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NONLINEAR SYSTEMS WITH GAUSSIAN INPUTS

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NONLINEAR SYSTEMS WITH GAUSSIAN INPUTS

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Abstract

Methods are developed for optimizing a general multi-input, single-output, nonlinear system whose inputs are Gaussian processes. The output of the nonlinear system is expressed as a sum of orthogonal functional polynomials of the inputs. The only statistical information needed for this optimization is composed of the first-order autocorrelations and crosscorrelations among the inputs, and the higher-order crosscorrelations between the inputs and the desired output.

Methods are developed also for optimizing simple single-input nonlinear systems whose input is Gaussian. The systems consist of combinations of linear systems and nonlinear no-memory devices. The systems have a fixed form with some undetermined parameters. The system is optimized by making an optimum choice of the values of these parameters. Methods are presented for determining these optimum values by measurements.

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I. INTRODUCTION

1.1 NONLINEAR SYSTEMS

A multi-input, single-output, nonlinear system, as its name implies, has several inputs and one output. Unless it is otherwise specified, the nonlinear systems discussed in this report are of this type. A block diagram of such a system is shown in Fig. 1.

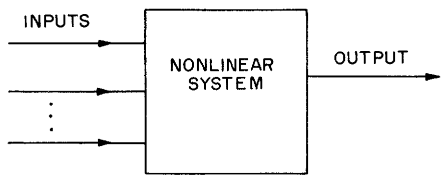

Fig. 1. Multi-input, single-output nonlinear system.

A nonlinear system operates on its inputs to give the present value of its output. The present value of the output is thus a nonlinear function of the inputs. If the present value of the output is a function of only the past and present values of the inputs, the system is called realizable; if the present value of the output is also a function of future values of the inputs, then the system is called unrealizable.

1.2 OPTIMUM NONLINEAR SYSTEMS

The purpose of this report is to develop a theory for determining optimum nonlinear systems of both the realizable and unrealizable types. An optimum nonlinear system is one that operates on a given set of inputs in such a manner that its output is as close as possible, in the mean-square sense, to a desired output. The inputs considered are all stationary, zero-mean, Gaussian, random variables. The design of the optimum nonlinear system will depend on the statistical relationships among the given inputs and the desired output.

As well as optimizing general nonlinear systems, we shall consider the problem of optimizing nonlinear systems of restricted forms. The restricted forms are chosen because the systems can be optimized and constructed simply.

1.3 AN APPLICATION FOR OPTIMUM NONLINEAR SYSTEMS

An application for an optimum nonlinear system might be found in weather forecasting. Suppose we wish to forecast some quantitative aspect of the weather, say, the visibility at an airport 24 hours hence. Suppose also that we are given related quantitative weather information, such as the past values of temperature, humidity, and wind velocity in neighboring towns.

The desired output is the future visibility at the airport; the actual output of the optimum nonlinear system is the prediction of that visibility; the inputs are the temperature, humidity, and wind velocities in neighboring towns. The statistical information needed for designing the optimum nonlinear system might (in more quantitative form) be that the visibility usually is poor after a temperature drop and an east wind. Because only past values of the inputs are used, the resulting nonlinear system will be realizable. In this example, the inputs, although random, would not be Gaussian.

1.4 GAUSSIAN INPUTS

The reason for restricting the random inputs to be of a specific type is that different optimization procedures will, in general, be necessary for different types of input. The reason for choosing the specific type of random input to be Gaussian is twofold. First, Gaussian processes are very common, because the sum of many random processes tends often to be Gaussian. Second, and perhaps more important, the Gaussian probability distribution makes the mathematics simple enough so that the optimization equations can be solved analytically.

1.5 CHARACTERIZATION OF NONLINEAR SYSTEMS

In order to optimize a system, it is necessary, first, to express the functional relationship between the inputs and the output of the system. Undetermined parameters in the functional relationship are then chosen to optimize the system.

For a multi-input, single-output, linear system the set of impulse responses, each of which corresponds to the response to an impulse at a different input terminal, completely characterizes the functional dependence of the output on the inputs. The present value of the output of a linear system is the sum of the convolutions of each of its inputs with the corresponding impulse response. It is the set of impulse responses that is chosen to optimize the linear system.

The functional relationships between the inputs and the output of a multi-input, single-output, nonlinear system cannot be characterized as simply as can that of the less general linear system. We shall present a characterization for a nonlinear system. This characterization has properties that make it particularly convenient in the optimization procedures.

1.6 BACKGROUND OF THIS RESEARCH

Two books by Wiener, one on linear systems (1) and one on nonlinear systems (2), give important background material for this study. In his book on linear systems Wiener solves the problem of optimizing a multi-input realizable linear system. The inputs are random but not necessarily Gaussian. The solution is not simple because it requires spectrum factorization and the solution of sets of linear equations. However, in this report use will be made of the fact that such a solution does exist.

In his book on nonlinear systems Wiener treats the problem of characterizing an arbitrary realizable single-input nonlinear system whose input is white Gaussian noise. That is, Wiener considers the problem in which both the output and the white Gaussian input of an arbitrary, realizable, single-input nonlinear system are given, and from this input and output the functional dependence of the output on the input is to be determined. Wiener's solution to this characterization problem is closely related to the optimization problem in which a desired output and a white Gaussian input are given, and we wish to determine a realizable single-input nonlinear system whose input is the given Gaussian input and whose output is as close as possible, in the mean-square sense, to the desired output. If we attempt to characterize, in Wiener's manner, an imaginary nonlinear system whose input is the given input and whose output is the desired output, then the resulting characterization will not represent the imaginary nonlinear system, but will represent the optimum realizable nonlinear system. Therefore, if the input is white Gaussian noise, Wiener's characterization method can be used for obtaining the optimum realizable single-input nonlinear system.

Barrett (3) and Zadeh (4) have considered the problem of optimizing a single-input nonlinear system for non-Gaussian inputs. In general, their optimization equations cannot be solved analytically, although machine computations could give solutions.

Bose (5) has used a novel expansion for a nonlinear system. Systems represented by his "gate-function" expansion can be optimized by simple measurements for any type of random input. His expansion and optimization procedure can be used for multi-input nonlinear systems.

II. CHARACTERIZATION OF A MULTI-INPUT NONLINEAR SYSTEM

2.1 INTRODUCTION

In this section we shall present several ways in which the relationship between the output and the inputs of a multi-input nonlinear system may be expressed. We shall start with one method that leads directly to a physical model for a nonlinear system. By a modification of this first method of characterization, we obtain a characterization in terms of functionals of the Volterra type. A brief history of the development and application of this functional representation will be given. Finally, we shall present both the method of characterization that will be used in the rest of this report and the important properties of this characterization.

2.2 CHARACTERIZATION OF THE PAST OF INPUTS

The output of a realizable multi-input nonlinear system depends in a nonlinear manner, on the past of its inputs. In order to describe how the output depends on the past of the inputs, we would like to be able to characterize the past of the inputs by some means that is more convenient, say, than a graph of each input over past time.

The past of the inputs can be characterized by an infinite set of coefficients in the following manner. We can expand the past of each input in terms of a complete orthonormal set of functions such as Laguerre functions. That is, if $\{h_j(T)\}$ is a complete set of functions with the property that

$$\int_0^{\infty} h_j(T)\, h_k(T)\, dT = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \tag{1}$$

then at any time t the past of each input $x_i(t-T)$ for $T \geq 0$ can be represented as

$$x_i(t-T) = \sum_{j=1}^{\infty} u_{i,j}(t)\, h_j(T) \qquad T \geq 0 \tag{2}$$

The coefficient $u_{i,j}(t)$ is the coefficient of the j^{th} Laguerre function (if $\{h_j(T)\}$ is a set of Laguerre functions) in the expansion of the past of the i^{th} input at time t. If there are N inputs $x_1, \dots x_N$, then at time t the infinite set of coefficients

$$\{u_{i,j}(t)\} \qquad \begin{matrix} i = 1, \dots N \\ j = 1, \dots \end{matrix}$$

is said to characterize the past of inputs because, by means of Eq. 2, we can express the past of each input in terms of the coefficients.

By multiplying both sides of Eq. 2 by $h_k(T)$, integrating on T, and then applying the orthonormality property of Eq. 1, we see that the Laguerre coefficient $u_{i,j}(t)$ is given by

$$u_{i,j}(t) = \int_0^\infty h_j(T)\, x_i(t-T)\, dT \tag{3}$$

2.3 CHARACTERIZATION OF A NONLINEAR SYSTEM

Since the output w(t) of a multi-input nonlinear system, as shown in Fig. 2, with inputs $\{x_1(t), \dots x_N(t)\}$ depends on the past of the inputs and, since at time t the past of the inputs can be characterized by the infinite set of coefficients $\{u_{i,j}(t)\}$, the output can be written as a nonlinear function of the coefficients $\{u_{i,j}(t)\}$; that is,

$$w(t) = F[u_{1,1}(t), u_{1,2}(t), \dots u_{2,1}(t), \dots] \tag{4}$$

The function F has no memory; that is, the present value of the output depends only on the present value of the coefficients $\{u_{i,j}(t)\}$, since all the information about the past of the inputs is contained in the present value of the coefficients.

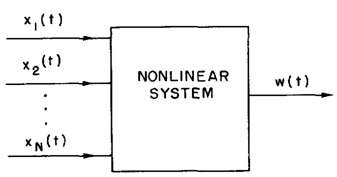

Fig. 2. Multi-input nonlinear system.

The function F characterizes the nonlinear system, for it tells how the output is produced from the inputs. It is only the nonlinear no-memory function F that is different for different nonlinear systems. Thus the function F is the parameter that is varied to optimize the nonlinear system.

One form in which the nonlinear no-memory function F could be expressed is a power series. By using this power series for F, Eq. 4 becomes

$$\begin{aligned} w(t) &= a + bu_{1,1}(t) + cu_{1,2}(t) + \dots + du_{2,1}(t) + \dots \\ &\quad + eu_{1,1}^2(t) + \dots + fu_{1,1}(t)\, u_{2,1}(t) + \dots \\ &\quad + gu_{1,3}(t)\, u_{2,4}(t)\, u_{3,5}(t) + \dots \end{aligned} \tag{5}$$

The constants (a, b, c, ...) depend upon the nonlinear no-memory function F. Thus the constants (a, b, c, ...) are the parameters that are different for different nonlinear systems and are the parameters that are varied to optimize the nonlinear system. The constants (a, b, c, ...) characterize the nonlinear system, for they are sufficient to describe how the output is formed from the inputs.

Both this method of characterizing a nonlinear system and the physical model of a nonlinear system given in the next section are due to Wiener (6) and have been discussed by Bose (5).

2.4 A PHYSICAL MODEL FOR A GENERAL MULTI-INPUT NONLINEAR SYSTEM

The previous method for characterizing a multi-input nonlinear system suggests the physical model of Fig. 3 for a general multi-input nonlinear system. In the model of

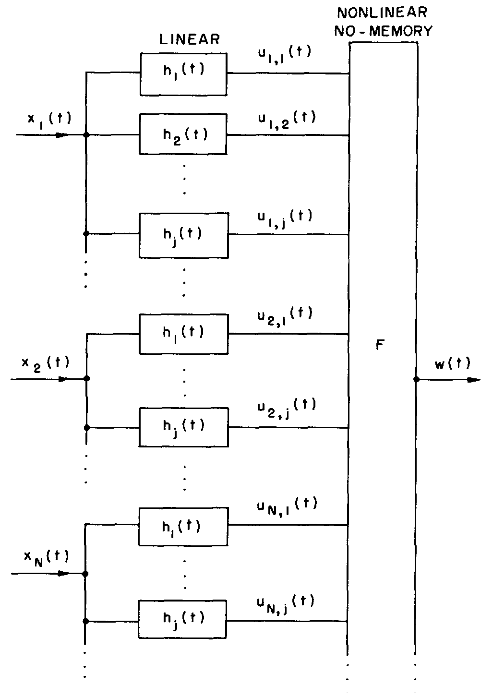

Fig. 3. Model for a general multi-input nonlinear system.

Fig. 3 each of the inputs $\{x_1(t), \ldots x_N(t)\}$ is passed through an infinite set of linear systems; each of their impulse responses is a different Laguerre function. The outputs of all these linear systems are the inputs to an infinite-input, single-output, nonlinear, no-memory device F. The output of the nonlinear system is w(t).

Since the output of a linear system is the convolution of the input with the impulse response of the linear system, the output at time t of the linear system in Fig. 3, whose impulse response is the j^{th} Laguerre function $h_j(t)$ and whose input is x_i, is, from Eq. 3, $u_{i,j}(t)$. Thus the present outputs of the linear systems in Fig. 3 are the coefficients $\{u_{i,j}(t)\}$ that characterize the past of the inputs. The box F performs the nonlinear no-memory operation on the coefficients $\{u_{i,j}(t)\}$ and characterizes the nonlinear system.

If a power series is used as in Eq. 5, then the box F performs only multiplication and addition of the coefficients $\{u_{i,j}(t)\}$. The constants (gains) in the multiplication and addition are the parameters that characterize the nonlinear system. Different nonlinear systems are all represented in the form shown in Fig. 3; these systems differ only in the gains in the power-series box F.

2.5 ANOTHER CHARACTERIZATION OF A NONLINEAR SYSTEM

We can obtain another method for characterizing a nonlinear system by grouping together some of the terms in Eq. 5. We shall group together those terms that are composed of one u, and then group together those terms that are products of two u's, and, in general, group together those terms that are products of n u's for each n.

Let us first consider the sum of terms in the right-hand side of Eq. 5 that consist of a single u multiplied by a constant, such as

$$au_{1,1}(t) + bu_{1,2}(t) + \dots cu_{2,1}(t) + \dots \tag{6}$$

From either Eq. 3 or Fig. 3, it is seen that each u is the output of a linear operation on an input. The sum of the outputs of linear operations on the same input is equivalent to the output of a single linear operation on that same input. This single linear operation, of course, depends upon the many linear operations with which it is equivalent. It also follows that the sum of the outputs of many linear operations on N inputs is equivalent to the output of N linear operations; one on each of the N inputs. Thus expression 6 is equivalent to

$$\sum_{i=1}^{N} \int_0^\infty K_{1,i}(T)\, x_i(t-T)\, dT \tag{7}$$

for some set of N kernels $\{K_{1,i}(T)\}$.

As an example, let us show that the expression

$$au_{1,1}(t) + bu_{1,2}(t) + cu_{2,1}(t)$$

can be written in the form of expression 7. Expanding each u in the form of Eq. 3, and rearranging terms, we obtain the desired form

$$\begin{aligned} au_{1,1}(t) + bu_{1,2}(t) + cu_{2,1}(t) &= a\int_0^\infty h_1(T)\, x_1(t-T)\, dT \\ &\quad + b\int_0^\infty h_2(T)\, x_1(t-T)\, dT + c\int_0^\infty h_1(T)\, x_2(t-T)\, dT \\ &= \int_0^\infty [ah_1(T)+bh_2(T)]\, x_1(t-T)\, dT + \int_0^\infty ch_1(T)\, x_2(t-T)\, dT \\ &= \int_0^\infty K_{1,1}(T)\, x_1(t-T)\, dT + \int_0^\infty K_{1,2}(T)\, x_2(t-T)\, dT \end{aligned} \tag{8}$$

where $K_{1,1}(T) = ah_1(T) + bh_2(T)$, and $K_{1,2}(T) = ch_1(T)$.

Let us now consider the sum of terms in the right-hand side of Eq. 5 that are composed of the product of two u's times a constant, such as

$$au_{1,1}^2(t) + bu_{1,1}(t)\, u_{1,2}(t) + \dots + cu_{1,1}(t)\, u_{2,1}(t) + \dots + du_{2,1}(t)\, u_{2,2}(t) + \dots \tag{9}$$

In considering the linear terms of expression 6, we divided the expression up into a single linear operation on each input (expression 7). By expressing each of the u's in expression 9 in the form of Eq. 3, it can be shown that expression 9 can be written as the sum of quadratic operations on all pairs of the inputs.

$$\sum_{i}\int_0^{\infty}\int_0^{\infty} K_{2,i}(T_1, T_2)\, x_{i_1}(t-T_1)\, x_{i_2}(t-T_2)\, dT_1 dT_2 \tag{10}$$

The sum on i is such that all pairs of inputs are considered once. That is, there will be a term in expression 10 that involves $x_1(t-T_1)\, x_1(t-T_2)$, and there will be another term that involves $x_1(t-T_1)\, x_2(t-T_2)$, and one term for each of the other pairs of inputs. The second-order kernels $\{K_{2,i}(T_1, T_2)\}$ will depend upon the constants (a, b, c, . . .).

As an example, let us show that the expression

$$au_{1,1}^2(t) + bu_{1,1}(t)\, u_{1,2}(t) + cu_{1,1}(t)\, u_{2,1}(t) \tag{11}$$

can be written in the form of expression 10. By expanding each u in the form of Eq. 3, expression 11 becomes

$$\begin{aligned} &au_{1,1}^2(t) + bu_{1,1}(t)\, u_{1,2}(t) + cu_{1,1}(t)\, u_{2,1}(t) \\ &= a\int_0^{\infty} h_1(T_1)\, x_1(t-T_1) \int_0^{\infty} h_1(T_2)\, x_1(t-T_2)\, dT_2 \\ &+ b\int_0^{\infty} h_1(T_1)\, x_1(t-T_1) \int_0^{\infty} h_2(T_2)\, x_1(t-T_2)\, dT_2 \\ &+ c\int_0^{\infty} h_1(T_1)\, x_1(t-T_1) \int_0^{\infty} h_1(T_1)\, x_2(t-T_2)\, dT_2 \end{aligned}$$

Writing these terms as double integrals and combining terms involving the product of the same pair of x's, we obtain the desired form:

$$\begin{aligned} &au_{1,1}^2(t) + bu_{1,1}(t)\, u_{1,2}(t) + cu_{1,1}(t)\, u_{1,2}(t) \\ &= \int_0^{\infty}\int_0^{\infty} [ah_1(T_1)h_1(T_2)+bh_1(T_1)h_2(T_2)]\, x_1(t-T_1)\, x_1(t-T_2)\, dT_1 dT_2 \\ &+ \int_0^{\infty}\int_0^{\infty} ch_1(T_1)\, h_1(T_2)\, x_1(t-T_1)\, x_2(t-T_2)\, dT_1 dT_2 \\ &= \int_0^{\infty}\int_0^{\infty} K_{2,1}(T_1, T_2)\, x_1(t-T_1)\, x_2(t-T_2)\, dT_1 dT_2 \\ &+ \int_0^{\infty}\int_0^{\infty} K_{2,2}(T_1, T_2)\, x_1(t-T_1)\, x_2(t-T_2)\, dT_1 dT_2 \end{aligned} \tag{12}$$

where

$$K_{2,1}(T_1, T_2) = ah_1(T_1)\, h_1(T_2) + bh_1(T_1)\, h_2(T_2) \tag{13}$$

and

$$K_{2,2}(T_1, T_2) = ch_1(T_1)\, h_1(T_2) \tag{14}$$

In general, the second-order kernels $K_{2,i}$ can be expressed as an infinite sum of terms of the form of the right-hand side of Eqs. 13 and 14.

By using the same methods, we could express the sum of terms composed of the product of n u's as

$$\sum_i \int_0^\infty \cdots \int_0^\infty K_{n,i}(T_1, \ldots T_n)\, x_{i_1}(t-T_1) \ldots x_{i_n}(t-T_n)\, dT_1 \ldots dT_n \tag{15}$$

The summation on i is such that all different products of n x's are considered. The n^{th}-order kernels $\{K_{n,i}(T_1, \ldots T_n)\}$ depend upon the constants (a, b, c, ...) in Eq. 5.

Since the output w(t) (Eq. 5) can be expressed as a sum over n of sums of terms composed of the product of n u's, and since these sums of products of n u's can be expressed in the form of expression 15, the output w(t) can be written as

$$w(t) = \sum_{n=0}^{\infty} \sum_i \int_0^\infty \cdots \int_0^\infty K_{n,i}(T_1, \ldots T_n)\, x_{i_1}(t-T_1) \ldots x_{i_n}(t-T_n)\, dT_1 \ldots dT_n \tag{16}$$

Here, the n = 0 term is a constant.

In expression 16, the infinite set of kernels $\{K_{n,i}(T_1, \ldots T_n)\}$ characterizes the nonlinear system in the same manner as the constants (a, b, c, ...) characterize the nonlinear system in Eq. 5.

2.6 FUNCTIONALS

The terms that we have used in the right-hand side of Eq. 16 are of a well-known mathematical form and are called functionals. A functional is a number that depends upon a function. For example, the present output r(t) of a linear system with impulse response K(t) and input x(t) is expressed as

$$r(t) = \int_0^\infty K(T)\, x(t-T)\, dT \tag{17}$$

and is a functional of the input. The present value of r(t) is a number depending upon the past of the input, and is a function. A functional may be contrasted with a function that is a number depending on a finite number of variables. An example of a function of an input x(t) is

$$g(t) = ax(t-T_1) + bx(t-T_2)$$

The present value of the function g(t) is a number that depends on only two past values of the input.

A functional may also be contrasted with an operator that transforms a function (the past of the inputs) into a function (the output of the system as a function of time).

A term of the form

$$\int_0^{\infty} \cdots \int_0^{\infty} K_n(T_1, \dots T_n)\, x_{i_1}(t-T_1) \dots x_{i_n}(t-T_n)\, dT_1 \dots dT_n \tag{18}$$

is called a homogeneous functional of degree n. Expression 18 is indeed a functional because its present value (a number) depends on the past of the inputs. It is homogeneous of degree n because it has the property that if each input is multiplied by the same constant A, then the functional is multiplied by A^n.

A finite sum of homogeneous functionals whose highest degree is n is called a functional polynomial of degree n. An example of a functional polynomial of degree 2 is

$$\int_0^{\infty} \int_0^{\infty} K_{2,3}(T_1, T_2)\, x_1(t-T_1)\, x_2(t-T_2)\, dT_1 dT_2 + \int_0^{\infty} K_{1,2}(T)\, x_2(t-T)\, dT \tag{19}$$

If the inputs are random, then both the homogeneous functionals and the functional polynomials are random variables because they both depend upon the random inputs. A system of functional polynomials in which functional polynomials of different degree are linearly independent (that is, the average value of their product is zero) is called a system of orthogonal functional polynomials. Such a system has this orthogonality (linear independence) for only certain input statistics.

In Eq. 16 we represented the output of a nonlinear system as an infinite sum of homogeneous functional polynomials. We shall later represent the output of a nonlinear system by an infinite sum of orthogonal functional polynomials. The orthogonality of the functional polynomials simplifies the optimization procedures.

2.7 BACKGROUND OF THE DEVELOPMENT AND APPLICATION OF FUNCTIONALS

As early as 1900, Volterra (7) discussed homogeneous functionals. In 1942, Wiener (8) used homogeneous functionals to describe the output of a nonlinear electrical circuit whose input was white Gaussian noise. In 1947, Cameron and Martin (9) developed a system of orthogonal functional polynomials for a single white Gaussian input.

Wiener (2, 6) has used orthogonal functional polynomials to characterize single-input nonlinear systems with a white Gaussian input.

Barrett (3) has given an excellent discussion on the application of functionals in the study of nonlinear systems. He presents the equations that must be solved in order to form a system of orthogonal functional polynomials for a single non-Gaussian input. Barrett's orthogonal functional polynomials for a single Gaussian input use the N-dimensional Hermite polynomials of Grad (9). For a single white Gaussian input, these orthogonal functional polynomials of Barrett reduce to those of Wiener (6).

Brilliant (10) and George (11) have discussed some of the convergence and cascading properties of the functional power-series representation for a single-input nonlinear system.

2.8 A SYSTEM OF ORTHOGONAL FUNCTIONAL POLYNOMIALS

We shall now define a system of orthogonal functional polynomials of a set of jointly Gaussian inputs $\{x_1(t), \ldots x_N(t)\}$. In general, different functional polynomials are required for different input statistics. The reason for introducing these functional polynomials is that in the rest of this report an infinite sum of these functional polynomials will be used to represent the output of a multi-input nonlinear system whose inputs are $\{x_1(t), \ldots x_N(t)\}$.

These functional polynomials are the same ones that Barrett (3) uses with a single Gaussian input. We shall use them with N Gaussian inputs and prove that they are indeed orthogonal.

Unfortunately, in order to describe these functional polynomials, a fair amount of notation has to be introduced. A functional polynomial of degree n is denoted by $G_n(t, K_{n,i}, \{x\}_i)$ and is called a G-functional. This functional polynomial is defined as

$$G_n(t, K_{n,i}, \{x\}_i) = \int \cdots \int K_{n,i}(T_1, \ldots T_n) \sum_{\nu=0}^{\left[\frac{n}{2}\right]} (-1)^\nu P_\nu\left[x_{i_1}(t-T_1), \ldots x_{i_n}(t-T_n)\right] dT_1 \ldots dT_n \tag{20}$$

The expression $\left[\frac{n}{2}\right]$ equals $\frac{n}{2}$ for n even and equals $\frac{n-1}{2}$ for n odd. Unless otherwise stated, the limits of integration are from minus infinity to infinity.

Implicit in the definition of $G_n(t, K_{n,i}, \{x\}_i)$ is a table that associates with each $\{x\}_i$ a corresponding set of n inputs $\{x_{i_1}, x_{i_2}, \ldots x_{i_n}\}$. Some of these n inputs may be the same ones.

The kernel $K_{n,i}(T_1, \ldots T_n)$ is arbitrary; it will be different for different nonlinear systems. The first subscript, n, of the kernel indicates how many T variables the kernel has; the second subscript, i, indicates which set of x's is used with the kernel.

The expression $P_\nu\left[x_{i_1}(t-T_1), \ldots x_{i_n}(t-T_n)\right]$ requires a definition of its own. The expression $P_\nu\left[x_{i_1}(t-T_1), \ldots x_{i_n}(t-T_n)\right]$ is a sum of terms. One such term is formed in three steps: (a) ν pairs of x's are formed without replacement from the set of x's $\{x_{i_1}(t-T_1), \ldots x_{i_n}(t-T_n)\}$. (Notice that there are many different ways of choosing these pairs.) (b) The statistical average of each pair is formed. (c) The product of the unpaired x's is multiplied by the product of the ν averaged pairs to form one of the terms of P_ν. To form another of the terms of P_ν, the same three steps are followed, except that a different set of pairs of x's is chosen. The sum of the terms formed by all possible different sets of pairs defines P_ν.

To give an example of a P_ν expression,

$$P_1[x_1(t-T_1), x_1(t-T_2), x_2(t-T_3)] = x_1(t-T_1)\,\overline{x_1(t-T_2)\,x_2(t-T_3)} + x_1(t-T_2)\,\overline{x_1(t-T_1)\,x_2(t-T_3)} + x_2(t-T_3)\,\overline{x_1(t-T_1)\,x_1(t-T_2)} \tag{21}$$

The bar indicates ensemble average. Since the input x processes are assumed to be stationary, we can define a correlation function as

$$R_{x_i x_j}(T) = \overline{x_i(t)\, x_j(t+T)} \tag{22}$$

By using this definition of correlation function, Eq. 21 becomes

$$P_1[x_1(t-T_1), x_1(t-T_2), x_2(t-T_3)] = x_1(t-T_1)\, R_{x_1x_2}(T_2-T_3) + x_1(t-T_2)\, R_{x_1x_2}(T_1-T_3) + x_2(t-T_3)\, R_{x_1x_1}(T_1-T_2) \tag{23}$$

As another example, the term $P_o[x_1(t-T_1), x_1(t-T_2), x_2(t-T_3)]$ involves no pairing because $\nu = 0$. Hence, it is given by

$$P_o[x_1(t-T_1), x_1(t-T_2), x_2(t-T_3)] = x_1(t-T_1)\, x_1(t-T_2)\, x_2(t-T_3) \tag{24}$$

As an example of a G-functional, we shall expand $G_3(t, K_{3,i}, \{x\}_i)$. We assume that we are given the fact that the i^{th} set of inputs is

$$\{x\}_i = \left(x_{i_1}, x_{i_2}, x_{i_3}\right) = (x_1, x_1, x_2) \tag{25}$$

From Eqs. 20 and 25, we obtain

$$G_3(t, K_{3,i}, \{x\}_i) = \iiint K_{3,i}(T_1, T_2, T_3) \sum_{\nu=0}^{1} (-1)\, P_\nu[x_1(t-T_1), x_1(t-T_2), x_2(t-T_3)]\, dT_1 dT_2 dT_3 \tag{26}$$

By substituting Eqs. 23 and 24 in Eq. 26, we obtain the desired expansion

$$G_3(t, K_{3,i}, \{x\}_i) = \iiint K_{3,i}(T_1, T_2, T_3) \Big[x_1(t-T_1) x_1(t-T_2) x_2(T-T_3) - x_1(t-T_1) R_{x_1x_2}(T_2-T_3) - x_1(t-T_2) R_{x_1x_2}(T_1-T_3) - x_2(t-T_3) R_{x_1x_1}(T_1-T_2)\Big]\, dT_1 dT_2 dT_3 \tag{27}$$

Notice that a G-functional depends upon the autocorrelation and crosscorrelation functions of the inputs.

The fact that $G_3(t, K_{3,i}, \{x\}_i)$ as given by Eq. 27 is indeed a functional polynomial of degree 3 may be seen by integrating on those T's that appear inside the correlation functions and by defining three new kernels:

$$K_{3,i}^{(1)}(T_1) = \iint K_{3,i}(T_1, T_2, T_3)\, R_{x_1x_2}(T_2-T_3)\, dT_2 dT_3 \tag{28}$$

$$K_{3,i}^{(2)}(T_2) = \iint K_{3,i}(T_1, T_2, T_3)\, R_{x_1x_2}(T_1-T_3)\, dT_1 dT_3 \tag{29}$$

$$K_{3,i}^{(3)}(T_3) = \iint K_{3,i}(T_1, T_2, T_3)\, R_{x_1x_1}(T_1-T_2)\, dT_1 dT_2 \tag{30}$$

Substituting Eqs. 28, 29, and 30 in Eq. 27, we obtain

$$G_3(t, K_{3,i}, \{x\}_i) = \iiint K_{3,i}(T_1, T_2, T_3)\, x_1(t-T_1)\, x_1(t-T_2)\, x_2(t-T_3)\, dT_1 dT_2 dT_3 - \int K_{3,i}^{(1)}(T)\, x_1(t-T)\, dT - \int K_{3,i}^{(2)}(T)\, x_1(t-T)\, dT$$
$$- \int K_{3,i}^{(3)}(T)\, x_2(t-T)\, dT \tag{31}$$

The right-hand side of Eq. 31 is recognized as a functional polynomial of degree 3, since it consists of a homogeneous functional of degree 3 plus three homogeneous functionals of degree 1.

The functional polynomial, $G_n(t, K_{n,i}, \{x\}_i)$, is, as the notation indicates, a function of time, a function of the set of inputs $\{x\}_i$, and a function of the kernel $K_{n,i}$. It also depends upon the autocorrelation and crosscorrelation functions of the inputs. We shall usually represent the output w(t) of a multi-input nonlinear system (Fig. 2) as an infinite sum of G-functionals of its inputs $\{x_1, \ldots x_N\}$, as follows:

$$w(t) = \sum_{n=0}^{\infty} \sum_{i} G_n(t, K_{n,i}, \{x\}_i) \tag{32}$$

The summation is made over all sets of the inputs (the summation on i), as well as over all degrees of polynomials (the summation on n).

It is the infinite set of kernels $\{K_{n,i}\}$ in Eq. 32 which is different for different nonlinear systems. The set of kernels $\{K_{n,i}\}$ is said to characterize the nonlinear system because by means of Eq. 32 we can express the output of a nonlinear system in terms of these kernels and the inputs. It is the kernels that are to be determined in optimizing a nonlinear system.

2.9 PROPERTIES OF G-FUNCTIONALS

We shall now present some important properties of G-functionals of Gaussian random processes. The proofs of these properties will be presented in section 2.10. The G-functionals were defined by Eq. 20.

The first important property is that G-functionals of different degree are orthogonal, that is, linearly independent. Therefore,

$$\overline{G_n(t, K_{n,i}, \{x\}_i)\, G_m(t, L_{m,j}, \{x\}_j)} = 0 \qquad n \neq m \tag{33}$$

The sets of random processes $\{x\}_i$ and $\{x\}_j$ together are jointly Gaussian. The kernels $K_{n,i}$ and $L_{m,j}$ are arbitrary. The bar indicates ensemble average.

The second important property of G-functionals concerns the average of the product of two G-functionals of the same degree. This average is given by

$$\overline{G_n(t, K_{n,i}, \{x\}_i)\ G_n(t, L_{n,j}, \{x\}_j)} = \int \dots \int K_{n,i}(T_1, \dots T_n)\ L_{n,j}(s_1, \dots s_n)$$

$$\times Q_n\left[x_{i_1}(t-T_1), \dots x_{i_n}(t-T_n); x_{j_1}(t-s_1), \dots x_{j_n}(t-s_n)\right]$$

$$\times dT_1 \dots dT_n\ ds_1 \dots ds_n \tag{34}$$

The expression $Q_n\left[x_{i_1}(t-T_1), \dots x_{i_n}(t-T_n); x_{j_1}(t-s_1), \dots x_{j_n}(t-s_n)\right]$ needs to be defined. Notice that Q_n is a function of two sets of variables, $\{x\}_i$ and $\{x\}_j$. The expression $Q_n\left[x_{i_1}(t-T_1), \dots x_{i_n}(t-T_n); x_{j_1}(t-s_1), \dots x_{j_n}(t-s_n)\right]$ is a sum of terms. To form one such term, we pair each member of the set $\{x\}_i$ with a member of the set $\{x\}_j$ and take the ensemble average of each pair. (This pairing can be done in many different ways.) The product of the n averaged pairs forms one term of Q_n. Each of the other terms is formed in the same manner, except that a different pairing arrangement is chosen. The sum defining Q_n is composed of terms, each of which requires a different pairing arrangement. Since there are n! different ways of pairing n things with n other things, the expression Q_n is the sum of n! terms.

As an example of a Q_n expression, let us take

$$Q_2[x_1(t-T_1), x_2(t-T_2); x_3(t-s_1), x_4(t-s_2)]$$

$$= \overline{x_1(t-T_1)\ x_3(t-s_1)}\ \overline{x_2(t-T_2)\ x_4(t-s_2)} + \overline{x_1(t-T_1)\ x_4(t-s_2)}\ \overline{x_2(t-T_2)\ x_3(t-s_1)} \tag{35}$$

Since the x processes are assumed stationary, we can use the correlation functions given in Eq. 22. With these correlation functions, Eq. 35 becomes

$$Q_2[x_1(t-T_1), x_2(t-T_2); x_3(t-s_1), x_4(t-s_2)]$$

$$= R_{x_1x_3}(T_1-s_1)\ R_{x_2x_4}(T_2-s_2) + R_{x_1x_4}(T_1-s_2)\ R_{x_2x_3}(T_2-s_1) \tag{36}$$

Notice that Q_2 is the sum of 2! terms.

As an illustration of the application of Eq. 34, we shall evaluate

$$\overline{G_2(t, K_{2,i}, \{x\}_i)\ G_2(t, L_{2,j}, \{x\}_j)} \tag{37}$$

In this example we shall assume that the jointly Gaussian set of processes $\{x\}_i$ and $\{x\}_j$ are given by

$$\{x\}_i = \left(x_{i_1}, x_{i_2}\right) = (x_1, x_2) \tag{38}$$

and

$$\{x\}_j = \left(x_{j_1}, x_{j_2}\right) = (x_3, x_4) \tag{39}$$

Substituting Eqs. 37, 38, and 39 in Eq. 34, we obtain

$$\overline{G_2(t, K_{2,i}, \{x\}_i)\, G_2(t, L_{2,j}, \{x\}_j)}$$
$$= \iiiint K_{2,i}(T_1, T_2)\, L_{2,j}(s_1, s_2)\, Q_2[x_1(t-T_1), x_2(t-T_2); x_3(t-s_1), x_4(t-s_2)]\, dT_1 dT_2\, ds_1 ds_2 \tag{40}$$

If we use the expression for Q_2 given by Eq. 36, then Eq. 40 becomes

$$\overline{G_2(t, K_{2,i}, \{x\}_i)\, G_2(t, L_{2,j}, \{x\}_j)} = \iiiint K_{2,i}(T_1, T_2)\, L_{2,j}(s_1, s_2)$$
$$\times \left[R_{x_1x_3}(T_1-s_1)\, R_{x_2x_4}(T_2-s_2) + R_{x_1x_4}(T_1-s_2)\, R_{x_2x_3}(T_2-s_1)\right] dT_1 dT_2\, ds_1 ds_2 \tag{41}$$

Several other properties of G-functionals can be derived in a simple manner from Eqs. 33 and 34. Since these equations will be proved valid for an arbitrary kernel $L_{m,j}$ and from the definition of G-functionals (Eq. 20) it can be shown that

$$G_m(t+a, L_{m,j}, \{x\}_j) = G_m(t, L'_{m,j}, \{x\}_j) \tag{42}$$

where

$$L'_{m,j}(s_1, \ldots s_m) = L_{m,j}(s_1+a, \ldots s_m+a)$$

then Eqs. 33 and 34 imply that the two following equations are true.

$$\overline{G_n(t, K_{n,i}, \{x\}_i)\, G_m(t+a, L_{m,j}, \{x\}_j)} = 0 \qquad m \neq n \tag{43}$$

$$\overline{G_n(t, K_{n,i}, \{x\}_i)\, G_n(t+a, L_{n,j}, \{x\}_j)} = \int \cdots \int K_{n,i}(T_1, \ldots T_n)\, L_{n,j}(s_1+a, \ldots s_n+a)$$
$$\times Q_n\left[x_{i_1}(t-T_1), \ldots x_{i_n}(t-T_n); x_{j_1}(t-s_1), \ldots x_{j_n}(t-s_n)\right] dT_1 \ldots dT_n\, ds_1 \ldots ds_n \tag{44}$$

A useful and equivalent form of Eq. 33 is

$$\overline{G_n(t, K_{n,i}, \{x\}_i)\, G_n(t, L_{m,j}, \{y\}_j)} = 0 \qquad n \neq m \tag{45}$$

The members of the sets of random processes $\{x\}_i$ and $\{y\}_j$ together are jointly Gaussian. The difference between Eq. 33 and Eq. 45 is merely a renaming of some of the random processes.

Similarly, an equivalent form of Eq. 34 is

$$\overline{G_n(t, K_{n,i}, \{x\}_i)\, G_n(t, L_{n,j}, \{y\}_j)} = \int \cdots \int K_{n,i}(T_1, \ldots T_n)\, L_{n,j}(s_1, \ldots s_n)$$
$$\times\, \overline{Q_n\left[x_{i_1}(t-T_1), \ldots x_{i_n}(t-T_n); y_{j_1}(t-s_1), \ldots y_{j_n}(t-s_n)\right]}\, dT_1 \ldots dT_n\, ds_1 \ldots ds_n \tag{46}$$

2.10 PROOFS OF PROPERTIES OF G-FUNCTIONALS

In this section we shall prove the two properties, Eqs. 33 and 34, of G-functionals presented in section 2.9.

We shall first prove the orthogonality property given by Eq. 33 and its equivalent, Eq. 45. The notation in the proof is a little simpler if we use the form of the orthogonality property given by Eq. 45.

In proving Eq. 45 we shall consider only the case $n > m$, since the proof of the other case, $n < m$, is similar. Since a functional polynomial of degree n such as G_n is a sum of homogeneous functionals of degree n or less, in order to prove Eq. 45 for $n > m$ it is sufficient to prove that

$$\overline{G_n(t, K_{n,i}, \{x\}_i) \int \cdots \int L_{k,j}(s_1, \ldots s_k)\, y_{j_1}(t-s_1) \ldots y_{j_k}(t-s_k)\, ds_1 \ldots ds_k} = 0 \qquad k = 0, \ldots n-1 \tag{47}$$

for arbitrary kernels $K_{n,i}$ and $L_{k,j}$.

We shall now prove Eq. 47. If we expand $G_n(t, K_{n,i}, \{x\}_i)$ as in Eq. 20 and interchange orders of integration and averaging, then Eq. 47 becomes

$$\int \cdots \int L_{k,j}(s_1, \ldots s_n)\, K_{n,i}(T_1, \ldots T_n) \sum_{\nu=0}^{\left[\frac{n}{2}\right]} (-1)^{\nu}$$
$$\times\, \overline{P_\nu\left[x_{i_1}(t-T_1), \ldots x_{i_n}(t-T_n)\right] y_{j_1}(t-s_1) \ldots y_{j_k}(t-s_k)}\, dT_1 \ldots dT_n\, ds_1 \ldots ds_n = 0 \qquad k = 0, \ldots n-1 \tag{48}$$

We shall prove Eq. 48 by showing that

$$\sum_{\nu=0}^{\left[\frac{n}{2}\right]} (-1)^{\nu}\, \overline{P_\nu\left[x_{i_1}(t-T_1), \ldots x_{i_n}(t-T_n)\right]\, y_{j_1}(t-s_1) \ldots y_{j_k}(t-s_k)} = 0 \qquad k = 0, \ldots n-1 \tag{49}$$

Recall that each term of $P_\nu\left[x_{i_1}(t-T_1), \ldots x_n(t-T_n)\right]$ contains the product of $n - 2\nu$ x's. Each term of the averaging operation in Eq. 49 is therefore the average of a product of Gaussian variables. The average of a product of Gaussian variables is the sum, over all ways of pairing, of products of averages of pairs of the Gaussian variables. We now divide the averaging operation in Eq. 49 into two parts: in part 1 each y is paired with an x for averaging; in part 2 at least two y's are paired for averaging. If in part 2 for a particular k we first average two y's that are paired and integrate on their corresponding s, then the remaining expression is again in the form of Eq. 49, but with k smaller by 2. Therefore if we prove the set of Eqs. 49 in order of increasing k, then for each k it is only necessary to prove that the terms from the averaging of part 1 equal zero, since the terms from the averaging of part 2 will have been proved equal to zero in the proof of the k - 2 equation of Eq. 49.

If we replace the averaging operation in Eq. 49 by the averaging of part 1 (indicated by a wavy line) as in the equation

$$\sum_{\nu=0}^{(n-k)/2} (-1)^{\nu}\, \widetilde{P_\nu\left[x_{i_1}(t-T_1), \ldots x_{i_n}(t-T_n)\right] y_{j_1}(t-s_1) \ldots y_{j_k}(t-s_k)} = 0 \tag{50}$$

then a proof of Eq. 50 will constitute a proof of Eq. 49. Since each term of $P_\nu\left[x_{i_1}(t-T_1), \ldots x_{i_n}(t-T_n)\right]$ contains $n - 2\nu$ x's, this pairing of all the k y's with k x's and the pairing of the remaining x's among themselves can exist if and only if k and n are either both even or odd and $\nu \leq \frac{1}{2}(n-k)$. Note that $(n-k)/2$ is an integer.

After the averaging of part 1, each term of the left-hand side of Eq. 49 consists of the product of $(n+k)/2$ averaged pairs: k of these are y's paired with x's and the $(n-k)/2$ remaining pairs are x's paired with x's. A typical term would be

$$\overline{x_{i_1}(t-T_1)\, y_{j_1}(t-s_1)}\; \overline{x_{i_2}(t-T_2)\, y_{j_2}(t-s_2)} \ldots \overline{x_{i_k}(t-T_k)\, y_{j_k}(t-s_k)}$$

$$\times \overline{x_{i_{k+1}}(t-T_{k+1})\, x_{i_{k+2}}(t-T_{k+2})} \ldots \overline{x_{i_{n-1}}(t-T_{n-1})\, x_{i_n}(t-T_n)}$$

Let us denote each different term that appears after the averaging in the left-hand side of Eq. 50 S_u. The subscript varies for different terms. By using this S_u notation we can write

$$\widetilde{P_\nu\left[x_{i_1}(t-T_1), \ldots x_{i_n}(t-T_n)\right] y_{j_1}(t-s_1) \ldots y_{j_n}(t-s_n)} = \sum_u b_\nu S_u \tag{51}$$

where b_ν is the number of times each S_u appears in the ν^{th} term. All S_u's appear the same number of times because of the symmetric nature of the averaging and the symmetric nature of the definition of P_ν. Substituting Eq. 32 in Eq. 50 and changing orders of summation, we obtain

$$\sum_u S_u \sum_{\nu=0}^{(n-k)/2} (-1)^{\nu} b_\nu = 0 \tag{52}$$

We shall prove Eq. 52 by showing that

$$\sum_{\nu=0}^{(n-k)/2} (-1)^{\nu} b_\nu = 0 \tag{53}$$

We must now evaluate b_ν, which is the number of times each S_u term appears in the left-hand side of Eq. 51. We recall from its definition that $P_\nu\left[x_{i_1}(t-T_1),\ldots x_{i_n}(t-T_n)\right]$ is a sum of terms and that each term of P_ν contains averaged pairs of x's multiplied by the remaining $n - 2\nu$ x's. From the definition of the averaging of part 1, it follows that a particular S_u will arise once from only those terms of P_ν in which each of the ν averaged pairs of x's is included in the $(n-k)/2$ averaged pairs of x's defining the particular S_u. Therefore b_ν is equal to the number of ways ν pairs can be chosen from $(n-k)/2$ pairs; hence, b_ν is the binomial coefficient $\binom{\frac{n-k}{2}}{\nu}$.

With this value of b_ν, Eq. 53 becomes

$$\sum_{\nu=0}^{(n-k)/2} (-1)\binom{\frac{n-k}{2}}{\nu} = 0 \tag{54}$$

But the left-hand side of Eq. 54 is recognized as the binomial expansion for $(1-1)^{(n-k)/2}$ which clearly is zero. Therefore we have proved Eq. 54, thereby proving the orthogonality property of Eqs. 33 and 45.

We shall now prove the second of the two important properties of G-functionals. That is, we shall prove Eq. 34 and its equivalent, Eq. 46. The notation in the proof is a little simpler if we use the form of the property given by Eq. 46.

If we expand $G_n(t, L_{n,j}, \{y\}_j)$ as in Eq. 20, then the left-hand side of Eq. 46 becomes

$$\overline{G_n(t, K_{n,i}, \{x\}_i)\, G_n(t, L_{n,j}, \{y\}_j)} = \overline{G_n(t, K_{n,i}, \{x\}_i) \int \cdots \int L_{n,j}(s_1, \ldots s_n) \sum_{\nu=0}^{\left[\frac{n}{2}\right]} (-1)^\nu P_\nu\left[y_{j_1}(t-s_1), \ldots y_{j_n}(t-s_n)\right] ds_1 \ldots ds_n} \tag{55}$$

From the definition of P_ν it may be seen that each term of the expansion for $G_n(t, L_{n,j}, \{y\}_j)$, except the term for which ν is zero, is a homogeneous functional of degree less than n. But by Eq. 47 each homogeneous functional of degree less than n is orthogonal to $G_n(t, K_{n,i}, \{x\}_i)$. Thus by applying this orthogonality and by using the definition of P_o, Eq. 55 becomes

$$\overline{G_n(t, K_{n,i}, \{x\}_i)\, G_n(t, L_{n,j}, \{y\}_j)} = \overline{G_n(t, K_{n,i}, \{x\}_i) \int \cdots \int L_{n,j}(s_1, \ldots s_n)\, y_{j_1}(t-s_1) \ldots y_{j_n}(t-s_n)\, ds_1 \ldots ds_n} \tag{56}$$

The averaging operation on the right-hand side of Eq. 56 involves the average of products of Gaussian variables $\{x\}_i$ and $\{y\}_j$. We now divide this averaging operation into two parts: in part 1 each y is paired with an x for averaging; in part 2 at least two y's are paired together for averaging. If in part 2 we first average the product of two y's that are paired together (say, for example, the last two y's) and integrate on their respective s, then that term has the form

$$\overline{G_n(t, K_{n,i}, \{x\}_i) \int \cdots \int L'_{n-2,j}(s_1, \ldots s_{n-2})\, y_{j_1}(t-s_1) \ldots y_{j_{n-2}}(t-s_{n-2})\, ds_1 \ldots ds_{n-2}} \tag{57}$$

where

$$L'_{n-2,j}(s_1, \ldots s_{n-2}) = \iint L_{n,j}(s_1, \ldots s_n)\, \overline{y_{j_{n-1}}(t-s_{n-1})\, y_{j_n}(t-s_n)}\, ds_{n-1} ds_n$$

From Eq. 47 it is seen that expression 57 equals zero. Therefore in performing the averaging on the right-hand side of Eq. 57 it is necessary to consider only the terms in which each of the n y's is paired with an x. However, in the expansion of Eq. 20 for $G_n(t, K_{n,i}, \{x\}_i)$, only the term in which $\nu = 0$ has at least n x's, and that term has exactly n x's. If we use the averaging of part 1 (indicated by a wavy line) and the $\nu = 0$ term of the expansion (Eq. 20) for $G_n(t, K_{n,i}, \{x\}_i)$, then Eq. 56 becomes

$$\overline{G_n(t, K_{n,i}, \{x\}_i)\, G_n(t, L_{n,j}, \{y\}_j)} = \int \cdots \int K_{n,i}(T_1, \ldots T_n)\, L_{n,j}(s_1, \ldots s_n) \times \widetilde{x_{i_1}(t-T_1) \ldots x_{i_n}(t-T_n)\, y_{j_1}(t-s_1) \ldots y_{j_n}(t-s_n)}\, dT_1 \ldots dT_n\, ds_1 \ldots ds_n \tag{58}$$

The expression

$$\widetilde{x_{i_1}(t-T_1) \ldots x_{i_n}(t-T_n)\, y_{j_1}(t-s_1) \ldots y_{j_n}(t-s_n)} \tag{59}$$

is, by the definition of the averaging of part 1, the sum over all ways of pairing of the product of n averaged pairs, each y being paired with an x. Since there are the same number of x's as y's, no x can be paired with another x. This definition of expression 59 is the same as the definition of $Q_n\left[x_{i_1}(t-T_1), \ldots x_{i_n}(t-T_n); y_{j_1}(t-s_1), \ldots y_{j_n}(t-s_n)\right]$.

By substituting this Q_n for expression 59, Eq. 58 becomes Eq. 46, which was to be proved.

III. OPTIMUM MULTI-INPUT NONLINEAR SYSTEMS

3.1 EIGHT OPTIMIZATION PROBLEMS

In this chapter we shall derive procedures for determining an optimum multi-input nonlinear system. More specifically, we shall be given a set of N, zero-mean, stationary, Gaussian, random inputs $\{x_1(t)\dots x_N(t)\}$ and a desired output z(t); we wish to determine a multi-input nonlinear system (Fig. 2) whose inputs are the given inputs and whose output w(t) minimizes the mean-square error, E, between the system ouput w(t) and the desired output z(t). This mean-square error, E, is given by

$$E = \overline{[w(t)-z(t)]^2} \tag{60}$$

Here, the bar indicates ensemble average.

There are, however, eight different cases to be considered. The desired output may be given in one of two forms; the allowable nonlinear system may be of two different types; and the set of inputs may be of two different types. These three parameters with two forms each give a total of 3^2 or eight different cases. We shall now discuss these different forms.

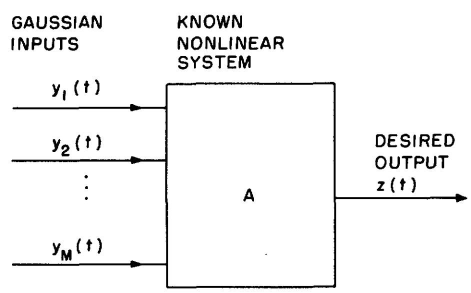

Fig. 4. System that produces the desired output.

One way in which the desired output z(t) may be given is as a known nonlinear functional of a set of M Gaussian inputs $\{y_1(t),\dots y_M(t)\}$ (see Fig. 4). In particular, the desired output z(t) can be expanded in a set of orthogonal G-functionals of the y's.

$$z(t) = \sum_{m=0}^{\infty} \sum_{j} G_m(t, L_{m,j}, \{y\}_j) \tag{61}$$

The kernels $\{L_{m,j}\}$ would be given.

An example, in which the desired output is given in the form of Eq. 61, is a chemical process in which the quality z(t) of the output is a known nonlinear functional of M fluctuating temperatures and pressures $\{y_1(t)\dots y_M(t)\}$ at different points in the system. Suppose we wish to estimate this output quality by making measurements on these temperatures and pressures, and suppose also that our measurements are inaccurate in the sense that we actually measure these temperatures and pressures plus random noise. That is, we measure $\{y_1(t)+n_1(t), \dots y_M(t)+n_M(t)\}$. The members of the set $\{n_1(t), \dots n_M(t)\}$ are random noises. The desired output (the quality z(t) of the output) is a known nonlinear functional (as in Eq. 61) of a set of inputs $\{y_1(t), \dots y_M(t)\}$; we wish to estimate the desired output by operating in a nonlinear manner on a set of inputs $\{y_1(t)+n_1(t), \dots y_M(t)+n_M(t)\}$.

The other type of desired output is merely one that is not produced in the previous manner. One reason for studying the case in which the desired output is formed by a known nonlinear operation on a set of Gaussian inputs is that the solution to that case is used as a tool in solving for the optimization in the other case.

The inputs can be either linearly independent of one another or dependent. Two inputs $x_i(t)$ and $x_j(t)$ are linearly independent if

$$\overline{x_i(t)\, x_j(t+T)} = 0 \qquad \text{for all } T$$

Since the inputs are Gaussian, linear independence implies statistical independence. The case in which the inputs are independent is used as a tool for solving the case in which the inputs are dependent.

The nonlinear systems to be optimized either may be restricted to be realizable in the sense that the present value of the output uses only past values of the inputs or may be permitted to be unrealizable in the sense that the present value of the output uses

Table 1. Eight optimization cases.

Case	Desired Output		Allowable Nonlinear System		Inputs	
	Produced from Gaussian Inputs	Not Produced from Gaussian Inputs	Unrealizable	Realizable	Independent	Dependent
1	✓		✓		✓	
2	✓		✓			✓
3	✓			✓	✓	
4	✓			✓		✓
5		✓	✓		✓	
6		✓		✓	✓	
7		✓	✓			✓
8		✓		✓		✓

future, as well as past, values of the inputs. The "unrealizable" case is of interest when we are allowed to approximate the delayed desired output instead of the desired output. The solution to the "unrealizable" case is used as a tool for solving the "realizable" case.

In Table 1 we list the eight cases arising from combinations of the different forms of desired output, inputs, and allowable systems. We shall now derive optimization procedures for each of these cases. Fortunately, some of these cases can be treated together. Some of the cases will be solved by reducing them to previously solved cases.

3.2 CASES 1 AND 2

In this section we shall consider optimizing a nonlinear system for cases 1 and 2 in Table 1. In these cases the desired output $z(t)$ is given as a known G-functional expansion of a set of M Gaussian inputs $\{y_1(t), \dots y_M(t)\}$ (Eq. 61). The nonlinear system to be optimized may be unrealizable (use future values of the inputs). As in Fig. 2, the inputs to the system to be optimized are N Gaussian inputs $\{x_1(t), \dots x_N(t)\}$ and the output is $w(t)$. We wish to choose the nonlinear system to minimize the mean-square error (Eq. 60). The inputs $\{x_1(t), \dots x_N(t)\}$ are independent in case 1 and dependent in case 2.

We shall first present the optimization procedure and then prove that it is correct. This optimization procedure was discovered by inspecting the results of a direct variational procedure for optimization.

The optimum nonlinear system for cases 1 and 2 is a cascade of two systems, B and C, and is shown in Fig. 5. The first system, B, is an N-input, M-output unrealizable linear system. The N-inputs of system B are, of course, the given N Gaussian inputs $\{x_1(t), \dots x_N(t)\}$. Each of the M outputs $\{u_1(t), \dots u_M(t)\}$ of system B is the optimum unrealizable linear mean-square approximation to the corresponding member of the Gaussian inputs $\{y_1(t), \dots y_M(t)\}$ from which the desired output $z(t)$ is formed. Notice that each u is Gaussian because it is formed by linear operations on Gaussian inputs.

The second system, C, is nonlinear and has M inputs and a single output. The M inputs to system C are $\{u_1(t), \dots u_M(t)\}$. The nonlinear system C operates on its inputs in such a manner that the G-functional expansion for its output $w(t)$ in terms of its inputs $\{u_1(t), \dots u_M(t)\}$ has the same kernels $\{L_{m,j}\}$ as does the G-functional expansion (Eq. 61) for the desired output $z(t)$ in terms of its inputs $\{y_1(t), \dots y_M(t)\}$. That is,

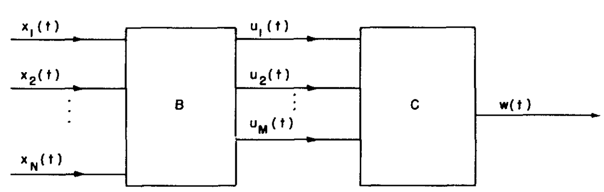

Fig. 5. Optimum nonlinear system of cases 1 and 2.

if the desired output z(t) is given by Eq. 61, then the output w(t) of system C is given by

$$w(t) = \sum_{m=0}^{\infty} \sum_{j} G_m(t, L_{m,j}, \{u\}_j) \tag{62}$$

We should note that the nonlinear system A of Fig. 4 and the nonlinear system C of Fig. 5 are not the same. The nonlinear operation described by a G-functional depends, as we have noted, not only on the kernel but also on the autocorrelations and crosscorrelations of its inputs; therefore, although the two systems have the same kernels in the G-functional expansion of their outputs, the fact that their inputs are different means that the systems are different.

There are two things yet to be done. One of them is to prove that the output w(t) (Fig. 5), formed by the previous procedure, is indeed the optimum unrealizable nonlinear approximation to the desired output z(t). The second is to present the procedures by which the linear system B can be determined. We shall treat the second problem in section 3. 3.

To prove that the output w(t) (Fig. 5), formed by the previous procedure, is the optimum unrealizable nonlinear approximation to the desired output z(t), we shall show that the mean-square error,

$$E_1 = \overline{[w(t)-z(t)]^2}$$

is less than or equal to the mean-square error

$$E_2 = \overline{[g(t)-z(t)]^2} \tag{63}$$

where g(t) is the output of any unrealizable nonlinear system with inputs $\{x_1(t), \dots x_N(t)\}$. That is, we wish to prove that $E_2 \geq E_1$.

We shall now express Eq. 63 in a more convenient form. If there exists a system whose output is g(t), there also exists a system whose output r(t) is given by

$$r(t) = g(t) - w(t) \tag{64}$$

for the output r(t) of this system is formed by merely subtracting the output w(t) from the output g(t). Since w(t) and g(t) are the outputs of unrealizable nonlinear systems with inputs $\{x_1(t), \dots x_N(t)\}$, then r(t) is also the output of an unrealizable nonlinear system with the same inputs.

By means of Eq. 64 we can express the error E_2 of Eq. 63 in terms of r(t) and w(t) instead of g(t):

$$E_2 = \overline{[r(t)+w(t)-z(t)]^2}$$

We can then regroup terms:

$$E_2 = \overline{\{r(t)-[z(t)-w(t)]\}^2}$$

and by expanding the square, we have

$$E_2 = \overline{r^2(t)} - 2\,\overline{r(t)[z(t)-w(t)]} + \overline{[z(t)-w(t)]^2} \tag{65}$$

We notice that the last term in Eq. 65 is E_1. Equation 65 can then be written as

$$E_2 = \overline{r^2(t)} - 2\,\overline{r(t)[z(t)-w(t)]} + E_1 \tag{66}$$

If it is true that

$$\overline{r(t)[z(t)-w(t)]} = 0 \tag{67}$$

for all unrealizable nonlinear systems with output r(t) and inputs $\{x_1(t), \dots x_N(t)\}$, then Eq. 66 becomes

$$E_2 = \overline{r^2(t)} + E_1 \tag{68}$$

Since the average of the square of an output such as $\overline{r^2(t)}$, is always greater than or equal to zero, then Eq. 68 implies that $E_2 \geqslant E_1$.

We shall prove that $E_2 \geqslant E_1$ by proving that Eq. 67 is true. Since r(t) is the output of an unrealizable nonlinear system with inputs $\{x_1(t) \dots x_N(t)\}$ it can be written as a sum of G-functionals of the inputs

$$r(t) = \sum_{n=0}^{\infty} \sum_i G_n(t, K_{n,i}, \{x\}_i) \tag{69}$$

for some set of kernels $\{K_{n,i}\}$. By using the G-functional expansions for z(t), w(t), and r(t) given by Eqs. 61, 62 and 69, Eq. 67 becomes

$$\overline{\sum_{n=0}^{\infty} \sum_i G_n(t, K_{n,k}, \{x\}_i)\left[\sum_{m=0}^{\infty} \sum_j G_m(t, L_{m,j}, \{y\}_j) - \sum_{m=0}^{\infty} \sum_j G_m(t, L_{m,j}, \{u\}_j)\right]} = 0 \tag{70}$$

Since the x's, y's, and u's are all Gaussian, we can apply the orthogonality of G-functionals of different degree given by Eq. 45. By using this orthogonality, Eq. 70 becomes

$$\overline{\sum_{n=0}^{\infty}\left\{ \sum_i G_n(t, K_{n,i}, \{x\}_i)\left[\sum_j G_n(t, L_{n,j}, \{y\}_j) - \sum_j G_n(t, L_{n,j}, \{u\}_j)\right]\right\}} = 0 \tag{71}$$

If we use Eq. 46 to evaluate these averages, and make use of the fact that the kernels $\{L_{n,j}\}$ are the same in the G-functional expansions of w(t) and z(t), then Eq. 71 becomes

$$\sum_{n=0}^{\infty} \sum_i \sum_j \int \cdots \int K_{n,i}(T_1, \dots T_n)\, L_{n,j}(s_1, \dots s_n)\Big\{Q_n\big[x_{i_1}(t-T_1), \dots x_{i_n}(t-T_n); y_{j_1}(t-s_1), \dots y_{j_n}(t-s_n)\big]$$
$$- Q_n\big[x_{i_1}(t-T_1), \dots x_{i_n}(t-T_n); u_{j_1}(t-s_1), u_{j_n}(t-s_n)\big]\Big\}\, dT_1 \dots dT_n\, ds_1 \dots ds_n = 0 \tag{72}$$

We shall prove Eq. 72 by proving that

$$Q_n\left[x_{i_1}(t-T_1), \dots x_{i_n}(t-T_n); y_{j_1}(t-s_1), \dots y_{j_n}(t-s_n)\right]$$

$$= Q_n\left[x_{i_1}(t-T_1), \dots x_{i_n}(t-T_n); u_{j_1}(t-s_1), \dots u_{j_n}(t-s_n)\right] \tag{73}$$

Recall that the expansion for Q_n is a sum of terms in which each of the first n of its variables is paired and averaged with one of the last n of its variables. To prove Eq. 73, it is therefore sufficient to show that

$$\overline{x_i(t-T)\, y_j(t-s)} = \overline{x_i(t-T)\, u_j(t-s)} \tag{74}$$

because if Eq. 74 is true, then each term of the expansion of the Q of the left-hand side of Eq. 73 is equal to the term corresponding to the same pairing in the expansion of the Q of the right-hand side of Eq. 73.

We shall now prove Eq. 74. Recall that $u_j(t-s)$ is defined as the optimum unrealizable linear approximation to $y_j(t-s)$ obtainable from the inputs $\{x_1(t)\dots x_N(t)\}$. That is, $u_j(t-s)$ minimizes the error defined by $\overline{[u_j(t-s)-y_j(t-s)]^2}$. Since $u_j(t-s)$ is optimum, it has the property that if we add to it any linear function of the x's such as

$$\epsilon \int_{-\infty}^{\infty} h(T)\, x_i(t-T)\, dT$$

where h(t) and ϵ are arbitrary, then the error must have zero derivative with respect to ϵ when $\epsilon = 0$. That is,

$$\frac{d}{d\epsilon}\left.\overline{\left[u_j(t-s) + \epsilon \int_{-\infty}^{\infty} h(T)\, x_i(t-T)\, dT - y_j(t-s)\right]^2}\right|_{\epsilon=0} = 0 \tag{75}$$

By performing the differentiation, Eq. 75 becomes

$$2\int_{-\infty}^{\infty} h(T)\, dT \left[\overline{x_i(t-T)u_j(t-s)}-\overline{x_i(t-T)y_j(t-s)}\right] = 0$$

Since the kernel h(T) is arbitrary, the quantity in brackets must be zero for all T. That is,

$$\overline{x_i(t-T)\, u_j(t-s)} = \overline{x_i(t-T)\, y_j(t-s)}$$

which was to be proved.

We have now proved that the optimization procedure shown in Fig. 5 is correct.

3.3 OPTIMUM MULTI-INPUT UNREALIZABLE LINEAR SYSTEMS

In this section we shall present methods for determining the N-input, M-output linear system B of Fig. 5. The N-inputs are the given inputs $\{x_1(t), \dots x_N(t)\}$. The

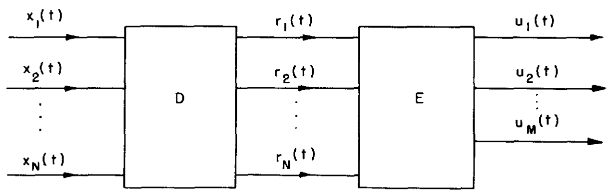

Fig. 6. Optimum multi-input multioutput unrealizable linear system.

M outputs $\{u_1(t), \ldots u_M(t)\}$ are the optimum unrealizable approximations to $\{y_1(t), \ldots y_M(t)\}$, respectively. We shall design system B as the cascade of two linear systems (Fig. 6), system D and system E.

System D has N inputs $\{x_1(t), \ldots x_N(t)\}$ and N white, linearly independent outputs $\{r_1(t), \ldots r_N(t)\}$. By linearly independent we mean that

$$\overline{r_i(t)\, r_j(t+T)} = 0 \qquad i \neq j \tag{76}$$

By white we mean that

$$\overline{r_i(t)\, r_i(t+T)} = \delta(T) \tag{77}$$

where $\delta(T)$ is an impulse. System D may be unrealizable in the sense that its present outputs depend upon future values of its inputs. System D is invertible in the sense that there exists an unrealizable linear system whose inputs are $\{r_1(t), \ldots r_N(t)\}$ and whose outputs are $\{x_1(t), \ldots x_N(t)\}$. We shall show how to design the system D.

System E is an N-input, M-output unrealizable linear system. Its inputs are $\{r_1(t), \ldots r_N(t)\}$. Its outputs, $\{u_1(t), \ldots u_M(t)\}$, are optimum unrealizable approximations to $\{y_1(t), \ldots y_N(t)\}$. The fact that the inputs to system E are the r's instead of the x's does not produce a poorer approximation to the y's, because system D is invertible and system E could internally produce the x's from the r's if necessary. The reason for this cascade arrangement of systems D and E is that it is easier to optimize a system whose inputs are the r's than one whose inputs are the x's.

Let us first consider how to design system D for the case in which the inputs are already independent. In this case system D only has to "whiten" each input. A random input can be whitened by a single-input, realizable, linear system that has a realizable linear inverse (12). The Fourier transform of the impulse response of the whitening system has for its poles the upper half-plane zeros of the power density spectrum of the input; it has for its zeros the upper half-plane poles of the power density spectrum of the input.

Thus, if $x_i(t)$ is a random input with autocorrelation function

$$R_{x_i x_i}(T) = \overline{x_i(t)\, x_i(t+T)} \tag{78}$$

and power density spectrum

$$S_{x_i x_i}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega T} R_{x_i x_i}(T)\, dT \qquad (79)$$

and if $S_{x_i x_i}(\omega)$ can be factored as

$$S_{x_i x_i}(\omega) = k^2 \frac{(\omega-a_1)\left(\omega-a_1^*\right)(\omega-a_2)\left(\omega-a_2^*\right)\ldots}{(\omega-\beta_1)\left(\omega-\beta_1^*\right)(\omega-\beta_2)\left(\omega-\beta_2^*\right)\ldots} \qquad (80)$$

in which the asterisk means complex conjugate, and a_1, a_2, ... β_1, β_2, ... all have imaginary parts that are greater than zero, then $x_i(t)$ can be whitened by a realizable, linear, invertible system with impulse response $f_i(t)$ whose Fourier transform $F_i(\omega)$ is given by

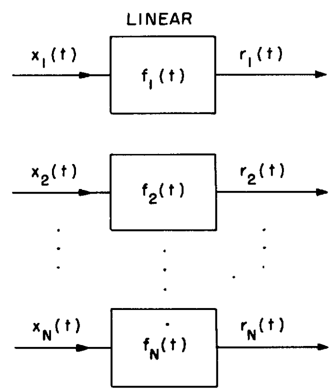

Fig. 7. System D for whitening independent inputs.

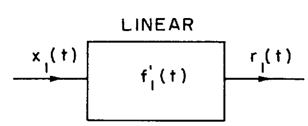

Fig. 8. Whitening system.

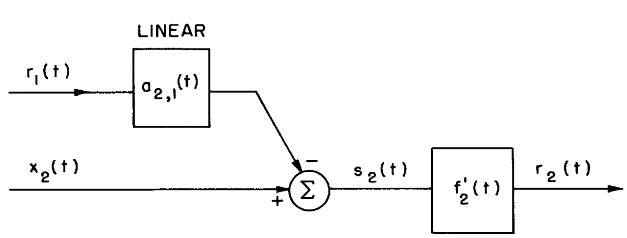

Fig. 9. Orthogonalizing system.

$$F_i(\omega) = \frac{1}{k} \frac{(\omega-\beta_1)(\omega-\beta_2)\,\ldots}{(\omega-a_1)(\omega-a_2)\,\ldots} \tag{81}$$

Since $f_i(t)$ is the inverse transform of $F_i(\omega)$, it is given by

$$f_i(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{+j\omega t}\, F_i(\omega)\, d\omega \tag{82}$$

Therefore, if the inputs $\{x_i(t), \ldots x_j(t)\}$ are independent, as they are for case 1 (Fig. 5), then system D has the simple form shown in Fig. 7. System D for this case consists of N single-input, single-output, linear systems with impulse responses $\{f_1(t), \ldots f_N(t)\}$ defined by Eqs. 78-82. Since each member of the set $\{f_i(t)\}$ is realizable and has a realizable inverse, system D of Fig. 7 is realizable and has a realizable inverse.

We now consider how to design system D when the inputs are dependent as in case 2 in Table 1. The method that we shall use is analogous to a "Schmidt orthogonalization procedure" (13). We treat the inputs in order. We first whiten input $x_1(t)$ by a single-input, realizable, linear system that has a realizable linear inverse. The impulse response of the whitening system is $f_1'(t)$ and its output is $r_1(t)$ (Fig. 8). The output $r_1(t)$ is white; that is,

$$\overline{r_1(t)\, r_1(t+T)} = \delta(T) \tag{83}$$

We next wish to form a signal $s_2(t)$ that is linearly independent of $r_1(t)$. The way in which we form $s_2(t)$ is to subtract from $x_2(t)$ the output of a linear operation on $r_1(t)$, as in Fig. 9. That is, we wish to form a signal $s_2(t)$ with the property that

$$\overline{s_2(t)\, r_1(t-T)} = 0 \tag{84a}$$

with $s_2(t)$ formed as

$$s_2(t) = x_2(t) - \int_{-\infty}^{\infty} a_{2,1}(t')\, r_1(t-t')\, dt' \tag{84b}$$

and in which the impulse response $a_{2,1}(t)$ of the linear system is to be determined.

We can determine $a_{2,1}(t)$ in the following manner. By multiplying both sides of Eq. 84b by $r_1(t-T)$ and then taking averages on both sides, we obtain

$$\overline{s_2(t)\, r_1(t-T)} = \overline{x_2(t)\, r_1(t-T)} - \int_{-\infty}^{\infty} a_{2,1}(t')\, \overline{r_1(t-t')\, r_1(t-T)}\, dt' \tag{85}$$

By substituting Eqs. 83 and 84a in Eq. 85 we obtain the desired impulse response,

$$a_{2,1}(T) = \overline{x_2(t)\, r_1(t-T)}$$

In general, $a_{2,1}(t)$ will be nonzero for $t < 0$ and hence will be unrealizable.

To form $r_2(t)$ (Fig. 9), we whiten $s_2(t)$ by a single-input, realizable, linear system that has a realizable linear inverse. The impulse response of the whitening system is $f_2'(t)$.

To form $r_i(t)$ we use a similar procedure. First, we form a signal $s_i(t)$ with the property that

$$\overline{s_i(t)\, r_j(t-T)} = 0 \qquad j < i$$

The signal $s_i(t)$ is formed by subtracting from $x_i(t)$ linear operations on the previous r's. Thus

$$s_i(t) = x_i(t) - \sum_{j=1}^{i-1} a_{i,j}(t')\, r_j(t-t')\, dt' \tag{86}$$

By multiplying both sides of Eq. 86 by $r_k(t-T)$ for $k < i$, averaging, and then applying Eqs. 76 and 77, we find that the impulse responses of the linear systems $\{a_{i,j}(t)\}$ are given by

$$a_{i,j}(T) = \overline{x_i(t)\, r_j(t-T)}$$

To form $r_i(t)$ we merely whiten $s_i(t)$ by a single-input, realizable, linear system that has a realizable linear inverse.

We have now shown how to derive a set of white, linearly independent signals, $\{r_1(t), \ldots r_N(t)\}$, by unrealizable linear operations on a set of dependent inputs $\{x_i(t), \ldots x_N(t)\}$. We must now show that by unrealizable linear operations on $\{r_1(t), \ldots r_N(t)\}$ we can obtain $\{x_i(t), \ldots x_N(t)\}$. From Eq. 86 we see that $x_i(t)$ is given by

$$x_i(t) = s_i(t) + \sum_{j=1}^{i-1} \int_{-\infty}^{\infty} a_{i,j}(t')\, r_j(t-t')\, dt' \tag{87}$$

We are given the set $\{r_1(t), \ldots r_N(t)\}$; we know the set of impulse responses $\{a_{i,j}(t)\}$; and we can form $s_i(t)$ by a realizable linear operation on $r_i(t)$ because $s_i(t)$ was whitened in an invertible manner. Therefore from Eq. 87 we see that we can form the set $\{x_1(t), \ldots x_N(t)\}$ by unrealizable, linear operations on $\{r_1(t), \ldots r_N(t)\}$.

We shall now show how to design the unrealizable linear system E whose inputs are $\{r_1(t), \ldots r_N(t)\}$ and whose outputs $\{u_1(t), \ldots u_M(t)\}$ are the optimum unrealizable linear mean-square approximations to $\{y_1(t), \ldots y_M(t)\}$. Since system E is linear, we can produce each output $u_j(t)$ as the sum of outputs of linear operations on each of the inputs as in Fig. 10. Each output is then given by

$$u_j(t) = \sum_{i=1}^{N} \int h_{j,i}(T)\, r_i(t-T)\, dT \tag{88}$$

where $\{h_{j,i}(t)\}$ are impulse responses that will be chosen to minimize the error

$$\overline{[u_j(t)-y_j(t)]^2}$$

Using Eq. 88, we see that this error is given by

$$\overline{\left[\sum_{i=1}^{N} \int h_{j,i}(T)\, r_i(t-T)\, dT - y_j(t)\right]^2}$$

If the impulse responses $\{h_{j,i}(t)\}$ are optimum, and if we add to one of them, say $h_{j,k}(t)$, an impulse response $\epsilon\, g_{j,k}(t)$, then the derivative of the error with respect to ϵ must be zero when $\epsilon = 0$. That is,

$$\frac{d}{d\epsilon} \overline{\left[\sum_{i=1}^{N} \int h_{j,i}(T)\, r_i(t-T) + \epsilon \int g_{j,k}(T)\, r_k(t-T)\, dT - y_j(t)\right]^2}\Bigg|_{\epsilon=0} = 0 \tag{89}$$

Performing this differentiation, we obtain

$$2\int g_{j,k}(T_1)\, dT_1 \left[\sum_{i=1}^{N} \int h_{j,i}(T_2)\, \overline{r_i(t-T_2)\, r_k(t-T_1)}\, dT_2 - \overline{y_j(t)\, r_k(t-T_1)}\right] = 0 \tag{90}$$

By using Eqs. 76 and 77 to evaluate the average of the product of two r's Eq. 90 becomes

$$2\int g_{j,k}(T_1)\, dT_1 [h_{j,k}(T_1) - \overline{y_j(t) r_k(t-T_1)}] = 0$$

Since the impulse response $g_{j,k}(T_1)$ is arbitrary, the term in brackets must be zero for all T_1. That is,

$$h_{j,k}(T_1) = \overline{y_j(t)\, r_k(t-T_1)} \tag{91}$$

Equation 91 gives us the set of impulse responses $\{h_{j,i}(t)\}$ that characterizes system E; each output $u_j(t)$ can be determined in terms of this set of impulse responses and the inputs $\{r_1(t), \ldots r_N(t)\}$ by Eq. 88.

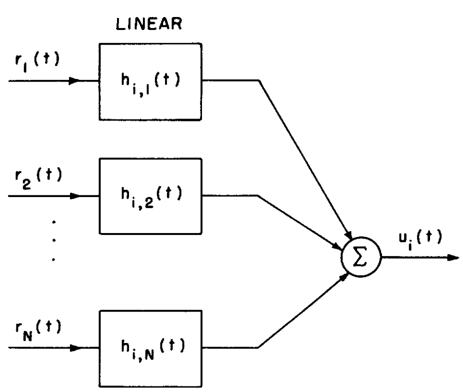

Fig. 10. Multi-input, single-output, linear system.

From Eq. 91 we see that these impulse responses depend only upon the crosscorrelation functions between the r's and y's. Since the r's depend in a linear manner on the x's, the crosscorrelation functions between the r's and the y's can be computed from the crosscorrelation functions between the x's and the y's.

3.4 CASES 3 AND 4

We shall now treat the optimization of a nonlinear system for cases 3 and 4 in Table 1. In these cases the desired output z(t) is given as a known G-functional expansion of a set of M Gaussian inputs $\{y_1(t), \dots y_M(t)\}$ (Eq. 61). The nonlinear system to be optimized must be realizable (use only past or present values of the inputs). As in Fig. 2, the inputs to the system to be optimized are N Gaussian inputs $\{x_1(t), \dots x_N(t)\}$, and the output is w(t). We wish to choose the nonlinear system to minimize E.

$$E = \overline{[w(t)-z(t)]^2}$$

The inputs $\{x_1(t), \dots x_N(t)\}$ are independent in case 3 and dependent in case 4.

Cases 3 and 4 differ from cases 1 and 2 only in the realizability constraint. Conceptually, the optimization procedure for cases 3 and 4 is very similar to that for cases 1 and 2. Just as in cases 1 and 2, the optimization procedure for cases 3 and 4 involves, at least conceptually, an optimum linear operation and a nonlinear operation that is defined in terms of the kernels of the G-functional expansion of the desired output (Eq. 61). Physically, however, these two operations for cases 3 and 4 cannot be separated into the cascade of two operations as they are for cases 1 and 2 in Fig. 5.

Again, this optimization procedure was discovered by inspecting the results of a direct variational procedure for optimization. We shall first present the optimization procedure and then prove that it is correct.

Conceptually, not physically, here are the two steps by which we can form w(t') — the optimum nonlinear approximation to the desired output at a fixed time t'. The first step is that at time t', by realizable linear operations on the past of the inputs $\{x_1(t), \dots x_N(t)\}$, we form M functions of t called $\{v_1(t|t'), \dots v_M(t|t')\}$. Each $\{v_j(t|t')\}$ is the optimum realizable linear approximation to the corresponding member of the set $\{y_j(t)\}$, the set of inputs from which the desired output z(t) is formed.

This approximation procedure is a little different from that which we normally consider a linear approximation. Normally, at an instant of time t' we think of making an approximation to the value of some signal at time $t' + \alpha$, where α is some fixed delay or advance. What we are doing with the function $v_j(t|t')$ is to approximate with a function of t to the whole past and future of the signal $v_j(t)$. That is, at time t' we form not a set of numbers but a set of functions of t.

The realizable linear operation that forms $v_j(t|t')$ is given by

$$v_j(t|t') = \sum_{i=1}^{N} \int_0^{\infty} h_{j,i}(s, t'-t)\, x_i(t'-s)\, ds \tag{92}$$

If t' is thought of as the present time, then each kernel $h_{j,i}(s,t'-t)$ can be thought of as the effect that an impulse that occurred s seconds ago in the input x_i produces in our present approximation to the value that y_j had t' − t seconds ago. The kernels $h_{j,i}(s,t'-t)$ are chosen to minimize the error

$$\overline{[v_j(t|t')-y_j(t)]^2} \tag{93}$$

In Eq. 93 the bar indicates ensemble average. The members of the set $\{v_j(t|t')\}$ are Gaussian because they are formed by linear operations on Gaussian inputs.

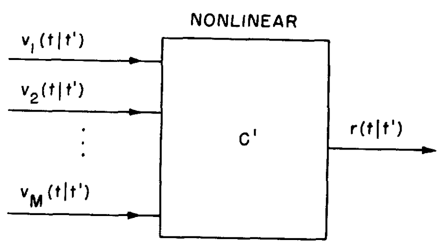

Fig. 11. Optimization procedure for cases 3 and 4.

The second step in this conceptual form of the optimization procedure is to use the set $\{v_1(t|t'), \ldots v_M(t|t')\}$ as the M inputs to a nonlinear system C' (Fig. 11), with t' held fixed, and t the independent variable. The nonlinear system C' operates on its inputs in such a manner that the G-functional expansion for its output $r(t|t')$ in terms of its inputs $\{v_1(t|t'), \ldots v_M(t|t')\}$ has the same kernels $\{L_{m,j}\}$ as does the G-functional expansion (Eq. 61) for the desired output z(t) in terms of its inputs $\{y_1(t), \ldots y_m(t)\}$. That is, if z(t) is given by Eq. 61, then the output $r(t|t')$ of system C' is given by

$$r(t|t') = \sum_{m=0}^{\infty} \sum_j G_m(t, L_{m,j}, \{v\}_j)$$

It will be shown that w(t'), the optimum realizable, nonlinear approximation to the desired output at a fixed time t', is given by

$$w(t') = r(t|t')\Big|_{t=t'} = \sum_{m=0}^{\infty} \sum_j G_m(t, L_{m,j}, \{v\}_j)\Big|_{t=t'} \tag{94}$$

Notice that for any value of t, the output $r(t|t')$ as shown in Fig. 11 will also depend upon t'.

To see how the two conceptual steps combine into a single physical operation, we examine a typical homogeneous functional that would be found in the expansion (Eq. 94) of w(t'). Let us pick, for example, the following homogeneous functional of degree 2.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L_{2,3}(T_1, T_2)\, v_1(t-T_1|t')\, v_2(t-T_2|t')\, dT_1 dT_2\Big|_{t=t'} \tag{95}$$

From Eq. 92 we see that

$$v_j(t-T|t')\Big|_{t=t'} = \sum_{i=1}^{N} \int_0^{\infty} h_{j,i}(s, T)\, x_i(t'-s)\, ds \tag{96}$$

By substituting Eq. 96 in expression 95 and by changing orders of integration, expression 95 becomes

$$\sum_{i=1}^{N} \sum_{k=1}^{N} \int_0^{\infty} \int_0^{\infty} x_i(t'-s_1) x_k(t'-s_2) \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{1,i}(s_1, T_1) h_{2,k}(s_2, T_2) L_{2,3}(T_1, T_2) \, dT_1 dT_2 \right] \times ds_1 ds_2 \tag{97}$$

The quantity in brackets in expression 97 is a second-order kernel of the variables s_1 and s_2. Expression 97 is therefore a sum of homogeneous functionals of degree 2 of the x inputs. In a similar manner it can be shown that all the terms in the expansion of w(t') (Eq. 94) can be realized by physical operations on the x inputs.

The optimum realizable linear kernels $\{h_{j,i}(s, t'-t)\}$ which minimize the error of expression 93 can be obtained by Wiener's (1) method for optimizing a realizable multi-input linear system. This method will not be described here.

We shall now prove that w(t') as given by Eq. 94 does indeed minmize the error E_1 between w(t') and the desired output z(t'):

$$E_1 = \overline{[w(t')-z(t')]^2}$$

The proof is very similar to the proof given for cases 1 and 2. The method of proof will be to show that if we add to w(t') an output s(t') of any realizable nonlinear system with inputs $\{x_1(t), \dots x_N(t)\}$, then the error E_2 defined by

$$E_2 = \overline{[w(t')+s(t')-z(t')]^2} \tag{98}$$

will be greater than E_1. That is, we shall prove that $E_2 \geq E_1$.

If we regroup the terms in Eq. 98 as

$$E_2 = \overline{\{s(t')-[z(t')-w(t')]\}^2}$$

and then expand the square, we obtain

$$E_2 = \overline{s^2(t')} - \overline{2s(t')[z(t')-w(t')]} + \overline{[z(t')-w(t')]^2} \tag{99}$$

The last term in Eq. 99 is recognized as E_1. If we can prove that

$$\overline{s(t')[z(t')-w(t')]} = 0 \tag{100}$$

then Eq. 99 would become

$$E_2 = \overline{s^2(t')} + E_1 \tag{101}$$

Since $\overline{s^2(t')}$ is always non-negative, Eq. 101 would show that $E_2 \geq E_1$.

We shall now prove that Eq. 100 is true. Since s(t') is the output of a realizable nonlinear system with inputs $\{x_1(t), \dots x_N(t)\}$, it can be written as a sum of G-functionals of the inputs.

$$s(t') = \sum_{n=0}^{\infty} \sum_{i} G_n(t', K_{n,i}, \{x\}_i) \tag{102}$$

for some set of kernels $\{K_{n,i}\}$. Since s(t') is realizable, each of the kernels $K_{n,i}(T_1, \ldots T_n)$ is zero if any of its T's is less than zero. By using the G-function expansions for z(t'), w(t'), and s(t') given by Eqs. 61, 94, and 102, Eq. 100 becomes

$$\overline{\sum_{n=0}^{\infty} \sum_{i} G_n(t', K_{n,i}, \{x\}_i) \left[\sum_{m=0}^{\infty} \sum_{j} G_m(t', L_{m,j}, \{y\}_j) - \sum_{m=0}^{\infty} \sum_{j} G_m(t, L_{m,j}, \{v\}_j) \Big|_{t=t'} \right]} = 0 \tag{103}$$

Since the x's, y's, and v's are all Gaussian, we can apply the orthogonality of G-functionals of different degree given by Eq. 45. By using this orthogonality, Eq. 103 becomes

$$\sum_{n=0}^{\infty} \left\{ \overline{\sum_{i} G_n(t', K_{n,i}, \{x\}_i) \left[\sum_{j} G_n(t', L_{n,j}, \{y\}_j) - \sum_{j} G_n(t, L_{n,j}, \{v\}_j) \Big|_{t=t'} \right]} \right\} = 0 \tag{104}$$

If we use Eq. 46 to evaluate these averages, and make use of the fact that the kernels $\{L_{n,j}\}$ are the same in the G-function expansion of w(t) and z(t), then Eq. 104 becomes

$$\sum_{n=0}^{\infty} \sum_{i} \sum_{j} \int \cdots \int K_{n,i}(T_1, \ldots T_n)\, L_{n,j}(s_1, \ldots s_n)$$

$$\Big\{ Q_n\left[x_{i_1}(t'-T_1), \ldots x_{i_N}(t'-T_n); y_{j_1}(t'-s_1), \ldots y_{j_n}(t'-s_n)\right]$$

$$- Q_n\left[x_{i_1}(t'-T_1), \ldots x_{i_n}(t'-T_n); v_{j_1}(t'-s_1|t'), \ldots v_{j_n}(t'-s_n|t')\right]\Big\}\, dT_1 \ldots dT_n\, ds_1 \ldots ds_n = 0 \tag{105}$$

We shall prove Eq. 105 by proving that

$$Q_n\left[x_{i_1}(t'-T_1), \ldots x_{i_n}(t'-T_n); y_{j_1}(t'-s_1), \ldots y_{j_n}(t'-s_n)\right]$$

$$= Q_n\left[x_{i_1}(t'-T_1), \ldots x_{i_n}(t'-T_n); v_{j_1}(t'-s_1|t'), \ldots v_{j_n}(t'-s_n|t')\right] \tag{106}$$

for all T's that are non-negative.

Recall that the expansion for Q_n is a sum of terms in which each of the first n of its variables is paired and averaged with one of the last n of its variables. To prove Eq. 106, it is sufficient to show that

$$\overline{x_i(t'-T)\, y_j(t'-s)} = \overline{x_i(t'-T)\, v_j(t'-s|t')} \tag{107}$$

because if Eq. 107 is true, then each term of the expansion of the Q of the left-hand side of Eq. 106 is equal to the term corresponding to the same pairing in the expansion of the Q of the right-hand side of Eq. 106.

We shall now prove Eq. 107. Recall that $v_j(t'-s|t')$ is defined as the optimum realizable linear approximation to $y_j(t'-s)$. By realizable, we mean that we use the inputs $\{x_i(T'), \dots x_N(T'_N)\}$ for $T' \leq t'$. Thus $v_j(t'-s|t')$ minimizes the error

$$\overline{[v_j(t'-s|t')-y_j(t'-s)]^2} \tag{108}$$

Since $v_j(t'-s|t')$ is optimum, it has the property that if we add to it any realizable linear function of the x's such as

$$\epsilon \int_0^\infty g(T)\, x_i(t'-T)\, dT$$

where g(T) and ϵ are arbitrary, then the error must have zero derivative with respect to ϵ when $\epsilon = 0$. That is,

$$\frac{d}{d\epsilon} \overline{\left[v_j(t'-s|t') + \epsilon \int_0^\infty g(T)\, x_i(t'-T)\, dT - y_j(t'-s)\right]^2}\Bigg|_{\epsilon=0} = 0$$

By performing the differentiation, this equation becomes

$$2 \int_0^\infty g(T)\, dT \left[\overline{x_i(t'-T)v_j(t'-s|t')} - \overline{x_i(t'-T)y_j(t'-s)}\right] = 0$$

Since the kernel g(T) is arbitrary, the quantity in brackets must be zero for $T \geq 0$. This proves Eq. 107.

We have now proved that the optimization procedure is correct.

3.5 EXAMPLE OF CASE 3

We shall present a simple example of an optimization procedure of the type of case 3. The example will be used to illustrate the following three points. It will be used to show how we can, by inspection, express the output of a simple power series device such as a single-input squaring device as a sum of G-functionals of its input. The example will illustrate the fact that two systems with the same kernels in the G-function expansion of their outputs will not, in general, be the same systems. It will also show that if the desired output is formed by a nonlinear no-memory operation on a Gaussian input, then the optimum realizable nonlinear system will consist of a cascade of an optimum linear system whose output approximates the input that produces the desired output and a nonlinear no-memory system.

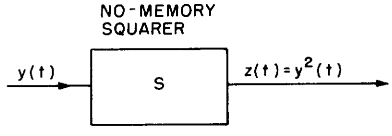

Fig. 12. System that produces the desired output.

In this example, the desired output z(t) is formed by squaring a single Gaussian input y(t), as shown in Fig. 12. Thus the desired output is given by

$$z(t) = y^2(t)$$

We are given a single Gaussian input x(t), and we wish to find a realizable nonlinear system to operate on x in such a manner that its output w(t) is as close as possible in the mean-square sense to z(t). This problem is of the type of case 3 because the desired output is produced in a known manner from a Gaussian variable; the optimum system must be realizable, and the single input x(t) is, of course, independent, since no other x's exist.

We are given the fact that the input y is the input x advanced by a known time α. That is,

$$y(t) = x(t+\alpha)$$

where $\alpha > 0$.

The autocorrelation $R_{xx}(T)$ of the x input is given as

$$R_{xx}(T) = \pi\, e^{-2\pi |T|} \tag{109}$$

The power density spectrum of x, $S_{x,x}(\alpha)$, which is the Fourier transform of the autocorrelation function, is therefore given by

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega T} R_{xx}(T)\, dT = \frac{1}{1 + \left(\frac{\omega}{2\pi}\right)^2} \tag{110}$$

The first thing we would like to do is to express the desired output z(t) as a sum of G-functionals of its input y(t). We note that we can express z(t) as a homogeneous functional in the form

$$z(t) = y^2(t) = \iint \delta(T_1)\, \delta(T_2)\, y(t-T_1)\, y(t-T_2)\, dT_1 dT_2 \tag{111}$$

We now form a G-functional of degree 2 of the input y that has the same kernel, $\delta(T_1)\,\delta(T_2)$, as the homogeneous functional of Eq. 111. From Eq. 20 we see that this G-functional is

$$G_2(t, \delta(T_1)\delta(T_2), \{y\}) = \iint \delta(T_1)\, \delta(T_2)\, y(t-T_1)\, y(t-T_2) - \iint \delta(T_1)\, \delta(T_2)\, \overline{y(t-T_1)\, y(t-T_2)}\, dT_1 dT_2 \tag{112}$$

We note that the right-hand side of Eq. 112 differs from the right-hand side of Eq. 111 only in the constant given by

$$\iint \delta(T_1)\, \delta(T_2)\, \overline{y(t-T_1)\, y(t-T_2)}\, dT_1 dT_2 = \overline{y^2(t)} \tag{113}$$

But a constant is a G-functional of degree zero. Therefore, we can express z(t) as the following sum of G-functionals:

$$z(t) = G_2(t, \delta(T_1)\delta(T_2), \{y\}) + \overline{y^2(t)} \tag{114}$$

Normally, to find w(t'), the optimum realizable nonlinear approximation to z(t'), we would first form $v(t|t')$, the optimum realizable linear approximation at time t' to the whole time function y(t); and then from Eq. 94 we would form

$$w(t') = \left[G_2[t, \delta(T_1)\delta(T_2), \{v\}] + \overline{y^2(t)}\right]\Big|_{t=t'} \tag{115}$$

Notice that we have used the same kernel $\delta(T_1)\ \delta(T_2)$ and constant $\overline{y^2(t)}$ as in Eq. 114. Let us expand Eq. 115 by means of the definition of the G-functionals given in Eq. 20:

$$\begin{aligned} w(t') &= \left[\iint \delta(T_1)\, \delta(T_2)\, v(t-T_1|t')\, v(t-T_2|t')\, dT_1 dT_2 - \iint \delta(T_1)\, \delta(T_2)\, \overline{v(T-T_1|t')\, v(t-T_2|t')}\, dT_1 dT_2 + \overline{y^2(t)}\right]\Big|_{t=t'} \\ &= v^2(t|t') - \overline{v^2(t'|t)} + \overline{y^2(t')} \end{aligned} \tag{116}$$

Let us define a new variable v'(t) as

$$v'(t) = v(t|t)$$

Thus v'(t) is the optimum realizable linear estimate of y(t) which can be made at time t. From Eq. 116 we see that w(t) is a nonlinear no-memory function of v'(t). Physically, we would form v'(t) as the output of a realizable linear system whose input is x. To

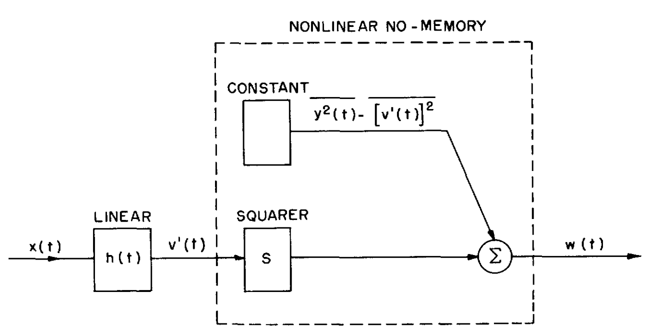

Fig. 13. Optimum nonlinear system.

form w(t), we would follow this system by a nonlinear no-memory system, as in Fig. 13.

Davenport and Root (14) have shown that the optimum realizable linear system with which to estimate y(t) from the past of x (where x and y satisfy Eqs. 108 and 110) is a no-memory gain of value $e^{-2\pi a}$. The linear output v'(t) is given by

$$v'(t) = e^{-2\pi a}\, x(t) \tag{117}$$

From Eqs. 109 and 117, we can evaluate $\overline{y^2(t)}$ and $\overline{[v'(t)]^2}$ as follows:

$$\overline{y^2(t)} = \overline{x^2(t)} = R_{xx}(0) = \pi$$

$$\overline{[v'(t)]^2} = \overline{[e^{-2\pi a}\, x(t)]^2} = e^{-4\pi a}\, \overline{x^2(t)} = \pi\, e^{-4\pi a}$$

The constant in Fig. 13 is

$$\overline{[y^2(t)]} - \overline{[v'(t)]^2} = \pi[1 - e^{-4\pi a}] \tag{118}$$

Thus if a is greater than zero, the constant (Eq. 118) is nonzero. Notice that the nonlinear part of Fig. 13 is composed of a nonzero constant plus a squaring device, while the nonlinear system of Fig. 12 is only a squaring device. The two systems are different even though the kernels of their G-functional expansions are the same.

3.6 CASES 5, 6, AND 7

We shall now derive the optimization procedures for cases 5, 6, and 7 in Table 1. In all these cases the desired output z(t) is not given as a known nonlinear operation on a set of Gaussian variables. In cases 5 and 6, the Gaussian inputs $\{x_1(t), \dots x_N(t)\}$ are linearly independent of each other; in case 7 the inputs are dependent. In cases 5 and 7 the allowable optimum nonlinear system may be unrealizable; in case 6 the optimum nonlinear system is restricted to be realizable.

We shall determine the optimum nonlinear system as the cascade of two systems D and F, as in Fig. 14. The first system, D, is linear and has N inputs and N outputs.

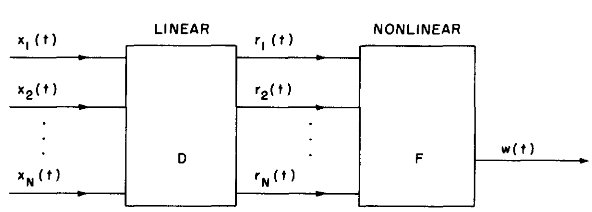

Fig. 14. Optimum nonlinear system for cases 5, 6, and 7.

Its N inputs are, of course, $\{x_1(t), \dots x_N(t)\}$. Its N outputs $\{r_1(t), \dots r_N(t)\}$ are white, Gaussian, and linearly independent of one another. In each of the three cases, system D is allowable and has a linear inverse that is allowable. Thus, for case 6, system D is realizable and has a realizable linear inverse.

The second system, F, is nonlinear and has N inputs $\{r_1(t), \dots r_N(t)\}$ and a single output w(t). System F is the optimum nonlinear system whose inputs are the r's. In cases 5 and 7 system F is unrealizable; in case 6 system F is realizable. The fact that the inputs to system F are the r's and not the x's does not make w(t) a poorer approximation to the desired output because system D is invertible and system F could always produce the x's internally if necessary. The cascade arrangement simplifies the optimization procedures.

In section 3.3 we have shown how to derive system D. For cases 5 and 6, in which the inputs are independent, system D is shown in Fig. 7. The system of Fig. 7 is realizable and has a realizable inverse. For case 7, in which the inputs are dependent, we have to use the more complicated "Schmidt-like" procedure described in section 3.3.

We shall now optimize system F of Fig. 14 by use of a direct variational procedure. We first expand the output w(t) as a sum of allowable G-functionals of the inputs $\{r_1(t), \ldots r_N(t)\}$.

$$w(t) = \sum_{n=0}^{\infty} \sum_i G_n(t, K_{n,i}, \{r\}_i)$$

We wish to find the set of kernels $\{K_{n,i}\}$ that minimizes the mean-square error given by

$$\overline{[w(t)-z(t)]^2} = \overline{\left[\sum_{n=0}^{\infty} \sum_i G_n(t, K_{n,i}, \{r\}_i) - z(t)\right]^2}$$

If the set of kernels $\{K_{n,i}\}$ is optimum, it has the property that if we add to the optimum w(t) any arbitrary allowable G-functional of the r's such as

$$\epsilon\, G_m(t, L_{m,j}, \{r\}_j)$$

then the error must have zero derivative with respect to ϵ when $\epsilon = 0$. That is,

$$\frac{d}{d\epsilon} \overline{\left[\sum_{n=0}^{\infty} \sum_i G_n(t, K_{n,i}, \{r\}_i) + \epsilon\, G_m(t, L_{m,j}, \{r\}_j) - z(t)\right]^2}\Bigg|_{\epsilon=0} = 0 \qquad (119)$$

The kernel $L_{m,j}$ is arbitrary with the restriction that for case 6, in which the system must be realizable, the kernel $L_{m,j}$ must use only past and present values of the inputs.

When we perform the differentiation, Eq. 119 becomes

$$\overline{2G_m(t, L_{m,j}, \{r\}_j)\left[\sum_{n=0}^{\infty} \sum_i G_n(t, K_{n,i}, \{r\}_i) - z(t)\right]} = 0$$

If we apply the orthogonality of G-functionals of different degree given by Eq. 33, we obtain

$$\overline{2G_m(t, L_{m,j}, \{r\}_j)\left[\sum_i G_m(t, K_{m,i}, \{r\}_i) - z(t)\right]} = 0 \qquad (120)$$

We now evaluate the averages in Eq. 120. From Eq. 34, we see that the averages of the pairs of G-functionals are given by

$$\overline{G_m(t, L_{m,j}, \{r\}_j) \sum_i G_m(t, K_{m,i}, \{r\}_i)} = \sum_i \int \cdots \int L_{m,j}(T_1, \ldots T_n)\, K_{m,i}(s_1, \ldots s_m)$$
$$\times\, Q_m\left[r_{j_1}(t-T_1), \ldots r_{j_m}(t-T_m); r_{i_1}(t-s_1), \ldots r_{i_m}(t-s_m)\right] dT_1 \ldots dT_m\, ds_1 \ldots ds_m \tag{121}$$

Recall that Q_m is a sum of terms in which each member of the set $\left\{r_{j_1}, \ldots r_{j_m}\right\}$ is paired and averaged with a member of the set $\left\{r_{i_1}, \ldots r_{i_m}\right\}$, and in which these averaged pairs are multiplied together. If $i \neq j$, then the set $\left\{r_{j_1}, \ldots r_{j_m}\right\}$ is different from the set $\left\{r_{i_1}, \ldots r_{i_m}\right\}$ and thus for every pairing arrangement at least one r will be paired with a different r for averaging. But since the r's are constructed to be independent of one another, the average of the product of two different r's is zero. Therefore

$$Q_m\left[r_{j_1}(t-T_1), \ldots r_{j_m}(t-T_m); r_{i_1}(t-s_1), \ldots r_{i_m}(t-s_m)\right] = 0 \qquad i \neq j \tag{122}$$

If $i = j$, then the set $\left\{r_{j_1}, \ldots r_{j_m}\right\}$ is the same as the set $\left\{r_{i_1}, \ldots r_{i_m}\right\}$. In that case, there may be several different pairing arrangements of Q_m in which no r is paired and averaged with a different r. If $b_k(m, j)$ is the number of times r_k appears in the set $\left\{r_{j_1}, \ldots r_{j_m}\right\}$, then the number of different pairing arrangements of Q_m in which no r is paired with a different r is given by

$$c_{m,j} = \prod_{k=1}^{N} \left[b_k(m, j)!\right] \tag{123}$$

because for each k there are $b_k(m, j)!$ permutations of the r's which do not change Q_m.

We can, without loss of generality, demand that each kernel $\{K_{m,i}(s_1, \ldots s_m)\}$ be symmetric in those s that correspond to s of the same r_k's. Without changing any of the G-functionals, we can substitute for an unsymmetric kernel a kernel that is symmetric in those s that correspond to s of the same r_k's and is formed by adding up kernels formed by all permutations of those s and dividing by the number of such permutations. We can also demand that the kernel $L_{m,j}(T_1, \ldots T_m)$ be similarly symmetric. For example, the expression

$$\iiint K_{3,3}(s_1, s_2, s_3)\, r_1(t-s_1)\, r_1(t-s_2)\, r_2(t-s_3)\, ds_1 ds_2 ds_3$$

in which the kernel is unsymmetric in s_1 and s_2, that is,

$$K_{3,3}(s_1, s_2, s_3) \neq K_{3,3}(s_1, s_2, s_3)$$

is equal to the expression

$$\iiint \frac{1}{2}\left[K_{3,3}(s_1, s_2, s_3)+K_{3,3}(s_2, s_1, s_3)\right] r_1(t-s_1)\, r_2(t-s_2)\, r_3(t-s_3)\, ds_1 ds_2 ds_3$$

in which the kernel

$$\frac{1}{2}\left[K_{3,3}(s_1, s_2, s_3)+K_{3,3}(s_2, s_1, s_3)\right]$$

is clearly symmetric in s_1, and s_2.

In Eq. 121 for $i = j$ and for kernels having the symmetry just discussed, we can replace

$$Q_m\left[r_{j_1}(t-T_1), \dots r_{j_m}(t-T_m); r_{j_1}(t-s_1), \dots r_{j_m}(t-s_m)\right] \tag{124}$$

by one of its pairing arrangements in which no r is paired with a different r, and then multiply this term by the number $c_{m,j}$ (Eq. 123) of pairing arrangements with the same property. One such pairing arrangement is clearly

$$\overline{r_{j_1}(t-T_1)\, r_{j_1}(t-s_1)}\; \overline{r_{j_2}(t-T_2)\, r_{j_2}(t-s_2)} \dots \overline{r_{j_m}(t-T_m)\, r_{j_m}(t-s_m)}$$

Thus we could replace expression 124 by

$$c_{m,j}\, \overline{r_{j_1}(t-T_1)\, r_{j_1}(t-s_1)}\; \overline{r_{j_2}(t-T_2)\, r_{j_2}(t-s_2)} \dots \overline{r_{j_m}(t-T_m)\, r_{j_m}(t-s_m)} \tag{125}$$

By using Eqs. 122 and expression 125, Eq. 121 becomes

$$\overline{G_m(t, L_{m,j}, \{r\}_j) \sum_i G_m(t, K_{m,i}, \{r\}_i)} = c_{m,j} \int \dots \int L_{m,j}(T_1, \dots T_m)\, K_{m,j}(s_1, \dots s_m)$$
$$\times \left[\overline{r_{j_1}(t-T_1) r_{j_1}(t-s_1)}\; \overline{r_{j_2}(t-T_2) r_{j_2}(t-s_2)} \dots \overline{r_{j_m}(t-T_m) r_{j_m}(t-s_m)}\right] dT_1 \dots dT_m\, ds_1 \dots ds_m \tag{126}$$

Since each of the inputs is white,

$$\overline{r_k(t-T)\, r_k(t-s)} = \delta(T-s) \tag{127}$$

Substituting Eq. 127 in Eq. 126 and then integrating on the s, we obtain

$$\overline{G_m(t, L_{m,j}, \{r\}_j) \sum_i G_m(t, K_{m,i}, \{r\}_i)}$$
$$= c_{m,j} \int \dots \int L_{m,j}(T_1, \dots T_m)\, K_{m,j}(T_1, \dots T_m)\, dT_1 \dots dT_m \tag{128}$$

We have now evaluated the averages of the pairs of G-functionals in Eq. 120. To evaluate the average of the G-functional with z(t) in Eq. 120, we expand the G-functional by means of Eq. 20 and obtain

$$\overline{z(t)\, G_m(t, L_{m,j}, \{r\}_j)} = \int \cdots \int L_{m,j}(T_1, \ldots T_m)\, \overline{z(t) \sum_{\nu=0}^{\left[\frac{m}{2}\right]} (-1)^{\nu} P_{\nu}\left[r_{j_1}(t-T_1), \ldots r_{j_m}(t-T_m)\right]}\, dT_1 \ldots dT_m \tag{129}$$

Substituting Eqs. 128 and 129 in Eq. 120, we obtain

$$2\int \cdots \int L_{m,j}(T_1, \ldots T_m)\left\{c_{m,j}K_{m,j}(T_1, \ldots T_m) - \overline{z(t) \sum_{\nu=0}^{\left[\frac{m}{2}\right]} (-1)^{\nu} P_{\nu}\left[r_{j_1}(t-T_1), \ldots r_{j_m}(t-T_m)\right]}\right\} dT_1 \ldots dT_m = 0 \tag{130}$$

For cases 5 and 7 in which the systems can be unrealizable, the kernel $L_{m,j}(T_1, \ldots T_m)$ is arbitrary for all T's, and hence the quantity in braces in Eq. 130 must be zero for all T's. Thus for cases 5 and 7, $K_{m,j}(T_1, \ldots T_m)$ is given by

$$K_{m,j}(T_1, \ldots T_m) = \frac{1}{c_{m,j}}\, \overline{z(t) \sum_{\nu=0}^{\left[\frac{m}{2}\right]} (-1)^{\nu} P_{\nu}\left[r_{j_1}(t-T_1), \ldots r_{j_m}(t-T_m)\right]} \tag{131}$$

For case 6 in which the systems must be realizable, the kernel $L_{m,j}(T_1, \ldots T_m)$ is arbitrary if all the T's are non-negative, and hence the quantity in braces in Eq. 130 must be zero if all the T's are non-negative. Thus for case 6

$$K_{m,j}(T_1, \ldots T_m) = \frac{1}{c_{m,j}}\, \overline{z(t) \sum_{\nu=0}^{\left[\frac{m}{2}\right]} (-1)^{\nu} P_{\nu}\left[r_{j_1}(t-T_1), \ldots r_{j_m}(t-T_m)\right]} \tag{132}$$

if all T's ≥ 0. For case 6, $L_{m,j}(T_1, \ldots T_m)$ will be zero if any T is negative, and this fact will satisfy Eq. 130. The realizability constraint means that for case 6

$$K_{m,j}(T_1, \ldots T_m) = 0 \qquad \text{if any } T < 0 \tag{133}$$

Equation 131 gives the optimum kernels for cases 5 and 7. Equations 132 and 133 give the optimum kernels for case 6. From the definition of P_ν as the sum of products and averages of its variables, we see that these kernels will depend upon the autocorrelation and crosscorrelation functions of the r's and the higher-order crosscorrelation functions between the r's and z(t). For example, if $K_{2,3}(T_1, T_2)$ operates on the set (r_1, r_2), then from Eq. 131 and the definitions of P_ν and $c_{m,j}$, we obtain

$$K_{2,3}(T_1, T_2) = \frac{1}{c_{2,3}}\, \overline{z(t)[r_1(t-T_1)r_2(t-T_1) - \overline{r_1(t-T_1)r_2(t-T_2)}]}$$

$$= \overline{z(t)\ r_1(t-T_1)\ r_2(t-T_2)} - \overline{z(t)}\ \overline{r_1(t-T_1)\ r_2(t-T_2)}$$

The number $c_{2,3} = 1$ because no input appears more than once in the set (r_1, r_2).

Since the r's are formed in a known linear manner from the x's, the higher-order crosscorrelation functions between the r's and z(t) can be computed from the higher-order crosscorrelation functions between the x's and z(t). Thus, to optimize a general nonlinear system for cases 5, 6, and 7 we must know all the higher-order crosscorrelation functions between the inputs $\{x_1(t), \ldots x_N(t)\}$ and the desired output z(t). That is, we must be given terms of the form

$$\overline{z(t)\, x_{i_1}(t-T_1)\, x_{i_2}(t-T_2) \ldots x_{i_n}(t-T_n)}$$

for all sets of the inputs and all n.

3.7 THEOREM RELATING OPTIMUM REALIZABLE SYSTEMS TO OPTIMUM UNREALIZABLE SYSTEMS

In this section we shall prove a theorem that is interesting in its own right and will be used in the optimization procedure for case 8.

The theorem can be stated: If $w_1(t)$ is the optimum, unrealizable, nonlinear approximation to z(t) in the mean-square sense, and if $w_2(t)$ is the optimum, realizable, nonlinear approximation to $w_1(t)$ in the mean-square sense, then $w_2(t)$ is the optimum, realizable, nonlinear approximation to z(t) in the mean-square sense.

The variable $w_1(t)$ can be thought of as the output of an optimum unrealizable nonlinear system whose desired output is z(t); the variable $w_2(t)$ can be thought of as the output of an optimum realizable nonlinear system whose desired output is $w_1(t)$. The inputs to these two systems are the same. This theorem states that the output of the optimum realizable nonlinear system whose desired output is z(t) and whose inputs are the same as those of the other two systems is actually $w_2(t)$.

To prove this theorem, we must show that the error E_1 defined by

$$E_1 = \overline{[w_2(t)-z(t)]^2}$$

is less than or equal to the error produced by the output of any realizable nonlinear system with the same inputs. We assume now that the allowable nonlinear systems are general enough so that the output of any realizable nonlinear system can be obtained as the sum of $w_2(t)$ plus the output g(t) of another realizable nonlinear system. Thus we wish to show that $E_1 \leqslant E_2$, where E_2 is defined by

$$E_2 = \overline{[w_2(t)+g(t)-z(t)]^2} \tag{134}$$

We now regroup terms in Eq. 134 as follows:

$$E_2 = \overline{\{g(t)-[z(t)-w_2(t)]\}^2}$$

Expanding the square, we obtain

$$E_2 = \overline{g^2(t)} - \overline{2g(t)[z(t)-w_2(t)]} + \overline{[z(t)-w_2(t)]^2} \tag{135}$$

The last term in Eq. 135 is recognized as E_1. If we can prove that

$$\overline{g(t)[z(t)-w_2(t)]} = 0 \tag{136}$$

is true, then Eq. 135 becomes

$$E_2 = \overline{g^2(t)} + E_1 \tag{137}$$

But $\overline{g^2(t)} \geq 0$, since the average of a squared real variable is always non-negative. Thus if Eq. 136 is true, Eq. 137 shows that $E_1 \leq E_2$. We shall now prove Eq. 136 which can be rewritten

$$\overline{g(t)\, z(t)} = \overline{g(t)\, w_2(t)} \tag{138}$$

The fact that $w_2(t)$ is the optimum realizable nonlinear approximation to $w_1(t)$ means that $w_2(t)$ minimizes the error

$$\overline{[w_2(t)-w_1(t)]^2}$$

In particular, if we add to $w_2(t)$ the output $\epsilon\, g(t)$, then derivative of the error with respect to ϵ will be zero when evaluated at $\epsilon = 0$. That is,

$$\frac{d}{d\epsilon} \overline{[w_2(t) + \epsilon\, g(t) - w_1(t)]^2}\Big|_{\epsilon=0} = 0$$

Performing this differentiation, we obtain

$$2\, \overline{g(t)[w_2(t)-w_1(t)]} = 0$$

This equation can be rewritten as

$$\overline{g(t)\, w_2(t)} = \overline{g(t)\, w_1(t)} \tag{139}$$

Similarly, the fact that $w_1(t)$ is the optimum unrealizable approximation to $z(t)$ means that $w_1(t)$ minimizes the error

$$\overline{[w_1(t)-z(t)]^2}$$

If we add to $w_2(t)$ the output $\epsilon\, f(t)$, where $f(t)$ is the output of any unrealizable system, then the derivative of the error with respect to ϵ will be zero when evaluated at $\epsilon = 0$. Since a realizable output is a special case of an unrealizable output, we may choose $f(t) = g(t)$, the realizable output used previously. This necessary condition on the derivative then becomes

$$\frac{d}{d\epsilon}\overline{[w_1(t)+\epsilon g(t)-z(t)]^2}\Big|_{\epsilon=0} = 0$$

Performing this differentiation, we obtain

$$2\,\overline{g(t)[w_1(t)-z(t)]} = 0 \tag{140}$$

Equation 140 can be rewritten as

$$\overline{g(t)\,w_1(t)} = \overline{g(t)\,z(t)} \tag{141}$$

Combining Eqs. 139 and 141, we obtain

$$\overline{g(t)\,w_2(t)} = \overline{g(t)\,w_1(t)} = \overline{g(t)\,z(t)} \tag{142}$$

which proves Eqs. 136 and 138 and completes the proof of the theorem.

This theorem can also be stated for linear systems, or for any class of systems that have the property that if a(t) and b(t) are allowable outputs then a(t) − b(t) is an allowable output. The proofs are the same.

3.8 CASE 8

We shall present the optimization procedure for case 8 in Table 1. Case 8 is the most general of all the cases because the desired output z(t) is not produced in a known manner from Gaussian variables; the optimum system is restricted to be realizable, and the inputs are dependent. This case is treated last because we need some of the results of the previous cases and the theorem of section 3.7 to derive the optimization procedure.

The optimization is carried out in two steps. First, we derive an optimum unrealizable nonlinear system whose inputs are the given dependent inputs $\{x_1(t), \dots x_N(t)\}$ and whose desired output is the given desired output z(t). We can derive this unrealizable system, for the problem is that of case 7 (Table 1) which we have solved. Let us call the output of this unrealizable system $w_1(t)$.

Next, we treat $w_1(t)$ as a desired output and derive an optimum realizable nonlinear system whose output is the best approximation to $w_1(t)$. We can derive this second system because our desired output $w_1(t)$ is produced in a known manner from Gaussian inputs $\{x_1(t), \dots x_N(t)\}$. The system to be optimized is restricted to be realizable, and its inputs $\{x_1(t), \dots x_N(t)\}$ are dependent. These three conditions are those of case 4 which we have solved. Let us call the output of this second system $w_2(t)$.

By the theorem of section 3.7, the optimum realizable approximation to the optimum unrealizable approximation is, in fact, the optimum realizable approximation to the desired output. Thus, the system that produces $w_2(t)$ is the optimum realizable nonlinear system for the desired output.

3.9 STATISTICAL INFORMATION USED IN OPTIMIZATION PROCEDURES

Let us review the statistical information that we must use in these different cases of optimization. In the first four cases, in which the desired output is given as a known nonlinear G-functional expansion of a set of Gaussian inputs, we performed, at least conceptually, some optimum linear operations, and then a nonlinear operation defined in terms of the kernels of the given G-functionals. The optimum linear operations require only a knowledge of first-order autocorrelations and first-order crosscorrelations. The nonlinear operations are well defined and need no statistical information.

In the last four cases, in which the desired output is not given as a G-functional of Gaussian inputs, the kernels that have to be determined depend upon higher-order crosscorrelations between the inputs $\{x_1(t), \dots x_N(t)\}$ and the desired output z(t). (Recall that case 8 depended upon case 7 and hence also used these higher-order crosscorrelations.) In particular, from Eq. 131 we see that a kernel of degree m depends upon

$$\overline{\frac{1}{c_{m,j}} z(t) \sum_{\nu=0}^{\left[\frac{m}{2}\right]} (-1) P_\nu\left[r_{j_1}(t-T_1), \dots r_{j_m}(t-T_m)\right]} \tag{143}$$

where the r's depend linearly on the x's.

In practice, we could never measure an infinite set of statistics such as the set of all higher-order crosscorrelations. It might therefore be of interest to see what restricted class of systems we can optimize if we know only crosscorrelations up to order M. We note from the definition of P_ν that the terms in expression 143 contain, at most, the product of m r's. Hence expression 143 involves crosscorrelations of order m or less, and thus an optimum kernel of degree m can be determined from crosscorrelations of order m or less. Since kernels of degree m determine G-functionals of degree m, it follows that if we know crosscorrelations of order M and less, then we can determine an optimum nonlinear system from the restricted class of nonlinear systems represented by sums of G-functionals of degree M and less.

IV. SIMPLE NONLINEAR SYSTEMS

4.1 INTRODUCTION

In Section III, we discussed general multi-input nonlinear systems. Unfortunately, there is a price we must pay for this generality. We have already noted that in order to optimize some of these general systems, we would have to know an infinite set of higher-order crosscorrelations. In practice these correlation functions would have to be measured, and hence we could never obtain an infinite set. Furthermore, even if we had all the statistical information that we needed, we would still be unable to construct a general nonlinear system. The model for a general multi-input nonlinear system shown in Fig. 3 contains an infinite number of linear systems and hence cannot be built. We might, therefore, be willing to give up some of the generality of these systems in exchange for the ease of determination and ease of construction of some simpler systems.

In section 3.9 we saw that by assuming no knowledge of crosscorrelations of higher order than M we could optimize only a nonlinear system whose G-function expansion was of no higher degree than M. However, with finite electrical systems we cannot at the present time construct arbitrary G-functionals of degree greater than one. Thus, by restricting the statistical information we have not ensured that we can construct the optimum system.

The approach that we shall take now is to consider nonlinear systems of restricted form that can be easily constructed. Each system will consist of a finite number of linear systems and a finite number of simple nonlinear no-memory devices such as squarers, multipliers, and constants (dc voltages). In each system of given form there will be some undetermined parameters such as the impulse responses of some of the linear systems. We shall derive methods by which we can determine these parameters in such a way as to minimize the mean-square error between the output of the system and some given desired output. Because these systems are restricted in form, their output will not be as good an approximation to the desired output as would the output of a more general nonlinear system.

Since the nonlinear devices will be simple, the optimization procedures will not involve crosscorrelations of higher order than 2. Instead of measuring the second-order crosscorrelation and then computing the optimum values of the parameters from the crosscorrelation, we shall describe measurements that will more directly yield these optimum values.

Each of the systems considered has a single white Gaussian input. The restriction that the input be Gaussian is necessary for the optimization procedures that will be presented. The restriction that the input be white causes no loss of generality because any non-white input can always be whitened realizably and reversibly.

4.2 G-FUNCTIONALS OF A SINGLE INPUT

In working with simple nonlinear systems with a single Gaussian input, we still find G-functionals useful. In this section we shall derive the form and properties of G-functionals of a single Gaussian input from the form and properties of G-functionals of many Gaussian inputs. These forms and properties are somewhat simpler in the single-input case.

Recall that both the definition (Eq. 20) and some of the properties (Eq. 34) of G-functionals involve summations over different pairings of the inputs. We have previously noted that a kernel can be made symmetric in those variables that are associated with the same input; therefore, with only a single input, we can, without loss of generality, require that each kernel be symmetric in all of its variables. For example, we require that

$$K_3(T_1, T_2, T_3) = K_3(T_3, T_2, T_1)$$

Since the kernels are symmetric for a single input, then any pairing arrangement has the same effect as any other similar pairing arrangement, and we can use one pairing arrangement and merely multiply by the number of similar pairing arrangements. For example,

$$\iiint K_3(T_1, T_2, T_3) \left[x(t-T_1)\overline{x(t-T_2)x(t-T_3)} + x(t-T_3)\overline{x(t-T_1)x(t-T_2)} + x(t-T_2)\overline{x(t-T_3)x(t-T_1)}\right] dT_1 dT_2 dT_3$$
$$= 3\iiint K_3(T_1, T_2, T_3)\, x(t-T_1)\, \overline{x(t-T_2)\, x(t-T_3)}\, dT_1 dT_2 dT_3$$

If we apply this technique to the definition of a G-functional given by Eq. 20, we obtain

$$G_n(t, K_n, x) = \sum_{\nu=0}^{\left[\frac{n}{2}\right]} a_{n-2\nu}^{(n)} \int \cdots \int K_n(T_1, \ldots T_n)\, x(t-T_1) \ldots x(t-T_{n-2\nu})$$
$$\times \overline{x(t-T_{n-2\nu+1})\, x(t-T_{n-2\nu+2})\, x(t-T_{n-2\nu+3})\, x(t-T_{n-2\nu+4}) \ldots x(t-T_{n-1})\, x(t-T_n)}\, dT_1 \ldots dT_n \tag{144}$$

The constant $a_{n-2\nu}^{(n)}$ is $(-1)^\nu$ times the number of different pairing arrangements in

$$P_\nu[x(t-T_1), \ldots x(t-T_n)]$$

and this number of pairing arrangements is, from the definition of P_ν, the number of different ways ν pairs can be formed from n x's and is equal to

$$\frac{n!}{2^\nu (n-2\nu)!\ \nu!}$$

Therefore $a^{(n)}_{n-2\nu}$ is given by

$$a^{(n)}_{n-2\nu} = \frac{(-1)^{\nu}\, n!}{2^{\nu}(n-2\nu)!\,\nu!} \tag{145}$$

In Eq. 144 we have omitted those subscripts which refer to sets of inputs, since there is only one input.

By applying the same technique to Eqs. 34 and 46, we obtain

$$\overline{G_n(t, K_n, x)\, G_n(t, L_n, x)} = n! \int \cdots \int K_n(T_1, \ldots T_n)\, L_n(s_1, \ldots s_n)$$
$$\times\, \overline{x(t-T_1)\, x(t-s_1)}\;\overline{x(t-T_2)\, x(t-s_2)} \cdots \overline{x(t-T_n)\, x(t-s_n)}\, dT_1 \ldots dT_n\, ds_1 \ldots ds_n \tag{146}$$

$$\overline{G_n(t, K_n, x)\, G_n(t, L_n, y)} = n! \int \cdots \int K_n(T_1, \ldots T_n)\, L_n(s_1, \ldots s_n)$$
$$\times\, \overline{x(t-T_1)\, y(t-s_1)}\;\overline{x(t-T_2)\, y(t-s_2)} \cdots \overline{x(t-T_n)\, y(t-s_n)}\, dT_1 \ldots dT_n\, ds_1 \ldots ds_n \tag{147}$$

The number n! is the number of different ways n things can be paired with n other things.

Of course, G-functionals of different degree are still linearly independent; we have not defined new G-functionals, we have merely rewritten the G-functionals in a form that is made possible by the single input.

If input x(t) is white Gaussian noise, that is,

$$\overline{x(t)\, x(t+T)} = \delta(T)$$

then Eqs. 144 and 146 take the following forms

$$G_n(t, K_n, x) = \sum_{\nu=0}^{\left[\frac{n}{2}\right]} a^{(n)}_{n-2\nu} \int \cdots \int K_n(T_1, \ldots T_n)\, x(t-T_1) \ldots x(t-T_{n-2\nu})$$
$$\times\, \delta(T_{n-2\nu+1}-T_{n-2\nu+2}) \cdots \delta(T_{n-1}-T_n)\, dT_1 \ldots dT_n \tag{148}$$

$$\overline{G_n(t, K_n, x)\, G_n(t, L_n, x)} = n! \int \cdots \int K_n(T_1, \ldots T_n)\, L_n(T_1, \ldots T_n)\, dT_1 \ldots dT_n \tag{149}$$

Equations 148 and 149 are the ones that Wiener (2) uses.

4.3 SYSTEM A

The first system that we shall consider has the form shown in Fig. 15 and is called system A. The output w(t) of system A is the sum of a constant c_o, plus the output of a linear system with impulse response $L_1(t)$, plus the squared output of N linear systems with impulse response $\{h_1(t), \ldots h_N(t)\}$, the squared outputs being multiplied by the gains $\{a_1, \ldots a_N\}$, respectively. The white Gaussian input to the system is x(t). The constant c_o, the impulse responses $L_1(t)$ and $\{h_i(t)\}$, and the gains $\{a_1, \ldots a_N\}$ are all to be determined to minimize the mean-square error between w(t) and the desired output z(t).

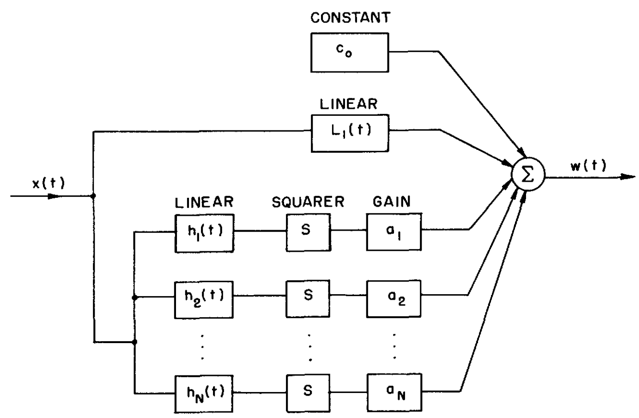

Fig. 15. System A.

By means of the constant c_o we can, of course, produce any G-functional of degree zero. With the linear system $L_1(t)$ we can produce any G-functional of degree 1 of the input x(t). The h's, the squarers, and the a's allow us to produce only a limited class of G-functionals of degree 2 of the input. System A is, then, just a bit more general than a linear system.

The arrangement of the h's, squarers, and a's in Fig. 15 is somewhat more general than at first it might appear. In particular, if we had N arbitrary linear systems with input x(t), and if we formed an output y(t) by summing the products of pairs of these outputs times arbitrary gains, then we could form this same output y(t) with the arrangement of the h's, squarers, and a's of Fig. 15 by proper choices of the impulse responses $\{h_i(t)\}$ and gains $\{a_i\}$. That is, if there are N arbitrary linear systems with impulse responses $\{g_i(t)\}$ and outputs $\{v_i(t)\}$ given by

$$v_i(t) = \int_0^{\infty} g_i(T)\, x(t-T)\, dT \tag{150}$$

and if we form y(t) as

$$y(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} b_{i,j} \, v_i(t) \, v_j(t) \tag{151}$$

where the constants $\{b_{i,j}\}$ are arbitrary, then by proper choice of the impulse responses $\{h_i(t)\}$ and the gains $\{a_i\}$ we can form y(t) as

$$y(t) = \sum_{i=1}^{N} a_i u_i^2(t) \tag{152}$$

where $u_i(t)$ is given by

$$u_i(t) = \int_0^{\infty} h_i(T) \, x(t-T) \, dT$$

To prove the statement just given we shall show how the $\{u_i(t)\}$ and $\{a_i\}$ are determined in terms of $\{v_i(t)\}$ and $\{b_{i,j}\}$. We can first assume with no loss of generality that $b_{i,j} = b_{j,i}$ because with no change in y(t) we can always make the $b_{i,j}$'s have that property. We now introduce some matrix notation. We form a column vector v given by

$$v = \begin{Bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_N(t) \end{Bmatrix}$$

and a symmetric $N \times N$ matrix b given by

$$b = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,N} \\ b_{2,1} & \cdots & & \\ \vdots & & & \\ b_{N,1} & \cdots & & b_{N,N} \end{bmatrix}$$

If v^T is the transpose of v, then from the definition of matrix multiplication, y(t) of Eq. 151 can be expressed as

$$y(t) = v^T \, b \, v \tag{153}$$

But according to matrix theory (see Hildebrand (15)) there always exists an $N \times N$ matrix Q with the following properties:

$$Q \, Q^T = Q^T \, Q = I \tag{154a}$$

$$Q \, b \, Q^T = c \tag{154b}$$

where I is the identity matrix, and c is an $N \times N$ diagonal matrix of the form

$$c = \begin{bmatrix} c_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & c_{2,2} & & & \\ \vdots & & & & \\ 0 & & & & c_{N,N} \end{bmatrix}$$

Using Eq. 154a, we can express Eq. 153 as

$$y(t) = v^T I \, b \, I \, v = v^T Q^T Q \, b \, Q^T Q \, v \tag{155}$$

Substituting Eq. 154b in Eq. 155, we obtain

$$y(t) = v^T Q^T c \, Q \, v \tag{156}$$

If we define a new vector d as

$$d = Q \, v \tag{157}$$

and recall from matrix theory that

$$d^T = (Qv)^T = v^T Q^T$$

then Eq. 156 becomes

$$y(t) = d^T c \, d \tag{158}$$

The summation indicated by Eq. 158 is

$$y(t) = \sum_{i=1}^{N} d_i^2 \, c_{i,i} \tag{159}$$

where d_i is the i^{th} element of the vector d. Equation 159 has the same form as our desired equation, Eq. 152. In Eq. 157 each element d_i is defined as a linear sum of the elements of v, each of which is the output of a linear system with impulse response $g_i(t)$ and input x(t), as shown in Eq. 150. Therefore d_i is the output of a linear system whose impulse response is the same sum of the $\{g_i(t)\}$ as d_i is of the corresponding elements of v. Each element $c_{i,i}$ is a sum of the $b_{i,j}$'s and hence is a number. Therefore, if we let $u_i(t) = d_i$ and if we let $a_i = c_{i,i}$ then Eq. 159 becomes Eq. 152, which was to be proved.

Without reducing the generality of system A (Fig. 15), we can normalize each of the impulse responses $\{h_i(t)\}$. Thus

$$\int_0^\infty h_i^2(t) \, dt = 1 \tag{160}$$

The gains $\{a_i\}$ can perform all of the necessary magnitude scaling.

In determining how to optimize system A (Fig. 15), it is convenient to express the output w(t) as a sum of G-functionals of the single input x(t). By inspection we can express w(t) as

$$w(t) = G_o + G_1(t, L_1, x) + G_2(t, L_2, x) \tag{161}$$

where

$$G_2(t, L_2, x) = \int_0^\infty \int_0^\infty \sum_{i=1}^{N} a_i h_i(T_1)\, h_i(T_2)[x(t-T_1)x(t-T_2) - \overline{x(t-T_1)x(t-T_2)}]\, dT_1 dT_2 \tag{162}$$

$$G_1(t, L_1, x) = \int_0^\infty L_1(t)\, x(t-T)\, dT \tag{163}$$

$$G_o = c_o + \int_0^\infty \int_0^\infty \sum_{i=1}^{N} a_i h_i(T_1)\, h_i(T_2)\, \overline{x(t-T_1)\, x(t-T_2)}\, dT_1 dT_2 \tag{164}$$

The fact that the input is white means that

$$\overline{x(t)\, x(t-T)} = \delta(T) \tag{165}$$

By using Eq. 165 and the normality of Eq. 160, Eqs. 162 and 164 can be expressed as

$$G_2(t, L_2, x) = \int_0^\infty \int_0^\infty \sum_{i=1}^{N} a_i h_i(T_1)\, h_i(T_2)\, x_i(t-T_1)\, x_i(t-T_2)\, dT_1 dT_2 - \sum_{i=1}^{N} a_i \tag{166}$$

$$G_o = c_o + \sum_{i=1}^{N} a_i \tag{167}$$

The kernel L_2 is given by

$$L_2(T_1, T_2) = \sum_{i=1}^{N} a_i h_i(T_1)\, h_i(T_2) \tag{168}$$

Notice that $L_2(T_1, T_2)$ is symmetric in T_1 and T_2.

We now express the mean-square error E in terms of these G-functionals.

$$E = \overline{[w(t)-z(t)]^2} = \overline{[G_o+G_1(t, L_1, x)+G_2(t, L_2, x)-z(t)]^2} \tag{169}$$

By expanding the square and applying the orthogonality of the G-functionals, Eq. 169 becomes

$$E = \overline{G_o^2} + \overline{G_1^2(t, L_1, x)} + \overline{G_2^2(t, L_2, x)} - \overline{2z(t)[G_o+G_1(t, L_1, x)+G_2(t, L_2, x)]} + \overline{z^2(t)} \tag{170}$$

Equation 170 can be rewritten as

$$E = \overline{[G_o-z(t)]^2} + \overline{[G_1(t, L_1, x)-z(t)]^2} + \overline{[G_2(t, L_2, x)-z(t)]^2} - \overline{2z^2(t)} \tag{171}$$

In Eq. 171 each G-functional appears in a different squared term and each term can be minimized separately. Although the constants $\{a_i\}$ appear in both G_o and $G_2(t, L_2, x)$, their effect on G_o can be completely compensated for by the constant c_o.

We shall now evaluate the optimum values of c_o, $L_1(t)$, $\{h_i(t)\}$, and $\{a_i\}$ by minimizing each of the terms in Eq. 171. The first term in Eq. 171 is the mean-square difference between a constant G_o and a random variable z(t). This term is clearly minimized by setting the constant G_o equal to the mean of the random variable.

$$G_o = \overline{z(t)} \tag{172}$$

Substituting the definition of G_o given by Eq. 167 in Eq. 172, we find that c_o is given by

$$c_o = \overline{z(t)} - \sum_{i=1}^{N} a_i \tag{173}$$

The constants $\{a_i\}$ will be determined later.

We minimize the second term of Eq. 171,

$$\overline{[G_1(t, L_1, x)-z(t)]^2}$$

by a direct variational procedure. We add an arbitrary realizable impulse response $\epsilon\, g(t)$ to the optimum impulse response $L_1(t)$ and set the derivative of the error with respect to ϵ equal to zero when $\epsilon = 0$.

$$\frac{d}{d\epsilon}\overline{\left\{\int_0^\infty [L_1(T)+\epsilon\, g(T)]\, x(t-T)\, dT - z(t)\right\}^2}\Bigg|_{\epsilon=0} = 0$$

Performing the differentiation and using the fact that the input is white, we obtain

$$2\int_0^\infty g(T)\, dT[L_1(T)-\overline{x(t-T)z(t)}] = 0$$

Since g(T) is arbitrary for $T \geqslant 0$, the term in brackets must be zero for $T \geqslant 0$. That is,

$$L_1(T) = \overline{x(t-T)\, z(t)} \qquad T \geqslant 0 \tag{174}$$

Since $L_1(T)$ is realizable,

$$L_1(T) = 0 \qquad T < 0 \tag{175}$$

Notice that $L_1(t)$ in Eq. 174 is just the crosscorrelation between the input and the desired output.

The minimization of the third term of Eq. 171,

$$\overline{[G_2(t, L_2, x)-z(t)]^2}$$

is a little more complicated. We first expand the square in

$$\overline{[G_2(t, L_2, x)-z(t)]^2} = \overline{G_2^2(t, L_2, x)} - \overline{2z(t)\ G_2(t, L_2, x)} + \overline{z^2(t)} \qquad (176)$$

By means of Eq. 149, we see that

$$\overline{G_2^2(t, L_2, x)} = 2\int_0^\infty \int_0^\infty L_2^2(T_1, T_2)\ dT_1 dT_2 \qquad (177)$$

By using the definition of L_2 given by Eq. 168, Eq. 177 becomes

$$\overline{G_2^2(t, L_2, x)} = 2\int_0^\infty \int_0^\infty \left[\sum_{i=1}^{N} a_i h_i(T_1)\ h_i(T_2)\right]^2 dT_1 dT_2 \qquad (178)$$

Since the average value of the product of a constant times a G-functional of degree 2 is zero, we may write the second term of Eq. 176 as

$$2\ \overline{z(t)\ G_2(t, L_2, x)} = 2\overline{[z(t)-\overline{z(t)}]\ G_2(t, L_2, x)} \qquad (179)$$

Since the average value of a constant times $z(t) - \overline{z(t)}$ is zero, we may write Eq. 179 as

$$2\ \overline{z(t)\ G_2(t, L_2, x)} = 2\overline{[z(t)-\overline{z(t)}]\left[G_2(t, L_2, x) + \sum_{i=1}^{N} a_i\right]} \qquad (180)$$

By using Eq. 166 for the G-functional, Eq. 180 becomes

$$2\ \overline{z(t)\ G_2(t, L_2, x)} = 2\int_0^\infty \int_0^\infty L_2(T_1, T_2)\overline{[x(t-T_1)x(t-T_2)][z(t)-\overline{z(t)}]}\ dT_1 dT_2 \qquad (181)$$

We can now define a symmetric kernel $K(T_1, T_2)$ as

$$K(T_1, T_2) = \frac{1}{2}\overline{[x(t-T_1)x(t-T_2)][z(t)-\overline{z(t)}]} \qquad (182)$$

Substituting this kernel in Eq. 181, and using the definition of L_2 given by Eq. 168, we obtain

$$2\ \overline{z(t)\ G_2(t, L_2, x)} = 4\int_0^\infty \int_0^\infty \sum_{i=1}^{N} a_i h_i(T_1)\ h_i(T_2)\ K(T_1, T_2)\ dT_1 dT_2 \qquad (183)$$

We can now substitute Eqs. 178 and 183 in Eq. 176 and rearrange terms to obtain

$$\overline{[G_2(t, L_2, x)-z(t)]^2} = 2\int_0^\infty \int_0^\infty \left[\sum_{i=1}^{N} a_i h_i(T_1)\ h_i(T_2) - K(T_1, T_2)\right]^2 dT_1 dT_2$$

$$-2\int_0^\infty \int_0^\infty K^2(T_1, T_2)\ dT_1 dT_2 + \overline{z^2(t)} \qquad (184)$$

In Eq. 184 only the term

$$2\int_0^\infty \int_0^\infty \left[\sum_{i=1}^{N} a_i h_i(T_1)\, h_i(T_2) - K(T_1, T_2)\right]^2 dT_1 dT_2 \tag{185}$$

contains the parameters $\{h_i(t)\}$ and $\{a_i\}$. Therefore we minimize the error of Eq. 170 if we choose $\{h_i(t)\}$ and $\{a_i\}$ so as to minimize expression 185. Now it follows directly from Hilbert-Schmidt theory (16) that there exist a set of normalized functions $\{\phi_i(t)\}$ called eigenfunctions and a corresponding set of numbers $\{\lambda_i\}$ called eigenvalues that satisfy the integral equation

$$\lambda_i \phi_i(T_2) = \int_0^\infty K(T_1, T_2)\, \phi_i(T_1)\, dT_1 \tag{186}$$

If the eigenvalues are ordered so that

$$\left|\lambda_i\right| \geq \left|\lambda_j\right| \qquad i < j \tag{187}$$

then, from Hilbert-Schmidt theory (16), it follows that expression 185 is minimized if we choose $\{a_i\}$ and $\{h_i(t)\}$ as follows:

$$\left.\begin{array}{ll} a_i = \lambda_i & i = 1, \dots N \\ h_i(t) = \phi_i(t) & i = 1, \dots N \end{array}\right\} \tag{188}$$

From Eqs. 182, 186, and 188 we see that the set $\{a_i\}$ and $\{h_i(t)\}$ must satisfy the equation

$$a_i h_i(T_2) = \int_0^\infty \frac{1}{2}\, \overline{x(t-T_1)\, x(t-T_2)[z(t)-\overline{z(t)}]}\, h_i(T_1)\, dT_1 \tag{189}$$

where the set a_i is the set of numbers of largest magnitude that satisfy Eq. 189.

Equations 173, 174, 175, and 189 give us the optimum values of the parameters of system A. The solution to the integral equation 189 presents the only difficulty.

4.4 EXPERIMENTAL PROCEDURE FOR OPTIMIZING SYSTEM A

We shall now present an experimental procedure for solving Eq. 189. After presenting the procedure we shall show that it is correct.

The experimental procedure for determining the impulse response $h_1(t)$ is an iterative one in which we start off with an arbitrary linear system with impulse response $f_1(t)$, and then by a series of measurements and a simple computation we determine a new normalized impulse response $f_2(t)$. A linear system with impulse response $f_2(t)$ is then substituted for the one with $f_1(t)$ and the procedure is repeated. The desired impulse response $h_1(t)$ is given by

$$h_1(t) = \lim_{n\to\infty} f_n(t)$$

The experimental system is shown in Fig. 16. The output of the system $f'_{n+1}(T)$ is the average value of the product of the output of a delay line, the output of a linear system with impulse response $f_n(t)$, and the desired output minus its mean $z(t) - \overline{z(t)}$. The procedure is to measure the output of the system $f'_{n+1}(T)$ as a function of the tap position T of the delay line. The next impulse response $f_{n+1}(t)$ is related to $f'_{n+1}(t)$ by

$$f_{n+1}(t) = \frac{f'_{n+1}(t)}{\left(\int_0^{\infty} [f'_{n+1}(T)]^2 \, dT\right)^{1/2}}$$

The gain a_1 is given by

$$a_1 = \lim_{n\to\infty} \frac{f'_{n+1}(t)}{2f_n(t)} \tag{190}$$

Equation 190 will not be a function of time.

With the determination of $h_1(t)$ and a_1 part of system A is known. The experimental procedure for determining each of the remaining impulse responses $\{h_i(t)\}$ and gains $\{a_i\}$

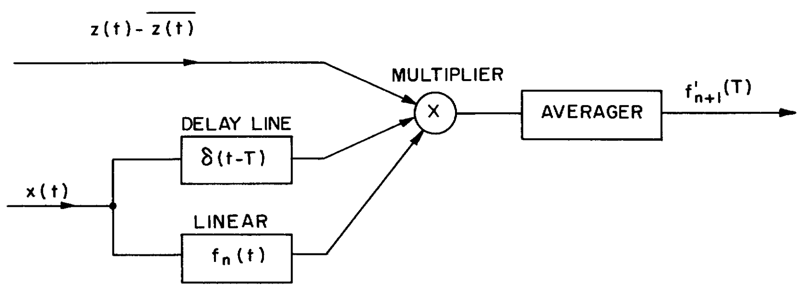

Fig. 16. Experimental system No. 1 for system A.

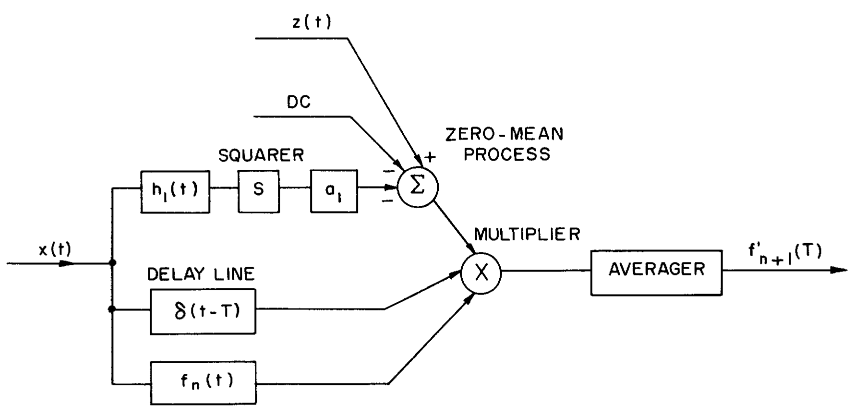

Fig. 17. Experimental system No. 2 for system A.

is the same as that for determining $h_1(t)$ and a_1, with the difference that the output of the part of system A that has already been determined is subtracted from the desired output z(t), and enough dc voltage is also subtracted to make the sum a zero-mean process. The experimental system for determining $h_2(t)$ is shown in Fig. 17.

To show that the procedure described above is correct, we have only to note that the output $f'_{n+1}(T)$ of Fig. 16 is given by

$$f'_{n+1}(T) = \overline{\int_0^\infty f_n(T_1)\, x(t-T_1)\, dT_1 \int_0^\infty \delta(T_2-T)\, x(t-T_2)\, dT_2 [z(t)-\overline{z(t)}]}$$
$$= \int_0^\infty f_n(T_1)\, \overline{x(t-T_1)\, x(t-T_1)[z(t)-\overline{z(t)}]}\, dT_1 \tag{191}$$

and that the output $f'_{n+1}(T)$ of Fig. 17 is given by

$$f'_{n+1}(T) = \overline{\int_0^\infty f_n(T_1)\, x(t-T_1)\, dT_1 \int_0^\infty \delta(T_2-T)\, x(t-T_2)\, dT_2 \left[z(t) - \overline{z(t)} - a_1 \int_0^\infty \int_0^\infty h_1(T_3)\, h_1(T_4)\, x(t-T_3)\, x(t-T_4)\, dT_3 dT_4 + a_1\right]}$$
$$= \int_0^\infty f_n(T_1) \left[\overline{x(t-T_1) x(t-T) [z(t)-\overline{z(t)}]} - 2a_1 h(T_1) h(T)\right] dT_1 \tag{192}$$

(What we actually measure are time averages, but these are equal to ensemble averages.) With the outputs $f'_n(T)$ defined by Eqs. 191 and 192, the iterative procedure just described is the same as the standard iterative procedure (see Hildebrand (17)) for solving the integral equation given by Eq. 189.

4.5 MEAN-SQUARE ERROR FOR SYSTEM A

In this section we shall derive an expression for the minimum mean-square error E between the desired output z(t) and the optimum output w(t) of system A. By definition (Eq. 169),

$$E = \overline{[w(t)-z(t)]^2}$$

We shall show, first, that E is given by

$$E = \overline{z^2(t)} - \overline{w^2(t)} \tag{193}$$

and then we shall evaluate $\overline{w^2(t)}$.

Equation 193 is true for any optimum system that has the property that if g(t) is a possible output, then c g(t) is a possible output. Here, c is an arbitrary constant. System A has this property because by multiplying c_o, $L_1(t)$, and the gains $\{a_i\}$ by c we can multiply the output by c. The cascade of, first, a linear system, whose impulse response we are free to choose, followed by an ideal clipper whose output is either +1 or −1 does not have this property.

To prove Eq. 193, we express our optimum output w(t) as

$$w(t) = c\,g(t) \tag{194}$$

and then choose c to minimize the error in the definition of E, by setting the derivative of the error with respect to c equal to zero. That is,

$$\frac{d}{dc}\overline{[c\,g(t)-z(t)]^2} = 0$$

Performing the differentiation, we obtain

$$c\,\overline{g^2(t)} - \overline{g(t)\,z(t)} = 0$$

Thus the optimum value of c is

$$c = \frac{\overline{g(t)\,z(t)}}{\overline{g^2(t)}} \tag{195}$$

Substituting Eqs. 195 and 194 in Eq. 169, we obtain

$$E = \overline{z^2(t)} - \frac{[\overline{g(t)z(t)}]^2}{\overline{g^2(t)}} \tag{196}$$

But from Eqs. 194 and 195 we notice that

$$\overline{w^2(t)} = \frac{\overline{g(t)\,z(t)}}{\overline{g^2(t)}} \tag{197}$$

Substituting Eq. 197 in Eq. 196, we obtain

$$E = \overline{z^2(t)} - \overline{w^2(t)}$$

which was to be proved.

Because of the orthogonality of G-functionals of different degree, the G-functional expansion for w(t) (Eq. 161) simplifies the derivation of $\overline{w^2(t)}$. In G-functional form, $\overline{w^2(t)}$, is given by

$$\overline{w^2(t)} = \overline{[G_o+G_1(t, L_1, x)+G_2(t, L_2, x)]^2}$$

$$= \overline{G_o^2} + \overline{G_1^2(t, L_1, x)} + \overline{G_2^2(t, L_2, x)} \tag{198}$$

because all of the cross products are zero. From Eq. 172 we see that $G_o = \overline{z(t)}$, hence

$$\overline{G_o^2} = [\overline{z(t)}]^2 \tag{199}$$

From Eqs. 163 and 174 we see that $G_1(t, L_1, x)$ is given by

$$G_1(t, L_1, x) = \int_0^\infty \overline{x(t-T)\ z(t)}\ x(t-T)\ dT$$

Hence, squaring both sides and averaging, we obtain

$$\overline{G_1^2(t, L_1, x)} = \overline{\int_0^\infty \overline{x(t-T_1)\ z(t)}\ x(t-T_1)\ dT_1 \int_0^\infty \overline{x(t-T_2)\ z(t)}\ x(t-T_2)\ dT_2}$$

$$= \int_0^\infty \int_0^\infty \overline{x(t-T_1)\ z(t)}\ \overline{x(t-T_2)\ z(t)}\ \overline{x(t-T_1)\ x(t-T_2)}\ dT_1 dT_2 \tag{200}$$

Since the input is white, Eq. 200 becomes

$$\overline{G_1^2(t, L_1, x)} = \int_0^\infty [\overline{x(t-T)z(t)}]^2\ dT \tag{201}$$

We can evaluate $\overline{G_2^2(t, L_2, x)}$ by means of Eq. 149.

$$\overline{G_2^2(t, L_2, x)} = 2 \int_0^\infty \int_0^\infty L_2^2(T_1, T_2)\ dT_1 dT_2 \tag{202}$$

By using the definition of $L_2(T_1, T_2)$ given by Eq. 168 and by rearranging terms, Eq. 202 becomes

$$\overline{G_2^2(t, L_2, x)} = 2 \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j \int_0^\infty h_i(T_1)\ h_j(T_1)\ dT_1 \int_0^\infty h_i(T_2)\ h_j(T_2)\ dT_2 \tag{203}$$

The optimum h's are eigenfunctions (Eq. 189), and eigenfunctions have the property (18) that

$$\int_0^\infty h_i(T)\ h_j(T)\ dT = 0 \qquad i \neq j \tag{204}$$

By applying the orthonormality of the h's given by Eqs. 160 and 204, Eq. 203 becomes

$$\overline{G_2^2(t, L_2, x)} = 2 \sum_{i=1}^{N} a_i^2 \tag{205}$$

From Eqs. 199, 201, and 205, we see that Eq. 198 for $\overline{w^2(t)}$ becomes

$$\overline{w^2(t)} = [\overline{z(t)}]^2 + \int_0^\infty [\overline{x(t-T)z(t)}]^2\ dT + 2 \sum_{i=1}^{N} a_i^2 \tag{206}$$

We see from Eq. 189 that the gains $\{a_i\}$ are eigenvalues of a second-order crosscorrelation (divided by 2)

$$\frac{1}{2}\ \overline{x(t-T_1)\ x(t-T_2)[z(t)-\overline{z(t)}]}$$

Substituting Eq. 206 in Eq. 193, we obtain the minimum error for system A:

$$E = \overline{z^2(t)} - \left[\overline{z(t)}\right]^2 - \int_0^\infty \left[\overline{x(t-T)z(t)}\right]^2 dT - 2 \sum_{i=1}^{N} a_i^2$$

4.6 EXAMPLE A_1

As an example of the optimization procedures discussed in the three previous sections, we now consider optimizing system A_1 of Fig. 18. System A_1 consists of a constant c_0, a linear system with impulse response $L_1(t)$, and a linear system with impulse response $h_1(t)$ whose output is squared and multiplied by a gain a_1. The input to system A_1 is the white Gaussian input x(t). The output is w(t). The constant c_0, the impulse responses $L_1(t)$ and $h_1(t)$, and the gain a_1 are all to be determined to minimize the mean-square difference between w(t) and the desired output z(t).

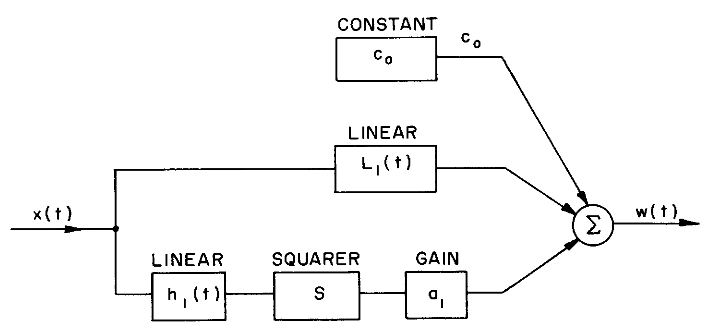

Fig. 18. System A_1.

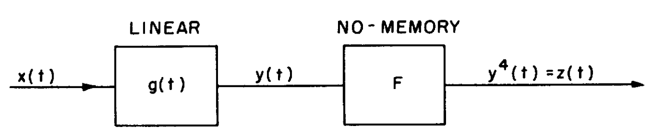

Fig. 19. Desired output for examples.

In this example, the desired output z(t) is produced in the manner shown in Fig. 19. This is done by raising to the 4^{th} power the output y(t) of a linear system whose input is x(t), and whose impulse response g(t) is given by

$$g(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{207}$$

In a real case we would be given x(t) and z(t), and we would then make measurements to determine the optimum values of the parameters in system A_1. In this example we shall compute analytically those quantities which, in practice, we would measure.

To facilitate these computations, we would like to express z(t) as a sum of

G-functionals of x(t). From Fig. 19 we see that we can express z(t) as

$$z(t) = \iiiint g(T_1)\, g(T_2)\, g(T_3)\, g(T_4)\, x(t-T_1)\, x(t-T_2)\, x(t-T_3)\, x(t-T_4)\, dT_1 dT_2 dT_3 dT_4 \tag{208}$$

By means of Eqs. 148 and 145 we can form a G-functional of degree 4 that has the same kernel as Eq. 208. This G-functional is

$$G_4[t, g(T_1)g(T_2)g(T_3)g(T_4), x] = \iiiint g(T_1)\, g(T_2)\, g(T_3)\, g(T_4)\, x(t-T_1)\, x(t-T_2)\, x(t-T_3)\, x(t-T_4)\, dT_1 dT_2 dT_3 dT_4$$
$$- 6 \iiiint g(T_1)\, g(T_2)\, g(T_3)\, g(T_3)\, dT_1 dT_2 dT_3 + 3 \iint g(T_1)\, g(T_1)\, g(T_2)\, g(T_2)\, dT_1 dT_2 \tag{209}$$

From the definition of g(t) (Eq. 207), we note that

$$\int_{-\infty}^{\infty} g^2(t)\, dt = \int_0^1 dt = 1$$

Using this expression in Eq. 209, we obtain

$$G_4[t, g(T_1)g(T_2)g(T_3)g(T_4), x] = \iiiint g(T_1)\, g(T_2)\, g(T_3)\, g(T_4)\, x(t-T_1)\, x(t-T_2)\, x(t-T_3)\, x(t-T_4)\, dT_1 dT_2 dT_3 dT_4$$
$$- 6 \iint g(T_1)\, g(T_2)\, x(t-T_1)\, x(t-T_2)\, dT_1 dT_2 + 3 \tag{210}$$

By means of Eqs. 148 and 145, we can form a G-functional of degree 2 whose kernel is the negative of the second-order kernel in the right-hand side of Eq. 210. The G-functional is

$$G_2[t, 6g(T_1)g(T_2), x] = 6 \iint g(T_1)\, g(T_2)\, x(t-T_1)\, x(t-T_2)\, dT_1 dT_2 - 6 \tag{211}$$

We can now form a G-functional of degree zero (a constant) which is the negative of the sum of the constants in the right-hand sides of Eqs. 210 and 211. We shall call this constant G_o'. It is given by

$$G_o' = 3 \tag{212}$$

From Eqs. 208, 210, 211, and 212 we see that z(t) is given by the sum of G-functionals:

$$z(t) = G_4[t, g(T_1)g(T_2)g(T_3)g(T_4), x] + G_2[t, 6g(T_1)g(T_2), x] + G_o' \tag{213}$$

We shall first determine the impulse response $L_1(t)$. Equation 174 states that $L_1(t)$ is given by

$$L_1(T) = \overline{x(t-T)\, z(t)}$$

But x(t−T) is a G-functional of degree 1, and z(t) has no G-functional of degree 1 in its expansion. Thus, x(t−T) is linearly independent of z(t), and hence

$$L_1(t) = 0$$

We can evaluate $h_1(t)$ and a_1 by inspection. Notice that if we choose $h_1(t) = g(t)$ and $a_1 = 6$, we can form a G-functional of degree 2 that will exactly equal the G-functional of degree 2 in the expansion of z(t) in Eq. 213. The iterative procedure described in section 4.4 would give us the same values.

By Eq. 173, the constant c_o is

$$c_o = \overline{z(t)} - a_1$$

The mean of z(t) is equal to G_o' because that is the only term in the G-functional expansion for z(t) (Eq. 213) that is not orthogonal to a constant. By Eq. 212, $G_o' = 3$. We have previously determined that $a_1 = 6$. Thus c_o becomes

$$c_o = 3 - 6 = -3$$

The G_o term in the expansion for w(t) will, of course, be equal to G_o' of Eq. 213.

The mean-square error between z(t) and w(t) is easy to compute. By optimizing system A_1, we have been able to produce an output w(t) that is composed of the zero- and second-degree G-functionals of the expansion for z(t) (Eq. 213). The instantaneous difference between z(t) and w(t) is, therefore, given by the remaining G-functional of degree 4 in the expansion for z(t). That is,

$$z(t) - w(t) = G_4[t, g(T_1)g(T_2)g(T_3)g(T_4), x]$$

The mean-square difference between z(t) and w(t) is

$$\overline{[z(t)-w(t)]^2} = \overline{G_4^2[t, g(T_1)g(T_2)g(T_3)g(T_4), x]}$$

By Eq. 149, this becomes

$$\overline{[z(t)-w(t)]^2} = 4!\left[\int g^2(t)\, dT\right]^4 = 24$$

We might wish to normalize this mean-square error by dividing by the mean square of the desired output z(t). We can evaluate $\overline{z^2(t)}$ by squaring and averaging Eq. 213, and using the fact that the G-functionals are orthogonal.

$$\overline{z^2(t)} = \overline{G_4^2[t, g(T_1)g(T_2)g(T_3)g(T_4)]} + \overline{G_2^2[t, g(T_1)g(T_2), x]} + \overline{[G_o']^2} \qquad (214)$$

If each of these terms is evaluated by means of Eq. 149, Eq. 214 becomes

$$\overline{z^2(t)} = 105$$

The normalized mean-square error is then given by

$$\frac{\overline{[z(t)-w(t)]^2}}{\overline{z^2(t)}} = \frac{24}{105} = 22.8 \text{ per cent}$$

4.7 SYSTEM B

In the previous sections we have shown that an experimental iterative procedure is necessary to optimize system A. We shall now consider a nonlinear system that can be optimized by direct measurements with no iteration. The cost of this simplification is a loss of efficiency of components. That is, for the same number of linear subsystems, system A would give a better approximation to the desired output than would the system that we shall now treat.

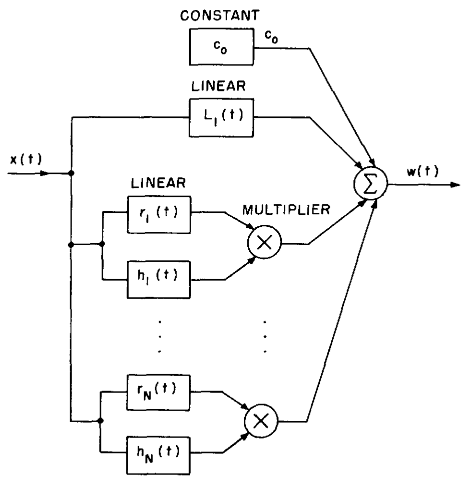

Fig. 20. System B.

We consider the optimization of system B which is shown in Fig. 20. System B consists of a constant c_o, a linear system with impulse response $L_1(t)$, and two sets of N linear systems whose impulse responses are $\{h_1(t), \ldots h_N(t)\}$ and $\{r_1(t), \ldots r_N(t)\}$. The output of each h system is multiplied by the output of the corresponding r system. The impulse responses $\{r_1(t), \ldots r_N(t)\}$ are given; they are not to be determined in the optimization procedure. The r's are an orthonormal set. That is, they have the following two properties:

$$\left.\begin{aligned} &\int_0^\infty r_i(t)\, r_j(t)\, dt = 0 \qquad i \neq j \\ &\int_0^\infty r_i(t)\, r_i(t)\, dt = 1 \end{aligned}\right\} \tag{215}$$

The constant c_o, the impulse response $L_1(t)$, and the set of impulse responses $\{h_1(t), \ldots h_N(t)\}$ are all to be determined so as to minimize the mean-square error between the output w(t) of system B and the desired output z(t). The input x(t) to system B is white and Gaussian.

By means of the undetermined constant c_o, system B can produce any zero-order output. By means of the undetermined impulse response $L_1(t)$, system B can perform any realizable linear operation on x(t). Because it contains only a limited number of linear systems whose outputs are multiplied together, system B can produce only a limited class of second-order operations on its input. If system B had an infinite number of such linear systems, it could produce a general second-order operation on x(t) in the manner shown for the general nonlinear system of Section II.

In section 3.3 we showed that for a given number of linear systems the form of system A is optimum. Therefore, for the same number of linear systems (number of h's plus number of r's), system B will produce a poorer approximation to z(t) than will system A.

System B has many parameters that have to be chosen to minimize the mean-square error between w(t) and z(t). The simultaneous choice of many parameters is often a formidable problem. However, by means of an orthogonal expansion for w(t) we shall be able to reduce this optimization problem to one of minimizing, separately, many different errors, each one of which involves only one parameter.

Before we determine the particular expansion for w(t), let us demonstrate that for any arbitrary orthogonal expansion for w(t), the mean-square error between z(t) and w(t) can be written as a sum of mean-square errors between z(t) and each of the terms of the expansion. Let w(t) be expressed as

$$w(t) = \sum_{i=1}^{N} g_i(t)$$

in which the g's are random signals with the property that

$$\overline{g_i(t)\, g_j(t)} = 0 \qquad i \neq j$$

Then the mean-square error can be expressed as

$$\overline{[w(t)-z(t)]^2} = \overline{\left[\sum_{i=1}^{N} g_i(t) - z(t)\right]^2}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \overline{g_i(t)\, g_j(t)} - 2 \sum_{i=1}^{N} \overline{g_i(t)\, z(t)} + \overline{z^2(t)} \tag{216}$$

By applying the orthogonality of the g's, we obtain

$$\overline{[w(t)-z(t)]^2} = \sum_{i=1}^{N} \overline{g_i^2(t)} - 2 \sum_{i=1}^{N} \overline{g_i(t)\, z(t)} + \overline{z^2(t)} \tag{217}$$

We now rearrange terms to obtain

$$\overline{[w(t)-z(t)]^2} = \sum_{i=1}^{N} \overline{[g_i(t)-z(t)]^2} - (N-1)\, \overline{z^2(t)} \tag{218}$$

The right-hand side of Eq. 218 is the sum of a constant which is independent of the g's plus the sum of the mean-square errors between z(t) and each of the terms of the orthogonal expansion for w(t). The form of Eq. 218 is particularly convenient if the undetermined parameters are such that each parameter affects only one g because, in that case, we can minimize each error term separately. We shall now determine such an orthogonal expansion for w(t).

The orthogonal expansion that we shall use for w(t) is a sum of G-functionals. The sum consists of one G-functional of degree zero, one G-functional of degree 1, plus many G-functionals of degree 2. The G-functionals of degree 2 are orthogonal to each other, and each G-functional of degree 2 can be optimized independently. The main problem is that of choosing the set of G-functionals of degree 2 which will have these properties.

The quadratic part of system B produces the following homogeneous functional of degree 2:

$$\iint \sum_{i=1}^{N} h_i(T_1)\, r_i(T_2)\, x(t-T_1)\, x(t-T_2)\, dT_1 dT_2 \tag{219}$$

The sum of the kernels of the G-functionals of degree 2 must have the same kernel,

$$\sum_{i=1}^{N} h_i(T_1)\, r_i(T_2) \tag{220}$$

as expression 219, for neither G-functionals of degree 1 nor G-functionals of degree zero contains homogeneous functionals of degree 2. We note from Eq. 149 that two G-functionals, $G_2(t, K_2, x)$ and $G_2(t, L_2, x)$, are orthogonal if

$$\iint K_2(T_1, T_2)\, L_2(T_1, T_2)\, dT_1 dT_2 = 0 \tag{221}$$

(Remember that kernels in Eq. 221 must be expressed in symmetric form.) Therefore, if we can express the kernel 220 as a sum of kernels that are mutually orthogonal in the sense of Eq. 221, then for each kernel in the sum there is a G-functional of degree 2 which uses that kernel. The sum of these G-functions produces expression 219 (plus a constant); and the G-functionals of degree 2 are orthogonal to each other. We wish to choose the kernels in the sum in such a manner that one and only one undetermined parameter appears in each kernel.

To divide up kernel 220, we first express each impulse response $h_i(t)$ as follows:

$$h_i(t) = h_i'(t) + \sum_{j=1}^{i} a_{i,j}\, r_j(t) \tag{222}$$

where $h_i'(t)$ is orthogonal to every r. That is,

$$\int h_i'(t)\, r_j(t)\, dt = 0 \tag{223}$$

The constants $\{a_{i,j}\}$ are undetermined in the same sense that $h_i(t)$ is undetermined. Thus, these constants will be determined to optimize the output. If j in Eq. 222 is allowed to run to N instead of to i, then clearly any arbitrary $h_i(t)$ can be written in the form of Eq. 223. However, this restriction on $h_i(t)$ places no restriction on the output w(t) because any part of w(t) formed by

$$\iint r_j(T_1)\, r_i(T_2)\, x(t-T_1)\, x(t-T_2)\, dT_1 dT_2 \qquad j > i$$

could be formed by letting $a_{j,i} = 1$ in expansion 222 for $h_j(t)$, as can be seen from expression 219. By using Eq. 222, kernel 220 becomes

$$\sum_{i=1}^{N} h_i'(T_1)\, r_i(T_2) + \sum_{i=1}^{N} \sum_{j=1}^{i} a_{i,j}\, r_j(T_1)\, r_i(T_2) \tag{224}$$

The symmetrized forms of each of the terms of expression 224 are orthogonal to each other in the sense of Eq. 221. This orthogonality is expressed by the following equations:

$$\iint \frac{1}{2}\left[h_i'(T_1)r_i(T_2)+r_i(T_1)h_i'(T_2)\right] \frac{1}{2}\left[h_j'(T_1)r_j(T_2)+r_j(T_1)h_j'(T_2)\right] dT_1 dT_2 = 0 \qquad i \neq j \tag{224a}$$

$$\iint \frac{1}{2}\left[h_i'(T_1)r_n(T_2)+r_n(T_1)h_i'(T_2)\right]$$

$$\times \frac{1}{2}\left[a_{i,j}\, r_j(T_1)r_i(T_2)+a_{i,j}\, r_i(T_1)r_j(T_2)\right] dT_1 dT_2 = 0 \tag{224b}$$

$$\iint \frac{1}{2}\left[a_{i,j}\, r_j(T_1)r_i(T_2)+a_{i,j}\, r_i(T_1)r_j(T_2)\right]$$

$$\times \frac{1}{2}\left[a_{m,n}\, r_m(T_1)r_n(T_2)+a_{m,n}\, r_n(T_1)r_m(T_2)\right] dT_1 dT_2 = 0 \qquad \text{if either } i \neq n \text{ or } j \neq m \tag{224c}$$

Equation 224a follows from the orthogonality of the r's (Eq. 215) and the orthogonality of h' with r (Eq. 223). Equation 224b follows from the orthogonality of h' with r (Eq. 223). Equation 224c follows from the orthogonality of the r's (Eq. 215) except when both i = n and j = m, in which case it follows from the fact that $a_{i,j} = 0$ for $j \geqslant i$.

By means of the kernels of expression 224, we can now express w(t) as a sum of orthogonal G-functionals as follows:

$$w(t) = G_o + G_1(t, L_1, x) + \sum_{i=1}^{N} G_2[t, h_i'(T_1)r_i(T_2), x]$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{i} a_{i,j} G_2[t, r_j(T_1)r_i(T_2), x] \tag{225}$$

The sum of the G-functionals of degree 2 is equal to expression 219 minus its mean. The constant G_o is equal to c_o plus the mean of expression 219. The G-functional of degree 1 is equal to the output of the linear system with impulse response $L_1(t)$.

The impulse responses $\{h_i'(t)\}$ and the gains $\{a_{i,j}\}$ have now replaced the impulse responses $\{h_i(t)\}$ and are, of course, undetermined parameters. Notice that each of these undetermined parameters appears in its own G-functional of degree 2.

We shall now optimize system B by determining the optimum values of the undetermined parameters. Recall from Eq. 218 that if the output w(t) can be expressed as the sum of orthogonal functions and if these functions can be adjusted independently, then each of the orthogonal functions should be adjusted to minimize the mean-square error between itself and the desired output.

To choose each gain $a_{i,j}$ we therefore adjust $a_{i,j}$ to minimize

$$\overline{\{a_{i,j} G_2[t, r_j(T_1)r_i(T_2), x] - z(t)\}^2}$$

Setting the derivative of this error with respect to $a_{i,j}$ equal to zero, we obtain

$$a_{i,j} = \frac{\overline{z(t)\, G_2[t, r_j(T_1)r_i(T_2), x]}}{\overline{G_2^2[t, r_j(T_1)r_i(T_2), x]}} \tag{226}$$

Evaluating these averages by means of the properties of the G-functionals given in Eqs. 148 and 149 and the properties of the r's given in Eqs. 215, we obtain two forms for Eq. 226:

$$a_{i,j} = \overline{z(t) \int r_j(T_1)\, x(t-T_1)\, dT_1 \int r_i(T_2)\, x(t-T_2)\, dT_2} \qquad i \neq j \tag{227a}$$

$$a_{i,i} = \overline{\frac{1}{2} z(t) \left[\int r_i(T)\, x(t-T)\, dT\right]^2} - \frac{1}{2}\overline{z(t)} \tag{227b}$$

The optimum impulse response $h_i'(t)$ is the one that minimizes the error:

$$\overline{\{G_2[t, h_i'(T_1)r_i(T_2), x] - z(t)\}^2} \tag{228}$$

subject to the constraint (Eq. 224) that h' be orthogonal to r. The easiest way to apply this constraint is to form a variable u(t) that is defined as

$$u(t) = z(t) - \sum_{i=1}^{N} \int \sum_{j=1}^{i} a_{i,j}\, r_j(T_1)\, x(t-T_1)\, dT_1 \int r_i(T_2)\, x(t-T_2)\, dT_2 \tag{229}$$

and then choose $h_i'(t)$ to minimize

$$\overline{\{G_2[t, h_i'(T_1) r_i(T_2), x] - u(t)\}^2} \tag{230}$$

The variable u(t) is just z(t) minus that part of the output of system B which has already been determined. Since we chose the a's to be optimum, we know that the impulse response $h_i'(t)$ that minimizes expression 230 subject to no constraint will indeed be orthogonal to all of the r's; this fact follows from the fact that any desired component of $h_i'(t)$ which is not orthogonal to the r's could have been produced by a different choice of gains $\{a_{i,j}\}$.

We shall now minimize the error of expression 230. We expand the square and use the properties of the G-functionals given by Eqs. 148 and 149, and use the orthogonality of h_i' and r to obtain the error of Eq. 228 in the following form:

$$\int [h_i'(T)]^2\, dT - 2 \iint h_i'(T_1)\, r(T_2)\, \overline{[u(t) - \overline{u(t)}]\, x(t-T_1)\, x(t-T_2)}$$
$$\times\, dT_1 dT_2 + \overline{u^2(t)} \tag{231}$$

If we now add an arbitrary realizable impulse response ϵ g(t) to $h_i'(t)$, then for the optimum $h_i'(t)$ the derivative of the error with respect to ϵ will be zero when $\epsilon = 0$. Applying this condition to the error of Eq. 231, we obtain

$$2 \int g(T)\, dT \left\{ h_i'(T) - \overline{[u(t) - \overline{u(t)}]\, x(t-T) \int r_i(T_1)\, x(t-T_1)\, dT_1} \right\} = 0$$

Since g(T) is arbitrary for $T \geq 0$, the quantity in braces must be zero for $T \geq 0$. Since $h_i'(T)$ is realizable, it must be zero for $T < 0$. Therefore, $h_i(T)$ is given by

$$h_i'(T) = \begin{cases} \overline{[u(t) - \overline{u(t)}]\, x(t-T) \int r_i(T_1)\, x(t-T_1)\, dT_1} & T \geq 0 \\ 0 & T < 0 \end{cases} \tag{232}$$

To choose the realizable impulse response $L_1(t)$, we minimize the error

$$\overline{[G_1(t, L_1, x) - z(t)]^2}$$

In the optimization of system A we have shown that the realizable impulse response that minimizes this error is

$$L_1(T) = \begin{cases} \overline{x(t-T)\, z(t)} & T \geq 0 \\ 0 & T < 0 \end{cases}$$

Clearly, the constant G_o that minimizes $\overline{[G_o - z(t)]^2}$ is

$$G_o = \overline{z(t)}$$

The constant c_o of system B serves two purposes. It produces the constants of the G-functionals of degree 2, and it produces the addition constant used in forming G_o. Thus c_o equals G_o plus the constants of the G-functionals of degree 2. From the definition of the G-functionals (Eq. 148), the orthogonality of h' and r (Eq. 223), and the orthogonality of the r's (Eq. 215), we find that the constants of the G-functionals of degree 2 are

$$-\sum_{i=1}^{N} a_{i,i}$$

Thus the constant c_o is given by

$$c_o = \overline{z(t)} - \sum_{i=1}^{N} a_{i,i}$$

The experimental procedures for determining the parameters of system B follow directly from the equations that define their optimum values. Thus, directly from Eq. 227a we see that the output of the system of Fig. 21 is the optimum value of gain $a_{i,j}$ for $i \neq j$. Similarly, from Eq. 227b we see that the output of the system of Fig. 22 is the gain $a_{i,i}$. From the definition of u(t) given in Eq. 229 and from the optimum value of $h_i'(T)$ given in Eq. 232, we see that the output of the system of Fig. 23, measured as a function of the delay T, is $h_i'(T)$.

Just as in system A, the mean-square error between w(t) and z(t) is the difference of the average values of the squares. That is,

$$\overline{[w(t)-z(t)]^2} = \overline{z^2(t)} - \overline{w^2(t)} \tag{233}$$

By using the orthogonal expansion of Eq. 225, we can express $\overline{w^2(t)}$ as

$$\overline{w^2(t)} = \overline{G_o^2} + \overline{G_i^2(t, L_1, x)} + \sum_{i=1}^{N} \overline{G_2^2[t, h_i'(T_1) r_i(T_2), x]}$$
$$+ \sum_{i=1}^{N} \sum_{j=1}^{i} a_{i,j}^2 \overline{G_2^2[t, r_j(T_1) r_i(T_2), x]} \tag{234}$$

From the properties of the G-functionals (Eq. 149) and the optimum values of the parameters which we have determined, we can evaluate each of the terms in Eq. 234. Then, substituting these terms in Eq. 233, we obtain the following error expression:

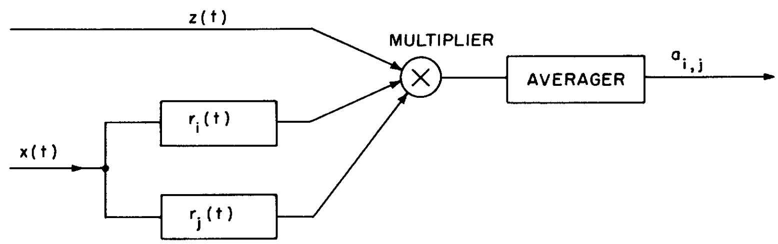

Fig. 21. Experimental system No. 1 for system B.

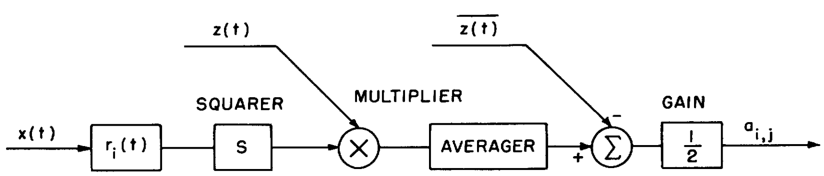

Fig. 22. Experimental system No. 2 for system B.

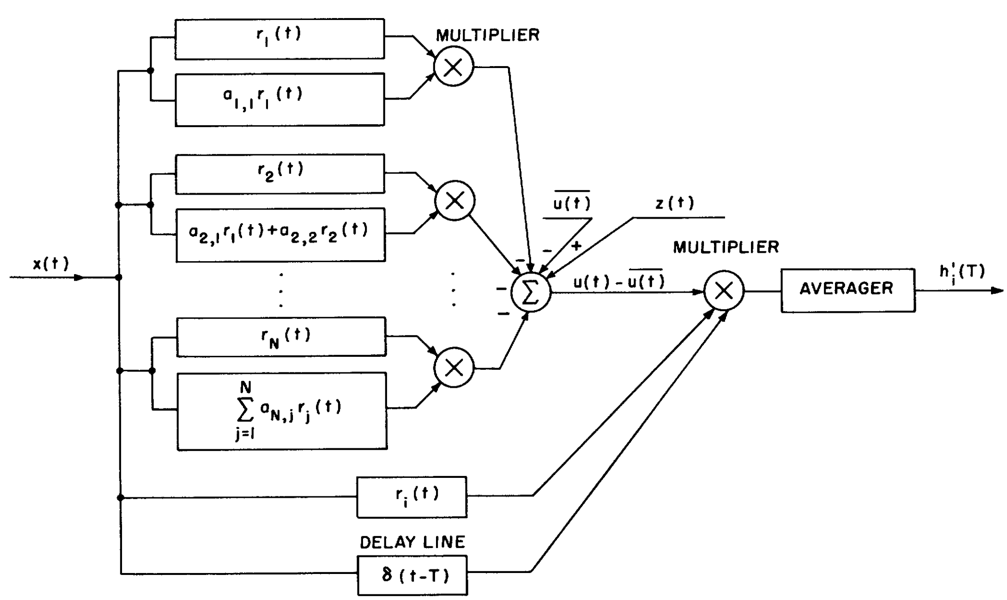

Fig. 23. Experimental system No. 3 for system B.

$$\overline{[w(t)-z(t)]^2} = \overline{z^2(t)} - [\overline{z(t)}]^2 - \int_0^\infty [\overline{x(t-T)z(t)}]^2 \, dT$$

$$- \sum_{i=1}^{N} \int [h_i'(T)]^2 \, dT$$

$$- \sum_{i=1}^{N} \sum_{j=1}^{i-1} \left[\iint r_j(T_1) \, r_i(T_2) \, \overline{z(t) \, x(t-T_1) \, x(t-T_2)} \, dT_1 dT_2 \right]^2$$

$$- \sum_{i=1}^{N} \frac{1}{2} \left[\iint r_i(T_1) \, r_i(T_2) \, \overline{z(t) \, x(t-T_1) \, x(t-T_2)} \, dT_1 dT_2 - \overline{z(t)} \right]^2 \qquad (235)$$

In a straightforward manner we could derive the expression for $\int [h_i'(t)]^2 \, dT$ in terms of second-order crosscorrelations between z(t) and x(t). However, the expression is very long and will not be presented here.

4.8 EXAMPLE B_1

As an example of the techniques discussed in the previous section we shall optimize system B_1 of Fig. 24. System B_1 consists of a constant c_o and three linear realizable

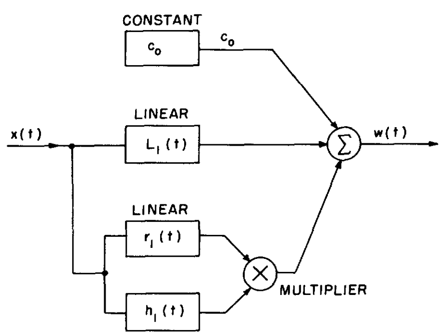

Fig. 24. System B_1.

systems with impulse responses $L_1(t)$, $h_1(t)$, and $r_1(t)$. The outputs of the linear systems with impulse responses $h_1(t)$ and $r_1(t)$ are multiplied together. The constant c_o, the impulse response $L_1(t)$, and the impulse response $h_1(t)$ are all to be determined. The impulse response $r_1(t)$, which is not to be determined, is given as

$$r(t) = \begin{cases} \frac{1}{\sqrt{2}} & 0 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

The input, x(t), is white and Gaussian. The output of system B is w(t).

The desired output z(t) is the same one used in example A_1 and is shown in Fig. 19. Recall that z(t) is formed by raising y(t) to the 4^{th} power, and y(t) is the output of a linear system with input x(t) and impulse response g(t) given by

$$g(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

To optimize $h_1(t)$, we first express $h_1(t)$ as

$$h_1(t) = a_{1,1}\, r_1(t) + h_1'(t) \tag{236}$$

From Eq. 227b, $a_{1,1}$ is given by

$$a_{1,1} = \frac{1}{2}\,\overline{z(t)\left[\int r_i(T_1)\, x(t-T_1)\, dT_1\right]^2} - \frac{1}{2}\,\overline{z(t)} \tag{237}$$

The averages in Eq. 237 can be evaluated by using the G-functional expansion for z(t) given by Eq. 213. By evaluating these averages, Eq. 237 becomes

$$a_{1,1} = 3 \tag{238}$$

To determine $h_1'(t)$, we form u(t) from Eq. 229.

$$u(t) = z(t) - a_{1,1}\left[\int r_1(s)\, x(t-s)\, ds\right]^2 \tag{239}$$

From Eq. 232 we see that $h_1'(T)$ is given by

$$h_1'(T) = \overline{[u(t)-\overline{u(t)}]\, x(t-T) \int r_1(s)\cdot x(t-s)\, ds} \tag{240}$$

Substituting Eq. 239 in Eq. 240 and evaluating the averages, we obtain

$$h_1'(t) = \begin{cases} \dfrac{6}{\sqrt{2}} & 0 \leq t \leq 1 \\ \dfrac{-6}{\sqrt{2}} & 1 < t \leq 2 \end{cases} \tag{241}$$

Notice that $h_1'(t)$ is orthogonal to $r_1(t)$.

From Eqs. 236, 238, and 241 we see that $h_1(t)$ is given by

$$h(t) = \begin{cases} \dfrac{9}{\sqrt{2}} \\ \dfrac{-3}{\sqrt{2}} \end{cases}$$

We have already seen that $L_1(t)$ is given by

$$L_1(T) = \begin{cases} \overline{x(t-T)\, z(t)} & T \geq 0 \\ 0 & T < 0 \end{cases}$$

However, z(t) is an even function of x and hence is uncorrelated with x. Thus the impulse response $L_1(t)$ should be zero for all time.

We have seen that the optimum value of the constant c_o is

$$c_o = \overline{z(t)} - a_{1,1}$$

In Eq. 212 we saw that the mean of z(t) is 3, and in Eq. 238 we saw that $a_{1,1} = 3$. Thus the optimum value of c_o is zero.

We can evaluate the mean-square error as the differences of the mean-square outputs.

$$\overline{[w(t)-z(t)]^2} = \overline{z^2(t)} - \overline{w^2(t)} \tag{242}$$

From Eq. 234 and the properties of the G-functionals, we find that $\overline{w^2(t)}$ is given by

$$\overline{w^2(t)} = [\overline{z(t)}]^2 + \int [h'(T)]^2 \, dT + a_{1,1}^2$$

By evaluating each of these terms, $\overline{w^2(t)}$ becomes

$$\overline{w^2(t)} = 54 \tag{243}$$

As shown in section 4.6, $\overline{z^2(t)}$ is given as

$$\overline{z^2(t)} = 105 \tag{244}$$

Substituting Eqs. 242 and 243 in Eq. 242, we obtain

$$\overline{[w(t)-z(t)]^2} = 51$$

The normalized error then becomes

$$\frac{\overline{[w(t)-z(t)]^2}}{\overline{z^2(t)}} = \frac{51}{105} = 49.5 \text{ per cent}$$

Notice that in system B_1 we have one more linear subsystem than we have in system A_1; and yet because one of its subsystems is fixed, system B_1 has a larger mean-square error than does system A_1.

V. CONCLUSIONS

In this report we have developed methods for optimizing general multi-input, single-output, nonlinear systems and methods for optimizing single-input nonlinear systems of restricted form. All of the inputs considered are Gaussian random processes. In all cases, an orthogonal expansion for the output of the nonlinear system has been a very valuable tool.

Theoretically, an infinite set of higher-order crosscorrelations must be known in order to optimize a general multi-input nonlinear system. In practice, however, good results can be obtained by using only a finite number of these higher-order crosscorrelations. In either case, an infinite number of linear subsystems is needed to build a general multi-input nonlinear system. For these reasons, we have treated some single-input, nonlinear systems of restricted form which can be optimized with a finite amount of statistical information and can be constructed with a finite number of simple subsystems. These systems of restricted form will produce a better approximation to the desired output than will the optimum linear system, but a poorer one than a general nonlinear system will produce.

Other restricted forms of nonlinear systems which can be easily optimized and constructed have already been presented by the author (19). Techniques could also be developed that would allow one to choose the best restricted form to use for a particular problem.

Because the optimization techniques developed in this report depend very strongly on the Gaussian nature of the inputs, these techniques are not directly applicable to problems with non-Gaussian inputs. However, for slightly non-Gaussian inputs (such as Gaussian processes that have been amplitude-limited), the inputs can be treated as Gaussian, and a nonlinear system can then be optimized by the methods presented in this report.

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