THEORY OF NOISY TWO-PORT NETWORKS

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E. Folke Bolinder

Abstract

A new geometric-analytic theory of noisy two-port networks is presented. It is based, geometrically, on the isometric sphere method, a generalization of the isometric circle method to three dimensions, and, analytically, on a three-dimensional conformal transformation which was originally derived by Poincaré and Picard. The transformation is used in a study of transformations of noise ensemble average ratios through bilateral two-port networks. The study is facilitated by interpreting the transformations as non-Euclidean movements in the Poincaré and Cayley-Klein models of three-dimensional hyperbolic space. By a simple extension, noise ensemble average transformations are performed in four-dimensional spaces by means of four-vectors which are analogous to the Stokes vectors used in optics and antenna theory.

The new theory has been used for studying several problems pertaining to noisy two-port networks: the Rothe and Dahlke method of splitting a noisy two-port network into noisy and noise-free parts, cascading of noisy two-port networks, noise tuning and noise matching, the wave representation of noisy two-port networks, and the optimum noise factor.

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I. INTRODUCTION

The rapid development of transistor amplifiers and longitudinal electron-beam amplifiers has led to the creation of linear noise theories. This report presents a new linear noise theory which is based on a conformal transformation method called "the isometric sphere method." This method is a straightforward extension of the isometric circle method to three dimensions. A further extension of the isometric sphere method introduces a four-vector that is analogous to the well-known Stokes vector that is used in optics and antenna theory.

Analytically, the theory of noisy two-port networks consists of a direct generalization of transformation equations for noise-free two-port networks. It has the advantage of yielding immediate geometric interpretations of noise transformations as transformations in three- and four-dimensional linear spaces.

In a recent paper (1) and in two recent notes (2, 3) it has been shown how an impedance transformation through a bilateral two-port network can be geometrically represented by a non-Euclidean movement in a Poincaré or a Cayley-Klein model of three-dimensional hyperbolic space. By using the Cayley-Klein model, points on the surface of the unit sphere, which correspond to impedance quantities, are transformed into points on the surface of the unit sphere. However, points inside the surface of the unit sphere are simultaneously transformed into points inside the surface. Now, a natural question is: "What physical interpretation can we give to these points inside the surface of the unit sphere?" The answer is that the points may be thought of as representing noise ensemble average ratios. Thus, transformations of noise ensemble average ratios through noise-free bilateral two-port networks can be geometrically represented by non-Euclidean transformations in models of three-dimensional hyperbolic space. The special cases of impedance transformations through noise-free bilateral two-port networks are obtained as non-Euclidean transformations of points on the absolute surfaces of the different models.

II. THE ISOMETRIC SPHERE METHOD

2.1 THE ISOMETRIC CIRCLE METHOD

The input voltage V' and the input current I' of a noise-free bilateral two-port network are linearly related to the output voltage V and the output current I:

$$\psi' = \begin{pmatrix} V' \\ I' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix} = T\psi; \qquad ad - bc = 1 \tag{1}$$

The input impedance $Z' = V'/I'$ is expressed in terms of the output impedance $Z = V/I$ by the linear fractional transformation:

$$Z' = \frac{aZ + b}{cZ + d} \tag{2}$$

The linear fractional transformation, which is sometimes called a homographic or a Möbius transformation, is the most general one-to-one and directly conformal transformation that transforms the entire complex plane into itself. The conformal property suggests the use of graphical methods. The isometric circle method (4, 2) is a method of this kind.

The isometric circle is defined as the circle that is the complete locus of points in the neighborhood of which lengths are unaltered in magnitude by the linear fractional transformation. The isometric circle of the direct transformation, C_d, has its center at $O_d = -d/c$ and radius $R_c = 1/|c|$; the isometric circle of the inverse transformation C_i, has its center at $O_i = a/c$ and the same radius. (See Fig. 1.) The isometric circle method is composed of the following constructions:

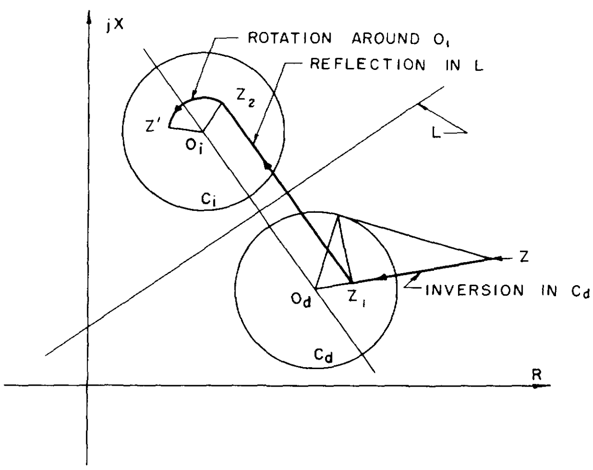

Fig. 1. Isometric circle method.

1. An inversion in the isometric circle of the direct transformation C_d, $Z \rightarrow Z_1$;

2. A reflection in the symmetry line L to the two circles, $Z_1 \rightarrow Z_2$; and

3. A rotation around the center O_i of the isometric circle of the inverse transformation C_i through an angle -2 arg(a+d), $Z_2 \rightarrow Z'$.

For lossless two-port networks, a+d is real and the third operation is eliminated and a "non-loxodromic" transformation is obtained. For lossy two-port networks, yielding "loxodromic" transformations, the third operation must be considered. The simplicity of the use of the isometric circle method in the complex plane is seriously hampered by the third operation, the rotation. It has been shown (5), however, that if the complex plane is stereographically mapped on the surface of the the unit sphere, which is again considered to be the absolute surface of the Cayley-Klein model of three-dimensional hyperbolic space, then the linear fractional transformation (Eq. 2) can be performed by two non-Euclidean reflections in that space.

2.2 THE ISOMETRIC SPHERE METHOD

If the isometric circles are replaced by (hemi) spheres and the symmetry line of the isometric circles is replaced by the symmetry plane of the (hemi) spheres, the isometric circle method can be generalized to "the isometric sphere method." The three operations of the isometric circle method are exchanged for the three operations;

1. An inversion in the isometric sphere of the direct transformation;

2. A reflection in the symmetry plane; and

3. A rotation through an angle -2 arg(a+d) around an axis that is perpendicular to the plane containing the original isometric circles and which passes through the center of the isometric sphere of the inverse transformation.

If we denote the rectangular coordinates of a point in the new three-dimensional space by (R_{cor}, X_{cor}, h), where $R_{cor} + jX_{cor} = Z_{cor}$, the analytic treatment of the different operations of the isometric sphere method are given by

$$\left.\begin{aligned} Z_{cor_1} + \frac{d}{c} &= \left(Z_{cor} + \frac{d}{c}\right)\frac{h'}{h} \\ Z_{cor_2} &= \frac{-\frac{a+d}{c} Z^*_{cor_1} + \frac{aa^* - dd^*}{cc^*}}{\frac{a^* + d^*}{c^*}} \\ Z'_{cor} - \frac{a}{c} &= \left(Z_{cor_2} - \frac{a}{c}\right) e^{-j2\arg(a+d)} \end{aligned}\right\} \tag{3}$$

and

$$h' = \frac{h}{cc^*s^2 + cd^*Z_{cor} + dc^*Z^*_{cor} + dd^*} \tag{4}$$

If we combine Eqs. 3 and 4 and set

$$s^2 = Z_{cor} Z^*_{cor} + h^2 \tag{5}$$

the isometric sphere method is expressed in analytic form by

$$\left.\begin{aligned} s'^2 &= \frac{aa^* s^2 + ab^* Z_{cor} + ba^* Z^*_{cor} + bb^*}{cc^* s^2 + cd^* Z_{cor} + dc^* Z^*_{cor} + dd^*} \\ Z'_{cor} &= \frac{ac^* s^2 + ad^* Z_{cor} + bc^* Z^*_{cor} + bd^*}{cc^* s^2 + cd^* Z_{cor} + dc^* Z^*_{cor} + dd^*} \end{aligned}\right\} \tag{6}$$

2.3 EXAMPLE OF THE USE OF THE ISOMETRIC SPHERE METHOD

A simple example of the use of the isometric sphere method is shown in Fig. 2.

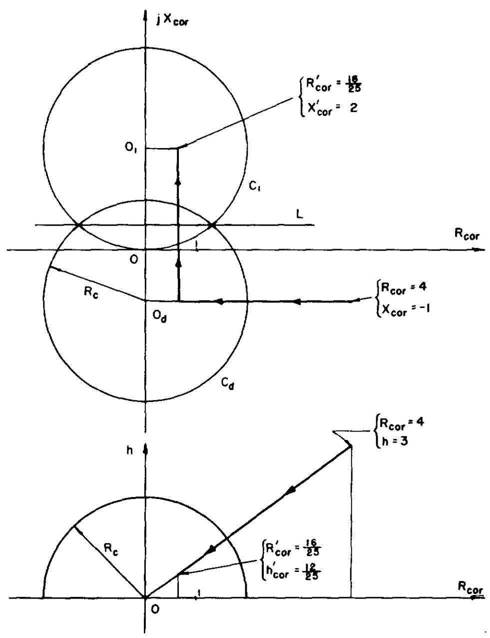

Fig. 2. Example of the use of the isometric sphere method.

The point $(R_{cor}, X_{cor}, h) = (4, -1, 3)$ is transformed through a lossless bilateral two-port network $(a = 1, = -j, c = -j/2, d = 1/2)$ to $(R'_{cor}, X'_{cor}, h') = (16/25, 2, 12/25)$.

III. ANALYTIC REPRESENTATIONS OF NOISY TWO-PORT NETWORKS

A study of the isometric sphere method reveals that for a noise process, transformed by a noise-free two-port network, Eqs. 1 and 2 are exchanged for the following equations:

$$Q' = \begin{pmatrix} Q_1' \\ Q_2' \\ Q_3' \\ Q_4' \end{pmatrix} = \begin{pmatrix} \overline{V'V'^*} \\ \overline{V'I'^*} \\ \overline{V'^*I'} \\ \overline{I'I'^*} \end{pmatrix} = \begin{pmatrix} aa^* & ab^* & ba^* & bb^* \\ ac^* & ad^* & bc^* & bd^* \\ ca^* & cb^* & da^* & db^* \\ cc^* & cd^* & dc^* & dd^* \end{pmatrix} \begin{pmatrix} \overline{VV^*} \\ \overline{VI^*} \\ \overline{V^*I} \\ \overline{II^*} \end{pmatrix} = LQ \tag{7}$$

$$s'^2 = \frac{Q_1'}{Q_4'} = \frac{aa^*s^2 + ab^*Z_{cor} + ba^*Z^*_{cor} + bb^*}{cc^*s^2 + cd^*Z_{cor} + dc^*Z^*_{cor} + dd^*} \tag{8a}$$

$$Z'_{cor} = \frac{Q_2'}{Q_4'} = \frac{ac^*s^2 + ad^*Z_{cor} + bc^*Z^*_{cor} + bd^*}{cc^*s^2 + cd^*Z_{cor} + dc^*Z^*_{cor} + dd^*} \tag{8b}$$

$$Z'^*_{cor} = \frac{Q_3'}{Q_4'} = \frac{ca^*s^2 + cb^*Z_{cor} + da^*Z^*_{cor} + db^*}{cc^*s^2 + cd^*Z_{cor} + dc^*Z^*_{cor} + dd^*} \tag{8c}$$

Asterisks indicate the complex conjugates of the designated quantities and bars indicate averages over ensembles of noise processes with identical statistical properties.

We recognize the conformal transformation (Eqs. 8) as the transformation (Eqs. 6) derived by the isometric sphere method.

For complete correlation between the voltage V and the current I, $h = h' = 0$, and we obtain $s^2 = ZZ^*$, with $Z_{cor} = Z$. Hence Eq. 8a reduces to $s'^2 = Z'Z'^*$, with $Z'_{cor} = Z'$; Eq. 8b reduces to Eq. 2, and Eq. 8c reduces to the complex conjugate of Eq. 2.

If we set

$$\left.\begin{aligned} P_1 &= \tfrac{1}{2}(Q_2 + Q_3) = \tfrac{1}{2}\left(\overline{VI^*} + \overline{V^*I}\right) \\ P_2 &= \tfrac{-j}{2}(Q_2 - Q_3) = \tfrac{-j}{2}\left(\overline{VI^*} - \overline{V^*I}\right) \\ P_3 &= \tfrac{1}{2}(Q_1 - Q_4) = \tfrac{1}{2}\left(\overline{VV^*} - \overline{II^*}\right) \\ P_4 &= \tfrac{1}{2}(Q_1 + Q_4) = \tfrac{1}{2}\left(\overline{VV^*} + \overline{II^*}\right) \end{aligned}\right\} \tag{9}$$

then Eqs. 7 and 8 correspond to

$$P' = \begin{pmatrix} P'_1 \\ P'_2 \\ P'_3 \\ P'_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = MP \tag{10}$$

and

$$\left.\begin{aligned} x' &= \frac{P'_1}{P'_4} = \frac{a_1 x + a_2 y + a_3 z + a_4}{d_1 x + d_2 y + d_3 z + d_4} \\ y' &= \frac{P'_2}{P'_4} = \frac{b_1 x + b_2 y + b_3 z + b_4}{d_1 x + d_2 y + d_3 z + d_4} \\ z' &= \frac{P'_3}{P'_4} = \frac{c_1 x + c_2 y + c_3 z + c_4}{d_1 x + d_2 y + d_3 z + d_4} \end{aligned}\right\} \tag{11}$$

in which the 16 real constants, a_1, a_2, ..., d_3, d_4, are all expressed (5) in the complex constants, a, b, c, and d.

If the two-port network contains inner noise sources Eq. 1 is modified. Thus we have

$$\psi'' = \psi^o + T\psi; \qquad \psi^o = \begin{pmatrix} V^o \\ I^o \end{pmatrix} \tag{12}$$

where V^o and I^o are two noise sources that have a certain correlation. Equation 7 is also modified:

$$Q'' = Q^o + LQ; \qquad Q = \begin{pmatrix} \overline{V^o V^{o*}} \\ \overline{V^o I^{o*}} \\ \overline{V^{o*} I^o} \\ \overline{I^o I^{o*}} \end{pmatrix} \tag{13}$$

Equations 7, 8, 10, 11, 12, and 13 constitute the basic formulas of the theory of noisy two-port networks. Equations 8a, 8b, and 8c are of special interest; and hence it is interesting to study how the great mathematicians Poincaré and Picard originally derived this transformation from the linear fractional transformation (Eq. 2).

IV. DERIVATIONS OF THE BASIC CONFORMAL TRANSFORMATION EQUATIONS

4.1 THE POINCARÉ METHOD

Poincaré (6) starts out with the following expression for a circle in the complex Z' -plane:

$$AZ'Z'^* + BZ' + B^*Z'^* + C = 0 \tag{14}$$

where A and C are real constants and B is a complex constant. (The circle is denoted by C' in Fig. 3.) He then transforms the circle by the linear fractional transformation (Eq. 2) into another circle C. Thus

$$\begin{aligned} & ZZ^*(A\,aa^* + B\,ac^* + B^*\,ca^* + C\,cc^*) \\ + \;& Z\,(A\,ab^* + B\,ad^* + B^*\,cb^* + C\,cd^*) \\ + \;& Z^*(A\,ba^* + B\,bc^* + B^*\,da^* + C\,dc^*) \\ + \;& 1\,(A\,bb^* + B\,bd^* + B^*\,db^* + C\,dd^*) = 0 \end{aligned} \tag{15}$$

Poincaré then generalizes to three dimensions. If we use our previous notation, the Poincaré extension consists in replacing ZZ^* by s^2, and Z by Z_{cor}. Thus Eq. 15 transforms into

$$\begin{aligned} & s^2\,(A\,aa^* + B\,ac^* + B^*\,ca^* + C\,cc^*) \\ + \;& Z_{cor}\,(A\,ab^* + B\,ad^* + B^*\,cb^* + C\,cd^*) \\ + \;& Z_{cor}^*\,(A\,ba^* + B\,bc^* + B^*\,da^* + C\,dc^*) \\ + \;& 1\,(A\,bb^* + B\,bd^* + B^*\,db^* + C\,dd^*) = 0 \end{aligned} \tag{16}$$

Equation 16 represents analytically the hemisphere S erected on the circle C. (See Fig. 3.) A point P on S has the coordinates (R_{cor}, X_{cor}, h). Similarly, a point P' with the coordinates (R'_{cor}, X'_{cor}, h') is situated on the hemisphere S' on C'. The equation for the second hemisphere is

$$As'^2 + BZ'_{cor} + B^*Z'^*_{cor} + C = 0 \tag{17}$$

If the constants A, B, and C are eliminated between Eqs. 16 and 17, the conformal transformation (Eqs. 8) is obtained.

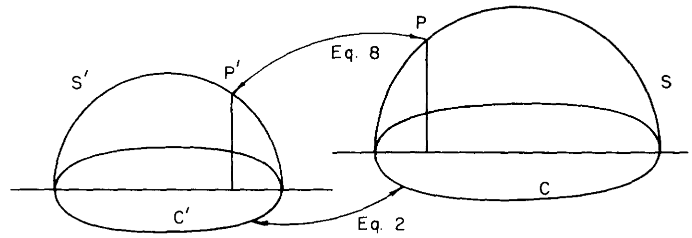

Fig. 3. Poincaré method.

Poincaré (6) showed that the transformation (Eqs. 8) can be considered to be a non-Euclidean transformation in a model of three-dimensional hyperbolic space that has the Z_{cor}-plane as the absolute surface. The model is now usually called "the Poincaré model" (1). Poincaré used this model, for example, in studying automorphic functions.

4.2 THE PICARD METHOD

The second method of deriving the conformal transformation (Eq. 8) is due to Picard (7, 8). Picard starts out with a linear fractional transformation

$$w' = \frac{aw + b}{cw + d} \tag{18}$$

where $w = u + jv$, $w' = u' + jv'$, and a, b, c, and d are real constants. He then calculates the square of the absolute value of w' and the real part u' of w':

$$\left.\begin{aligned} u'^2 + v'^2 &= \frac{a^2(u^2 + v^2) + 2abu + b^2}{c^2(u^2 + v^2) + 2cdu + d^2} \\ u' &= \frac{ac(u^2 + v^2) + (ad + bc)\,u + bd}{c^2(u^2 + v^2) + 2cdu + d^2} \end{aligned}\right\} \tag{19}$$

We recognize that if we consider $Z_{cor} = R_{cor}$, and a, b, c, and d real in Eq. 8, then Eq. 8, reduces to the form of Eq. 19. Actually, Picard obtained Eq. 8 from Eq. 19 by studying the transformation of Hermitian forms by linear fractional transformations, and by setting $uu^* + v^2 = s^2$, $u = Z_{cor}$, and $v = h$.

Obviously, Picard starts out in a plane that is perpendicular to the Z_{cor}-plane and cuts this plane along the R_{cor}-axis. He then extends the transformation to three dimensions. In current technical language, we may say that while Poincaré extends a study of noise-free lossy two-port networks to a study of noisy lossy networks, Picard extends a study of noisy resistive networks (with real correlation between the voltage and the current) to a study of noisy lossy two-port networks.

Thus we find that the extension of the isometric circle method to the isometric sphere method, which constitutes the basis of the new theory of noisy two-port networks, is an extension in the sense of Poincaré's method.

The conformal transformation (Eq. 8) has been used extensively by Fricke and Klein (9) in a study of automorphic functions, and it has also been derived by Hurwitz and Courant (10).

V. GEOMETRIC REPRESENTATIONS

5.1 CONNECTIONS BETWEEN THE POINCARÉ AND CAYLEY-KLEIN MODELS OF THREE-DIMENSIONAL HYPERBOLIC SPACE

Before we apply the basic transformations, Eqs. 8 and 11, in the theory of noisy two-port networks, let us look into their geometric representations.

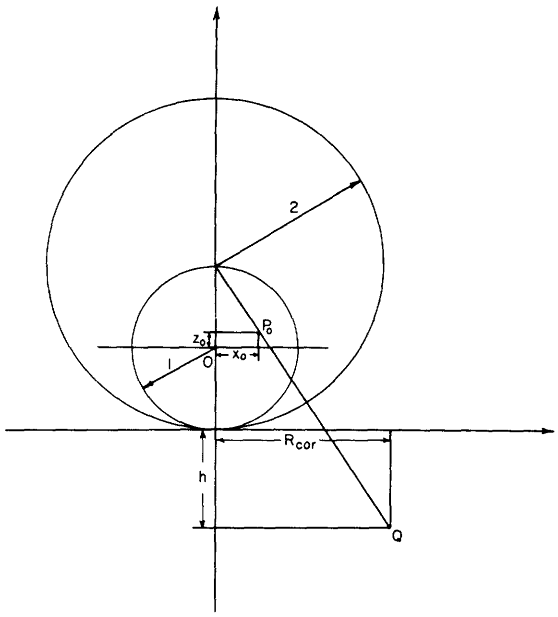

Fig. 4. Geometric connection between the Poincaré models of three-dimensional hyperbolic space.

We know (from Poincaré) that Eq. 8 may be considered as a non-Euclidean transformation in a three-dimensional hyperbolic space with the Z_{cor}-plane as the absolute surface. This model (turned upside down and in double scale) is transformed into a unit sphere by a simple inversion, as shown in Fig. 4. A point Q, with coordinates (R_{cor}, X_{cor}, h), is transformed into a point P_o, with coordinates (x_o, y_o, z_o), by means of the equations

$$\left.\begin{aligned} x_o &= \frac{2R_{cor}}{R_{cor}^2 + X_{cor}^2 + (h+1)^2} \\ y_o &= \frac{2X_{cor}}{R_{cor}^2 + X_{cor}^2 + (h+1)^2} \\ 1 - z_o &= \frac{2(h+1)}{R_{cor}^2 + X_{cor}^2 + (h+1)^2} \end{aligned}\right\} \tag{20}$$

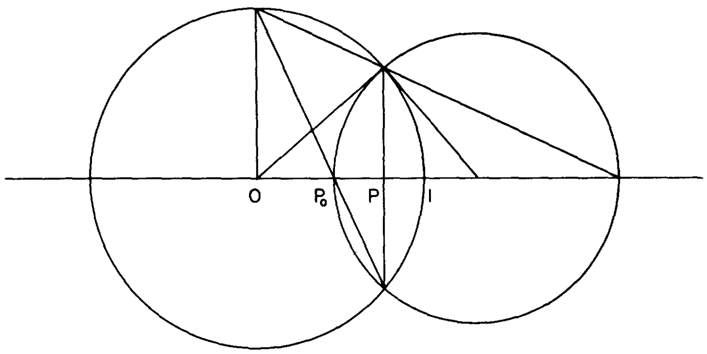

Fig. 5. Darboux transformation.

A semicircle perpendicular to the Z_{cor}-plane is transformed into an arc of a circle orthogonal to the surface of the unit sphere. Thus we obtain the Poincaré model of three-dimensional hyperbolic space with the unit sphere as the absolute surface. We now transform this model into a Cayley-Klein model of three-dimensional hyperbolic space (also with the unit sphere as the absolute surface) by the "Darboux transformation" through which every pair of points that is inverse with respect to the sphere is transformed into the pole of its symmetry plane (11). The transformation is equivalent to the transformation $\mathscr{B}$ that was introduced by Deschamps (12) in the two-dimensional case. A point P_o with the coordinates (x_o, y_o, z_o) is transformed into a point P with the coordinates (x, y, z) (see Fig. 5) by means of the following equations:

$$\left.\begin{aligned} x &= \frac{2x_o}{x_o^2 + y_o^2 + z_o^2 + 1} \\ y &= \frac{2y_o}{x_o^2 + y_o^2 + z_o^2 + 1} \\ z &= \frac{2z_o}{x_o^2 + y_o^2 + z_o^2 + 1} \end{aligned}\right\} \qquad (21)$$

A combination of Eqs. 20 and 21 yields the following transformation equations:

$$\left.\begin{aligned} x &= \frac{2R_{cor}}{R_{cor}^2 + X_{cor}^2 + h^2 + 1} \\ y &= \frac{2X_{cor}}{R_{cor}^2 + X_{cor}^2 + h^2 + 1} \\ z &= \frac{R_{cor}^2 + X_{cor}^2 + h^2 - 1}{R_{cor}^2 + X_{cor}^2 + h^2 + 1} \end{aligned}\right\} \qquad (22)$$

which represent the analytic connection between the point Q in the Poincaré model and the point P in the Cayley-Klein model (13, 14). The inverse transformation of Eq. 22 is obtained directly:

$$\left.\begin{aligned} R_{cor} &= \frac{x}{1 - z} \\ X_{cor} &= \frac{y}{1 - z} \\ h &= (\pm) \frac{\sqrt{1 - x^2 - y^2 - z^2}}{1 - z} \end{aligned}\right\} \qquad (23)$$

By using Eq. 5, we obtain

$$z = \frac{s^2 - 1}{s^2 + 1}; \qquad s^2 = \frac{1 + z}{1 - z} \qquad (24)$$

Equations 22, 23, and 24 are all easily derived from Eqs. 8 and 9. This means that while Eq. 8 corresponds to a non-Euclidean movement in the Poincaré model of three-dimensional hyperbolic space, with the Z_{cor}-plane as the absolute surface, Eq. 11 corresponds to a non-Euclidean movement in the Cayley-Klein model of three-dimensional hyperbolic space, with the unit sphere as the absolute surface.

Equation 23 transforms a hemisphere, orthogonal to the Z_{cor}-plane, which has its center at the point (e, f), and a radius r,

$$R^2_{cor} + X^2_{cor} + h^2 - 2eR_{cor} - 2fX_{cor} + c^2 + f^2 - r^2 = 0$$

into a plane that cuts the unit sphere of the Cayley-Klein model:

$$ux + vy + wz + 1 = 0 \qquad (25)$$

where

$$\left.\begin{aligned} u &= -\frac{2e}{1 + e^2 + f^2 - r^2} \\ v &= -\frac{2f}{1 + e^2 + f^2 - r^2} \\ w &= \frac{1 - e^2 - f^2 + r^2}{1 + e^2 + f^2 - r^2} \end{aligned}\right\} \qquad (26)$$

5.2 EXAMPLE

The two isometric spheres and the symmetry plane used in the example that illustrated the isometric sphere method (see Fig. 2) are transformed into the planes:

$$y + 2z - 1 = 0$$

$$4y - z - 1 = 0$$

$$2y + z - 1 = 0$$

The planes all cut along a straight line that is parallel to the x-axis and cuts the yz-plane in the point $\left(0, \frac{1}{3}, \frac{1}{3}\right)$. The straight line cuts the surface of the unit sphere in two points that constitute the image points obtained by stereographically mapping the fixed points in the Z_{cor}-plane on the unit sphere. (The fixed points are marked by crosses in Fig. 2.) In the example that has been chosen the transformation is a non-Euclidean rotation around this straight line, i.e., it is an elliptic transformation (4, 5).

In the (R_{cor}, X_{cor}, h)-space an elliptic transformation corresponds to a movement of points on circles perpendicular to a fixed semicircle that passes orthogonally through the fixed points in the Z_{cor}-plane. In the hyperbolic case a stretching is obtained along circles through the fixed points. A transformation through a bilateral lossy two-port network will correspond, therefore, to a loxodromic movement on a horned cyclide (9, 15).

VI. NOISY TWO-PORT NETWORK THEORY

6.1 THE THEORY OF ROTHE AND DAHLKE

In a recent theory of noisy two-port networks Rothe and Dahlke (16) split a noisy two-port network into a noise-free part and a noisy part. The noisy part, which consists basically of a complex noise voltage source V and a complex noise current source I

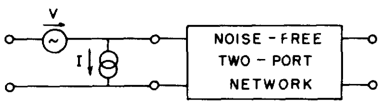

Fig. 6. Splitting of a noisy two-port network into noisy and noise-free parts.

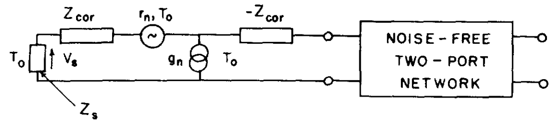

Fig. 7. Equivalent noisy two-port network of Rothe and Dahlke.

(see Fig. 6), is represented by an equivalent circuit that contains an equivalent noise resistance r_n, an equivalent noise conductance g_n, and a complex correlation impedance $Z_{cor} = R_{cor} + jX_{cor}$. (See Fig. 7.) Following Rothe and Dahlke, we split the noise voltage V into two components:

$$V = V_1 + Z_{cor} I \tag{27}$$

where

$$\overline{V_1 I^*} = 0 \tag{28}$$

Thus V_1 and $Z_{cor}I$ are uncorrelated. By introducing a complex correlation coefficient,

$$\gamma = \frac{\overline{VI^*}}{\sqrt{\overline{VV^*}\ \overline{II^*}}} = \frac{Q_2}{\sqrt{Q_1\ Q_4}} \tag{29}$$

Rothe and Dahlke show that the complex correlation impedance Z_{cor} can be expressed as

$$Z_{cor} = \gamma \cdot \sqrt{\frac{\overline{VV^*}}{\overline{II^*}}} \tag{30}$$

which is the expression for the correlation impedance that has been used before; that is,

$$Z_{cor} = \frac{Q_2}{Q_4} = \frac{P_1 + jP_2}{P_4 - P_3} \tag{31}$$

If we use the equations

$$\left.\begin{aligned}\overline{V_1V_1^*} &= 4kT_o\Delta f\, r_n\\ \overline{II^*} &= 4kT_o\Delta f\, g_n\end{aligned}\right\} \tag{32}$$

in which k is Boltzmann's constant, T_o the absolute temperature, and Δf the bandwidth, and remember that $s^2 = Q_1/Q_4$, Eq. 27 will yield

$$s^2 = \frac{r_n}{g_n} = Z_{cor}Z^*_{cor} \tag{33}$$

A comparison of Eq. 33 with Eq. 5 shows that

$$h = \sqrt{\frac{r_n}{g_n}} \tag{34}$$

Therefore the connection between the Rothe and Dahlke theory and the new geometric-analytic theory is given by the following equations:

$$Q = 4kT_o\Delta f\, g_n \begin{pmatrix} \frac{r_n}{g_n} + Z_{cor}Z^*_{cor} \\ Z_{cor} \\ Z^*_{cor} \\ 1 \end{pmatrix} \tag{35}$$

and

$$P = 4kT_o\Delta f\, g_n \begin{pmatrix} R_{cor} \\ X_{cor} \\ \frac{1}{2}\left(\frac{r_n}{g_n} + Z_{cor}Z^*_{cor} - 1\right) \\ \frac{1}{2}\left(\frac{r_n}{g_n} + Z_{cor}Z^*_{cor} + 1\right) \end{pmatrix} \tag{36}$$

Equation 36 checks with Eq. 22.

A noise ensemble average transformation through a noise-free two-port network is performed by transforming the complex four-vector Q by Eq. 7, or by transforming the real four-vector P by Eq. 10. The 4×4 matrix L in Eq. 7 is constructed in a regular and simple manner by using the elements of the 2×2 matrix T in Eq. 1. The matrix L is termed the Kronecker or tensor product (17) of T^* by T and it is denoted $T \times T^*$. Thus, Eq. 13 can be written as

$$Q'' = Q^o + Q' = Q^o + T \times T^* Q \tag{37}$$

The 4×4 matrix M in Eq. 10 belongs to the G_+ subgroup of the proper Lorentz group (5).

6.2 CASCADING OF NOISY TWO-PORT NETWORKS

If we insert the Q-vectors (Eq. 35) in Eq. 37 we obtain

$$\left.\begin{aligned} r''_n &= r^o_n + r'_n + \frac{g^o_n\, g'_n \left| Z'_{cor} - Z^o_{cor} \right|}{g^o_n + g'_n} \\ g''_n &= g^o_n + g'_n \\ Z''_{cor} &= \frac{g^o_n Z^o_{cor} + g'_n Z'_{cor}}{g^o_n + g'_n} \end{aligned}\right\} \tag{38}$$

From Eqs. 37 and 35 we also obtain

$$\left.\begin{aligned} g'_n &= cc^* r_n + \left| cZ_{cor} + d \right|^2 g_n \\ r'_n &= \frac{\left| ad - bc \right|^2 r_n g_n}{cc^* r_n + \left| cZ_{cor} + d \right|^2 g_n} \\ Z'_{cor} &= \frac{ac^* r_n + (aZ_{cor} + b)(c^* Z^*_{cor} + d^*) g_n}{cc^* r_n + \left| cZ_{cor} + d \right|^2 g_n} \end{aligned}\right\} \tag{39}$$

Equations 38 and 39 are formulas obtained by Dahlke (18). In Dahlke's formulas $a_{11} = d$, $a_{12} = c$, $a_{21} = b$, and $a_{22} = a$.

6.3 NOISE TUNING AND NOISE MATCHING

If a signal source with impedance $Z_s = R_s + jX_s$ is connected to the input of the noisy two-port network (see Fig. 7), then noise tuning and noise matching (16) is obtained when

$$\left.\begin{aligned} X_s &= -X_{cor} \\ R_s^2 &= R^2_{cor} + h^2 = R^2_{cor} + \frac{r_n}{g_n} \end{aligned}\right\} \tag{40}$$

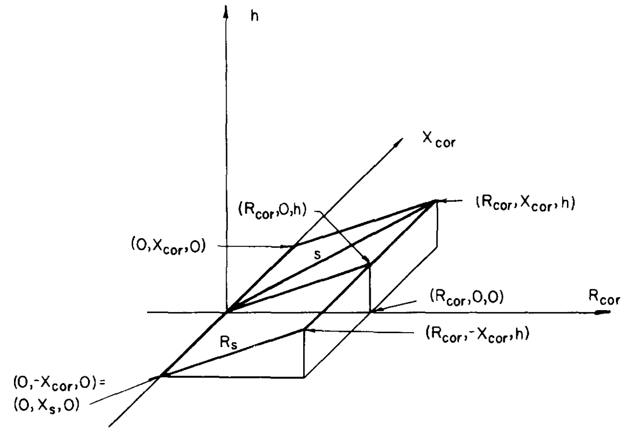

Fig. 8. Noise tuning and noise matching.

Equations 40 constitute a generalization of ordinary tuning and matching to three dimensions. (See Fig. 8.)

6.4 WAVE REPRESENTATION

Instead of the voltage-current representation used in previous sections it is sometimes more convenient to use a wave representation.

We split the reflected wave at the output of the two-port network into two components

$$a_2 = a_{2n} + \Gamma_{cor}\, a_1 \tag{41}$$

In Eq. 41 a_1 is the transmitted wave and Γ_{cor} is a complex correlation reflection coefficient, defined by

$$\Gamma_{cor} = \frac{\overline{a_2\, a_1^*}}{\overline{a_1\, a_1^*}} \tag{42}$$

In Eq. 41 a_{2n} and $\Gamma_{cor}\, a_1$ are uncorrelated, so that

$$\overline{a_{2n}\, a_1^*} = 0 \tag{43}$$

If we set

$$\begin{aligned} \overline{a_2\, a_2^*} &= k\Delta f\, T_2 \\ \overline{a_{2n}\, a_{2n}^*} &= k\Delta f\, T_{2n} \\ \overline{a_1\, a_1^*} &= k\Delta f\, T_1 \end{aligned} \tag{44}$$

where T_2, T_{2n}, and T_1 are noise temperatures, Eq. 41 yields

$$T_2 = T_{2n} + \left|\Gamma_{cor}\right|^2 T_1 \tag{45}$$

The four-vector Q can now be written in the wave representation as

$$Q_w = \begin{pmatrix} Q_{w1} \\ Q_{w2} \\ Q_{w3} \\ Q_{w4} \end{pmatrix} = \begin{pmatrix} \overline{a_2a_2^*} \\ \overline{a_2a_1^*} \\ \overline{a_1a_2^*} \\ \overline{a_1a_1^*} \end{pmatrix} = k\Delta f\ T_1 \begin{pmatrix} \frac{T_{2n}}{T_1} + \left|\Gamma_{cor}\right|^2 \\ \Gamma_{cor} \\ \Gamma^*_{cor} \\ 1 \end{pmatrix} \tag{46}$$

The reflected and transmitted waves at the input of a noise-free two-port network are expressed in the transmitted and reflected waves at the output by

$$\psi'_w = \begin{pmatrix} a'_2 \\ a'_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} = T_w\ \psi_w \tag{47}$$

By using the chain matrix, we obtain

$$T_w = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = STS^{-1} \tag{48}$$

For a noise process we have

$$Q'_w = \begin{pmatrix} \overline{a'_2a'^*_2} \\ \overline{a'_2a'^*_1} \\ \overline{a'_1a'^*_2} \\ \overline{a'_1a'^*_1} \end{pmatrix} = \begin{pmatrix} AA^* & AB^* & BA^* & BB^* \\ AC^* & AD^* & BC^* & BD^* \\ CA^* & CB^* & DA^* & DB^* \\ CC^* & CD^* & DC^* & DD^* \end{pmatrix} \begin{pmatrix} \overline{a_2a_2^*} \\ \overline{a_2a_1^*} \\ \overline{a_1a_2^*} \\ \overline{a_1a_1^*} \end{pmatrix} = L_wQ_w \tag{49}$$

where

$$L_w = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} aa^* & ab^* & ba^* & bb^* \\ ac^* & ad^* & bc^* & bd^* \\ ca^* & cb^* & da^* & db^* \\ cc^* & cd^* & dc^* & dd^* \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{50}$$

L_w is the product of the three Kronecker products $S \times S$, $T \times T$, and $S^{-1} \times S^{-1}$.

If we set

$$\left.\begin{aligned} V &= \frac{1}{\sqrt{2}}(a_1 + a_2) \\ I &= \frac{1}{\sqrt{2}}(a_1 - a_2) \end{aligned}\right\} \qquad \left.\begin{aligned} a_2 &= \frac{1}{\sqrt{2}}(V - I) \\ a_1 &= \frac{1}{\sqrt{2}}(V + I) \end{aligned}\right\} \tag{51}$$

we obtain

$$\begin{aligned} P_{z1} &= \frac{1}{2}\left(\overline{VI^*} + \overline{V^*I}\right) = -\frac{1}{2}\left(\overline{a_2a_2^*} - \overline{a_1a_1^*}\right) = -P_{\Gamma 3} \\ P_{z2} &= -\frac{j}{2}\left(\overline{VI^*} - \overline{V^*I}\right) = -\frac{j}{2}\left(\overline{a_2a_1^*} - \overline{a_1a_2^*}\right) = P_{\Gamma 2} \\ P_{z3} &= \frac{1}{2}\left(\overline{VV^*} - \overline{II^*}\right) = \frac{1}{2}\left(\overline{a_2a_1^*} + \overline{a_1a_2^*}\right) = P_{\Gamma 1} \\ P_{z4} &= \frac{1}{2}\left(\overline{VV^*} + \overline{II^*}\right) = \frac{1}{2}\left(\overline{a_2a_2^*} + \overline{a_1a_1^*}\right) = P_{\Gamma 4} \end{aligned} \tag{52}$$

Thus Eq. 49 can be interpreted geometrically as a movement in a Poincaré model of three-dimensional hyperbolic space that has the complex reflection-coefficient plane for its absolute surface. The corresponding Cayley-Klein model, which has the unit sphere as the absolute surface, is obtained from the Cayley-Klein model that was treated in previous sections by a 90° rotation around the y-axis.

Wave representations of noisy two-port networks have been studied by Bauer and Rothe (19, 20).

VII. FOUR-DIMENSIONAL TREATMENT

7.1 COHERENCY MATRIX

If we postmultiply the vector ψ' in Eq. 1 by its transposed conjugate value, indicated by the symbol $\dagger$, we obtain

$$\psi'\psi'^{\dagger} = T\psi(T\psi)^{\dagger} = T\psi\psi^{\dagger}T^{\dagger} \tag{53}$$

or

$$\begin{pmatrix} V'V'^{*} & V'I'^{*} \\ V'^{*}I' & I'I'^{*} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} VV^{*} & VI^{*} \\ V^{*}I & II^{*} \end{pmatrix}\begin{pmatrix} a^{*} & c^{*} \\ b^{*} & d^{*} \end{pmatrix} \tag{54}$$

Equation 54, which was derived by Cauer (21), can be considered, for example, to be the transformation equation for the components and the magnitude of the associate vector of a spinor (22, 23). With a certain correlation between the voltage and the current we can write Eq. 54 in the form

$$\begin{pmatrix} Q_1' & Q_2' \\ Q_3' & Q_4' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}\begin{pmatrix} a^{*} & c^{*} \\ b^{*} & d^{*} \end{pmatrix} \tag{55}$$

or

$$C' = \begin{pmatrix} P_4' + P_3' & P_1' + jP_2' \\ P_1' - jP_2' & P_4' - P_3' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} P_4 + P_3 & P_1 + jP_2 \\ P_1 - jP_2 & P_4 - P_3 \end{pmatrix}\begin{pmatrix} a^{*} & c^{*} \\ b^{*} & d^{*} \end{pmatrix} = TCT^{\dagger} \tag{56}$$

7.2 OPTICAL ANALOGY

At this point, it is interesting to compare some of the derived equations with analogous equations in matrix optics. In optics the voltage-current vector ψ corresponds to the Maxwell vector that is composed of two complex perpendicular field-strength components E_x and E_y. The complex four-vector Q in Eq. 7 is analogous to the complex Stokes vector, and the real four-vector P in Eq. 9 is analogous to the real Stokes vector. The Stokes vector was introduced by Stokes (24), in 1852, for representing partially polarized light. The complex 2×2 matrix T in Eq. 1 is analogous in optics to the Jones matrix (25); the complex 4×4 matrix in Eq. 7 and the real 4×4 matrix in Eq. 9 are analogous to 4×4 matrices used by Soleillet (26), Perrin (27), Chandrasekhar (28), Mueller (29, 30), Parke (31-34), McCormick (35), Manasse (36), and others. The Kronecker (tensor) product $T \times T^{*}$ was introduced into optics by McCormick (35).

The 2×2 Hermitian matrix C in Eq. 56 is analogous to the "density matrix" used by von Neumann (37) in quantum mechanics, and to the "coherency matrix" used by Wiener in optics (38, 39, 40). The coherency matrix has found extensive application in communication theory (41, 42).

It can be proved by means of the Schwartz inequality that the determinants of the coherency matrices C and C' of Eq. 56 are equal to or greater than zero; that is,

$$\det C = D^2 \geq 0, \qquad \det C' = D'^2 \geq 0 \tag{57}$$

We can rewrite Eq. 56 in the form

$$\sigma_1 P_1' + \sigma_2 P_2' + \sigma_3 P_3' + \sigma_4 P_4' = T(\sigma_1 P_1 + \sigma_2 P_2 + \sigma_3 P_3 + \sigma_4 P_4)\, T^\dagger \tag{58}$$

in which

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{59}$$

are the Pauli matrices (43, 44). By using the Pauli matrices, Eqs. 55 and 56 can be interpreted as transformations of Q four-vectors and P four-vectors in four-dimensional spaces.

The determinant D^2 of the coherency matrix C is an important quantity in the theory of noisy two-port networks. We obtain

$$D^2 = Q_1 Q_4 - Q_2 Q_3 = P_4^2 - P_3^2 - P_2^2 - P_1^2 = (4kT_o \Delta f)^2\, r_n g_n \tag{60}$$

Then, from Eqs. 5, 8, 34, and 60, we find

$$h = \frac{D}{Q_4} = \frac{D}{P_4 - P_3} = \sqrt{\frac{r_n}{g_n}} \tag{61}$$

7.3 GEOMETRIC INTERPRETATION OF HERMITIAN FORMS

If, in Eq. 1, instead of postmultiplying ψ' with its transposed conjugate value, we premultiply with the same value, we obtain

$$\psi'^\dagger \psi' = (T\psi)^\dagger\, T\psi = \psi^\dagger T^\dagger T \psi \tag{62}$$

Here $T^\dagger T$ is a Hermitian matrix which we denote by

$$T^\dagger T = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \tag{63}$$

Equations 62 and 63 yield a Hermitian form

$$q(\psi) = A\, V\, V^* + B\, V^* I + B^* V I^* + C\, I\, I^* \tag{64}$$

In a geometric interpretation of Hermitian forms, Deschamps (44) sets

$$T^\dagger T = -\sigma_1 X - \sigma_2 Y - \sigma_3 Z + \sigma_4 T \tag{65}$$

and thus establishes a one-to-one correspondence between the Hermitian matrices and the vectors of a four-dimensional Minkowski space with real coordinates (X, Y, Z, T).

7.4 FOUR-DIMENSIONAL DERIVATION OF THE NOISE FACTOR

Let us assume that we have split a noisy two-port network into noisy and noise-free parts, as in Fig. 7. We then attach a noisy impedance $Z_s = R_s + jX_s$ at the input of the network. The excess-noise factor, by definition, is

$$F_z = \frac{Q_1'}{\overline{V_s V_s^*}} = \frac{L_a Q}{\overline{V_s V_s^*}} \tag{66}$$

where L_a is a four-vector composed of the first row of the 4×4 matrix L in Eq. 7. Thus

$$L_a = (aa^*, ab^*, ba^*, bb^*) \tag{67}$$

and

$$\overline{V_s V_s^*} = 4kT\Delta f\, R_s \tag{68}$$

Whence the excess-noise factor is proportional to the scalar product of two four-vectors. For the series impedance Z_s, we have $a = 1$, $b = Z_s$, $c = 0$, and $d = 1$, so that

$$F_z = \frac{Q_1 + Z_s^* Q_2 + Z_s Q_3 + Z_s Z_s^* Q_4}{4kT\Delta f\, R_s} \tag{69}$$

which is a well-known expression (16, 45, 46). The optimum excess-noise factor is obtained in the usual manner by equating the partial differentiations with respect to X_s and R_s to zero. Whence we obtain

$$X_{s,\,opt} = -\frac{Q_{2i}}{Q_4} = -\frac{P_2}{P_4 - P_3}, \qquad Q_2 = Q_{2r} + jQ_{2i}$$

$$R_{s,\,opt} = \frac{\sqrt{Q_1 Q_4 - Q_{2i}^2}}{Q_4} = \frac{\sqrt{P_1^2 + D^2}}{P_4 - P_3} \tag{70}$$

$$F_{z,\,opt} = \frac{1}{2kT\Delta f}\left(\sqrt{Q_1 Q_4 - Q_{2i}^2} + Q_{2r}\right) = \frac{1}{2kT\Delta f}\left(\sqrt{P_1^2 + D^2} + P_1\right)$$

7.5 THEORY OF LONGITUDINAL ELECTRON BEAMS

In a theory of longitudinal electron beams, Haus (47, 48), and Haus and Robinson (49) define the self-power density spectrum (SPDS) of the kinetic noise-voltage modulation Φ, the SPDS of the noise-current modulation Ψ, and the cross-power density spectrum (CPDS) between the kinetic voltage and current modulations θ. These variables can be combined to form a Q-vector as follows:

$$Q = 4\pi \, \Delta f \begin{pmatrix} \Phi \\ \theta \\ \theta^* \\ \Psi \end{pmatrix} \tag{71}$$

VIII. CONCLUSION

From the standpoint of projective geometry, the geometric-analytic theory of noisy two-port networks shows that (a) bilateral lossless two-port networks are essentially one-dimensional in structure and can be described by real numbers, (b) bilateral lossy two-port networks are essentially two-dimensional and can be described by complex numbers, (c) noisy two-port networks are, on an ensemble average ratio basis, essentially three-dimensional and can be described by combinations made up of a complex number and a real number, or of three real numbers, and (d) noisy two-port networks are, on an ensemble average basis, essentially four-dimensional and can be described by combinations made up of a complex number and two real numbers, or of four real numbers.

The fact that Eq. 8 is a conformal transformation makes extensions to higher dimensional spaces possible. Thus, for example, the three-dimensional (R_{cor}, X_{cor}, h)-space can be stereographically mapped on the surface of a unit hypersphere (50). However, such transformations have not been treated in this report.

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