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ON THE STATICS AND DYNAMICS
OF MAGNETIC DOMAIN BOUNDARIES

J. V. HARRINGTON

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Division 2

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ABSTRACT

A semiclassical analysis of the variation of spin orientation in 180° and 90° domain walls is presented which takes into account both the rotational or latitude angle $\theta$ and the precessional or longitude angle $\phi$ of the macroscopic magnetization vector in ferromagnetic single crystals. The Euler equations for the crystal are derived for both $\phi$ and $\theta$ variations, and solutions of these equations indicate the possibility of a change in spin orientation along a 180° Bloch wall resulting from a precession in $\phi$. This possibility is verified experimentally by some Bitter powder pattern results in which 180° wall breakup is recorded in a single crystal window frame of silicon iron under the action of a moderate vertically applied magnetic field. Further static solutions of the Euler equation for 360° walls or two adjacent 180° walls of similar rotation driven together by an applied field are obtained. It is shown that when this field is reduced to the order of the coercive force, the equilibrium spacing between 180° walls is about ten wall thicknesses. Some experimental verification of this is obtained from recent observations of Williams on double walls in thin films of nickel iron.

By including the kinetic energy term characteristic of gyroscopic motion in the Lagrangian and assuming a Rayleigh dissipation function, the Euler equations for wall motion are obtained. It is shown that the assumption of Rayleigh dissipation leads to a more reasonable variation of wall velocity with losses in the material. In particular, the velocity singularities predicted by the Landau-Lifshitz formula for very low and very high dissipation are eliminated and a very general upper bound to the domain wall velocity is predicted, based only on known constants of the ferromagnetic lattice. Thus,

$$\frac{V_{\text{max}}}{H} = \frac{\gamma}{2} \sqrt{\frac{A}{K_1}}$$

where $\gamma$ is the magnetomechanical ratio, and $\sqrt{A/K_1}$ is essentially the domain wall thickness. This upper bound is consistent with present wall velocity data in iron and magnetite but differs with spotty data on nickel ferrite. A transient solution for the general equation of wall motion is obtained where the effects of wall mass are included. An interpretation of wall mass in terms of the macroscopic inertial moments of the spin system is given.

*This report is identical with a thesis of the same title submitted in partial fulfillment of the requirements for the degree of Doctor of Science in the Department of Electrical Engineering at the Massachusetts Institute of Technology.
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CHAPTER I
INTRODUCTION

1.1 SURVEY

Although the theory of the magnetic domain wall dates back some twenty years to the work of Bloch, Landau and Lifshitz, it is predicated even to the present day on simple two-dimensional models of spin variations through a domain boundary layer. The simplicity of the analyses made inevitably leads to inaccurate and incomplete results both for static wall conditions and for the dynamics of wall motion. There is a further difficulty in the dynamic case which arises with regard to the all-important damping mechanism chosen. The one which prevails today in the theoretical studies of wall motion is the empirical one proposed by Landau and Lifshitz. It will be shown that its character leads to a physically unreasonable relationship between wall velocity and magnetic system losses.

It is the purpose of this investigation to evolve a more complete three-dimensional treatment of the theory of magnetic domain boundaries. The treatment is essentially a classical one and in the dynamic case employs a Rayleigh dissipation function which is believed to be a more reasonable damping mechanism than those previously employed. This procedure, like others which have been applied to the problem, is justified by the result it achieves. In the statics case, the mathematical formulation obtained permits the calculation of spin orientation not only through but also along the domain wall. Such spin transitions have been observed by Williams in a study of domain boundaries in perminvar and by the present author in some observations made on silicon iron single crystals. A further new result is the theoretical treatment of the 360° wall. These results also appear to be consistent with some double wall observations made recently in thin magnetic films. The results on wall velocity are particularly interesting in that a more acceptable dependence of velocity on system losses is predicted and, more important, an upper bound in wall velocity which is a function only of known fundamental constants of the ferromagnetic lattice is obtained. In general, a more satisfactory picture of the very complex motion of the macroscopic magnetization vectors during domain boundary transitions is produced.

The general structure of this study is as follows. In Chapter II, the various energy contributions to the total energy in the ferromagnetic lattice are considered. A variational treatment of this total energy yields the basic three-dimensional Euler-Lagrange differential equations for the ferromagnetic material. The remaining chapters deal principally with application of these equations. In Chapter III, static applications are considered: first, to the usual 90° and 180° walls; second, to more complex domain boundaries. Dynamic applications of the basic equations are considered in Chapter IV. In particular, the motion of 180° domain walls is analyzed. Wall velocity relationships are obtained and compared with the Landau-Lifshitz results. An upper bound on this velocity is derived. Some consideration is given to wall acceleration and wall mass effects. A second dynamics application of the basic equation is made in

*Superscripts refer to references listed on page 65.
Chapter V, this time to ferromagnetic resonance phenomena. The object here is to establish that the Rayleigh damping assumption leads to resonance spectra which are consistent with the experimental data thus far obtained. Finally, in Chapter VI, some experimental domain pattern observations are presented, some of which support the occurrence of a spin transition along a domain wall as predicted by the results in Chapter III.

1.2 HISTORY

The first of the important papers on the subject of magnetic domains was the 1907 paper of Pierre Weiss, in which he introduced magnetic domains to account for gross magnetization changes in ferromagnetic media, even though at any given point in the medium the macroscopic magnetization vector was assumed to have a constant amplitude. In this same paper, Weiss postulated the so-called molecular field to account for the strong interaction between adjacent atomic dipoles. Some time later, Heisenberg showed that the nature of this molecular field was quantum mechanical and that it was essentially due to the so-called exchange energy between electron spins and not to any magnetic field as such. The exchange energy is basically electrostatic in nature.

The first description of the boundary or transition region between domains was given by Bloch in 1932, in which he derived expressions for both the energy and the width of some simple domain walls. He used a variational method to minimize the free energy of the ferromagnetic crystal and to calculate these quantities, and this is a method that has been used by all investigators since then. Bloch, however, merely considered the statics of the domain walls. In 1935, Landau and Lifshitz, in a now classic paper, not only considered the energy and thickness of domain boundaries but also derived an expression—the so-called Landau-Lifshitz equation—in which a description of the dynamics of magnetization changes is contained. This equation has proved to be an extremely useful one and is the theoretical basis on which most gyromagnetic phenomena are analyzed. It is basically an equation of motion for an elementary gyroscope. In some recent work, equations due to Bloch and Gilbert have been used which are based on somewhat different damping mechanisms. In 1944, both Lifshitz and Néel included magnetoelastic energy in the calculation of some domain wall configurations. In 1950, Lilley extended this work and made some very careful calculations of the energy and widths of domain boundaries in the principal ferromagnetic metals, i.e., cobalt, iron and nickel.

In addition to these theoretical treatises on energy and thickness of domain walls, a great deal of very comforting experimental evidence has been obtained on the existence of domains, principally by Elmore, Williams, and others who improved a powder pattern technique originally devised by Bitter.

Evidence of the gyroscopic nature of the elementary magnetic dipoles was first deduced from the famous Einstein-deHaas experiment. However, with the advent of microwave techniques, additional evidence from ferromagnetic resonance experiments has been obtained to support this gyro-like behavior of the magnetization vector in a ferromagnetic medium.

An additional area of activity also concerned with dynamic magnetic effects has been the measuring of the velocity with which domain walls move under the action of an external field.
Galt\textsuperscript{14} has investigated this for single crystals of nickel ferrite; Williams, Shockley and Kittel\textsuperscript{15}, and Rodbell and Bean\textsuperscript{16,17} for single crystals of silicon iron; and Galt, et al\textsuperscript{18} and Epstein\textsuperscript{19} for single crystals of magnetite.

One concludes that there is a substantial amount of theoretical as well as experimental work that well establishes the basis for and existence of magnetic domains. The basis for the formation of magnetic domains in ferromagnetic materials is the minimization of the free energy in the magnetic system. Of these free-energy components, the four which seem to be principal contributors are: the magnetostatic energy; the exchange energy, which is a function of the misalignment of adjacent spin moments; the anisotropy energy, which is based on the occurrence of easy and hard directions of magnetization in a crystalline lattice; and the magnetostrictive energy in which the energy storage is principally elastic, i.e., is due to the state of strain in the lattice.

The manner in which these various energies adjust is to make the total free energy in the lattice a minimum. For many sample shapes, where the ferromagnetic material does not have a uniaxial anisotropy, this implies the absence of magnetic poles on the specimen boundaries, since the occurrence of magnetic poles on any of the sample boundaries produces a very large magnetostatic energy contribution. Calculations made by Kittel\textsuperscript{20,21} and others on specimens of simple geometry show that, in general, when the domain structure is arranged to eliminate these free poles, a much lower energy configuration results. This leads to the formation, at the boundaries, of so-called domains of closure which are quite commonly encountered in ferromagnetic materials.

This general concept of the minimization of energy is, of course, a fairly common one in physics but, as mentioned earlier, it was first applied to the domain wall problem by Bloch in 1932 and has been used in one form or another by most investigators since. The method, however, has been used only in simple two-dimensional cases. It is proposed in this investigation to do a comprehensive three-dimensional variational treatment of the free energy in a ferromagnetic material. The inclusion of a gyroscopic-kinetic energy contribution from the spin system will permit a complete description of the dynamic behavior of Bloch walls without the necessity of later invoking the Landau-Lifshitz equation as is usually done. This treatment leads to the usual spin relationships in both 180° and 90° walls, but may describe more complex spin variations as well. It leads to a motional equation which is qualitatively of the right sort to describe the known behavior of individual domain walls moving through ferromagnetic media.
CHAPTER II
CONSIDERATIONS OF THE MAGNETIC SYSTEM

2.1 INTRODUCTION

This chapter has as its object the development of the basic equations of motion for the macroscopic magnetization vector in a ferromagnetic material. The starting point is a description of the principal energy contributions to the free energy of the lattice. The first part of this description follows closely a summary given by Kittel in his 1949 review article on ferromagnetic domains. The introduction of a kinetic energy term and a Rayleigh dissipation term then permit the application of a general variational method to minimize the total free energy. The desired equations of motion are the result.

2.2 EXCHANGE ENERGY

This is perhaps the most important of any of the energy contributions to the ferromagnetic lattice in the sense that the exchange forces are considered to be the basis for ferromagnetism. It is known from gyromagnetic and ferromagnetic resonance measurements that the major contribution to the magnetic moment in a ferromagnetic crystal comes from the uncompensated spin moments of the electrons in ferromagnetic atoms and very little from the orbital moments. It is believed that the spin moments of neighboring electrons are coupled via the so-called exchange energy which in ferromagnetic media tends to align the electron spins. The origin of this energy is the quantum-mechanical exchange integral which in itself is a consequence of the Pauli exclusion principle. The exchange energy concept was first given by Heisenberg to explain the very strong Weiss molecular field. One can obtain a rough idea of the origin of exchange energy by observing that, on the average, electrons with parallel spins are near each other for smaller durations than would classically be expected. Therefore, their average repulsion is less than would be expected classically. This, in turn, leads to an apparent energy of interaction which is the exchange energy.

Van Vleck\(^2\) points out that, while it is generally agreed that the exchange forces between electrons provide the coupling between the elementary magnets in ferromagnetism, no complete quantum-mechanical theory for ferromagnetism exists. This is because the accurate solution of the eigenvalue problem for a real ferromagnetic lattice is so complex that usually simplifying models are assumed for which some approximate calculations can be made. Unfortunately, each of the models thus far invented has both advantages and disadvantages, and each appears to be suitable for work or results in a given area. We will not go further into the theoretical arguments from which exchange energy can be derived, but we will simply assume that the interactions between nearest neighbors only are important for exchange energy and that all such interactions are equal; i.e., there is no anisotropy in the exchange energy. Then the following simple relation holds:

\[ W_{\text{ex}} = -2JS^2 \sum \cos \phi_{ij}, \]
where

\[ J = \text{exchange integral between neighboring spins}, \]
\[ S = \text{net spin angular momentum of atom}, \]
\[ \phi_{ij} = \text{angle between direction of spin momentum vectors of atoms } i \text{ and } j. \]

It is usually assumed that the angle \( \phi \) between neighboring spins is small; this will later be borne out by calculations on the variation of spin through a domain wall. The cosine term can then be replaced by the first two terms of its Taylor series, and hence the part of the energy which varies with the angle is given by

\[ \Delta W_{\text{ex}} = JS^2 \sum \phi_{ij}^2, \]

Another expression for the exchange energy density more useful for our later purposes may be obtained by assuming a small spatial variation in magnetization over nearest neighbors, so that the direction cosines of the spin moment at a given lattice point may be expressed in terms of the first few terms of the Taylor series for the direction cosines at a neighboring lattice point. From these the exchange energy between neighboring lattice points can be calculated, and when this is properly summed over, say, a body centered cubic lattice, the resultant expression is:

\[ W_{\text{ex}} = A \left[ (\nabla \alpha_1)^2 + (\nabla \alpha_2)^2 + (\nabla \alpha_3)^2 \right], \]

where

\[ A = 2JS^2/a^2 = \text{exchange constant}, \]
\[ \alpha_1, \alpha_2, \alpha_3 = \text{direction cosines of macroscopic magnetization vector relative to the crystalline axes}, \]
\[ a = \text{lattice constant}. \]

This symmetrical differential form for the exchange energy density in terms of the derivatives of the direction cosines at a given point was first given by Landau and Lifshitz. However, it is a form that is believed to have application beyond the exchange energy basis on which it is derived. It would appear to express in a very general way the energy of interaction between neighboring vector quantities where the neighboring elements are similar and exert an effect on one another proportional to the angular displacement between them. Consequently, the expression may be a very basic expression which might describe a number of different kinds of angular interaction in addition to the exchange interaction.

There are a number of different ways of determining the exchange energy constant. The two principal ways are:

(a) From the measured Curie temperature of the material, knowing that the exchange energy is roughly equal to the thermal energy at the Curie temperature,

(b) To relate the experimental value of a constant in the Bloch 3/2 power law* for the temperature dependence of the saturation magnetization at low temperatures to the effective exchange integral.

---

*For ferrites it is not certain that the 3/2 power law is applicable.
These two methods appear to yield results in fair agreement and, for the case of iron, which will be dealt with mostly, we will use a value of \( A = 2 \times 10^{-6} \) ergs per centimeter. This appears to be typical of the 3 to 4 per cent silicon iron materials that we will later discuss.

2.3 ANISOTROPY ENERGY

The next important energy contribution is the crystalline anisotropy energy which is revealed by the existence of easy and hard directions of magnetization in a crystal. The origin of anisotropy energy is even more obscure than that of the exchange energy, and it appears to be a result of spin-orbit coupling in the magnetic lattice. It is not a consequence of the exchange coupling because, as indicated previously, this coupling is isotropic; that is, the lattice itself can be rotated without changing the exchange energy. The theory of crystalline anisotropy as summarized by Van Vleck is one that deals primarily with spin-orbit interaction and is a fairly complex theory, even when simplifications are made. In general, however, the magnetic anisotropy energy appears to be large for crystals having a magnetic lattice of low symmetry, while the anisotropy is rather low for lattices of high symmetry. Examples of the two cases are cobalt, having high anisotropy and a hexagonal structure, and iron, having not as high anisotropy and a rather symmetrical body centered cubic lattice. In Fig. 1 is shown the temperature dependence of the anisotropy energies of some common ferromagnetic materials. It is observed that the change with temperature is very marked and difficult to correlate with any known law. Hence,

![Graphs showing temperature dependence of anisotropy energy for various ferromagnetic materials.](image-url)

*Fig. 1. Variation of anisotropy constants with temperature. (Adapted from Fig. 19 of Ref. 20.)*
for our purposes we will assume that, while the theoretical origin of anisotropy energy is again obscure, its occurrence is well known, and it can be characterized by the following expressions for materials exhibiting uniaxial (Co) and cubic (Fe, Ni) symmetry.

For uniaxial symmetry:

\[ E_K = K''_1 \sin^2 \theta + K''_2 \sin^4 \theta \]

where

\[ \theta = \text{angle between magnetization vector and easy axis}, \]

\[ K''_1 = 4.1 \times 10^6 \text{ ergs/cc} \]
\[ K''_2 = 1.0 \times 10^6 \text{ ergs/cc} \]

for cobalt at room temperature.

For cubic symmetry:

\[ E_K = K'_1 \left( \alpha_1^2 \alpha_2^2 + \alpha_2^2 \alpha_3^2 + \alpha_3^2 \alpha_4^2 \right) + K'_2 \alpha_1^2 \alpha_2^2 \alpha_3^2 \]

where

\[ \alpha_1, \alpha_2, \alpha_3 = \text{direction cosines of magnetization vector referred to cube edges}, \]

\[ K'_1 = 4.2 \times 10^5 \text{ ergs/cc} \]
\[ K'_2 = 1.5 \times 10^5 \text{ ergs/cc} \]
\[ K''_1 = -3.4 \times 10^5 \text{ ergs/cc} \]
\[ K''_2 = 5.0 \times 10^4 \text{ ergs/cc} \]

for iron at room temperature.

It should be noted that the kind of anisotropy energy we have been considering is really crystalline in nature. There is another type that is a function of the geometry of the specimen and may be called shape anisotropy. The reason for the preferred axis under these conditions is really the demagnetizing energy due to surface pole distributions when an asymmetrically shaped specimen is used; for instance, in rod-like specimens there will be a preferred axis along the long dimension—the reason being that the demagnetizing energy is much less in this direction than it would be in the short dimension of the specimen. Whether this is truly an anisotropy in the crystalline sense or not, its effect is important and will be referred to later in connection with the net anisotropy expression used for thin sheets.

2.4 MAGNETOSTATIC ENERGY

The next form of energy to be considered is the magnetostatic energy. This is one of the simplest and best understood, being simply the energy of interaction between a dipole and an applied field. It is classically given by simply \(-M \cdot \bar{H}\) or \(-MH \cos \phi\), where \(\phi\) is the angle between the magnetization vector \(M\) and the applied field \(H\). It is an energy that characterizes the interaction between a dipole and an impressed magnetic field, although it should be pointed out that, in general, \(\bar{H}\) is considered to be an effective field, one component of which is the external applied field. When the magnetostatic energy of interest is the self energy of a permanent magnet in its own field, the same formula applies, but the energy is diminished by a factor of one-half.
2.5 MAGNETOELASTIC ENERGY

The magnetoelastic energy or magnetostrictive energy results from the interaction between the magnetization and the mechanical strain of the lattice. The energy is stored as either a distortion or stress in the crystal. The magnetoelastic energy is defined as zero in an unstrained lattice. The relationships that give the elastic energy density in a crystal are not conceptually involved but are algebraically lengthy because the elastic moduli are, in general, different along the different crystalline axes. We will not repeat the usual expressions but instead will extract two results from those given previously by Kittel. The first is that one of the effects of the magnetoelastic energy in iron crystals is simply to add an increment, $\Delta K$, to the first-order anisotropy constant, $K'_1$, previously described. That is, there is an interaction between the anisotropy energy and the magnetoelastic energy such that the existence of the magnetostrictive terms increases the effective anisotropy of the crystal. The second result is that under certain simplified symmetry conditions, such as with so-called "isotropic magnetostriction," where the magnetostrictive constants are equal, the magnetostrictive energy may be expressed as

$$f_{ne} = \frac{3}{2} \lambda T \sin^2 \theta ,$$

where

- $\theta$ = angle between the magnetization vector and the tension $T$,
- $T$ = uniform tensile stress applied to the crystal,
- $\lambda$ = magnetostrictive constant.

Again, mathematically, the effect of magnetostriction in this case is shown to be equivalent to adding to the crystalline anisotropy energy a term equal to $C \sin^2 \theta$, a form which we will find later is of importance. The principal reason for the inclusion of the magnetostrictive energy is that, in the classical treatment of the $180^\circ$ wall in iron, the two $90^\circ$ halves can be shown to be not infinitely far apart as they would be in the usual development where crystalline anisotropy alone is considered.

2.6 KINETIC ENERGY

At this point, we depart from the usual treatment of free energy in the material and introduce a kinetic energy term which assumes that the macroscopic spin moment motion is gyroscopic in the classical sense. Since the basic phenomenon is really quantum-mechanical, some justification or discussion of this assumption should perhaps be made. As Kittel$^{25}$ points out in connection with ferromagnetic resonance effects, one would expect the classical theory to be applicable, since the number of quantum states in the entire system is very large. Van Vleck$^{26}$ has shown in a rigorous quantum-mechanical derivation of ferromagnetic resonance equations that the simpler classical theory on which Kittel first derived those equations is correct, and concludes that there is no loss of rigor by employment of the classical theory. The strong connection between classical and quantum expressions for the ferromagnetic free energy is also noted$^{27}$ by Bloembergen. Our contention now is that the dynamics of spin variation in ferromagnetic resonance phenomena is essentially the same as the classical gyroscopic motion with which we are concerned in domain wall motion, and hence our classical description of the kinetic
energy of the macroscopic spin system is theoretically reasonable. Its ultimate justification, of course, will depend on the correctness of the description of wall motion to which it leads. The inclusion of the kinetic energy term in the free energy expression makes it possible to derive equations of motion for the Bloch walls without later requiring the use of the Landau-Lifshitz equation. In other words, it leads to a somewhat more orderly treatment of the wall motion.

The classical kinetic energy expression for gyroscope motion is composed of two terms. The first is the conventional inertial energy, and the second, an energy component characteristic of the gyroscopic spin. Thus,

\[ f_{KE} = \frac{I_1}{2} \left( \dot{\alpha}_1^2 + \dot{\alpha}_2^2 + \dot{\alpha}_3^2 \right) + \frac{I_3}{2} f(\alpha_4, \alpha_5, \alpha_6, \Omega), \]

where

- \( I_1 \) = inertial moment about coordinate axes,
- \( I_3 \) = inertial moment about spin axis,
- \( \dot{\alpha}_j \) = time derivative of direction cosines,
- \( \Omega \) = spin angular velocity.

We will investigate these gyroscopic terms in more detail later when we work in spherical coordinates. However, for the present, when the independent variables are the direction cosines, the gyroscopic terms become somewhat more cumbersome to deal with and will not be included.

### 2.7 DISSIPATION FUNCTION

A final factor which must be included for the sake of generality is a dissipation factor. This is especially true when we are considering motion which is known to be as well damped as that of domain wall motion. In order to be consistent with our classical variational treatment, we will assume that the dissipation may be given by a classical Rayleigh dissipation term where the dissipation is merely proportional to the first power of the velocity of motion. Thus,

\[ f_D = F/2 (\dot{\alpha}_1^2 + \dot{\alpha}_2^2 + \dot{\alpha}_3^2) \]

That is, we are assuming here that the losses are proportional to the velocity of the macroscopic magnetization vector, and that there are no losses associated with gyroscopic spin. Since the only losses are associated with the motion about the non-spin axes of the elementary gyroscopes, the dissipation term given here will not be affected by later inclusion of gyroscopic-kinetic terms.

### 2.8 ENERGY MINIMIZATION

Thus the list of principal contributions to the free energy of the lattice that we will consider is:

- \( f_{ex} = A \left[ (\nabla \alpha_1)^2 + (\nabla \alpha_2)^2 + (\nabla \alpha_3)^2 \right] \)
- \( f_{an} = K_1' \left[ \alpha_4^2 \alpha_2^2 + \alpha_5^2 \alpha_3^2 + \alpha_6^2 \alpha_1^2 \right] \) for cubic symmetry
  - \( = \left[ A' \alpha_4^2 + B' \alpha_5^2 + C' \alpha_6^2 \right] \) for non-cubic symmetry,
$$f_{\text{mag}} = -MH\left[\alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3\right],$$

$$f_{\text{kin}} = \frac{1}{2} \left[\dot{\alpha}_1^2 + \dot{\alpha}_2^2 + \dot{\alpha}_3^2\right] + \frac{1}{2} \left[\dot{\alpha}_1 \Omega\right],$$

$$f_{D} = \frac{F}{2} \left[\dot{\alpha}_1^2 + \dot{\alpha}_2^2 + \dot{\alpha}_3^2\right],$$

where $\beta_1, \beta_2, \beta_3$ are the direction cosines for the H field. We observe then that in summing these up we will obtain a Lagrangian of the following form:

$$L = f\left(\frac{\partial \alpha_j}{\partial x}, \frac{\partial \alpha_j}{\partial y}, \frac{\partial \alpha_j}{\partial z}, \frac{\partial \alpha_j}{\partial t}, x, y, z, t\right), \quad j = 1, 2, 3,$$

and this integrated over the volume of the crystal and over the time of interest gives the total energy to be minimized. Thus,

$$W_T = \int \int \int L \, dx \, dy \, dz \, dt.$$

A variational treatment for an integral of this sort, when both spatial and time derivatives of the dependent variables are involved, reduces to a set of Euler-Lagrange equations for each of the variables, $\alpha_j$, which have the following form:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\alpha}_j}\right) + \sum_k \frac{\partial}{\partial x_k} \left(\frac{\partial L}{\partial \alpha_j}\right) - \frac{\partial L}{\partial \alpha_j} + \frac{\partial f_D}{\partial \dot{\alpha}_j} = 0, \quad j = 1, 2, 3; \quad k = 1, 2, 3,$$

where

$$\alpha_j = \alpha_j(x_1, x_2, x_3, t).$$

A complete discussion of minimization problems of this sort is given by Goldstein. It should be remembered that the proper boundary conditions must be satisfied if satisfaction of the Euler-Lagrange equations is to imply minimum energy in the crystal. In our case, this means that at the boundaries the variation of $\alpha_1, \alpha_2$ and $\alpha_3$ must be zero. Under certain conditions of surface pole distribution, this requirement may not be a good assumption, but for the present let us just bear this in mind and assume that the variation is zero at the boundaries. Then, after applying this to our free energy expression, inserting the Rayleigh dissipation term and remembering that the direction cosines are subject to a constraint:

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1,$$

we obtain, for cubic symmetry,

$$I_1 \frac{\partial^2 \alpha_1}{\partial t^2} + F \frac{\partial \alpha_1}{\partial t} + 2\Lambda \nabla^2 (\alpha_1) - 2K_1 \alpha_1 (1 - \alpha_1^2) + MH \beta_1 + \Lambda \alpha_1 = 0,$$

$$I_1 \frac{\partial^2 \alpha_2}{\partial t^2} + F \frac{\partial \alpha_2}{\partial t} + 2\Lambda \nabla^2 (\alpha_2) - 2K_1 \alpha_2 (1 - \alpha_2^2) + MH \beta_2 + \Lambda \alpha_2 = 0.$$
\[ I_1 \frac{\partial^2 \alpha_3}{\partial t^2} + F \frac{\partial \alpha_3}{\partial t} + 2 \Lambda \nabla^2 (\alpha_3) - 2K_1' \alpha_3 (1 - \alpha_3^2) + MH\beta_3 + \Lambda \alpha_3 = 0 \]

where $\beta_{1,2,3}$ are the direction cosines for the H field, and $\Lambda$ is a Lagrangian multiplier. We have for the moment omitted the gyroscopic energy terms. These equations may be considered as wave equations of a sort for the wall motion in ferromagnetic single crystals. However, unlike the usual wave equations, these have a static solution which gives the stable configuration for the wall when it is not moving. They are, unfortunately, nonlinear equations even to a greater degree than the above form indicates. It is usual in variational problems to find that the Lagrangian multiplier $\Lambda$ is a constant. If that were so in this problem, we would have very powerful and very useful equations in the above set. Unfortunately, $\Lambda$ can be a function of $x$, $y$, $z$ and $t$ and, even in the static case, it may be shown to have a rather complex expression. While the symmetrical expressions are possibly of theoretical interest, the non-constancy of the $\Lambda$ term makes them somewhat unwieldy to handle, and it is preferable to work in terms of other variables. This will be done in the subsequent section.

### 2.9 Domain Wall Equations

In the preceding development, it took four variables; i.e., $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\Lambda$ to characterize the behavior of the amplitude-constant magnetization vector, and it is known that two should suffice. In order to make simpler the inclusion of gyroscopic effects, we will select these two variables as the well-known Euler angles. These are defined in Fig. 2.* The direction cosines are given in terms of them by the following expressions:

\[
\begin{align*}
\alpha_1 &= \cos \theta \\
\alpha_2 &= \sin \theta \sin \phi \\
\alpha_3 &= \sin \theta \cos \phi
\end{align*}
\]

When these conversion equations are applied to the expression for the exchange energy in terms of the direction cosines, we obtain

\[
f_{ex} = \Lambda [(\nabla \theta)^2 + \sin^2 \theta \left( \nabla \phi \right)^2]
\]

where

\[
(\nabla \theta)^2 = (\nabla \theta \cdot \nabla \theta) = \left( \frac{\partial \theta}{\partial x} \right)^2 + \left( \frac{\partial \theta}{\partial y} \right)^2 + \left( \frac{\partial \theta}{\partial z} \right)^2
\]

*In a strict sense, $\phi$ is not an Euler angle but differs from the corresponding Euler angle by a fixed angle. The difference is not significant for our purposes.
This is a polar expression for the exchange energy which appears in Becker and Döring. The anisotropy energy when converted for the cubic case reduces to:

\[ f_{an} = \frac{K'_1}{4} [\sin^2 2\theta + \sin^4 \theta \sin^2 2\phi] \]

and, for the case where we have a uniaxial anisotropy, is very simply given by

\[ f_{an} = K'_1 \sin^2 \theta \]

The magnetostrictive energy need not be converted, since the second form given for it is already suitable for our needs.

The magnetostatic energy is directly given by

\[ f_{\text{mag}} = -MH \cos \theta \]

where we are now assuming that \( H \) is applied parallel to the axis from which \( \theta \) is measured. Thus, in a 180° wall, the applied field and the reference axis will be parallel to the wall. A corresponding expression for field components perpendicular to the wall could be quite easily written.

The kinetic terms now become

\[ f_{\text{kin}} = \frac{1}{2} \left( (\dot{\phi})^2 + \sin^2 \theta (\dot{\phi})^2 \right) + \frac{1}{2} \left( (\dot{\theta})^2 + \dot{\phi} \cos \theta \right) \]

where the second part of this expression is the gyroscopic energy or the energy due to the spin moment of the elementary gyroscopic. The additional coordinate used here is the angle \( \phi \), which is the angle of spin of the gyroscope about its own axis. These expressions for inertial and gyroscopic energy may be readily derived by methods similar to those described by Goldstein.

Similarly, the Rayleigh dissipation function becomes

\[ f_D = \frac{F}{2} \left( (\dot{\theta})^2 + (\dot{\phi})^2 \sin^2 \theta \right) \]

If now the variational procedure is followed with these energy expressions, we obtain the following three Euler–Lagrange equations: the first from a variation in \( \theta \), the second from a variation in \( \phi \), and the third from a variation in \( \psi \).

\[ I_1 \frac{\partial^2 \theta}{\partial t^2} - I_1 \sin \theta \cos \theta \left( \frac{\partial \phi}{\partial t} \right)^2 + I_3 \left[ \dot{\psi} + \dot{\phi} \cos \theta \right] \frac{\partial \phi}{\partial t} \sin \theta + F \frac{\partial \theta}{\partial t} = T_\theta \]

\[ \frac{d}{dt} \left[ I_1 \sin^2 \theta \frac{\partial \phi}{\partial t} + I_3 (\dot{\psi} + \dot{\phi} \cos \theta) \cos \theta \right] + F \sin^2 \theta \frac{\partial \phi}{\partial t} = T_\phi \]

\[ \frac{d}{dt} I_3 (\dot{\psi} + \dot{\phi} \cos \theta) = 0 \]

where

\[ T_\theta = 2A \nabla^2 \theta + 2A \sin \theta \cos \theta \left( \nabla \phi \right)^2 + \frac{\partial K}{\partial \theta} + MH \sin \theta \]

\[ T_\phi = -2A \sin^2 \theta \nabla^2 \phi + 2A \sin \theta \cos \theta \left( \nabla \theta \cdot \nabla \phi \right) + \frac{\partial K}{\partial \phi} \]

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The third equation suggests that the component of spin about the spin axis of the gyroscope is conserved; i.e., we have applied no torque nor do we have any loss occurring that would change this component of spin. This is consistent with the assumption of constant amplitude of magnetization. We let \((\dot{\psi} + \dot{\phi} \cos \Theta) = \Omega = \text{constant}\) and insert this expression back in the first two and eliminate \(\dot{\psi}\). The resulting two equations completely describe the motion of the magnetization vector in the ferromagnetic medium.

\[
I_1 \frac{\partial^2 \Theta}{\partial t^2} - I_1 \sin \Theta \cos \Theta \left( \frac{\partial \phi}{\partial t} \right)^2 + I_3 \Omega \sin \Theta \frac{\partial \phi}{\partial t} + F \frac{\partial \Theta}{\partial t} = T_{\Theta},
\]

\[
\frac{d}{dt} \left[ I_1 \sin^2 \Theta \frac{\partial \phi}{\partial t} + I_3 \Omega \cos \Theta \right] + F \sin^2 \Theta \frac{\partial \phi}{\partial t} = T_{\phi}.
\]

For the remainder of this paper, we will be concerned with solutions of these two equations and will not refer further to the symmetrical form. These are fairly general equations and contain both the static and dynamic descriptions of the domain wall behavior. At first, they appear to be fairly complicated expressions that would yield little of direct informational value. However, the two equations can be readily solved and do yield some significant results that agree with known domain wall behavior. In the next chapter we will consider some simple static solutions in order to understand better the significance of these two equations. In succeeding chapters, we will consider the dynamic solutions.
3.1 INTRODUCTION

In this chapter, we will obtain certain solutions of the Euler-Lagrange equations for the ferromagnetic crystal under the conditions that all the time derivatives are zero; i.e., the domain walls are stationary. The conditions that must be satisfied then are that

\[ T_\theta = T_\phi = 0, \]

or

\[ 2A \nabla^2 \theta = 2A \sin \theta \cos \theta (\nabla \phi)^2 + \frac{\partial K}{\partial \theta} + MH \sin \theta, \]

\[ 2A \sin^2 \theta \nabla^2 \phi = -4A \sin \theta \cos \theta (\nabla \theta \cdot \nabla \phi) + \frac{\partial K}{\partial \phi}, \]

where \( K = K(\phi, \theta) \) is a generalized anisotropy function having magnetocrystalline, magnetostatic and magnetoelastic components.

As examples, solutions will be given for the well-known \( 180^\circ \) and \( 90^\circ \) walls under certain anisotropy conditions. Then two new cases will be considered:

1. A complex \( 180^\circ \) wall in which both \( \theta \) and \( \phi \) variations are allowed,
2. An analysis of wall structure for a double \( 180^\circ \) or \( 360^\circ \) wall in which the effect of the magnetostatic term (\( MH \sin \theta \)) is important.

It is evident by reference to the equations that must be solved that the first step is to arrive at the anisotropy function to be used. At first this may seem to be a fairly simple and straightforward process. A few examples, however, will show that this is not necessarily the case.

3.2 SIMPLE 180° WALL - CUBIC ANISOTROPY

In this case, the terms involving \( \phi \) can be dropped, and the anisotropy for the case of cubic symmetry, e.g., iron, would be given by \( K = K_1 \sin^2 \theta \). The coordinate axes are then directions of minimum energy. If the wall is considered to be a plane wall with no change in spin orientation in either the \( x \)- or \( y \)-directions, the differential equation then becomes

\[ 2A \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial K}{\partial \theta} = 4K_1 \sin \theta \cos \theta, \]

which can be readily shown to have a solution

\[ z = \frac{A}{2K_1} \log \tan \theta + C_1. \]

This is essentially the result first obtained by Bloch.\(^1\)

It is somewhat disconcerting to discover that \( z \to \infty \), for \( \theta = 0, \pi/2 \); hence, our solution provides only for a \( 90^\circ \) wall. The reason for this is that the axes are directions of minimum
anisotropy energy and a transition of 90° is sufficient to provide a change from one easy direction to the adjacent. The classical conclusion from this, arrived at by Néel and Lifshitz, is that some other energy is required to bring the two 90° sections together to provide a 180° wall; i.e., without some additional energy being effective, there could be an infinite distance between the two 90° sections, which is obviously not the case as revealed by domain pattern measurements.

The usual way out of this is to invoke the magnetoelastic energy of the crystal, which has been shown previously to contribute a term proportional to \( \sin^2 \theta \). The general anisotropy term then becomes \( K = B \sin^2 \theta + K'_1 \sin^2 2\theta \), and the solution for the wall equation as given by Kittel under these conditions is

\[
\sinh z \sqrt{\frac{K'_1 (1 + P)}{A}} = -\sqrt{\frac{1 + P}{P}} \cot \theta,
\]

where

\[
P = \frac{B}{K'_1} \approx 2 \times 10^{-3} \text{ for iron.}
\]

It is evident from Fig. 3, which is a plot of this relation, that a very small amount of magnetostrictive energy is sufficient to bring the two sections together to form the 180° wall. Further data on variation of wall width with magnetostrictive energy is given by Lilley.

3.3 REMARKS ON ANISOTROPY

The effectiveness of the magnetostrictive term in reducing the distance between 90° sections may be understood by considering a plane wall in a thin sheet of material in which there is no variation of magnetization through the thickness dimension (perpendicular to plane of sheet). At the center of the wall, if the strain is in the direction of magnetization, the strain is then in a direction perpendicular to the plane of the sheet, whereas, in the domains on either side of the wall, the strain is parallel to the wall and in the plane of the sheet. The two domains on either side of the wall are large and effectively constrain the potential deformation in the wall region. The stress then produced in the wall and, hence, the magnetoelastic energy, is a function of the vertical component of magnetization. This same effect, i.e., an energy dependent on the vertical component of magnetization, could be brought about by the magnetostatic energy associated with the poles that must appear at the points where the wall intersects the surface of the crystal. This magnetostatic energy is proportional to the square of the pole density which, in turn, is proportional to the vertical component of magnetization. Hence, energy terms proportional to \( \sin^2 \theta \) would result which would have about the same functional effect on 180° walls as does the magnetostrictive energy.
The importance of this magnetostatic energy across the wall itself has recently been studied by Néel. He considered the effect of the magnetostatic energy in determining both wall thickness and spin orientation in thin films and predicts that, when the film becomes thin enough, the increased magnetostatic energy may actually cause a rotation of the wall so that all magnetization vectors lie within the plane of the film. There has also been some recent work by Kaczer in which similar considerations were made. Here, however, the object was to determine the effect of this magnetostatic energy on the coercive force of thin sheets. Kaczer showed that the energy is not negligible when compared to the anisotropy energy, and that the thickness of the Bloch wall is a function of the thickness of the sheet.

It is obvious at this point that the anisotropy term in our general equation is not really derived from any one source but that it is, rather, an effective anisotropy that could include both the crystalline anisotropy contributions and shape anisotropy contributions which, in turn, may be due to either demagnetizing effects or to magnetostrictive effects. Since the anisotropy term is, in general, a complicated term, the possibility arises of expediting our analytical work by approximating the actual function by a simpler one having the same minimum energy positions. We can assume, in other words, that the various contributions to the anisotropy energy act to give this approximate over-all behavior and then determine simple functions of $\theta$ and $\phi$ which will agree at the minimum energy points (easy axes) with the true function. It will differ at intermediate points. There would appear to be little point in making a very rigorous analysis of the anisotropy contribution, since we do not have rigorous expressions for all the contributing terms. We can support some of these contentions by the following example. Remember that, in the preceding case, the effect of adding the magnetostrictive energy to the magnetocrystalline energy was to remove the null at the 90° point which, in effect, allowed 180° transitions to occur. Suppose we considered only the term that had nulls along the desired axis ($\theta = 0, \pi$) and kept the same maximum value $K_1$ for the anisotropy, and asked how the solution using it alone would compare with the more complicated, but more nearly correct, one. The variation of $\theta$ through the wall, in the simple case, is readily obtained as $\sqrt{K_1/A} z = \log \tan \theta/2$, and this is compared graphically with the one previously obtained in Fig. 2.

It is seen that the general character of the transition is much the same, although the detailed contours, especially at the center of the wall, are somewhat different. However, for about the same values of the exchange and anisotropy constants, the wall energies are almost the same, and the wall thickness is of the same order of magnitude. So, except for the question of the detailed variation of spin orientation through the wall, which is usually not of prime interest, one can determine almost as much about the wall by employing the simple anisotropy function as one can by the more nearly correct one. Following this argument, we claim that, for sheets of ferromagnetic material, in which there is no variation in magnetization throughout the thickness, the effective anisotropy term is one that is proportional to the component of magnetization normal to the surface such that this is not an easy direction of magnetization. Thus, our approximate anisotropy expression is

$$K = K_1 \alpha_3^2 = K_1 \sin^2 \theta \cos^2 \phi.$$  

For $\phi = 0$, this reduces to the simple 180° wall case previously discussed.
In a recent article, VanVleck,\textsuperscript{32} in a discussion of anisotropic exchange, points out that, in non-cubic crystals, the energy of anisotropy has macroscopically the form

\[ K = A \alpha_1^2 + B \alpha_2^2 + C \alpha_3^2, \]

whereas, for cubic structures, where \( A = B = C \) and \( \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \), the lowest-order anisotropy energy is of the form

\[ K = K_1 [\alpha_1^2 \alpha_2^2 + \alpha_2^2 \alpha_3^2 + \alpha_3^2 \alpha_1^2]. \]

If we can extrapolate from this and assume that these are basic first-order forms, then, in the macroscopic case, where the cubic magnetocrystalline anisotropy is modified by the addition of other energy contributions to form the net anisotropy function for the sample, we should expect that a reasonable first-order form for the net energy would be

\[ K = A \alpha_1^2 + B \alpha_2^2 + C \alpha_3^2, \]

which agrees with our previous development. Then, in polar coordinates, the lowest-order non-cubic anisotropy function becomes

\[ K = A \cos^2 \Theta + B \sin^2 \Theta \sin^2 \phi + C \sin^2 \Theta \cos^2 \phi. \]

### 3.4 SIMPLE 90° WALL

Now, let us take \( K = K_1 \sin^2 \Theta \cos^2 \phi \) and consider solutions in which \( \Theta \) is fixed, but \( \phi \) can vary. Where \( \Theta \) is to be considered fixed, the first of the basic equations vanishes, and the second reduces to

\[ 2A \sin^2 \Theta \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial K}{\partial \phi}. \]

When the assumed anisotropy is substituted in this, we obtain

\[ 2A \sin^2 \Theta \frac{\partial^2 \phi}{\partial x^2} = -K_1 \sin^2 \Theta \cos \phi \sin \phi, \]

where \( x \) is measured normal to wall. This has a first integral

\[ A \left( \frac{\partial \phi}{\partial x} \right)^2 = K_1 \cos^2 \phi, \]

and a solution

\[ \sqrt{\frac{K_1}{A}} x = \log \cos \frac{\phi}{2}. \]

Thus, the \( \phi \) variation is much the same as the one previously deduced for \( \Theta \) alone, except that \( \phi \) starts from \( \pi/2 \), goes through zero, and ends up at \( -\pi/2 \), which is consistent with the assumed anisotropy function, since the values \( \Theta = \pm \pi/2 \) are the low energy directions. If we assume the initial value for \( \Theta \) is 45°, then the initial and final vectors differ by 90° and, in the case of iron, lie along easy directions in the plane of the sheet. We see that the variation of the magnetization
vector through what is now a 90° wall is sort of a conical variation where a 180° change in the variable $\phi$ only provides a 90° change in the direction of the net magnetization. One of the interesting things about this picture of a 90° wall is that the normal magnetization component at the center of the wall is only some 70 per cent of that at the center of the normal 180° wall because of the 45° tilt of the vector throughout the wall itself. In some observations of 90° domain walls, it was noticed that the 90° walls were not quite so well defined or not quite so bright under the action of the colloidal magnetite particles as the 180° walls. The reason for this may well be the fact that the orthogonal component of magnetization in this 90° wall is less by our model than would be the corresponding component in a 180° wall. By taking $\Theta = 90°$, we obtain a description of a 180° wall which reduces, as it should, to the same description that we obtained previously.

The accompanying diagram (Fig. 4) shows better the rotation of the magnetization vector at successive points through the wall. One might note in this diagram that, since the $\phi$ rotation occurs about the normal to the wall, the component of $M$ along the normal must be constant, a condition that Néel points out must be true in any simple wall.

Fig. 4. Change in spin orientation through a 90° wall.
3.5 COMPLEX 180° WALL

Now let us consider a 180° wall in which both θ and φ variations are allowable. If we attempt this study using the same anisotropy term as for the 90° wall, we encounter some difficulty because the minimum energy positions for φ are at ±π/2, which means that the vector would lie in the plane for values of θ different from zero. Obviously, since this is not observed experimentally, the form of this anisotropy function is not correct for our 180° wall investigation. It is evident that the sin²θ term is appropriate, since we are assuming that in the adjacent domains the directions of magnetization are given by θ = 0, π. However, in the domain boundary where θ is different from zero, the minimum energy is necessarily at φ = 0, π. These conditions are satisfied by assuming

\[ K = K_1 \sin^2 \theta (1 + a \sin^2 \phi) \]

where

\[ a > 0 \]

which is one of the simplest functions that will give the required anisotropy behavior for both θ and φ variations in the case we are about to study.

An alternative way of obtaining this anisotropy expression is to start from the first-order anisotropy expression for non-cubic symmetry given previously. Thus,

\[ K = A \cos^2 \theta + B \sin^2 \theta \sin^2 \phi + C \sin^2 \theta \cos^2 \phi \]

which reduces to

\[ K = A + \sin^2 \theta [(C - A) + (B - C) \sin^2 \phi] \]

or, ignoring the constant value and taking

\[ B > C > A \]

then

\[ K = K_1 \sin^2 \theta [1 + a \sin^2 \phi] \]

where

\[ K_1 = C - A \]

\[ a = \frac{B - C}{C - A} \]

If we insert this anisotropy expression in our basic wall equations we obtain:

\[ 2A \nabla^2 \theta = 2A \sin \theta \cos \theta (\nabla \phi)^2 + 2K_1 \sin \theta \cos \theta (1 + a \sin^2 \phi) + MH \sin \theta \]

\[ 2A \sin^2 \theta \nabla^2 \phi = 2A \sin \theta \cos \theta (\nabla \theta \cdot \nabla \phi) + 2aK_1 \sin^2 \theta \sin \phi \cos \phi \]

Since we are interested in a plane 180° wall where θ varies in a direction normal to the wall...
(y-axis) and $\phi$ is to vary along the wall itself (x-axis), then the derivatives of the two angles are orthogonal and $\nabla \theta \cdot \nabla \phi = 0$. Hence, for no applied field,

$$2A \frac{\partial^2 \theta}{\partial y^2} = 2A \sin \theta \cos \theta \left( \frac{\partial \phi}{\partial x} \right)^2 + 2K_1 \sin \theta \cos \theta \left( 1 + a \sin^2 \phi \right) ,$$

$$2A \frac{\partial^2 \phi}{\partial x^2} = 2a K_1 \sin \phi \cos \phi .$$

The second equation may be directly integrated, giving

$$\frac{\partial \phi}{\partial x} = \sqrt{\frac{aK_1}{A}} \sin \phi ,$$

and

$$y - y_0 = \frac{2}{\sqrt{\frac{aK_1}{A}}} \log \tan \frac{\phi}{2} .$$

When this expression for $\partial \phi/\partial x$ is inserted in the first equation we find

$$2A \frac{\partial^2 \theta}{\partial y^2} = 2K_1 \sin \theta \cos \theta \left[ 1 + 2a \sin^2 \phi \right] ,$$

which may also be directly integrated with respect to $x$, since $\phi$ is a function of $x$ only, to obtain

$$\frac{\partial \theta}{\partial y} = \sqrt{\frac{K_1 \left( 1 + 2a \sin^2 \phi \right)}{A}} \sin \theta ,$$

and

$$y - y_0 = \frac{A}{\sqrt{K_1 \left( 1 + 2a \sin^2 \phi \right)}} \log \tan \frac{\theta}{2} .$$

Thus, the two solutions for $\theta$ and $\phi$ have the same general form as that previously obtained for $\theta$ alone in a simple 180° wall, except that now the possibility of a reversal of spin orientation along the wall itself is included. That is, if we examine the equation for $\phi$ and remember that the x-direction is the direction parallel to the wall, we note that the angle $\phi$ can vary from 0 to 180° along the wall, or the magnetization vector which points up out of the plane in one section of the wall may reverse to point down into the plane on the further section. The transition between these two conditions is a transition that resembles very much the transition in $\theta$ in the direction transverse to the wall (y-direction). This change in $\phi$ is not exactly a wall in the domain boundary sense, but is a transition in spin orientation that behaves very much like one.

It is interesting to note that the width of the $\phi$ transition is approximately

$$\delta \phi = \sqrt{\frac{A}{K_1 a}} .$$

Thus, this width may be greater or less than that of the wall proper, depending on whether $a$ is less or greater than unity. It is difficult to obtain a numerical estimate for $a$, since all the
energies that contribute to its value are not readily determined. Our experiments, to be described later, indicate that \( \phi \) transition widths are somewhat wider than wall widths, hence \( a < 1 \).

We note from the equation for the \( \Theta \) variation that the wall thickness proper is approximately equal to

\[
\delta \Theta = \frac{A}{\sqrt{K (1 + 2a \sin^2 \phi)}}.
\]

Where \( \phi \) is zero, this thickness is the ordinary thickness for a 180° wall. However, for \( \phi \) in the transition region where it is varying from 0 to \( \pi \), the anisotropy is higher, and hence the wall thickness is narrower. In other words, the \( \Theta \) contour is constricted in the vicinity of the change in spin orientation. The geometrical picture of this is attempted in Fig. 5, where both the constriction in width and the variation in \( \phi \) are shown.

In this same region, the magnetization vector at places within the wall has a component normal to the wall direction. The question then arises as to whether the additional magneto-static energy that is produced by virtue of this is included in our analysis. The answer is that it is included and is a reason for the increased general anisotropy energy when \( \phi \neq 0, \pi \). It is instructive to determine the pole distribution (\( \nabla \cdot \vec{M} \)) within this complex wall. Thus,

\[
\nabla \cdot \vec{M} = \frac{\partial}{\partial x} (M \cos \Theta) + \frac{\partial}{\partial y} (M \sin \Theta \sin \phi) + \frac{\partial}{\partial z} (M \sin \Theta \cos \phi).
\]

Our results show that \( \phi \) is a function only of \( x \) and that \( \Theta \) for small \( a \) has only a second order dependence on \( x \); hence,

\[
\nabla \cdot \vec{M} \approx M \cos \Theta \sin \phi \frac{\partial \Theta}{\partial y},
\]
or

\[
\nabla \cdot \vec{M} \approx M \sqrt{\frac{K_1}{A}} \sin \Theta \cos \Theta \sin \phi; \quad a \ll 1.
\]

\[\text{Fig. 5. Spin orientation in a 180° wall with } \phi \text{ transition.}\]
Thus, in the normal wall where $\phi = 0, \pi$, $\nabla \cdot \vec{M}$ is everywhere zero and at the wall boundaries $\theta = 0, \pi$, $\nabla \cdot \vec{M}$ is also zero as required for the continuity of the normal component of magnetization on either side of the wall. However, within the wall in the region of the $\phi$ transition, a dipole layer exists and gives rise to an additional magnetostatic energy which is reflected in the higher anisotropy energy for $\phi \neq 0, \pi$.

The wall energy per unit length may be readily calculated as follows. The exchange energy is given by

$$f_{\text{ex}} = A \left( \frac{\partial \theta}{\partial y} \right)^2 + A \sin^2 \theta \left( \frac{\partial \phi}{\partial x} \right)^2$$

$$= K_1 \left( 1 + 2a \sin^2 \phi \right) \sin^2 \theta + aK_1 \sin \phi \sin^2 \theta$$

$$= K_1 \left( 1 + 3a \sin^2 \phi \right) \sin^2 \theta,$$

and the anisotropy energy by

$$f_{\text{an}} = K_1 \sin^2 \theta \left( 1 + a \sin^2 \phi \right).$$

The total energy is then

$$f_{\text{tot}} = f_{\text{ex}} + f_{\text{an}} = K_1 \sin^2 \theta \left( 2 + 4a \sin^2 \phi \right).$$

Note that no longer is $f_{\text{ex}} = f_{\text{an}}$ everywhere in the wall, as is the case for a simple wall, but that the exchange energy is somewhat larger. Then the wall energy per unit length, $\sigma_w$, is

$$\sigma_w = \int_{-\infty}^{\infty} \left( f_{\text{ex}} + f_{\text{an}} \right) dx = \int_{0}^{\pi} \frac{2K_1 \sin^2 \theta \left( 1 + 2a \sin^2 \phi \right)}{\sin^2 \theta + aK_1 \left( 1 + 2a \sin^2 \phi \right)} d\theta$$

$$\sigma_w = 4K_1 \sqrt{\frac{1 + 2a \sin^2 \phi}{A}}.$$

This is much the same as the well-known expression for $180^\circ$ wall energy, with the exception of the $\sin^2 \phi$ term. Here we observe that the energy per unit length increases when $\phi$ is in the transition region, which is to be expected because the anisotropy energy here is much greater. This makes such transitions energetically unfavorable, all other conditions being equal, since more energy is inherent in such a reversal than would be obtained if no $\phi$ variation occurred at all. Hence, we suspect that, in a completely unstrained and free crystal, these are not likely to occur, although in crystals that are not well annealed and are not in a minimum energy state, we may expect to find them. Domain pattern measurements which are reported in later chapters, bear this out. The variation in spin orientation along the wall predicted by these equations is well borne out by measurements, especially in Figs. 15 and 16, which show a $180^\circ$ wall section with and without a small vertical field applied. We note that the small gaps in the $180^\circ$ wall become much larger gaps, indicating a reversal in spin direction when a vertical field is applied.
The vertical field here is assumed to reinforce the magnetization components pointing out of the plane and to diminish the components pointing into the plane to the extent that the colloid concentration is increased in the first case and reduced in the second. Thus, the occurrence of these regions is considered to be evidence of spin variation along the wall. This will be discussed at somewhat greater length later, but it is of interest to mention it here as support for these equations.

3.6 DOUBLE 180° BOUNDARIES

Having shown that if the $\phi$ variation does occur it requires additional energy and occupies only a short region, of the order of a wall thickness, so that $\phi = 0$ in most of the 180° wall; then, let us neglect, for the purposes of this next section, the $\phi$ variation, and consider that the only magnetization change through the wall is the result of a change in $\theta$. Also, let us now consider the effect of the magnetostatic energy term $MH \sin \theta$, which we have previously excluded. It should be stressed that this is a bulk magnetostatic energy, sometimes called the Zeeman $^{32}$ energy, which is the result of the interaction of an applied field $H$ and the volume magnetization. It is different from the magnetostatic energy considered as the contributor to the anisotropy term, which was essentially a dipole interaction or boundary effect. We ask, under these conditions, whether the basic differential equation given by

$$2A \partial^2 \theta = 2K_1 \sin \theta \cos \theta + MH \sin \theta$$

has any static solutions; that is, in the presence of an applied uniform field $H$, is it possible to obtain solutions to our basic equations, which do not require a time variation. If we proceed in a straightforward manner and take $\theta$ as a function of $x$ alone, multiply these by $d\theta/dx$ $dx$ and integrate, we obtain as a first integral

$$2A \int \left( \frac{d^2 \theta}{dx^2} \right) \left( \frac{d\theta}{dx} \right) dx = \int \left( 2K_1 \sin \theta \cos \theta + MH \sin \theta \right) d\theta ,$$

and

$$A \left( \frac{d\theta}{dx} \right)^2 = K_1 \sin^2 \theta - MH \cos \theta + C .$$

The constant $C$ we evaluate by assuming $d\theta/dx = 0$, at $\theta = 0$. Under these conditions, $C = MH$ and we obtain

$$dx = \frac{A}{\sqrt{K_1 \sin^2 \theta + MH (1 - \cos \theta)}} \cdot d\theta .$$

If we let

$$q = 1 - \cos \theta ,$$

and

$$h = \frac{MH}{K_1} ,$$

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then

\[ \int \sqrt{\frac{K_1}{A}} \, dx = \int \frac{d\Phi}{\sqrt{\sin^2 \Phi + h (1 - \cos \Phi)}} \]

or

\[ \sqrt{\frac{K_1}{A}} (x - x_o) = \int \frac{dq}{q \sqrt{2 - q} (2 + h - q)} \]

which can be integrated directly (Peirce 182)* to obtain

\[ \sqrt{\frac{K_1}{A}} (x - x_o) = \frac{-1}{\sqrt{2(2 + h)}} \log \left[ \frac{2 \sqrt{2(2 + h)} (2 - q) (2 + h - q) + 4(2 + h)}{q} \right] \]

The equation is a complicated one and its behavior is best characterized by the plot shown in Fig. 6. It is evident that the value of \( \Theta = 0 \) occurs infinitely far from the transition because of the boundary conditions assumed. However, the angle \( \Theta = \pi \), for finite \( h \), occurs relatively close to the transition but does not occur with zero slope. More accurately at \( \Theta = \pi \), \( d\Theta/dx = \sqrt{MH/A} \). The indication is that the wall here is not strictly a 180° wall, since lack of tangency

![Fig. 6. \( \Theta \) vs \( x \) for various \( h \).](image)

at \( \Theta = \pi \) is evidence that the rotation continues. Thus, we are led to guess that what we have is really only half a wall, the other half being a similar structure which is added on to make a 360° wall. The two termini at \( \Theta = 0 \) and \( 2\pi \) are infinitely far from one another, as we would expect must be the case. While we have called this a 360° wall, a more accurate description would be two 180° walls that have been pushed together under the influence of an applied field. The rotation of spin orientation through each of these walls is in the same direction (\( d\Theta/dx \) has same sign in both boundaries) so that the effect of the applied field is to cause each 180° section to be "wound" tighter; i.e., \( d\Theta/dx \) is increased.

*Peirce 182 refers to Table 182 of B.O. Peirce, *A Short Table of Integrals.*
If we set $x = 0$, for $\theta = \pi$, then the equation for $\theta$ progressing through a double wall is

$$
\frac{K_4}{A} x = \frac{\pm q}{\sqrt{2(2 + h)}} \log \left[ \frac{2 \sqrt{(2 + h)(2 + h - q)(2 + h - q) + 4(2 + h)}}{qh} - \frac{(4 + h)}{h} \right]
$$

for $q = 1 - \cos \theta$ and $0 < \theta < 2\pi$.

When this is plotted as in Fig. 7, we observe that the spacings of the two halves do indeed diminish as the applied field increases. We note that our simple theory requires the two halves to be infinitely far apart in the absence of any applied field $H$. Since it is only at very large distances that, as each half unwinds, it will cease exerting an opposing force on its neighbor.

It is of interest to calculate the spacing between these neighboring walls as a function of the applied field. This is most easily done by considering the center of each $180^\circ$ half to occur at $\theta = \pi/2$ and $\theta = 3\pi/2$. If we then take the spacing $d$ as the value between the $\pi/2$ and $3\pi/2$ positions of the spins in the wall, we see that the spacing $d = 2x$, where $x$ is evaluated for $\theta = \pi/2$ ($q = 1$) in the foregoing equation.

Thus,

$$
d = \frac{A}{\sqrt{K_4}} \frac{1}{\sqrt{1 + h}} \log \left[ \frac{4 \sqrt{(1 + h/2)(1 + h) + 4 + 3h}}{h} \right].
$$

This is a cumbersome expression for calculation purposes, and we are led to simplify it somewhat, based on the knowledge that the parameter $h$, which is the ratio of the magnetostatic to the anisotropy energy, must in any practical case be very small. For instance, for iron, the product $MH$ for applied fields of the order of several oersteds is about 1000 ergs per cubic centimeter, whereas $K$ is on the order of 500,000 ergs per cubic centimeter. Hence, the ratio is of the order of 0.005 and is much less than 1. Under these conditions,

$$
d \approx \frac{A}{\sqrt{K_4}} \log \frac{8}{h} \text{ for } h \ll 1.
$$

Thus, when $H$ is zero, the two halves are infinitely far from one another and, as $H$ becomes progressively larger, the two halves approach one another logarithmically. The actual magnitude of the separation for any given value of $H$ is shown in Fig. 8. We note that, for an applied field of about an oersted, where $h$ is of the order of 0.001, the two halves are separated by about 9 wall thicknesses. Consequently, even very small fields may bring the two sections quite close together. Furthermore, we would expect that, if the effective field holding them together were a coercive field, i.e., were due to the coercive force, then for $H$ between 0.01 and 1 oersted, the two sections would be spaced on the order of 5 to 10 wall thicknesses. This is substantiated by some data (Fig. 9) recently published by Williams on the occurrences of double walls in thin nickel-iron films. The double boundaries in these films are quite marked and remain parallel over relatively long intervals without terminating in some complex closure domain structure as is common in sheet specimens. It seems reasonable, therefore, to expect our plane wall theory...
Fig. 7. Spin relationships in double 180° wall for different applied fields.

Fig. 8. $d/\sqrt{A/K_1}$ vs $h$. 
Fig. 9. Domain patterns on 147 Å Fe-Ni-Mo film, showing double 180° boundaries (after Williams).
to apply to this situation. It would appear that the two sections are separated by about 5 to 10 wall thicknesses, which is predicted by the preceding theory, assuming that the nature of the holding field is coercive. Williams also obtained some convincing experimental proof of the $180^\circ$ character of each half of the double boundary by causing the walls to move apart to form parallel and antiparallel domains.

### 3.7 DOUBLE WALL ENERGY

The calculation of free energy in a double wall is a more complicated algebraic procedure, although the method is fairly straightforward. The wall energy density may be written as

$$ f_w = f_{ex} + f_{an} + f_{mag} $$

$$ = A \left( \frac{\partial \Theta}{\partial x} \right)^2 + K_1 \sin^2 \Theta + MH (1 - \cos \Theta) $$

but in Sec. 3.6 we found

$$ A \left( \frac{\partial \Theta}{\partial x} \right)^2 = K_1 \sin^2 \Theta + MH (1 - \cos \Theta) $$

$$ \therefore f_w = 2 K_1 \sin^2 \Theta + 2 MH (1 - \cos \Theta) $$

and the wall energy per unit area is then

$$ \sigma_w = \int_{-\infty}^{\infty} 2 [K_1 \sin^2 \Theta + MH (1 - \cos \Theta)] dx $$

$$ = 2 \int_{0}^{\pi} \sqrt{A(K_1 \sin^2 \Theta + MH (1 - \cos \Theta))} d\Theta $$

$$ = 2 \sqrt{AK_1} \int_{0}^{\pi} \sin^2 \Theta + h (1 - \cos \Theta) d\Theta $$

Setting $q = 1 - \cos \Theta$,

$$ \sigma_w = 2 \sqrt{K_1 A} \int_{0}^{2} \sqrt{\frac{z + h - q}{z - q}} dq $$

which, with the aid of Peirce 114 and 113, can be integrated. After inserting the limit, we find for the wall energy

$$ \sigma_w = 2 \sqrt{K_1 A} \left[ \sqrt{2 + h} + h \log \frac{\sqrt{h}}{\sqrt{2 + h} - \sqrt{2}} \right] $$

On the basis of the same argument as before — that $h$ must necessarily be very much less than 1, since the magnetostatic energy is, in general, very much less than the anisotropy energy — this expression reduces further. Thus, for $h \ll 1$ so that $\sqrt{1 + h} \approx 1 + h/2$,
\[ \sigma_w = 2 \sqrt{K/4} \left[ 2 \sqrt{\frac{1}{4} + \frac{h}{2}} + h \log \frac{\sqrt{h}}{\sqrt{2} \left( \sqrt{1 + \frac{h}{2}} - 1 \right)} \right], \]

or

\[ \sigma_w = 4 \sqrt{K/4} \left[ 1 + \frac{h}{4} + \frac{h}{2} \log \frac{4}{\sqrt{2}h} \right], \]

and finally,

\[ \sigma_w = 4 \sqrt{K/4} \left[ 1 + 0.77 h - \frac{h \log h}{4} \right]. \]

This behavior is shown in Fig. 10, and it is seen that the wall energy, under the conditions of small \( h \), increases slightly with applied field as one would expect, since the magnetostatic energy increases linearly with the applied field but is nevertheless very small compared to other energy contributions.

### 3.8 POSSIBLE EXPERIMENT

The separation between the two \( 180^\circ \) walls under the action of a field which tends to make them coalesce is a simple function of the magnitude of the applied field, leading one to believe that an experiment in which transverse wall separation is measured as a function of the applied field would yield some interesting data on wall thickness and possible other characteristics.

For instance, since

\[ d = \sqrt{\frac{K}{A}} \log \frac{8}{h}, \]
then
\[ \frac{d(d)}{dh} = -\sqrt{\frac{K_1}{A}} \left(\frac{1}{h}\right) , \]
or
\[ \frac{d(d)}{dH} = -\sqrt{\frac{K_1}{A}} \left(\frac{1}{H}\right) . \]

so that, from a plot of \( d(d)/dH \) vs \( 1/H \), one should be able to determine the wall thickness parameter \( \sqrt{K_1 A} \). Such an experiment requires the existence of two neighboring walls, which are planar in all respects and are completely free to move, influenced only by the applied field. This would be so in a simple crystal such as the one with which Schockley, Bozorth and Williams first demonstrated simple wall motion. It is not so in crystals such as those on which the data given later in this paper were taken. The region in which simple 180° walls were obtained was short and was terminated on either side by very complex domain structures; that is, by domains of closure which tended to restrict the motion of the 180° walls. Some of the walls did not move at all under the application of the applied field. Evidently, they were restrained by the forces at the wall termini. A better crystal would have yielded some interesting data, but the experiment is meaningless unless such a simple parallel wall structure can be obtained.
CHAPTER IV
DYNAMICS OF BLOCH WALLS

4.1 INTRODUCTION

The basic paper on domain wall dynamics is the 1935 Landau-Lifshitz paper in which a formula for domain wall velocity was first presented. The Becker calculation for wall velocity and wall mass which was made later agrees with this in a general way, although the model upon which it was based is a simple two-dimensional one. In general, most of the treatment given wall dynamics in the literature is based on consideration of the rotational motion of the magnetization vector. Our intent is to be more rigorous than this and to include both the rotational and precessional motions which, as we will show in this chapter, lead to important new conclusions concerning wall velocity.

An examination of the Euler equations which characterize the motion will show them to be rather complicated, with closed solutions at first glance appearing unlikely. Accordingly, we will first obtain solutions for the simple two-dimensional case to obtain results consistent with the present literature. Then the more realistic case where the precessional motion is included will be solved which, with the assumption of Rayleigh dissipation, will lead to the determination of an upper bound on wall velocity. Finally, transient solutions for the general equation will be found, where the effect of spin system inertial moments or wall mass will also be included.

4.2 EQUATION OF MOTION - ROTATION ONLY

Under the conditions that the motion in the \( \phi \) direction can be neglected and that inertial effects will be deferred until later, the differential equation of motion given in Chapter II reduces to

\[
F \frac{d \Theta}{dt} = -2A \nabla^2 \Theta + 2K_1 \sin \Theta \cos \Theta + MH \sin \Theta
\]

It may be shown by direct substitution that a solution for this equation is

\[
\log \tan \frac{\Theta}{2} = ax + by + cz + dt
\]

where

\[
a^2 + b^2 + c^2 = \frac{K_4}{A}
\]

and

\[
d = \frac{MH}{F}
\]

If we consider the motion to occur in the \( z \)-direction only, then the solution can be written as

\[
\log \tan \frac{\Theta}{2} = z \sqrt{\frac{K_4}{A}} + \frac{MH}{F} t
\]

We interpret the result as an equation characteristic of a wall (\( \Theta \)-variation) which is fixed in
shape and translated uniformly with time under the action of an applied field (Fig. 11). It follows that the velocity with which the wall moves is given by

$$v = \frac{MH}{F} \sqrt{\frac{A}{K_4}}$$

The result is quite reasonable and agrees with calculations of Landau and Lifshitz and Becker which indicate that the wall velocity under these simple conditions is directly proportional to the applied field. The other factors enter in an equally reasonable manner.

It is evident that the anisotropy energy and the exchange energy combine to keep the wall shape fixed and that the magnetostatic energy supplied by the external field is balanced by what might be called the "viscous motional" energy to regulate the velocity of the wall. This linear dependence of wall velocity on applied field agrees with experiments using small applied fields; however, we observe that any coercive effects are missing, since the velocity becomes zero only when the applied $H = 0$, which is not in accordance with the known coercive effects. That is, for the sake of generality we should probably interpret $H$ in the above equation as

$$H = H_{applied} - H_{coercive}$$

This balance between the large anisotropy and exchange energy to keep the wall shape constant and the balance between the two smaller energies to regulate velocity are related to a really fundamental magnetic phenomenon; namely, that a very small field of the order of tenths of oersteds (just above the coercive force) can change or reverse the magnetization in a ferromagnetic material in which the Weiss molecular field due to exchange is of the order of $10^7$ oersteds. It is now evident that this is possible because the two fields really have completely different effects on the domain boundary. One acts to form the wall and the other pushes the wall against viscous damping.
4.3 EQUATION OF MOTION – ROTATION PLUS PRECESSION

The equation of motion previously derived must necessarily be a simplification or an approximation to the actual motion, since it is based on changes occurring only in the $O$-direction (rotation), which is the only direction usually considered in the literature. One would expect that the variation of the magnetization vector in $\phi$ must also be important and should be included, since the application of a lateral field, that is, a field along the wall, must certainly cause precession of the spin moments.

To include the precessional motion, we start with the general equations developed in Chapter II. Thus, for the ferromagnetic single crystal,

\[ I_3 \frac{\partial^2 \Theta}{\partial t^2} - I_4 \sin \Theta \cos \Theta \frac{\partial^2 \phi}{\partial t^2} + I_3 \Omega \sin \Theta \frac{\partial \phi}{\partial t} + F \frac{\partial \Theta}{\partial t} = T_\Theta, \]

\[ \frac{d}{dt} (I_4 \sin^2 \Theta \frac{\partial \phi}{\partial t} + I_3 \Omega \cos \Theta) + F \sin^2 \Theta \frac{\partial \phi}{\partial t} = T_\phi, \]

where

\[ T_\Theta = -2AV^2\Theta + 2A \sin \Theta \cos \Theta (\nabla \phi)^2 + \frac{\partial K}{\partial \Theta} + MH \sin \Theta, \]

\[ T_\phi = -2A \sin^2 \Theta \nabla^2 \phi + 4A \sin \Theta \cos \Theta (\nabla \Theta \cdot \nabla \phi) + \frac{\partial K}{\partial \phi}. \]

If for the moment we assume that the inertial terms involving second-order time derivatives are small and remember that the spin angular velocity $\Omega$ is considered to be much larger than any precessional velocity so that $\Omega >> \partial \phi / \partial t$, the equations simplify to

\[ I_3 \Omega \sin \Theta \frac{\partial \phi}{\partial t} + F \frac{\partial \Theta}{\partial t} = T_\Theta, \]

\[ -I_3 \Omega \sin \Theta \frac{\partial \Theta}{\partial t} + F \sin^2 \Theta \frac{\partial \phi}{\partial t} = T_\phi. \]

We further recall that the spin moment $I_3 \Omega$ is related to the saturation magnetization by the magnetomechanical ratio $\gamma$. Thus,

\[ I_3 \Omega = \frac{M}{\gamma}, \]

where

\[ \gamma = \frac{g e}{2mc} = 1.76 \times 10^7 \frac{1}{\text{oe-sec}} \text{ for a free electron (} g = 2 \text{)}. \]

The torque expressions may also be considerably simplified by setting $\nabla \phi = 0$ which is valid in the low energy case where there are no $\phi$ transitions; then, as before, taking $K = K_1 \sin^2 \Theta (1 + a \sin^2 \phi)$, we obtain

\[ \frac{M}{\gamma} \sin \Theta \frac{\partial \phi}{\partial t} + F \frac{\partial \Theta}{\partial t} = -2AV^2 \Theta + 2K_1 \sin \Theta \cos \Theta (1 + a \sin^2 \phi) + MH \sin \Theta, \]

\[ -\frac{M}{\gamma} \sin \Theta \frac{\partial \Theta}{\partial t} + F \sin^2 \Theta \frac{\partial \phi}{\partial t} = aK_1 \sin^2 \Theta \sin^2 \phi. \]
Finally, after eliminating first the terms involving $\partial \phi / \partial t$ and secondly, those involving $\partial \theta / \partial t$, we find

$$
\left( F + \frac{M^2}{\gamma^2 F} \right) \frac{\partial \theta}{\partial t} = -2AV^2 \theta + 2K_1 \sin \theta \cos \theta (1 + a \sin^2 \phi) + M \left[ H - \frac{ak}{M} \left( \frac{M}{\gamma F} \right) \sin^2 \phi \right] \sin \theta ,
$$

\begin{align*}
\left( F + \frac{M^2}{\gamma^2 F} \right) \frac{\partial \phi}{\partial t} &= \frac{M^2}{\gamma^2 F} \left\{ \gamma H + \left( \frac{ak}{M} \right) \left( \frac{M}{\gamma F} \right) \sin^2 \phi + \frac{\gamma}{\sin \theta M} \left[ -2AV^2 \theta + 2K_1 \sin \theta \cos \theta (1 + a \sin^2 \phi) \right] \right\} .
\end{align*}

If $a << 1$ (as we will show subsequently), then the first of these equations becomes

$$
\left( F + \frac{M^2}{\gamma^2 F} \right) \frac{\partial \theta}{\partial t} = -2AV^2 \theta + 2K_1 \sin \theta \cos \theta + MH \sin \theta ,
$$

which is almost identical with the equation of motion obtained previously by considering rotational motion only. Now, however, the viscous drag coefficient is augmented by a term $\gamma M^2 / F$, which is obviously contributed by the precessional motion. When $T_\phi \sim 0$, as is the case for small $a$ or as is equally true when a $\phi$ transition occurs producing a balance between the exchange and anisotropy forces to make $T_\phi$ zero, as we found in Chapter II, then we obtain

$$
\sin \theta \frac{\partial \phi}{\partial t} = \left( \frac{M}{\gamma F} \right) \frac{\partial \theta}{\partial t} .
$$

This is a fairly fundamental relationship between the rates of rotation and precession with the coefficient $(M/\gamma F)$ being approximately their ratio.

It was shown earlier that the Rayleigh dissipation factor from which the damping is derived is given by

$$
f_D = \frac{1}{2} F \left\{ \left( \frac{\partial \theta}{\partial t} \right)^2 + \sin^2 \theta \left( \frac{\partial \phi}{\partial t} \right)^2 \right\} .
$$

We can now express this entirely in terms of $\partial \theta / \partial t$ as

$$
f_D = \frac{1}{2} F \left\{ \left( \frac{\partial \theta}{\partial t} \right)^2 + \left( \frac{M}{\gamma F} \right)^2 \left( \frac{\partial \theta}{\partial t} \right)^2 \right\} = \frac{1}{2} \left[ F + \frac{M^2}{\gamma^2 F} \right] \left( \frac{\partial \theta}{\partial t} \right)^2 ,
$$

to obtain the new damping coefficient in a somewhat more direct way.

The nature of this damping term is reasonable, since in the absence of losses the application of a lateral field will simply cause the spins to precess at a constant angle $\theta$, and the effective $F$ will be infinite. There can be no rotation unless there are losses present. Similarly, if very large losses are present, the wall velocity should go to zero; that is, the effective $F$ should become very large. It is reasonable to expect a minimum viscous drag between these two extremes (Fig. 12) which the equation shows. The net

![Fig. 12. Effective damping coefficient vs $M/\gamma F$.](image)
wall velocity may, by analogy with the previous expression, be written

\[
\frac{dz}{dt} = \frac{MH}{F + \frac{M^2}{\gamma F} K_1} \left( \frac{\gamma H}{(\gamma F + M) K_1} \right) A
\]

and it is evident that the velocity is a maximum when \( M/\gamma F = 1 \). Hence,

\[
\left. \frac{dz}{dt} \right|_{\text{max}} = \frac{(\gamma H)}{2} d_w
\]

where \( d_w = \text{wall thickness} \approx \sqrt{A/K_1} \). This is a rather remarkable and very general result which implies an upper bound to the 180° plane wall velocity which is independent of any arbitrary constants such as \( F \) and is given entirely by fundamental parameters of the ferromagnetic solid.

If \( 10^{-5} < d_w < 10^{-4} \) cm, as it is for most magnetic materials, then since

\[
\gamma = 1.7 \times 10^7 \ \frac{1}{\text{oe-sec}}
\]

we obtain

\[
v_{\text{max}} \sim 100 \text{ to } 1000 \ \text{cm/sec/oe}
\]

It is interesting to compare this theoretical result with recent wall velocity measurements made in single crystal window frames.

**Iron**

- at 90°K \( v = 5.7 \ \text{cm/sec/oe} \) (Stewart)\textsuperscript{34}
- at 300°K \( v = 6.3 \ \text{cm/sec/oe} \) (Stewart)\textsuperscript{34}

**Magnetite**

- at 300°K \( v \sim 1200 \ \text{cm/sec/oe} \) (Galt)\textsuperscript{18}
- at 300°K \( v = 620 \ \text{cm/sec/oe} \) (Epstein)\textsuperscript{19}

**Nickel Iron Ferrite**

- at 77°K \( v = 160 \ \text{cm/sec/oe} \) (Galt)\textsuperscript{14}
- at 200°K \( v = 26,150 \ \text{cm/sec/oe} \) (Galt)\textsuperscript{14}

Our very general bound would appear to fit well the measured velocity data for iron — to be of the same order of magnitude as the measured low-temperature velocities in magnetite and nickel ferrite but to be sharply violated by the high-temperature velocity data of the nickel ferrite. It should be stressed that wall velocity data on silicon iron is in fair supply. Furthermore, readily applied powder pattern techniques insure that the data recorded are for the motion of a single 180° wall. In the much harder ferrites, it is much more difficult to apply the powder pattern technique, and some uncertainty as to the actual domain structure in the magnetite and nickel iron ferrite samples may exist. The ferrite data given above were taken on only one sample in each case, hence further measurements are probably in order. Thus, with the exception of the one measurement that may be due to a more complex domain structure, the theoretical upper
bound that we have found for the wall velocity appears to be obeyed, which should lend credence to our analysis.

If now we rewrite the equations for the precessional velocity $d\phi/dt$ under the condition that the anisotropy field is negligible, we find

$$d\phi/dt = \frac{(M/\gamma_F)}{(\gamma_F + \gamma_H/M)} \gamma_H$$

When this is plotted as shown in Fig. 13, we observe that $d\phi/dt$ becomes constant at the Larmor frequency $\gamma_H$ for sufficiently high values of $(M/\gamma_F)$. The precessional rate has been determined from ferromagnetic resonance experiments to be very nearly the Larmor rate in most ferromagnetic materials. We conclude that in these materials $(M/\gamma_F) >> 1$, so that we are in the region where $d\phi/dt$ does not depend in a first-order way on the losses. The wall velocity $dz/dt$ is then given approximately by

$$\frac{dz}{dt} = (\gamma_F/M) \gamma_H \sqrt{\frac{A}{K_1}}$$

or the velocity increases with increasing loss. This is perhaps something that has not been generally realized heretofore. The reduction of the precession frequency with increasing loss appears to be a very reasonable result, and experimental evidence to confirm it may lie in the ferromagnetic resonance data taken by Bloembergen on Supermalloy.

4.4 COMPARISON WITH PREVIOUS VELOCITY EXPRESSIONS

The basic wall velocity expression used today is the one first derived by Landau and Lifshitz and re-derived in a simpler fashion by others. The starting point for the derivation is the well-known Landau-Lifshitz equation of motion for the magnetization vector $M$ under the influence of a field $H$. Thus,

$$\frac{d\vec{M}}{dt} = -\gamma(\vec{M} \times \vec{H}) - \frac{\lambda}{|\vec{M}|^2} [\vec{M} \times \vec{M} \times \vec{H}]$$

This is simply the usual equation of motion for a gyroscope, with the addition of an "erecting" torque given by the triple vector product term so designed as to conserve the amplitude of the magnetization vector. $\lambda$ is an arbitrary constant which depends on the damping losses and has the dimensions of a relaxation frequency. The wall velocity expression that is derived from this is

$$\frac{v}{H} = (\gamma M^2 + \lambda^2) \sqrt{\frac{A}{K_1}} \lambda M$$
It is usually assumed that \( \lambda \ll \gamma M \), so that

\[
\frac{v}{H} \approx \frac{\gamma M}{\lambda} \frac{A}{\sqrt{K_1}}.
\]

If we rewrite this velocity expression in the form

\[
\frac{v}{H} = \left[ \frac{\gamma M}{\lambda} + \frac{\lambda}{\gamma M} \right] \gamma \frac{A}{\sqrt{K_1}} \quad \text{[L-L]}
\]

and compare it with the velocity equation we had previously obtained,

\[
\frac{v}{H} = \frac{1}{\left[ \frac{M}{\gamma F} + \frac{\gamma F}{M} \right]} \gamma \frac{A}{\sqrt{K_1}} \quad \text{[R]}
\]

Some rather striking differences emerge. Reference to Fig. 14, in which these are plotted, will show that the expression derived on the basis of Landau-Lifshitz (L-L) damping predicts velocities extending from very large values down to a minimum of \( 2\gamma H \sqrt{A/K_1} \), while the Rayleigh damping (R) leads to the reciprocal prediction of very small velocities up to a maximum of \( \gamma H/2 \sqrt{A/K_1} \). If, as shown earlier, we take the (R) limit as 100 to 1000 cm/sec/oe, then the (L-L) minimum velocity is 400 to 4000 cm/sec/oe. The measured data, although not abundant, would tend to support, say, 1000 cm/sec/oe as a maximum, not a minimum, velocity; hence, our supposition of Rayleigh dissipation more closely approximates the actual physical behavior.

![Fig. 14. Variation of wall velocity with damping.](image)
It should also be noted that, because it is always true that
\[ |\gamma M \frac{\lambda}{\lambda} + \frac{\lambda}{\gamma M} > |\frac{1}{\gamma F^2} + \frac{M}{\gamma F} | \] ,
there can be no equivalence relationship between the two arbitrary constants \( \lambda \) and \( F \) for any real values of these quantities. The two velocity expressions (L-L) and (R) are simply not equivalent for any loss situation.

4.5 WALL ACCELERATION

Now that steady-state solutions to the dynamical domain wall equations have been deduced, it is appropriate to seek transient solutions where the inertial or "wall mass" effects are to be included. Under the conditions that the spin angular velocity \( \Omega \) is very much greater than either the precessional or rotational velocity, and that \( \nabla \phi \) and \( \partial K/\partial \phi \) are zero, the differential equations describing the motion of a 180° plane wall become

\[
\begin{align*}
I_1 \frac{\partial^2 \Theta}{\partial t^2} + I_2 \Omega \sin \Theta \frac{\partial \Theta}{\partial t} + F \frac{\partial \Theta}{\partial t} = -2A \nabla^2 \Theta + 2K_1 \sin \Theta \cos \Theta + M_1 \sin \Theta \quad (1), \\
I_1 \sin^2 \Theta \frac{\partial^2 \phi}{\partial t^2} - I_3 \Omega \sin \Theta \frac{\partial \Theta}{\partial t} + F \sin^2 \Theta \frac{\partial \phi}{\partial t} = 0 \quad (2),
\end{align*}
\]

or from (2),

\[
\rho \frac{\partial \Theta}{\partial t} = \sin \Theta \left[ \tau_1 \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial \phi}{\partial t} \right],
\]

where

\[
\tau_1 = \frac{I_1}{F},
\]

\[
\rho = \frac{M}{\gamma F} = \frac{I_3 \Omega}{F}.
\]

By differentiating this, and eliminating the derivative of \( \Theta \) in the first equation, we obtain

\[
\tau_1 \frac{\partial^3 \phi}{\partial t^3} + 2 \tau_1 \frac{\partial^2 \phi}{\partial t^2} + (1 + \rho^2) \frac{\partial \phi}{\partial t} = \rho^2 \gamma H + \rho \left[ -2A \nabla^2 \Theta + \left( 2K_1 - \frac{I_1}{\rho^2} \left( \tau_1 \phi + \phi^2 \right) \right) \sin \Theta \cos \Theta \right].
\]

The terms within the bracket determine the wall shape, as we saw earlier. Assuming that the anisotropy constant \( K_1 \) is very much greater than the kinetic term \( [(I_1/\rho^2) \left( \tau_1 \phi + \phi^2 \right)] \), the wall shape is constant and the bracketed terms sum to zero. Hence, \( \partial \phi/\partial t \) is determined by:

\[
\tau_1 \frac{\partial \phi}{\partial t} + 2 \tau_1 \phi + (1 + \rho^2) \phi = \rho^2 \gamma H.
\]

If at \( t = 0 \), \( \phi = \phi^* = 0 \), and the applied field is a step function of magnitude \( H \), this has the solution

\[
\frac{\partial \phi}{\partial t} = \frac{\rho^2 \gamma H}{1 + \rho^2} \left[ 1 - e^{-t/\tau_1} \left( \cos \frac{\rho}{\tau_1} t + \frac{t}{\rho} \sin \frac{\rho}{\tau_1} t \right) \right].
\]
For low damping, $p >> 1$, this becomes
\[ \frac{\partial \phi}{\partial t} \approx \gamma H \left[ 1 - e^{-t/T_1} \cos \frac{\rho}{T_1} \right] ; \quad p >> 1 , \]
indicating an oscillatory approach to the Larmor frequency $\gamma H$. For high damping, $p << 1$, we find
\[ \frac{\partial \phi}{\partial t} \approx \rho^2 \gamma H \left[ 1 - \frac{t}{T_1} e^{-t/T_1} \right] ; \quad p << 1 , \]
which indicates a damped approach to a smaller frequency $\rho^2 \gamma H$.

With $\partial \phi/\partial t$ determined, we can rewrite the first of our motional equations to obtain a differential equation for $\theta$ alone. Thus,
\[ I_1 \frac{\partial^2 \theta}{\partial t^2} + F \frac{\partial \theta}{\partial t} = MH \sin \theta \left( 1 - \frac{1}{\gamma H} \frac{\partial \phi}{\partial t} \right) - 2A \frac{\partial^2 \theta}{\partial x^2} + 2K_1 \sin \theta \cos \theta . \]

From our earlier experience with equations of this type, we are tempted to seek solutions of the form
\[ C_2 z(t) + C_2 x = \log \tan \frac{\theta}{2} , \]
where $C_2$ is a constant and $z(t)$ is a function of time only. In the simpler cases considered earlier, $z(t) = t$, but now the presence of higher-order $\theta$ derivatives and the more complex driving function would suggest generalizing this. Thus,
\[ \frac{\partial \theta}{\partial x} = C_2 \sin \theta , \]
and
\[ \frac{\partial^2 \theta}{\partial x^2} = C_2 ^2 \sin \theta \cos \theta . \]

Also,
\[ \frac{\partial \theta}{\partial t} = C_2 \sin \theta z'(t) , \]
\[ \frac{\partial^2 \theta}{\partial t^2} = C_2 \sin \theta z''(t) + C_2 ^2 (z'(t))^2 \sin \theta \cos \theta . \]

Using these in the equation of motion, we find, after regrouping terms,
\[ \{C_2 I_1 z'' + C_2 Fz' - MH (1 - \frac{1}{\gamma H} \phi') \} \sin \theta = (-2AC_2 ^2 + 2K_1 - 1_4 C_2 ^2 z'^2) \sin \theta \cos \theta , \]
which is satisfied for all values of $\theta$ when the bracketed terms are zero, or when

First,
\[ C_2 = \sqrt{\frac{K_1}{\sqrt{A + \frac{1}{2} I_4 z'^2}}} \approx \sqrt{\frac{K_1}{A}} , \]
since the exchange energy constant $A$ is undoubtedly large compared to any kinetic energy term, and

Second,

$$C_2^1 \frac{\partial^2 z}{\partial t^2} + C_2^1 F \frac{\partial z}{\partial t} = MH \left(1 - \frac{1}{\gamma H} \frac{\partial \phi}{\partial t}\right),$$

or

$$\sqrt{\frac{K_1}{A}} \frac{\partial^2 z}{\partial t^2} + \sqrt{\frac{K_1}{A}} F \frac{\partial z}{\partial t} = \frac{MH}{1 + \rho^2} \left[1 + \rho e^{-t/\tau_1} \left(\sin \frac{\rho}{\tau_1} t + \rho \cos \frac{\rho}{\tau_1} t\right)\right],$$

or

$$\tau_1 \frac{\partial^2 z}{\partial t^2} + \frac{\partial z}{\partial t} = \sqrt{\frac{A}{K_1}} \frac{\rho}{1 + \rho^2} \gamma H \left[1 + \rho e^{-t/\tau_1} \left(\sin \frac{\rho}{\tau_1} t + \rho \cos \frac{\rho}{\tau_1} t\right)\right].$$

The first of these conditions leads to $C_2$ being the reciprocal of the wall thickness parameter and implies that the wall shape is constant during the motion. The variable $x$ can be thought of as measuring the distance from the center of the wall to any other part of the wall.

The second of these conditions leads to the linear equation of motion for the wall when its shape is considered constant and $z(t)$ is the distance measured perpendicular to the wall between the origin and the center of the wall. It resembles strongly the usual phenomenological equation written for $180^\circ$ wall motion, viz:

$$m \frac{d^2 z}{dt^2} + \beta \frac{dz}{dt} + \alpha z = 2MH,$$

except that the linear term is missing and an additional transient driving force term contributed by the precessional motion is present.

By analogy we can very nicely find the wall mass as

$$m = \sqrt{\frac{K_1}{A}} I_1 = \text{mass/unit area},$$

or we can interpret wall mass simply as the inertial moment of the spin system per unit volume divided by the wall thickness parameter.

The transient solution for the linear equation of motion may be readily found by transformation techniques to be

$$\frac{dz}{dt} = \sqrt{\frac{A}{K_1}} \frac{\rho}{1 + \rho^2} \gamma H \left[1 + e^{-t/\tau_1} \left(\rho \sin \frac{\rho}{\tau_1} t - \cos \frac{\rho}{\tau_1} t\right)\right].$$

For small damping, $\rho \gg 1$, this simplifies to

$$\frac{dz}{dt} \approx \sqrt{\frac{A}{K_1}} \frac{\gamma H}{\rho} \left[1 + \rho e^{-t/\tau_1} \sin \frac{\rho}{\tau_1} t\right]; \quad \rho \gg 1,$$

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and for large damping, \( \rho \ll 1 \), it becomes

\[
\frac{dz}{dt} \approx \frac{A}{\sqrt{K_1}} \rho \gamma H \left[ 1 - e^{-t/\tau_1} \right] ; \quad \rho \ll 1 ,
\]

giving the expected oscillatory response for little damping and exponential rise for large damping. There is something special about the situation for small damping in that, for sufficiently large \( \rho \), the equation predicts very large initial oscillation which may even produce negative values for the velocity.

Transient wall velocities of this precise character have not been observed either because they occur so rapidly (\( I_\perp \) is very small) that the usual observational means do not permit it or they are obscured by other effects associated with coercive force which our equations have not taken into account. These coercive effects are those which are represented by the \( \alpha z \) term in the empirical equation of motion. These effects are quite complex and are considered to be produced primarily by the growth of Néel spikes from the domain wall in question to neighboring inclusions. Néel spikes are visible in some of the domain patterns given in Chapter VI. Their affinity for the wall is such that, for small wall motion, the restoring force would be expected to be proportional to the displacement (\( \alpha = \text{constant} \)). For larger motions, the spikes snap off, removing that particular force on the wall, while other spikes may form elsewhere on the wall. \( \alpha z \) represents an average force over the entire wall which, with increasing displacement, would initially increase and then decrease to some lesser value given by the coercive force.

While little is known about the nature of the restoring term \( \alpha z \), and it is difficult to account for it in our theory in order to obtain theoretical velocity responses more nearly representative of those observed, we nevertheless have obtained from our theory a possible explanation for the fast initial "spike" on the velocity response. Our theory suggests that, in the case of low damping, the initial wall velocity can be very much higher than the average because the damping contributed by the precessional motion to the rotational motion is very small during the precessional acceleration time.

This may be seen somewhat more clearly by the following argument which, although not so exact as the previous argument, better illustrates the physical principles involved in the complex motion of the magnetization vector.

We saw earlier that

\[
I_\perp \sin \theta \left[ \dot{\phi} + \frac{F}{I_\perp} \dot{\phi} \right] = \frac{M}{\gamma} \dot{\phi} ,
\]

or

\[
\tau_1 e^{-t/\tau_1} \frac{\partial}{\partial t} \left( e^{t/\tau_1} \dot{\phi} \right) = \rho \frac{\dot{\phi}}{\sin \theta} .
\]

It was also shown that in the steady state

\[
F(1 + \rho^2) \frac{\partial \theta}{\partial t} = MH \sin \theta ,
\]

or

\[
\frac{\dot{\theta}}{\sin \theta} = \text{const} .
\]
Assuming that this applies approximately during the precessional acceleration, we can integrate the preceding equation to obtain

\[ \dot{\varphi} \approx \rho \left( 1 - e^{-t/\tau_1} \right) \frac{\dot{\varphi}}{\sin \Theta} . \]

The motional equation for the \( \Theta \) variable was given by

\[ I_1 \ddot{\Theta} + F \dot{\Theta} + \frac{M}{\gamma} \sin \Theta \dot{\varphi} = T_\Theta = MH \sin \Theta . \]

Using the approximate expression for \( \dot{\varphi} \) in this, we find

\[ I_1 \ddot{\Theta} \left( F + \frac{M^2}{\gamma^2 F} \left( 1 - e^{-t/\tau_1} \right) \right) \dot{\Theta} \approx MH \sin \Theta . \]

Thus, the damping factor at \( t = 0 \) is \( F \) and increases with time constant \( \tau_1 \) to its final value \( F + M^2/\gamma^2 F \) as the precessional velocity increases. Since \( M^2/\gamma^2 F \gg F \) for small damping, it is evident that a large initial spike could be produced by the small initial damping, changing to a very large effective rotational damping following acceleration.

4.6 COMMENTS ON WALL MASS

We found in the previous section by analogy with the usual phenomenological equation for wall motion that the wall mass is given by

\[ m = \sqrt{\frac{K_4}{A}} \cdot I_1 , \]

where \( I \), the inertial moment per unit volume is a fundamental property of the spin system and \( \sqrt{K_4/A} \) is essentially the reciprocal of the wall width. It was also shown that a more accurate relation for wall width and, by implication, wall energy includes a correction term dependent on wall velocity. Thus, the dynamic wall energy is slightly greater than the static energy by virtue of an increased kinetic energy component contributed by the motion of the spin system within the wall.

Thus, the augmented wall thickness \( \delta_w' \) is given by

\[ \delta_w' = \sqrt{\frac{A}{K_4}} \sqrt{1 + \frac{I_1}{A} \frac{z'^2}{2}} , \]

and the dynamic wall energy \( \sigma_w' \), by

\[ \sigma_w' = \sqrt{AK_4} \sqrt{1 + \frac{I_1}{A}} \frac{z'^2}{2} . \]

If we take

\[ \frac{dz'}{dt} = z' = \frac{\rho}{\sqrt{K_4 \left( 1 + \frac{z'^2}{2} \right) \gamma H}} , \]

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then

\[ \delta_w' = \frac{1}{\sqrt{K}} \sqrt{1 + \frac{1}{2} \frac{1}{K} \left( \frac{\rho}{1 + \rho} \gamma H \right)^2} , \]

and

\[ \sigma_w' = \frac{1}{\sqrt{AK}} \sqrt{1 + \frac{1}{2} \frac{1}{K} \left( \frac{\rho}{1 + \rho} \gamma H \right)^2} . \]

If the correction term is small, then

\[ \sigma_w' \approx \frac{1}{\sqrt{AK}} \left[ 1 + \frac{1}{4} \frac{1}{A} z'^2 \right] = \frac{1}{\sqrt{AK}} \cdot \frac{1}{4} \sqrt{\frac{1}{A}} 1 z'^2 , \]

or

\[ \sigma_w' = \frac{1}{4} m v^2 , \]

indicating that the increased wall energy is half the kinetic energy of the wall. This is a proof of a relationship assumed by Becker in his calculation of wall mass, although his assumption omitted the factor of one-half which is evidently the result of averaging the motion through the wall.

In the absence of any analysis such as we have given, the basic method for calculating wall mass is due to Becker. Investigators such as Galt and Kittel, who have since studied the problem, have used essentially the same method. We intend to repeat some of this reasoning and to compare it with the theoretical results that we have obtained.

The basis of the method is the assumption of an effective field \( H_E \) which is perpendicular to the plane of the wall, about which the magnetization vectors within the wall precess with the Larmor frequency. Thus, we write that

\[ \gamma H_E = \frac{\partial \Theta}{\partial t} = \frac{\partial \Theta}{\partial z} \cdot \frac{\partial z}{\partial t} , \]

giving the basic relation due to Becker that

\[ H_E = \frac{v}{\gamma} \frac{\partial \Theta}{\partial z} , \]

where the Z-direction is normal to the wall.

The next step is to argue that the \( H_E \) field is much larger within the wall than the applied external field \( H_0 \) and that, as a consequence of this, the Landau-Lifshitz equation may be written as shown by Kittel:

\[ \mathbf{H} \cdot \frac{\partial \mathbf{M}}{\partial t} \approx \lambda H_E^2 , \]

where

\[ \mathbf{H} = H_0 + H_E . \]
Now we return to the simple theory of the domain wall to show that

$$\frac{\partial \Theta}{\partial z} = \sqrt{\frac{K(\Theta)}{A}} = \sqrt{\frac{K_1}{A}} \sin \Theta$$

Using this, we get an expression for $H_E$ which can be substituted in the reduced Landau-Lifshitz equation, both sides of which may then be integrated to obtain a relation as given by Galt:

$$v = \frac{2M y^2}{\lambda} \int_0^{2\pi} \sqrt{K(\Theta)} \, d\Theta$$

This is Galt's Eq. (11). Remembering that $K(\Theta)$ in our case was assumed to be $K \sin^2 \Theta$ we find that Galt's Eq. (12) reduces to

$$v = \frac{M y^2}{\lambda} \sqrt{\frac{A}{K_1}} H_0$$

which is essentially the same as the velocity equation that we have derived, provided our $F$ is set equal to $y^2/\lambda$. If we go on now to calculate the wall mass by a continuation of this method, we find that the next step is to assume that the inertial energy in the wall, namely, $1/2 \, m v^2$, is equal to the increased energy in the wall due to the $H_E$ field. Thus,

$$\frac{1}{2} m v^2 = \frac{1}{8\pi} \int \int \int H_E^2 \, dv$$

When the value for $H_E$ given previously is inserted in this expression and the integration carried out, we find that the wall mass is given by

$$m = \frac{1}{2\pi y^2} \sqrt{\frac{K_1}{A}}$$

The surprising thing about this result is that, if $\sqrt{A/K_1}$ is interpreted as the wall thickness, then the mass is given entirely in terms of $y$ and the wall thickness and, according to this theory, would appear not to be an independent wall parameter. This is certainly not the case in the analysis that we have made, and it would appear much more reasonable that it not be a function of $y$ and other wall energies but simply be a characteristic of the spin inertial moment. In other words, it is felt that the interpretation of wall mass as a consequence of spin system inertia is much more reasonable than the one usually given. Further, the derivation of wall velocity by considering the basic energies effective in the ferromagnetic crystal would appear to be much more orderly and reasonable than the one based on an assumed transverse field, even though the final results would appear to be similar. In a physical sense, it seems unreasonable to expect a Larmor precession in the $\Theta$-direction about some field when the applied field is orthogonal to this such that precession, as we have previously shown, must occur in the $\phi$-direction. It is the precession in the $\phi$-direction which is uniform, whereas the rotation in the $\Theta$-direction is a consequence more of the losses in the material than of any Larmor rotational effects, although, by postulating the proper variable $H_E$ field, evidently one can arrive at a similar result.
4.7 VECTOR EQUATIONS OF MOTION

A clearer view of the difference between the Landau-Lifshitz and Rayleigh damping mechanisms may be obtained by reformulating our set of simultaneous differential equations into more concise vector form. A suitable form for this was first given by Gilbert. His equation was also based on a Lagrangian formulation of the free energy contributions and considered the damping to be of the Rayleigh type. Gilbert's treatment of the problem, however, is somewhat different from ours in that he did not include any of the kinetic energy terms in the free energy expression. Instead, he calculated an "effective" damping field due to the dissipation which was subtracted from the static internal field $H$ to obtain an effective dynamic internal field $H_{\text{eff}}$. Then

$$H_{\text{eff}} = H - \alpha \frac{\partial M}{\partial t}.$$  

Here, $H$ includes the contributions from the external, demagnetizing and anisotropy field, and $\alpha$ is a constant proportional to the dissipation in the material. This expression for the effective dynamic field is then inserted in the usual gyroscopic equation of motion,

$$\frac{\partial M}{\partial t} = -\gamma M \times H_{\text{eff}},$$

to obtain the vector equation of motion proposed by Gilbert as

$$\frac{\partial M}{\partial t} = -\gamma M \times \left(H - \alpha \frac{\partial M}{\partial t}\right).$$  

That this is characteristic of Rayleigh damping may be shown by expressing the component of $M$ in spherical coordinates and expanding the vector equation to obtain two simultaneous equations of motion which are the same as our basic motional equations. Thus, if

$$M = iM \cos \theta + jM \sin \theta \sin \phi + kM \sin \theta \cos \phi,$$

and

$$H = iH + jO + kO,$$

then

$$\frac{\partial \theta}{\partial t} = \alpha \gamma M \sin \theta \frac{\partial \phi}{\partial t},$$

$$\frac{M}{\gamma} \sin \theta \frac{\partial \phi}{\partial t} + \alpha M^2 \frac{\partial \theta}{\partial t} = -MH \sin \theta,$$

which are the same as the relations we had previously derived, provided one takes

$$\alpha = \frac{F}{|M|^2}.$$

*Unfortunately, little of this work of Gilbert's appears in the general scientific literature. The source of the material given above was an unpublished American Research Foundation Report by Gilbert, dated May 1956, which became available after most of this work had been done.*
Thus, our set of motional equations could be expressed in the form

\[ \frac{\partial \mathbf{M}}{\partial t} = -\gamma \mathbf{M} \times \mathbf{H} - \frac{\mathbf{F}}{|\mathbf{M}|^2} \frac{\partial \mathbf{M}}{\partial t} \]

Gilbert also showed that this may be transformed to the Landau-Lifshitz form, although with an important change in constants. Thus,

\[ \frac{\partial \mathbf{M}}{\partial t} = -\gamma \mathbf{M} \times \mathbf{H} + \frac{\gamma \mathbf{F}}{|\mathbf{M}|^2} \left( \mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t} \right) \]

and

\[ \mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t} = -\gamma \mathbf{M} \times (\mathbf{M} \times \mathbf{H}) + \frac{\gamma \mathbf{F}}{|\mathbf{M}|^2} \mathbf{M} \times \left( \mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t} \right) \]

Using the vector identity for the vector triple product, and remembering that since \(|\mathbf{M}|\) is conserved, then \((\mathbf{M} \cdot \partial \mathbf{M}/\partial t) = 0\), we find

\[ \mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t} = \left( \mathbf{M} \cdot \frac{\partial \mathbf{M}}{\partial t} \right) \mathbf{M} - (\mathbf{M} \cdot \mathbf{M}) \frac{\partial \mathbf{M}}{\partial t} = -|\mathbf{M}|^2 \frac{\partial \mathbf{M}}{\partial t} \]

Combining these equations, we obtain

\[ \frac{\partial \mathbf{M}}{\partial t} = -\gamma (\mathbf{M} \times \mathbf{H}) - \frac{\gamma \mathbf{F}}{|\mathbf{M}|^2} \mathbf{M} \times (\mathbf{M} \times \mathbf{H}) - (\gamma \mathbf{F}/|\mathbf{M}|)^2 \frac{\partial \mathbf{M}}{\partial t} \]

When the losses are small so that the third term on the right-hand side is negligible, this reduces to the usual Landau-Lifshitz form, where

\[ \lambda = \gamma^2 |\mathbf{F}| \]

However, when the losses are appreciable,

\[ \frac{\partial \mathbf{M}}{\partial t} = \frac{-\gamma}{1 + (\lambda/|\mathbf{M}|)^2} (\mathbf{M} \times \mathbf{H}) - \frac{\lambda}{1 + (\lambda/|\mathbf{M}|)^2} \frac{4}{|\mathbf{M}|^2} \mathbf{M} \times (\mathbf{M} \times \mathbf{H}) \]

While this is superficially still of the Landau-Lifshitz form, there are important changes in the constants which will lead to significantly different conclusions when the equation is applied to physical problems. For instance, the new magnetomechanical ratio \( \gamma' = \gamma/1 + (\lambda/\gamma|\mathbf{M}|)^2 \) is now a function of the losses present and is smaller than \( ge/2mc \). Similarly, the loss constant \( \lambda' = \lambda/1 + (\lambda/\gamma|\mathbf{M}|)^2 \) which has a maximum value \( \lambda'_{\text{max}} = \gamma|\mathbf{M}|/2 \) at \( \lambda/\gamma|\mathbf{M}| = 1 \). Thus, the effective damping term is bounded, and this is what leads to the prediction of a maximum wall velocity, as we found earlier.

There is a still more general formula for the vector equation of motion which we now propose that will also include the inertial terms. Thus,
\[ \frac{\partial \mathbf{M}}{\partial t} = -\gamma \mathbf{M} \times \left[ \frac{\mathbf{H}}{|\mathbf{M}|^2} - \frac{\mathbf{F}}{|\mathbf{M}|^2} \frac{\partial \mathbf{M}}{\partial t} - \frac{I_1}{|\mathbf{M}|^2} \frac{\partial^2 \mathbf{M}}{\partial t^2} \right] \]

expands to

\[ \frac{d}{dt} \left( I_1 \sin^2 \theta \frac{\partial \phi}{\partial t} + M \cos \theta \right) + F \sin^2 \theta \frac{\partial \phi}{\partial t} = 0 \]

and

\[ I_1 \frac{\partial^2 \phi}{\partial t^2} - I_1 \sin \theta \cos \theta \left( \frac{\partial \phi}{\partial t} \right)^2 + F \frac{\partial \theta}{\partial t} + \frac{M}{\gamma} \sin \theta \frac{\partial \phi}{\partial t} = MH \sin \theta \]

Hence, it completely describes the equations of motion that we had earlier derived and represents a vector motional equation for the macroscopic magnetization vector in which the damping and inertial effects are consistently introduced in the Lagrangian sense.

At this point, one should be reminded that \( I_1 \) was defined on the basis of a classical model of gyroscopic motion. A more exact physical interpretation of \( I_1 \) on the basis of the quantum mechanics is obscure. Consequently, it is probably best to regard the above vector equation of motion as a general dynamical form where \( I_1 \) is simply a parameter associated with one of the higher-order terms.
CHAPTER V
FERROMAGNETIC RESONANCE RELATIONSHIPS

5.1 INTRODUCTION

A great wealth of information about ferromagnetic materials has been obtained in recent years through ferromagnetic resonance experiments. The experiments have confirmed that the gyromagnetic factor $g$ has ordinarily a value of approximately 2, with small but undoubtedly significant variations. The measured shapes of the absorption and effective permeability spectra have been found to be in good agreement with theoretical calculations based on a simple gyrosopic equation of motion modified by either Landau-Lifshitz or Bloch type damping terms. It is hoped that our equations of motion derived through a Lagrangian formulation of the free energy in the ferromagnetic lattice, where the damping is of the Rayleigh type, will lead to ferromagnetic resonance relationships that will also agree with this wealth of experimental data. The purpose of this chapter is to prove that this is the case. In other words, while our chief interest is in domain wall motion, the basic equations of motion for the macroscopic magnetization vector must be the same in the two cases, and our results must also lead to a satisfactory theoretical description of ferromagnetic resonance phenomena. The chief difference between domain wall motion and ferromagnetic resonance experiments insofar as the motion of the macroscopic magnetization vector is concerned is that in resonance experiments a usually high constant field is applied orthogonal to a small alternating field so that there is essentially only a precessional motion ($\phi$) about the DC field with a very small excursion in rotation ($\theta$). In domain wall motion, on the other hand, the rotational motion is very large ($\Delta \theta = \pi$), and the exciting field is only a moderate DC field.

5.2 LANDAU-LIFSHITZ AND BLOCH EQUATIONS

The two phenomenological equations of motion that have been used to describe the motion of a magnetic moment $\vec{M}$ in an effective field $\vec{H}$ are the Landau-Lifshitz formula,

$$\frac{d\vec{M}}{dt} = \gamma(\vec{M} \times \vec{H}) - \frac{\lambda}{|\vec{M}|^2} \vec{M} \times (\vec{M} \times \vec{H}) \quad ,$$

and the Bloch form,

$$\frac{d\vec{M}}{dt} = \gamma(\vec{M} \times \vec{H}) - \frac{\vec{M} - \vec{M}_0}{\tau_2} + \left(\frac{|\vec{M}|}{|\vec{H}|}\right) \frac{\vec{H}}{\tau_1} .$$

$\vec{M}_0$ is the initial value for the magnetization vector. $\lambda$ is an arbitrary parameter having the effect of a relaxation frequency, and $\tau_1$ and $\tau_2$ are phenomenological relaxation times which, for convenience, are usually taken equal in ferromagnetic resonance applications. It should be noted that Bloch first developed this equation to describe nuclear magnetic resonance, and it was later applied to ferromagnetic resonance by Bloembergen.

If an effective static field $H_z$ is assumed parallel to the $z$-axis and the motion under the influence of time varying fields $h_x e^{i\omega t}$, $h_y e^{i\omega t}$ and $h_z e^{i\omega t}$ is studied, the solution of the Landau-Lifshitz equation after suitable linearizing assumptions are made is found to be of the form

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where \( b \) and \( h \) are the flux density and field time-varying vectors, and \( ||T|| \) is the so-called complex permeability tensor. The linearizing assumptions that are made to obtain this linear tensor relation are based on the time-varying component's being much smaller than the static components, i.e.,

\[
H_z \gg h_x, h_y, h_z, \\
M \gg m_x, m_y, m_z.
\]

The small alternating components of the magnetization and effective field are designated by \( m \) and \( h \) with the appropriate subscript.

Higher-order time-varying terms such as \( m_x^2, m_y^2, \) etc., are also neglected.

The permeability tensor is given by

\[
||T|| = \begin{pmatrix}
\mu' - j\mu'' & -j(k' - jk'') & 0 \\
\mu(k' - jk'') & \mu' - j\mu'' & 0 \\
0 & 0 & \tau
\end{pmatrix},
\]

where

\[
\mu' = 1 + 2\pi \frac{M_z}{H_z} \left[ \frac{-(\omega + \omega_0)\omega_0 \tau^2 + 1}{(-\omega + \omega_0)^2 \tau^2 + 1} + \frac{(\omega + \omega_0)\omega_0 \tau^2 + 1}{(\omega + \omega_0)^2 \tau^2 + 1} \right],
\]

\[
\mu'' = 2\pi \frac{M_z}{H_z} \left[ \frac{\omega \tau}{(-\omega + \omega_0)^2 \tau^2 + 1} + \frac{\omega \tau}{(\omega + \omega_0)^2 \tau^2 + 1} \right],
\]

\[
k' = 2\pi \frac{M_z}{H_z} \left[ \frac{(-\omega + \omega_0)\omega_0 \tau^2 + 1}{(-\omega + \omega_0)^2 \tau^2 + 1} - \frac{(\omega + \omega_0)\omega_0 \tau^2 + 1}{(\omega + \omega_0)^2 \tau^2 + 1} \right],
\]

\[
k'' = 2\pi \frac{M_z}{H_z} \left[ \frac{\omega \tau}{(-\omega + \omega_0)^2 \tau^2 + 1} - \frac{\omega \tau}{(\omega + \omega_0)^2 \tau^2 + 1} \right],
\]

and where

\[
\omega_0^2 = \gamma^2 H_z^2 + \frac{1}{\tau^2}.
\]

Because of the linearizing assumptions made, the two forms of damping become equivalent, as shown by Bloembergen, provided one takes

\[
\lambda \frac{|H_z|}{|M|} = \frac{\tau}{\tau_1}, \quad \text{and} \quad \tau_1 = \tau_2 = \tau,
\]

so that the expressions for \( \mu', \mu'', k' \) and \( k'' \) apply equally well to both Landau-Lifshitz and Bloch damping.
It should be noted, however, that these expressions do not give the permeability terms usually measured in ferromagnetic resonance experiments. What is usually measured is some simple function of the tensor components. For example, the complex propagation constant in an infinite ferromagnetic material of the type we are discussing may be written:

\[ \Gamma = \alpha + j\beta \]

where

\[ \alpha = \omega \sqrt{\varepsilon \mu_0 \mu_r} \]
\[ \beta = \omega \sqrt{\varepsilon \mu_0 \mu_L} \]

and where the effective permeability \( \mu_L \) and effective absorption coefficient \( \mu_r \) are given by

\[ \mu_r = \frac{\sqrt{\mu_r^2 + \mu_s^2} - \mu_s}{2} \]
\[ \mu_L = \frac{\sqrt{\mu_r^2 + \mu_s^2} + \mu_s}{2} \]

It is these effective permeability components which, since they appear naturally in the expression for the propagation constant, are the quantities usually measured.

5.3 RESONANCE EQUATIONS FOR RAYLEIGH DAMPING

If, for the sake of simplicity, we assume a single domain sample, then the equations of motion may be written

\[ F \frac{\partial \phi}{\partial t} + \frac{M}{\gamma} \sin \theta \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial \theta} (-\vec{M} \cdot \vec{H}) \]
\[ -\frac{M}{\gamma} \sin \theta \frac{\partial \phi}{\partial t} + F \sin^2 \theta \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial \phi} (-\vec{M} \cdot \vec{H}) \]

where \( \vec{H} \) is the effective internal magnetic field and may include components due to both anisotropy and demagnetizing fields. If we assume a large static field in the x-direction and a small alternating field in the z-direction corresponding to the usual ferromagnetic resonance experiment, then

\[ \vec{H} = H_x i_1 + H_z i_3 \sin\omega t \]

also,

\[ \vec{M} = M_x i_1 + M_y i_2 + M_z i_3 \]

where

\[ M_x = M \cos \theta \]
\[ M_y = M \sin \theta \sin \phi \]
\[ M_z = M \sin \theta \cos \phi \]
so that
\[ \mathbf{M} \cdot \mathbf{H} = M_H x \cos \Theta - M_H z \sin \omega t \sin \Theta \cos \phi . \]

Using this as the driving energy in the equations of motion, we obtain
\[ F \frac{\partial \Theta}{\partial t} + \frac{M}{\gamma} \sin \Theta \frac{\partial \phi}{\partial t} = M_H x \sin \Theta - M_H z \sin \omega t \cos \Theta \cos \phi , \]
\[ -\frac{M}{\gamma} \sin \Theta \frac{\partial \phi}{\partial t} + F \sin^2 \Theta \frac{\partial \phi}{\partial t} = M_H z \sin \omega t \sin \Theta \sin \phi , \]
which, when solved for the angular velocities, yield
\[ \frac{\partial \phi}{\partial t} = \omega_r - \frac{\rho}{\gamma H_x} \gamma h_z, \]
where
\[ \rho = \frac{M}{\gamma F}. \]

Under this same assumption of \( H_x \gg h_z \), the rotational motion is constrained to very small angles by the large static field, so that we may take \( \sin \Theta \approx \Theta \) and \( \cos \Theta \approx 1 \). When we write \( \phi = \omega_r t + \phi_o \) and make the obvious reductions, we obtain the linear equation of motion for \( \Theta \) as
\[ \frac{d \Theta}{d t} = \frac{\omega_r}{\rho} \Theta - \frac{\gamma h_z}{2} \rho \frac{1}{\sqrt{1 + \rho^2}} \left\{ \sin [(\omega + \omega_r) t + \phi_o - \tan^{-1} \rho] + \sin [(\omega - \omega_r) t - \phi_o + \tan^{-1} \rho] \right\}. \]

This has a steady state solution of the form
\[ \Theta = A \exp \left[j(\omega + \omega_r) t + \phi_o \right] + B \exp \left[j(\omega - \omega_r) t - \phi_o \right], \]
where
\[ A = \frac{\gamma h_z}{2} \frac{\rho}{1 + \rho^2} \left[ \frac{1 - j \rho}{\omega_r / \rho - j (\omega + \omega_r)} \right], \]
\[ B = \frac{\gamma h_z}{2} \frac{\rho}{1 + \rho^2} \left[ \frac{1 + j \rho}{\omega_r / \rho - j (\omega - \omega_r)} \right]. \]

Since we now know both \( \Theta \) and \( \phi \) as functions of time, we can evaluate the alternating z-component of the magnetization and then the complex permeability that the alternating field will "see." Thus,
\[ m_z = M \sin \Theta \cos \phi \approx M \Theta \cos (\omega_r t + \phi_o) \].
Since \( \Theta \) has frequency components \( \omega \pm \omega_r \), we would expect that the product \( \sin \Theta \cos \phi \), and hence \( m_z \), would have components \( \omega \), \( \omega + 2\omega_r \) and \( \omega - 2\omega_r \). The latter two may be assumed to lie outside the detector passband, so that the component of \( m_z \) which varies at frequency \( \omega \) is given by

\[
m_z = \frac{\gamma h_x}{2} M \frac{\rho}{1 + \rho^2} \left[ \frac{1 - j\rho}{\omega_r \rho - j(\omega + \omega_r)} + \frac{1 + j\rho}{\omega_r \rho - j(\omega - \omega_r)} \right],
\]

which can be manipulated into the form

\[
m_z = \frac{\hbar}{2} M \left[ \frac{\tau^2 \omega_r (\omega + \omega_r) + 1}{\tau^2 (\omega + \omega_r)^2 + 1} + \frac{\tau^2 \omega_r (-\omega + \omega_r) + 1}{\tau^2 (-\omega + \omega_r)^2 + 1} + j \frac{\tau \omega}{1 + \tau^2 (\omega + \omega_r)^2} + j \frac{\tau \omega}{1 + \tau^2 (-\omega + \omega_r)^2} \right],
\]

where

\[
\frac{1}{\tau} = \frac{\omega_r}{\rho} = \frac{\rho}{1 + \rho^2} \gamma H_x.
\]

Then, since

\[
\mu = \left( 1 + 4\pi \frac{m_z}{\hbar} \right) = \mu' + j\mu'',
\]

we find

\[
\mu' = 1 + 2\pi \frac{M}{H_x} \left[ \frac{\tau^2 \omega_r (\omega + \omega_r) + 1}{\tau^2 (\omega + \omega_r)^2 + 1} + \frac{\tau^2 \omega_r (-\omega + \omega_r) + 1}{\tau^2 (-\omega + \omega_r)^2 + 1} \right],
\]

and

\[
\mu'' = 2\pi \frac{M}{H_x} \left[ \frac{\tau \omega}{\tau^2 (\omega + \omega_r)^2 + 1} + \frac{\tau \omega}{\tau^2 (-\omega + \omega_r)^2 + 1} \right].
\]

### 5.4 COMPARISON OF DAMPING FORMS

These are identical to the expressions given by Bloembergen for the complex permeabilities on the principal diagonal of the complex permeability tensor for either Landau-Lifshitz or Bloch type damping. Hence, we conclude that, under the small rotation conditions \( \Delta \Theta \ll 1 \) applying to the usual ferromagnetic resonance experiment, all three types of damping are essentially equivalent. Therefore, the wealth of experimental resonance data which supports (L-L) damping could also support Rayleigh damping. This does not imply, however, that these damping mechanisms are equivalent for domain wall motion where \( \Delta \Theta \) is large. Indeed, we have shown in Chapter IV that the Rayleigh damping leads to reasonable variations of wall velocity with varying losses, whereas the (L-L) damping produces singularities for either very low or very high damping which are not physically reasonable.

Actually the correspondence between (L-L) and (R) type damping in ferromagnetic resonance is somewhat better in the sense that \( \lambda \) and \( \rho \) are simply related, whereas their equivalence with \( \tau \) involves \( H_z \), which is ordinarily the variable parameter in the experiment. Nevertheless, there
is some opinion\textsuperscript{38} that \( \tau \) is somewhat less sensitive to frequency than is \( \lambda \). Thus, since

\[
\lambda \left( \frac{H_z}{M} \right) = \frac{1}{\tau},
\]

and

\[
\frac{\rho}{1 + \rho^2} \gamma H_z \tau = \frac{1}{\tau},
\]

then

\[
\frac{\lambda \gamma M}{\rho} = \frac{\rho}{1 + \rho^2}
\]

is the desired equivalence between the (L-L) constant \( \lambda \) and the (R) constant \( \rho \).

It is known that the line width \( \Delta H \) measured between half-amplitude points in the absorption characteristic is related to the constant \( \lambda \) as follows:

\[
\Delta H = \frac{\lambda}{\gamma M} = \frac{\lambda}{\gamma M}
\]

where \( H_\tau \) is the effective DC field at resonance. Then

\[
\frac{\Delta H}{2H_\tau} = \frac{\lambda}{\gamma M}
\]

where \( H_\tau \) is the effective DC field at resonance. Then

\[
\frac{\Delta H}{2H_\tau} = \frac{\rho}{1 + \rho^2}
\]

or

\[
\frac{H_\tau}{\Delta H} \approx \frac{\rho}{2} \quad \text{for} \quad \rho >> 1.
\]

Hence, \( \rho/2 \) is roughly the "Q" of the absorption spectrum and is a convenient way for evaluating \( \rho \).

There is perhaps an additional relationship that should be demonstrated between the expressions for the resonant frequency in the several cases. Thus, for (L-L) and (B),

\[
\omega_0^2 = \gamma^2 H_z^2 + \frac{4}{\gamma^2} \tau^2
\]

\[
\omega_0^2 = \gamma^2 H_z^2 + \frac{\rho^2}{(1 + \rho^2)^2} \gamma^2 H_z^2
\]

or

\[
\omega_0 = \frac{\rho^2}{1 + \rho^2} \gamma H_z \sqrt{1 + \frac{3}{\rho^2} + \frac{4}{\rho^4}}
\]

For the situation where \( \rho > 3 \), \( \omega_o \) and \( \omega_\tau \) become very nearly the same. Since \( \rho = 3 \) corresponds to a very broad line width, \( \Delta H = 0.6H_\tau \), the equivalence is quite good in any relatively low loss case.
5.5 GENERALIZED RAYLEIGH DAMPING

One of the features of the Bloch equation of motion (p.48) is the possibility of different relaxation times for the longitudinal and transverse components of the magnetization vector. Proponents of this idea assert that interactions between the spin system and lattice vibrations determine $\tau_1$, the longitudinal relaxation time. An intermediate quantity $\tau_2^*$ is determined by magnetic and exchange interactions within the system of spins. Then the transverse relaxation time $\tau_2$ is determined by the relation

$$\frac{1}{\tau_2} = \frac{1}{2\tau_1} + \frac{1}{\tau_2^*}.$$

It is $\tau_2$ which is measured in ferromagnetic resonance experiments.

The possibility of unequal relaxation frequencies need not be lost by employing a Rayleigh damping function, since this situation may be represented by writing

$$2F = F_1 \left(\frac{\partial \theta}{\partial t}\right)^2 + F_2 \sin^2 \Theta \left(\frac{\partial \phi}{\partial t}\right)^2.$$

If the complex permeabilities are computed by using this more general expression, essentially the same general relationships are obtained as given on page 52 with slightly different definitions for the relaxation time $\tau$ and resonant frequency $\omega_T$. Thus,

$$\tau' = \frac{\alpha \rho'}{\omega_T},$$

$$\omega_T' = \frac{\rho'}{\rho' + \frac{1}{\alpha}} \gamma H_z,$$

where

$$\alpha = \sqrt{\frac{F_1}{F_2}},$$

$$\rho' = \frac{\lambda}{\gamma \sqrt{F_1 F_2}}.$$

The line width expression may then be written

$$\frac{\Delta H}{2H_T} = \frac{1}{a} \frac{\rho'}{1 + \rho'^2},$$

which, for $\rho' \gg 1$, becomes

$$\frac{\Delta H}{2H_T} \approx \frac{1}{a \rho'} = \frac{\gamma F_2}{M}.$$

Hence, the statement that for the usual low loss ferromagnetic resonance experiment the relaxation time measured is the transverse or precessional one.
If a corresponding expression for wall velocity is derived for the two relaxation time case, we find

\[
\frac{v}{H} = \frac{\gamma \sqrt{\frac{A}{K_1}}}{\left(\frac{\gamma F_1}{M} + \frac{M}{\gamma F_2}\right)} = \frac{1}{a(\frac{1}{\rho} + \rho')} \frac{\gamma}{\sqrt{K_1}} \frac{\sqrt{A}}{\sqrt{F_1}}
\]

so that

\[
\frac{v_{\text{max}}}{H} = \sqrt{\frac{F_2}{F_1}} \frac{\gamma}{\sqrt{2}} \frac{\sqrt{A}}{\sqrt{K_1}}
\]

While this expression is perhaps of theoretical interest, there appears to be little experimental justification for assuming \( F_1 \) very different from \( F_2 \), although no really conclusive experiments have been made.
CHAPTER VI
DOMAIN PATTERN OBSERVATIONS

6.1 EXPERIMENTAL TECHNIQUES

The material on which the domain patterns to be reported were observed was approximately 3 per cent silicon iron single crystal sheet furnished by Dr. A. Dunn of the General Electric Company. The crystals were actually produced by the strain-anneal process and were, perhaps, not quite so uniform for simple domain pattern observations as crystals grown by other processes. The material was furnished in the form of 1-inch by 5-inch sheets approximately 40 thousandths of an inch thick. They were oriented by x-ray diffraction techniques, and small window frames approximately 1/2 inch square by about 1/8 inch in thickness were cut such that the boundaries conformed to 100 planes. The x-ray data showed that the plane of the sheet was tilted from the 100 plane by about 3° to 5° in various places across the sample. Following cutting, the samples were very deeply etched and polished and then annealed for 4 to 7 hours at about 1100°C in pure hydrogen. The etching technique and, indeed, the entire powder pattern technique follows very closely the procedure developed by Williams, et al., as reported in the literature. Following the annealing process, the surface was further polished with a very fine metallographic paper and then electrolytically polished as follows.

A bath composed of 80 grams of an 85 per cent solution of phosphoric acid and 20 grams of solid chromium trioxide was made. A cathode consisting of a piece of heavy copper sheet was immersed in the bath which was contained in a small laboratory beaker. The specimen was used as the anode and was electropolished with a current on the order of 5 amperes when specimen dimensions were as previously described. The time of polishing varied from 1 to 4 minutes depending upon the degree of polishing required. It was found that the temperature of the bath approached that given by Williams (about 90°C) if the bath were electrolyzed using a copper anode and cathode for several minutes before the sample was inserted. The current was obtained from a 12-volt battery charger and was adjusted by a simple variac at the charger input. It was found that, if current substantially higher than the 5 amperes were used, the surface was badly pitted in the polishing process. Similarly, if very low currents were used, somewhat unsatisfactory results were also obtained. In addition, the polishing process was considerably longer. In general, the polishing was stopped when visual examination showed that a sufficiently scratch-free surface had been obtained.

Following the etching process, the specimen was very carefully and thoroughly washed, since it was found later that any of the etching bath remaining on the specimen would cause the colloid later used to coagulate in very short order. A solution of colloidal magnetite was made essentially according to the directions of Elmore and Williams, although the procedure of necessity was modified somewhat by D. Wickham, who prepared the colloid actually used in these experiments.

The first batch of colloidal magnetite behaved very satisfactorily for several weeks, although after that time the particles rapidly lost their resolving power. The second batch was checked after a week or two for pH, and it was found the the pH had changed somewhat from 7 to 6. This
was brought back by the addition of a few drops of pyridine to the 7 level, and the magnetite kept considerably longer. It is evidently important to prolong the life of the colloidal suspension to maintain the pH at neutral, and the addition of the organic base pyridine appears to be a satisfactory way of accomplishing this.

Immediately after the specimen had been etched and washed, it was placed in a small brass tank, covered with a few drops of the colloidal suspension, and then covered with a microscope cover glass such that the glass was very close — as close as possible — to the specimen surface. If the suspension were not immediately applied following etching, some oxidation and discoloring of the specimen surface would occur. The powder patterns were then observed, using a binocular microscope having magnifying powers from 36 to 96 diameters. A microscope camera of a very simple cut-film type, which simply mounted over one of the eye pieces, was used to photograph the powder patterns.

At least two microscope lights were used to cross light the specimen being observed in order that certain of the domain boundaries be visible at all times. It was found that when one light was used it was possible for certain orientations of the pattern with respect to the light source to render the domain boundaries invisible. With the cross lighting used, this was no longer observed and, hence, the film pictures obtained are true reproductions of the powder patterns. At first, the film used to photograph these patterns was Super XX 3-1/4 by 4-1/4 cut film which required an exposure under the lighting and stop conditions used of some 5 to 10 minutes. It was observed that the photographic results obtained under these conditions were considerably poorer than the patterns visible to the eye. In the latter stages of the investigation, some very much faster film (Kodak Royal X Pan) was obtained which permitted the exposure time to be reduced to 10 seconds. Under these conditions, photographs that more nearly represented the patterns as seen by the eye were obtained. The reason for this very substantial improvement was associated with the blurring due to building vibrations, which was quite serious over the 5-minute exposure time but which was not serious during the 10-second exposure time. The measurements taken in the presence of strong vertical fields were taken with the specimen lying flat on the surface of the 2-inch pole piece of the large electromagnet with the microscope and camera assembly mounted above this. The field at the center of the pole piece was calibrated by using a little exploring coil attached to a recording flux meter. The vertical and horizontal components of this field were obtained as a function of the driving current in the electromagnet.

6.2 EXPERIMENTAL RESULTS

The experimental results of principal interest in this report are the ones that demonstrate wall breakup under the influence of an applied vertical field. This is because these results confirm the essentially three-dimensional character of the spin variation in the domain wall, which is important to the theoretical arguments previously given. However, some other results and observations that are particularly interesting are also included here. Figures 15 and 16 show a portion of a $180^\circ$ domain wall observed in one specimen with and without the application of a small vertical field. The field was applied at first to check the direction of the spin orientation of the wall, with the surprising results that the wall broke up into light and dark areas, as shown in Fig. 16, indicating that the direction of spin varied along the wall. It will be noted that there are
two walls in Fig. 15, whereas in Fig. 16 the second one has evidently moved, the first remaining in roughly the same position. It is very difficult to apply a supposedly vertical field that does not also have a small horizontal component. Since the coercive force in this material is of the order of a few tenths of an oersted, it is likely that the horizontal component will exceed this and cause motion of at least one of the walls. Evidently, this is what occurred.

In the observations recorded here and in other observations made visually, no particular regularity in the light and dark areas along the wall was observed. The distribution of the spin reversals is apparently arbitrary, and these can occur in as general a way as the walls themselves appear to occur. To check the behavior of this wall breakup under more controlled vertical fields, the specimen was placed on the pole piece of a large electromagnet, and domain patterns for various positive and negative magnetizing currents were taken. At fields much less than the order of 50 gauss, the $180^\circ$ walls in certain portions of the sample appeared to break up into sections, as previously described. For higher vertical fields, the walls tended to move somewhat and to form dagger domains near inclusions. The daggers tended to remain when the field was lowered back to the small value and the wall moved back to its original position. In general, under the higher vertical field, the gaps in the wall tended to disappear, and more complex dagger and spike patterns were obtained in their place. This is noted in Figs. 17, 18 and 19, which are taken for two parallel $180^\circ$ walls at vertical fields of +55, –55, and +235 gauss, respectively. It is apparent that the dagger patterns for the +55 and –55 gauss fields are different and are oppositely directed. At the higher positive vertical field of 235 gauss, the wall separation is different from that in the first two cases, and the spike pattern is also substantially different. The structure of the spikes has now become more complex such that they themselves exhibit a tree-like structure. It should be noted that the specimen and the entire viewing system were in the same relative position to one another and that any difference in the patterns shown here was that due solely to the application of the nominally vertical field. It is interesting to note that the colloid in the regions away from the domain boundaries appears to collect in striations that are orthogonal to what we would expect the direction of the magnetization in the domain to be. This is to be expected when one realizes that the surface, although polished, is still microscopically scratched to an extent that it is probably large compared to the diameter of the colloidal particles (0.1). The flux jumps these scratches in the direction of the magnetization but is not present in the scratches that are parallel to the direction of the magnetization. Hence, the collection of the colloid is transverse to this direction. This effect was actually used and exploited by Williams and Shockley in their so-called scratch tests to actually determine field orientation in magnetic domains. Our results simply substantiate this effect and confirm, incidentally, that the boundaries we are looking at are $180^\circ$ boundaries.

H. J. Williams has observed similar phenomena in experiments with a ring of polycrystalline Perminvar which, through a magnetic anneal, behaved as a magnetic single crystal. The same general breakup of a $180^\circ$ wall into alternate regions where colloid did and did not collect was observed under the action of a moderate vertical field. With the application of a stronger field, this structure changed to one where the spin orientation was uniform along the wall, and this was maintained as the field was reduced to low values.
This agrees in a general way with our observations, although we found that, while for higher fields the gap structure disappeared, a complex dagger-like structure occurred instead, which tended to persist as the field was reduced.

Fig. 15. 180° domain walls in SiFe window frame, no vertical field.
Fig. 16. Breakup of 180° domain walls in SiFe window frame in presence of small vertical field.
Fig. 17. 180° domain boundaries in SiFe window frame. Vertical field +55 gauss, magnification 96x
Fig. 18. 180° domain boundaries in SiFe window frame. Vertical field – 55 gauss, magnification 96x.
Fig. 19. 180° domain boundaries in SiFe window frame. Vertical field +235 gauss, magnification 96x.
REFERENCES


