with a function of time, and, in the form given in equation (46), the β_i terms are not applied with any such function. To avoid this

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*For example, a term such as $\sqrt{1-\mu^2}$ could be written as $1 - \frac{\mu^2}{2} - \frac{\mu^4}{8} - \frac{\mu^6}{16} \ldots$ It should be noted, however, that some operations, such as squaring both sides of an equation to remove a square root, and later taking the square root of the answer, may introduce extraneous results.

difficulty, equation (46) may be rearranged to the following form:

$$\sum_{i=1}^{m} \phi_i \beta_i \Psi(t) = \sum_{i=1}^{n} \theta_i \alpha_i f_i(t) \tag{47}$$

Now to obtain the viscoelastic solution, the operator equivalents of the elastic constants are substituted in equation (47).

In order to derive the form of the solution when these operators are substituted into equation (47), consider first a typical term

$$\left(\prod_{j=1}^{k} \gamma_j \right) \nu(t) \tag{48}$$

where

γ_j $j = 1, 2, \cdots K$ are elastic constants which have operator equivalents of the form

$$\gamma_{j\,equiv.} = \left[\gamma_j(0) - \int_0^{t^-} (\quad) \frac{\partial \gamma_j(t-\tau)}{\partial \tau} d\tau \right] \tag{49}$$

and $\nu(t)$ is either an $f_i(t)$ or $\Psi(t)$.

Substituting the operator equivalents (49) into the typical term (48) one obtains the following multiple convolution integrals:

$$\int_{0^+}^{t} \nu(t-\tau)\frac{\partial}{\partial\tau}\int_{0^+}^{\tau}\gamma_1(\tau-\lambda)\frac{\partial}{\partial\lambda}\int_{0^+}^{\lambda}\cdots\int_{0^+}^{\xi}\gamma_{k-1}(\xi-\eta)\frac{\partial\gamma_k(\eta)}{\partial\eta}d\eta$$

$$+\gamma_{k-1}(\xi)\gamma_k(0)d\xi+\cdots\cdots+\gamma_1(\tau)\gamma_2(0)\cdots\gamma_k(0)\,d\tau$$

$$+\nu(t)\gamma_1(0)\gamma_2(0)\cdots\gamma_k(0) \tag{50}$$

It is convenient to rewrite (50) in the form

$$\int_{0^+}^{t}\nu(t-\tau)\frac{\partial\gamma(\tau)}{\partial\tau}d\tau + \nu(t)\gamma(0) \tag{51}$$

where

$$\gamma(\tau)=\int_{0^+}^{\tau}\gamma_1(\tau-\lambda)\frac{\partial}{\partial\lambda}\int_{0^+}^{\lambda}\cdots\int_{0^+}^{\xi}\gamma_{k-1}(\xi-\eta)\frac{\partial\gamma_k(\eta)}{\partial\eta}d\eta$$

$$+\gamma_{k-1}(\xi)\gamma_k(0)d\xi+\cdots+\gamma_1(\tau)\gamma_2(0)\cdots\gamma_k(0) \tag{52}$$

With the results of equation (50) and the notation of equation (51), the general form for the corresponding viscoelastic solution can be written as follows, after substituting the operator equivalents for the elastic constants into equation (47):

$$\sum_{i=1}^{m} \phi_i \left\{ \int_{0^+}^{t} \Psi(t-\tau) \frac{\partial \beta_i(\tau)}{\partial \tau} d\tau + \Psi(t) \beta_i(0) \right\}$$

$$= \sum_{i=1}^{n} \theta_i \left\{ \int_{0^+}^{t} f_i(t-\tau) \frac{\partial \alpha_i(\tau)}{\partial \tau} d\tau + f_i(t) \alpha_i(0) \right\} \qquad (53)$$

Equation (53) is a Voltera integral equation; the solution of this equation yields $\Psi(t)$, the desired stress or displacement for the viscoelastic body. It should be pointed out again that the $\alpha_i(t)$ and $\beta_i(t)$ terms are multiple convolution integrals.

Equation (53) is in a convenient form for numerical solution, as will be illustrated when presenting two relevant examples in the following chapters. There are two principle phases to this numerical solution. First of all, the terms $\alpha_i(t)$ and $\beta_i(t)$ must be evaluated at certain values of t. Two alternative approaches for evaluating these terms will be presented in Chapter IV. The first technique utilizes only numerical integration. The second is exact, but depends on expressing the relevant relaxation functions in terms of Dirichlet series. After obtaining these terms, and knowing the $f_i(t)$ at appropriate discrete values, the integral equation (53) can be solved by a numerical step-out procedure.

The above approach has three main advantages. First of all, the Laplace transform is not used, and thus it is not necessary that all of the equations and boundary conditions have Laplace transforms. Secondly, the application of the above method, although possibly appearing complex because of its abstract form in the above presentation, is straight-forward. This will be apparent when the examples are presented. Thirdly, due to the general approaches used to evaluate the multiple convolution integrals, and since the integral equation is solved numerically, the relaxation or creep functions which appear in the solution can be kept realistic and representative of real materials.

Before presenting the techniques for solving equation (53), and two examples of the use of the method, it is worthwhile to note that it is not necessary to use the specific operator equivalents (the hereditary form) used above, although it would seem to be the most convenient form. Any of the forms previously discussed could be used, although the numerical procedures for solving the resulting equations would vary depending on the form selected.

CHAPTER IV

SOLUTION OF THE GENERAL INTEGRAL EQUATION

The solution of the general integral equation, equation (53) of the previous chapter, must proceed with two principle phases. First the multiple convolution integrals $\alpha_i(t)$ and $\beta_i(t)$ must be obtained at appropriate values of t, and then, using these values, the integral equation is solved by a step out procedure. Two different approaches for evaluating the multiple convolution integrals will be presented. The method of solution of the total integral equation will then be discussed, and the implications of using each technique on the solution of the total integral equation will then be discussed.

IV-1. Numerical Evaluation of the Multiple Integrals

The typical term $\alpha_i(t)$ or $\beta_i(t)$ has been given in equation (52) of the previous chapter. To evaluate such a term numerically, assume first that each $\gamma_i(t)$ is known at appropriate values of t (recall that $\gamma_i(t)$ is a creep or relaxation function). Consider the innermost integration:

$$I_1(\xi) = \int_{0^+}^{\xi} \gamma_{k-1}(\xi-\eta)\frac{\partial \gamma_k(\eta)}{\partial \eta}\,d\eta + \gamma_{k-1}(\xi)\gamma_k(0) \qquad (54)$$

Let this integral be divided into n_1 intervals:

$$I_1(\xi)=\sum_{i=1}^{n_1}\int_{t_{n_1}-t_{n_1-i+1}}^{t_{n_1}-t_{n_1-i}}\gamma_{k-1}(\xi-\eta)\frac{\partial\gamma_k(\eta)}{\partial\eta}d\eta+\gamma_{k-1}(\xi)\gamma_k(0) \qquad (55)$$

where $t_0 = 0^+$ and $t_{n_1} = \xi$. For $\gamma_{k-1}(\xi-\eta)$ a continuous function and the interval $t_{n_1-i+1}-t_{n_1-i}$ small enough, $\gamma_{k-1}(\xi-\eta)$ may be approximated by a constant, say $\frac{1}{2}\left[\gamma_{k-1}(t_{n_1-i+1})+\gamma_{k-1}(t_{n_1-i})\right]$ and (55) may be written

$$I_1(\xi)=\sum_{i=1}^{n_1}\frac{1}{2}\left[\gamma_{k-1}(t_{n_1-i+1})+\gamma_{k-1}(t_{n_1-i})\right]\int_{t_{n_1}-t_{n_1-i+1}}^{t_{n_1}-t_{n_1-i}}\frac{\partial\gamma_k(\eta)}{\partial\eta}d\eta+\gamma_{k-1}(t_{n_1})\gamma_k(0) \qquad (56)$$

or, since the integral of a derivative is just the function evaluated at the limits, this is:

$$I_1(\xi)=\sum_{i=1}^{n_1}\frac{1}{2}\left[\gamma_{k-1}(t_{n_1-i+1})+\gamma_{k-1}(t_{n_1-i})\right]\left[\gamma_k(t_{n_1}-t_{n_1-i})-\gamma_k(t_{n_1}-t_{n_1-i+1})\right]+\gamma_{k-1}(t_{n_1})\gamma_k(0) \qquad (57)$$

which gives an approximate expression for the integral (54). If the n_1 intervals are chosen equal, then the approximation equation (57) is equivalent to using the trapezoidal rule in conjunction with first order central difference derivative approximations for $\gamma_k(t)$, except at the end points 0^+ and t_{n_1}, where first order forward or backward differences, respectively, are used. Note that in the form of expression (57) the spacing does not enter explicitly.

Next consider a two-fold convolution from equation (52):

$$\int_{0^+}^{\rho} \gamma_{k-2}(\rho-\xi)\frac{\partial}{\partial\xi}\int_{0^+}^{\xi}\gamma_{k-1}(\xi-\eta)\frac{\partial\gamma_k(\eta)}{\partial\eta}\,d\eta + \gamma_{k-1}(\xi)\gamma_k(0)\,d\xi + \gamma_{k-2}(\rho)\gamma_{k-1}(0)\gamma_k(0) \qquad (58)$$

If the inner integral is approximated using expression (57) at all necessary values of t, then the outside integral can be evaluated in the same manner. However, in the general case a sum of n_1 terms will be needed to evaluate (54) for each time t_j used in evaluating the outer integral. Clearly to evaluate the total result where ρ is divided into n_2 intervals will take $n_2 \times n_1$ terms of the type in the sum of expression (57). Repeating this procedure for m integrations will require $\prod_{j=1}^{m} n_j$ terms to be evaluated. Unless each n_j is small, this would require a prodigious number of computations. To avoid this, let ρ and ξ be divided into the same equal intervals. Then each successive evaluation of the inner integral requires only a single additional computation. In this way the evaluation of m integrations requires only the order of $\sum_{j=1}^{m} n_j$ terms.

Following the above discussion, the double convolution integral, expression (58), can be written:

$$\begin{aligned} I_2(\rho) = \sum_{j=1}^{n_2} \tfrac{1}{2}\left[\gamma_{k-2}(t_{n_2-j+1}) + \gamma_{k-2}(t_{n_2-j})\right] \Big\{ & \sum_{i=1}^{j} \tfrac{1}{2}\left[\gamma_{k-1}(t_{j-i+1}) + \gamma_{k-1}(t_{j-i})\right]\left[\gamma_k(t_i) - \gamma_k(t_{i-1})\right] \\ & + \gamma_{k-1}(t_j)\gamma_k(0) - \sum_{i=1}^{j-1} \tfrac{1}{2}\left[\gamma_{k-1}(t_{j-i}) + \gamma_{k-1}(t_{j-i-1})\right]\left[\gamma_k(t_i) - \gamma_k(t_{i-1})\right] - \gamma_{k-1}(t_{j-1})\gamma_k(0)\Big\} + \gamma_{k-2}(t_{n_2})\gamma_{k-1}(0)\gamma_k(0) \\ = \sum_{j=1}^{n_2} \tfrac{1}{2}\left[\gamma_{k-2}(t_{n_2-j+1}) + \gamma_{k-2}(t_{n_2-j})\right] & \left\{I_1(t_j) - I_1(t_{j-1})\right\} + \gamma_{k-2}(t_{n_2})\gamma_{k-1}(0)\gamma_k(0) \end{aligned} \qquad (59)$$

Similarly, m fold multiple convolution integrals may be approximately evaluated.

The obvious shortcoming of the above approach is that with equal spacings the evaluation of many-fold convolution integrals at long times will require n_j to become very large, and hence the number of computations will become prohibitively large. To avoid this, the following scheme has been found to work reasonably well:

Equal spacing is used to evaluate $\gamma(t)$ up to some t_n. The spacing is then doubled, and all of the even values of t and the corresponding values of $\gamma(t)$ are retained and used to calculate $\gamma(t)$ up to the new t_n, which is double the original t_n. Further discussion of this approach is included later in this chapter when numerical examples are presented.

It should be noted that no functional expression is necessary for $\gamma_j(t)$ when using the above numerical scheme.

IV-2. Exact Evaluation of the Multiple Integrals

Although the above numerical evaluation of the multiple convolution integrals has been found to work reasonably well (as will be shown subsequntly), it is apparent that an approach that would yield an explicit solution for the $\alpha_i(t)$ and $\beta_i(t)$ terms, which could be evaluated exactly for any time t, would be desirable.

To achieve this result, and at the same time to maintain generality in the representation of the appropriate relaxation functions, the following technique has been developed. Assume that each $\gamma_i(t)$ can be represented by a Dirichlet series:

$$\gamma_i(t) = \sum_{j=1}^{n} G_i^j e^{-t\delta^j} \tag{60}$$

where the G_i^j's and δ^j's are constants (some G_i^j may be zero, and one δ^j may be zero). This representation is sufficient to accurately characterize real materials (although n may be as large, or larger, than ten), as has been demonstrated by Schapery [109] using irreversible thermodynamic arguments. In addition, Schapery has demonstrated a simple collocation scheme to calculate the coefficients G_i^j (a version of this will be used in the example in Chapter V, and also in curve-fitting later in this chapter).

Consider now a single convolution integral, the innermost integral of the general term given in equation (52):

$$I_1(\xi) = \int_{0^+}^{\xi} \gamma_{k-1}(\xi-\eta) \frac{\partial \gamma_k(\eta)}{\partial \eta} d\eta + \gamma_{k-1}(\xi)\gamma_k(0) \tag{61}$$

With the representation of equation (60) for $\gamma_{k-1}(t)$ and $\gamma_k(t)$, this becomes:

$$I_1(\xi) = \int_{0^+}^{\xi} \left(\sum_{j=1}^{n} G_{k-1}^{j} e^{(-\xi+\eta)\delta^j}\right)\left(-\sum_{i=1}^{n} G_k^i \delta^i e^{-\eta\delta^i}\right) d\eta$$

$$+ \sum_{j=1}^{n} G_{k-1}^{j} \left(\sum_{i=1}^{n} G_k^i\right) e^{-\xi\delta^j} \tag{62}$$

Rearranging the summations, equation (62) may be written

$$I_1(\xi) = \sum_{j=1}^{n} G_{k-1}^{j} e^{-\xi\delta^j} \left\{ \sum_{i=1}^{n} G_k^i \left[1 - \delta^i \int_{0^+}^{\xi} e^{-\eta(\delta^i - \delta^j)} d\eta\right]\right\} \tag{63}$$

The integrals in equation (63) may be evaluated, but the result varies depending on whether i = j:

$$\int_{0^+}^{\xi} e^{-\eta(\delta^i-\delta^j)} d\eta \quad \begin{aligned} &= \xi && i = j \\ &= \frac{-1}{\delta^i - \delta^j}\left(e^{-\xi(\delta^i-\delta^j)} - 1\right) && i \neq j \end{aligned} \tag{64}$$

Substituting the result expressed in (64) into (63) yields:

$$I_1(\xi) = \sum_{j=1}^{n} G_{k-1}^{j} \left\{ G_k^j e^{-\xi\delta^j} - G_k^j \delta^j \xi e^{-\xi\delta^j} + \sum_{\substack{i=1 \\ i\neq j}}^{n} G_k^i \left[\frac{-\delta^j}{\delta^i - \delta^j} e^{-\xi\delta^j} + \frac{\delta^i}{\delta^i - \delta^j} e^{-\xi\delta^i}\right]\right\} \tag{65}$$

Equation (65) can be rearranged and written in the following relatively simple form:

$$I_1(\xi) = \sum_{j=1}^{n} \left\{ {}_1B_1^j + {}_1B_2^j \xi \right\} e^{-\xi\delta^j} \tag{66}$$

where

$${}_1B_1^j = G_{k-1}^j G_k^j + G_{k-1}^j \sum_{i=1}^{n} G_k^i \frac{-\delta^j}{\delta^i - \delta^j}(1 - \delta_{ij})$$

$$+ G_k^j \sum_{i=1}^{n} G_{k-1}^i \frac{\delta^j}{\delta^j - \delta^i}(1 - \delta_{ij}) \tag{67}$$

$$ {}_1B_2^j = -\delta^j G_{k-1}^j G_k^j \tag{68} $$

and

$$ \delta_{ij} = \text{Kronecker delta function} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \tag{69} $$

Next consider the innermost two-fold convolution integral of equation (52):

$$ I_2(\rho) = \int_{0^+}^{\rho} \gamma_{k-2}(\rho-\xi)\frac{\partial}{\partial\xi}\int_{0^+}^{\xi} \gamma_{k-1}(\xi-\eta)\frac{\partial\gamma_k(\eta)}{\partial\eta}d\eta + \gamma_{k-1}(\xi)\gamma_k(0)d\xi + \gamma_{k-2}(\rho)\gamma_{k-1}(0)\gamma_k(0) \tag{70} $$

Using the result expressed in equation (66), and the form (60) for $\gamma_{k-2}(t)$, equation (70) can be written as follows:

$$ I_2(\rho) = \int_{0^+}^{\rho}\left(\sum_{j=1}^{n} G_{k-2}^j e^{-(\rho-\xi)\delta^j}\right)\left(-\sum_{i=1}^{n} {}_1B_1^i \delta^i e^{-\xi\delta^i}\right)d\xi $$

$$ + \sum_{j=1}^{n} G_{k-2}^j\left[\sum_{i=1}^{n} {}_1B_1^i\right]e^{-\rho\delta^j} + \int_{0^+}^{\rho}\left(\sum_{j=1}^{n} G_{k-2}^j e^{-(\rho-\xi)\delta^j}\right)\left(\frac{\partial\left[\sum_{i=1}^{n}\xi\, {}_1B_2^i e^{-\xi\delta^i}\right]}{\partial\xi}\right)d\xi \tag{71} $$

Comparing equation (71) and equation (62), it is clear that $I_2(\rho)$ is of the same form as $I_1(\rho)$ plus the last integral term in equation (71). Consequently, $I_2(\rho)$ can be written:

$$ I_2(\rho) = \sum_{j=1}^{n}\left\{ {}_1B_1^j + {}_1B_2^j\rho\right\}e^{-\rho\delta^j} + \int_{0^+}^{\rho}\left(\sum_{j=1}^{n} G_{k-2}^j e^{-(\rho-\xi)\delta^j}\right)\left(\frac{\partial\left[\sum_{i=1}^{n}\xi\, {}_1B_2^i e^{-\xi\delta^i}\right]}{\partial\xi}\right)d\xi \tag{72} $$

where ${}_1B^j_1$ and ${}_1B^j_2$ are defined as in equations (67) and (68), letting now

$$G^j_{k-1} = G^j_{k-2} \tag{73}$$

$$G^j_k = {}_1B^j_1 \tag{74}$$

The integral in equation (72) can be evaluated by rearranging the summations and carrying out the indicated differentiation and integrations. The result, for only the integral term, may be written:

$$\sum_{j=1}^{n} G^j_{k-2}\,{}_1B^j_2\,\rho\, e^{-\rho\delta^j} - \sum_{j=1}^{n} G^j_{k-2}\,{}_1B^j_2\,\frac{\delta^j}{2}\,\rho^2 e^{-\rho\delta^j}$$

$$-\sum_{j=1}^{n} G^j_{k-2}\left[\sum_{\substack{i=1\\ i\neq j}}^{n} {}_1B^j_2\,\frac{\delta^j}{(\delta^i-\delta^j)^2}\right] e^{-\rho\delta^j} + \sum_{j=1}^{n} {}_1B^j_2\left[\sum_{\substack{i=1\\ i\neq j}}^{n} G^i_{k-2}\,\frac{\delta^i}{(\delta^j-\delta^i)^2}\right] e^{-\rho\delta^i} \tag{75}$$

$$+\sum_{j=1}^{n} {}_1B^j_2\left[\sum_{\substack{i=1\\ i\neq j}}^{n} G^i_{k-2}\,\frac{\delta^j}{\delta^j-\delta^i}\right] \rho\, e^{-\rho\delta^j}$$

Using the results of equation (75) substituted into equation (72), the total result $I_2(\rho)$ can again be written in the following relatively simple form:

$$I_2(\rho) = \sum_{j=1}^{n}\left\{ {}_2B^j_1 + {}_2B^j_2\,\rho + {}_2B^j_3\,\rho^2 \right\} e^{-\rho\delta^j} \tag{76}$$

where

$${}_2B^j_1 = {}_1B^j_1 - G^j_{k-2}\sum_{i=1}^{n} {}_1B^j_2\,\frac{\delta^j}{(\delta^i-\delta^j)^2}\,(1-\delta_{ij}) \tag{77}$$

$$+\,{}_1B^j_2\sum_{i=1}^{n} G^j_{k-2}\,\frac{\delta^i}{(\delta^j-\delta^i)^2}\,(1-\delta_{ij})$$

$$ {}_2B_2^j = {}_1B_2^j + G_{k-2}^j \, {}_1B_2^j + {}_1B_2^j \sum_{i=1}^{n} G_{k-2}^i \frac{\delta^j}{(\delta^j-\delta^i)}(1-\delta_{ij}) \tag{78} $$

$$ {}_2B_3^j = -G_{k-2}^j \, {}_1B_2^j \, \frac{\delta^j}{2} \tag{79} $$

A third convolution will clearly follow a similar pattern. The necessary manipulations are quite cumbersome. The results are listed below for three, four, and five-fold convolutions, which have been obtained by the author:

$$ I_3(\nu) = \int_{0^+}^{\nu} \gamma_{k-3}(\nu-\rho)\frac{\partial I_2(\rho)}{\partial \rho}\,d\rho + \gamma_{k-3}(\nu)\, I_2(0) \tag{80} $$

$$ = \sum_{j=1}^{n} \left\{ {}_3B_1^j + {}_3B_2^j\,\nu + {}_3B_3^j\,\nu^2 + {}_3B_4^j\,\nu^3 \right\} e^{-\nu\delta^j} \tag{81} $$

where

$$ G_{k-1}^j = G_{k-3}^j \tag{82} $$

$$ G_k^j = {}_2B_1^j \tag{83} $$

$$ {}_1B_2^j = {}_2B_2^j \tag{84} $$

$$ \begin{aligned} {}_3B_1^j = {}_2B_1^j - G_{k-3}^j \sum_{i=1}^{n} {}_2B_3^i \frac{2!\,\delta^j}{(\delta^i-\delta^j)^3}(1-\delta_{ij}) \\ + {}_2B_3^j \sum_{i=1}^{n} G_{k-3}^i \frac{2!\,\delta^i}{(\delta^j-\delta^i)^3}(1-\delta_{ij}) \end{aligned} \tag{85} $$

$$ {}_3B_2^j = {}_2B_2^j + {}_2B_3^j \sum_{i=1}^{n} G_{k-3}^i \frac{2!\,\delta^i}{(\delta^j-\delta^i)^2}(1-\delta_{ij}) \tag{86} $$

$$ {}_3B_3^j = {}_2B_3^j + {}_2B_3^j \sum_{i=1}^{n} G_{k-3}^i \frac{\delta^j}{\delta^j - \delta^i}(1-\delta_{ij}) + {}_2B_3^j G_{k-3}^j \tag{87} $$

$$ {}_3B_4^j = -{}_2B_3^j G_{k-3}^j \frac{\delta^j}{3} \tag{88} $$

$$ I_4(\lambda) = \int_{0^+}^{\lambda} \gamma_{k-4}(\lambda-\nu) \frac{\partial I_3(\nu)}{\partial \nu} d\nu + \gamma_{k-4}(\lambda) I_3(0) \tag{89} $$

$$ = \sum_{j=1}^{n} \left\{ {}_4B_1^j + {}_4B_2^j \lambda + {}_4B_3^j \lambda^2 + {}_4B_4^j \lambda^3 + {}_4B_5^j \lambda^4 \right\} e^{-\lambda \delta^j} \tag{90} $$

where

$$ G_{k-1}^j = G_{k-4}^j \tag{91} $$

$$ G_k^j = {}_3B_1^j \tag{92} $$

$$ {}_1B_2^j = {}_3B_2^j \tag{93} $$

$$ {}_2B_3^j = {}_3B_3^j \tag{94} $$

$$ {}_4B_1^j = {}_3B_1^j - G_{k-4}^j \sum_{i=1}^{n} {}_3B_4^i \frac{3!\,\delta^j}{(\delta^i - \delta^j)^4}(1-\delta_{ij}) + {}_3B_4^j \sum_{i=1}^{n} G_{k-4}^i \frac{3!\,\delta^i}{(\delta_j - \delta_i)^4}(1-\delta_{ij}) \tag{95} $$

$$ {}_4B_2^j = {}_3B_2^j + {}_3B_4^j \sum_{i=1}^{n} G_{k-4}^i \frac{3!\,\delta^i}{(\delta^j-\delta^i)^3}(1-\delta_{ij}) \tag{96} $$

$$ {}_4B_3^j = {}_3B_3^j + {}_3B_4^j \sum_{i=1}^{n} G_{k-4}^i \frac{3!\,\delta^i}{2!(\delta^j-\delta^i)^2}(1-\delta_{ij}) \tag{97} $$

$$ {}_4B_4^j = {}_3B_4^j + {}_3B_4^j \sum_{i=1}^{n} G_{k-4}^i \frac{\delta^j}{\delta^j-\delta^i}(1-\delta_{ij}) + {}_3B_4^j\, G_{k-4}^j \tag{98} $$

$$ {}_4B_5^j = -{}_3B_4^j\, G_{k-4}^j \,\frac{\delta^j}{4} \tag{99} $$

$$ I_5(\chi) = \int_{0^+}^{\chi} \gamma_{k-5}(\chi-\lambda)\frac{\partial I_4(\lambda)}{\partial\lambda}\,d\lambda + \gamma_{k-5}(\chi)\, I_4(0) \tag{100} $$

$$ = \sum_{j=1}^{n}\left[\sum_{i=1}^{6}\left\{ {}_5B_i^j\, \chi^{i-1}\right\}\right] e^{-\chi\delta^j} \tag{101} $$

where

$$ G_{k-1}^j = G_{k-5}^j \tag{102} $$

$$ G_k^j = {}_4B_1^j \tag{103} $$

$$ {}_{1}B_2^j = {}_{4}B_2^j \tag{104} $$

$$ {}_{2}B_3^j = {}_{4}B_3^j \tag{105} $$

$$ {}_{3}B_4^j = {}_{4}B_4^j \tag{106} $$

$$ {}_{5}B_1^j = {}_{4}B_1^j - G_{k-5}^j \sum_{i=1}^{n} {}_{4}B_5^i \frac{4!\,\delta^j}{(\delta^i-\delta^j)^5}(1-\delta_{ij}) + {}_{4}B_5^j \sum_{i=1}^{n} G_{k-5}^i \frac{4!\,\delta^i}{(\delta^j-\delta^i)^5}(1-\delta_{ij}) \tag{107} $$

$$ {}_{5}B_2^j = {}_{4}B_2^j + {}_{4}B_5^j \sum_{i=1}^{n} G_{k-5}^i \frac{4!\,\delta^i}{(\delta^j-\delta^i)^4}(1-\delta_{ij}) \tag{108} $$

$$ {}_{5}B_3^j = {}_{4}B_3^j + {}_{4}B_5^j \sum_{i=1}^{n} G_{k-5}^i \frac{4!\,\delta^i}{2!\,(\delta^j-\delta^i)^3}(1-\delta_{ij}) \tag{109} $$

$$ {}_{5}B_4^j = {}_{4}B_4^j + {}_{4}B_5^j \sum_{i=1}^{n} G_{k-5}^i \frac{4!\,\delta^i}{3!(\delta^j-\delta^i)^2}(1-\delta_{ij}) \tag{110} $$

$$ {}_{5}B_5^j = {}_{4}B_5^j + {}_{4}B_5^j G_{k-5}^j + {}_{4}B_5^j \sum_{i=1}^{n} G_{k-5}^i \frac{\delta^j}{\delta^j-\delta^i}(1-\delta_{ij}) \tag{111} $$

$$ {}_{5}B_6^j = -{}_{4}B_5^j G_{k-5}^j \frac{\delta^j}{5.} \tag{112} $$

Further multiple integrals follow by analogy with the above, since there is an obvious sequence of results, and it is thus not necessary to actually carry out any further integrations rigorously.

The general result, then, for m-fold convolutions, can be written in the following relatively simple form:

$$I_m(t) = \sum_{j=1}^{n} \left[\sum_{i=1}^{m+1} {}_mB_i^j \, t^{i-1} \right] e^{-t\delta^j} \tag{113}$$

which can be evaluated for any time t, and is <u>exact</u> for the representation given in equation (60).

IV-3. Comparison of the Techniques.

The two methods for evaluating multiple convolution integrals have been programmed as subroutines INTEGR, both of which are included in the Appendix. To compare the two techniques, a five-fold multiple convolution integral of the form given in equation (52) has been evaluated using both techniques. The $\gamma_i(t)$'s which were used were all given by the following equation:

$$\gamma_i(t) = \sum_{j=1}^{5} G_i^j \, e^{-t\delta^j} \tag{114}$$

where

$$G_i^1 = 5.0$$

$$G_i^j = -1.0 \qquad j = 2,3,4,5$$

$$\delta^1 = 0.$$

$$\delta^2 = 1.0$$

$$\delta^3 = \sqrt{10}/10.$$

$$\delta^4 = .1$$

$$\delta^5 = \sqrt{10}/100.$$

The result of these integrations ($I_5(t)$) is given in Figure 4. A comparison of the numerical values obtained at various times, and the per cent difference, is given in Table 1. The numerical evaluation was performed using an initial spacing of .2 seconds, for 50 equal spacings, and then doubling the interval, as previously described. The exact evaluations used an equal $\log_{10} t$ spacing of .0625.

It is clear from Table 1 that both techniques give essentially the same result in this case, and that thus either technique is suitable for evaluating this particular multiple convolution integral.

IV-4. Solution of the Integral Equation.

The general integral equation (53) of the previous chapter can be solved numerically once the $\alpha_i(t)$ and $\beta_i(t)$ terms have been evaluated at appropriate values

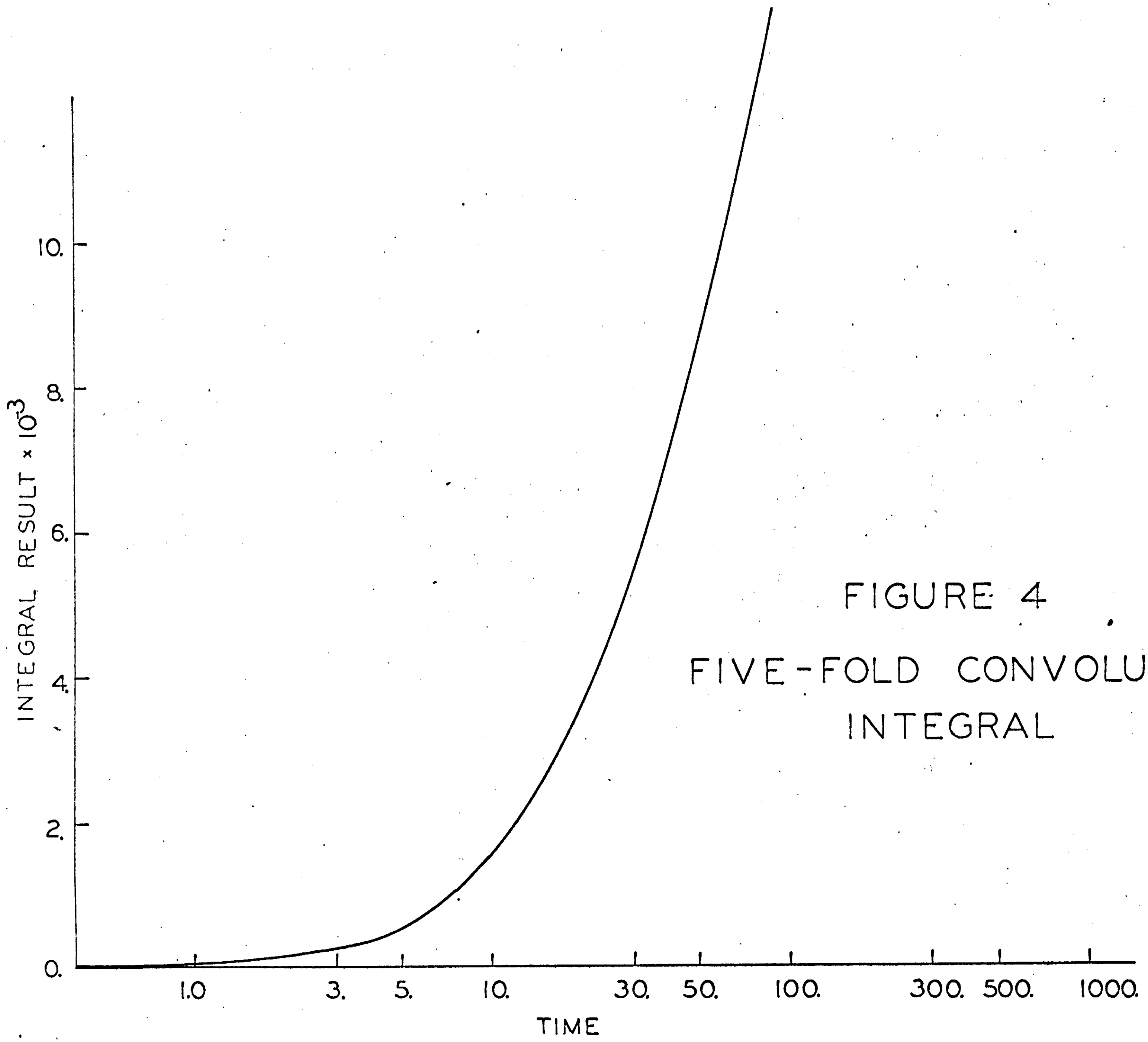


FIGURE 4

FIVE-FOLD CONVOLUTION INTEGRAL

TABLE I

COMPARISON OF FIVE-FOLD MULTIPLE CONVOLUTION INTEGRAL RESULTS

Time	Numerical Evaluation	Exact Evaluation	Per Cent Difference
.10	2.00	2.03	1.5
1.0	24.5	24.6	.4
1.54	51.5	51.5	0.0
5.623	554.	555.	.2
11.55	1760.	1760.	0.0
23.71	4460	4470	.2
31.62	6050.	6050.	0.0
42.17	7860	7870.	.1
64.00	10,700.	10,700.	0.0
100.00	13,300.	----	---

(Accuracy of Table is 3 figures due to necessity of interpolating times.)

of time. In the following it is assumed that this has been done.

To obtain the solution, the integrals on the left side of equation (53) are divided into finite sums. The integrals on the right may presumably be evaluated at any time t by either numerical or direct integration (depending on the method used to evaluate $\alpha_i(t)$ and on the form of $f_i(t)$), and thus can be denoted simply I(t). That is:

$$I(t) = \sum_{i=1}^{n} \theta_i \left\{ \int_{0^+}^{t} f_i(t-\tau) \frac{\partial \alpha_i(\tau)}{\partial \tau} d\tau + f_i(t) \alpha_i(0) \right\} \tag{115}$$

If, for example, the integrals are evaluated numerically using the same procedure used in evaluating $\alpha_i(t)$, then this becomes:

$$I(t_{n_1}) = \sum_{i=1}^{n} \theta_i \left\{ \sum_{j=1}^{n_1} \tfrac{1}{2} \left[f_i(t_{n_1-j+1}) + f_i(t_{n_1-j}) \right] \left[\alpha_i(t_{n_1} - t_{n_1-j}) - \alpha_i(t_{n_1} - t_{n_1-j+1}) \right] + f_i(t_{n_1}) \alpha_i(0) \right\} \tag{116}$$

Dividing the integrals on the left of equation (53) into the same finite sum used above, the general integral equation may be written:

$$\sum_{i=1}^{m} \phi_i \left\{ \sum_{j=1}^{n_1} \tfrac{1}{2} \left[\Psi(t_{n_1-j+1}) + \Psi(t_{n_1-j}) \right] \left[\beta_i(t_{n_1} - t_{n_1-j}) - \beta_i(t_{n_1} - t_{n_1-j+1}) \right] + \Psi(t_{n_1}) \beta_i(0) \right\} = I(t_{n_1}) \tag{117}$$

Rearranging the summations, and separating $\Psi(t_{n_1})$, the equation becomes:

$$\Psi(t_{n_1})\sum_{i=1}^{m}\frac{\phi_i}{2}\left[\beta_i(t_{n_1}-t_{n_1-1})+\beta_i(0)\right]+\Psi(t_{n_1-1})\sum_{i=1}^{m}\frac{\phi_i}{2}\left[\beta_i(t_{n_1}-t_{n_1-1})-\beta_i(0)\right]$$

$$+\sum_{j=2}^{n_1}\left[\Psi(t_{n_1-j+1})+\Psi(t_{n_1-j})\right]\left[\sum_{i=1}^{m}\frac{\phi_i}{2}\left[\beta_i(t_{n_1}-t_{n_1-j})-\beta_i(t_{n_1}-t_{n_1-j+1})\right]\right]=I(t_{n_1}) \tag{118}$$

This equation is now solved to give a recurrence relation for $\Psi(t_{n_1})$ which allows each successive value of $\Psi(t_j)$ to be obtained once the previous values have been obtained:

$$\Psi(t_{n_1})=\frac{\left[\begin{array}{l} I(t_{n_1})-\Psi(t_{n_1-1})\sum_{i=1}^{m}\frac{\phi_i}{2}\left[\beta_i(t_{n_1}-t_{n_1-1})-\beta_i(0)\right] \\ -\sum_{j=2}^{n_1}\left[\Psi(t_{n_1-j+1})+\Psi(t_{n_1-j})\right]\left[\sum_{i=1}^{m}\frac{\phi_i}{2}\left[\beta_i(t_{n_1}-t_{n_1-j})-\beta_i(t_{n_1}-t_{n_1-j+1})\right]\right]\end{array}\right]}{\sum_{i=1}^{m}\frac{\phi_i}{2}\left[\beta_i(t_{n_1}-t_{n_1-1})+\beta_i(0)\right]} \tag{119}$$

Note that the spacing is again not included explicitly, and thus, if appropriate values of $\beta_i(t_j)$ and $I(t_{n_1})$ are available, a variable spacing can be used.

To examine the error propagation in the solution (equation (119)), consider the terms on the right side of equation (119) with the following reasonable simplification that the $\phi_i\beta_i(t)$ terms are of the same order of magnitude, and that hence the summations on i can be dropped in the following. Then the solution can be written

$$\psi(t_n) = \frac{2I(t_n)}{\beta(t_n - t_{n-1}) + \beta(0)} - \psi(t_{n-1}) \frac{\beta(t_n - t_{n-1}) - \beta(0)}{\beta(t_n - t_{n-1}) + \beta(0)}$$

$$- \sum_{j=2}^{n} \left[\psi(t_{n-j+1}) + \psi(t_{n-j}) \right] \frac{\beta(t_n - t_{n-j}) - \beta(t_n - t_{n-j+1})}{\beta(t_n - t_{n-1}) + \beta(0)} \tag{120}$$

in which it is clear that each of the previous terms add much less than their full value (and their error) into the next $\psi(t_n)$ being solved for. Since the solution does not depend strongly on the previous values, it is expected that the error in each interval will be decreased when this result is used to obtain new results, and that the error will attenuate.

IV-5. Implications of the Technique Used to Evaluate the Convolution Integrals

As noted above, the method used in solving the integral equation does not require equally spaced intervals. However, if the multiple convolution integrals are evaluated numerically at equally spaced intervals, then of necessity the integral equation will have to be solved at these same equally spaced intervals. When the interval is doubled in the numerical integrations, then the interval can also be doubled in the equation solution. With the exact evaluation of the convolution integrations, however, the result can be easily evaluated at any time t, and hence a variable spacing can be used.

The exact evaluation of the multiple convolution integrals offers two other distinct advantages. First of all, since each $\beta_i(t)$ is of the form given in equation (113), the summations on i can be carried out before the $\beta_i(t)$ terms are evaluated. That is, the terms

$$\sum_{i=1}^{m} \phi_i \, \beta_i(t_n)$$

can be written as

$$\sum_{i=1}^{m} \phi_i \, \beta_i(t_n) = \sum_{j=1}^{n} \left\{ \sum_{\ell=1}^{q+1} \left[\sum_{i=1}^{m} ({}_pB_\ell^j)_i \, \phi_i \right] t_n^{\ell-1} \right\} e^{-t_n \delta^j} \qquad (121)$$

where q is the maximum number of convolution integrations of any β_i. The result in equation (121) can be expressed as:

$$\rho(t_n) = \sum_{j=1}^{n} \left\{ \sum_{\ell=1}^{q+1} C_\ell^j \, t_n^{\ell-1} \right\} e^{-t_n \delta^j} \qquad (122)$$

and with this notation the solution equation (119) becomes more simply (and more easily evaluated):

$$\psi(t_{n_1}) = \frac{\left[\begin{array}{l} 2I(t_{n_1}) - \psi(t_{n_1-1})\left[\rho(t_{n_1}-t_{n_1-1})-\rho(0)\right] \\ -\sum_{j=2}^{n_1}\left[\psi(t_{n_1-j+1})+\psi(t_{n_1-j})\right]\left[\rho(t_{n_1}-t_{n_1-j})-\rho(t_{n_1}-t_{n_1-j+1})\right] \end{array}\right]}{\rho(t_{n_1}-t_{n_1-1})+\rho(0)} \qquad (123)$$

The second advantage of the exact evaluation procedure is that it provides a fairly direct check on the solution of the integral equation. To perform the check, a Dirichlet series must first be fitted to the numerical solution. For the examples considered in this dissertation, a simple collocation procedure has been used (the collocation is performed by a single matrix multiplication, in a subroutine CVEFIT which is included in the appendix). Such a Dirichlet series can be integrated exactly such that

$$\int_{0^+}^{t} \psi(t-\tau)\frac{\partial\rho(\tau)}{\partial\tau}d\tau + \psi(t)\rho(0) \qquad (124)$$

can then be evaluated at any time t. A comparison of the left-hand side of the original equation (expression (124)) with the original right-hand side (I(t)) serves as a check on the solution.

IV-6. Numerical Example.

The numerical solution of the general integral equation has been programmed for the case that I(t) is expressible in the form of equation (113). If the convolution integrals are evaluated numerically, then the subroutine SOLVIT is used. If the convolution integrals are in the form of (113), then the subroutine SOLVE is used.

As a comparison of the results using these techniques and of the results versus known exact solutions, the following integral equation has been solved to obtain $\Psi(t)$ by both techniques:

$$\int_{0^+}^{t} \Psi(t-\tau)\,\frac{\partial \beta(\tau)}{\partial \tau}\,d\tau + \Psi(t)\beta(0) = \alpha(t) \tag{125}$$

where

$\beta(t) =$ four-fold convolution of $\gamma_i(t)$

$\alpha(t) =$ five-fold convolution of $\gamma_i(t)$

$\gamma_i(t)$ is given in equation (114).

The exact solution to this equation is just $\gamma_i(t)$, that is,

$$\Psi(t) = \gamma_i(t) = 5 - e^{-t} - e^{-\frac{\sqrt{10}}{10}t} - e^{.1t} - e^{\frac{\sqrt{10}}{100}t} \tag{126}$$

which is plotted in Figure 5.

Table II compares the exact solution with that obtained using the numerical integration procedure. Table III compares the exact solution with that obtained using the exact integration approach. Table IV gives the check discussed above for the exact integration solution. Clearly the errors are small enough to be disregarded in any engineering application, since the largest error (recorded in the check of the left-side of the equation versus the right side) is less than one and one-half per cent.

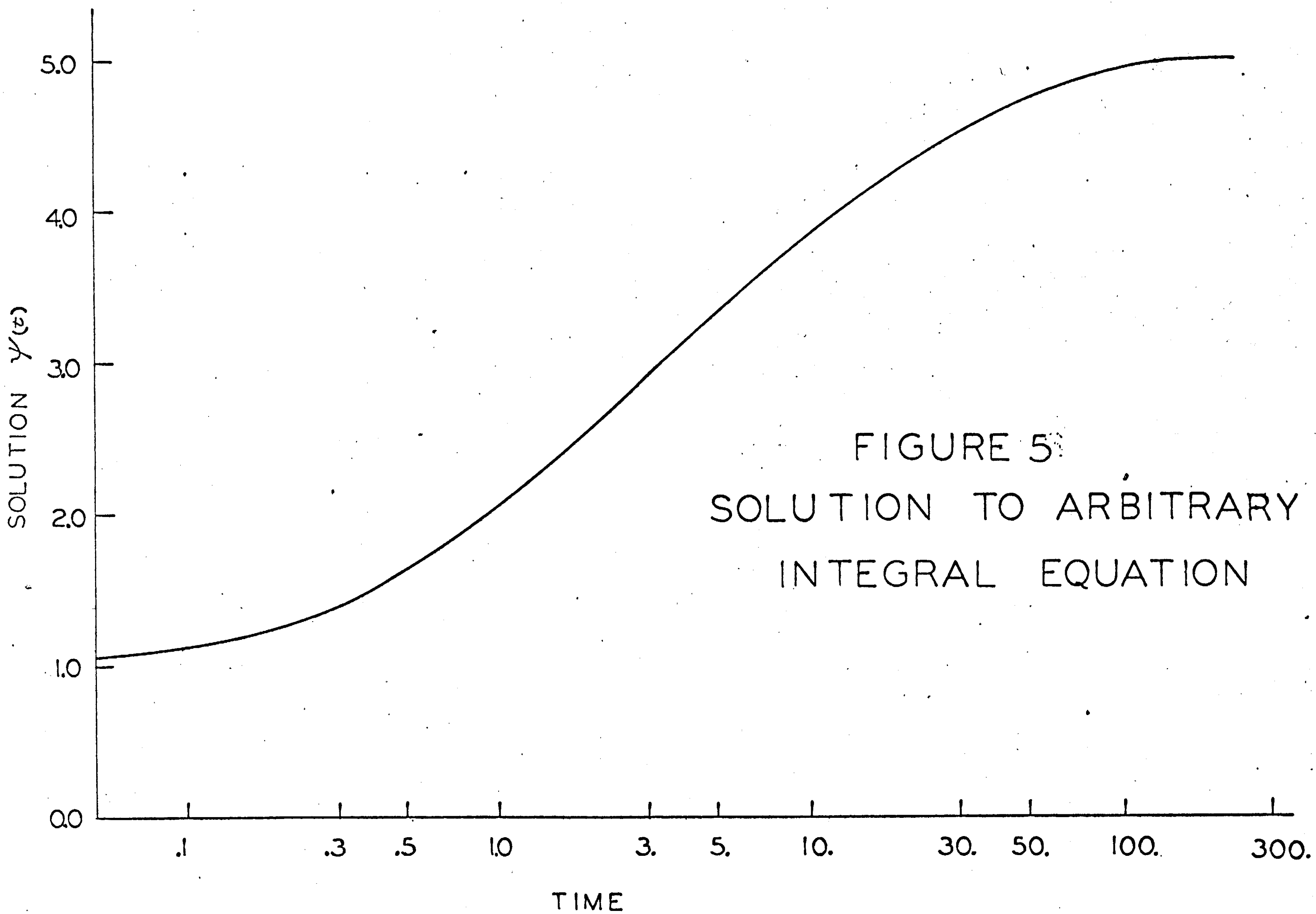


FIGURE 5
SOLUTION TO ARBITRARY INTEGRAL EQUATION

TABLE II

ERRORS IN SOLUTION OF INTEGRAL
EQUATION - NUMERICAL INTEGRATION

Time	Exact	Numerical	% Error
.10	1.1393967	1.1393967	0.00000
.20	1.2686605	1.2686596	+.00007
.50	1.6041727	1.6041784	-.00036
1.00	2.0295115	2.0295115	0.00000
1.36	2.2621241	2.2621269	-.00010
1.68	2.4321442	2.4321270	+.00071
2.00	2.5759306	2.5759268	+.00015
2.40	2.7275772	2.7275639	+.00049
3.20	2.9658089	2.9657574	.00173
4.0	3.1479130	3.1477318	.00570
5.76	3.4394388	3.4390802	.01043
7.04	3.5961704	3.5964375	-.00743
8.00	3.6941786	3.6949921	-.02220
10.88	3.9221535	3.9226503	-.01275
14.08	4.1030493	4.1027336	+.00732
16.00	4.1888266	4.1887054	+.002865
20.48	4.3461809	4.3460541	+.002991
25.60	4.4773235	4.4787922	.03127
32.00	4.5956745	4.6001835	-.10009
44.80	4.7461500	4.7467737	-.01306
55.04	4.8204975	4.8129482	+.15560
64.0	4.8661919	4.8673563	-.02466

TABLE III

ERRORS IN SOLUTION OF INTEGRAL EQUATION -- EXACT INTEGRATION

Time	Exact	Calculated	% Error
.100	1.1393967	1.1516581	+1.076
.205	1.2753115	1.2786722	+ .264
.316	1.4073467	1.4053669	- .142
.649	1.7464705	1.7463741	- .057
1.00	2.0295172	2.0308418	+ .064
2.05	2.5978622	2.6006689	+ .108
3.16	2.9560595	2.9558239	- .008
6.49	3.5334749	3.5357094	+ .067
10.00	3.8608513	3.8567371	- .106
20.54	4.3478355	4.3631077	+ .345
31.62	4.5897446	4.5771999	- .272
64.94	4.8702040	4.8823967	+ .248
100.0	4.9576244	4.9496946	- .161
205.4	4.9984865	5.0082741	+ .200
316.2	4.9999542	4.9904718	- .190

TABLE IV

COMPARISON OF LEFT- AND RIGHT-HAND SIDES OF INTEGRAL EQUATION

Time	Left	Right	% Difference
.100	2.0586	2.0313	1.326
.205	3.4180	3.3711	1.372
.316	5.1875	5.1172	1.355
.649	12.7891	12.6406	1.173
1.00	24.8047	24.5625	.976
2.05	87.6211	87.1002	.594
3.16	195.777	194.988	.403
6.49	712.642	710.867	.250
10.00	1426.69	1423.90	.196
20.54	3780.07	3780.05	.001
31.62	6042.92	6050.34	.123
64.94	10,767.4	10,774.7	.068
100.0	13,332.1	13,336.5	.033
205.4	15,388.7	15,426.2	.244
316.2	15,583.5	15,612.5	.186

CHAPTER V

DEFLECTION OF A VISCOELASTIC CANTILEVER BEAM

As a first illustration of the methods of analysis described in the previous chapters, the analysis of the deflection of a viscoelastic cantilever beam under the action of a time-varying point load applied at the unsupported end will be presented. The analysis will be presented for a beam with arbitrary linear viscoelastic characterization for the equivalent elastic shear modulus and elastic bulk modulus. A specific example will then be presented in which the equivalent modulii are characterized by the behavior of simple models. With this characterization, an explicit solution can be obtained using the Laplace transform. This solution is presented, and the error in the numerical solution is thus obtained and presented for this specific case. A second example using more realistic relaxation functions is then presented, and several implications of the results are discussed.

V-1. Formulation of the General Solution.

The geometry of the beam is presented in Figure 6. With the boundary conditions

$$\left. \begin{matrix} u_1 \\ u_2 \end{matrix} \right|_{\substack{x_1=\ell \\ x_2=0}} = 0 \qquad \left. \frac{\partial u_1}{\partial x_2} \right|_{\substack{x_1=\ell \\ x_2=0}} = 0 \qquad (127)$$

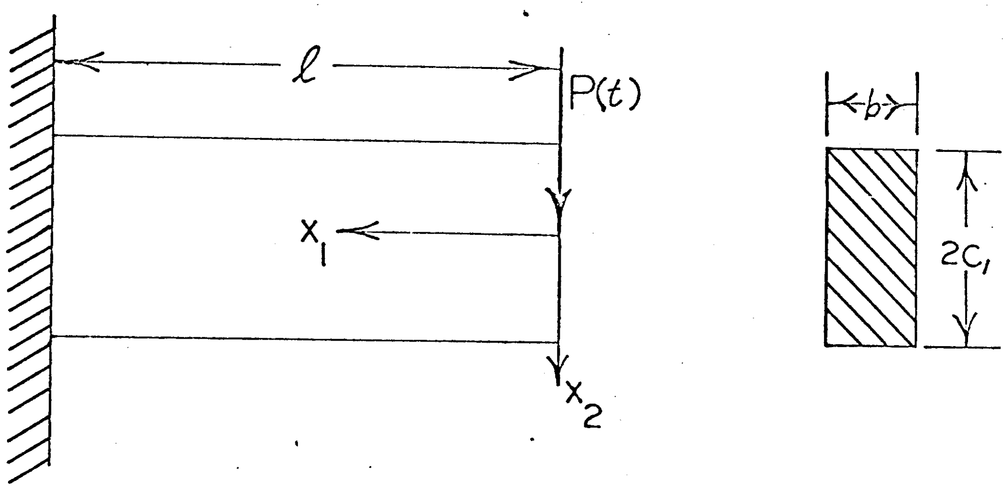


FIGURE 6

GEOMETRY OF CANTILEVER BEAM

the solution for the deflection in the X_2 direction for an elastic beam is given [127] as

$$u_2(t)\Big|_{X_2=0} = \frac{\theta_1 K P(t) + \theta_2 G P(t)}{\phi_1 K G} \tag{128}$$

where

G = elastic shear modulus

K = bulk modulus

$\theta_1 = 3(X_1^3 - 3\ell^2 X_1 + 2\ell^3) + 27 c_1^2(\ell - X_1)/2$

$\theta_2 = X_1^3 - 3\ell^2 X_1 + 2\ell^3$

$\phi_1 = 54I$

I = moment of inertia of the beam

Equation (128) is of the general form of equation (46) where now

$$\begin{aligned} \psi(t) &= u_2(t)\Big|_{X_2=0} \\ \alpha_1 &= K \\ \alpha_2 &= G \\ \beta_1 &= KG \\ f_1(t) &= f_2(t) = P(t) \end{aligned} \tag{129}$$

Consequently, the corresponding viscoelastic solution for the cantilever beam can be written immediately as follows:

$$\phi_1\left[\int_{0^+}^{t} U_2(t-\tau)\frac{\partial \beta_1(\tau)}{\partial \tau}d\tau + U_2(t)\beta_1(0)\right]$$

$$= \sum_{i=1}^{2}\theta_i\left[\int_{0^+}^{t} P(t-\tau)\frac{\partial \alpha_i(\tau)}{\partial \tau}d\tau + P(t)\alpha_i(0)\right] \tag{130}$$

where

$$\beta_1(t) = \int_{0^+}^{t} K_r(t-\lambda)\frac{\partial G_r(\lambda)}{\partial \lambda}d\lambda + K_r(t)G_r(0) \tag{131}$$

$$\alpha_1(t) = K_r(t) \tag{132}$$

$$\alpha_2(t) = G_r(t) \tag{133}$$

and $G_r(t)$ and $K_r(t)$ are defined in terms of the following constitutive equations:

$$\sigma(t) = 3\int_{0^-}^{t} K_r(t-\tau)\frac{\partial e(\tau)}{\partial \tau}d\tau \tag{134}$$

$$S_{ij}(t) = 2\int_{0^-}^{t} G_r(t-\tau)\frac{\partial \epsilon_{ij}(\tau)}{\partial \tau}d\tau \tag{135}$$

The 3 and 2 in equations (134) and (135), respectively, are used in these equations so that the "equivalent elastic modulii" will be just operators, without multiplicative constants, since for the elastic case $\sigma = 3Ke$ and $S_{ij} = 2G\epsilon_{ij}$.

V-2. First Numerical Example, Exact Solution Known.

The solution of the general equation (130) for the deflection of a viscoelastic cantilever beam has been programmed for both techniques discussed in the previous chapter. These programs are presented in the appendix.

As a first illustration of the solution, consider a load function

$$P(t) = \frac{e^{-t/10\tau_1} - e^{-t/\tau_1}}{.9} \qquad (136)$$

as shown in Figure 7, and relaxation functions

$$G_r(t) = G_o\, e^{-t/\tau_1} \qquad (137)$$

$$K_r(t) = K_o \frac{e^{-t/\tau_1} - e^{-t/10\tau_1}}{.9} \qquad (138)$$

which are shown in Figure 8. The relations (137) and (138) were selected in order that an exact solution could be easily obtained. As shown in Figure 8, the bulk modulus becomes negative (which is physically

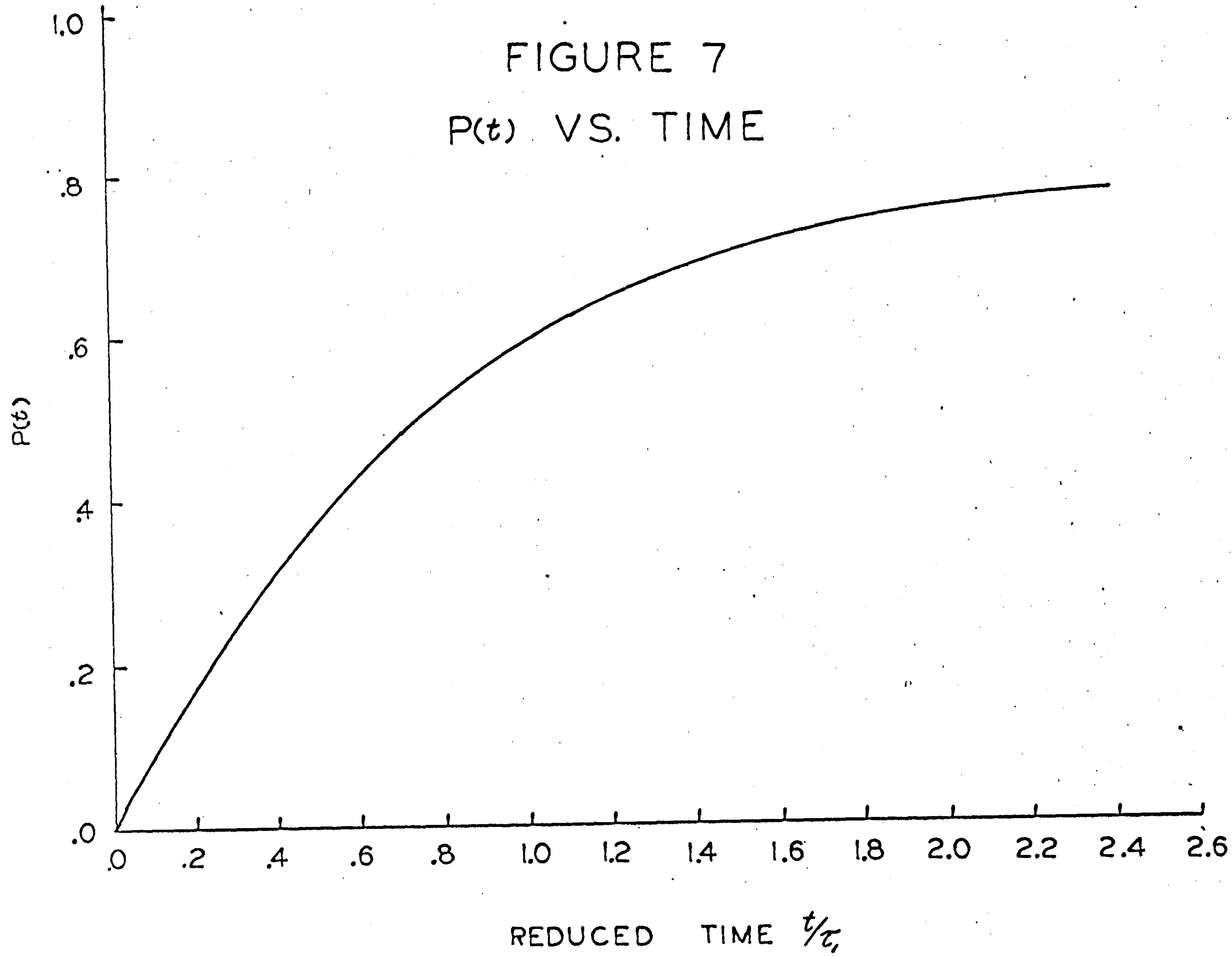


FIGURE 7

P(t) VS. TIME

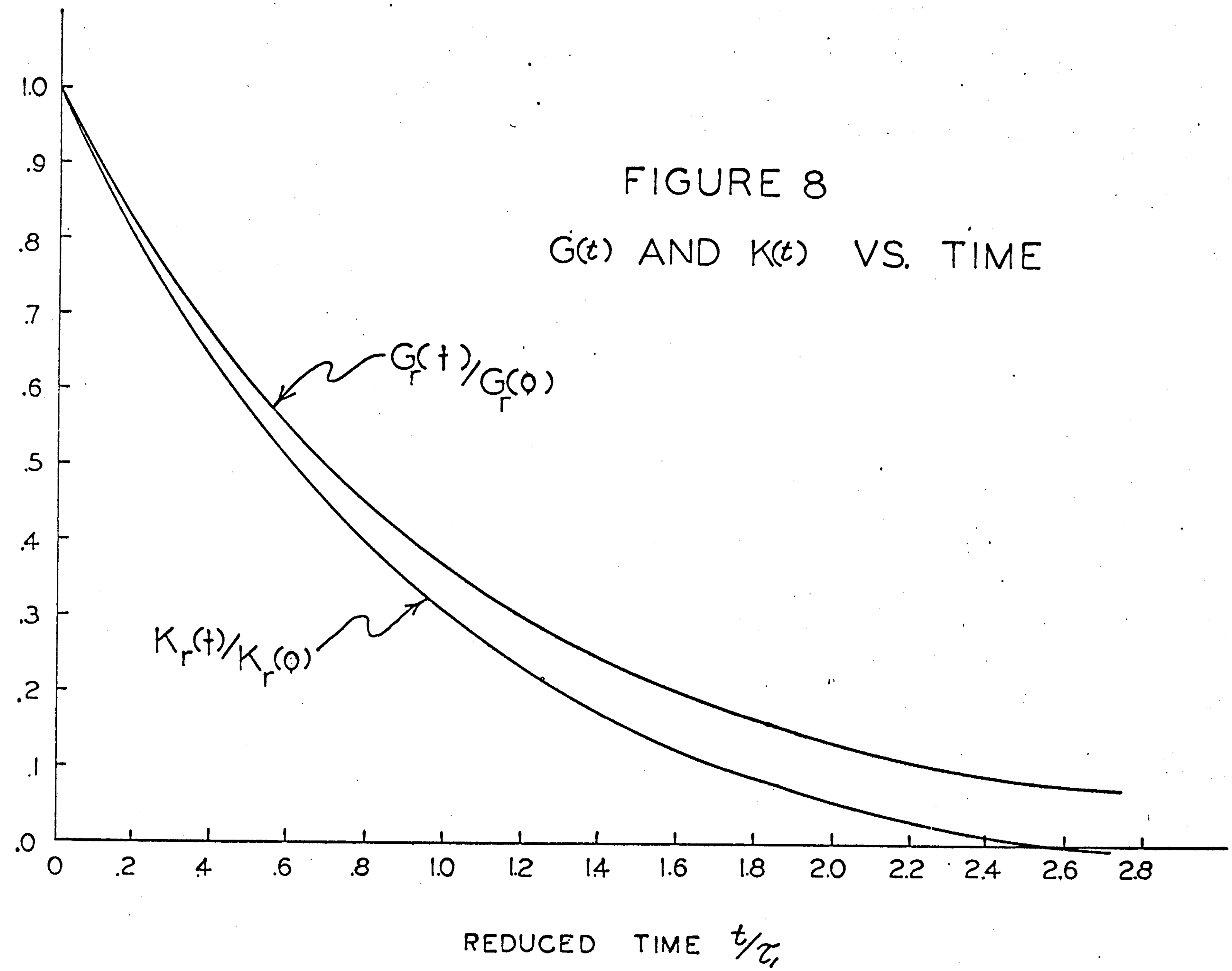


FIGURE 8

G(t) AND K(t) VS. TIME

impossible) before $t/\tau_1 = 2.6$. For this reason the results will be presented only up to $t/\tau_1 = 2.40$ seconds.

Transforming both sides of equation (130) using the Laplace transform, one obtains the following relationship:

$$\frac{\phi_1 U_2^*(s)\, s^3 G_0 K_0}{(s+\frac{1}{\tau_1})^2\,(s+\frac{1}{10\tau_1})} = \frac{\theta_1\, s^2 K_0}{(s+\frac{1}{\tau_1})^2(s+\frac{1}{10\tau_1})^2} + \frac{\theta_2\, s\cdot G_0}{(s+\frac{1}{\tau_1})^2(s+\frac{1}{10\tau_1})} \qquad (139)$$

Solving for $U_2^*(s)$:

$$U_2^*(s) = \frac{\theta_1}{\phi_1 G_0\, s(s+\frac{1}{10\tau_1})} + \frac{\theta_2}{\phi_1\, s^2 K_0} \qquad (140)$$

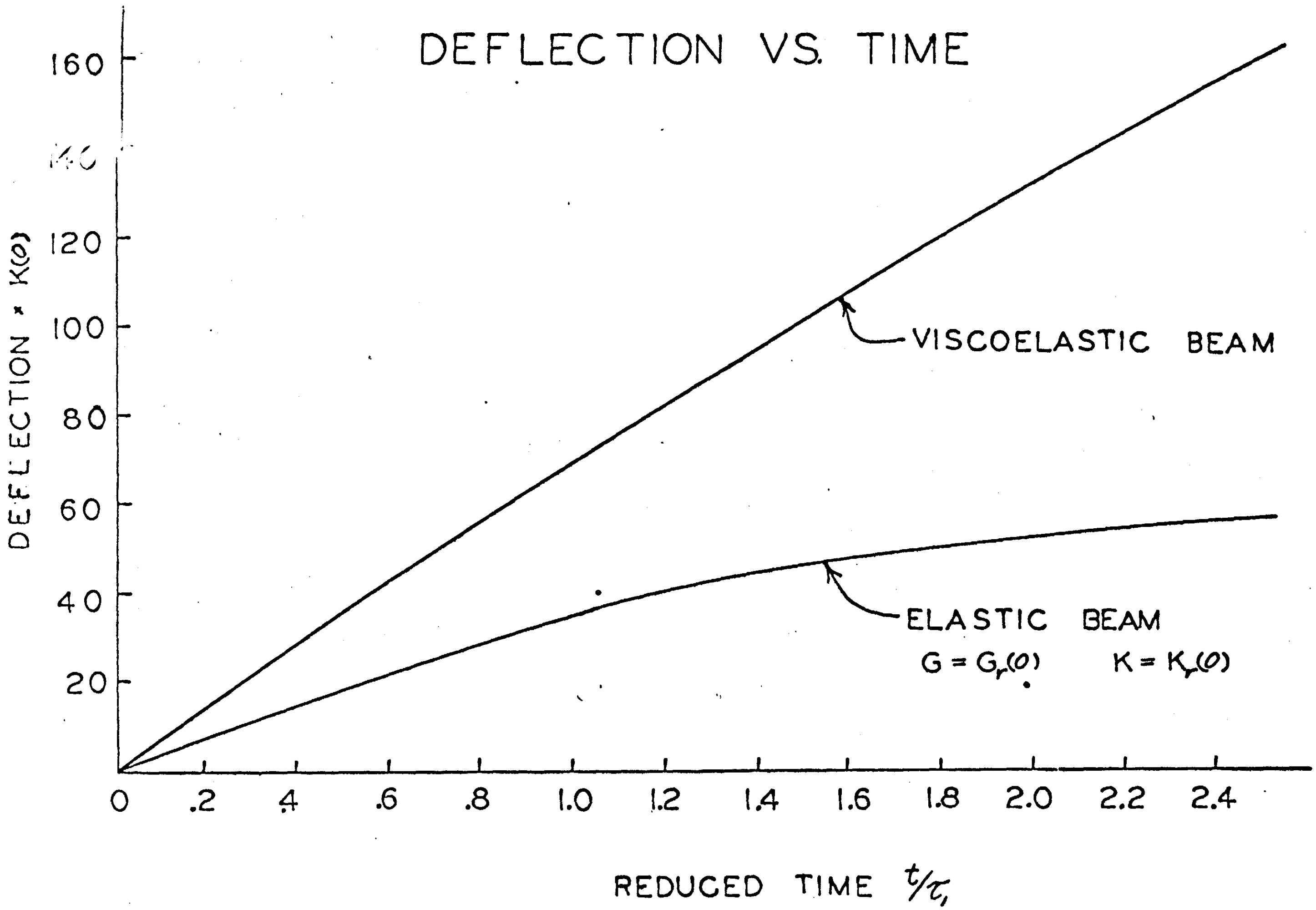
Performing now the inverse Laplace transform, the solution $U_2(t)$ is obtained as:

$$U_2(t) = \frac{\theta_1}{\phi_1 G_0}\left(e^{-t/10\tau_1} - H(t)\right) + \frac{\theta_2}{\phi_1 K_0}\, t \qquad (141)$$

This solution is plotted in Figure 9 for the particular case of

$$\begin{aligned} &\ell = 20 && b = .354 \\ &X_1 = 0 && I = 18 \\ &c_1 = 4.24 && \\ &G_0 = K_0 && \end{aligned} \qquad (142)$$

FIGURE 9


DEFLECTION VS. TIME
160
140
120
100
80
60
40
20
DEFLECTION × K(0)
VISCOELASTIC BEAM
ELASTIC BEAM
G = G_r(0) K = K_r(0)
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
REDUCED TIME t/τ₁

The deflection of an elastic beam with $G = G_r(0)$, $K = K_r(0)$, is also plotted in Figure 9 for comparison.

Equation (130) has been solved numerically for the above input, by both techniques, and these results are compared in Tables V and VI. The results were obtained only up to $t/\tau_1 = 2.40$ at which time the bulk modulus becomes negative. The errors shown in these tables are quite small. In Table VII the result of fitting the solution obtained using the exact integration procedure with a Dirichlet series is compared with the exact solution. The errors are still small, although at very short times some error is noted. This error in fitting the numerical solution shows up markedly in Table VIII, where the left-hand and right-hand sides of the original integral equation are compared. Although the error throughout most of the solution is less than one per cent, it increases markedly, in this checking procedure, at the end-points. A more careful curve-fitting scheme, for instance a least squares fit, would probably decrease this error, since the original numerical solution has been shown to be quite accurate.

V-3. <u>Second Numerical Solution.</u>

A second solution has been obtained for a beam with the same geometry used in the above example. In

TABLE V

DEFLECTION OF A VISCOELASTIC CANTILEVER BEAM, ERRORS, NUMERICAL INTEGRATION TECHNIQUE

Time	Exact	Numerical	% Error
0.	0.00	0.00	0.00
.10	7.0573	7.0508	.09
.20	14.0607	14.0477	.09
.30	21.0107	20.9914	.09
.40	27.9081	27.8823	.09
.50	34.7532	34.7211	.09
.60	41.5465	41.5081	.09
.70	48.2886	48.2439	.09
.80	54.9800	54.9290	.09
.90	61.6221	61.5640	.09
1.00	68.1493	68.2128	.09
1.20	81.2487	81.1729	.09
1.40	94.0916	94.0035	.09
1.60	106.7454	106.6451	.09
1.80	119.2139	119.1016	.09
2.00	131.5005	131.3763	.09
2.20	143.6094	143.4730	.09
2.40	155.5431	155.3953	.10

TABLE VI

DEFLECTION OF A VISCOELASTIC CANTILEVER BEAM, ERRORS, EXACT INTEGRATION TECHNIQUE

Time	Exact	Numerical	% Error
0.	0.	-.00007	-----
.0316	2.23754	2.23746	.003
.10	7.0573	7.0570	.004
.154	10.8453	10.8446	.006
.205	14.4340	14.4327	.009
.274	19.1979	19.1948	.016
.365	25.5120	25.505	.03
.487	33.8640	33.8471	.05
.649	44.8823	44.8415	.09
.750	51.6336	51.5703	.12
1.00	68.2129	68.0588	.23
1.33	89.8439	89.4667	.42
1.78	117.869	116.938	.79
2.37	153.846	151.534	1.50

TABLE VII

DEFLECTION OF A VISCOELASTIC CANTILEVER BEAM, ERRORS, FITTED SOLUTION

Time	Exact	Numerical	% Error
0.	0.	.059	----
.0316	2.23754	2.2813	-1.95
.10	7.0573	7.1211	- .90
.154	10.8453	10.9219	- .71
.205	14.4340	14.5195	- .59
.274	19.1979	19.2734	- .39
.365	25.5120	25.5430	- .12
.487	33.8640	33.8086	.16
.649	44.8823	44.7422	.31
.750	51.6336	51.4727	.31
1.00	68.2129	68.1172	.14
1.33	89.8439	89.9844	- .16
1.78	117.869	118.1289	- .22
2.37	153.846	152.9883	.56

TABLE VIII

DEFLECTION OF A VISCOELASTIC CANTILEVER BEAM, COMPARISON OF LEFT- AND RIGHT-HAND SIDES OF EQUATION

Time	Left	Right	% Difference
.01	731.	681.	6.8
.0316	2144.	2104.	1.9
.10	6222.	6174.	0.8
.154	9020.	8961.	0.7
.205	11352.	11294.	0.5
.274	14003.	13964.	0.3
.365	16817.	16826.	0.1
.487	19502.	19581.	0.4
.649	21603.	21730.	0.6
.750	22248.	22368.	0.5
1.00	22283.	22290.	0.0
1.33	20040.	19867.	0.8
1.78	15009.	14906.	0.7
2.37	7434.	8142.	9.5

this case, the load used was a step function, that is:

$$P(t) = H(t) \tag{143}$$

and the relaxation functions were described by the following Dirichlet series:

$$\frac{G_r(t)}{K_r(0)} = .2 + .5e^{-t/\tau_1} + .2e^{-t/10\tau_1} + .1e^{-t/100\tau_1} \tag{144}$$

$$\frac{K_r(t)}{K_r(0)} = .5 + .2e^{-t/\tau_1} + .2e^{-t/10\tau_1} + .1e^{-t/100\tau_1} \tag{145}$$

These relaxation functions are plotted in Figure 10. Also plotted in Figure 10 are $G_r(t)/K_r(0)$ and $K_r(t)/K_r(0)$ without the short time relaxation behavior of the e^{-t/τ_1} term, that is:

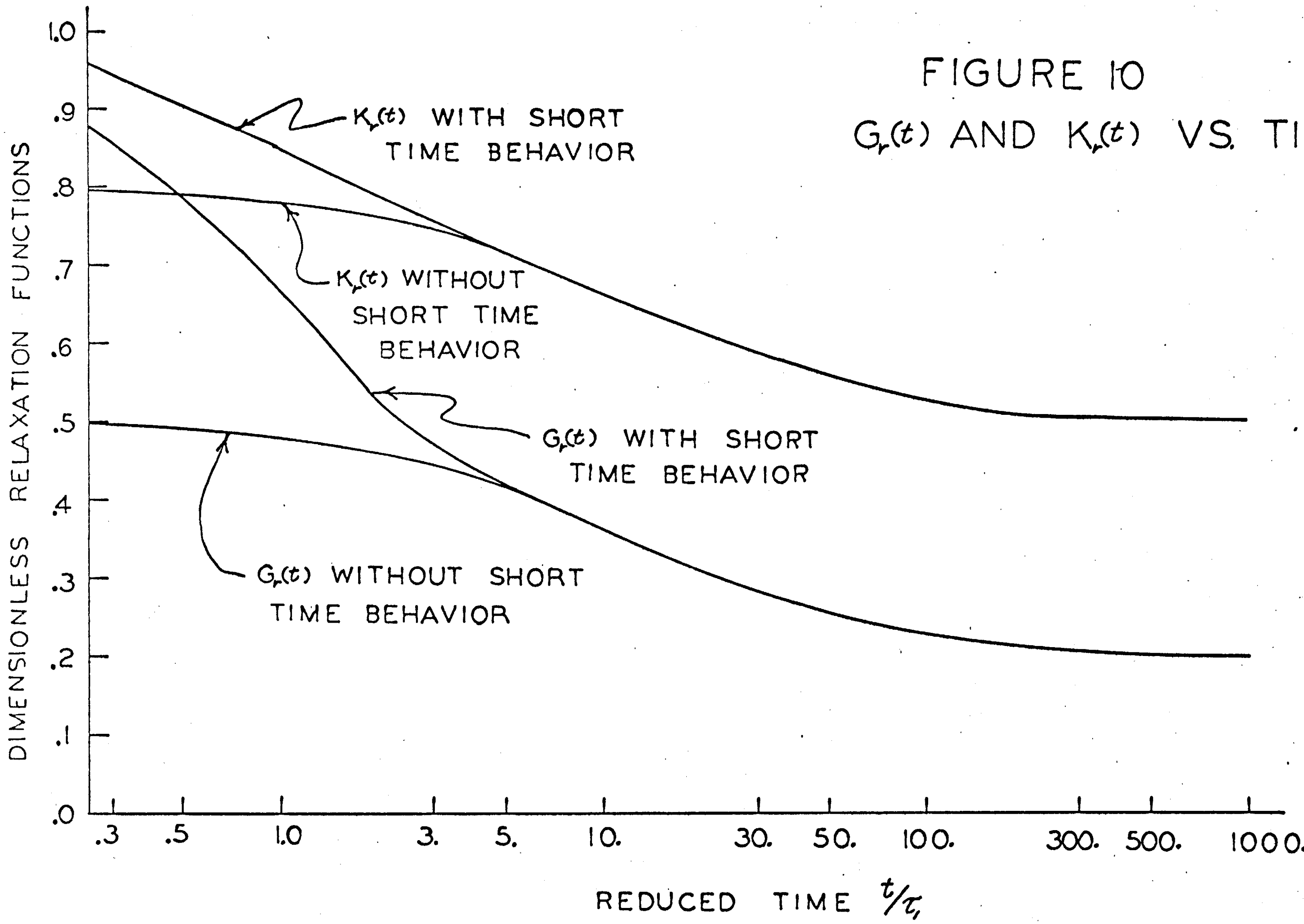
$$\frac{G_r(t)}{K_r(0)} = .2 + .2e^{-t/10\tau_1} + .1e^{-t/100\tau_1} \tag{146}$$

$$\frac{K_r(t)}{K_r(0)} = .5 + .2e^{-t/10\tau_1} + .1e^{-t/100\tau_1} \tag{147}$$

The solution for the end deflection using both sets of relaxation functions has been obtained using both numerical techniques. Both solutions are plotted in Figure 11, and numerical values are compared in Table IX. Clearly the solutions converge when $t/\tau_1 > 40$. This behavior has a practical implication: Short-time behavior cannot appreciably affect long-time results. Consequently, if one is interested in long-time results,

FIGURE 10

$G_r(t)$ AND $K_r(t)$ VS. TIME



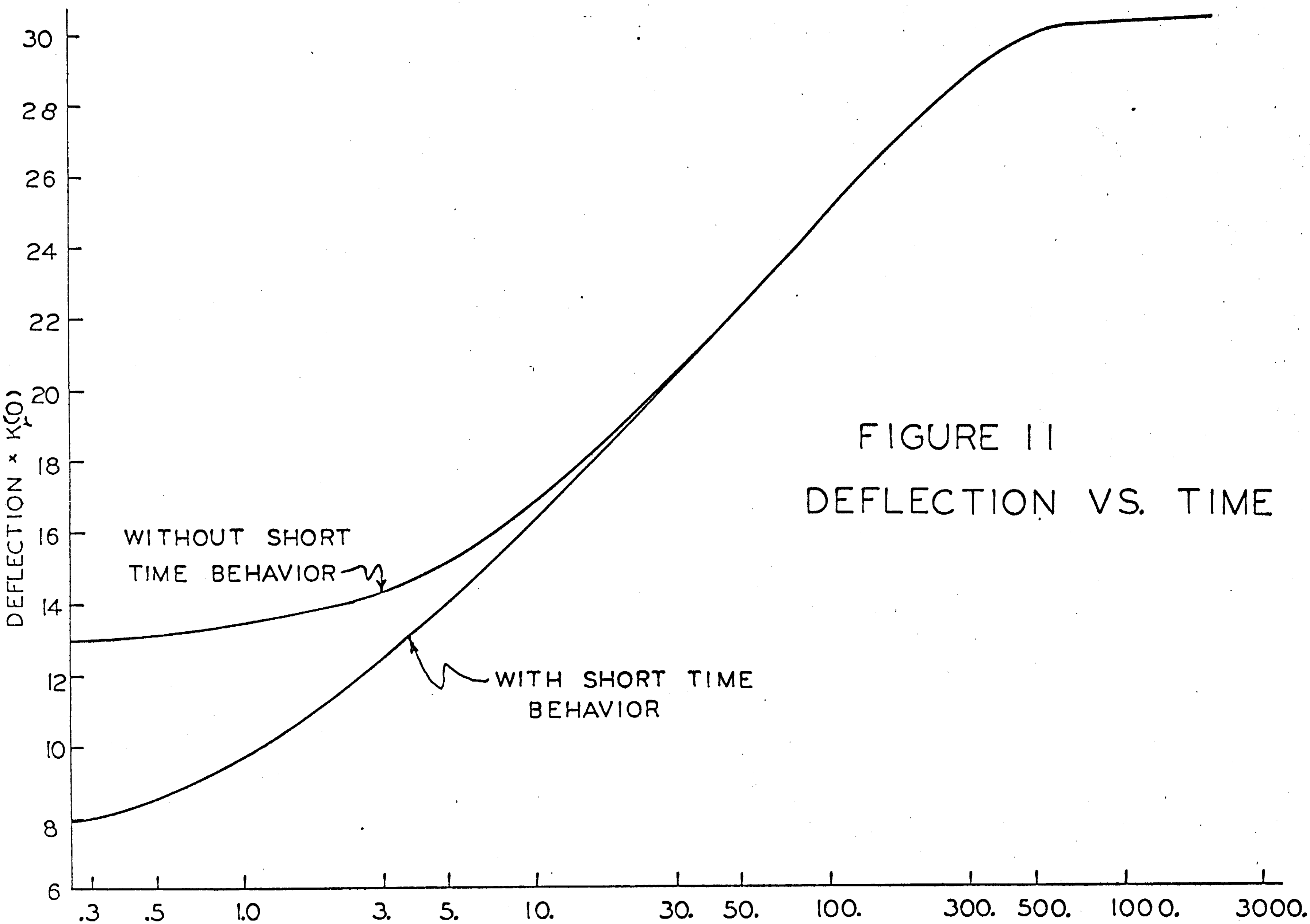


FIGURE 11
DEFLECTION VS. TIME

the very rapidly varying short-time behavior can be neglected, and consequently greater time spacings can be used, thus saving computational effort.

In Table X the solution obtained, for the relaxation functions given in equations (144) and (145), by both techniques, as well as the fitted solution of the exact integration technique, are compared. The solutions quite obviously agree. In Table XI the left- and right-hand sides of the original integral equation are compared by means of the fitted solution. Fairly good agreement is shown.

TABLE IX

CONVERGENCE OF SOLUTIONS WITH
AND WITHOUT SHORT TIME BEHAVIOR

Time	Solution 1 (with Fast Time Behavior)	Solution 2	% Difference
0.	7.08	12.92	82.5
.2	7.69	13.07	70.0
.5	8.51	13.20	55.0
1.0	9.65	13.45	39.3
1.5	10.57	13.69	29.7
2.5	11.95	14.12	18.2
4.0	13.33	14.80	11.0
5.0	13.99	15.20	8.6
8.0	15.41	16.26	5.5
10.0	16.14	16.88	4.6
16.0	17.84	18.42	3.2
20.0	18.72	19.25	2.8
40.0	21.50	21.80	1.4
80.0	24.02	24.20	0.7
160.0	26.63	26.80	0.6
320.0	----	29.1	----
640.0	----	30.3	----

TABLE X

COMPARISON OF SOLUTIONS FOR CANTILEVER BEAM

Time	Solution 1 (Numerical Integration)	Solution 2 (Exact Integration)	Solution 3 (Fitted)
0.	7.084	7.084	7.108
.1	7.396	7.396	7.426
1.0	9.650	9.[illegible]	9.673
10.0	16.139	16.[illegible]	16.066
100.0	24.796	23.[illegible]	23.981
1000.0	30.433	29.[illegible]	29.806
10000.0	----	30.[illegible]	30.468
100000.0	----	30.[illegible]	30.486

TABLE XI

COMPARISON OF LEFT- AND RIGHT- HAND
SIDES OF INTEGRAL EQUATION

Time	Left	Right	% Difference
0.0	69086.	68860.	0.3
.10	67215.	66949.	0.4
1.0	55821.	55741.	0.1
10.0	40435.	40928.	1.2
100.0	30464.	32164.	5.6
1000.0	28742.	29630.	3.1
10000.0	29525.	29630.	0.4
100000.0	29600.	29630.	0.1

CHAPTER VI

ANALYSIS OF A THREE-LAYER VISCOELASTIC HALF-SPACE

In this chapter, a second illustration of the methods of analysis described in Chapters III and IV, the analysis of a three-layer linear viscoelastic half-space under a uniformly distributed circular load will be presented. This problem demonstrates the capability of both of the previously described approaches for solving the general integral equation on an involved problem. This problem, furthermore, demonstrates the relative simplicity of the present approach in formulating the general solution compared to other methods of solution.

In addition to the above motivation for this example, the analysis contained in this chapter has direct application in the study of layered highway systems, and is thus of considerable practical engineering interest. For this reason, and because most of the following is unavailable elsewhere, the analysis will be presented in a reasonably detailed fashion.

The elastic analysis for layered systems has been formulated by several authors [21,58,117], using basically Burmister's approach [21]. An explicit statement of the constants involved, however, has not been presented for the three-layer system for any except the first layer, and these are not in a suitable form for the present analysis.

The geometry of the system is shown in Figure 12. The load is distributed over a circle of radius a and is normal to the surface. Each of the layers is assumed to be infinite in horizontal extent. The lower layer is assumed to be infinite in vertical extent. Each layer has distinct physical properties, which will be considered to be functions of time.

In the following analysis, Poisson's ratio has been taken equal to 1/2 in each layer (Bulk modulus infinite). This assumption has been made because of the simplifications that result. Just as in the available elastic analyses[21,33,59], however, it is expected that this assumption will not cause very large errors, and it does decrease the algebra considerably.

The other constitutive relation necessary for each layer will be assumed in terms of a viscoelastic equivalent to the elastic compliance. That is, for the i-th layer:

$$\frac{1}{E_i}\,(\text{equivalent}) = \left[D_{r_i}(0) - \int_0^{t} (\quad) \frac{\partial D_{r_i}(t-\tau)}{\partial \tau}\, d\tau \right] \qquad (148)$$

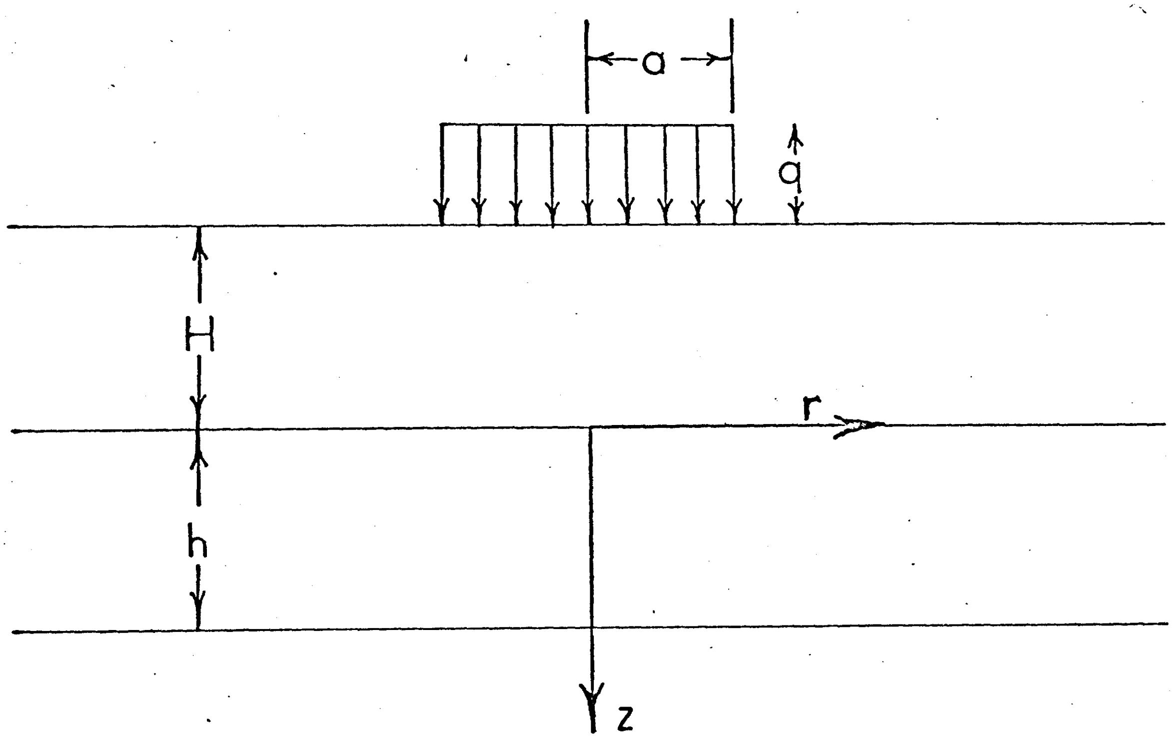


FIGURE 12

CROSS-SECTION OF THREE-LAYER SYSTEM

In the following, $D_{r_i}(t)$ will be denoted simply $D_i(t)$, since it is clear from the context what is implied.

The relationships will be obtained in terms of compliances, rather than elastic modulii, for two reasons. First of all, more data is generally available on creep than on relaxation behavior. Secondly, it is preferable to keep the number of convolution integrations needed on the left-hand side of equation (53) as small as possible, even at the expense of the number of integrations on the right-hand side, since those on the left enter more directly into the numerical solution, and thus errors in these integrations should preferably be minimized. Also, the multiple integrations on the left side must be evaluated at more times when using the exact integrations approach and one thus desires to keep the function representation (equation (113)) as short as possible.

VI-1. Derivation of the Elastic Solution for All Stresses and Displacements.

Assuming an axi-symmetric load distribution, the equations of equilibrium, compatibility, stress, and displacement are given in cylindrical coordinates for a general incompressible symmetrical elastic body in the following form:

Equilibrium:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (149)$$

In the following, $D_{r_i}(t)$ will be denoted simply $D_i(t)$, since it is clear from the context what is implied.

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Equilibrium:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{149}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0 \tag{150}$$

Compatibility:

$$\nabla^4 \varphi = 0 \tag{151}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

Stress Components:

$$\sigma_z = \frac{\partial}{\partial z}\left[1.5\,\nabla^2\varphi - \frac{\partial^2 \varphi}{\partial z^2}\right] \tag{152}$$

$$\sigma_r = \frac{\partial}{\partial z}\left[.5\,\nabla^2\varphi - \frac{\partial^2 \varphi}{\partial r^2}\right] \tag{153}$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left[.5\,\nabla^2\varphi - \frac{1}{r}\frac{\partial \varphi}{\partial r}\right] \tag{154}$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[.5\,\nabla^2\varphi - \frac{\partial^2 \varphi}{\partial z^2}\right] \tag{155}$$

Displacement Components:

$$w = \frac{1.5}{E}\left[\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right] \tag{156}$$

$$u = -\frac{1.5}{E}\frac{\partial^2 \varphi}{\partial r^2} \tag{157}$$

$$u = \frac{1.5}{E} J_1(mr)\left[Am^2e^{mz} + Bm^2e^{-mz} + Cme^{mz}(1+mz) + Dme^{-mz}(mz-1)\right] \tag{163}$$

If now each layer of the layered system is considered to have a solution of the form given in equations (159) through (163), and the constants for each of these solutions are evaluated from the boundary conditions given below, then the problem of an elastic layered system is solved. An n-layer system will have 4n constants A_i, B_i, C_i, D_i, which must be evaluated from the boundary conditions

VI-1.1 Boundary Conditions

The boundary conditions for the lower layer include that all stresses and displacements go to zero when z becomes infinite. From this it is immediately evident that the constants A and C must be zero for this layer. At the surface the boundary conditions are that the shearing stress must be zero:

$$\tau_{rz}\Big|_{z=-H} = 0 \tag{164}$$

and that the normal stress is given, for a uniform circular load of magnitude q and radius a as:

$$\sigma_z\Big|_{z=-H} = -qa\int_0^\infty J_0(mr)\,J_1(ma)\,dm \tag{165}$$

It will be convenient to use an incremental load

$$\sigma_z \Big|_{z=-H} = -J_0(mr)\,J_1(ma) \qquad (166)$$

and then integrate the final expressions from 0 to ∞ with respect to m, and multiply this result by qa, which will then yield the same result.

The remaining boundary conditions involve continuity at the interfaces between the layers. At each interface four conditions must be imposed. Assuming continuity of the displacements, vertical stress, and shear stress across an interface, the boundary conditions between layers i and i + 1 are:

$$W_i = W_{i+1} \qquad (167)$$

$$U_i = U_{i+1} \qquad (168)$$

$$\sigma_{z_i} = \sigma_{z_{i+1}} \qquad (169)$$

$$\tau_{rz_i} = \tau_{rz_{i+1}} \qquad (170)$$

For an n layer system, equations (167) to (170) yield 4n-4 equations. In addition, two equations (164) and (166) are available for the surface layer, and two constants in the bottom layer are zero. Thus a total of 4n-2 equations in 4n-2 unknowns must be solved. For a three-layer system this will be ten equations in ten unknowns. These ten equations are listed below for a three-layer system under the incremental normal load $-J_0(mr)J_1(ma)$. In these equations, the thickness of the first layer has been taken as unity to non-dimensionalize distances.

$$-mJ_0(mr)\left[A_1 m^2 e^{-m} + B_1 m^2 e^{m} - C_1 m^2 e^{-m} - D_1 m^2 e^{m}\right] = -J_0(mr)J_1(ma) \tag{171}$$

$$mJ_1(mr)\left[A_1 m^2 e^{-m} - B_1 m^2 e^{m} + C_1 m(1-m)e^{-m} + D_1 m(1+m)e^{m}\right] = 0 \tag{172}$$

$$A_1 + B_1 = A_2 + B_2 \tag{173}$$

$$A_1 m - B_1 m + C_1 + D_1 = A_2 m - B_2 m + C_2 + D_2 \tag{174}$$

$$\frac{1.5}{E_1}\left[A_1 - B_1\right] = \frac{1.5}{E_2}\left[A_2 - B_2\right] \tag{175}$$

$$\frac{1.5}{E_1}\left[A_1 m + B_1 m + C_1 - D_1\right] = \frac{1.5}{E_2}\left[A_2 m + B_2 m + C_2 - D_2\right] \tag{176}$$

$$A_2 m e^{mh} + B_2 m e^{-mh} + C_2 mh e^{mh} + D_2 mh e^{-mh}$$
$$= B_3 m e^{-mh} + D_3 mh e^{-mh} \qquad (177)$$

$$A_2 m e^{mh} - B_2 m e^{-mh} + C_2 (1+mh) e^{mh} + D_2 (1-mh) e^{-mh}$$
$$= -B_3 m e^{-mh} + D_3 (1-mh) e^{-mh} \qquad (178)$$

$$\frac{1.5}{E_2}\left[A_2 m e^{mh} - B_2 m e^{-mh} + C_2 mh e^{mh} - D_2 mh e^{-mh}\right]$$
$$= \frac{1.5}{E_3}\left[-B_3 m e^{-mh} - D_3 mh e^{-mh}\right] \qquad (179)$$

$$\frac{1.5}{E_2}\left[A_2 m e^{mh} + B_2 m e^{-mh} + C_2 (1+mh) e^{mh} - D_2 (1-mh) e^{-mh}\right]$$
$$= \frac{1.5}{E_3}\left[B_3 m e^{-mh} - D_3 (1-mh) e^{-mh}\right] \qquad (180)$$

The ten constants A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, B_3, D_3 can be obtained by solving equations (171) to (180). For the present perposes, it is important to keep the elastic constants separate from the geometrical constants. An efficient approach to solving equations (171) to (180) with respect to obtaining the constants in a suitable form is to solve equations (171) and (172) for A_1 and B_1 in terms of C_1 and D_1, then use these expressions to solve equations (173) to (176) for A_2, B_2, C_2, and D_2 in terms of C_1 and D_1. Next, equations (177) and (178) are solved for B_3 and D_3 in terms of C_1 and D_1, using the results from equations (173) to (176). Finally all these expressions are substituted into equations (179) and (180) to yield two simultaneous equations for the constants C_1 and D_1. After obtaining these two constants, the other eight constants may be obtained immediately by back substitution.

If the elastic constants are kept always separate from the geometrical terms, then C_1 and D_1 can be written in the following form:

$$C_1 = \frac{J_1(ma)}{m^2} \frac{\sum_{i=1}^{9} q_{3,1,i} \alpha_{1,i}}{\sum_{i=1}^{9} \theta_i \alpha_{1,i}} \tag{181}$$

$$D_1 = \frac{J_1(ma)}{m^2} \frac{\sum_{i=1}^{9} q_{4,1,i} \alpha_{1,i}}{\sum_{i=1}^{9} \theta_i \alpha_{1,i}} \tag{182}$$

where the $q_{3,1,i}$, $q_{4,1,i}$, and θ_i terms are constants involving only the geometrical variables and the $\alpha_{1,i}$ terms are products of four elastic compliances. The geometrical constants are given in Table XII, and the $\alpha_{1,i}$'s are listed below:

$$\alpha_{1,1} = \frac{1}{E_1^2 E_3^2} \tag{183}$$

$$\alpha_{1,2} = \frac{1}{E_1^2 E_2 E_3} \tag{184}$$

$$\alpha_{1,3} = \frac{1}{E_1 E_2 E_3^2} \tag{185}$$

$$\alpha_{1,4} = \frac{1}{E_1 E_2^2 E_3} \tag{186}$$

$$\alpha_{1,5} = \frac{1}{E_1^2 E_2^2} \tag{187}$$

$$\alpha_{1,6} = \frac{1}{E_1 E_2^3} \tag{188}$$

$$\alpha_{1,7} = \frac{1}{E_2^2 E_3^2} \tag{189}$$

$$\alpha_{1,8} = \frac{1}{E_2^3 E_3} \tag{190}$$

$$\alpha_{1,9} = \frac{1}{E_2^4} \tag{191}$$

Now by back-substituting, the other eight constants can immediately be found in a form similar to equations (181) and (182):

$$A_1 = \frac{J_1(ma)}{m^3} \frac{\sum_{i=1}^{9} q_{1,1,i}\,\alpha_{1,i}}{\sum_{i=1}^{9} \theta_i\,\alpha_{1,i}} \tag{192}$$

$$B_1 = \frac{J_1(ma)}{m^3} \frac{\sum_{i=1}^{9} q_{2,1,i}\,\alpha_{1,i}}{\sum_{i=1}^{9} \theta_i\,\alpha_{1,i}} \tag{193}$$

$$A_2 = \frac{J_1(ma)}{m^3} \frac{\sum_{i=1}^{18} q_{1,2,i}\,\alpha_{2,i}}{\sum_{i=1}^{9} \theta_i\,\alpha_{2,i}} \tag{194}$$

$$B_2 = \frac{J_1(ma)}{m^3} \frac{\sum_{i=1}^{18} q_{2,2,i}\,\alpha_{2,i}}{\sum_{i=1}^{9} \theta_i\,\alpha_{2,i}} \tag{195}$$

$$C_2 = \frac{J_1(ma)}{m^2} \frac{\sum_{i=1}^{18} q_{3,2,i}\,\alpha_{2,i}}{\sum_{i=1}^{9} \theta_i\,\alpha_{2,i}} \tag{196}$$

$$D_2 = \frac{J_1(ma)}{m^2} \frac{\sum_{i=1}^{18} q_{4,2,i}\,\alpha_{2,i}}{\sum_{i=1}^{9} \theta_i\,\alpha_{2,i}} \tag{197}$$

$$B_3 = \frac{J_1(ma)}{m^3} \frac{\sum_{i=1}^{18} q_{2,3,i}\,\alpha_{2,i}}{\sum_{i=1}^{9} \theta_i\,\alpha_{2,i}} \tag{198}$$

$$D_3 = \frac{J_1(ma)}{m^2} \frac{\sum_{i=1}^{18} q_{4,3,i}\,\alpha_{2,i}}{\sum_{i=1}^{9} \theta_i\,\alpha_{2,i}} \tag{199}$$

The geometrical constants are given in Table XII. The $\alpha_{2,i}$'s are products of five elastic compliances:

$$\alpha_{2,i} = \alpha_{1,i}/E_2 \qquad i = 1 \cdots 9 \tag{200}$$

$$\alpha_{2,i} = \alpha_{1,i-9}/E_1 \qquad i = 10 \cdots 18 \tag{201}$$

Since the constants are now known, the expressions for the stresses and displacements, equations (159) to (163), can be rewritten in terms of the geometry and the elastic compliances in the following simplified form:

$$\sigma_{z_i} = J_0(mr)\,J_1(ma)\,\frac{\sum_{j=1}^{18} \phi_{1,i,j}\,\alpha_{i,j}}{\sum_{j=1}^{9} \theta_j\,\alpha_{i,j}} \tag{202}$$

$$\tau_{rz_i} = J_1(mr)\,J_1(ma)\,\frac{\sum_{j=1}^{18} \phi_{2,i,j}\,\alpha_{i,j}}{\sum_{j=1}^{9} \theta_j\,\alpha_{i,j}} \tag{203}$$

$$\sigma_{r_i} = J_0(mr)\,J_1(ma)\,\frac{\sum_{j=1}^{18} \phi_{3,i,j}\,\alpha_{i,j}}{\sum_{j=1}^{9} \theta_j\,\alpha_{i,j}} + \frac{J_1(mr)\,J_1(ma)}{mr}\,\frac{\sum_{j=1}^{18} \phi_{4,i,j}\,\alpha_{i,j}}{\sum_{j=1}^{9} \theta_j\,\alpha_{i,j}} \tag{204}$$

$$W_i = \frac{J_0(mr)\,J_1(ma)}{m}\,\frac{\sum_{j=1}^{18} \phi_{5,i,j}\,\alpha_{i,j}/E_i}{\sum_{j=1}^{9} \theta_j\,\alpha_{i,j}} \tag{205}$$

$$U_i = \frac{J_1(mr)\,J_1(ma)}{m}\,\frac{\sum_{j=1}^{18} \phi_{6,i,j}\,\alpha_{i,j}/E_i}{\sum_{j=1}^{9} \theta_j\,\alpha_{i,j}} \tag{206}$$

where

$$\phi_{m,i,j} = \sum_{K=1}^{4} q_{K,i,j} \lambda_{m,K} \qquad \begin{matrix} m = 1 \cdots 6 \\ i = 1 \cdots 3 \\ j = 1 \cdots 18 \end{matrix} \tag{207}$$

$$\alpha_{3,j} = \alpha_{2,j} \qquad j = 1 \cdots 18 \tag{208}$$

$$\alpha_{1,j} = 0 \qquad j = 10 \cdots 18 \tag{209}$$

and the $\lambda_{m,K}$'s are defined in Table XII.

A subroutine entitled CNSTNT has been written which calculates the $\phi_{m,i,j}$ and θ_j terms for a given geometry. This program has been used in conjunction with the original ten boundary conditions and arbitrary input geometry to check the above derivation.

To obtain the elastic solution under a uniform circular load, the above stresses and displacements must be integrated from zero to infinity with respect to m, and multiplied by qa. For example, the normal stress at any off-set r is given, for a uniform circular load of radius a and intensity q, as follows:

$$\sigma_{z_i} = qa \int_0^{\infty} J_0(mr) J_1(ma) \frac{\sum_{j=1}^{18} \phi_{1,i,j} \alpha_{i,j}}{\sum_{j=1}^{9} \theta_j \alpha_{i,j}} dm \tag{210}$$

TABLE XII

CONSTANTS FOR THE THREE-LAYER HALF-SPACE SOLUTION

Define

$$C_1 = A_1A_5 - B_1B_5$$

$$C_2 = A_2A_5 + A_1A_6 - B_2B_5 - B_1B_6$$

$$C_3 = A_3A_5 + A_1A_7 - B_3B_5 - B_1B_7$$

$$C_4 = A_4A_5 + A_3A_6 + A_2A_7 + A_1A_8 - B_4B_5 - B_3B_6 - B_2B_7 - B_1B_8$$

$$C_5 = A_2A_6 - B_2B_6$$

$$C_6 = A_4A_6 + A_2A_8 - B_4B_6 - B_2B_8$$

$$C_7 = A_3A_7 - B_3B_7$$

$$C_8 = A_4A_7 + A_3A_8 - B_4B_7 - B_3B_8$$

$$C_9 = A_4A_8 - B_4B_8$$

Then for

$A_1 = g_{45}$	$B_1 = g_{49}$
$A_2 = g_{46}$	$B_2 = g_{50}$
$A_3 = g_{47}$	$B_3 = g_{51}$
$A_4 = g_{48}$	$B_4 = g_{52}$
$A_5 = g_{65}$	$B_5 = g_{61}$
$A_6 = g_{66}$	$B_6 = g_{62}$
$A_7 = g_{67}$	$B_7 = g_{63}$
$A_8 = g_{68}$	$B_8 = g_{64}$

$$\theta_i = C_i \qquad i = 1 \cdots 9$$

TABLE XII (continued)

for	$A_1 = g_{49}$	$B_1 = g_{41}$
	$A_2 = g_{50}$	$B_2 = g_{42}$
	$A_3 = g_{51}$	$B_3 = g_{43}$
	$A_4 = g_{52}$	$B_4 = g_{44}$
	$A_5 = g_{57}$	$B_5 = g_{65}$
	$A_6 = g_{58}$	$B_6 = g_{66}$
	$A_7 = g_{59}$	$B_7 = g_{67}$
	$A_8 = g_{60}$	$B_8 = g_{68}$

$$q_{3,1,i} = C_i \qquad i = 1 \cdots 9$$

for	$A_1 = g_{61}$	$B_1 = g_{45}$
	$A_2 = g_{62}$	$B_2 = g_{46}$
	$A_3 = g_{63}$	$B_3 = g_{47}$
	$A_4 = g_{64}$	$B_4 = g_{48}$
	$A_5 = g_{41}$	$B_5 = g_{57}$
	$A_6 = g_{42}$	$B_6 = g_{58}$
	$A_7 = g_{43}$	$B_7 = g_{59}$
	$A_8 = g_{44}$	$B_8 = g_{60}$

$$q_{4,1,i} = C_i \qquad i = 1 \cdots 9$$

$$q_{1,1,i} = g_1\ \theta_i + g_3\ q_{3,1,i} + g_4\ q_{4,1,i} \qquad i = 1 \cdots 9$$
$$q_{2,1,i} = g_2\ \theta_i + g_5\ q_{3,1,i} + g_6\ q_{4,1,i} \qquad i = 1 \cdots 9$$
$$q_{1,2,i} = g_7\ \theta_i + g_9\ q_{3,1,i} + g_{11}\ q_{4,1,i} \qquad i = 1 \cdots 9$$
$$q_{2,2,i} = q_{1,2,i} \qquad i = 1 \cdots 9$$
$$q_{3,2,i} = -g_2\ \theta_i + g_{13}\ q_{3,1,i} + g_{15}\ q_{4,1,i} \qquad i = 1 \cdots 9$$
$$q_{4,2,i} = g_1\ \theta_i + g_{17}\ q_{3,1,i} + g_{19}\ q_{4,1,i} \qquad i = 1 \cdots 9$$
$$q_{1,3,i} = 0 \qquad i = 1 \cdots 18$$
$$q_{2,3,i} = g_{29}\ \theta_i + g_{31}\ q_{3,1,i} + g_{33}\ q_{4,1,i} \qquad i = 1 \cdots 9$$
$$q_{3,3,i} = 0 \qquad i = 1 \cdots 18$$
$$q_{4,3,i} = g_{21}\ \theta_i + g_{23}\ q_{3,1,i} + g_{25}\ q_{4,1,i} \qquad i = 1 \cdots 9$$

$$q_{1,1,i} = q_{2,1,i} = q_{3,1,i} = q_{4,1,i} = 0 \qquad i = 10 \cdots 18$$
$$q_{1,2,i} = g_8\ \theta_{i-9} + g_{10}\ q_{3,1,i-9} + g_{12}\ q_{4,1,i-9} \qquad i = 10 \cdots 18$$
$$q_{2,2,i} = -q_{1,2,i} \qquad i = 10 \cdots 18$$
$$q_{3,2,i} = g_2\ \theta_{i-9} + g_{14}\ q_{3,1,i-9} + g_{16}\ q_{4,1,i-9} \qquad i = 10 \cdots 18$$
$$q_{4,2,i} = -g_1\ \theta_{i-9} + g_{18}\ q_{3,1,i-9} + g_{20}\ q_{4,1,i-9} \qquad i = 10 \cdots 18$$
$$q_{2,3,i} = g_{30}\ \theta_{i-9} + g_{32}\ q_{3,1,i-9} + g_{34}\ q_{4,1,i-9} \qquad i = 10 \cdots 18$$
$$q_{4,3,i} = g_{22}\ \theta_{i-9} + g_{24}\ q_{3,1,i-9} + g_{26}\ q_{4,1,i-9} \qquad i = 10 \cdots 18$$

where

$s = mh$ $\qquad Z_3 = e^{-2m}$

$Z_o = e^{m}$ $\qquad Z_4 = e^{2s}$

$Z_1 = e^{-m}$ $\qquad Z_5 = e^{s}$

$Z_2 = e^{2m}$ $\qquad Z_6 = e^{-s}$

$g_1 = Z_o/2$

$g_2 = Z_1/2$

$g_3 = (2m - 1)/2$

$g_4 = -Z_2/2$

$g_5 = Z_3/2$

$g_6 = (1 + 2m)/2$

$g_7 = (g_1 + g_2)/2$

$g_8 = (g_1 - g_2)/2$

$g_9 = (g_3 + g_5)/2$

$g_{10} = (g_3 - g_5)/2$

$g_{11} = (g_4 + g_6)/2$

$g_{12} = (g_4 - g_6)/2$

$g_{13} = .5 - g_5$

$g_{14} = .5 + g_5$

$g_{15} = .5 - g_6$

$g_{16} = -g_{15}$

$g_{17} = .5 + g_3$

$g_{18} = -g_{17}$

$g_{19} = .5 + g_4$

$g_{20} = .5 - g_4$

$g_{21} = g_{27}\, g_7 - g_{28}\, g_2 + g_1$

$g_{22} = g_{27}\, g_8 + g_{28}\, g_2 - g_1$

$g_{23} = g_{27}\, g_9 + g_{28}\, g_{13} + g_{17}$

$$g_{24} = g_{27}\, g_{10} + g_{28}\, g_{14} + g_{18}$$

$$g_{25} = g_{27}\, g_{11} + g_{28}\, g_{15} + g_{19}$$

$$g_{26} = g_{27}\, g_{12} + g_{28}\, g_{16} + g_{20}$$

$$g_{27} = 2 \cdot Z_4$$

$$g_{28} = (1 + 2mh)Z_4$$

$$g_{29} = g_{35}\, g_7 + g_7 - g_{36}\, g_2$$

$$g_{30} = g_{35}\, g_8 - g_8 + g_{36}\, g_2$$

$$g_{31} = g_{35}\, g_9 + g_9 + g_{36}\, g_{13}$$

$$g_{32} = g_{35}\, g_{10} - g_{10} + g_{36}\, g_{14}$$

$$g_{33} = g_{35}\, g_{11} + g_{11} + g_{36}\, g_{15}$$

$$g_{34} = g_{35}\, g_{12} - g_{12} + g_{36}\, g_{16}$$

$$g_{35} = (1-2S)Z_4$$

$$g_{36} = -2S^2 Z_4$$

$$g_{37} = Z_5$$

$$g_{38} = Z_6$$

$$g_{39} = (1. + S)Z_5$$

$$g_{40} = -(1. - S)Z_6$$

$$g_{41} = g_{37}\, g_7 + g_{38}\, g_7 - g_{39}\, g_2 + g_{40}\, g_1$$

$$g_{42} = -\, g_{38}\, g_{29} - g_{40}\, g_{21}$$

$$g_{43} = g_{37}\, g_8 - g_{38}\, g_8 + g_{39}\, g_2 - g_{40}\, g_1$$

$$g_{44} = -\, g_{38}\, g_{30} - g_{40}\, g_{22}$$

$$g_{45} = g_{37}\, g_9 + g_{38}\, g_9 + g_{39}\, g_{13} + g_{40}\, g_{17}$$

$$g_{46} = -\, g_{38}\, g_{31} - g_{40}\, g_{23}$$

$$g_{47} = g_{37} g_{10} - g_{38} g_{10} + g_{39} g_{14} + g_{40} g_{18}$$

$$g_{48} = - g_{38} g_{32} - g_{40} g_{24}$$

$$g_{49} = g_{37} g_{11} + g_{38} g_{11} + g_{39} g_{15} + g_{40} g_{19}$$

$$g_{50} = - g_{38} g_{33} - g_{40} g_{25}$$

$$g_{51} = g_{37} g_{12} - g_{38} g_{12} + g_{39} g_{16} + g_{40} g_{20}$$

$$g_{52} = - g_{38} g_{34} - g_{40} g_{26}$$

$$g_{53} = Z_5$$

$$g_{54} = - Z_6$$

$$g_{55} = SZ_5$$

$$g_{56} = -SZ_6$$

$$g_{57} = g_{53} g_7 + g_{54} g_7 - g_{55} g_2 + g_{56} g_1$$

$$g_{58} = - g_{54} g_{29} - g_{56} g_{21}$$

$$g_{59} = g_{53} g_8 - g_{54} g_8 + g_{55} g_2 - g_{56} g_1$$

$$g_{60} = - g_{54} g_{30} - g_{56} g_{22}$$

$$g_{61} = g_{53} g_9 + g_{54} g_9 + g_{55} g_{13} + g_{56} g_{17}$$

$$g_{62} = - g_{54} g_{31} - g_{56} g_{23}$$

$$g_{63} = g_{53} g_{10} - g_{54} g_{10} + g_{55} g_{14} + g_{56} g_{18}$$

$$g_{64} = - g_{54} g_{32} - g_{56} g_{24}$$

$$g_{65} = g_{53} g_{11} + g_{54} g_{11} + g_{55} g_{15} + g_{56} g_{19}$$

$$g_{66} = - g_{54} g_{33} - g_{56} g_{25}$$

$$g_{67} = g_{53} g_{12} - g_{54} g_{12} + g_{55} g_{16} + g_{56} g_{20}$$

$$g_{68} = - g_{54} g_{34} - g_{56} g_{26}$$

TABLE XII (continued)

$E_Z = mZ$

$E_{Z1} = e^{mZ}$

$E_{Z2} = e^{-mZ}$

$\lambda_{1,1} = -E_{Z1}$

$\lambda_{1,2} = -E_{Z2}$

$\lambda_{1,3} = -E_Z\ E_{Z1}$

$\lambda_{1,4} = -E_Z\ E_{Z2}$

$\lambda_{2,1} = -\lambda_{1,1}$

$\lambda_{2,2} = \lambda_{1,2}$

$\lambda_{2,3} = \lambda_{2,1} - \lambda_{1,3}$

$\lambda_{2,4} = -\lambda_{1,2} + \lambda_{1,4}$

$\lambda_{3,1} = \lambda_{2,1}$

$\lambda_{3,2} = -\lambda_{2,2}$

$\lambda_{3,3} = 2\lambda_{3,1} - \lambda_{1,3}$

$\lambda_{3,4} = 2\lambda_{2,2} - \lambda_{1,4}$

$\lambda_{4,1} = \lambda_{1,1}$

$\lambda_{4,2} = \lambda_{1,2}$

$\lambda_{4,3} = -\lambda_{2,3}$

$\lambda_{4,4} = \lambda_{2,4}$

$\lambda_{5,1} = -1.5\ E_{Z1}$

$\lambda_{5,2} = 1.5\ E_{Z2}$

$\lambda_{5,3} = -1.5\ E_Z\ E_{Z1}$

$\lambda_{5,4} = -1.5\ \lambda_{1,4}$

$\lambda_{6,1} = 1.5\ E_{Z1}$

$\lambda_{6,2} = 1.5\ E_{Z2}$

$\lambda_{6,3} = 1.5\ \lambda_{2,3}$

$\lambda_{6,4} = -1.5\ \lambda_{2,4}$

VI-2. The Viscoelastic Solution.

For the viscoelastic case, the time variation of the loading must be specified. In this case, the normal stress boundary condition will be taken as:

$$\sigma_z \Big|_{z=-1} = qa \int_0^\infty J_0(mr)\, J_1(ma) dm \;\; H(t) \qquad (211)$$

Again the incremental load

$$\sigma_z \Big|_{z=-1} = J_0(mr)\, J_1(ma)\; H(t) \qquad (212)$$

will be considered, and then the final result will be integrated from 0 to ∞ with respect to m, and then multiplied by qa, to yield the viscoelastic solution under a uniform circular load.

Since in the elastic solutions, equations (202) to (206), the Bessel functions appear as multipliers to the summation-over-summation terms, and since these Bessel functions vary only with m for a given geometry, it will be useful to treat the elastic solutions in the following forms:

Define:

$$\Psi_{k,i}(m,t) = \frac{\sum_{j=1}^{18} \phi_{k,i,j}\, \beta_{i,j}'\, H(t)}{\sum_{j=1}^{9} \theta_j\, \alpha_{i,j}} \qquad (213)$$

where

$$\beta_{i,j} = \alpha_{i,j} \qquad K \leq 4 \tag{214}$$

$$\beta_{i,j} = \alpha_{i,j}/E_i \qquad K > 4 \tag{215}$$

$$\Theta_1(m) = J_0(mr)\, J_1(ma) \tag{216}$$

$$\Theta_2(m) = J_1(mr)\, J_1(ma) \tag{217}$$

Then the time-varying elastic solutions are given as follows:

$$\sigma_{z_i}(t) = qa\int_0^\infty \Theta_1(m)\, \psi_{1,i}(t,m)\, dm \tag{218}$$

$$\tau_{rz_i}(t) = qa\int_0^\infty \Theta_2(m)\, \psi_{2,i}(t,m)\, dm \tag{219}$$

$$\sigma_{r_i}(t) = qa\int_0^\infty \left[\Theta_1(m)\psi_{3,i}(t,m) + \frac{\Theta_2(m)}{mr}\psi_{4,i}(t,m)\right] dm \tag{220}$$

$$W_i(t) = qa\int_0^\infty \frac{\Theta_1(m)}{m}\, \psi_{5,i}(t,m)\, dm \tag{221}$$

$$u_i(t) = qa\int_0^\infty \frac{\Theta_2(m)}{m}\, \psi_{6,i}(t,m)\, dm \tag{222}$$

Clearly, to obtain the viscoelastic solution, all that is needed is to obtain the corresponding $\psi_{k,i}(t,m)$ for the viscoelastic case, since the $\theta_j(m)$ terms do not vary in time. But the $\psi_{k,i}(t,m)$ terms for the elastic case are in the general form of equation (46) of Chapter III, and thus an integral equation for $\psi_{k,i}(t,m)$, for a given value of m, can be written immediately. From the solution of this equation for appropriate m, the total solution can be obtained by numerical integration of the equations (218) to (222).

Following equation (53), the integral equation for $\psi_{k,i}(t,m)$ for the viscoelastic case can be written

$$\sum_{j=1}^{9}\theta_j(m)\left[\int_{0^+}^{t}\psi_{k,i}(m,t-\tau)\frac{\partial\alpha_{i,j}(\tau)}{\partial\tau}d\tau+\psi_{k,i}(m,t)\alpha_{i,j}(0)\right]$$

$$=\sum_{j=1}^{18}\phi_{k,i,j}(m)\beta_{ij}(t) \tag{223}$$

in which $\alpha_{i,j}(t)$ is a three-fold convolution integral of the following form (for $\alpha_{i,j}=1/E_sE_tE_uE_v$ in the elastic case):

$$\alpha_{i,j}(t)=\int_{0^+}^{t}D_s(t-\tau)\frac{\partial}{\partial\tau}\int_{0^+}^{\tau}D_t(\tau-\lambda)\frac{\partial}{\partial\lambda}\int_{0^+}^{\lambda}D_u(\lambda-\rho)\frac{\partial D_v(\rho)}{\partial\rho}d\rho$$

$$+D_u(\lambda)D_v(0)d\lambda\quad+D_t(\tau)D_u(0)D_v(0)d\tau \tag{224}$$

$$+D_s(t)D_t(0)D_u(0)D_v(0)$$

and

$$\alpha_{3,j}(t) = \alpha_{2,j}(t) = \int_{0^+}^{t} D_w(t-\xi)\frac{\partial \alpha_{1,\ell}(\xi)}{\partial \xi}d\xi + D_w(t)\alpha_{1,\ell}(0) \qquad (225)$$

with $D_w(t) = D_2(t)$ and $\ell = j$ for $j \leq 9$

and $D_w(t) = D_1(t)$ and $\ell = j-9$ for $j > 9$

$$\beta_{i,j}(t) = \alpha_{i,j}(t) \qquad \text{for } k \leq 4 \qquad (226)$$

$$\beta_{i,j}(t) = \int_{0^+}^{t} D_i(t-\xi)\frac{\partial \alpha_{i,j}(\xi)}{\partial \xi}d\xi + D_i(t)\alpha_{i,j}(0) \qquad (227)$$

for $k > 4$

The above integral equations for $\psi_{k,i}(m,t)$ have been programmed for solution by both of the numerical approaches described in Chapter IV. The programs are given in the appendix.

VI-2.1 Integration on m

Once $\psi_{k,i}(m,t)$ has been obtained for appropriate values of m and t, the total result is obtained by integrating with respect to m. In the present analysis the integral equation (223) was solved for thirteen values of m (m = 0, .2, .4, .7, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0). Intermediate values of $\psi_{k,i}(m,t)$ were then obtained by approximating the curve between three consecutive points by a parabola [108], and then evaluating this parabola at values of m spaced .1 m apart. These results were multiplied by the $\Theta_j(m)$ terms, (which are more rapidly varying with respect to m), and then the total integral calculated using Simpson's rule, which is based on approximating the integral between three consecutive points by a second degree polynomial. For the 91 points spaced .1 m apart used in the present analysis, the total integral can then be calculated with the following formula:

$$\int_0^{9m} f(m)dm \approx \frac{.1}{3}\Big[f(0) + 4f(.1m) + 2f(.2m) + \ldots + 4f(8.9m) + f(9m)\Big] \tag{228}$$

This procedure is carried out by a subroutine entitled TERPO, given in the appendix. The remainder of the integral, form 9. m to ∞, was considered negligible.

VI-2.2 Evaluation of the Bessel Functions

The Bessel functions that occur in the solution can be evaluated by use of the infinite series

$$J_N(x) = \sum_{K=0}^{\infty} \frac{(-1)^K \left(\frac{x}{2}\right)^{2K+N}}{K!\,(K+N)!} \tag{229}$$

where N is either zero or one. A previously prepared program, using a finite number of the above series terms [42], was modified for use in the present analysis. For values of the argument X greater than 12, the appropriate asymptotic expansions were inserted into the program used in reference [42]:

$$J_1(X) = \sqrt{\frac{2}{\pi X}} \cos\left(X - \frac{3\pi}{4}\right) \qquad X > 12 \tag{230}$$

$$J_0(X) = \sqrt{\frac{2}{\pi X}} \cos\left(X - \frac{\pi}{4}\right) \qquad X > 12 \tag{231}$$

The total program is given in the appendix as a function subprogram entitled BESSEL.

VI-2.3 Total Solution

The total solution obtained using both techniques discussed in Chapter IV has been programmed. The programs are presented in the appendix. Numerical examples and comparisons are given below.

VI-2.4 Numerical Examples

To illustrate the effectiveness of the computer programs, and to give a particular example of the results, a three-layer half-space with the following geometry and material characterization has been analysed:

$$\frac{a}{h} = 1.0 \tag{232}$$

$$\frac{H}{h} = 1.0 \tag{233}$$

$$D_i(t) = \sum_{j=1}^{6} G_i^j e^{-t/\tau_j} \tag{234}$$

where

$G_1^1 = -.05$	$G_2^1 = -.10$	$G_3^1 = -.05$
$G_1^2 = -.10$	$G_2^2 = -.15$	$G_3^2 = -.05$
$G_1^3 = -.32$	$G_2^3 = -.10$	$G_3^3 = -.05$
$G_1^4 = -.32$	$G_2^4 = -.15$	$G_3^4 = -.05$
$G_1^5 = -.19$	$G_2^5 = -.10$	$G_3^5 = 0.0$
$G_1^6 = 1.0$	$G_2^6 = 1.0$	$G_3^6 = 1.0$

$\tau_1 = 1.0$ $\qquad \tau_5 = 100.$

$\tau_2 = \sqrt{10.}$ $\qquad \tau_6 = \infty$

$\tau_3 = 10.$

$\tau_4 = 10.\sqrt{10.}$

The compliance of each layer is plotted in Figure 13. The results for the normal stress σ_z for one point in each of the three layers are given in Figure 14. All three points were selected along the axis of the load. Figure 15 presents the results for the shear stress τ_{rz} at one point with off-set of $r/a = 1.0$ for each of the three layers. Figure 16 presents the results for the vertical deflection w at one point for each of the three layers, all of which are along the axis of the load. Figure 17 presents the results for the radial deflection u at an off-set of $r/a = 1.0$ for one point in each layer. And Figure 18 presents the results for the radial stress σ_r along the axis of the load for one point in each of the layers.

Since all of the compliances tend to unity at large times, the solutions should all tend to the solution for a homogeneous incompressible elastic half-space. The results have all been compared, at long times, to the homogeneous half-space solutions (from reference [3]). Very good agreement (generally less than a one per cent difference) were found with these solutions.

The results plotted in Figures 14 through 18 were obtained using the exact integration technique. The solutions at various times are tabulated in Tables XIII through XXVII, and compared, at these times, with the solutions obtained using the numerical integration procedure. None of the differences shown are large enough to show up on the plots of Figures 15 through 18. For the solutions that are very small in absolute values (noteably the radial stress in the third layer at the second interface) some fairly large per cent differences are noted. This is due to round-off errors, particularly in the subroutine INTEGR for the exact convolution integrations (at short times only). These could be eliminated through the use of double precision coding, at the loss of execution time, but since the errors are only significant as the stresses or displacements tend to zero, which is of the least interest, this does not seem necessary.

Obviously either technique works adequately in the usual case. It should be noted that the procedure utilizing the exact integration technique (and thus using a log spacing in time) required only approximately one-third the execution time in this analysis.

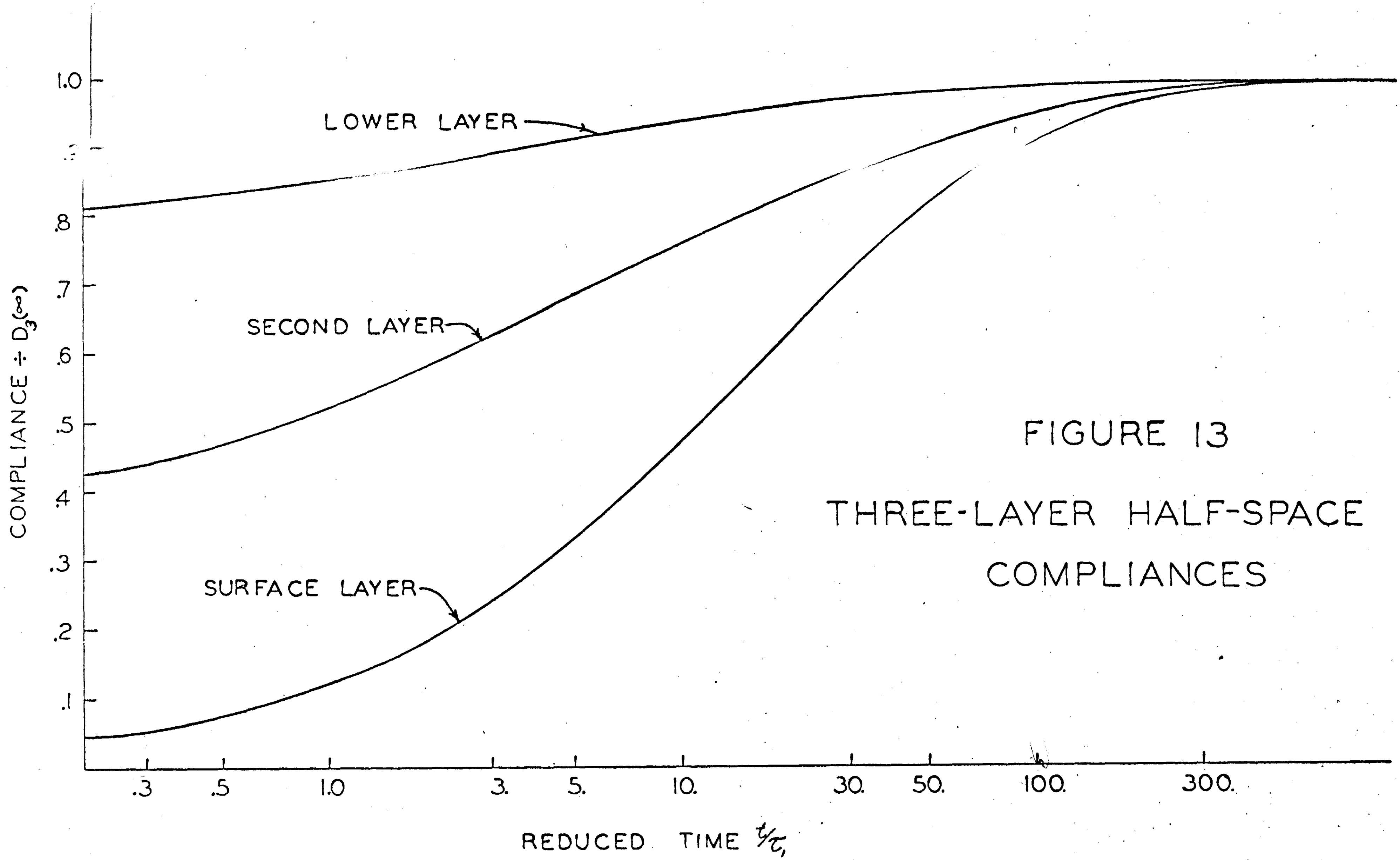


FIGURE 13

THREE-LAYER HALF-SPACE COMPLIANCES

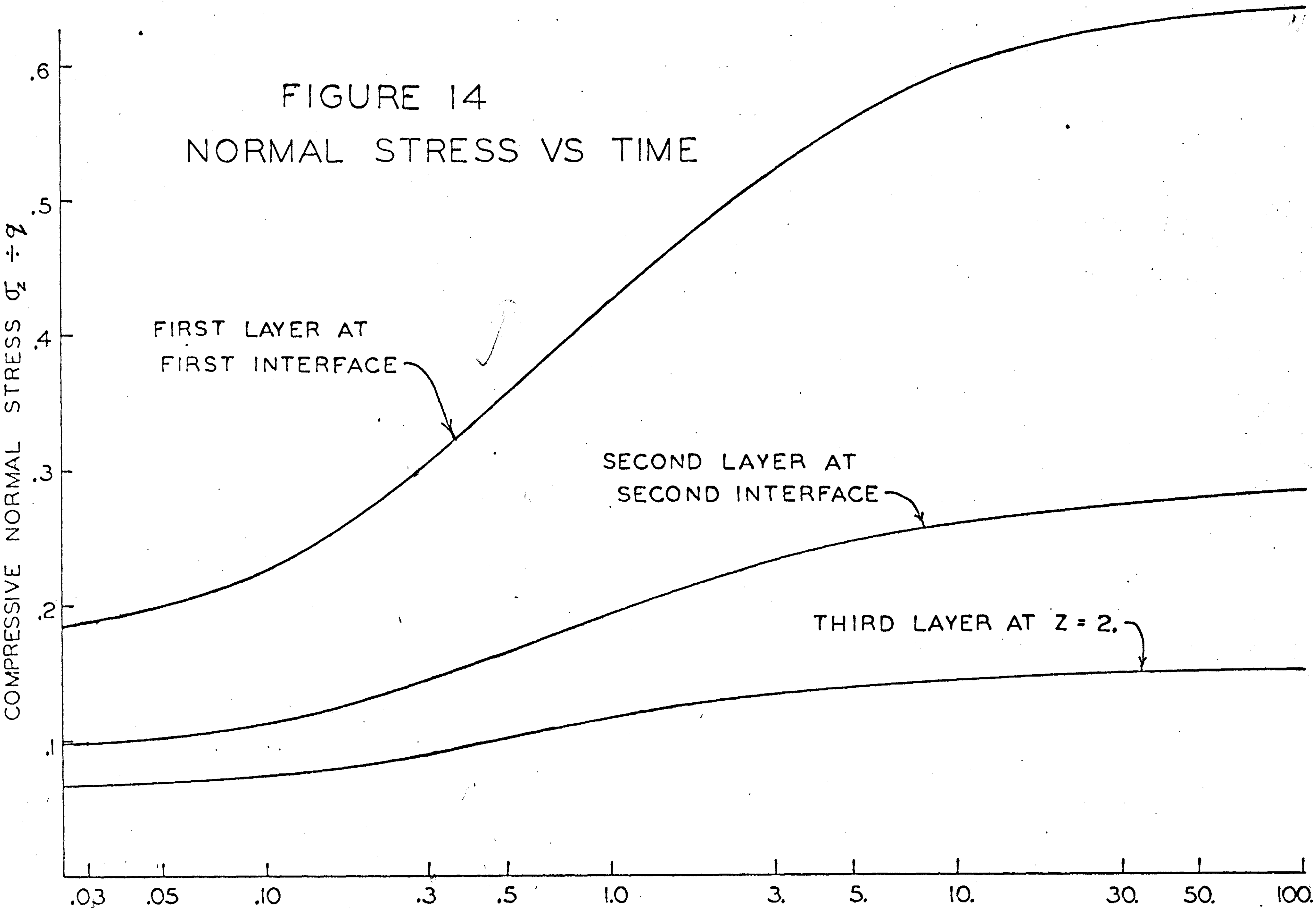


FIGURE 14
NORMAL STRESS VS TIME

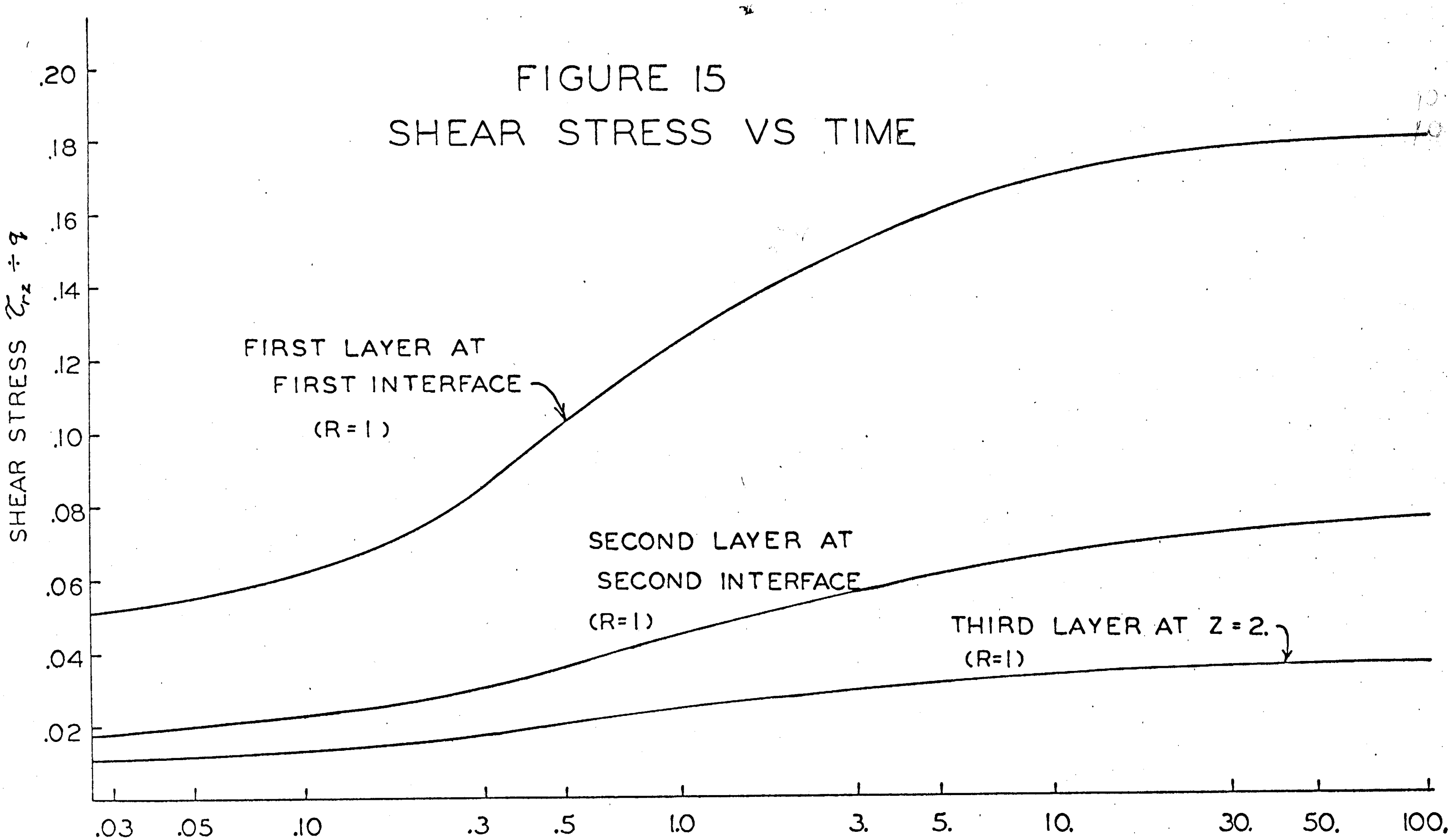


FIGURE 15
SHEAR STRESS VS TIME
SHEAR STRESS $\tau_{rz} \div q$
.20
.18
.16
.14
.12
.10
.08
.06
.04
.02
FIRST LAYER AT
FIRST INTERFACE
(R = 1)
SECOND LAYER AT
SECOND INTERFACE
(R=1)
THIRD LAYER AT Z = 2.
(R=1)
.03
.05
.10
.3
.5
1.0
3.
5.
10.
30.
50.
100.

FIGURE 16

VERTICAL DEFLECTION VS. TIME

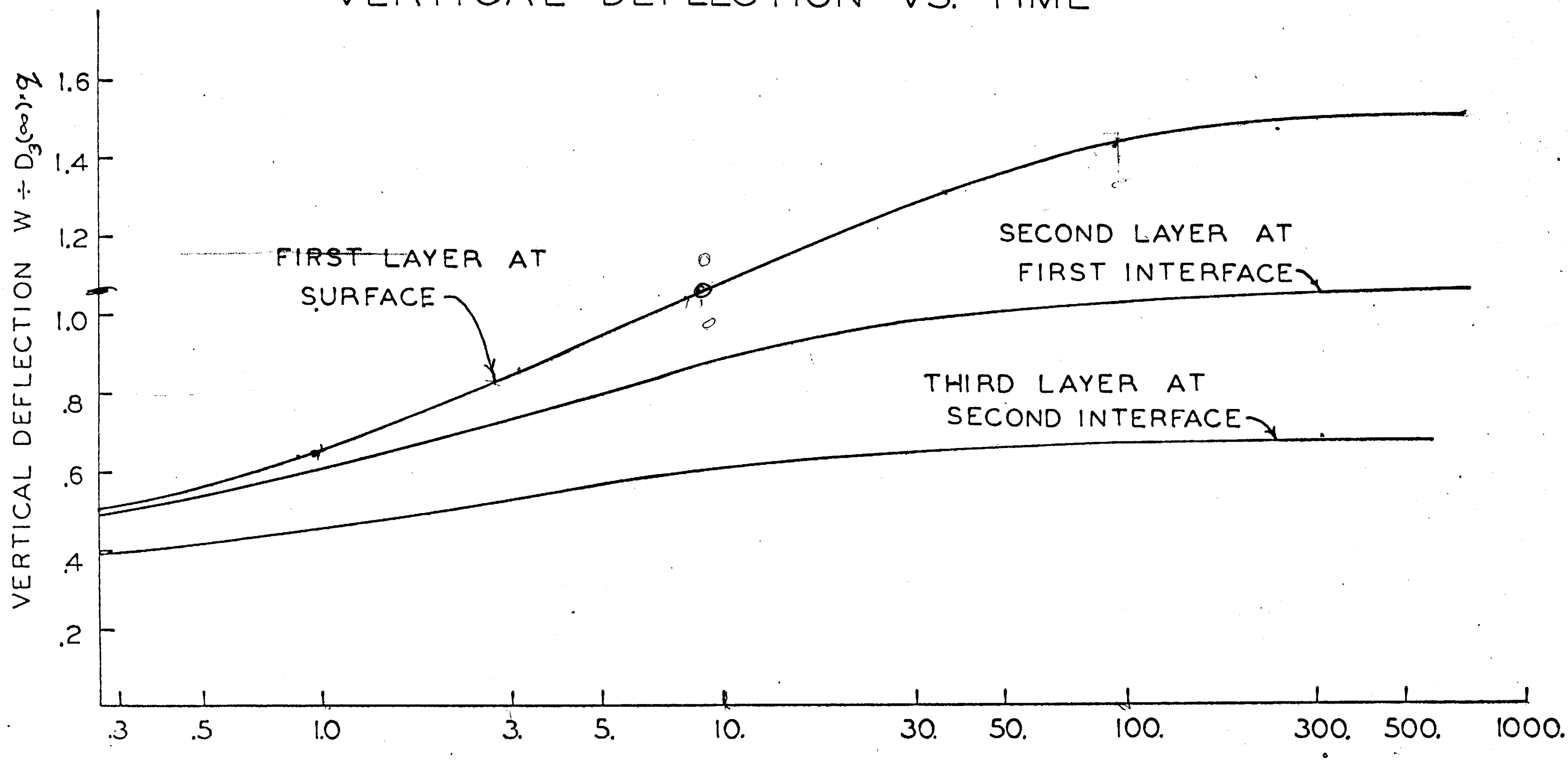
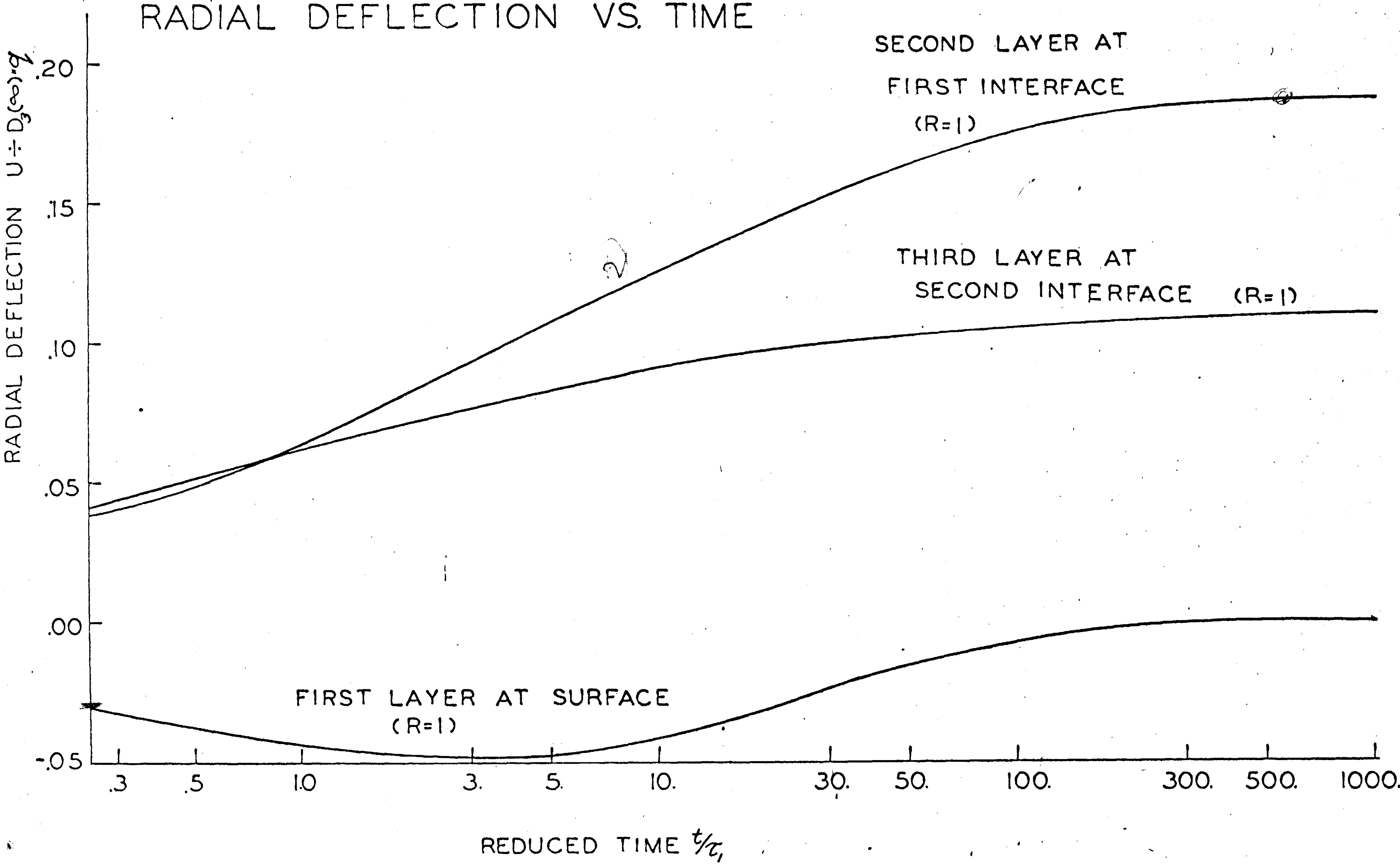


FIGURE 17

RADIAL DEFLECTION VS. TIME



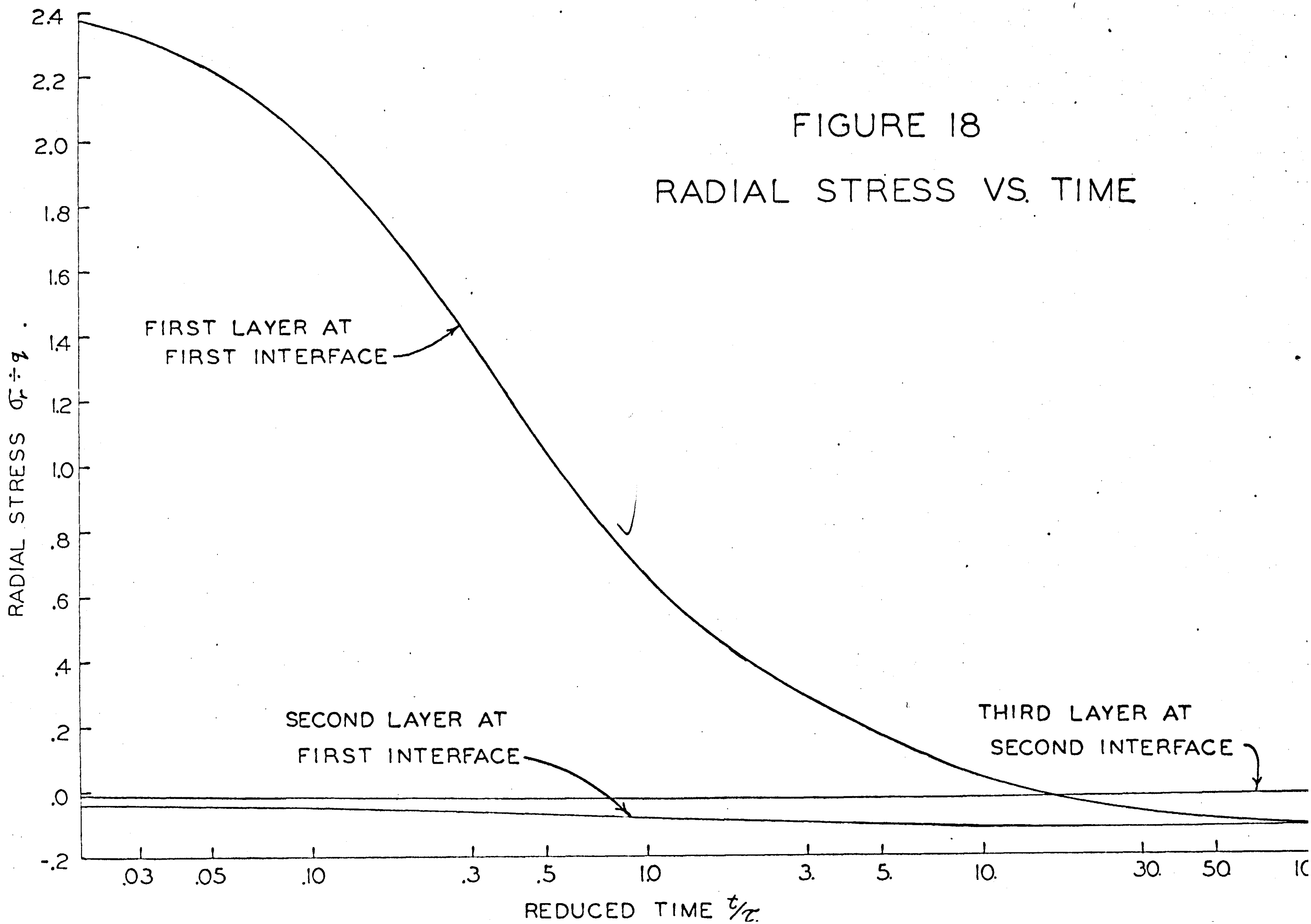


FIGURE 18
RADIAL STRESS VS. TIME
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
.0
-.2
RADIAL STRESS $\sigma_r \div q$
FIRST LAYER AT
FIRST INTERFACE
SECOND LAYER AT
FIRST INTERFACE
THIRD LAYER AT
SECOND INTERFACE
.03
.05
.10
.3
.5
1.0
3.
5.
10.
30.
50.
10
REDUCED TIME t/τ

TABLE XIII

COMPARISON OF NORMAL STRESS RESULTS FOR FIRST LAYER AT FIRST INTERFACE

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
0.	-.1724	-.1726	.12
.05	-.2009	-.2011	.10
.10	-.2263	-.2264	.04
.25	-.2876	-.2877	.03
.35	-.3191	-.3191	.00
.50	-.3564	-.3563	.03
.65	-.3851	-.3852	.03
.75	-.4007	-.4008	.03
.875	-.4173	-.4173	.00
1.00	-.4312	-.4313	.02
1.25	-.4533	-.4535	.04
1.55	-.4733	-.4732	.02
1.80	-.4861	-.4862	.02
2.05	-.4968	-.4970	.04
3.20	-.5302	-.5302	.00
4.20	-.5485	-.5484	.02
5.00	-.5595	-.5595	.00
10.00	-.5958	-.5960	.03

TABLE XIV

COMPARISON OF NORMAL STRESS RESULTS
FOR SECOND LAYER AT SECOND INTERFACE

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
0.	-.0925	-.0924	.11
.05	-.1043	-.1043	.0
.10	-.11476	-.11463	.11
.25	-.1393	-.1391	.14
.35	-.1516	-.1514	.13
.50	-.1658	-.1657	.06
.65	-.1767	-.1766	.06
.75	-.1826	-.1826	.0
.875	-.1889	-.1888	.05
1.00	-.19415	-.19420	.03
1.25	-.2027	-.2026	.05
1.55	-.2106	-.2104	.09
1.80	-.2157	-.2156	.05
2.05	-.2200	-.2200	.0
2.50	-.2264	-.2264	.0
3.20	-.2341	-.2338	.13
4.20	-.2414	-.2414	.0
5.00	-.2461	-.2458	.12
10.00	-.2607	-.2606	.04

TABLE XV

COMPARISON OF NORMAL STRESS RESULTS
FOR THIRD LAYER AT Z = 2.0 H

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
0.	-.06401	-.06361	.62
.05	-.07063	-.07021	.59
.10	-.07632	-.07596	.47
.25	-.08934	-.08899	.39
.35	-.09561	-.09533	.29
.50	-.1027	-.1024	.30
.65	-.1079	-.1078	.09
.75	-.1107	-.1106	.09
.875	-.1136	-.1135	.09
1.00	-.1159	-.1159	.00
1.25	-.1197	-.1197	.00
1.55	-.1231	-.1231	.00
1.80	-.1253	-.1253	.00
2.05	-.1271	-.1272	.08
3.20	-.1328	-.1327	.08
4.20	-.1357	-.1357	.00
5.00	-.1374	-.1373	.07
10.00	-.1425	-.1426	.07

TABLE XVI

COMPARISON OF SHEAR STRESS RESULTS
FOR FIRST LAYER, AT INTERFACE
AND UNIT OFF-SET

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
.0	-.04509	-.04512	.07
.05	-.05421	-.05424	.06
.10	-.06236	-.06239	.05
.25	-.08209	-.08207	.02
.35	-.09218	-.09216	.02
.50	-.1040	-.1040	.0
.65	-.1129	-.1129	.0
.75	-.1176	-.1177	.09
.875	-.1225	-.1226	.08
1.00	-.1266	-.1266	.0
1.25	-.1328	-.1328	.0
1.55	-.1381	-.1381	.0
1.80	-.1414	-.1413	.07
2.05	-.1441	-.1441	.0
2.50	-.1479	-.1479	.0
3.20	-.1524	-.1524	.0
4.20	-.1571	-.1571	.0
5.00	-.1599	-.1599	.0
10.00	-.1697	-.1698	.06

TABLE XVII

COMPARISON OF SHEAR STRESS RESULTS
FOR SECOND LAYER AT SECOND INTERFACE
AND UNIT OFF-SET

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
0.	-.01624	-.01682	3.56
.05	-.01932	-.01985	2.74
.10	-.02208	-.02253	2.16
.25	-.02878	-.02908	1.04
.35	-.03226	-.03250	.74
.50	-.03642	-.03657	.41
.65	-.03966	-.03979	.33
.75	-.04144	-.04160	.38
.875	-.04336	-.04348	.28
1.00	-.04500	-.04510	.22
1.25	-.04766	-.04779	.28
1.55	-.05011	-.05023	.24
1.80	-.05174	-.05185	.21
2.05	-.05314	-.05325	.21
3.20	-.05757	-.05771	.24
4.20	-.06021	-.06027	.10
5.00	-.06169	-.06181	.20
10.00	-.06723	-.06714	.13

TABLE XVIII

COMPARISON OF SHEAR STRESS RESULTS FOR THIRD LAYER AT Z = 2.0 AND UNIT OFF-SET

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
0.	-.01004	-.01011	.70
.05	-.01170	-.01176	.51
.10	-.01317	-.01321	.31
.25	-.01665	-.01666	.06
.35	-.01841	-.01841	.00
.50	-.02045	-.02044	.05
.65	-.02200	-.02200	.00
.75	-.02283	-.02283	.00
.875	-.02371	-.02369	.08
1.00	-.02445	-.02445	.00
1.25	-.02563	-.02563	.00
1.55	-.02670	-.02669	.04
1.80	-.02740	-.02739	.04
2.05	-.02798	-.02798	.00
3.20	-.02985	-.02981	.14
4.20	-.03081	-.03080	.03
5.00	-.03141	-.03139	.06
10.00	-.03325	-.03328	.09

TABLE XIX

COMPARISON OF VERTICAL DEFLECTION RESULTS FOR FIRST LAYER AT SURFACE

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
0.0	.3646	.3664	.60
.05	.3967	.3980	.33
.10	.4250	.4256	.14
.25	.4925	.4923	.04
.35	.5274	.5274	.0
.50	.5698	.5697	.02
.65	.6040	.6037	.05
.75	.6236	.6235	.02
.875	.6455	.6457	.03
1.00	.6649	.6652	.05
1.25	.6986	.6985	.01
1.55	.7326	.7325	.01
1.80	.7569	.7571	.03
2.05	.7787	.7789	.03
2.50	.8129	.8128	.01
3.20	.8572	.8568	.05
4.20	.9078	.9076	.02
5.00	.9413	.9414	.01
10.00	1.079	1.079	.0

TABLE XX

COMPARISON OF VERTICAL DEFLECTION RESULTS
FOR SECOND LAYER AT FIRST INTERFACE

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
0.	.3573	.3588	.42
.05	.3871	.3885	.36
.10	.4132	.4145	.31
.25	.4744	.4748	.08
.35	.5052	.5060	.16
.50	.5418	.5407	.20
.65	.5705	.5704	.02
.75	.5866	.5865	.02
.875	.6041	.6042	.02
1.00	.6194	.6199	.08
1.25	.6452	.6452	.00
1.55	.6704	.6703	.01
1.80	.6877	.6877	.00
2.05	.7029	.7028	.01
3.20	.7549	.7556	.09
4.20	.7861	.7863	.03
5.00	.8058	.8056	.02
10.00	.8788	.8829	.61

TABLE XXI

COMPARISON OF VERTICAL DEFLECTION RESULTS
FOR THIRD LAYER AT SECOND INTERFACE

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
0.	.3059	.3095	1.17
.05	.3266	.3282	.49
.10	.3444	.3461	.49
.25	.3850	.3852	.05
.35	.4048	.4048	.00
.50	.4277	.4261	.33
.65	.4452	.4444	.18
.75	.4548	.4551	.07
.875	.4651	.4646	.11
1.00	.4740	.4744	.08
1.25	.4887	.4888	.02
1.55	.5027	.5023	.08
1.80	.5123	.5124	.02
2.05	.5205	.5201	.08
3.20	.5476	.5490	.25
4.20	.5632	.5634	.04
5.00	.5728	.5729	.02
10.00	.6063	.6105	.69

TABLE XXII

COMPARISON OF RADIAL DEFLECTION RESULTS FOR FIRST LAYER AT SURFACE AND UNIT OFF-SET

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
0.	-.01773	-.01773	.00
.05	-.02078	-.02078	.00
.10	-.02348	-.02348	.00
.25	-.02993	-.02991	.07
.35	-.03316	-.03315	.03
.50	-.03691	-.03690	.03
.65	-.03970	-.03970	.00
.75	-.04116	-.04117	.02
.875	-.04267	-.04267	.00
1.00	-.04388	-.04390	.05
1.25	-.04569	-.04570	.02
1.55	-.04713	-.04713	.00
1.80	-.04791	-.04791	.00
2.05	-.04847	-.04847	.00
3.20	-.04917	-.04917	.00
4.20	-.04863	-.04863	.00
5.00	-.04787	-.04782	.10
10.00	-.04129	-.04129	.00

TABLE XXIII

COMPARISON OF RADIAL DEFLECTION
RESULTS FOR SECOND LAYER AT
FIRST INTERFACE AND UNIT OFF-SET

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
0.0	.02174	.02185	.51
.05	.02590	.02598	.31
.10	.02966	.02975	.30
.25	.03906	.03909	.08
.35	.04412	.04415	.07
.50	.05043	.05042	.02
.65	.05561	.05563	.04
.75	.05860	.05863	.05
.875	.06193	.06193	.00
1.00	.06490	.06488	.03
1.25	.07002	.07006	.06
1.55	.07516	.07518	.03
1.80	.07882	.07880	.03
2.05	.08210	.08212	.02
2.50	.08723	.08719	.05
3.20	.09384	.09385	.01
4.20	.1013	.1014	.10
5.00	.1062	.1062	.00
10.00	.1258	.1258	.00

TABLE XXIV

COMPARISON OF RADIAL DEFLECTION RESULTS
FOR THIRD LAYER AT SECOND INTERFACE
AND UNIT OFF-SET

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
0.	.02571	.02640	2.68
.05	.02964	.03025	2.03
.10	.03310	.03361	1.51
.25	.04137	.04172	.84
.35	.04558	.04586	.61
.50	.05055	.05073	.36
.65	.05440	.05457	.31
.75	.05651	.05665	.25
.875	.05880	.05889	.17
1.00	.06077	.06087	.16
1.25	.06401	.06408	.11
1.55	.06709	.06712	.04
1.80	.06916	.06921	.07
2.05	.07098	.07105	.10
3.20	.07699	.07708	.12
4.20	.08061	.08063	.02
5.00	.08277	.08289	.08
10.00	.09088	.09086	.02

TABLE XXV

COMPARISON OF RADIAL STRESS RESULTS FOR FIRST LAYER AT FIRST INTERFACE

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
0.	2.497	2.495	.08
.05	2.224	2.223	.05
.10	1.995	1.994	.05
.25	1.500	1.501	.07
.35	1.277	1.278	.08
.50	1.041	1.041	.00
.65	.8786	.8780	.07
.75	.7973	.7960	.16
.875	.7166	.7160	.08
1.00	.6527	.6516	.17
1.25	.5579	.5579	.09
1.55	.4780	.4786	.13
1.80	.4301	.4298	.07
2.05	.3909	.3904	.13
2.50	.3349	.3350	.03
3.20	.2697	.2694	.11
4.20	.2046	.2016	1.49
5.00	.1656	.1649	.42
10.00	.03975	.03793	4.58

TABLE XXVI

COMPARISON OF RADIAL STRESS RESULTS
FOR SECOND LAYER AT FIRST INTERFACE

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
0.	-.03894	-.03927	.85
.05	-.04313	-.04340	.63
.10	-.04692	-.04709	.36
.25	-.05647	-.05653	.11
.35	-.06172	-.06176	.06
.50	-.06841	-.06856	.22
.65	-.07404	-.07403	.01
.75	-.07732	-.07732	.00
.875	-.08099	-.08097	.02
1.00	-.08424	-.08426	.02
1.25	-.08971	-.08973	.02
1.55	-.09488	-.09488	.00
1.80	-.09828	-.09810	.18
2.05	-.1011	-.1010	.10
2.50	-.1050	-.1049	.10
3.20	-.1091	-.1092	.09
4.20	-.1128	-.1128	.00
5.00	-.1147	-.1147	.00
10.00	-.1189	-.1189	.00

TABLE XXVII

COMPARISON OF RADIAL STRESS RESULTS
FOR THIRD LAYER AT SECOND INTERFACE

Time t/τ_1	Numerical Integration Solution	Exact Integration Solution	Per Cent Difference
0.	-.02125	-.01568	26.1
.05	-.02203	-.01771	20.5
.10	-.02265	-.01813	19.9
.25	-.02388	-.02057	13.8
.35	-.02444	-.02170	11.2
.50	-.02496	-.02294	8.1
.65	-.02533	-.02382	6.0
.75	-.02554	-.02423	5.1
.875	-.02569	-.02467	4.0
1.00	-.02582	-.02505	3.0
1.25	-.02607	-.02556	2.0
1.55	-.02646	-.02593	2.0
1.80	-.02645	-.02611	1.3
2.05	-.02641	-.02621	.8
2.50	-.02661	-.02624	1.4
3.20	-.02690	-.02613	2.9
4.20	-.02633	-.02585	1.8
5.00	-.02631	-.02560	2.7
10.00	-.02398	-.02407	.4

CHAPTER VII

NON-LINEAR VISCOELASTICITY

This chapter presents a review of the pertinent literature on non-linear viscoelasticity with respect to a consideration of the practical implications for stress and displacement analysis. In particular, the various physically meaningful types of non-linearity are discussed with respect to the possibility of extening the techniques already discussed in this thesis to these certain non-linear problems, or of the applicability of other practical means of analysis.

The discussion is divided into four principle areas: ageing effects, thermoviscoelasticity, finite strain and geometrical non-linearities, and material non-linearities. A correspondence between a certain type of non-linear elasticity problem and a certain form of material non-linearity is illustrated in the last section where the analysis of an infinite linear viscoelastic plate on a non-linear viscoelastic foundation is presented.

VII-1. Ageing Effects

The constitution of many materials (for example, concrete) is a function of the age of the material (i.e. the time since the material was formed) during the time of interest. Thus the creep compliance or relaxation function of such a material is a function of two times, the time ($t-t_k$) since loading, and the time (t) with respect to the time when the material was formed:

$$\gamma_i(t) = f(t-t_k, t) \tag{235}$$

The effect of t_k on $\gamma_i(t)$ may be linear or non-linear, but in either case this "ageing" effect introduces additional complexity into a structural analysis. Reference [66] illustrates the effect of ageing on the creep behavior of concrete specimens.

The structural analysis of materials which exhibit "ageing" effects has been largely ignored in the literature. This is in spite of the fact that many materials do exhibit "ageing." However, although the behavior is exhibited, the response $\gamma_i(t)$ for a material that ages, although being a function of the age since forming as well as the duration of load, varies much more slowly for a variation in t_k than for a variation in $t-t_k$. That is, "ageing" effects generally occur over relatively long times, while relaxation or creep effects are often rapidly changing over short times. The practical implication of this is that if the response time of interest is relatively short, then the creep or relaxation function can be approximated by a particular linear viscoelastic function at the time of (say) loading t_K. That is, for a load applied at time t_K:

$$\gamma_i(t) \approx f(t-t_K, t_K) \tag{236}$$

This approximation will be acceptable as long as $t-t_K$ is small relative to some "characteristic ageing time." More explicitly, the above approximation should be adequate as long as the difference

$$f(t-t_K, t_K) - f(t-t_K, t) \tag{237}$$

remains sufficiently small.

If one finds, however, that the approximation expressed by equation (236) is not sufficiently close to the real materials behavior (that is, for long times of loading, if the difference (237) is greater than is considered allowable), then an analysis must be performed which considers the ageing effects explicitly. Little is available in the literature to guide such an analysis (see, however, reference [10] for concrete applications). However, the numerical approach in Chapter IV can be, in theory, used to carry out such analyses with only the changes to be discussed below.

The evaluation of the convolution integrals, which are now of the following form:

$$I(t_j) = \int_{t_\ell}^{t_j} \gamma_{k-1}(t_j-\tau, t_\ell+\tau)\, \frac{\partial \gamma_k(\tau, t_\ell+\tau)}{\partial \tau}\, d\tau + \gamma_{k-1}(t_j-t_\ell, t_j)\, \gamma_k(0, t_j) \tag{238}$$

(where t_j is the time of interest and t_ℓ is the time of loading), can be carried out as before by dividing the integrals into finite sums:

$$I(t_j) = \sum_{i=1}^{n_1} \frac{1}{2}\left[\gamma_{K-1}(t_j - t_i, t_\ell + t_i) + \gamma_{K-1}(t_j - t_{i-1}, t_\ell + t_{i-1})\right] \times \left[\gamma_K(t_i, t_\ell + t_i) - \gamma_K(t_{i-1}, t_\ell + t_{i-1})\right] + \gamma_{K-1}(t_j - t_\ell, t_j)\,\gamma_K(0, t_j) \quad (239)$$

Every term in the sum of equation (239) is of the form $f(t-t_K, t)$, and thus is presumably known, so that the integral can be approximated using only discrete knowledge of $\gamma_{K-1}(t-t_K, t)$ and $\gamma_K(t-t_K, t)$. In an analogous way, the solution to the integral equation can be readily obtained numerically.

VII-2. Thermoviscoelasticity

In all of the applications previously discussed it has been assumed that either the properties of the material did not vary with temperature (a very poor assumption for most materials displaying viscoelastic properties) or else that isothermal conditions exist. This section discusses the analysis of linear viscoelastic materials under variable temperature conditions, that is, thermoviscoelasticity.

The analysis under varying temperature fields presents no unusual problems if the physical properties of the material are assumed independent of temperature, as shown by Sternberg [124] (1958). However, if the more realistic assumption of temperature-dependent properties is imposed, there appears to be no general method of solution of the equations [139].

The general problem of temperature dependent properties has been considered by Morland and Lee [84] and by Muki and Sternberg [86]. In both of these papers, the assumption of "thermorheologically simple" materials, originally proposed by Leaderman [67], was invoked. Since this assumption is representative of a large number of viscoelastic materials, the following discussion will also employ that assumption.

"Thermorheologically simple" materials are materials whose characteristic functions (creep and relaxation functions) obey the following law:

$$\gamma_i(t, T_1) = \gamma_i(\xi, T_0) \tag{240}$$

where

ξ = "reduced time" = $t/a(T_1)$

T_0 = reference temperature

T_1 = any other temperature

$a(T_1)$ = experimentally determined shift factor, a function of the temperature T_1 referred to the reference temperature T_0

As shown by Muki and Sternberg [86], the general constitutive equations under transient temperatures, for a "thermorheologically-simple" material, can then be written as follows:

$$S_{ij}(t) = \int_{0^-}^{t} G_r(\xi - \xi') \frac{\partial \epsilon_{ij}(\tau)}{\partial \tau} d\tau \tag{241}$$

$$\sigma(t) = \int_{0^-}^{t} K_r(\xi - \xi') \frac{\partial}{\partial \tau} \{ e(\tau) - 3\alpha_0 \theta(\tau) \} d\tau \tag{242}$$

where

$$\xi = \int_0^t \frac{du}{a(T(u))} \qquad \xi' = \int_0^\tau \frac{du}{a(T(u))} \tag{243}$$

$$\theta(t) = \frac{1}{\alpha_0} \int_{T_0}^{T(t)} \alpha(\tau) d\tau \qquad \alpha_0 = \alpha(T_0) \tag{244}$$

and $\alpha(T)$ is the temperature dependent coefficient of thermal expansion.

If the coefficient of thermal expansion is taken constant over the range of temperature $T(t) - T_0$, then:

$$\theta(t) = T(t) - T_0 \tag{245}$$

and equations (241) and (242) can be written in the following manner:

$$S_{ij}(t) = \left\{ G_r \text{[illegible]} + \int_{0^+}^{t} (\text{[illegible]}) \frac{\partial G_r(\xi - \xi')}{\partial \tau} d\tau \right\} \epsilon_{ij}(t) \tag{246}$$

$$\sigma(t)=\left\{K_r(0)-\int_0^{t^-}(\quad)\frac{\partial K_r(\xi-\xi')}{\partial\tau}d\tau\right\}\left[e(t)-3\alpha_o(T(t)-T_o)\right] \tag{247}$$

which give operators analogous to the elastic operators for the transient temperature case.

It is exceedingly important to note that the constitutive equations (246) and (247) will vary spatially under transient temperature conditions even for an initially isotropic body.

For the case that $T(t) = T_o$, $a(T) = 1$ and $\xi = t$ so that the equations reduce to the case considered in the previous chapters. If $T(t) = T_1 =$ constant, then $\xi = t/a(T_1)$, and the creep or relaxation functions are all "shifted" by an amount $\log_{10} a(T_1)$. However, they still can be handled as simple linear viscoelastic functions and a simple correspondence between elastic and viscoelastic problems still exists.

For the case that T(t) is not constant, two possibilities exist. First, the temperature of the body, while varying, may be uniformly varying throughout the body. In this case there is no spatial variation of the constitutive equations (246) and (247), and the following operators can again be used as "equivalent elastic constants":

$$2G=\left\{G_r(0)-\int_0^{t^-}(\quad)\frac{\partial G_r(\xi-\xi')}{\partial\tau}d\tau\right\} \tag{248}$$

$$3K = \left\{ K_r(0) - \int_0^{t^-} (\ ) \frac{\partial K_r(\xi - \xi')}{\partial \tau} d\tau \right\} \tag{249}$$

Just as previously discussed, the bulk behavior may reasonably be considered constant with respect to time (but not with respect to temperature) in some applications (see reference [32]), or infinite in others, as a fairly reasonable further simplification. Use of the above operators will permit the formulation of the solution to this type of thermoviscoelastic problem in terms of integral equations of the general form (53). Evaluation of the multiple convolution integrals can be handled numerically as previously described. For example, a single general convolution integral becomes:

$$I_1(t) = \int_{0^+}^{t} \gamma_{k-1}(\xi - \xi') \frac{\partial \gamma_k(\xi)}{\partial \tau} d\tau + \gamma_{k-1}(\xi)\gamma_k(0) \tag{250}$$

which can be written as the following finite sum:

$$I_1(t_n) = \sum_{i=1}^{n} \tfrac{1}{2}\left[\gamma_{k-1}(\xi_n - \xi_i) + \gamma_{k-1}(\xi_n - \xi_{i-1}) \right] \times \left[\gamma_k(\xi_i) - \gamma_k(\xi_{i-1}) \right] + \gamma_{k-1}(\xi)\gamma_k(0) \tag{251}$$

For any ξ_i, $\gamma_k(\xi_i)$ or $\gamma_{k-1}(\xi_i)$ can be obtained by integrating equations (243), (exactly or numerically), and solving for t. This value of t can then be used to evaluate $\gamma_k(\xi_i)$ or $\gamma_{k-1}(\xi_i)$, and in this way the above numerical integration can be carried out. Although the bookkeeping would be somewhat complex, the principle is relatively straightforward.

The second case with T(t) varying is the case that the temperature varies non-uniformly throughout the body. In this case, since the temperature history varies from spatial point to spatial point, the constitutive equations (246) and (247) vary spatially also. In this case there seems to be no method in general to use in approaching the problem. It would seem, however, that the application of finite element techniques such as are now beginning to see wider usage offers a reasonable path to follow. Presumably one could approach the problem step-wise in time, and for any given time t the temperature and temperature history of each of the nodes of each of the elements could be used to calculate element properties at that time, and thus the necessary stiffnesses or flexibilities could be calculated. For sufficiently small elements and steps in time, one would expect this procedure to yield realistic answers.

With regard to more rigorous approaches, Muki and Sternberg [86] have managed to solve the problems of the thermal stresses in an infinite thermovisco-elastic slab, and the stresses in a thermoviscoelastic sphere. Morland and Lee [84] have also managed to solve the problem of a hollow viscoelastic cylinder reinforced with an elastic case under steady state conditions. Their methods of solution, however, seem to offer little hope for obtaining a general method of analysis, especially under transient temperature conditions.

VII-3. Finite Strain and Geometrical Non-Linearity

In all of the previous discussions and examples, the tacit assumption that the deformations could be represented by the linear infinitesimal strain tensor has been made. However, if the strains are large (usually a strain greater than ten per cent is considered too large for the use of the linear infinitesimal strain tensor), then a finite strain formulation must be invoked. The theory has been discussed by Eringen [29] and by Pipkin [96].

The theoretical groundwork for small strains superposed on finite strains for materials with memory has been considered by Lianis [78] and by Pipkin and Rivlin [97]. Strains of this magnitude are quite uncommon in work involving concrete, asphalt, or even soils. Usually separation (failure) of the body would occur before such strains are reached. Except in the analysis of rubber-like materials, there would thus seem to be limited application of the theories of finite strain within the realm of common viscoelastic materials. However, if such large strains are to be considered, then Biot's approach using incremental deformations [17] appears more practical than attempts to solve such problems directly. The use of finite element techniques also offers hope for attacking these finite strain problems.

A somewhat similar non-linearity occurs when the deformations cause large displacements which cannot be ignored when considering the equilibrium equations. Buckling problems are generally of this type, and also bending problems for beams and plates, where a small load causing small strains may cause large deflections. For this type of problem, a correspondence between the solution for an elastic body and the solution for a viscoelastic body exists in the same sense as previously discussed. Examples of this type of problem are Lee and Rogers' solution for the finite deflection of a viscoelastic cantilever beam [107] (also considered by Schapery [112]), Baltrukonis and Vaishnav's solution [13] for the creep-bending of a viscoelastic beam-column, and Anderson's solution [6] for the buckling of shallow viscoelastic arches.

VII-4. Material Non-Linearities

Although it would seem that large strain non-linearities are not often a major cause for concern in most analyses, the possibility that the material exhibits non-linear responses at strain levels corresponding to small strain still exists. As pointed out by Arutyunyan [10], for example, linear behavior can be expected for concrete up to about one-half the ultimate strength. Above this, however, the response becomes non-linear. This is still generally at very low strain levels (less than one per cent).

Possible approaches for solving boundary-value problems in the regions of small strain with physical non-linearity will be discussed below. Although a sizeable amount of work has been expended on formulating acceptable characterizations for physical non-linearity, little has been done to date with respect to solving boundary value problems.

VII-4.1. Non-Linearities and the Theory of Plasticity

Before considering the general characterization of non-linear materials with memory, it is appropriate to consider the realm of application of such theories. As will be shown below, such theories generally result in constitutive relations that are cumbersome from the point of view of both the analyst and the experimentalist. For engineering applications, it is thus desirable, when sufficient accuracy can be maintained, to consider possible simplifications.

It is possible, for certain materials, to use the theory of plasticity when large strains or marked non-linearities exist. Reference [35] presents stress-strain curves for polyethylene for four different strain rates, varying from .022 inches per inch per minute to .260 inches per inch per minute (a variation of over 100 times) for strains up to .40 inches per inch. The data is clearly non-linear. However, the maximum variation in the curves for the different strain rates is less than ten per cent. Furthermore, the curves can all be approximated very nearly by bi-linear stress-strain curves, composed of a linear-elastic segment up to approximately .08 inches per inch strain, and then a perfectly plastic stress-strain curve. Clearly, for most applications, the assumption that the material has no time variation but does "go plastic" above eight per cent strain should yield results sufficiently accurate, for engineering purposes, for those applications where large strains are expected. (Metals generally show approximately the same amount of strain rate effects as the polyethylene in reference [35].)

VII-4.2. Non-Linear Creep Analysis

Many materials, notably concrete at stresses above one-half the ultimate strength and metals at high temperatures, can be characterized accurately by non-linear creep laws for constant stresses. The most usual form of such relations is:

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E_o} + \frac{\sigma}{k}^{m} \tag{252}$$

Such non-linear creep laws have been used successfully to analyze the creep buckling of columns. Hoff [52] has presented a survey of the approaches used on this problem. T. H. Lin [79], in 1956, and Pian [94], in 1958, have also presented such analyses.

Other similar approaches are also common, (see, for example, references [10,66]), and have been shown to give good results for constant stress applications. It is important to note, however, that a direct use of equations such as (252) under variable stress conditions may lead to erroneous results.

VII-4.3. General Non-Linear Analysis

As mentioned above, a considerable amount of work has been expended on developing constitutive relations for non-linear viscoelastic materials. In particular, Green and Rivlin [38] in 1957, Eringen and Grot [30] in 1965, Lianis [77] in 1965, Rivlin [103] in 1965, and T. Tokuoka [129] in 1961 have presented theoretical developments for general non-linear materials with memory.

The general result deduced in the above papers, for the case of small strain, is that the stress-strain relationships can be represented by multiple-integrals involving stress- or strain-rates, and certain kernel functions. For the one-dimensional case, such a representation becomes:

$$
\begin{aligned}
S_{ij}(t) &= \int_{0^-}^{t} G_1(t-\tau_1) \frac{\partial \epsilon_{ij}(\tau_1)}{\partial \tau_1} d\tau_1 \\
&+ \int_{0^-}^{t}\int_{0^-}^{t} G_2(t-\tau_1, t-\tau_2) \frac{\partial \epsilon_{ij}(\tau_1)}{\partial \tau_1} \frac{\partial \epsilon_{ij}(\tau_2)}{\partial \tau_2} d\tau_1 d\tau_2 \\
&+ \int_{0^-}^{t}\int_{0^-}^{t}\int_{0^-}^{t} G_3(t-\tau_1, t-\tau_2, t-\tau_3) \frac{\partial \epsilon_{ij}(\tau_1)}{\partial \tau_1} \frac{\partial \epsilon_{ij}(\tau_2)}{\partial \tau_2} \frac{\partial \epsilon_{ij}(\tau_3)}{\partial \tau_3} d\tau_1 d\tau_2 d\tau_3 \\
&+ \cdots
\end{aligned}
\tag{253}
$$

where the kernel functions $G_1(\ )$, $G_2(\ )$, $G_3(\ )$, ... are symmetric functions of their arguments. It is readily apparent that the experimental determination of the kernels (relaxation functions) requires a large number of independent tests. $G_1(t_1)$ is a linear material function described by a single curve with respect to a single time coordinate, while $G_2(t_1, t_2)$ is a second order function describable by a surface with respect to two time coordinates, while $G_3(t_1, t_2, t_3)$ is described by a hypersurface with respect to three time coordinates, etc. [32]. The experimental determination of $G_1(\ )$, $G_2(\ )$, and $G_3(\ )$ has been discussed by Ward and Onat [134] in 1963.

Some attempts have been made, for one-dimensional cases, to determine the kernel functions experimentally. Examples of such attempts are given by Ward and Onat [134] in 1963, Hadley and Ward [41] in 1965, Leaderman, McCrackin, and Nakada [69] in 1963, and Onaron and Findley [88] in 1965. Onat [89] has also recently discussed the problems and approaches of such experimental studies.

The possibility of solving boundary value problems for bodies governed by constitutive equations such as equation (253) seems even more formidable than the experimental problem of determining the appropriate kernel functions. Some investigators have made progress along these lines, however. Appleby and Lee [8] have shown that for short times a third-order theory (through the triple integral of equation (253)) can be simplified to include only single integrals, although a large number of these integrals will occur. Huang and Lee [55] have also considered the problems of incompressible non-linear viscoelastic materials under small finite deformation and for short time ranges. By means of the equations they have derived, they were able to analyze a pressurized viscoelastic hollow cylinder with an elastic case (for short times) by utilizing some fairly involved numerical analysis.

Other approaches are also possible. Vaishnav and Dafermos [133] have managed to analyze an infinitely long, thick-walled, non-linearly viscoelastic cylinder with an elastic case by expressing the constitutive equation in non-linear differential form. With the assumption of an incompressible material, they were able to carry out an analysis using fairly representative material properties for the quasi-static case. The analysis, however, required extremely tedious and careful numerical solutions.

VII-4.4. A Simplified Non-Linear Constitutive Equation

It would appear that the general constitutive equation (253) suffers from excessive generality. In order to arrive at somewhat simpler relationships, Schapery [111,113,114] has invoked irreversible thermodynamics. Halpin [43] has derived equivalent simplified relationships by considering the kinetic theories of elastic and viscoelastic responses. In both cases, constitutive equations of the following form have been theorized:

$$S_{ij}(t) = \int_{0^-}^{t} G_r(t-\tau) \frac{\partial f(\epsilon_{ij}(\tau))}{\partial \tau} d\tau \qquad (254)$$

where $f(\epsilon_{ij}(\tau))$ is some non-linear function of the strain $\epsilon_{ij}(\tau)$.

Although the constitutive equation (254) is certainly not sufficiently general to apply to all non-linear materials, there seems to be ample evidence that it can accurately describe the non-linear response of many viscoelastic materials. Halpin's paper [43] presents some experimental evidence of this, as do two of Schapery's works [111,113]. In addition, Leaderman [68] presents some experimental verification.

The advantages of a constitutive law of the type given in equation (254) are obvious. First of all, only one kernel function G(t) must be determined for the uniaxial case, and only two such functions for the three-dimensional case. Furthermore, these kernel functions are just the relaxation functions of linear viscoelasticity, and thus experimental techniques for their determination are known. In addition, the analysis of bodies for which the constitutive relation (254) holds seems relatively straight-forward, since there is a correspondence between a certain type of non-linear elasticity problem and this type of non-linear viscoelasticity problem. To see this, we write equation (254) in the following operational form:

$$S_{ij}(t) = \left\{ -\int_0^{t^+} (\quad) \frac{\partial G_r(t-\tau)}{\partial \tau} d\tau \right\} f(\epsilon_{ij}(t)) \tag{255}$$

Clearly then there is a correspondence between the operator within the brackets of equation (255) and the modulus G in the following non-linear elasticity relationship:

$$S_{ij} = G\, f(\epsilon_{ij}) \tag{256}$$

Hence if a boundary value problem can be solved for a body obeying the non-linear elastic law of equation (256), then the non-linear viscoelastic solution can be obtained by means of the techniques of Chapter IV. This correspondence is illustrated below on the problem of determining the deflection of an infinite linear viscoelastic plate on a non-linear viscoelastic (Winkler) foundation.

VII-4.4.1. Deflection of an Infinite Linear Viscoelastic Plate on a Non-Linear Viscoelastic Foundation

The geometry to be considered in this example is illustrated in Figure 19. It consists of a plate, infinite in horizontal extent, supported by a foundation which supplies only a vertical reaction. To illustrate the non-linear elastic--non-linear viscoelastic correspondence described in the previous section, the deflection of an incompressible linear viscoelastic plate on a foundation supplying a non-linear viscoelastic vertical reaction will be analysed under the action of a single load of magnitude P at the origin of coordinates.

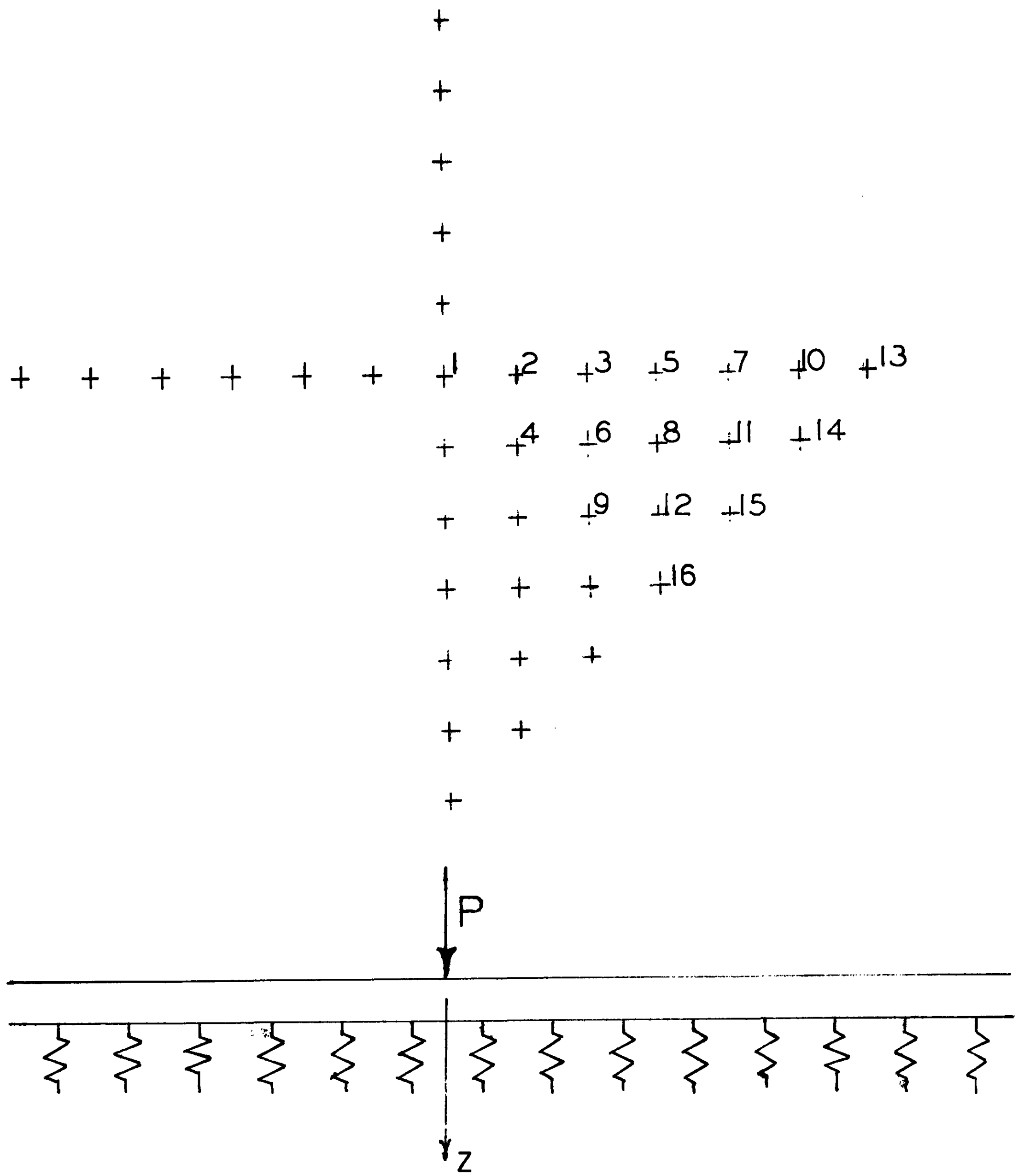


FIGURE 19 GEOMETRY OF INFINITE PLATE ON WINKLER FOUNDATION

The solution for the deflection of a linear elastic plate on a non-linear elastic foundation has been given elsewhere by the author. This solution was obtained by means of a finite element analysis of the plate, since an exact solution of the non-linear problem has not been found. If the plate is divided into appropriate finite elements, and the flexibility coefficients for each node are calculated, then the equations of vertical equilibrium for each of the nodes provides a sufficient number of equations to determine the deflections at these nodes. Since the problem is axially symmetric, only the nodes numbered in Figure 19 need to be considered. If the flexibility coefficients are denoted Ea_{ij} (Ea_{ij} gives the force at node i due to a unit deflection at node j), then the equilibrium equations to be considered can be written in matrix form as follows (the details for calculating the flexibility coefficients have been given in reference [12] and will not be repeated here):

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & \\ a_{21} & a_{22} & \cdots & \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix} = \begin{bmatrix} \frac{P}{E} - \frac{Kf(W_1)}{E} \\ -\frac{Kf(W_2)}{E} \\ \vdots \\ -\frac{Kf(W_n)}{E} \end{bmatrix} \qquad (257)$$

where

W_i = deflection of the i th node

E = Young's modulus of the plate

and the foundation reaction is given by the following (non-linear) expression:

$$f_i = K f(w_i) \tag{258}$$

As has been illustrated in reference [12], the above system of simultaneous non-linear equations can be solved for the nodal deflections by using a perturbation about the linear solution. First the forces applied to the plate due to the deflection are added to both sides of equation (257) to yield the following form:

$$\begin{bmatrix} a_{11}+\frac{Kf(W_1)}{EW_1} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}+\frac{Kf(W_2)}{EW_2} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}+\frac{Kf(W_n)}{EW_n} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix} = \begin{bmatrix} P/E \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{259}$$

If w_i is known, then $f(w_i)$ can be calculated, and the square matrix in equation (259) can be inverted to yield the w_i's. Clearly an iterative technique is suggested. In reference [12], the following procedure was found to work quite adequately.

First, the linear part of $f(w_i)$ is used so that the terms $f(w_i)/w_i$ may be immediately calculated. Using these results, the equations (259) may be solved to yield a first (linear) approximation for the w_i's. This approximation is then used to calculate the $f(w_i)/w_i$ terms, and a second approximation is then obtained by resolving equations (259). This procedure is repeated until the relative changes in each w_i are less than a prescribed amount.

Consider now a plate composed of an incompressible linear viscoelastic material with an "equivalent compliance" given by the following operator:

$$\left(\frac{1}{E}\right)_{\text{equivalent}} = \left\{ D(0) - \int_0^{t^-} (\ )\, \frac{\partial D(t-\tau)}{\partial \tau}\, d\tau \right\} \qquad (260)$$

and a foundation which yields a non-linear vertical reaction of the form suggested in the previous section, that is:

$$f_i(t) = \left\{ K(0) - \int_0^{t^-} (\ )\, \frac{\partial K(t-\tau)}{\partial \tau}\, d\tau \right\} f(w_i(t)) \qquad (261)$$

The following "equivalent foundation modulus" is suggested by equations (258) and (261):

$$K \text{ equivalent} = \left\{ K(0) - \int_0^{t^-} (\quad) \frac{\partial K(t-\tau)}{\partial \tau} d\tau \right\} \tag{262}$$

Replacing 1/E and K by their equivalent operator expressions, the matrix equations (259), which express the equilibrium of the nodes, can be written as follows:

$$\begin{bmatrix} a_{11} + \frac{g_1(t)}{W_1} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} + \frac{g_2(t)}{W_2} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} + \frac{g_n(t)}{W_n} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix} = \begin{bmatrix} PD(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{263}$$

where

$$g_i(t) = \int_{0^+}^{t} f(W_i(t-\tau)) \frac{\partial}{\partial \tau} \int_{0^+}^{\tau} K(\tau-\lambda) \frac{\partial D(\lambda)}{\partial \lambda} d\lambda + K(\tau) D(0) d\tau \tag{264}$$
$$+ f(W_i(t)) K(0) D(0)$$

and it is assumed that the load P is applied as a step function in time.

The matrix equation (263) gives a set of n simultaneous non-linear integral equations in the n unknown

w_i's. They can be solved using the same perturbation technique discussed above for the non-linear equations in combination with the technique for the numerical solution of the integral equations as previously discussed. For clarity, the i th equation will be considered in the following discussion.

Denote the inner convolution integral of $g_i(t)$ as $\alpha(t)$. That is:

$$\alpha(t) = \int_{0^+}^{t} K(t-\lambda)\frac{\partial D(\lambda)}{\partial \lambda}\,d\lambda + K(t)D(0) \tag{265}$$

In the numerical example to be presented below, K(t) and D(t) are taken in the form of Dirichlet series, and $\alpha(t)$ is then calculated exactly using the subroutine INTEGR.

With $\alpha(t)$ now assumed known for any value of t, $g_i(t_m)$ can be approximated by the following finite sum:

$$g_i(t_m) \approx \sum_{j=1}^{m}\frac{1}{2}\left[f(w_i(t_{m-j+1})) + f(w_i(t_{m-j}))\right]\left[\alpha(t_m - t_{m-j}) - \alpha(t_m - t_{m-j+1})\right] + f(w_i(t_m))\,\alpha(0) \tag{266}$$

Separating the terms involving $w_i(t_m)$, $g_i(t_m)$ can be divided into the following form:

$$g_i(t_m) = \frac{1}{2}\left[\alpha(t_m - t_{m-1}) + \alpha(0)\right]f(w_i(t_m)) + \frac{1}{2}\left[\alpha(t_m - t_{m-1}) - \alpha(0)\right]f(w_i(t_{m-1})) + \sum_{j=2}^{m}\frac{1}{2}\left[f(w_i(t_{m-j+1})) + f(w_i(t_{m-j}))\right]\left[\alpha(t_m - t_{m-j}) - \alpha(t_m - t_{m-j+1})\right] \tag{267}$$

Substituting the above expression for $g_i(t_m)$ into the matrix equation (263) and rearranging, the following set of non-linear (algebraic) equations are obtained:

$$\begin{bmatrix} a_{11}+\frac{[\alpha(0)+\alpha(t_m-t_{m-1})]f(W_1(t_m))}{2W_1} & a_{12} & \cdots \\ a_{21} & a_{22}+\frac{[\alpha(0)+\alpha(t_m-t_{m-1})]f(W_2(t_m))}{2W_2} & \cdots \\ \vdots & \vdots & \\ a_{n1} & a_{n2} & \cdots \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix} \tag{268}$$

$$= \begin{bmatrix} PD(t_m)-\frac{1}{2}[\alpha(t_m-t_{m-1})-\alpha(0)]f(W_1(t_{m-1}))-\sum_{j=2}^{m}\frac{1}{2}[f(W_1(t_{m-j+1}))+f(W_1(t_{m-j}))][\alpha(t_m-t_{m-j})-\alpha(t_m-t_{m-j+1})] \\ -\frac{1}{2}[\alpha(t_m-t_{m-1})-\alpha(0)]f(W_2(t_{m-1}))-\sum_{j=2}^{m}\frac{1}{2}[f(W_2(t_{m-j+1}))+f(W_2(t_{m-j}))][\alpha(t_m-t_{m-j})-\alpha(t_m-t_{m-j+1})] \\ \vdots \\ -\frac{1}{2}[\alpha(t_m-t_{m-1})-\alpha(0)]f(W_n(t_{m-1}))-\sum_{j=2}^{m}\frac{1}{2}[f(W_n(t_{m-j+1}))+f(W_n(t_{m-j}))][\alpha(t_m-t_{m-j})-\alpha(t_m-t_{m-j+1})] \end{bmatrix}$$

The set of simultaneous (non-linear) equations (268) can be solved using the same perturbation technique described above, where now one must iterate at each time t_i. Note that the right-hand side of equation (268) contains only known constants, and terms of the form $f(w_i(t_{m-j}))$. Since $w_i(t_{m-j})$ has been calculated at a previous step, $f(w_i(t_{m-j}))$ can be calculated directly.

The above procedure has been programmed and a program listing is given in the appendix. To illustrate the results, K(t) and D(t) have been assumed in the following form:

$$K(t) = 250.(1 + e^{-t/\tau_1} + e^{-t/\sqrt{10}\tau_1} + e^{-.1t/\tau_1}) \qquad (269)$$

$$D(t) = 10^{-5}(.2 - .1e^{-t/\tau_1} - .05e^{-t/\sqrt{10}\tau_1} \qquad (270)$$

These functions are plotted in Figures 20 and 21. The results for a plate of two inch thickness, with a load of 16000 pounds, are plotted in Figures 22 and 23. The function f(w(t)) has been taken as follows:

$$f(w(t)) = w(t) - 1.6\,[w(t)]^2 \quad \text{(linear dimensions expressed in inches)} \qquad (273)$$

In Figure 22 the maximum deflection is plotted as a function of time. For comparison purposes, the linear viscoelastic solution, and the non-linear elastic and linear elastic solutions using the zero time compliance and foundation reaction, are also plotted. Clearly the non-linear behavior has a major influence on the maximum deflection in this particular case. Figure 23 presents a plot of the deflection profile for $t/\tau_1 = 0.0$, $t/\tau_1 = 1.0$, and $t/\tau_1 = 10.0$. The magnitude of the deflections change markedly, but the general shape appears to remain similar.

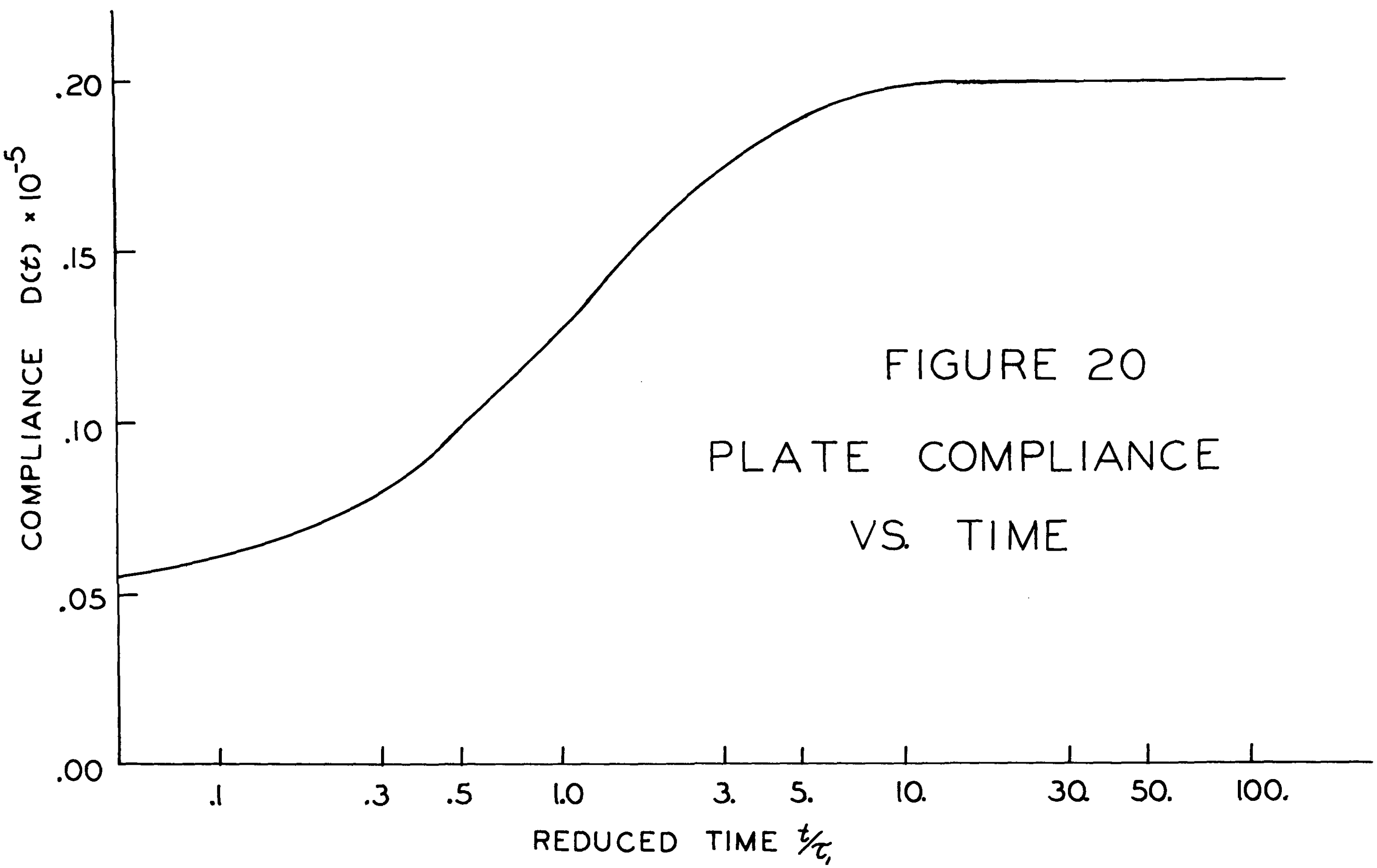


FIGURE 20
PLATE COMPLIANCE
VS. TIME

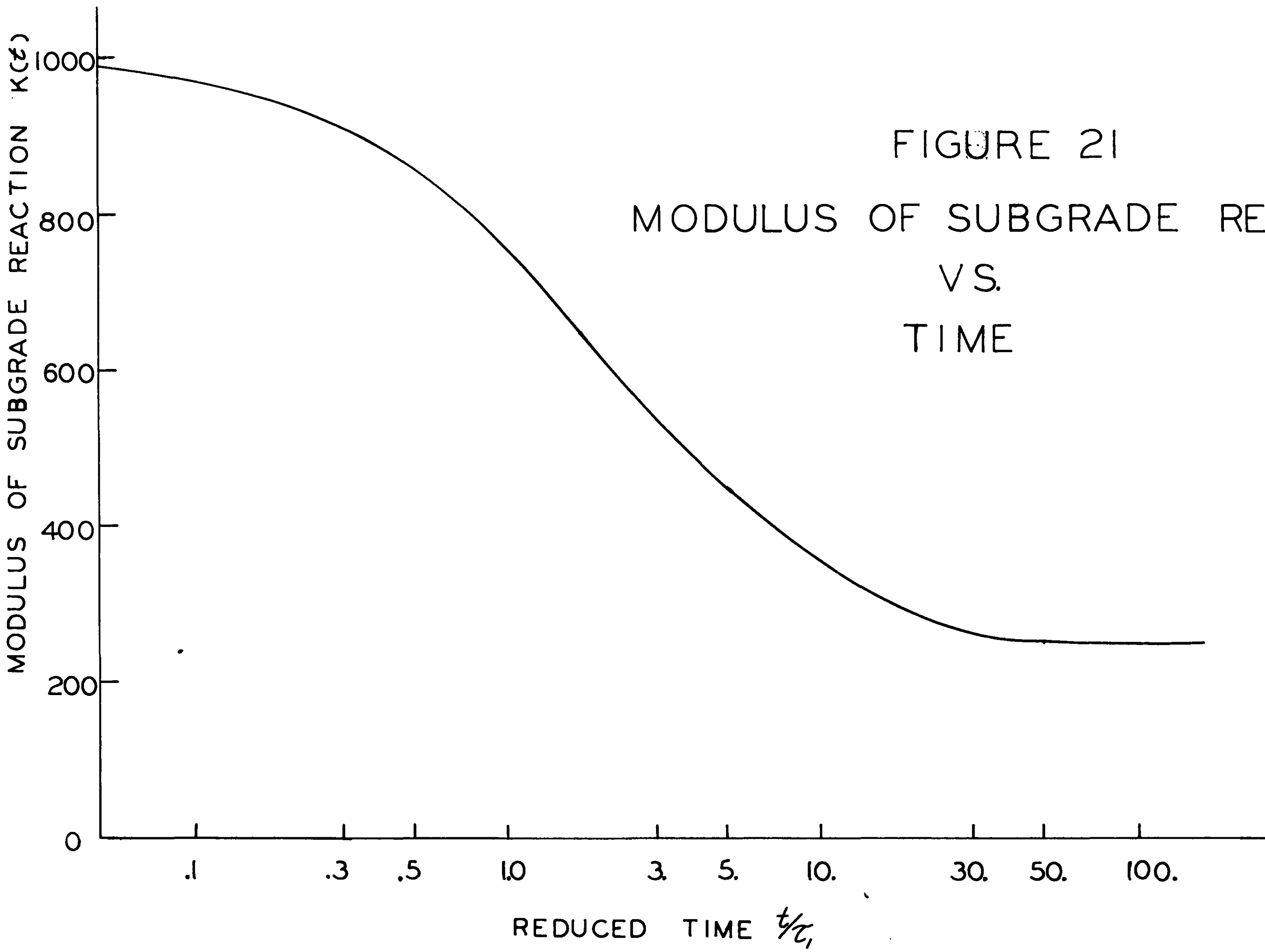


FIGURE 21
MODULUS OF SUBGRADE REACTION
VS.
TIME

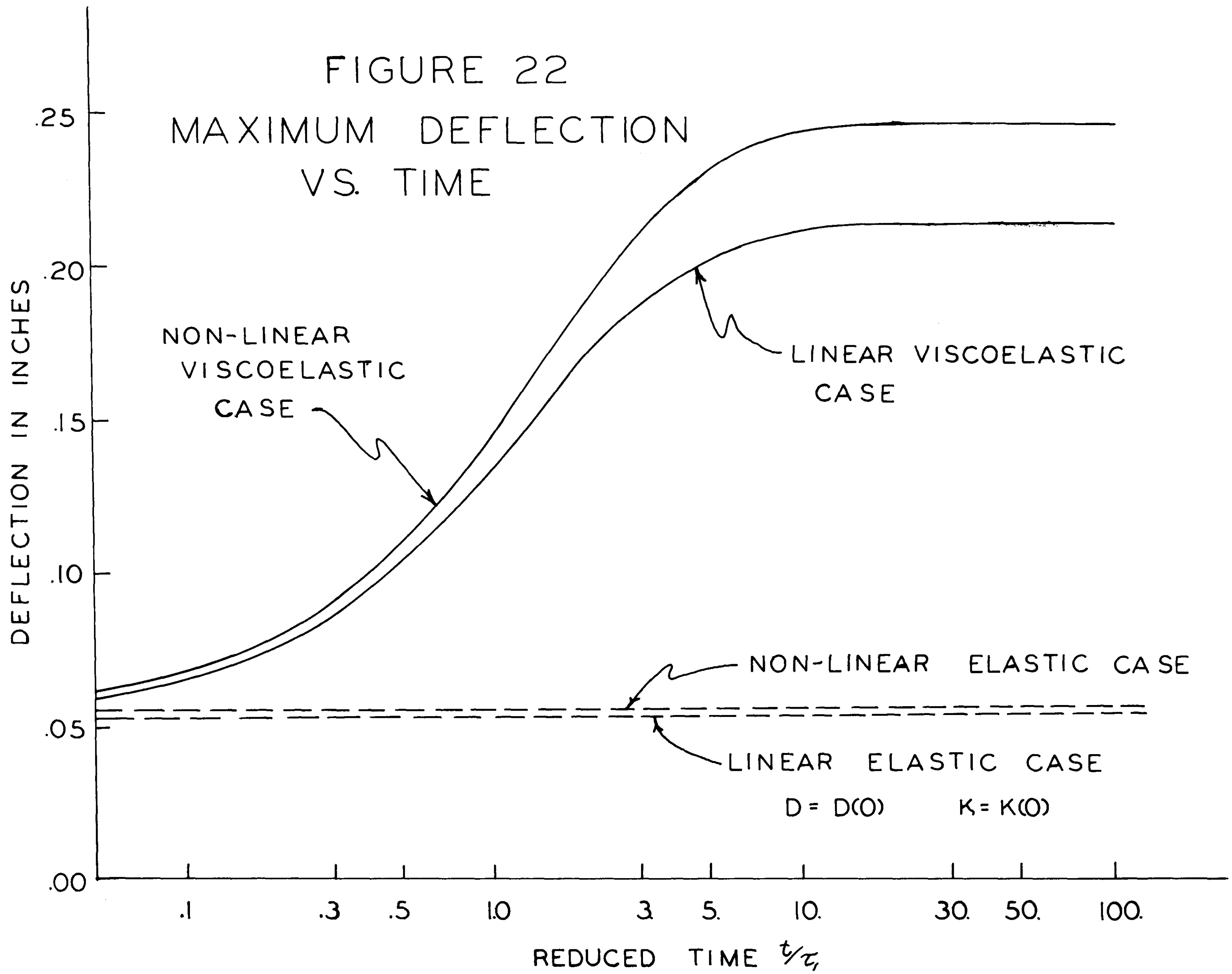


FIGURE 22
MAXIMUM DEFLECTION VS. TIME

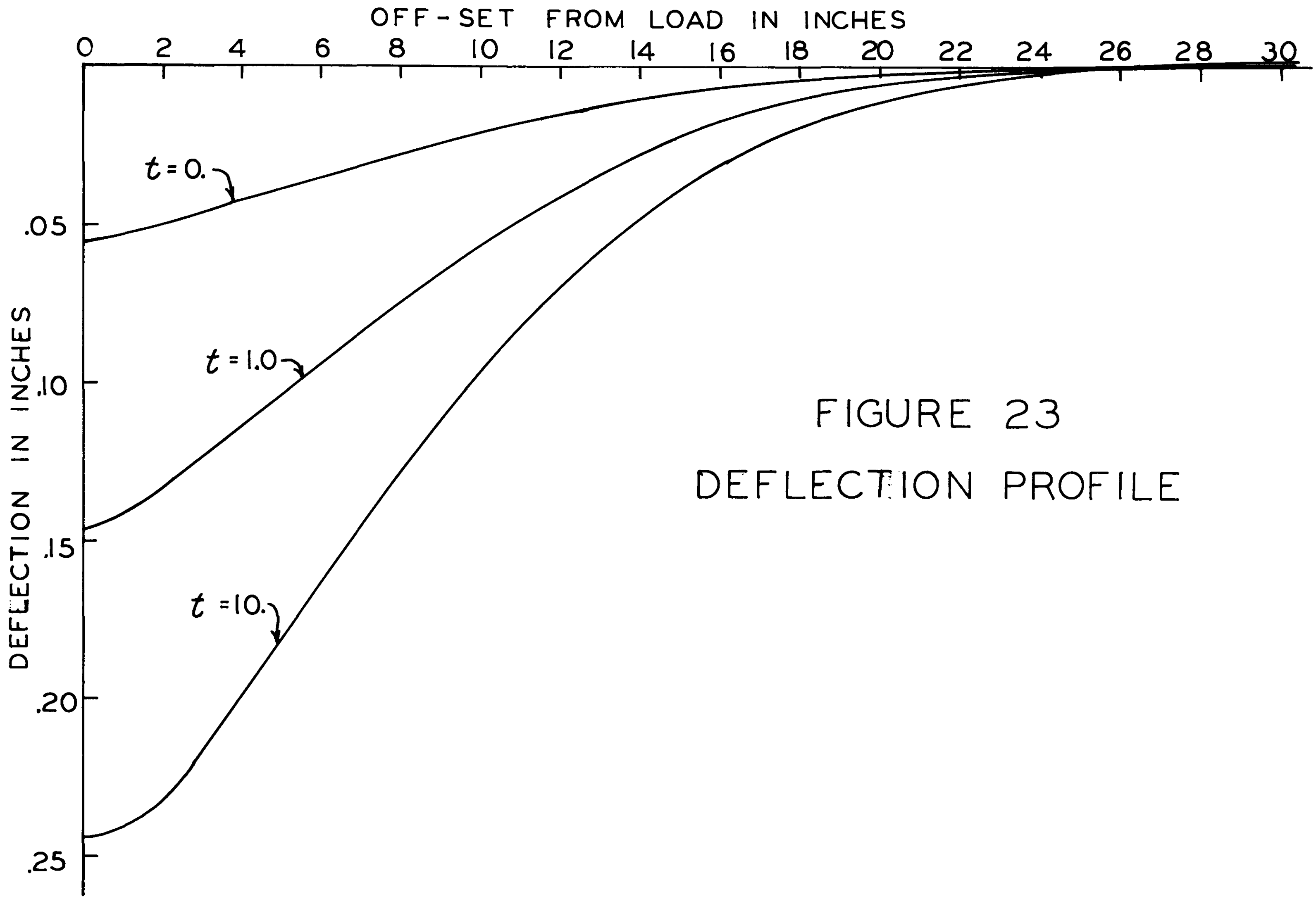


FIGURE 23
DEFLECTION PROFILE

VII-4.5. Concluding Remarks

Material non-linearities have been briefly considered in this section. Although a considerable amount of work has been expended in recent years on developing constitutive equations for non-linear viscoelastic materials, it would appear that the more general approaches are too cumbersome for reasonable application. Furthermore, until the rational basis for such non-linear viscoelastic constitutive equations are developed and verified more extensively through experiments, their use seems of doubtful value.

Until such work has been carried out, the use of the more firmly grounded theories of plasticity, linear viscoelasticity, and creep is indicated for most applications. In those cases where the use of these theories does not seem appropriate, then an experimental consideration of appropriate constitutive relations may be necessary. In this case, simplifications such as the one considered in section 4.4. of this chapter will decrease the complexity of the structural analysis.

CHAPTER VIII

CONCLUSIONS

The method of analysis presented in this thesis for stresses and displacements in linear viscoelastic bodies has three principle advantages.

1. The Laplace transform is not needed, and thus it is not necessary that all of the equations and boundary conditions have Laplace transforms.*

2. The application of the above method is rather straight-forward, and requires only a few steps for the problems where the equivalent elastic solution can be written in the form of equation (46).

3. The method of solution of the general equation, using either technique to evaluate the multiple convolution integrals, allows realistic material representations to be used.

The example in Chapter V concerning the deflection of a viscoelastic cantilever beam illustrates that where exact solutions can be found, the method presented herein gives equivalent results, and that the numerical techniques used can yield extremely accurate solutions.

The example in Chapter VI, the analysis of the stresses and displacements of a three-layered viscoelastic half-space under a circular load, illustrates the applicability of the technique to problems involving

*Subject to the limitations discussed in Chapter III.

different types of linear viscoelastic materials, and the straight-forwardness of its application. The feasibility of evaluating many-fold multiple convolution integrals by both techniques is also apparent. Furthermore, the analysis should be of engineering value in foundation and pavement design.

Reasonable approaches to certain non-linear problems have been suggested in Chapter VII. In particular, a correspondence between a certain type of non-linear elastic problem and non-linear viscoelastic problem has been formulated. The use of this correspondence principle to determine the deflection of a linear viscoelastic plate on a non-linear viscoelastic foundation illustrates the ease of such analysis when used together with the techniques discussed in this thesis for linear viscoelastic analysis.

CHAPTER IX

FUTURE RESEARCH

The method of analysis presented in this thesis appears to be easily applied, and quite accurate. Furthermore, it would seem that it could be applied to a large number of problems. For this reason the possibility of generating packaged computer programs for the evaluation of the multiple convolution integrals and for the numerical solution of the integral equation warrants future consideration.

Also, the use of the technique on those problems where the time variations of the loading are very rapid (assuming that inertia terms are then likely to have to be included) would warrant some investigation. Although there have been no signs of problems to be encountered in such applications in the present work, such rapid variations in loadings could possibly cause numerical difficulties.

Further investigation of the methods of analysis for non-linear problems, considered briefly in Chapter VII, should also be considered.

APPENDICES

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MAIN PROGRAM FOR CANTILEVER BEAM

ANALYSIS USING NUMERICAL INTEGRATION

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C
C
C
C      THIS IS THE MAIN PROGRAM TO ANALYSE A LINEAR VISCOELASTIC BEAM
C      USING THE NUMERICAL INTEGRATION OF THE CONVOLUTION INTEGRALS.
C      THE NECESSARY SUBROUTINES ARE TIME1, VALUE, INTEGR, SOLVIT, AND
C      REJECT.  THE INPUT RELAXATION FUNCTIONS ARE TAKEN AS DIRICHLET
C      SERIES FOR CONVENIENCE AND FOR COMPARISON WITH THE EXACT SOLUTION
C      AND WITH THE SOLUTION OBTAINED USING THE OTHER  INTEGRATION
C      TECHNIQUE.
       DIMENSION G(20),AK(20),P(20),EX(61),ERR(61),PH(18),TH(9),E(7,61),
      1GAM(61,7,18)
       COMMON BETA(61),B(8,20),DELTA(20),T(61),MN,SI(61)
C      THE LOOP THROUGH 1000 ALLOWS SEVERAL SETS OF DATA TO BE RUN.
       DO 1000 JJJ=1,100
C      N = NUMBER OF TERMS IN DIRICHLET SERIES
C      NNN = NUMBER OF STEPS IN EACH INTEGRATION LOOP BEFORE DOUBLING
C      N8 = NUMBER OF TIMES THE LOOP (FOR INTEGRATION. IS TO BE DOUBLED
C      DEL = INITIAL SPACING OF TIME
       READ(5,1) N,NNN,N8,DEL
C      NX IS A DUMMY FOR THE INPUT INTO THE SUBROUTINE TIME1
       NX=0
     1 FORMAT(3I5,F10.5)
C      MN CONTROLS THE BEGINNING OF SEVERAL DO LOOPS WHICH VARY DEPENDING
C      ON WHETHER IT IS THE FIRST OR SUBSEQUENT TIMES THROUGH THE LOOP
       MN=1
C      N1 IS THE EQUIVALENT TO MN NOT IN COMMON
       N1=1
C      AL = LENGTH OF THE BEAM
C      C = HALF THE DEPTH OF THE BEAM
C      X = DISTANCE FROM THE FREE END THE DEFLECTION IS DESIRED
       READ(5,11) AL,C,X
    11 FORMAT(6F10.5)
C      AI = MOMENT OF INERTIA OF THE BEAM
       AI=2.*(C**3)/3.
C      THE PH( ) TERMS ARE THE PHI S OF THE TEXT
       PH(2)=(X**3)-3.*AL*AL*X&2.*(AL**3)
       PH(1)=3.*PH(2)&27.*C*C*(AL-X)/2.
C      TH(1) IS THETA 1 OF THE TEXT
       TH(1)=54.*AI
C      ALAM1 AND ALAM2 ARE CONSTANTS FOR THE EXACT SOLUTION (FOR CASE
C      THAT IT IS KNOWN.
       ALAM1=PH(1)/TH(1)
       ALAM2=PH(2)/TH(1)
C      THE VECTOR G CONTAINS THE CONSTANTS OF THE DIRICHLET SERIES
C      REPRESENTATION FOR THE SHEAR RELAXATION MODULUS
       READ(5,11)(G(J),J=1,N)
C      THE VECTOR AK( ) CONTAINS THE CONSTANTS FOR THE DIRICHLET SERIES
C      REPRESENTATION OF  THE BULK RELAXATION MODULUS
       READ(5,11)(AK(J),J=1,N)
C      THE VECTOR P( ) CONTAINS THE  CONSTANTS FOR THE LOAD SERIES
       READ(5,11)(P(J),J=1,N)
       WRITE(6,2)(G(J),J=1,N)
       WRITE(6,2)(AK(J),J=1,N)
       WRITE(6,2)(P(J),J=1,N)
     2 FORMAT(1H /6H INPUT/(6F10.5))
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      C     AK1 IS USED TO NON-DIMENSIONALIZE THE RESULTS WITH RESPECT TO THE
      C     BULK MODULUS AT ZERO TIME
.0024       AK1=0.
.0025       DO 12 J=1,N
.0026    12 AK1=AK1&AK(J)
.0027       DO 13 J=1,N
.0028       G(J)=G(J)/AK1
.0029    13 AK(J)=AK(J)/AK1
      C     SOLUTION MUST BE MULTIPLIED BY  1/AK1
      C     STATEMENT 10 IS THE BEGINNING OF THE REPEATED (LOOP) PART OF THE
      C     PROGRAM.  THE SUBROUTINE TIME1 COMPUTES THE RELEVANT TIMES AND IF
      C     THIS IS THE FIRST ENTRANCE TO STATEMENT 10, THEN THE RELAXATION
      C     TIMES ARE COMPUTED AND STORED IN THE DELTA( ) VECTOR
.0030    10 CALL TIME1(NNN,DEL,NX)
      C     THE LOAD IS EVALUATED AT EACH OF THE TIMES, USING THE SUBROUTINE
      C     VALUE (AND THE DUMMY ARRAY B( ,) )AND PRINTED OUT
.0031       DO 14 J=1,N
.0032    14 B(1,J)=P(J)
.0033       CALL VALUE(N,1,NNN)
.0034       WRITE(6,4)
.0035       WRITE(6,3)(T(L),BETA(L),L=1,NNN)
.0036     3 FORMAT(2E15.8)
.0037     4 FORMAT(1H /12H INPUT CURVE)
      C     THE P VALUES ARE STORED IN THE ARRAY E( , ), IN THE FIRST COLUMN
.0038       DO 5 I=1,NNN
.0039     5 E(1,I)=BETA(I)
      C     THE VALUES OF THE BULK RELAXATION MODULS ARE COMPUTED USING THE
      C     SUBROUTINE VALUE, THEN PRINTED OUT AND THEN STORED IN E(2, )
.0040       DO 15 J=1,N
.0041    15 B(1,J)=AK(J)
.0042       CALL VALUE(N,1,NNN)
.0043       WRITE(6,4)
.0044       WRITE(6,3)(T(L),BETA(L),L=1,NNN)
.0045       DO 16 I=1,NNN
.0046    16 E(2,I)=BETA(I)
      C     THE CONVOLUTION OF THE LOAD AND THE BULK RELAXATION MODULUS IS
      C     COMPUTED NUMERICALLY, USING SUBROUTINE INTEGR, AND STORED IN THE
      C     ARRAY GAM(( , ,1)
.0047       CALL INTEGR(N1,NNN,E,GAM,1,2)
      C     THE SHEAR RELAXATION MODULUS IS COMPUTED USING THE SUBROUTINE
      C     VALUE, THEN PRINTED, AND THEN STORED IN THE ARRAY E(2,I).  THE
      C     BULK RELAXATION FUNCTION IS SAVED AND STORED IN BETA() TEMPORARILY
.0048       DO 17 J=1,N
.0049    17 B(1,J)=G(J)
.0050       CALL VALUE(N,1,NNN)
.0051       WRITE(6,4)
.0052       WRITE(6,3)(T(L),BETA(L),L=1,NNN)
.0053       DO 18 I=1,NNN
.0054       SAVE=E(2,I)
.0055       E(2,I)=BETA(I)
.0056    18 BETA(I)=SAVE
      C     THE CONVOLUTION OF P AND THE RELAXATION MODULUS IN SHEAR  IS
      C     COMPUTED AND STORED IN  THE ARRAY GAM( , ,2)
.0057       CALL INTEGR(N1,NNN,E,GAM,2,2)
      C     THE BULK RELAXATION MODULUS IS TRANSFERRED BACK TO E(1, ), AND
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      C
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C
C     THE CONVOLUTION OF  THE TWO RELAXATION MODULII IS COMPUTED AND
C     STORED IN GAM( , ,3)
      DO 19 I=1,NNN
   19 E(1,I)=BETA(I)
      CALL INTEGR(N1,NNN,E,GAM,3,2)
C     THE CONVOLUTION RESULT OF THE RELAXATION MODULII IS TRANSFERRED
C     TO GAM( ,2,3) FROM GAM( ,3,1)
      DO 20 I=1,NNN
   20 GAM(I,3,1)=GAM(I,2,3)
C     THE INTEGRAL EQUATION IS SOLVED NUMERICALLY USING THE SUBROUTINE
C     SOLVIT.  THE RESULT IS STORED IN SI( ).
      CALL SOLVIT(NNN,PH,TH,GAM,2,1,2,3)
      WRITE(6,7)
    7 FORMAT(1H /9H SOLUTION)
C     THE EXACT SOLUTION IS CALCULATED AND STORED IN THE VECTOR EX( ),
C     AND THE PER CENT ERROR  IN THE NUMERICAL SOLUTION IS CALCULATED
C     AND STORED IN ERR( )
      DO 22 I=1,NNN
      EX(I)=    ALAM1*(EXP(-.1*T(I))-1.)*(-10.)&ALAM2*T(I)
      IF(I-1)23,23,24
   24 ERR(I)=(EX(I)-SI(I))/EX(I)*100.
      GO TO 22
   23 ERR(I)=0.
   22 CONTINUE
      WRITE(6,21)(T(L),SI(L),EX(L),ERR(L),L=1,NNN)
   21 FORMAT(4E15.8)
C     N8 IS ZERO ONLY WHEN THE LOOP HAS BEEN DOUBLED N8 (ORIGINAL) TIMES
      IF(N8)8,9,8
    8 N8=N8-1
C     THE SUBROUTINE REJECT  SAVES THE APPROPRIATE VALUES TO REDOUBLE
C     THE SOLUTION LOOP
      CALL REJECT(NNN,GAM)
      N1=MN
C     THE SPACING IS DOUBLED
      DEL=DEL*2.
      NX=5
      GO TO 10
    9 CONTINUE
 1000 CONTINUE
      END
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MAIN PROGRAM FOR CANTILEVER BEAM ANALYSIS USING EXACT INTEGRATION

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      C
      C
      C
      C     THIS PROGRAM IS TO ANALYSE THE VISCOELASTIC CANTILEVER BEAM USING
      C     THE EXACT INTEGRATION PROCEDURE.  THE INPUT FUNCTIONS ARE IN THE
      C     FORM OF DIRICHLET SERIES.  THE NECESSARY SUBROUTINES ARE TIME,
      C     SOLVE, INTEGR, VALUE, AND CVEFIT.  THE SOLUTION IS COMPARED TO THE
      C     EXACT SOLUTION WHERE APPLICABLE.  THE SOLUTION IS FITTED WITH A
      C     SERIES REPRESENTATION AND THEN THE ORIGINAL LEFT-HAND SIDE OF THE
      C     INTEGRAL EQUATION IS COMPUTED AND PRINTED FOR COMPARISON WITH THE
      C     RIGHT HAND SIDE AS A CHECK.
            DIMENSION G(8,20),D(8,20),EX(200),H(8,20),ARRAY(12,50),ERR(200)
            COMMON X(20),BB(8,20),  T(201) ,DELTA(20),BETA(201),P(8,20),
           1SI(201)
      C     NNN IS THE NUMBER OF STEPS TO BE COMPUTED IN THE NUMERICAL
      C     SOLUTION.  THESE ARE LOG STEPS, THE SIZE OF THEM BEING DETERMINED
      C     BY SUBROUTINE TIME.
            NNN=98
      C     ARRAY IS THE INVERSE OF THE COLLOCATION MATRIX FOR THE DELTA S
      C     COMPUTED USING SUBROUTINE TIME.  THIS ARRAY IS USED IN SUBROUTINE
      C     CVEFIT.
            READ(5,1)((ARRAY(I,J),I=1,12),J=1,12)
            WRITE(6,15)((ARRAY(I,J),I=1,12),J=1,12)
          1 FORMAT(4E15.8)
         15 FORMAT(12H INPUT ARRAY/(4E15.8))
      C     THE LOOP THROUGH 1000 ALLOWS MULTIPLE SETS OF DATA TO BE EXECUTED.
            DO 1000 III=1,100
      C     C1 = HALF THE DEPTH OF THE BEAM
      C     AL = LENGTH OF THE BEAM
      C     X1 = DISTANCE FROM THE FREE END THAT THE DEFLECTION IS DESIRED
            READ(5,2)C1,AL,X1
            WRITE(6,16)C1,AL,X1
         16 FORMAT(14H BEAM GEOMETRY/(3F10.5))
          2 FORMAT(3F10.5)
      C     AI = MOMENT OF INERTIA OF THE BEAM
            AI=2.*(C1**3)/3.
      C     T1 AND T2 CORRESPOND TO THE PHI S OF THE TEXT
            T2=(X1**3)-3.*AL*AL*X1&2.*(AL**3)
            T1=3.*T2&27.*C1*C1*(AL-X1)/2.
      C     PH CORRESPONDS TO THETA(1) OF THE TEXT
            PH=54.*AI
      C     ALAM AND ALAM1 ARE CONSTANTS IN THE EXACT SOLUTION
            ALAM=T1/PH
            ALAM1=T2/PH
      C     THE LOOP UP TO 37 ZERO S THE ARRAYS TO BE USED SUBSEQUENTLY
            DO 37 I=1,8
            DO 37 J=1,20
            D(I,J)=0.
            G(I,J)=0.
            BB(I,J)=0.
         37 H(I,J)=0.
      C     THE INPUT SERIES REPRESENTATIONS FOR THE RELAXATION FUNCTIONS AND
      C     THE LOAD, ALL OF LENGTH N, ARE READ INTO THE G( , ) ARRAY.  G(1, )
      C     IS THE SHEAR RELAXATION MODULUS, G(2, ) IS THE BULK RELAXATION
      C     MODULUS, AND G(3, ) IS THE LOAD FUNCTION
            READ(5,3)N,((G(J,I),I=1,N),J=1,3)
      C
      C
      C
      C
      C
      C
      C
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C
C
C
    3 FORMAT(I10/(6F10.1))
      WRITE(6,17)N,((G(J,I),I=1,N),J=1,3)
   17 FORMAT(7H CURVES/I10/(6F10.3))
C     THE SHEAR RELAXATION MODULUS AND THE BULK RELAXATION MODULUS ARE
C     NON-DIMENSIONALIZED BY DIVIDING BY THE BULK RELAXATION FUNCTION
C     AT ZERO TIME
      SUM=0.
      DO 40 I=1,N
   40 SUM=SUM&G(2,I)
      DO 41 J=1,N
      DO 41 I=1,2
   41 G(I,J)=G(I,J)/SUM
C     SOLUTION MUST BE MULTIPLIED BY 1/SUM
C     SUBROUTINE TIME CALCULATES THE NNN APPROPRIATE TIME VALUES,
C     AND  THE RELAXATION TIMES (DELTA( ))
      CALL TIME(NNN)
C     THE EXACT SOLUTIONS ARE CALCULATED AND STORED IN THE VECTOR EX( )
      DO 42 I=1,NNN
   42 EX(I)=ALAM*(-EXP(-.1*T(I))&1.)/.1&ALAM1*T(I)
C     THE PER CENT ERRORS WILL BE STORED IN THE VECTOR ERR( ) ( WHEN
C     EXACT SOLUTION IS APPLICABLE)
      ERR(1)=0.
C     SUBROUTINE VALUE IS USED TO CALCULATE VALUES FOR BOTH RELAXATION
C     MODULII AND FOR THE LOAD,  SO THAT THIS DATA CAN BE PRINTED OUT
      DO 4 I=1,3
      DO 5 J=1,N
    5 B(1,J)=G(I,J)
      CALL VALUE(N,1,NNN)
      WRITE(6,24)
    4 WRITE(6,6)(T(L),BETA(L),L=1,NNN)
    6 FORMAT(2E15.8)
C     THE CONVOLUTION OF THE TWO RELAXATION MODULII IS CALCULATED AND
C     THEN PRINTED.  THIS IS A TWO STEP OPERATION--FIRST THE RESULT IS
C     FOUND USING SUBROUTINE INTEGR, AND THEN THIS RESULT IS EVALUATED
C     USING SUBROUTINE VALUE.
      CALL INTEGR(G,N,1,0)
      CALL VALUE(N,2,NNN)
      WRITE(6,35)
      WRITE(6,6)(T(L),BETA(L),L=1,NNN)
C     THE CONVOLUTION OF THE RELAXATION MODULII IS MULTIPLIED BY PH AND
C     STORED IN THE ARRAY D( , ) FOR FUTURE USE
      DO 7 I=1,2
      DO 7 J=1,N
    7 D(I,J)=B(I,J)*PH
C     THE BULK RELAXATION MODULUS AND THE LOAD SERIES ARE TRANSFERRED
C     INTO  THE ARRAY H( , ).  THEN THE CONVOLUTION OF THESE TWO SERIES
C     IS CALCULATED  USING SUBROUTINE INTEGR.
      DO 8 I=1,2
      DO 8 J=1,N
    8 H(I,J)=G(I&1,J)
      CALL INTEGR(H,N,1,0)
C     THE RESULT OF THE LAST CONVOLUTION INTEGRATION IS MULTIPLIED BY T1
C     AND STORED IN THE ARRAY BB( , )
      DO 9 I=1,2
      DO 9 J=1,N
    9 BB(I,J)=B(I,J)*T1
C
C
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      C
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      C
      C     THE SHEAR RELAXATION MODULUS AND  THE LOAD SERIES ARE TRANSFERRED
      C     INTO THE ARRAY H( , ) AND THEN THE CONVOLUTION OF THESE TWO SERIES
      C     IS CALCULATED USING SUBROUTINE INTEGR.
.0060       DO 10 J=1,N
.0061       H(1,J)=G(1,J)
.0062    10 H(2,J)=G(3,J)
.0063       CALL INTEGR(H,N,1,0)
      C     THE RESULT OF THE LAST CONVOLUTION IS MULTIPLIED BY T2 AND ADDED
      C     TO THE RESULT STORED IN BB( , )
.0064       DO 11 I=1,2
.0065       DO 11 J=1,N
.0066    11 BB(I,J)=BB(I,J)&T2*B(I,J)
      C     THE KERNAL OF THE INTEGRAL ON THE LEFT SIDE OF THE INTEGRAL
      C     EQUATION IS EVALUATED AND PRINTED
.0067       DO 36 I=1,N
.0068       DO 36 J=1,2
.0069    36 B(J,I)=D(J,I)
.0070       CALL VALUE(N,2,NNN)
.0071       WRITE(6,38)
.0072       WRITE(6,6)(T(L),BETA(L),L=1,NNN)
.0073    38 FORMAT(1H /25H INTEGRAL BEFORE SOLUTION)
      C     THE INTEGRAL EQUATION IS SOLVED USING SUBROUTINE SOLVE
.0074       CALL SOLVE(N,2,2,NNN)
      C     THE ERROR IN THE SOLUTION IS CALCULATED AND STORED IN ERR( )
.0075       DO 43 I=2,NNN
.0076    43 ERR(I)=(EX(I)-SI(I))/EX(I)*100.
.0077       WRITE(6,25)
.0078       WRITE(6,50)(T(L),SI(L),  EX(L),ERR(L),L=1,NNN)
.0079    50 FORMAT(4E15.8)
      C     THE SOLUTION IS FITTED WITH A DIRICHLET SERIES USING SUBROUTINE
      C     CVEFIT, THEN THIS SERIES IS EVALUATED USING SUBROUTINE VALUE,
      C     THEN THIS SOLUTION IS COMPARED TO THE EXACT SOLUTION, AND THEN
      C     THESE RESULTS ARE PRINTED
.0080       CALL CVEFIT(ARRAY)
.0081       N=12
.0082       DO 12 J=1,N
.0083    12 B(1,J)=X(J)
.0084       CALL VALUE(N,1,NNN)
.0085       DO 44 I=2,NNN
.0086    44 ERR(I)=(EX(I)-BETA(I))/EX(I)*100.
.0087       WRITE(6,26)
.0088       WRITE(6,50)(T(L),BETA(L),EX(L),ERR(L),L=1,NNN)
      C     THE FITTED SOLUTION IS STORED IN G(8, ), AND THE KERNAL FUNCTION
      C     OF THE LEFT-HAND INTEGRAL IS STORED IN G(1, ) AND G(2, ).  THEN
      C     THE TOTAL LEFT-HAND SIDE IS CALCULATED USING SUBROUTINE INTEGR
      C     AND EVALUATED USING SUBROUTINE VALUE, AND THEN THESE RESULTS ARE
      C     PRINTED FOR COMPARISON WITH THE RIGHT-HAND SIDE OF THE EQUATION
.0089       DO 22 J=1,N
.0090       DO 23 I=1,2
.0091    23 G(I,J)=D(I,J)
.0092    22 G(3,J)=X(J)
.0093       CALL INTEGR(G,N,2,1)
.0094       CALL VALUE(N,3,NNN)
.0095       WRITE(6,29)
      C
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      C
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      C
            WRITE(6,6)(T(L),BETA(L),L=1,NNN)
      C     THE ORIGINAL RIGHT-HAND SIDE OF THE INTEGRAL EQUATION IS EVALUATED
      C     AND PRINTED
            DO 31 J=1,N
            DO 31 I=1,2
         31 B(I,J)=BB(I,J)
            CALL VALUE(N,2,NNN)
            WRITE(6,30)
            WRITE(6,6)(T(L),BETA(L),L=1,NNN)
       1000 CONTINUE
         24 FORMAT(1H /26H VALUES OF INPUT FUNCTIONS)
         25 FORMAT(1H /30H SOLUTION OF INTEGRAL EQUATION)
         26 FORMAT(1H /37H FITTED SOLUTION OF INTEGRAL EQUATION)
         29 FORMAT(1H /36H LEFT HAND SIDE OF ORIGINAL EQUATION)
         30 FORMAT(1H /37H RIGHT HAND SIDE OF ORIGINAL EQUATION)
         35 FORMAT(1H /20H INTEGRAL OF G AND K)
            END
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MAIN PROGRAM FOR ANALYSIS OF PLATE

ON NON-LINEAR FOUNDATION

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      C
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      C     THIS IS THE MAIN PROGRAM FOR THE ANALYSIS OF A LINEAR VISCOELASTIC
      C     PLATE ON A NON-LINEAR VISCOELASTIC (WINKLER) FOUNDATION.  THE
      C     NECESSARY SUBROUTINES ARE TIME, VALUE,AND INTEGR.  THE CONVOLUTION
      C     INTEGRAL OF THE PLATE COMPLIANCE AND THE FOUNDATION RELAXATION
      C     FUNCTION IS CALCULATED EXACTLY FOR THE DIRICHLET SERIES
      C     REPRESENTATIONS USING THE SUBROUTINE INTEGR.  THE NUMERICAL
      C     SOLUTION OF THE INTEGRAL IS OBTAINED AT N70 VALUES OF TIME, USING
      C     A LOG TIME SPACING.  AT EACH STEP IN TIME THE SOLUTION IS ITERATED
      C     TO OBTAIN THE NON-LINEAR SOLUTION.
            DIMENSION A(40,40), WX(15,100),    G(8,20),D(100),PL(15),X(40)
            COMMON T(100),DELTA(20),BETA(100),B(20,20),SI(100)
      C     NNNN = THE NUMBER OF SETS OF DATA
            READ(5,200) NNNN
        200 FORMAT(I10)
      C     THE LOOP THROUGH 100 IS EXECUTED FOR EACH SET OF DATA
            DO 100 IIII=1,NNNN
      C     N = THE NUMBER OF GRIDS FROM CENTER TO OUTSIDE
      C     W = THE WIDTH OF EACH GRID, WHICH WILL BE COMPUTED IF NOT GIVEN
      C     CK1 = NON-LINEAR PART OF SOIL MODULUS
      C     P = LOAD
      C     N9 = MAXIMUM NUMBER OF ITERATIONS ALLOWED
      C     U = POISSONS RATIO, TAKEN  AS .5 IN THIS  ANALYSIS
            READ(5,10)N,W,  H,U,   CK1,P,N9
         10 FORMAT(I5,4E11.5/F10.2,I10)
            WRITE(6,10)N,W,H,U,CK1,P,N9
      C     NN IS THE NUMBER OF TERMS IN THE DIRICHLET SERIES REPRESENTATION
      C     OF THE COMPLIANCE AND FOUNDATION FUNCTIONS
            READ(5,10)NN
      C     N70 IS THE NUMBER OF TIME STEPS TO BE EXECUTED
            READ(5,10)N70
      C     THE CONSTANTS FOR THE PLATE COMPLIANCE SERIES ARE READ INTO G(1, )
            READ(5,140)(G(1,I),I=1,NN)
      C     THE CONSTANTS FOR THE FOUNDATION RELAXATION FUNCTION ARE READ INTO
      C     THE VECTOR G(2, )
            READ(5,140)(G(2,I),I=1,NN)
            WRITE(6,10)NN
            WRITE(6,10)N70
            WRITE(6,140)((G(J,I),I=1,NN),J=1,2)
        140 FORMAT(6F10.5)
      C     THE SUBROUTINE TIME CALCULATES THE N70 VALUES OF TIME AND THE
      C     RELAXATION TIMES OF THE SERIES REPRESENTATIONS.
            CALL TIME(N70)
      C     THE VALUES OF THE PLATE COMPLIANCE AT EACH OF THE TIMES IS CALC-
      C     ULATED AND STORED IN  THE VECTOR D( ) AFTER BEING PRINTED OUT. THE
      C     EVALUATION OF THE SERIES IS PERFORMED IN THE SUBROUTINE VALUE.
            DO 141 I=1,NN
        141 B(1,I)=G(1,I)
            CALL VALUE(NN,1,N70)
            WRITE(6,160)(T(L),BETA(L),L=1,N70)
            DO 142 I=1,N70
        142 D(I)=BETA(I)
            WRITE(6,161)
      C
      C
      C
      C
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      C
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C
C     THE CONVOLUTION OF THE COMPLIANCE AND THE FOUNDATION RELAXATION
C     FUNCTION IS PERFORMED USING THE SUBROUTINE INTEGR, AND WRITTEN OUT
C     AND STORED IN THE B( , ) ARRAY.
      CALL INTEGR(G,NN,1,0)
      WRITE(6,160)((B(I,J),I=1,2),J=1,NN)
C     COMPUTE TOTAL NUMBER OF GRID POINTS
      SUT=0.
      DO 211 J=1,NN
  211 SUT=SUT&G(2,J)
      DO 164 I=1,2
      DO 164 J=1,NN
  164 G(I,J)=B(I,J)
      ND2=N/2
      IR=0
      IF(ND2*2-N)120,11,120
  120 ND2=ND2&1
      DO 12 J=1,ND2
   12 IR=IR&2*J-1
      GO TO 13
   11 DO 14 I=1,ND2
   14 IR=IR&2*I
C     COMPUTE GRID WIDTH IF NOT SPECIFIED, BASED ON AN APPROXIMATE
C     RADIUS OF RELATIVE STIFFNESS.
   13 IF(W)121,121,15
C     RL = RADIUS OF RELATIVE STIFFNESS
  121 RL=(((H*H*H)/(12.*(1.-U*U)*SUT)/D(1))**.25)
      AN=N
      W=7.0*RL/AN
   15 BE=W/H
      BI=W*H*H*H/12.
C     THE  FLEXIBILITY COEFFICIENTS DIVIDED BY THE PLATE MODULUS ARE
C     NOW CALCULATED USING A MOMENT DISTRIBUTION PROCEDURE.
C     COMPUTE MOMENT DISTRIBUTION FACTORS
      IF(BE-1.6)122,122,16
  122 IF(BE-.625)17,123,123
  123 BET =(W*H)**3
      BI1=7.16*(W*W&H*H)*BI*(1.&U)
      BET =BET /BI1
      GO TO 18
   16 W1=W
      H1=H
      GO TO 19
   17 W1=H
      H1=W
   19 BET =W*H1*H1*H1*(1.-.63*H1/W1)/(6.*(1.&U)*BI)
   18 BL=2./(4.&BET )
C     COMPUTE MOMENTS DUE TO UNIT DEFLECTION AT POINT O
  160 FORMAT(2E15.8)
      T1=.5-BL
      T2=T1*T1
      BT2=BL*T2
      B2=BL*BL
      B3=BL*B2
      B3T2=B3*T2
      BT4=BL*T2*T2
      B5=B3*B2
C
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      C
            B4=B2*B2
            B2T2=B2*T2
            BT=BL*T1
            B3T=B3*T1
            BT3=BT*T2
            B2T=B2*T1
            B2T3=BT3*BL
            B4T=B4*T1
            N3=3*N
            DO 1 I=1,N3
            DO 1 J=1,N3
          1 A(I,J)=0.0
            A(N,N)=6.*(-8.&6.*BL&12.*BT2&1.5*B3&10.*B3T2&8.*BT4&B5)
            A(N,N&2)=6.*(-1.5*BL&.0625*B5&3.*B3T2&2.*BT4)
            A(N,N&1)=6.*(2.-.75*B2-.75*B4-1.5*B2T2&3.*BT&3.*B3T&3.*BT3)
            A(N,N&3)=6.*(.75*B2&.5625*B4&1.5*B2T2&3.*BT3)
            A(N&1,N&3)=6.*(B2T*1.5&B2T3&B4T)
            A(N&2,N&2)=6.*(-3.*BT2-.5*B3T2-4.*BT4)
            A(N&1,N&2)=6.*(-1.5*BT-.375*B3T-1.5*BT3-2.25*B2T2)
            A(N,N&4)=6.*(-.375*B3-.25*B5-2.*B3T2)
            A(N,N&5)=.375*(3.*B4-B5)
            A(N&1,N&1)=6.*(-3.*B2T-2.75*B4T-6.*B2T3)
            A(N&1,N&4)=6.*(-1.125*B3T&.375*B4T)
            A(N&2,N&3)=6.*(2.5*B2T2-1.5*B3T2-1.5*BT3&2.*B2T3)
            WRITE(6,20) N,IR,  U,H,W,P,RL,CK
         20 FORMAT(24H1TOTAL NUMBER OF GRIDS =I2/19H NUMBER OF POINTS =I3/
           1         17H POISSONS RATIO =F6.3/21H PAVEMENT THICKNESS =F10.3/17
           2 WIDTH OF GRIDS =E11.4/7H LOAD =E11.4/31H RADIUS OF RELATIVE STIF
           3NESS =E11.4/17H EQUATION FOR K =F5.0,12H*(1.0-1.6*W))
            DO 2 I=1,5
            J=N-I
            IJ=I&N
            A(N,J)=A(N,IJ)
            A(J,N)=A(N,IJ)
          2 A(IJ,N)=A(N,IJ)
            N1=N&1
            NM=N-1
            DO 3 I=1,4
            J=N-I
            IJ=I&N
            A(IJ,N1)=A(N1,IJ)
            A(NM,J)= A(N1,IJ)
            A(IJ,NM)=A(N1,IJ)
            A(N1,J) =A(N1,IJ)
            A(J,NM) =A(N1,IJ)
            A(NM,IJ)=A(N1,IJ)
          3 A(J,N1) =A(N1,IJ)
            N2=N&2
            NM=N-2
            A(N2,NM)=A(N2,N2)
            A(NM,NM)=A(N2,N2)
            A(NM,N2)=A(N2,N2)
            N3=N&3
            NM1=N-3
            A(N3,N2)  =A(N2,N3)
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C
C
      A(N3,NM)   =A(N2,N3)
      A(N2,NM1)  =A(N2,N3)
      A(NM1,NM)  =A(N2,N3)
      A(NM,NM1)  =A(N2,N3)
      A(NM1,N2)  =A(N2,N3)
      A(NM,N3)   =A(N2,N3)
      NJ=1
C     CREATE ARRAY BY SUPERIMPOSING A MATRIX OVER EACH POINT ON GRID
      NNN=2*N
      NR=N
      NS=N-1
      DO 4 I=1,IR
      C=1.0
      IF(NNN-NS-1)125,5,125
  125 NS=NS&1
      IF(NR-N)124,124,6
  124 C=.5
      IF(NJ-1)126,127,126
  127 C=.125
  126 NJ=0
      GO TO 6
    5 NR=NR&1
      NS=NR
      NNN=NNN-1
      C= .5
    6 K=0
      L=0
      NN=N
      LL=1
      DO 4 II=1,IR
      NRK=NR&K
      NSL=NS&L
      NRL=NR&L
      NSK=NS&K
      NRMK=NR-K
      NSMK=NS-K
      NSML=NS-L
      NRML=NR-L
      B(II,I)=(A(NRK,NSL)&A(NRL,NSK)&A(NRMK,NSL)&A(NRL,NSMK)&A(NRMK,NSML
     1)&A(NRML,NSMK)&A(NRK,NSML)&A(NRML,NSK))*C
      IF(NN-K-1)128,8,128
  128 K=K&1
      GO TO 4
    8 K=LL
      L=LL
      NN=NN-1
      LL=LL&1
    4 CONTINUE
C     PUT B MATRIX (EQUATIONS) IN A MATRIX, AND CREATE CONSTANTS COLUMN
      LLL=0
      P=P*W*W*W/BI
      ALAM=(W**5)/BI
C     AT THIS POINT THE FLEXIBILITY COEFFICIENTS HAVE BEEN CALCULATED
C     AND THE SOLUTION OF THE MATRIX EQUATIONS AT THE N70 TIMES BEGINS.
      DO 143 KN=1,N70
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C
      BL1=1.0
      LLL=0
C     COMPUTE APPROPRIATE INTEGRALS
      DO 146 J=1,KN
      T1=T(KN)-T(J)
      BETA(J)=0.
      DO 146 I=1,N
  146 BETA(J)=BETA(J)&(G(1,I)&G(2,I)*T1)*EXP(-DELTA(J)*T1)
C     THE EFFECTIVE LOADS ON EACH NODE ARE CALCULATED AND STORED IN
C     PL( ) AND PRINTED OUT.
      IF(KN-1)151,151,152
  152 DO 153  I=1,IR
      WW=WX(I,KN-1)*(1.+CK1*WX(I,KN-1))
      PL(I)=-.5*WW*(BETA(KN-1)-BETA(KN))
      IF(K-2)153,153,154
  154 DO 155 J=3,KN
      WW=WX(I,J-1)*(1.+CK1*WX(I,J-1))
      WWW=WX(I,J-2)*(1.+CK1*WX(I,J-2))
  155 PL(I)=PL(I)-.5*(WW+WWW)*(BETA(J-2)-BETA(J-1))
  153 PL(I)=PL(I)*ALAM
  151 CONTINUE
      WRITE(6,162)(PL(L),L=1,IR)
  162 FORMAT(6E15.8)
   21 LJK=-1
      LLL=LLL&1
C     THE (FLEXIBILITY) ARRAY IS TRANSFERRED TO THE B ARRAY FOR SOLUTION
      DO 22 I=1,IR
      DO 22 J=1,IR
   22 A(I,J)=B(I,J)
      IF(BL1)129,129,569
C     THE  TERMS ON THE DIAGONAL MUST BE CALCULATED
C     ENTER HERE IF ON SECOND,ETC.,ITERATION
  129 DO 24 I=1,IR
      X(I)=A(I,IR+1)
      IF(KN-1)148,148,149
  149 A(I,I)=A(I,I)-ALAM*.5*(BETA(KN)+BETA(KN-1))*(1.+CK1*A(I,IR+1))
      GO TO 24
  148 A(I,I)=A(I,I)-ALAM*BETA(1)*(1.+CK1*A(I,IR+1))
   24 A(I,IR+1)=PL(I)
      A(1,IR+1)=A(1,IR+1)-P*D(KN)
      GO TO 25
C     ENTER HERE IF ON FIRST TIME THROUGH
  569 IF(KN-1)23,23,144
  144 BL1=-1.0
      DO 147 I=1,IR
      A(I,I)=A(I,I)-ALAM*.5*(BETA(KN)+BETA(KN-1))*(1.+CK1*WX(I,KN-1))
      X(I)=0.0
  147 A(I,IR+1)=PL(I)
      A(1,IR+1)=-P*D(KN)+A(1,IR+1)
      GO TO 25
   23 BL1=-1.0
      DO 26 I=1,IR
      A(I,I)=A(I,I)-ALAM*BETA(1)
      X(I)=0.0
   26 A(I,IR&1)=0.0
      A(1,IR+1)=-P*D(1)
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   25 NM1=IR-1
      ERR=.001
      N1=IR&1
C     SOLVE EQUATIONS USING GAUSSIAN ELIMINATION
      DO 34 K=1,NM1
      BL=A(K,K)
      IF(ABS(BL) -  ERR) 130,130,28
  130 K1=K&1
      DO 29 I=K1,IR
      IF(ABS(A(I,K)) -  ERR)29,29,30
   29 CONTINUE
      WRITE(6,51) ERR
C     IF ERR IS PRINTED, MATRIX IS SINGULAR
   51 FORMAT(1H F16.8)
      GO TO 100
   30 DO 32 J=K,N1
      BL=A(K,J)
      A(K,J)=A(I,J)
   32 A(I,J)=BL
      BL=A(K,K)
   28 DO 33 I=K,N1
   33 A(K,I)=A(K,I)/BL
      K1=K&1
      DO 34 I=K1,IR
      BL=A(I,K)
      DO 34 J=K,N1
   34 A(I,J)=A(I,J)-BL*A(K,J)
      A(IR,N1)=A(IR,N1)/A(IR,IR)
      DO 35 KK=1,NM1
      K=IR-KK
      K1=K&1
      DO 37 J=K1,IR
   37 A(K,N1)=A(K,N1)-A(K,J)*A(J,N1)
C     CHECK THE RELATIVE CHANGES IN EACH OF THE DEFLECTIONS COMPARED
C     TO THE PREVIOUS ITERATION, STORING  1 IN LJK IF THE CHANGE IS TOO
C     LARGE.
C     CONTINUE ITERATING ONLY IF HAVE NOT ITERATED N9 TIMES YET
      IF(ABS((X(K)-A(K,N1))/A(K,N1)) -  .001)35,35,132
  132 LJK=1
   35 CONTINUE
      WRITE(6,36)LLL,(A(I,N1),I=1,IR)
   36 FORMAT(1H I10/(1H 6E15.8))
      IF(LJK)111,133,133
C     LJK WILL BE NEGATIVE ONLY WHEN ALL THE RELATIVE CHANGES ARE LESS
C     THAN .001
  133 IF(LLL-N9)21,134,134
  134 WRITE(6,44) LLL
   44 FORMAT(22H NO CONVERGENCE AFTER I2,8H CYCLES.)
  111 WRITE(6,215)T(KN)
  215 FORMAT(8H TIME = E15.8)
      WRITE(6,114)
  114 FORMAT(36H0  DEFLECTION     DISTANCE FROM LOAD)
      DO 112 IX =1,N
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      AIX=IX-1
      S=W*AIX
  113 FORMAT(E16.8,5X,F7.3)
      BXX=A(IX,N1)
  112 WRITE(6,113) BXX,S
      DO 156 I=1,IR
  156 WX(I,KN)=A(I,IR+1)
  143 CONTINUE
  100 CONTINUE
      END
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MAIN PROGRAM FOR HALF-SPACE

ANALYSIS USING NUMERICAL INTEGRATION

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C     THIS IS THE MAIN PROGRAM FOR THE ANALYSIS OF A THREE LAYER HALF-
C     SPACE (LINEAR VISCOELASTIC) UNDER A UNIFORM CIRCULAR LOAD, FOR THE
C     CASE THAT THE MULTIPLE CONVOLUTION INTEGRALS ARE EVALUATED BY
C     NUMERICAL INTEGRATION.  THE NECESSARY SUBROUTINES ARE TIME1, VALUE
C     INTEGR (NUMERICAL), SOLVIT, TERPO, AND THE FUNCTION SUBPROGRAM
C     BESSEL.  ALSO REQUIRED IS SUBROUTINE CNSTNT.  THE INPUT
C     TO BE READ IS IST, H, A, R, DEL, ZZ, ILAYER,
C     IDEFLE, IDOUBL, N, NNN, AND G( , ). IST IS A DUMMY WHICH TOGETHER
C     WITH IDEFLE DETERMINES WHICH STRESS OR DISPLACEMENT IS TO BE
C     CALCULATED.  IST IS 1 FOR EITHER NORMAL STRESS OR NORMAL DEFLEC-
C     TION, IS 2 FOR SHEAR STRESS OR RADIAL DEFLECTION, OR IS 3 FOR
C     RADIAL STRESS.  H IS THE THICKNESS OF THE SECOND LAYER (THE THICK-
C     NESS OF THE FIRST LAYER IS TAKEN AS UNITY).  A IS THE RADIUS OF
C     THE LOADED AREA.  R IS THE OFF-SET AT WHICH THE STRESS OR DEFLEC-
C     TION IS TO BE CALCULATED.  DEL IS THE INITIAL SPACING IN TIME.
C     ZZ IS THE DEPTH AT WHICH THE STRESS OR DISPLACEMENT IS DESIRED.
C     ILAYER IS THE LAYER OF INTEREST (1,2,OR 3).  IDEFLE IS 1 IF A
C     DEFLECTION IS TO BE CALCULATED, ZERO OTHERWISE.  IDOUBLE IS THE
C     NUMBER OF TIMES THE INTERVAL OF TIME IS TO BE DOUBLED.  N IS THE
C     NUMBER OF TERMS IN THE DIRICHLET SERIES REPRESENTATION OF THE
C     INPUT CREEP FUNCTIONS (SERIES HAVE BEEN USED HERE, BUT ARE NOT
C     NECESSARY WHEN USING THIS TECHNIQUE).  NNN IS THE NUMBER OF TIME
C     POINTS TO BE USED IN EACH LOOP.  G( , ) CONTAINS THE CONSTANTS
C     FOR THE SERIES REPRESENTATION OF THE CREEP FUNCTIONS.  ROW 1 CON-
C     TAINS THE CREEP FUNCTION FOR THE FIRST LAYER, ROW TWO THE CREEP
C     FUNCTION FOR THE SECOND LAYER, AND ROW 3 THE CREEP FUNCTION FOR
C     THE LOWER LAYER.
C     THE OUTPUT FROM THIS PROGRAM IS THE VALUE OF THE DESIRED STRESS OR
C     DISPLACEMENT FOR THE DESIRED TIMES (ASSUMING A LOAD OF UNIT INTEN-
C     SITY.
      DIMENSION E1(61),E2(61),E3(61),GAM(61,7,18),E(7,61),G(3,20),
     1EM(13),BESSS(91)              ,SII(13,61),SIII(13,61),S(13),BESS(91),
     2PH(18),PHJ(18),TH(9)
      COMMON BETA(61),B(8,20),DELTA(20),T(61),MN,SI(61),WI
C     THIS LOOP ALLOWS MULTIPLE SETS OF DATA TO BE HANDLED.
      DO 1000 III=1,100
      READ(5,51)IST,H,A,R,DEL,ZZ
      READ(5,20) ILAYER,IDEFLE,IDOUBL
      WRITE(6,101)IST,H,A,R,DEL,ZZ
  101 FORMAT(7H IST = I5/26H SECOND LAYER THICKNESS = E15.8/
     118H RADIUS OF LOAD = E15.8/11H OFF-SET = E15.8/
     219H INITIAL SPACING = E15.8/9H DEPTH = E15.8)
      WRITE(6,102)ILAYER,IDEFLE,IDOUBL
  102 FORMAT(11H LAYER NO. I3/10H IDEFLE = I3/
     133H NO. OF TIMES DOUBLING INTERVAL =I3)
   51 FORMAT(I5/5F10.5)
C     THE DUMMY IOWA IS SET EQUAL TO 1,2,3,5, OR 6 DEPENDING ON WHICH
C     STRESS OR DEFLECTION IS DESIRED.  THIS IS FOR INPUT INTO THE
C     SUBROUTINE CNSTNT.
      IF(IDEFLE)52,52,53
   52 IOWA=IST
      GO TO 54
   53 IOWA=4&IST
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      C
      C
      C
   54 CONTINUE
      C    IDB IS A DUMMY SET EQUAL TO ZERO BEFORE THE FIRST DOUBLING LOOP,
      C    BUT MADE POSITIVE THEREAFTER.
           IDB=0
           READ(5,20)N,NNN
   20 FORMAT(5I5)
           READ(5,40)((G(I,J),J=1,N),I=1,3)
   40 FORMAT(6F10.5)
           WRITE(6,2)((G(I,J),J=1,N),I=1,3)
    2 FORMAT(27H INPUT RELAXATION FUNCTIONS/(6F10.5))
      C    N10 IS USED TO BEGIN CERTAIN DO LOOPS.  IT IS 1 FOR THE FIRST
      C    DOUBLING LOOP, AND EQUAL TO NNN/2&2 THEREAFTER.
           N10=1
      C    NX IS A DUMMY USED AS INPUT TO THE SUBROUTINE TIME1.  IF IT IS
      C    ZERO, THEN THE INVERSES OF THE RELAXATION TIMES WILL BE COMPUTED
      C    AND STORED IN DELTA( ).  IF IT IS NON-ZERO (EVERY LOOP EXCEPT THE
      C    FIRST) THE DELTA( ) VECTOR IS NOT RECOMPUTED.
           NX=0
      C    STATEMENT 69 BEGINS THE LOOP WHICH IS REPEATED EACH DOUBLING.
      C    FIRST THE TIMES AND DELTA( ) VECTOR ARE COMPUTED.
   69 CALL TIME1(NNN,DEL,NX)
      C    THE SERIES REPRESENTATIONS OF EACH OF THE CREEP FUNCTIONS IS TRANS
      C    FERRED TO THE B( , ) ARRAY AND EVALUATED AT EACH TIME USING THE
      C    SUBROUTINE VALUE.  THEN THESE RESULTS ARE STORED IN E1( ), E2( )
      C    OR E3( ).
           DO 41 J=1,3
           DO 42 I=1,N
   42 B(1,I)=G(J,I)
           CALL VALUE(N,1,NNN)
           DO 43 I=1,NNN
           IF(J-2)44,45,46
   44 E1(I)=BETA(I)
           GO TO 43
   45 E2(I)=BETA(I)
           GO TO 43
   46 E3(I)=BETA(I)
   43 CONTINUE
   41 CONTINUE
      C    THE VECTOR EM( ) PROVIDES INTERMEDIATE STORAGE FOR THE VALUES OF
      C    THE DUMMY INTEGRATION VARIABLE M THAT WILL BE USED.  THESE VALUES
      C    ARE 0., .2, .4, .7, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0.
           EM(10)=6.0
           EM(11)=7.0
           EM(12)=8.0
           EM(13)=9.0
      C    THE LOOP TO STATEMENT 3 IS EXECUTED FOR EACH OF THE POSSIBLE
      C    COMBINATIONS OF THE FIRST FOUR CREEP FUNCTIONS FOR THE MULTIPLE
      C    CONVOLUTION INTEGRALS.
           DO 3 I=1,9
      C    EACH VALUE OF THE APPROPRIATE  CREEP FUNCTION IS STORED IN THE
      C    THE PROPER ROW OF THE E( , ) ARRAY.
           DO 19 J=1,NNN
      C    THESE TESTS DIRECT THE FLOW TO THE PROPER ARRANGEMENT OF CREEP
      C    FUNCTIONS.
           IF(I-2)4,5,15
      C
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C
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   15 IF(I-4)6,7,16
   16 IF(I-6)8,9,17
   17 IF(I-8)10,11,12
C     SOME OF THE EM( ) VECTOR VALUES ARE FILLED IN THIS PHASE ALSO.
    4 EM(I)=0.
      E(1,J)=E2(J)
      E(2,J)=E2(J)
      E(3,J)=E2(J)
      E(4,J)=E2(J)
      GO TO 19
    5 EM(I)=.2
      E(1,J)=E2(J)
      E(2,J)=E2(J)
      E(3,J)=E2(J)
      E(4,J)=E3(J)
      GO TO 19
    6 EM(I)=.4
      E(1,J)=E1(J)
      E(2,J)=E2(J)
      E(3,J)=E2(J)
      E(4,J)=E2(J)
      GO TO 19
    7 EM(I)=.7
      E(1,J)=E1(J)
      E(2,J)=E2(J)
      E(3,J)=E2(J)
      E(4,J)=E3(J)
      GO TO 19
    8 EM(I)=1.0
      E(1,J)=E2(J)
      E(2,J)=E2(J)
      E(3,J)=E3(J)
      E(4,J)=E3(J)
      GO TO 19
    9 EM(I)=2.0
      E(1,J)=E1(J)
      E(2,J)=E2(J)
      E(3,J)=E3(J)
      E(4,J)=E3(J)
      GO TO 19
   10 EM(I)=3.0
      E(1,J)=E1(J)
      E(2,J)=E1(J)
      E(3,J)=E2(J)
      E(4,J)=E2(J)
      GO TO 19
   11 EM(I)=4.0
      E(1,J)=E1(J)
      E(2,J)=E1(J)
      E(3,J)=E2(J)
      E(4,J)=E3(J)
      GO TO 19
   12 EM(I)=5.0
      E(1,J)=E1(J)
      E(2,J)=E1(J)
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      C
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.0100       E(3,J)=E3(J)
.0101       E(4,J)=E3(J)
.0102    19 CONTINUE
      C     AT THIS POINT, FOR THE PARTICULAR I BEING EXECUTED, HAVE STORED
      C     THE PROPER FIRST FOUR CREEP FUNCTIONS IN THE FIRST FOUR ROWS OF
      C     THE E( , ) ARRAY.  THE REMAINING ROWS OF E( , ) WILL BE FILLED OR
      C     NOT FILLED DEPENDING ON WHICH LAYER AND OR WHETHER A STRESS OR
      C     DEFLECTION IS DESIRED.  THEN THE MULTIPLE CONVOLUTION INTEGRALS
      C     WILL BE CALCULATED ACCORDINGLY, USING THE SUBROUTINE INTEGR.
.0103       IF(ILAYER-2)22,23,23
      C     IF IN THE FIRST LAYER, NEED ADD ANOTHER CREEP FUNCTION ONLY IF
      C     DOING A DEFLECTION.
.0104    22 IF(IDEFLE)24,24,25
      C     IF NOT DOING A DEFLECTION, BUT IN FIRST LAYER, THEN HAVE ONLY 9
      C     THREE-FOLD CONVOLUTION INTEGRATIONS IN ALL.  OBTAIN THE I TH ONE
      C     AT THIS POINT USING SUBROUTINE INTEGR, STORING THE RESULT IN
      C     GAM( , ,I).
.0105    24 CALL INTEGR(NIO,NNN,E,GAM,I,3)
      C     MAX IS THE NUMBER CREEP FUNCTIONS INCLUDED IN THE 'DENOMINATOR'
      C     MULTIPLE CONVOLUTION INTEGRALS, MIN THE NUMBER IN THOSE OF THE
      C     'NUMERATOR' AND IMX IS THE NUMBER OF DIFFERENT INTEGRALS IN THE
      C     'NUMERATOR.'
.0106       MAX=4
.0107       IMX=9
.0108       MIN=4
      C     M6 EQUAL TO IMX.
.0109       M6=9
.0110       GO TO 50
      C     IF IN FIRST LAYER AND DOING A DEFLECTION, MUST ADD THE CREEP FUN-
      C     TION OF THE FIRST LAYER  TO THE E( , ) ARRAY.  THE 'NUMERATOR' HAS
      C     ONE MORE INTEGRATION THAN THE 'DENOMINATOR' IN THIS CASE, SO MIN
      C     IS ONE GREATER THAN MAX.
.0111    25 MIN=5
.0112       MAX=4
.0113       IMX=9
.0114       DO 38 J=1,NNN
.0115    38 E(5,J)=E1(J)
.0116       CALL INTEGR(NIO,NNN,E,GAM,I,4)
.0117       M6=9
.0118       GO TO 50
      C     IF ENTERING STATEMENT 23, AM DOING SECOND OR THIRD LAYER.
.0119    23 IF(IDEFLE)26,26,27
      C     IF DOING A DEFLECTION, THEN MUST PUT EITHER THE CREEP FUNCTION OF
      C     THE SECOND LAYER OR THIRD LAYER INTO THE E( , ) ARRAY.  THIS IS
      C     PUT INTO ROW SIX BECAUSE ROW FIVE MUST BE FILLED (BELOW) WHETHER
      C     DOING A STRESS OR A DEFLECTION.
.0120    27 MIN=6
.0121       MAX=5
.0122       IMX=18
.0123       M6=18
.0124       IF(ILAYER-2)28,28,29
.0125    28 DO 30 J=1,NNN
.0126    30 E(6,J)=E2(J)
.0127       GO TO 31
.0128    29 DO 32 J=1,NNN
      C
      C
      C
      C
      C
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C
C
   32 E(6,J)=E3(J)
      GO TO 31
   26 MIN=5
      MAX=5
      IMX=18
      M6=18
C     THE 'NUMERATOR' FOR THE SECOND AND THIRD LAYER RESULTS CONTAINS
C     18 CIFFERENT INTEGRALS.  THE FIRST NINE ARE THE SAME AS THOSE IN
C     THE 'DENOMINATOR' IF DOING A STRESS.  THE SECOND NINE HAVE THE
C     FIFTH CREEP FUNCTION EQUAL TO E1( ) RATHER THAN E2( ).
C     THE LOOP TO 33 PLACES E2( ) IN ROW 5 OF E( , ) AND THEN THE FIRST
C     NINE INTEGRATIONS ARE CARRIED OUT.  IF A DEFLECTION IS BEING DONE,
C     THE 'DENOMINATOR' INTEGRALS WILL BE STORED IN THE GAM( , ,MI)
C     ARRAY AS WELL AS THE NUMERATOR RESULTS.
   31 DO 33 J=1,NNN
   33 E(5,J)=E2(J)
      MI=MIN-1
      CALL INTEGR(N10,NNN,E,GAM,I,MI)
C     NOW ROW 5 OF E( , ) IS REPLACED WITH E1( ), AND THE SECOND 9
C     INTEGRALS ARE CALCULATED.
      DO 34 J=1,NNN
   34 E(5,J)=E1(J)
      II=I&9
      CALL INTEGR(N10,NNN,E,GAM,II,MI)
   50 CONTINUE
    3 CONTINUE
C     AT THIS POINT ALL OF THE RELEVANT CONVOLUTION INTEGRALS HAVE BEEN
C     CALCULATED AND STORED IN THE GAM( , , ) ARRAY.
      MN=N10
C     THE LOOP THROUGH STATEMENT 1111 SOLVES THE INTEGRAL EQUATION
C     FOR EACH OF THE 13 VALUES OF THE DUMMY INTEGRATION VARIABLE M.
      DO 1111 K=1,13
      EMM=EM(K)
C     THE CONSTANTS FOR THE NUMERATOR ( STORED IN THE VECTOR PH( ) AND
C     PHJ( ))AND FOR THE DENOMINATOR (STORED IN THE VECTOR TH( )) ARE
C     COMPUTED FOR THIS VALUE OF M.
      CALL CNSTNT(EMM,H,ZZ,IOWA,PH,PHJ,TH,ILAYER)
      IF(IDB)302,302,303
C     ON ALL EXCEPT THE FIRST TIME THROUGH (WHEN IDB IS ZERO) EVERY
C     OTHER OF THE LATEST VALUES OF THE SOLUTION VECTOR FOR THIS M MUST
C     BE STORED IN THE FIRST MN1 LOCATIONS OF THE SOLUTION VECTOR SI( ).
C     THESE RESULTS HAVE BEEN STORED IN THE KTH ROW OF THE ARRAY SII( ,)
  303 MN1=N10-1
      DO 301 JJ=1,MN1
      KK=2*JJ-1
  301 SI(JJ)=SII(K,KK)
C     THE SCLUTION IS CALCULATED FOR THIS VALUE OF M AND STORED IN THE
C     VECTOR SI( )
  302 CALL SOLVIT(NNN,PH,TH,GAM,IMX,9,MIN,MAX)
C     THE RESULTS FOR THIS VALUE OF M ARE TRANSFERRED INTO THE KTH ROW
C     OF THE ARRAY SII( , ).
      DO 57 I=1,NNN
   57 SII(K,I)=SI(I)
      IF(IST-3)1111,58,58
C
C
C
C
C
C
C
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      C
      C     WHEN DOING THE RADIAL STRESS (IST EQUAL TO 3), MUST SOLVE TWO SETS
      C     OF INTEGRAL EQUATIONS.  THE CONSTANTS FOR THIS CASE ARE IN THE
      C     VECTORS PHJ( ) AND TH( ).   THE PREVIOUS SOLUTIONS ARE IN THE
      C     ARRAY SIII( , ) AND THE NEW SOLUTIONS WILL BE STORED THERE.
.0158     58 IF(IDB)304,304,305
.0159    305 MN1=N10-1
.0160        DO 306 JJ=1,MN1
.0161        KK=2*JJ-1
.0162    306 SI(JJ)=SIII(K,KK)
.0163    304 CALL SOLVIT(NNN,PHJ,TH,GAM,IMX,9,MIN,MAX)
.0164        DO 60 I=1,NNN
.0165     60 SIII(K,I)=SI(I)
.0166   1111 CONTINUE
      C     IF ON THE FIRST TIME THROUGH, MUST COMPUTE THE APPROPRIATE BESSEL
      C     TERM MULTIPLIERS.  IF ON OTHER THAN FIRST DOUBLING LOOP, TRANSFER
      C     DIRECTLY TO THE INTEGRATION WITH RESPECT TO M.  THIS IS DONE BE-
      C     GINNING WITH STATEMENT 70 UNLESS ARE DOING RADIAL STRESS IN WHICH
      C     CASE IT IS DONE BEGINNING WITH STATEMENT 272.
.0167        IF(IDB)269,269,270
.0168    270 IF(IST-2)70,70,272
      C     ENTER STATEMENT 269 ONLY ON FIRST DOUBLING LOOP (IDB = 0).
      C     DEPENDING WHICH STRESS OR DISPLACEMENT IS BEING DONE, A DIFFERENT
      C     BESSEL MULTIPLIER IS USED.  IF DOING A DEFLECTION, THE BESSEL
      C     TERMS ARE ALSO DIVIDED BY M (WHICH IS THE PURPOSE OF DIVIDE)
.0169    269 DIVIDE=1.
.0170        IF(IST-2)78,79,78
      C     THE FIRST BESSEL TERM IS J(1) FOR SHEAR STRESS OR RADIAL DEFLEC-
      C     TION.  IF IS J(0) OTHERWISE.  IDEX STORES 1 OR 0 ACCORDINGLY.
      C     IF R IS ZERO, THE FIRST BESSEL TERM IS ZERO IF J(1) AND 1 IF J(0)
      C     AND THE TM1 TERM IS SET ACCORDINGLY.
.0171     79 IDEX=1
.0172        TM1=0.
.0173        GO TO 80
.0174     78 IDEX=0
.0175        TM1=1.
.0176     80 IF(IDEFLE)81,81,82
      C     IF DOING A STRESS, THEN THE LIMIT OF J1(MA) AS M TENDS TO ZERO IS
      C     ZERO.
.0177     81 BESS(1)=0.
.0178        GO TO 83
      C     IF DOING A DEFLECTION, THEN THE LIMIT OF J1(MA)/M AS M TENDS TO
      C     ZERO IS A/2.
.0179     82 BESS(1)=A/2.
.0180     83 DDD=0.
      C     DDD IS EQUAL TO M. THE BESSEL MULTIPLIERS ARE CALCULATED AT 91
      C     POINTS SPACED .1 M APART FOR USE IN SUBROUTINE TERPO.
.0181        DO 86 I=2,91
.0182        DDD=DDD&.1
.0183        RM=R*DDD
 0184        AM=A*DDD
      C     IF R IS ZERO, THE FIRST TERM NEED NOT BE CALCULATED USING THE
      C     FUNCTION SUBPROGRAM.
 0185        IF(RM-.0001)84,84,85
      C     THE BESSEL TERMS ARE CALCULATED USING THE FUNCTION SUBPROGRAM
      C     BESSEL, THEN MULTIPLIED TOGETHER AND IF DOING A DEFLECTION ARE
      C     DIVIDED BY M.  THE RESULT IS STORED IN THE VECTOR BESS( ).
      C
      C
      C
      C
      C
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C
C
   85 TM1=BESSFL(IDEX,RM)
   84 TM2=BESSEL(1,AM)
      IF(IDEFLE)86,86,87
   87 DIVIDE=DDD
   86 BESS(I)=TM1*TM2/DIVIDE
C     IF DOING RADIAL STRESS (IST=3) MUST COMPUTE A SECOND BESSEL
C     MULTIPLIER.  THIS IS STORED IN THE VECTOR BESSS( ) AND IS COMPUTED
C     IN AN ANALOGOUS MANNER.
      IF(IST-3)70,71,71
C     THE LIMIT OF J1(MR)J1(MA)/MR AS M TENDS TO ZERO IS ALWAYS ZERO.
   71 BESSS(1)=0.
      DDD=0.
      RR=R
      DO 77 I=2,91
      DDD=DDD&.1
      RM=R*DDD
      AM=A*DDD
C     THE LIMIT OF J1(MR)J1(MA)/MR AS R TENDS TO ZERO IS MJ1(MA)/2.M
      IF(RR-.0001)271,271,76
  271 TM1=DDD/2.
      R=1.
      GO TO 577
   76 TM1=BESSFL(1,RM)
  577 TM2=BESSEL(1,AM)
   77 BESSS(I)=TM1*TM2/R/DDD
C     CONTROL ENTERS AT STATEMENT 272 ONLY WHEN DOING RADIAL STRESS.
C     IN THIS CASE, MUST CARRY OUT TWO SEPARATE INTEGRATIONS WITH RES-
C     PECT TO M, AND ADD THE RESULTS TOGETHER.
C     THE INTEGRATION MUST BE EXECUTED AT EACH OF THE NEWLY CALCULATED
C     VALUES OF TIME (NNN SUCH VALUES OR NNN-MN&1 VALUES.)
  272 DO 72 I=MN,NNN
C     THE 13 SOLUTION VALUES (13 VALUES OF M) ARE TRANSFERRED (FOR ONE
C     TIME) INTO THE VECTOR S( ), FROM THE ARRAY SII( , ).
      DO 73 J=1,13
   73 S(J)=SII(J,I)
C     THE SOLUTION FOR THIS INTEGRAL EQUATION (AND THE MULTIPLIER
C     BESS( )) IS CALCULATED USING SUBROUTINE TERPO AND TRANSFERRED INTO
C     WII.
      CALL TERPO(S,BESS)
      WRITE(6,701)WI
      WII=WI
C     THE 13 VALUES FROM SIII( , ) ARE TRANSFERRED INTO S( ) AND THE
C     SOLUTION WITH BESSS( ) IS CALCULATED AND ADDED INTO WII.  THIS IS
C     THEN MULTIPLIED BY A AND PRINTED OUT WITH THE TIME (THE TOTAL
C     SOLUTION FOR THE RADIAL STRESS AT THIS TIME).
      DO 74 J=1,13
   74 S(J)=SIII(J,I)
      CALL TERPO(S,BESSS)
      WRITE(6,701)WI
  701 FORMAT(E15.8)
      WII=WI&WII
      WII=WII*A
   72 WRITE(6,63)T(I),WII
      GO TO 75
C
C
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C
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C
C
C     CONTROL ENTERS AT STATEMENT 70 FOR ALL EXCEPT RADIAL STRESS.  THE
C     INTEGRATION ON M IS NOW CARRIED OUT AT EACH OF THE NEWLY CONSID-
C     ERED TIMES.
   70 DO 61 I=MN,NNN
C     THE 13 VALUES OF THE SOLUTION AT EACH TIME (FOR THIRTEEN VALUES
C     OF M) ARE TRANSFERRED INTO THE VECTOR S( ).
      DO 62 J=1,13
   62 S(J)=SII(J,I)
C     NONE OF THESE SOLUTIONS CHANGE SIGN AFTER THE FIRST 3 POINTS SO
C     IF ANY ARE FOUND THAT DO CHANGE SIGN AT LARGE M (DUE TO ROUND-OFF
C     ERRORS IN THE SUBROUTINE CNSTNT) THEY ARE ZEROED.
      DO 705 J=4,13
      IF(S(J)*S(J-1))706,706,705
  706 S(J)=0.
  705 CONTINUE
C     THE TOTAL SOLUTION IS COMPUTED USING SUBROUTINE TERPO.  IT IS THEN
C     MULTIPLIED BY A AND PRINTED WITH THE TIME.
      CALL TERPO(S,BESS)
      WI=WI*A
   61 WRITE(6,63)T(I),WI
   63 FORMAT(8H TIME = E15.8,12H SOLUTION = E15.8)
C     NOW MUST REJECT APPROPRIATE VALUES AND RETURN TO THE BEGINNING OF
C     THE DOUBLING LOOP (STATEMENT 69) IF HAVE NOT DOUBLED A SUFFICIENT
C     NUMBER OF TIMES.
C     IDB IS INCREASED BY 1 (MAKING IT POSITIVE AFTER THE FIRST LOOP)
C     AND N10 IS COMPUTED FOR THE SECOND AND SUBSEQUENT LOOPS..  NM1 AND
C     NX ARE GIVE APPROPRIATE VALUES ALSO.
   75 N10=NNN/2&2
      MN1=N10-1
      IDB=IDB&1
      NX=1
      IF(IDOUBL-IDB)67,68,68
C     THE INTERVALS OF TIME ARE DOUBLED.
   68 DEL=DEL*2.
C     THE RELEVANT VALUES OF THE GAM( , , ) ARRAY AND THE VECTORS E1( )
C     E2( ), AND E3( )  ARE SAVED.
      DO 64 I=2,MN1
      K=2*I-1
      DO 66 J=1,18
      DO 66 L=1,7
   66 GAM(I,L,J)=GAM(K,L,J)
      E1(I)=E1(K)
      E2(I)=E2(K)
   64 E3(I)=E3(K)
      GO TO 69
   67 CONTINUE
 1000 CONTINUE
      END
C
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MAIN PROGRAM FOR HALF-SPACE

ANALYSIS USING EXACT INTEGRATION

TC

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         C
         C
         C
         C      THIS IS THE MAIN PROGRAM FOR THE ANALYSIS OF A LINEAR VISCOELASTIC
         C      THREE-LAYER HALF-SPACE UNDER A UNIFORM CIRCULAR LOAD, FOR THE CASE
         C      THAT THE MULTIPLE CONVOLUTION INTEGRALS ARE EVALUATED EXACTLY.
         C      THE NECESSARY SUBROUTINES ARE CNSTNT, TIME, SOLVE, TERPO, AND
         C      INTEGR (EXACT).  ALSO NECESSARY IS THE FUNCTION SUBPROGRAM BESSEL.
         C      THE INPUT IS IST,H,A,R,ZZ,ILAYER,IDEFLE, NJJJ,DELTX,DELXX, AND THE
         C      VECTORS E1( ), E2( ), AND E3( ).  IST IS A DUMMY WHICH, TOGETHER
         C      WITH IDEFLE DETERMINES WHICH STRESS OR DISPLACEMENT IS DESIRED.
         C      IST IS 1 FOR NORMAL STRESS OR NORMAL DEFLECTION, IS 2 FOR SHEAR
         C      STRESS OR RADIAL DEFLECTION, AND IS 3 FOR RADIAL STRESS.  H IS THE
         C      THICKNESS OF THE SECOND LAYER (THE THICKNESS OF THE FIRST LAYER IS
         C      ONE).  A IS THE RADIUS OF THE LOAD.  R IS THE OFF-SET AT WHICH THE
         C      STRESS OR DISPLACEMENT IS DESIRED.  ZZ IS THE DEPTH AT WHICH THE
         C      SOLUTION IS DESIRED.  ILAYER IS THE LAYER OF INTEREST (1,2, OR 3)
         C      IDEFLE IS POSITIVE IF A DEFLECTION IS TO BE DONE, ZERO OTHERWISE.
         C      NJJJ IS AN INPUT TO THE SUBROUTINE SOLVE, AND IS EXPLAINED IN
         C      DETAIL THERE.  DELTX AND DELXX ARE INPUTS TO THE SUBROUTINE TIME
         C      AND ARE EXPLAINED IN DETAIL THERE.  N AND NNN ARE ALSO INPUT.  N
         C      IS  THE NUMBER OF TERMS IN THE DIRICHLET SERIES REPRESENTATIONS OF
         C      THE INPUT CREEP FUNCTONS.  NNN IS THE NUMBER OF POINTS IN TIME AT
         C      WHICH THE SOLUTION IS DESIRED.  THE VECTORS E1( ), E2( ), AND E3()
         C      CONTAIN THE CONSTANTS FOR THE SERIES REPRESENTATIONS OF THE CREEP
         C      FUNCTIONS FOR THE FIRST, SECOND, AND THIRD LAYERS RESPECTIVELY.
         C      THE RESULT OF THE PROGRAM IS THE DESIRED STRESS OR DISPLACEMENT
         C      AT EACH OF THE NNN TIMES.
S.0001          DIMENSION E1(12),E2(12),E3(12),EM(13),G(7,12,18),GG(7,12,9),
               1E(8,12),PH(18),PHJ(18),TH(9),SII(13,201),SIII(13,201),S(13),
               1BESS(91),BESSS(91)
S.0002          COMMON X(20),BB(8,20),  T(201) ,DELTA(20),BETA(201),B(8,20),
               1SI(201),WI,DELTX,DELXX,NJ,NJJ
         C      THE LOOP THROUGH 1000 ALLOWS MULTIPLE SETS OF DATA TO BE RUN.
S.0003          DO 1000 III=1,100
S.0004          READ(5,52)IST,H,A,R,ZZ
S.0005       52 FORMAT(I5/5F10.5)
S.0006          READ(5,20) ILAYER,IDEFLE
S.0007          WRITE(6,210)IST,ILAYER,IDEFLE,H,A,R,ZZ
S.0008      210 FORMAT(7H IST = I10/10H ILAYER = I10/10H IDEFLE = I10/
               15H H = F10.5/5H A = F10.5/5H R = F10.5/6H ZZ = F10.5)
         C      IOWA IS GIVEN THE VALUE 1,2, 3, 5, OR 6, DEPENDING ON WHICH STRESS
         C      OR DEFLECTION IS DESIRED.  THIS DUMMY IS USED AS INPUT TO THE
         C      SUBROUTINE CNSTNT.
S.0009          IF(IDEFLE)55,55,53
S.0010       55 IOWA=IST
S.0011          GO TO 54
S.0012       53 IOWA=4&IST
S.0013       54 CONTINUE
S.0014          READ(5,20)NJJJ
S.0015          READ(5,1)DELTX,DELXX
         C      NJ AND NJJ ARE INPUTS TO THE SUBROUTINE SOLVE.  THEY HAVE NO SIG-
         C      NIFICANCE IN THE PRESENT USE OF THAT SUBROUTINE AND ARE GIVEN
         C      ARBITRARY VALUES.
S.0016          NJ=10
S.0017          NJJ=8
         C
         C
         C
         C
         C
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C
C
C
      READ(5,20)N,NNN
   20 FORMAT(5I5)
      READ(5,1)(E1(I),I=1,N)
      READ(5,1)(E2(I),I=1,N)
      READ(5,1)(E3(I),I=1,N)
      WRITE(6,2)(E1(I),I=1,N)
      WRITE(6,2)(E2(I),I=1,N)
      WRITE(6,2)(E3(I),I=1,N)
    1 FORMAT(6F10.5)
    2 FORMAT(22H INPUT CREEP FUNCTIONS/(6F10.5))
C     THE APPROPRIATE NNN VALUES OF TIME ARE CALCULATED AND STORED IN
C     THE VECTOR T( ) USING SUBROUTINE TIME.  ALSO CALCULATED WITH THIS
C     SUBROUTINE ARE THE INVERSES OF THE RELAXATION TIMES, WHICH ARE
C     STORED IN THE VECTOR DELTA( ).
      CALL TIME(NNN)
C     THE VECTOR EM( ) SERVES AS INTERMEDIATE STORAGE OF THE VALUES OF
C     THE DUMMY INTEGRATION VARIABLE M FOR WHICH THE INTEGRAL EQUATION
C     IS SOLVED.  THESE VALUES OF M ARE 0.0, .2, .4, .7, 1., 2., 3., 4.,
C     5., 6., 7., 8., AND 9.
      EM(10)=6.0
      EM(11)=7.0
      EM(12)=8.0
      EM(13)=9.0
C     THE LOOP FROM HERE TO THREE ARRANGES EACH OF THE POSSIBLE COMBIN-
C     ATIONS OF THE FIRST FOUR CREEP FUNCTIONS FOR THE MULTIPLE
C     CONVOLUTION INTEGRATIONS AND COMPUTES THE THREE-FOLD INTEGRAL OF
C     THESE FOUR FUNCTIONS.
      DO 3 I=1,9
C     EACH OF THE CONSTANTS  (N OF THEM) MUST BE TRANSFERRED INTO THE
C     APPROPRIATE ROW OF THE ARRAY E( , ).
      DO 19 J=1,N
C     THERE ARE NINE COMBINATIONS OF THESE RELAXATION FUNCTIONS.
      IF(I-2)12,11,15
   15 IF(I-4)10,9,16
   16 IF(I-6)8,7,17
   17 IF(I-8)6,5,4
C     SOME OF THE M VALUES ARE STORED DURING THIS ARRANGEMENT.
    4 EM(I)=5.0
      E(1,J)=E1(J)
      E(2,J)=E1(J)
      E(3,J)=E3(J)
      E(4,J)=E3(J)
      GO TO 19
    5 EM(I)=4.0
      E(1,J)=E1(J)
      E(2,J)=E1(J)
      E(3,J)=E2(J)
      E(4,J)=E3(J)
      GO TO 19
    6 EM(I)=3.0
      E(1,J)=E1(J)
      E(2,J)=E1(J)
      E(3,J)=E2(J)
      E(4,J)=E2(J)
      GO TO 19
C
C
C
C
C
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C
C
    7 EM(I)=2.0
      E(1,J)=E1(J)
      E(2,J)=E2(J)
      E(3,J)=E3(J)
      E(4,J)=E3(J)
      GO TO 19
    8 EM(I)=1.0
      E(1,J)=E2(J)
      E(2,J)=E2(J)
      E(3,J)=E3(J)
      E(4,J)=E3(J)
      GO TO 19
    9 EM(I)=.70
      E(1,J)=E1(J)
      E(2,J)=E2(J)
      E(3,J)=E2(J)
      E(4,J)=E3(J)
      GO TO 19
   10 EM(I)=.40
      E(1,J)=E1(J)
      E(2,J)=E2(J)
      E(3,J)=E2(J)
      E(4,J)=E2(J)
      GO TO 19
   11 EM(I)=.20
      E(1,J)=E2(J)
      E(2,J)=E2(J)
      E(3,J)=E2(J)
      E(4,J)=E3(J)
      GO TO 19
   12 EM(I)=0.0
      E(1,J)=E2(J)
      E(2,J)=E2(J)
      E(3,J)=E2(J)
      E(4,J)=E2(J)
   19 CONTINUE
C     THE ITH INTEGRAL IS CALCULATED AS A SERIES OF N EXPONENTIAL TERMS
C     EACH MULTIPLIED BY A THIRD DEGREE POLYNOMIAL.  THE CONSTANTS ARE
C     TRANSFERRED INTO G( , ,I).
      CALL INTEGR(E,N,3,0)
      DO 21 L=1,N
      DO 21 J=1,4
   21 G(J,L,I)=B(J,L)
    3 CONTINUE
  102 FORMAT(24H INTEGRAL RESULT FOLLOWS/(E15.8))
C     IF ARE IN FIRST LAYER, HAVE ONLY 9 DIFFERENT MULTIPLE INTEGRALS
C     IN THE 'NUMERATOR'.  IF IN THE SECOND OR THIRD LAYER, HAVE 18 SUCH
C     DIFFERENT INTEGRATIONS.
      IF(ILAYER-2)22,23,23
C     IF IN THE FIRST LAYER, THEN THE 'NUMERATOR' AND 'DENOMINATOR' EACH
C     HAVE ONLY 9 SEPARATE INTEGRAL RESULTS.
   22 IF(IDEFLE)24,24,25
C     IF DOING A STRESS, THE NUMFRATOR AND DENOMINATOR INTEGRAL RESULTS
C     ARE THE SAME.  CONSEQUENTLY, THE RESULTS STORED IN G( , , ) ARE
C     ALSO TRANSFERRED INTO GG( , , ).
C
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C
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C
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C
C
C
   24 DO 38 I=1,9
      DO 38 L=1,N
      DO 38 J=1,4
   38 GG(J,L,I)=G(J,L,I)
C     N9 = NUMBER OF INTEGRAL RESULTS IN THE 'NUMERATOR'.  N7 TELLS HOW
C     MANY TERMS IN THE POLYNOMIALS MULTIPLYING THE EXPONENTIALS IN THE
C     'NUMERATOR' WHILE N8 CONTAINS HOW MANY FOR THE 'DENOMINATOR'.
      N7=4
      N8=4
      N9=9
      GO TO 50
C     WHEN DOING A DEFLECTION IN THE FIRST LAYER, THE 'NUMERATOR' INTE-
C     GRATIONS CONTAIN ONE ADDITIONAL INTEGRATION INVOLVING E1( ).  THUS
C     THE PRESENT CONTENTS OF G( , , ) ARE FIRST TRANSFERRED TO GG( , ,)
C     WHICH IS THE DENOMINATOR ARRAY, THEN THE ADDITIONAL INTEGRATION
C     IS CARRIED OUT BY PUTTING E1( ) IN E(8, ) (EIGHTH ROW OF E( , ) )
C     AND USING THE SPECIAL OPTION OF SUBROUTINE INTEGR FOR EXECUTING
C     ONE ADDITIONAL INTEGRATION GIVEN THE RESULTS OF PREVIOUS INTEGRA-
C     TIONS OF SERIES.  THE FINAL RESULT IS STORED BACK IN G( , , ).
   25 DO 26 J=1,N
   26 E(8,J)=E1(J)
      DO 111 I=1,9
      DO 111 L=1,N
      DO 111 J=1,4
  111 GG(J,L,I)=G(J,L,I)
      DO 27 I=1,9
      DO 28 L=1,N
      DO 28 K=1,4
   28 E(K,L)=G(K,L,I)
      CALL INTEGR(E,N,4,1)
      DO 29 L=1,N
      DO 29 J=1,5
   29 G( J,L,I)=B(J,L)
   27 CONTINUE
      N7=5
      N8=4
      N9=9
      GO TO 50
C     WHEN IN THE SECOND OR THIRD LAYER, THE 'NUMERATOR' AND 'DENOMIN-
C     ATOR' CONTAIN ONE ADDITIONAL INTEGRATION.  IN ADDITION, THE 'NUM-
C     ERATOR' CONTAINS 9 ADDITIONAL INTEGRAL RESULTS.  TO CALCULATE
C     THESE, USE IS AGAIN MADE OF THE SPECIAL OPTION FOR EXECUTING A
C     SINGLE ADDITIONAL INTEGRATION USING SUBROUTINE INTEGR.  FIRST THE
C     EIGHTH ROW OF E( , ) IS FILLED WITH E1( ) AND USING THE RESULTS
C     STORED IN G( , , ) THE TENTH THROUGH EIGHTEENTH INTEGRAL RESULTS
C     ARE FOUND USING SUBROUTINE INTEGR.  THEN THESE RESULTS ARE STORED
C     IN G( , , ).  NEXT THE EIGHTH ROW OF E( , ) IS REPLACED WITH E2( )
C     AND INTEGRAL RESULTS ONE TO NINE ARE CALCULATED.  THESE ARE ALSO
C     STORED IN G( , , ).
   23 DO 30 I=1,9
   32 DO 35 J=1,N
   35 E(8,J)=E1(J)
      IJ=I&9
   34 DO 36 J=1,N
      DO 36 K=1,4
C
C
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C
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      C
      C
S.0134      36 E(K,J)=G(K,J,I)
S.0135    1001 CALL INTEGR(E,N,4,1)
S.0136         DO 37 L=1,N
S.0137         DO 37 J=1,5
S.0138      37 G(J,L,IJ)=B(J,L)
S.0139         IF(IJ-9)30,30,31
S.0140      31 DO 33 J=1,N
S.0141      33 E(8,J)=E2(J)
S.0142         IJ=I
S.0143         GO TO 1001
S.0144      30 CONTINUE
S.0145         N8=5
S.0146         N9=18
S.0147         IF(IDEFLE)39,39,40
      C        IF DOING A STRESS, THE DENOMINATOR INTEGRAL RESULTS ARE THE SAME
      C        AS THE FIRST NINE 'NUMERATOR' RESULTS, AND THUS THESE ARE TRANS-
      C        FERRED INTO GG( , , ).
S.0148      39 DO 41 I=1,9
S.0149         DO 41 J=1,5
S.0150         DO 41 L=1,N
S.0151      41 GG(J,L,I)=G(J,L,I)
S.0152         N7=5
S.0153         GO TO 50
      C        IF A DEFLECTION IS DESIRED, THE 'NUMERATOR' INTEGRAL RESULTS MUST
      C        BE INTEGRATED WITH EITHER E2( ) OR E3( ) YET.  FIRST THE PRESENT
      C        FIRST NINE INTEGRAL RESULTS ARE TRANSFERRED INTO THE DENOMINATOR
      C        ARRAY GG( , , ).  THEN THE INTEGRATION OF THE NUMERATOR RESULTS
      C        AND E2( ) OR E3( ) IS CARRIED OUT BY STOREING E2( ) OR E3( ) IN
      C        THE EIGHTH ROW OF E( , ) AND USING SUBROUTINE INTEGR WITH THE
      C        SINGLE ADDITIONAL INTEGRATION OPTION.  THE RESULTS ARE STORED BACK
      C        IN THE G( , , ) ARRAY.
S.0154      40 IF(ILAYER-2)42,42,43
S.0155      42 DO 44 J=1,N
S.0156      44 E(8,J)=E2(J)
S.0157         GO TO 45
S.0158      43 DO 46 J=1,N
S.0159      46 E(8,J)=E3(J)
S.0160      45 DO 112 I=1,9
S.0161         DO 112 L=1,N
S.0162         DO 112 J=1,5
S.0163     112 GG(J,L,I)=G(J,L,I)
S.0164         DO 47 I=1,18
S.0165         DO 48 J=1,N
S.0166         DO 48 L=1,5
S.0167      48 E(L,J)=G(L,J,I)
S.0168         CALL INTEGR(E,N,5,1)
S.0169         DO 49 L=1,N
S.0170         DO 49 J=1,6
S.0171      49 G( J,L,I)=B(J,L)
S.0172      47 CONTINUE
S.0173         N7=6
S.0174      50 CONTINUE
      C        ALL NECESSARY INTEGRALS ARE NOW STORED.  THE NUMERATOR RESULTS
      C        ARE STORED IN THE G ARRAY, DENOMINATOR RESULTS IN GG ARRAY
S.0175         NNX=NNN
      C
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      C
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C
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C
C     THE LOOP TO STATEMENT 56 SOLVES THE INTEGRAL EQUATION FOR EACH
C     OF THE THIRTEEN VALUES OF M.
      DO 56 K=1,13
      EMM=EM(K)
C     THE CONSTANTS IN THE INTEGRAL EQUATION ARE CALCULATED FOR THIS
C     VALUE OF M USING THE SUBROUTINE CNSTNT.  THE RESULTS ARE STORED
C     IN THE VECTORS PH( ), PHJ( ), AND TH( ).
      CALL CNSTNT(EMM,H,ZZ,IOWA,PH,PHJ,TH,ILAYER)
C     THE TOTAL RIGHT HAND SIDE OF THE INTEGRAL EQUATION IS REDUCED TO
C     A SERIES OF EXPONENTIALS EACH MULTIPLIED BY A POLYNOMIAL CONTAIN-
C     ING N7 TERMS.  THE CONSTANTS IN THIS SERIES REPRESENTATION ARE ALL
C     STORED IN THE BB( , ) ARRAY.
      DO 58 J=1,N
      DO 58 L=1,N7
      BB(L,J)=0.
      DO 58 I=1,N9
   58 BB(L,J)=BB(L,J)&PH(I)*G(L,J,I)
C     THE KERNAL OF THE INTEGRAL OF THE LEFT-HAND SIDE OF THE INTEGRAL
C     EQUATION IS REDUCED TO A SERIES OF EXPONENTIALS EACH MULTIPLIED BY
C     A POLYNOMIAL CONTAINING N8 TERMS.  THE CONSTANTS IN THIS SERIES
C     REPRESENTATION ARE ALL STORED IN THE B( , ) ARRAY.
      DO 59 J=1,N
      DO 59 L=1,N8
      B(L,J)=0.
      DO 59 I=1,9
   59 B(L,J)=B(L,J)&TH(I)*GG(L,J,I)
   57 CONTINUE
C     THE INTEGRAL EQUATION IS SOLVED FOR THIS VALUE OF M USING SUBROU-
C     TINE SOLVE.  THE RESULTS ARE STORED IN THE VECTOR SI( ).
      CALL SOLVE(N,N8,N7,NNX,NJJJ)
C     THE RESULT IN SI( ) IS TRANSFERRED INTO THE KTH ROW OF SII( , ).
      DO 60 I=1,NNN
   60 SII(K,I)=SI(I)
      IF(IST-3)56,61,61
C     IF DOING RADIAL STRESS (IST=3), THEN MUST SOLVE A SECOND INTEGRAL
C     EQUATION FOR EACH M.  THIS IS DONE IN THE SAME WAY AS THE FIRST
C     ONE.  THE CONSTANTS ARE ALREADY AVAILABLE, IN PHJ( ) AND TH( ).
C     THE FINAL RESULT IS STORED IN SIII( , ).
   61 DO 63 J=1,N
      DO 63 L=1,N8
      BB(L,J)=0.
      DO 63 I=1,N9
   63 BB(L,J)=BB(L,J)&PHJ(I)*G(L,J,I)
      CALL SOLVE(N,N8,N8,NNX,NJJJ)
      DO 64 I=1,NNN
   64 SIII(K,I)=SI(I)
   56 CONTINUE
C     NEXT THE BESSEL MULTIPLIERS MUST BE CALCULATED.  THESE VARY
C     DEPENDING ON WHICH STRESS OR DEFLECTION IS BEING DONE.
C     THE BESSEL MULTIPLIERS ARE DIVIDED BY M FOR DEFLECTION ONLY.  THE
C     VARIABLE DIVIDE IS UNITY UNLESS DOING A DEFLECTION.
      DIVIDE =1.
      IF(IST-2)78,79,78
C     IDEX IS A DUMMY USED FOR SELECTING EITHER J0(MR) OR J1(MR).
   79 IDEX=1
C
C
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      C
      C
      C
      C     TM1 IS A DUMMY USED TO STORE THE FIRST BESSEL TERM.   SINCE J1(MR)
      C     IS ZERO FOR R=0, AND J0(MR) IS 1 FOR R=0, TM1 IS SET ACCORDINGLY.
            TM1=0.
            GO TO 80
         78 IDEX=0
            TM1=1.
         80 IF(IDEFLE)81,81,82
      C     THE LIMIT OF J1(MA) AS M TENDS TO ZERO IS 0. SO THE FIRST TERM FOR
      C     ALL STRESSES IS ZERO.
         81 BESS(1)=0.
            GO TO 83
      C     THE LIMIT  OF J1(MA)/M AS M TENDS TO ZERO IS A/2. SO BESS(1) IS
      C     A/2 FOR DEFLECTIONS.
         82 BESS(1)=A/2.
      C     DDD TAKES ON THE VALUES OF M.  91 VALUES OF THE BESSEL MULTIPLIERS
      C     ARE COMPUTED, AT VALUES OF M .1 M APART.
         83 DDD=0.
            DO 86 I=2,91
            DDD=DDD&.1
            RM=R*DDD
            AM=A*DDD
            IF(RM-.0001)84,84,85
         85 TM1=BESSEL(IDEX,RM)
         84 TM2=BESSEL(1,AM)
            IF(IDEFLE)86,86,87
         87 DIVIDE=DDD
         86 BESS(I)=TM1*TM2/DIVIDE
            IF(IST-3)70,71,71
      C     IF DOING RADIAL STRESS, MUST COMPUTE A SECOND SET OF BESSEL MUL-
      C     TIPLIERS, WHICH ARE STORED IN BESSS( ).
      C     THE LIMIT OF J1(MR)J1(MA)/MR IS ZERO AS M TENDS TO ZERO.
         71 BESSS(1)=0.
            DDD=0.
            RR=R
            DO 77 I=2,91
            DDD=DDD&.1
            RM=R*DDD
            AM=A*DDD
      C.    THE LIMIT OF J1(MR)J1(MA)/MR AS R TENDS TO ZERO IS MJ1(MA)/2.M
            IF(RR-.0001)250,250,799
        250 R=1.
            TM1=DDD/2.
            GO TO 76
        799 TM1=BESSEL(1,RM)
         76 TM2=BESSEL(1,AM)
         77 BESSS(I)=TM1*TM2/R/DDD
      C     TWO DIFFERENT INTEGRATIONS ON M ARE CARRIED OUT WHEN DOING THE
      C     RADIAL STRESS.  FIRST, AT EACH VALUE OF TIME, 13 VALUES ARE TRANS-
      C     FERRED  FROM SII( , ) INTO THE VECTOR S( ).  THESE RESULTS ARE
      C     USED WITH BESS( ) IN SUBROUTINE TERPO TO COMPUTE THIS INTEGRAL
      C     RESULT.  THIS IS STORED IN WII.  THEN 13 VALUES (FOR THE SAME
      C     TIME) ARE TRANSFERRED FROM SIII( , ) INTO S( ) AND USED WITH
      C     BESSS( ) TO COMPUTE THE SECOND INTEGRAL RESULT.  THIS IS ADDED
      C     INTO WII, THE TOTAL RESULT MULTIPLIED BY A, AND THEN THIS ANSWER
      C     IS PRINTED ALONG WITH THE CORRESPONDING TIME.
      C
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C
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C
      DO 72 I=1 ,NNN
      DO 73 J=1,13
   73 S(J)=SII(J,I)
      CALL TERPO(S,BESS)
      WII=WI
      WRITE(6,102)WI
      DO 74 J=1,13
   74 S(J)=SIII(J,I)
      CALL TERPO(S,BESSS)
      WRITE(6,102)WI
      WII=WI&WII
      WII=WII*A
   72 WRITE(6,93)T(I),WII
      GO TO 1000
C     CONTROL ENTERS HERE FOR ALL BUT RADIAL STRESS FOR THE FINAL INTE-
C     GRATION ON M.  THIS IS DONE AT EACH OF THE NNN VALUES OF TIME.
   70 DO 91 I=1,NNN
C     THE 13 VALUES OF THE SOLUTION AT THE 13 VALUES OF M ARE TRANS-
C     FERRED INTO THE S( ) VECTOR FROM SI( , ).
      DO 92 J=1,13
   92 S(J)=SII(J,I)
C     AFTER THE FIRST THREE VALUES, NONE OF THE SOLUTIONS CHANGE SIGN.
C     IF THEY DO, DUE TO ROUND-OFF ERRORS IN SUBROUTINE CNSTNT, THEY ARE
C     SET TO ZERO.
      DO 783 J=4,13
      IF(S(J-1)*S(J))784,784,783
  784 S(J)=0.
  783 CONTINUE
C     THE SOLUTION IS CALCULATED FOR THIS TIME USING SUBROUTINE TERPO
C     AND THE CONTENTS OF S( ) BESS( ).  THIS RESULT IS MULTIPLIED BY A
C     AND PRINTED ALONG WITH THE CORRESPONDING TIME.
      CALL TERPO(S,BESS)
      WI=WI*A
   91 WRITE(6,93)T(I),WI
   93 FORMAT(8H TIME = E15.8,12H SOLUTION = E15.8)
 1000 CONTINUE
      END
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FUNCTION SUBPROGRAM BESSEL

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      C
      C
      C
            FUNCTION BESSEL(NN,S)
      C     THIS IS A FUNCTION SUB-PROGRAM TO CALCULATE BESSEL FUNCTIONS OF
      C     THE ZEROETH AND FIRST ORDER, OF THE FIRST KIND.  THE INPUT IS NN,
      C     AND S.  NN IS THE ORDER DESIRED(EITHER ZERO OR ONE) AND S IS THE
      C     ARGUMENT OF THE BESSEL FUNCTION.  IF THE ARGUMENT IS LESS THAN OR
      C     EQUAL TO 12. THE FUNCTION IS EVALUATED  USING THE INFINITE SERIES
      C     REPRESENTATION.  IF THE ARGUMENT IS GREATER THAN 12., THEN THE
      C     ASYMPTOTIC EXPANSION FORMULAS ARE USED.  THE OUTPUT IS THE SINGLE
      C     NUMBER STORED IN BESSEL.
            COMMON X(20),BB(8,20),  T(201) ,DELTA(20),BETA(201),B(8,20),
           1ST(201),WI,DELIX,DELXX,NJ,NJJ
            N=NN
            KK=N
      C     THE SIZE OF THE ARGUMENT DETERMINES WHETHER THE ASYMPTOTIC EXPAN
      C     SIONS CAN BE USED.
            IF(S-12.)16,16,17
      C     THE FORM OF THE ASYMPTOTIC EXPANSION DEPENDS ON WHICH FUNCTION  IS
      C     TO BE EVALUATED.
         17 IF(N)18,19,18
         19 PHI=S-3.14159*.75
            GO TO 20
         18 PHI=S-3.14159*.75
         20 BES=((2./3.14159/S)**.5)*COS(PHI)
            GO TO 15
      C     THE PROGRAM FROM HERE TO THE END IS THE SAME AS GIVEN IN THE
      C     REFERENCE CITED IN THE TEXT.
         16 IF(N)2,1,2
          1 BESSEL=1.
            GO TO 6
          2 FACT=N
          3 N=N-1
            IF(N-1)5,5,4
          4 XN=N
            FACT=FACT*XN
            GO TO 3
          5 XFACT=FACT
            BESSEL=((S/2.)**KK)/XFACT
          6 K=1
          7 EXP=2*K&KK
            EXP=EXP/2.
            K1=K
            K2=K&KK
            FACT1=K1
          8 K1=K1-1
            IF(K1-1)10,10,9
          9 XN=K1
            FACT1=FACT1*XN
            GO TO 8
         10 XFACT1=FACT1
            FACT2=K2
         11 K2=K2-1
            IF(K2-1)13,13,12
         12 XN=K2
            FACT2=FACT2*XN
      C
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      C
      C
            GO TO 11
   13 XFACT2=FACT2
      SUM1=((S/2.)**EXP)/XFACT1
      SUM2=((S/2.)**EXP)/XFACT2
      SUM=((-1.)**K)*SUM1*SUM2
      BES=BESSEL&SUM
      IF(ABS(BES-BESSEL)-.0001)15,15,14
   14 BESSEL=BES
      K=K&1
      GO TO 7
   15 BESSEL=BES
      RETURN
      END
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SUBROUTINE TERPO

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      C
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      C
0001        SUBROUTINE TERPO(S,BESS)
      C     THIS SUBROUTINE IS USED TO INTERPOLATE VALUES OF THE SOLUTION AS
      C     A FUNCTION OF THE DUMMY INTEGRATION VARIABLE M, THEN MULTIPLY
      C     THESE VALUES BY THE PROPER BESSEL TERMS (THE CAPITOL THETA TERMS
      C     IN THE TEXT) AND THEN INTEGRATE THE RESULTS USING SIMPSONS RULE
      C     NUMERICAL INTEGRATION PROCEDURE, FOR THE THREE-LAYER HALF-SPACE
      C     ANALYSES.  THE INPUT IS THE VECTOR S( ) CONTAINING THIRTEEN VALUES
      C     OF THE FUNCTION PSI(T,M) OF THE TEXT, AT THE VALES OF M OF 0.,.2,
      C     .4,.7,1.0,2.0,3.0,4.0,5.0,6.0,7.0,8.0,AND 9.0.  ALSO INPUT IS THE
      C     VALUE OF THE APPROPRIATE BESSEL TERM MULTIPLIER AT 91 POINTS
      C     SPACED .1 M APART, WHICH IS STORED IN THE VECTOR BESS( ).  THE
      C     OUTPUT IS THE SINGLE NUMBER WI,(THE RESULT OF THE INTEGRATION)
      C     THE SOLUTION FOR THE TIME OF THE INPUT S( ).
0002        DIMENSION S(13),BESS(91),FUN(91)
0003        COMMON X(20),BB(8,20), T(201) ,DELTA(20),BETA(201),B(8,20),
           1SI(201),WI,DELTX,DELXX,NJ,NJJ
      C     THE VECTOR FUN( ) IS USED TO STORE THE ORIGINAL POINTS AND THE
      C     INTERPOLATED VALUES OF THE FUNCTION DESCRIBED BY THE CONTENTS OF S
      C     FIRST THE INPUT VALUES ARE STORED IN THE APPROPRIATE LOCATIONS
      C     OF FUN( ).
0004        FUN(1)=S(1)
0005        FUN(3)=S(2)
0006        FUN(5)=S(3)
0007        FUN(8)=S(4)
0008        FUN(11)=S(5)
0009        K=11
0010        DO 1 I=6,13
0011        K=K&10
0012      1 FUN(K)=S(I)
      C     THE INTERPOLATION IS PERFORMED BY FITTING A PARABOLA TO THREE CON-
      C     SECUTIVE POINTS, AND THEN EVALUATING THIS PARABOLA AT THE INTER-
      C     MEDIATE POINTS.  THE EQUATION OF THE PARABOLA IS A*X*X&V*X&C.
      C     THE CENTER VALUE IS USED AS OM IN ALL CASES.
      C     NY IS A DUMMY USED TO DIRECT THE FLOW TO TAKE CARE OF THE THREE
      C     DIFFERENT SPACINGS OF THE THREE POINTS.
0013     11 NY=-1
      C     IN ALL, 91 VALUES OF FUN( ) ARE FOUND, SPACED .1M APART
0014        YI=S(2)
0015        YL=S(1)
0016        YR=S(3)
0017        H=.2
0018      2 C=YI
0019        A=(YL-2.*YI&YR)/2./H/H
0020        V=A*H&(C-YL)/H
0021        IF(NY)3,4,5
0022      3 FUN(2)=A*.01-V*.1&C
0023        FUN(4)=A*.01&V*.1&C
0024        NY=0
0025        YI=S(4)
0026        YL=S(3)
0027        YR=S(5)
0028        H=.3
0029        GO TO 2
0030      4 FUN(6)=A*.04-V*.2&C
      C
      C
      C
      C
      C
      C
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C
C
      FUN(7)=A*.01-V*.1&C
      FUN(9)=A*.01&V*.1&C
      FUN(10)=A*.04&V*.2&C
   13 NY=1
      KK=10
      K=5
    8 YL=S(K)
      YR=S(K&2)
      YI=S(K&1)
      KK=KK&1
      H=1.
      GO TO 2
    5 DX=-1.0
      DO 6 I=1,19
      DX=DX+.1
      KK=KK&1
    6 FUN(KK)=A*DX*DX+V*DX+C
      K=K&2
      IF(K-13)8,7,7
C     AT THIS POINT THE INTERPOLATED VALUES HAVE ALL BEEN STORED IN FUN
C     AND THE INTEGRATION OF THE PRODUCTS FUN(I)*BESS(I) IS NOW CARRIED
C     OUT.
    7 WI=0.
      DO 70 J=2,88,2
   70 WI=WI&4.*BESS(J)*FUN(J)&2.*BESS(J&1)*FUN(J&1)
      WI=WI&BESS(1)*FUN(1)&4.*BESS(90)*FUN(90)
      WI=WI&BESS(91)*FUN(91)
      WI=WI*.1/3.
      RETURN
      END
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<u>SUBROUTINE VALUE</u>

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      C
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      C
            SUBROUTINE VALUE(N,M,NNN)
      C     THIS SUBROUTINE EVALUATES THE GENERAL RESULT OF THE EXACT MULTIPLE
      C     CONVOLUTION INTEGRATIONS, WHICH ARE EXPRESSED AS SERIES.  THE
      C     INPUT IS N, THE LENGTH OF THE SERIES, M WHICH IS THE NUMBER OF
      C     CONSTANTS FOR EACH RELAXATION TIME (FOR INSTANCE, IF TERMS UP TO
      C     AND INCLUDING T**5 ARE INCLUDED, THEN M IS 6) ,AND NNN, THE NUMBER
      C     OF TIMES AT WHICH THE EVALUATION IS DESIRED.  THE SERIES IS
      C     INPUT THROUGH COMMON STORAGE IN THE B( , ) ARRAY.  ALSO INPUT BY
      C     MEANS OF COMMON ARE THE TIMES T( ), AND THE RELAXATION TIMES
      C     DELTA( ).  THE OUT-PUT IS STORED IN THE VECTOR BETA( ).
            DIMENSION T1(20)
            COMMON X(20),BB(8,20),  T(201) ,DELTA(20),BETA(201),B(8,20),
           1SI(201),WI,DELTX,DELXX,NJ,NJJ
      C     THE VECTOR T1( ) STORES PRODUCTS OF TIMES.  T1(1) IS T**0, T1(2)
      C     IS T**1, T1(3) IS T**2, ETC.
            T1(1)=1.
      C     THE LOOP THROUGH 4 IS EXECUTED FOR EACH TIME DESIRED
            DO 4 L=1,NNN
      C     THE SOLUTION VECTOR IS ZEROED
            BETA(L)=0.
      C     THE PRODUCTS OF T(L) ARE CALCULATED AND STORED IN T1( ).
            DO 5 I=2,M
          5 T1(I)=T1(I-1)*T(L)
      C     THE TERMS MULTIPLYING EACH EXPONENTIAL TERM ARE CALCULATED AND
      C     STORED IN SUM, THEN MULTIPLIED BY THE EXPONENTIAL TERM AND STORED
      C     IN THE SOLUTION LOCATION BETA(L).
            DO 18 J=1,N
            SUM=0.
            DO 9 I=1,M
          9 SUM=SUM&B(I,J)*T1(I)
         18 BETA(L)=BETA(L)&SUM*EXP(-DELTA(J)*T(L))
          4 CONTINUE
            RETURN
            END
      C
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      C
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SUBROUTINE CNSTNT

```
DECK
    C
    C
    C
          SUBROUTINE CNSTNT(XM,HH,ZZZ,IOWA,PH,PHJ,TH,ILAYER)
    C     THIS SUBROUTINE CALCULATES THE CONSTANTS FOR THE THREE LAYER HALF-
    C     SPACE, USING THE EQUATIONS PRESENTED IN THE TEXT.  THE NOTATION
    C     USED IS ESSENTIALLY THE SAME THROUGH-OUT AS THE TEXT.  THE INPUT
    C     IS XM=EM=M, THE DUMMY INTEGRATION VARIABLE, HH = H, THE THICKNESS
    C     OF THE SECOND LAYER EXPRESSED AS MULTIPLES OF THE FIRST LAYER
    C     THICKNESS, ZZZ=ZZ=Z OF TEXT, THE DEPTH OF INTEREST, IOWA= INTEGER
    C     1 OR 2 OR 3 OR ... OR 6 DEPENDING ON WHICH PHI S ARE DESIRED (THAT
    C     IS, WHICH STRESS OR DISPLACEMENT IS  BEING CONSIDERED--IOWA WILL
    C     BE 1 FOR NORMAL STRESS, 2 FOR SHEAR STRESS, 3 FOR RADIAL STRESS,
    C     5 FOR VERTICAL DEFLECTION, OR 6 FOR RADIAL DEFLECTION), ILAYER=
    C     THE LAYER OF INTEREST.  ALSO READ IN ARE THE VECTORS PH( ), PHJ( )
    C     AND TH( ).  THESE ARE READ IN ONLY SO THE RESULTS, WHICH ARE
    C     STORED IN THESE VECTORS WILL BE RETURNED TO THE MAIN PROGRAM (TO
    C     SAVE COMMON STORAGE).
          DIMENSION PH(18),PHJ(18),TH(9)
          COMMON X(20),BB(8,20),  T(201) ,DELTA(20),BETA(201),B(8,20),
         1SI(201),WI,DELTX,DELXX,NJ,NJJ
    C     ALL THE OPERATIONS ARE EXECUTED IN DOUBLE PRECISION SINCE IT WAS
    C     FOUND THAT THIS IS NECESSARY TO MAINTAIN REASONABLE ACCURACY AT
    C     LARGE VALUES OF M.
          DOUBLE PRECISION S,EM,H,ZZ,C(9),V(9),PHI(6,3,18),ALAM(6,4),
         1Q(4,3,18),Z,Z1,Z2,Z3,Z4,Z5,Z6,A1,A2,A3,A4,A5,A6,A7,A8,B1,B2,B3,
         2B4,B5,B6,B7,B8,Q3,Q4,EZ,EZ1,EZ2,G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,
         3G11,G12,G13,G14,G15,G16,G17,G18,G19,G20,G21,G22,G23,G24,G25,G26,
         3G27,G28,G29,G30,G31,G32,G33,G34,G35,G36,G37,G38,G39,G40,G41,G42,
         4G43,G44,G45,G46,G47,G48,G49,G50,G51,G52,G53,G54,G55,G56,G57,G58,
         5G59,G60,G61,G62,G63,G64,G65,G66,G67,G68
    C     THE NOTATION IN ALL THE FOLLOWING IS THE SAME AS THE TEXT, WITH
    C     Z = ZZ AND M = EM, AND AN OCCASIONAL INTERMEDIATE VARIABLE DEFINED
    C     TO SAVE EXECUTION TIME.
          EM=XM
          H=HH
          ZZ=ZZZ
          S=EM*H
          Z=DEXP(EM)
          Z1=DEXP(-EM)
          Z2=DEXP(2.*EM)
          Z3=DEXP(-2.*EM)
          G1=Z/2.
          G2=Z1/2.
          G3=(-1.&2.*EM)/2.
          G4=-Z2/2.
          G5=Z3/2.
          G6=(1.&2.*EM)/2.
          G7=(G1&G2)/2.
          G8=(G1-G2)/2.
          G9=(G3&G5)/2.
          G10=(G3-G5)/2.
          G11=(G4&G6)/2.
          G12=(G4-G6)/2.
          G13=.5-G5
          G14=.5+   G5
          G15=.5-   G6
    C
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    C
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C
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C
      G16=-G15
      G17=.5+    G3
      G18=-G17
      G19=.5+    G4
      G20=.5-    G4
      Z4=DEXP(2.*S)
      G27=2.*Z4
      G28=(1.&2.*EM*H)*Z4
      G21=G27*G7-G28*G2&G1
      G22=G27*G8&G28*G2-G1
      G23=G27*G9&G28*G13&G17
      G24=G27*G10&G28*G14&G18
      G25=G27*G11&G28*G15&G19
      G26=G27*G12&G28*G16&G20
      G35=(1.-2.*S)*Z4
      G36=-2.*S*S*Z4
      G29=G35*G7&G7-G36*G2
      G30=G35*G8-G8&G36*G2
      G31=G35*G9&G9&G36*G13
      G32=G35*G10-G10&G36*G14
      G33=G35*G11&G11&G36*G15
      G34=G35*G12-G12&G36*G16
      L=0
      Z5=DEXP(S)
      Z6=DEXP(-S)
      G53=Z5
      G54=-Z6
      G55=S*Z5
      G56=-S*Z6
      G37=G53
      G38=G54
      G39=G55
      G40=G56
    3 G41=G37*G7&G38*G7-G39*    G2&G40    *G1
      G42=-(G38*G29&G40*G21)
      G43=G37*G8-G38*G8&G39    *G2-G40    *G1
      G44=-(G38*G30&G40*G22)
      G45=G37*G9&G38*G9&G39*G13&G40*G17
      G46=-(G38*G31&G40*G23)
      G47=G37*G10-G38*G10&G39*G14&G40*G18
      G48=-(G38*G32&G40*G24)
      G49=G37*G11&G38*G11&G39*G15&G40*G19
      G50=-G38*G33-G40*G25
      G51=G37*G12-G38*G12&G39*G16&G40*G20
      G52=-G38*G34-G40*G26
      IF(L)1,1,2
    1 L=5
      G57=G41
      G58=G42
      G59=G43
      G60=G44
      G61=G45
      G62=G46
      G63=G47
      G64=G48
C
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C
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C
      G65=G49
      G66=G50
      G67=G51
      G68=G52
      G38=-G38
      G39=(1.&S)*Z5
      G40=-(1.-S)*Z6
      GO TO 3
    2 A1=G45
      A2=G46
      A3=G47
      A4=G48
      A5=G65
      A6=G66
      A7=G67
      A8=G68
      B1=G49
      B2=G50
      B3=G51
      B4=G52
      B5=G61
      B6=G62
      B7=G63
      B8=G64
    8 C(1)=A1*A5-B1*B5
      C(2)=A2*A5&A1*A6-B2*B5-B1*B6
      C(3)=A3*A5&A1*A7-B3*B5-B1*B7
      C(4)=A4*A5&A3*A6&A2*A7&A1*A8-B4*B5-B3*B6-B2*B7-B1*B8
      C(5)=A2*A6-B2*B6
      C(6)=A4*A6&A2*A8-B4*B6-B2*B8
      C(7)=A3*A7-B3*B7
      C(8)=A4*A7&A3*A8-B4*B7-B3*B8
      C(9)=A4*A8-B4*B8
      IF(L)4,5,6
    6 DO 7 I=1,9
C     THE V(I) TERMS ARE THE THETA(I) TERMS OF THE TEXT
    7 V(I)=C(I)
      A1=G49
      A2=G50
      A3=G51
      A4=G52
      A5=G57
      A6=G58
      A7=G59
      A8=G60
      B1=G41
      B2=G42
      B3=G43
      B4=G44
      B5=G65
      B6=G66
      B7=G67
      B8=G68
      L=0
      GO TO 8
C
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C
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C
    5 L=-5
      DO 9 I=1,9
    9 Q(3,1,I)=C(I)
      A1=G61
      A2=G62
      A3=G63
      A4=G64
      A5=G41
      A6=G42
      A7=G43
      A8=G44
      B1=G45
      B2=G46
      B3=G47
      B4=G48
      B5=G57
      B6=G58
      B7=G59
      B8=G60
      GO TO 8
    4 DO 10 I=1,9
   10 Q(4,1,I)=C(I)
      DO 11 I=1,9
      Q3=Q(3,1,I)
      Q4=Q(4,1,I)
      Q(1,1,I)=V(I)*G1&G3*Q3&G4*Q4
      Q(2,1,I)=V(I)*G2&G5*Q3&G6*Q4
      Q(1,2,I)=V(I)*G7&G9*Q3&G11*Q4
      Q(2,2,I)=Q(1,2,I)
      Q(3,2,I)=-V(I)    *G2&G13*Q3&G15*Q4
      Q(4,2,I)= V(I)    *G1&G17*Q3&G19*Q4
      Q(4,3,I)= V(I)*G21&G23*Q3&G25*Q4
      Q(2,3,I)=V(I)*G29&G31*Q3&G33*Q4
      J=I&9
      Q(1,2,J)=V(I)*G8&G10*Q3&G12*Q4
      Q(2,2,J)=-Q(1,2,J)
      Q(3,2,J)= V(I)    *G2&G14*Q3&G16*Q4
      Q(4,2,J)=-V(I)    *G1&G18*Q3&G20*Q4
      Q(4,3,J)= V(I)*G22&G24*Q3&G26*Q4
   11 Q(2,3,J)=V(I)*G30&G32*Q3&G34*Q4
      EZ=EM*ZZ
      EZ1=DEXP(EZ)
      EZ2=DEXP(-EZ)
C     THE ALAM(I,J) TERMS ARE THE LAMDA(I,J) S OF THE TEXT
      ALAM(1,1)=-EZ1
      ALAM(1,2)=-EZ2
      ALAM(1,3)=-EZ*EZ1
      ALAM(1,4)=-EZ*EZ2
      ALAM(2,1)=-ALAM(1,1)
      ALAM(2,2)=ALAM(1,2)
      ALAM(2,3)=ALAM(2,1)-ALAM(1,3)
      ALAM(2,4)=-ALAM(1,2)&ALAM(1,4)
      ALAM(3,1)=ALAM(2,1)
      ALAM(3,2)=-ALAM(2,2)
      ALAM(3,3)=2.*ALAM(3,1)-ALAM(1,3)
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      ALAM(3,4)=2.*ALAM(2,2)-ALAM(1,4)
      ALAM(4,1)=ALAM(1,1)
      ALAM(4,2)=ALAM(1,2)
      ALAM(4,3)=-ALAM(2,3)
      ALAM(4,4)=ALAM(2,4)
      ALAM(5,1)=-1.5*EZ1
      ALAM(5,2)=1.5*EZ2
      ALAM(5,3)=-1.5*EZ*EZ1
      ALAM(5,4)=-1.5*ALAM(1,4)
      ALAM(6,1)=1.5*EZ1
      ALAM(6,2)=1.5*EZ2
      ALAM(6,3)=1.5*ALAM(2,3)
      ALAM(6,4)=-1.5*ALAM(2,4)
      DO 107 I=10,18
      TH(I-9)=V(I-9)
C     THE UNDEFINED Q( ,1, ) S ARE ZEROED.
      DO 107 J=1,4
  107 Q(J,1,I)=0.
C     THE PHI S ARE CALCULATED  FOR ALL POSSIBILITIES.
      DO 106 J=1,6
      DO 106 I=1,18
      DO 106 K=1,3
      PHI(J,K,I)=0.
      DO 106 M=1,4
  106 PHI(J,K,I)=PHI(J,K,I)&Q(M,K,I)*ALAM(J,M)
C     THE PROPER PHI S ARE STORED IN PH( ) FOR RETJRN TO THE MAIN PRO -
C     GRAM.  SINCE THE RADIAL STRESS INVOLVES TWO SETS OF PHI S, ONE SET
C     IS ALWAYS STORED IN THE PHJ( ) VECTOR FOR RETURN TO THE MAIN
C     PROGRAM.  THE THETA( ) S ARE ALSO RETURNED TO THE MAIN PROGRAM,
C     IN THE TH( ) VECTOR.
      DO 50 I=1,18
      PH(I)=PHI(IOWA,ILAYER,I)
   50 PHJ(I)=PHI(4,ILAYER,I)
      RETURN
      END
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SUBROUTINE REJECT

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C
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C
      SUBROUTINE REJECT(NNN,GAM)
C     THIS SUBROUTINE SAVES THE VALUES OF THE ARRAY GAM( , , ), AND OF
C     THE VECTOR SI( ), WHICH WILL BE NEEDED IN THE NEXT TIME THROUGH
C     THE LOOP SOLVING THE INTEGRAL EQUATION, FOR THE CASE THAT THE
C     CONVOLUTION INTEGRALS ARE EVALUATED NUMERICALLY.
      DIMENSION GAM(61,7,18)
      COMMON BETA(61),B(8,20),DELTA(20),T(61),MN,SI(61)
C     MN IS THE NUMBER OF VALUES , OF NNN POSSIBLE VALUES, WHICH ARE TO
C     BE SAVED AND RESTORED.
      MN=NNN/2&2
      MN1=MN-1
      DO 1 I=2,MN1
      K=2*I-1
      SI(I)=SI(K)
      DO 2 J=1,18
      DO 2 L=1,7
    2 GAM(I,L,J)=GAM(K,L,J)
    1 CONTINUE
      RETURN
      END
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SUBROUTINE CVEFIT

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C
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      C
      SUBROUTINE CVEFIT(ARRAY)
C     THIS SUBROUTINE COMPUTES A DIRICHLET SERIES APPROXIMATION  TO AN
C     INPUT CURVE DESCRIBED BY TWELVE POINTS STORED IN  THE VECTOR X( ).
C     THE FITTING IS PERFORMED BY MEANS OF A SINGLE MATRIX
C     MULTIPLICATION .  THE PRE-MULTIPLIER IS THE ARRAY NAMED ARRAY,
C     WHICH IS READ IN, AND THE POST-MULTIPLIER IS THE VECTOR X( ).
C     ARRAY IS THE INVERSE OF THE COLLOCATION MATRIX OBTAINED BY WRITING
C     12 EQUATIONS EQUATING THE SERIES REPRESENTATION AT EACH OF 12
C     POINTS TO THE INPUT CURVES VALUE AT THESE 12 POINTS.  X( ) CON-
C     TAINS THESE TWELVE POINTS FOR THE INPUT CURVE.  THE MATRIX ARRAY
C     WAS OBTAINED USING THE GAUSSIAN ELIMINATION PROCEDURE ON 12 RIGHT
C     HAND SIDES WHICH COLLECTIVELY MADE UP AN IDENTITY MATRIX.
      DIMENSION Y(12),ARRAY(12,12)
      COMMON X(20),BB(8,20),  T(201) ,DELTA(20),BETA(201),B(8,20),
     1SI(201)
      DO 1 I=1,12
      Y(I)=0.
C     THE CONSTANTS FOR THE SERIES REPRESENTATION ARE CALCULATED AND
C     STORED IN THE  VECTOR Y( ).
      DO 1 J=1,12
    1 Y(I)=Y(I)&ARRAY(I,J)*X(J)
C     THE CONSTANTS ARE TRANSFERRED INTO THE X VECTOR.
      DO 2 I=1,12
    2 X(I)=Y(I)
      RETURN
      END
```

SUBROUTINE TIME1

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C
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.0001         SUBROUTINE TIME1(NNN,DEL,NX)
        C     THIS SUBROUTINE COMPUTES THE TIMES AND RELAXATION TIME INVERSES
        C     FOR THE CASE THAT THE CONVOLUTION INTEGRALS ARE EVALUATED NUMERI-
        C     CALLY.  THE INPUT IS NNN, DEL, AND NX.  NNN IS THE NUMBER OF
        C     POINTS FOR WHICH THE TIME IS TO BE COMPUTED.  DEL IS THE SPACING
        C     OF THESE NNN POINTS OF TIME.  NX IS ZERO IF THE DELTA( ) VECTOR,
        C     WHICH CONTAINS THE INVERSES OF THE RELAXATION TIMES, IS TO BE
        C     COMPUTED, WHILE IF THEY HAVE PREVIOUSLY BEEN COMPUTED NX IS NON-
        C     ZERO.
.0002         COMMON BETA(61),B(8,20),DELTA(20),T(61),MN,SI(61),WI
.0003         N=12
        C     FIRST THE NNN TIMES ARE COMPUTED, WITH T(1) ALWAYS ZERO.
.0004         T(1)=0.
.0005         NNNN=NNN-1
.0006         DO 7 K=1,NNNN
.0007       7 T(K&1)=T(K)&DEL
.0008         IF(NX)1,2,1
        C     IF NX IS ZERO, THEN THE DELTA( ) VECTOR IS TO BE COMPUTED.  EACH
        C     SUCCESSIVE DELTA(J) IS 1 DIVIDED BY THE SQUARE ROOT OF TEN TIMES
        C     THE PREVIOUS DELTA(J), EXCEPT DELTA(2) IS 10. AND DELTA(1) IS ZERO
.0009       2 DELTA(1)=0.
.0010         DELTA(2)=10.
.0011         DO 6 J=3,N
.0012       6 DELTA(J)=DELTA(J-1)/(10.**.5)
.0013       1 CONTINUE
.0014         RETURN
              END
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SUBROUTINE INTEGR (NUMERICAL)

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C
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C
      SUBROUTINE INTEGR(N,N1,E,GAM,II,MMM)
C     THIS SUBROUTINE COMPUTES THE MULTIPLE CONVOLUTION INTEGRALS NUMER-
C     ICALLY.  THE INPUT IS N, N1, E( , ), GAM( , , ), II, AND MMM.
C     N IS EITHER 1 OR N1/2&2 DEPENDING ON WHETHER THIS IS THE FIRST
C     TIME THROUGH THIS ROUTINE OR NOT.  N1 IS THEN NUMBER OF POINTS IN
C     TIME FOR WHICH THE MULTIPLE CONVOLUTION INTEGRATIONS ARE TO BE
C     CALCULATED.  E( , ) CONTAINS THE VALUES OF THE EACH OF THE RELAX-
C     ATION FUNCTIONS OR CREEP FUNCTIONS AT EACH OF THE N1 TIMES.  EACH
C     ROW OF E CONTAINS ONE OF THESE FUNCTIONS.  GAM( , , ) IS THE SOLU-
C     TION ARRAY--THE NUMERICAL VALUES OF THE MULTIPLE CONVOLUTION INTE-
C     GRALS.  THE FIRST TIME THROUGH THIS ROUTINE THEY ARE INITIALLY
C     UNKNOWN AT ALL TIMES.  EACH SUCCESSIVE TIME THROUGH, THE FIRST
C     N-1 VALUES (FROM PREVIOUS CALCULATIONS) ARE STORED IN GAM( , , ).
C     II IS THE THIRD SUBSCRIPT OF THE GAM( , , ) ARRAY TO BE COMPUTED.
C     MMM IS THE NUMBER OF INTEGRATIONS INVOLVED.
      DIMENSION E(7,61),GAM(61,7,18)
      COMMON BETA(61),B(8,20),DELTA(20),T(61),MN,SI(61),WI
C     THE LOOP TO STATEMENT 1 STORES THE FIRST RELAXATION FUNCTION IN
C     GAM( ,1,II)
      DO 1 I=N,N1
    1 GAM(I,1,II)=E(1,I)
C     THE LOOP FROM HERE  TO 2 IS EXECUTED FOR EACH INTEGRATION.
      DO 2 I=1,MMM
C     THIS LOOP IS EXECUTED FOR EACH POINT IN TIME FOR WHICH THE RESULTS
C     ARE NEEDED.
      DO 50 J=N,N1
C     THE INTEGRAL TO BE EVALUATED ON THIS CYCLE (GAM(J,I&1,II)) IS
C     ZEROED.
      GAM(J,I&1,II)=0.
      I1=J-1
      IF(J-1)51,52,51
C     IF J IS EQUAL TO 1, AM AT ZERO TIME AND THE INTEGRAL RESULT CAN
C     BE EVALUATED DIRECTLY (JUST THE INITIAL CONDITIONS).
   52 GAM(1,I&1,II)=GAM(1,I,II)*E(I&1,1)
      GO TO 50
C     THE GENERAL TERM IS CALCULATED BY COMPUTING THE SUM DESCRIBED IN
C     THE TEXT AND ADDING THE INITIAL CONDITIONS.  X STORES THE AVERAGED
C     RELAXATION OR CREEP FUNCTION, AND XX STORES THE DIFFERENCE OF THE
C     GAM( , , ) TERMS, WHICH ARE EITHER PREVIOUSLY OBTAINED INTEGRAL
C     RESULTS OR E(1, ).
   51 DO 60 K=2,J
      JA=J-K&1
      X=(E(I&1,JA)&E(I&1,JA&1))/2.
      XX=GAM(K,I,II)-GAM(K-1,I,II)
   60 GAM(J,I&1,II)=GAM(J,I&1,II)&X*XX
C     THE INITIAL CONDITIONS ARE ADDED ON.
   62 GAM(J,I&1,II)=GAM(J,I&1,II)&E(I&1,J)*GAM(1,I,II)
   50 CONTINUE
    2 CONTINUE
      RETURN
      END
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SUBROUTINE SOLVIT

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C
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C
      SUBROUTINE SOLVIT(NNN,PH,TH,GAM,N,M,N1,M1)
C     THIS SUBROUTINE SOLVES THE GENERAL INTEGRAL EQUATION FOR THE CASE
C     THAT THE MULTIPLE CONVOLUTION INTEGRALS HAVE BEEN EVALUATED NUMER-
C     ICALLY AND STORED IN THE ARRAY GAM( , , ).  THE INPUT IS NNN,
C     PH( ), TH( ), GAM( , , ), N, M, N1,M1.  NNN IS THE NUMBER OF
C     POINTS IN TIME TO BE CONSIDERED.  PH( ) AND TH( ) ARE THE
C     CONSTANTS MULTIPLYING THE MULTIPLE CONVOLUTION INTEGRALS IN THE
C     NUMERATOR AND DENOMINATOR RESPECTIVELY.  GAM( , ,) CONTAINS THE
C     RESULTS OF THE NUMERICAL EVALUATION OF THE MULTIPLE CONVOLUTION
C     INTEGRATIONS.  N IS THE NUMBER OF TERMS
C     IN THE NUMERATOR, AND M IS THE NUMBER OF TERMS IN THE DENOMINATOR.
C     N1 IS THE SECOND (MIDDLE) SUBSCRIPT OF THE GAM( , , ) ARRAY FOR
C     THE NUMERATOR MULTIPLE CONVOLUTION INTEGRATION RESULTS.  M1 IS THE
C     SECOND SUBSCRIPT OF THE GAM( , , ) ARRAY FOR THE DENOMINATOR
C     MULTIPLE CONVOLUTION INTEGRAL RESULTS.  THE RESULT OF THIS SUB-
C     ROUTINE IS THE SOLUTION TO THE INTEGRAL EQUATION AT THE APPRO-
C     PRIATE TIMES, STORED IN THE VECTOR SI( ).  ALSO INPUT TO THE
C     SUBROUTINE THROUGH COMMON STORAGE IS MN, WHICH IS 1 IF THIS IS THE
C     FIRST TIME THROUGH THE ROUTINE, AND IS NNN/2&2 OTHERWISE.  IT IS
C     USED TO  MAKE POSSIBLE THE CALCULATION OF THE NEXT SET OF SOLU-
C     TIONS WHEN DOUBLING THE SIZE OF INTERVALS.  IN THESE CASES THE
C     MN-1 VALUES OF SI( ) THAT ARE NEEDED ARE ALSO BROUGHT INTO THE
C     ROUTINE THROUGH COMMON STORAGE.
      DIMENSION PH(18),TH(9),GAM(61,7,18)
      COMMON BETA(61),B(8,20),DELTA(20),T(61),MN,SI(61),WI
C     THE LOOP FROM HERE TO STATEMENT 1 IS REPEATED NNN TIMES OR NNN-MN
C     TIMES.
      DO 1 I=MN,NNN
C     ANUM AND DNUM ARE INTERMEDIATE VARIABLES FOR STORING THE NUMERATOR
C     AND DENOMINATOR OF THE SOLUTION AT THE POINT BEING CONSIDERED.
      ANUM=0.
      DNUM=0.
C     THE RIGHT HAND SIDE OF THE INTEGRAL EQUATION IS CALCULATED AND
C     STORED IN ANUM.
      DO 2 J=1,N
    2 ANUM=ANUM&PH(J)*GAM(I,N1,J)
      IF(I-1)3,3,4
C     IF THIS IS THE FIRST SOLUTION POINT (I=1) THEN THE DENOMINATOR
C     ONLY NEEDS TO BE CALCULATED BEFORE COMPUTING THE ANSWER.
    3 DO 5 J=1,M
    5 DNUM=DNUM&TH(J)*GAM(1,M1,J)
      GO TO 6
C     AFTER THE FIRST POINT, THE SOLUTION MUST BE OBTAINED BY THE FINITE
C     DIFFERENCE APPROXIMATION.  THE DENOMINATOR IS CALCULATED AND
C     STORED IN DNUM.  THEN THE FIRST PREVIOUS SOLUTION TIMES THE APPRO-
C     PRIATE TERMS IS SUBTRACTED FROM ANUM.
    4 DO 7 J=1,M
      DNUM=DNUM&TH(J)*(GAM(1,M1,J)&GAM(2,M1,J))*.5
      ANUM=ANUM-.5*TH(J)*(GAM(2,M1,J)-GAM(1,M1,J))*SI(I-1)
      SUM=0.
      IF(I-2)7,7,8
C     FOR ALL BUT THE SECOND POINT IN TIME THE OTHER SOLUTION POINTS
C     THAT HAVE BEEN OBTAINED MUST BE MULTIPLIED BY THE APPROPRIATE
C     TERMS OF THE GAM( , , ) ARRAY AND THE TH( ) VECTOR AND SUBTRACTED
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C
C     FROM THE NUMERATOR (ANUM).  FIRST THESE TERMS ARE COMPUTED AND
C     STORED IN  THE TERM SUM, AND THEN SUM IS MULTIPLIED BY THE APPRO-
C     PRIATE TH( ) TERM.
    8 DO 9 K=3,I
      L=I-K&1
      LL=L&1
    9 SUM=SUM-.5*(SI(LL)&SI(L))*(GAM(K,M1,J)-GAM(K-1,M1,J))
      ANUM=ANUM&SUM*TH(J)
    7 CONTINUE
C     THE SOLUTION IS COMPUTED AND STORED IN SI(I).
    6 SI(I)=ANUM/DNUM
    1 CONTINUE
      RETURN
      END
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SUBROUTINE TIME

DECK

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C
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C
      SUBROUTINE TIME(NNN)
C     THIS SUBROUTINE CALCULATES THE TIMES THAT THE SOLUTION, FOR THE
C     CASE THAT THE INTEGRATIONS ARE PERFORMMED EXACTLY, ARE DESIRED.
C     IT ALSO CALCULATES THE INVERSE OF THE RELAXATION TIMES ( THE DELTA
C     TERMS OF THE TEXT) AND STORES THIS RESULT IN THE VECTOR DELTA( ).
C     THE INPUT CONSISTS OF NNN=NUMBER OF TIMES DESIRED.  ALSO, DELTX
C     AND DELXX ARE REQUIRED, WHICH ARE IN COMMON STORAGE.  DELXX
C     SPECIFIES THE LOGARITHMIC INCREMENT OF TIME (IT HAS BEEN TAKEN AS
C     .0625 IN THE APPLICATIONS IN THIS THESIS) AND DELTX SPECIFIES THE
C     THE LOG OF THE FIRST FINITE TIME MINUS DELTX (TAKEN AS -2.0625
C     OR -2.5625 DEPENDING ON THE SIZE OF  SHORT TIME VARIATION IN  THE
C     RESPONSE THAT WAS EXPECTED)
      COMMON X(20),BB(8,20),  T(201) ,DELTA(20),BETA(201),B(8,20),
     1SI(201),WI,DELTX,DELXX,NJ,NJJ
      N=12
C     THE FIRST TIME IS SET EQUAL TO ZERO, AND THEN THE OTHER NNN-1
C     TIMES ARE CALCULATED BY RAISING 10. TO THE DELT POWER, WHERE DELT
C     IS INCREMENTED BY DEL AT EACH STEP.
      DELT=DELTX
      DEL=DELXX
      T(1)=0.
      NNNN=NNN-1
      DO 7 K=1,NNNN
      DELT=DELT&DEL
    7 T(K&1)=10.**DELT
C     THE FIRST DELTA IS SET EQUAL TO ZERO, THE SECOND EQUAL TO 10., AND
C     10 ADDITIONAL ONES ARE CALCULATED BY SUCCESSIVELY DIVIDING BY THE
C     SQUARE ROOT OF TEN.
      DELTA(1)=0.
      DELTA(2)=10.
      DO 6 J=3,N
    6 DELTA(J)=DELTA(J-1)/(10.**.5)
      RETURN
      END
C
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SUBROUTINE SOLVE

DECK

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C
C
C
      SUBROUTINE SOLVE(N,M,MM,NNN,NJJJ)
C     THIS SUBROUTINE IS USED TO SOLVE THE GENERAL INTEGRAL EQUATION FOR
C     THE CASE THAT THE MULTIPLE CONVOLUTION INTEGRALS ARE EVALUATED
C     EXACTLY.  N IS INPUT AS THE NUMBER OF RELAXATION TIMES IN THE
C     ORIGINAL SERIES REPRESENTATIONS.  THE LENGTH OF THE VECTORS
C     OF CONSTANTS FOR EACH OF THE MULTIPLE CONVOLUTION RESULTS FOR EACH
C     RELAXATION TIME ( THAT IS, IF THE NUMERATOR RESULT INCLUDES TERMS
C     UP TO AND INCLUDING T**4, THEN ITS LENGTH IS 5) IS INPUT AS THE
C     NUMBERS M AND MM.  THE LENGTH OF
C     THE VECTOR FOR THE KERNAL FUNCTION IS M, WHILE THE LENGTH FOR THE
C     RIGHT HAND SIDE IS MM.  NNN IS THE NUMBER OF TIME STEPS.  NJJJ IS
C     THE NUMBER OF TERMS (MAXIMUM) TO BE INCLUDED IN  THE CALCULATION
C     OF THE NEXT SOLUTION (THIS WILL BE EXPLAINED BELOW).  ALSO AS IN-
C     PUT ARE THE ARRAYS B( , ) AND BB( , ) WHICH ARE THE RESULTS FOR
C     THE KERNAL FUNCTION AND RIGHT-HAND SIDES RESPECTIVELY, AND ARE IN
C     COMMON STORAGE.  THE TIMES AND RELAXATION TIMES ARE IN THE VECTORS
C     T( ) AND DELTA( ) RESPECTIVELY, IN COMMON. THE PROGRAM SELECTS 12
C     POINTS FROM THE SOLUTION VECTOR (SI( )) AND STORES THEM IN THE
C     X( ) VECTOR.  THESE TWELVE POINTS ARE SELECTED FOR POSSIBLE USE IN
C     FITTING A DIRICHLET SERIES TO THE RESULTS, USING THE SUBROUTINE
C     CVFFIT.  SINCE THE LOCATION OF THE PROPER POINTS IN THE SOLUTION
C     VECTOR SI( ) DEPENDS ON THE TIMES CALCULATED IN THE SUBROUTINE
C     TIME, TWO NUMBERS, NJ AND NJJ ENTER THE PROGRAM (THROUGH COMMON
C     STORAGE).
      DIMENSION T1(20)
      COMMON X(20),BB(8,20),  T(201) ,DELTA(20),BETA(201),B(8,20),
     1SI(201),WI,DELTX,DELXX,NJ,NJJ
C     THE FIRST POINT, T = 0.0, IS CALCULATED FIRST.  IT REQUIRES ONLY
C     THE FIRST COLUMN OF THE ARRAYS B( , ) AND BB( , ).
      BETA(1)=0.
      SUMM=0.
      DO 2 I=1,N
      SUMM=SUMM&BB(1,I)
    2 BETA(1)=BETA(1)&B(1,I)
      SI(1)=SUMM/BETA(1)
C     THE VECTOR T1( ) IS USED TO STORE PRODUCTS OF TIMES.  T1(1) IS
C     T**0, T1(2) IS T**1, T1(3) IS T**2, ETC.
      T1(1)=1.
C     SINCE THE TIME SPACING IS LOGARITHMIC, SUCCESSIVE ANSWERS DEPEND
C     LESS AND LESS ON THE FIRST ANSWERS.  FOR THIS REASON, IT IS POSSI-
C     BLE TO NEGLECT SOME TERMS WHEN COMPUTING THE RESULTS.  IN GENERAL
C     NJJJ TERMS OF THE SOLUTION VECTOR WILL BE USED TO CALCULATE THE
C     NEXT TERM, AFTER THE FIRST NJJJ TERMS HAVE BEEN CALCULATED.  THIS
C     ALLOWS SUCCESSIVE STEPS TO TAKE A CONSTANT AMOUNT OF EXECUTION
C     TIME, RATHER THAN A CONTINUALLY INCREASING AMOUNT.  FURTHERMORE,
C     THE APPROXIMATION INVOLVED IS WELL WITHIN THE APPROXIMATION THAT
C     IS MADE USING THE INTERVAL OF SOME OR MOST OF THE OTHER SOLUTION
C     POINTS, DUE TO THE LOG SPACING.  IN THE ANALYSES REPORTED IN THE
C     TEXT, NJJJ HAS ALWAYS BEEN TAKEN AS 31, WHICH SEEMS TO BE ADEQUAT-
C     LY LARGE.  N5, N6, AND N4 ARE INTEGERS USED TO PROPERLY SELECT THE
C     POINTS OF THE SOLUTION VECTOR TO BE USED.  THEY ARE TAKEN AS 1,1,
C     AND 4 UNTIL NJJJ SOLUTION POINTS HAVE BEEN OBTAINED.
      N5=1
      N6=1
      N4=4
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C
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C
C     THE LOOP UP TO 3 CALCULATES THE NNN SOLUTIONS  (EXCEPT FOR T=0.)
      DO 3 K=2,NNN
C     IF K IS GREATER THAN NJJJ, THEN INCREMENT N5 AND N4 BY 1, AND
C     PUT A NEGATIVE NUMBER IN N6.
      IF(K-NJJJ)7,7,13
   13 N6=-5
      N5=N5&1
      N4=N4&1
C     THE TIME OF THIS SOLUTION IS STORED IN T2
    7 T2=T(K)
      K1=K-1
C     THE LOOP UP TO 4 CALCULATES THE VALUES OF THE KERNAL FUNCTION
C     (WHICH IS A RESULT OF  MULTIPLE CONVOLUTION INTEGRATIONS AND IS
C     STORED IN THE ARRAY B( , )) NECESSARY FOR THE NEXT SOLUTION.  THEY
C     ARE AT THE TIMES T2-T(L) WHERE L GOES FROM ZERO TO K.  IF K IS
C     GREATER THAN NJJJ, THEN K-NJJJ POINTS ARE SKIPPED.  THESE ARE THE
C     TIMES T2-T(L) CORRESPONDING TO T( ) SMALL, EXCEPT INCLUDING ALWAYS
C     ZERO TIME.  THE VALUE OF L IS SELECTED THUS EQUAL TO LL EXCEPT
C     AT THE FIRST POINT, WHEN IT IS SET EQUAL TO 1 (T=0) AND N6 IS MADE
C     POSITIVE.
      DO 4 LL=N5,K
      L=LL
      IF(N5*N6-1)6,8,8
    6 L=1
      N6=1
    8 BETA(L)=0.
C     THE LOOP TO 5 STORES THE PROPER PRODUCTS OF THE TIME IN THE VECTOR
C     T1( ).
      DO 5 I=2,M
    5 T1(I)=T1(I-1)*(T2-T(L))
C     THE TERM MULTIPLYING EACH EXPONENTIAL TERM IS CALCULATED AND
C     STORED IN SUM, THEN MULTIPLIED BY THE EXPONENTIAL TERM AND ADDED
C     INTO BETA(L).
      DO 18 J=1,N
      SUM=0.
      DO 9 I=1,M
    9 SUM=SUM&B(I,J)*T1(I)
   18 BETA(L)=BETA(L)&SUM*EXP(-DELTA(J)*(T2-T(L)))
    4 CONTINUE
C     FROM HERE TO STATEMENT 21 CALCULATES THE RIGHT-HAND SIDE RESULT
C     FROM THE INPUT ARRAY BB( , ) ANALOGOUS TO THE ABOVE CALCULATIONS
C     FOR THE KERNAL FUNCITON, EXCEPT AT ONLY THE ONE TIME T2, AND
C     STORES THE RESULT IN SUMM.
      DO 23 I=2,MM
   23 T1(I)=T1(I-1)*T2
      SUMM=0.
      DO 21 J=1,N
      SUM=0.
      DO 22 I=1,MM
   22 SUM=SUM+BB(I,J)*T1(I)
   21 SUMM=SUMM+SUM*EXP(-DELTA(J)*T2)
C     THE NUMERATOR OF THE SOLUTION IS NOW CALCULATED AND STORED IN BUM.
C     THE TERMS IN THIS NUMERATOR VARY DEPENDING ON THE SIZE OF K.
      BUM=SUMM-.5*SI(K-1)*(BETA(K-1)-BETA(K))
C
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C
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      C
      C
      C
            IF(K-2)10,10,11
         11 IF(N4-K)15,15,14
         15 DO 12 LL=N4,K
            L=LL
         12 BUM=BUM-.5*(SI(L-2)&SI(L-1))*(BETA(L-2)-BETA(L-1))
         14 BUM=BUM-.5*(SI(1)&SI(N4-2))*(BETA(1)-BETA(N4-2))
      C     THE SOLUTION AT THIS TIME IS CALCULATED AND STORED IN  SI(K)
         10 SI(K)=BUM/(.5*(BETA(K)&BETA(K-1)))
          3 CONTINUE
      C     THE SOLUTION AT ZERO TIME IS STORED IN X(1)
            X(1)=SI(1)
      C     THE SOLUTION CORRESPONDING TO DELTA(J)*T(J)=1. FOR EACH DELTA(J)
      C     IS CALCULATED AND STORED IN X(J) FOR USE IN THE SUBROUTINE CVEFIT.
      C     THIS IS TRUE BECAUSE THE INPUT NJ AND NJJ ARE SELECTED APPRO-
      C     PRIATELY.
            K=NJ
            DO 20 I=2,12
            K=K&NJJ
         20 X(I)=SI(K)
            RETURN
            END
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SUBROUTINE INTEGR (EXACT)

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      C
S.0001        SUBROUTINE INTEGR(G,N,ITEST,ISIB)
      C       THIS SUBROUTINE PERFORMS THE EXACT INTEGRATIONS FOR THE CASE THAT
      C       THE CREEP OR RELAXATION FUNCTIONS CAN BE REPRESENTED BY DIRICHLET
      C       SERIES.  THE INPUT IS THE ARRAY G( , ), AND THE INTEGERS N, ITEST
      C       AND ISIB.  THE ARRAY G( , ) CONTAINS THE RELAXATION FUNCTIONS FOR
      C       THE MULTIPLE CONVOLUTION INTEGRATIONS IN THE FORM OF SERIES.  EACH
      C       COLUMN OF G( , ) CONTAINS THE CONSTANTS FOR ONE  OF THE SERIES.
      C       N IS THE NUMBER OF TERMS IN THE SERIES REPRESENTATIONS.  ITEST IS
      C       THE NUMBER OF MULTIPLE CONVOLUTION INTEGRATIONS INVOLVED.  THE .
      C       MAXIMUM NUMBER FOR THIS PROGRAM IS 6 (THAT IS, THE INTEGRATION OF
      C       7 RELAXATION OR CREEP FUNCTIONS..  ISIB IS A DUMMY WITH THE VALUE
      C       OF EITHER ZERO OR ONE.  IF ISIB=0, THEN THE MULTIPLE CONVOLUTION
      C       INTEGRATIONS ARE TO BE PERFORMED FROM THE BEGINNING.  IF ISIB=1,
      C       THEN THE RESULT OF ITEST-1 INTEGRATIONS IS STORED IN G( , ), AND
      C       THE ONE NEW RELAXATION OR CREEP FUNCTION SERIES IS STORED IN G(8,)
      C       AND IN THIS CASE ONLY ONE INTEGRATION IS PERFORMED.  THE RESULT
      C       FROM THIS PROGRAM, STORED IN THE ARRAY B( , ), IS A FINITE SERIES
      C       OF EXPONENTIALS EACH MULTIPLIED BY A FINITE POLYNOMIAL.  THE CON-
      C       STANTS OF THESE POLYNOMIALS ARE STORED IN THE COLUMNS OF B( , ).
      C       THE DELTA TERMS (THE INVERSE OF THE RELAXATION TIMES) IS INPUT
      C       TO THIS PROGRAM THROUGH STORAGE IN THE VECTOR DELTA( ), WHICH IS
      C       CALCULATED IN THE SUBROUTINE TIME.  THE NOTATION OF THIS PROGRAM
      C       IS DIFFERENT THAN THAT OF THE TEXT.
S.0002        DIMENSION G(8,20),DEL(20),AK(20),AL(20),AM(20),AP(20),AR(20),
             1D(20),C(20),B1(20),C1(20),D2(20),D1(20),E1(20),C2(20),B2(20),
             1E3(20),F3(20),C3(20),B3(20),D3(20),        AS(20),AT(20),F4(20),
             2H4(20),C4(20),B4(20),D4(20),E4(20),H5(20),P5(20)
S.0003        COMMON X(20),BB(8,20),  T(201) ,DELTA(20),BETA(201),P(8,20),
             1SI(201),UI,DELTX,DELXX,NJ,NJJ
S.0004        NN=ITEST&1
      C       THE DELTA( ) TERMS ARE TRANSFERRED TO THE VECTOR DEL( ).
S.0005        DO 250 I=1,N
S.0006    250 DEL(I)=DELTA(I)
      C       ISIG, ISIG1, ISIG2, ISIG3, AND ISIG4 ARE DUMMY VARIABLES USED TO
      C       DETERMINE WHEN THE PROPER NUMBER OF INTEGRATIONS HAVE BEEN PER-
      C       FORMED.
S.0007        ISIG=1
S.0008        ISIG1=1
S.0009        ISIG2=1
S.0010        ISIG3=1
S.0011        ISIG4=1
S.0012        IF(ISIB)200,200,201
S.0013    200 ISIG=0
      C       IF ISIB IS ZERO, THEN ITEST INTEGRATIONS ARE PERFORMED, AND ALL
      C       THE ISIG  S ARE ZEROED UP TO THE LAST ONE.
S.0014        IF(ITEST-1)202,202,203
S.0015    203 ISIG1=0
S.0016        IF(ITEST-2)202,202,204
S.0017    204 ISIG2=0
S.0018        IF(ITEST-3)202,202,205
S.0019    205 ISIG3=0
S.0020        IF(ITEST-4)202,202,206
S.0021    206 ISIG4=0
      C
      C
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      C
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      C
      C
      C     THE SERIES REPRESENTATIONS OF THE FIRST TWO RELAXATION OR CREEP
      C     FUNCTIONS ARE STORED IN THE VECTORS AK( ), AND AL( ).
        202 DO 1 J=1,N
            AK(J)=G(1,J)
          1 AL(J)=G(2,J)
            GO TO 7
      C     IF ONLY ONE CERTAIN ADDITIONAL MULTIPLE CONVOLUTION INTEGRATION
      C     IS TO BE PERFORMED, THEN ONLY A SINGLE ISIG IS ZEROED.
        201 IF(ITEST-2)207,208,209
        207 ISIG=0
            GO TO 215
        208 ISIG1=0
            GO TO 215
        209 IF(ITEST-4)210,211,212
        210 ISIG2=0
            GO TO 215
        211 ISIG3=0
            GO TO 215
        212 IF(ITEST-5)213,213,215
        213 ISIG4=0
      C     IF ONLY ONE CERTAIN ADDITIONAL CONVOLUTION INTEGRATION IS TO BE
      C     PERFORMED, THEN THE RESULT OF THE LAST INTEGRATION MUST BE STORED
      C     IN THE VECTORS AK( ), AM( ), AP( ), AR( ), AS( ), AND AT( ).  SOME
      C     OF THESE MAY NOT BE USED.  THE NEW SERIES IS STORED IN THE VECTOR
      C     AL( ) ALSO.
        215 DO 216 J=1,N
            AK(J)=G(1,J)
            AM(J)=G(2,J)
            AP(J)=G(3,J)
            AR(J)=G(4,J)
            AS(J)=G(5,J)
            AT(J)=G(6,J)
        216 AL(J)=G(8,J)
      C     STATEMENT SEVEN BEGINS THE FIRST INTEGRATION, AND ALSO BEGINS THE
      C     EVALUATION OF THE CONSTANTS RELATED TO A FIRST INTEGRATION FOR THE
      C     LATER INTEGRATIONS (SEE TEXT).
          7 DO 2 J=1,N
      C     THE RESULT OF THE FIRST INTEGRATION WILL BE STORED INTHE VECTORS
      C     C( ), AND D( ).  THE VARIABLES ADUM1 AND ADUM2 ARE USED FOR INTER-
      C     MEDIATE STORAGE.
            D(J)=-DEL(J)*AL(J)*AK(J)
            C(J)=AL(J)*AK(J)
            ADUM1=0.
            ADUM2=0.
            DO 3 I=1,N
            IF(I-J)21,3,21
         21 ADUM1=ADUM1-DEL(J)*AK(I)/(DEL(I)-DEL(J))
            ADUM2=ADUM2&DEL(J)*AL(I)/(DEL(J)-DEL(I))
          3 CONTINUE
          2 C(J)=C(J)&AL(J)*ADUM1&AK(J)*ADUM2
      C     IF ISIG IS EQUAL TO 1, THEN EITHER THIS IS THE SECOND OR MORE TIME
      C     THROUGH THIS PATH OR ELSE A SINGLE INTEGRATION IS TO BE DONE,
      C     WHERE THERE WERE PREVIOUSLY DONE INTEGRATIONS.
            IF(ISIG-1)6,9,9
      C     THE FIRST TIME THROUGH, THE RESULTS OF THE FIRST INTEGRATION ARE
      C     STORED IN AK( ) AND AM( ), AND THE NEXT SERIES IS STORED IN AL( ).
      C
      C
      C
      C
      C
      C
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      C
      C
      C       THE RESULTS ARE ALSO STORED IN THE B( , ) ARRAY.
S.0058      6 DO 5 J=1,N
S.0059        B(1,J)=C(J)
S.0060        B(2,J)=D(J)
S.0061        AK(J)=C(J)
S.0062        AL(J)=G(3,J)
S.0063      5 AM(J)=D(J)
      C       ISIG IS SET EQUAL TO 1 SO THAT THE BRANCH TO SIX WILL NOT BE TAKEN
      C       AGAIN, AND IF MORE INTEGRATIONS ARE TO BE DONE CONTROL RETURNS TO
      C       SEVEN.  IF NO MORE ARE TO BE DONE, THE SUBROUTINE IS ENDED.
S.0064        ISIG=1
S.0065        IF(ITEST-1)7,151,7
      C       CONTROL ENTERS 9 IF IT IS NOT THE END OF THE FIRST INTEGRATION.
      C       THE SECOND INTEGRATION IS CARRIED OUT, AND THE RESULTS STORED IN
      C       THE VECTORS B1( ), D1( ), AND C1( ).  ADUM1, ADUM2,  AND ADUM3 ARE
      C       USED FOR INTERMEDIATE STORAGE.
S.0066      9 DO 8 J=1,N
S.0067        B1(J)=AL(J)*AM(J)
S.0068        D1(J)=-B1(J)*DEL(J)/2.
S.0069        ADUM1=0.
S.0070        ADUM2=0.
S.0071        ADUM3=0.
S.0072        DO 10 I=1,N
S.0073        IF(I-J)22,10,22
S.0074     22 ADUM1=ADUM1&AL(I)*DEL(J)/(DEL(J)-DEL(I))
S.0075        ADUM2=ADUM2-AM(I)*DEL(J)/((DEL(I)-DEL(J))**2)
S.0076        ADUM3=ADUM3&AL(I)*DEL(I)/((DEL(J)-DEL(I))**2)
S.0077     10 CONTINUE
S.0078        B1(J)=B1(J)&AM(J)*ADUM1
S.0079      8 C1(J)=AL(J)*ADUM2&AM(J)*ADUM3
      C       CONTROL BRANCHES TO 11 OR 23 DEPENDING ON WHICH INTEGRATION HAS
      C       BEEN COMPLETED.
      C       THE REMAINDER OF THE PROGRAM FOLLOWS THE SAME TYPE OF LOGIC.  THE
      C       INTEGRATIONS ARE SUCCESSIVELY CARRIED OUT, RETURNING ALWAYS TO
      C       STATEMENT SEVEN IF NOT A SUFFICIENT NUMBER HAVE BEEN EXECUTED.
      C       WHEN THE APPROPRIATE NUMBER HAVE BEEN CALCULATED THEN THE CONTROL
      C       IS SENT TO STATEMENT  151 AND THE PROGRAM IS ENDED.
S.0080        IF(ISIG1-1)11,23,23
S.0081     11 DO 12 J=1,N
S.0082        B(1,J)=C(J)&C1(J)
S.0083        B(2,J)=D(J)&B1(J)
S.0084        B(3,J)=D1(J)
S.0085        AK(J)=C(J)&C1(J)
S.0086        AL(J)=G(4,J)
S.0087        AM(J)=D(J)&B1(J)
S.0088     12 AP(J)=D1(J)
S.0089        ISIG1=1
S.0090        IF(ITEST-2)7,151,7
S.0091     23 DO 13 J=1,N
S.0092        D2(J)=AL(J)*AP(J)
S.0093        E1(J)=-D2(J)*DEL(J)/3.
S.0094        ADUM1=0.
S.0095        ADUM2=0.
S.0096        ADUM3=0.
S.0097        ADUM4=0.
      C
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C
C
      DO 14 I=1,N
      IF(I-J)24,14,24
   24 ADUM=DEL(I)-DEL(J)
      ADUM1=ADUM1-AP(I)*2.*DEL(J)/(ADUM**3)
      ADUM2=ADUM2&AL(I)*(-2.)*DEL(I)/(ADUM**3)
      ADUM3=ADUM3&AL(I)*2.*DEL(I)/(ADUM**2)
      ADUM4=ADUM4&AL(I)*DEL(J)/(-ADUM)
   14 CONTINUE
      C2(J)=AL(J)*ADUM1&AP(J)*ADUM2
      B2(J)=AP(J)*ADUM3
   13 D2(J)=D2(J)&AP(J)*ADUM4
      IF(ISIG2-1)55,26,26
   55 DO 15 J=1,N
      AK(J)=C(J)&C1(J)&C2(J)
      AM(J)=B(J)&B1(J)&B2(J)
      AP(J)=D1(J)&D2(J)
      AR(J)=E1(J)
      AL(J)=G(5,J)
      B(1,J)=AK(J)
      B(2,J)=AM(J)
      B(3,J)=AP(J)
   15 B(4,J)=AR(J)
      ISIG2=1
      IF(ITEST-3)7,151,7
   26 DO 27 J=1,N
      E3(J)=AL(J)*AP(J)
      F3(J)=-E3(J)*DEL(J)/4.
      ADUM1=0.
      ADUM2=0.
      ADUM3=0.
      ADUM4=0.
      ADUM5=0.
      DO 28 I=1,N
      IF(I-J)29,28,29
   29 ADUM=DEL(J)-DEL(I)
      ADUM1=ADUM1-AR(I)*6.*DEL(J)/(ADUM**4)
      ADUM2=ADUM2&AL(I)*6.*DEL(I)/(ADUM**4)
      ADUM3=ADUM3&AL(I)*6.*DEL(I)/(ADUM**3)
      ADUM4=ADUM4&AL(I)*3.*DEL(I)/(ADUM**2)
      ADUM5=ADUM5&AL(I)*DEL(J)/ADUM
   28 CONTINUE
      C3(J)=AL(J)*ADUM1&AR(J)*ADUM2
      B3(J)=AP(J)*ADUM3
      D3(J)=AR(J)*ADUM4
   27 E3(J)=E3(J)&AP(J)*ADUM5
      IF(ISIG3-1)33,31,31
   33 DO 32 J=1,N
      AK(J)=C(J)&C1(J)&C2(J)&C3(J)
      AL(J)=G(6,J)
      AM(J)=D(J)&B1(J)&B2(J)&B3(J)
      AP(J)=D1(J)&D2(J)&D3(J)
      AR(J)=E1(J)&E3(J)
      AS(J)=F3(J)
      B(1,J)=AK(J)
      B(2,J)=AM(J)
C
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      C
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      C
            B(3,J)=AP(J)
            B(4,J)=AP(J)
         32 B(5,J)=AS(J)
            ISIG3=1
            IF(ITEST-4)7,151,7
         31 DO 35 J=1,N
            F4(J)=AL(J)*AS(J)
            H4(J)=-F4(J)*DEL(J)/5.
            ADUM1=0.
            ADUM2=0.
            ADUM3=0.
            ADUM4=0.
            ADUM5=0.
            ADUM6=0.
            DO 38 I=1,N
            IF(I-J)37,38,37
         37 ADUM=DEL(J)-DEL(I)
            ADUM1=ADUM1+AS(I)*24.*DEL(J)/(ADUM**5)
            ADUM2=ADUM2&AL(I)*24.*DEL(I)/(ADUM**5)
            ADUM3=ADUM3&24.*AL(I)*DEL(I)/(ADUM**4)
            ADUM4=ADUM4&12.*AL(I)*DEL(I)/(ADUM**3)
            ADUM5=ADUM5&4.*AL(I)*DEL(I)/(ADUM**2)
            ADUM6=ADUM6&AL(I)*DEL(J)/ADUM
         38 CONTINUE
            C4(J)=AL(J)*ADUM1&AS(J)*ADUM2
            B4(J)=AS(J)*ADUM3
            D4(J)=AS(J)*ADUM4
            E4(J)=AS(J)*ADUM5
         35 F4(J)=F4(J)&AS(J)*ADUM6
            IF(ISIG4-1)220,221,221
        220 DO 39 J=1,N
            AK(J)=C(J)&C1(J)&C2(J)&C3(J)&C4(J)
            AM(J)=D(J)&B1(J)&B2(J)&B3(J)&B4(J)
            AP(J)=D1(J)&D2(J)&D3(J)&D4(J)
            AR(J)=E1(J)&E3(J)&E4(J)
            AS(J)=F3(J)&F4(J)
            AT(J)=H4(J)
            B(1,J)=AK(J)
            B(2,J)=AM(J)
            B(3,J)=AP(J)
            B(4,J)=AR(J)
            B(5,J)=AS(J)
         39 B(6,J)=AT(J)
            ISIG4=1
            IF(ITEST-5)7,151,7
        221 DO 222 J=1,N
            H5(J)=AL(J)*AT(J)
            P5(J)=-H5(J)*DEL(J)/6.
            ADUM1=0.
            ADUM2=0.
            ADUM3=0.
            ADUM4=0.
            ADUM5=0.
            ADUM6=0.
            ADUM7=0.
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            DO 223 I=1,N
            IF(I-J)224,223,224
        224 ADUM=DEL(J)-DEL(I)
            ADUM1=ADUM1-AT(I)*120.*DEL(J)/(ADUM**6)
            ADUM2=ADUM2&AL(I)*120.*DEL(I)/(ADUM**6)
            ADUM3=ADUM3&AL(I)*120.*DEL(I)/(ADUM**5)
            ADUM4=ADUM4&AL(I)*60.*DEL(I)/(ADUM**4)
            ADUM5=ADUM5&AL(I)*20.*DEL(I)/(ADUM**3)
            ADUM6=ADUM6&AL(I)*5.*DEL(I)/(ADUM**2)
            ADUM7=ADUM7&AL(I)*DEL(J)/ADUM
        223 CONTINUE
            B(1,J)=AL(J)*ADUM1&AT(J)*ADUM2&C(J)&C1(J)&C2(J)&C3(J)&C4(J)
            B(2,J)=AT(J)*ADUM3&D(J)&B1(J)&B2(J)&B3(J)&B4(J)
            B(3,J)=AT(J)*ADUM4&D1(J)&D2(J)&D3(J)&D4(J)
            B(4,J)=AT(J)*ADUM5&E1(J)&E3(J)&E4(J)
            B(5,J)=AT(J)*ADUM6&F3(J)&F4(J)
            B(6,J)=H5(J)&AT(J)*ADUM7&H4(J)
        222 B(7,J)=P5(J)
        151 CONTINUE
            RETURN
            END
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BIOGRAPHY

James Edward Ashton was born July 2, 1942, in Davenport, Iowa. He attended Central High School in Davenport and graduated in 1960. He then entered the University of Iowa, College of Civil Engineering. At Iowa he was active in cross-country and received two varsity letters in this sport. He was an instructor for three semesters in "Digital Computer Programming," and was elected into memberships of Tau Beta Pi, Chi Epsilon, Sigma Xi, Omicron Delta Kappa, and Phi Eta Sigma Honorary Fraternities, and was a member of Theta Tau Professional Engineering Fraternity. In June of 1964, he received a B.S.C.E. with Highest Distinction.

In September of 1964, the author entered the Massachusetts Institute of Technology, Civil Engineering Department, on a National Science Foundation Fellowship. He received his S.M. in June, 1965, after writing a thesis entitled: Deflection Curve for an Infinite Plate on a Non-Linear Elastic (Winkler) Base. He expects his Ph.D. in January, 1967, with a combined major field of materials and structural mechanics.

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