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Abstract

Conventional wisdom holds that dependence among geological prospects increases exploration risk. However, dependence also creates the option to truncate exploration if early results are discouraging. We show that the value of this option creates incentives for explorationists to plunge into dependence; i.e., to assemble portfolios of highly correlated exploration prospects. Risk-neutral and risk-averse investors are distinguished not by the plunging phenomenon, but by the threshold level of dependence that triggers such behavior. Aversion to risk does not imply aversion to dependence. Indeed the potential to plunge may be larger for risk-averse investors than for risk-neutral investors.

Keywords: portfolio choice, diversification, risk aversion, real options, petroleum exploration, information spillovers

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1. Introduction

Consider an investor who holds the right to explore N petroleum prospects. Exploration is risky. Probability of success on the i^{th} prospect is denoted p_i, and the value of a success is V_i. We assume the cost of exploration, C, to be the same for each prospect; and without further loss of generality set C=1. Thus, the expected value of the i^{th} prospect is then:

$$E_i = p_i V_i - 1.$$

The risk and return of this portfolio, and therefore its value to the investor, depends on the expected values of its components, but also on the investor's risk tolerance and the extent to which the individual exploration outcomes are interrelated. In this paper we assume the prospects are interrelated via positive dependence, and that the investor's preferences can be represented by a mean-variance utility function, $U(\cdot)$.

By positive dependence, we mean that the probability of success on any one prospect is directly related to the outcome of exploration on the others. If $S_i = 0, 1$ denotes failure or success on the i^{th} prospect, then the outcome of an exploration sequence can be represented by the random vector $S = (S_1, S_2, \ldots, S_N)$, with joint probability function given by $f(S) = f(S_1, S_2, \ldots, S_N)$. We further assume the N prospects are exchangeable (i.e., statistically indistinguishable) which means that f(S) is symmetric in its arguments.[1] This allows us to drop subscripts and write $p_i = p_j = p$ and $V_i = V_j = V$, for all i and j.

[1] In Smith and Thompson (2004), we examine some implications for sequential investment strategies of heterogeneity among the N prospects.

We also assume—and this is critical—that the prospects can be exploited sequentially; the outcome of the first prospect can be observed before investing in the second, etc. The investor therefore holds a set of N options, each of which corresponds to the decision whether or not to explore a given prospect. Positive dependence creates information spillovers, and the decision to exercise each option is informed by the outcomes of options that have been exercised previously.

We assume that each prospect would be explored on its own merits, if not part of a portfolio. That is, if there were no information spillovers, all N prospects would be explored. In the case of risk neutrality, this simply means that the expected value of each prospect is positive—they are all "in the money." Given the existence of information spillovers, a passive (but not unprofitable) strategy would therefore be to explore all N prospects, regardless of intervening exploration outcomes. We represent the monetary return to the passive strategy by the random variable $\Pi°$, with mean value $E[\Pi°] = N(pV-1) \geq 0$. An active strategy, in contrast, would take stock of intervening exploration successes and failures, update probabilities accordingly, and terminate the sequence when the expected utility of continuing to explore becomes negative. We represent the monetary return to the active strategy by the random variable Π^* with mean $E[\Pi^*]$. It then follows that $E[U(\Pi°)] \leq E[U(\Pi^*)]$.

We will show that if positive dependence is strong enough, the preceding inequality is strict, $E[U(\Pi°)] < E[U(\Pi^*)]$; i.e., active management commands a premium. However, our primary purpose is not to demonstrate the superiority of active management, but to examine the impact on portfolio value of the degree of dependence among prospects. Since part of management's job is to identify prospects and assemble the portfolio, and since many prospects are available at any given time—some interdependent, others not—the degree of dependence among prospects included in the portfolio represents a choice that is part of the utility maximization process.[2]

We will also show, under a broad range of assumptions regarding the degree of risk inherent in exploration, and regardless of the investor's degree of risk aversion, that the agent would choose to assemble a portfolio of dependent prospects. Relative to a comparable portfolio of independent (i.e., geologically diversified) prospects, a portfolio of dependent prospects has higher expected utility and therefore higher value. Moreover, we find that strong incentives exist for "plunging" behavior; i.e., making portfolio selections that *maximize* the degree of dependence among prospects.

Our findings might appear to defy the conventional wisdom that "dependencies increase the exploration risk," but in fact the two are entirely consistent.[3] Increasing the degree of dependence, while holding constant the marginal probability of success, creates a mean-preserving spread in the distribution of exploration outcomes. Dependence causes good or bad outcomes to cluster together, which creates volatility. The variance of the total number of successes rises but the mean remains constant—at least if the passive strategy is employed. By using information spillovers to truncate ill-advised exploration investments, active management is able to transform the extra volatility created by dependence into added portfolio value.

This points to the central question of our research: If we fix N, V, and p (which ensures that the intrinsic value of the portfolio is held constant), how much dependence is

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[2] Higher dependence is obtained by assembling prospects that are more closely related in geological terms; lower (or zero) dependence is obtained by assembling prospects that are geologically unrelated.
[3] The quotation is from Delfiner (2000), page 5. The argument that dependence increases exploration risk has also been set forth by Murtha (1996, p. 41-42) and Erdogan et. al. (2001, p. 3).

"optimal," in the sense of maximizing an investor's expected utility? Under what conditions would an investor prefer to diversify holdings and thereby minimize (or eliminate) positive dependence? Under what conditions would it be better to concentrate holdings in related assets and thereby increase (or maximize) dependence? And, finally, to what extent should risk-neutral and risk-averse agents be expected to behave differently in this regard?

2. Related Literature

Our work relates to several strands of previous research. Starting with Peterson (1975), Stiglitz (1975), and Gilbert (1979, 1981), several important implications of information externalities in private exploration have already been examined.[4] These earlier studies focused primarily on questions of economic efficiency and identified potential distortions created by information spillovers. They demonstrated (from the social point of view) that either too much or too little exploration could result, depending on how much of the information gleaned from exploration conducted by one party spills over to benefit other owners of property located in the vicinity.[5] Grenadier (1999) took this idea further via a model that applies to oil exploration (as well as other competitive settings) in which proprietary information is revealed indirectly by one party's investment decisions. A similar idea, where private information is likewise conveyed via investment decisions, was developed by Thijssen, Huisman, and Kort (2001). In both of those papers, the research focus remains on the welfare implications of potential

[4] Allais's (1957) pioneering work on the economics of mineral exploration in the Sahara Desert had already dealt with the problem of modeling exploration outcomes on adjoining tracts; but by defining the tracts to be sufficiently large, he was able to reasonably assume that the exploration outcomes on adjacent tracts would be independent. In that instance, there would be no spillovers.

[5] In addition to the efficiency effects of what we may call "local information externalities", Stiglitz (1975) and Gilbert (1978,1979, 1981) explore the social value of global exploration information pertaining to the total remaining stock of a depletable resource.

distortions caused by the externality. In contrast, we examine the impact of information spillovers and risk aversion on the composition of privately assembled asset portfolios.

Other papers have examined certain "portfolio" aspects of capital budgeting and project selection, especially in the sphere of research and development. Until relatively recently, these consisted mostly of attempts to produce an efficient frontier in the manner of Markowitz, by which is indicated the combination of projects that would minimize the variance of outcomes subject to a constraint on total expected return. If the separate research projects are deemed to be independent, this approach is straightforward, but then the impact of information spillovers has been omitted. Galligan (1991) and Erdogan (2001) exemplify this branch of research, in which possible interdependencies among projects under consideration are simply neglected. Other studies have employed linear programming and integer programming approaches to select projects, subject to resource constraints, that maximize total expected return without regard for the variance.[6] These methods assume implicitly that the projects under consideration are additive with no substantial interactions. Chien (2002), on the other hand, cited project interactions as a primary cause of the difference between the preference for a portfolio of R&D projects as a whole and the preference for the individual projects, and described four types of project interactions that might be taken into account.[7]

Within the literature on real options, some types of interactions among multiple options have been studied intensively. Roberts and Weitzman (1981) considered the

[6] Gear, Lockett, and Pearson (1971) review and summarize some representative models of this type.
[7] There exists an entirely different approach to portfolio decisions, typified by Linton, Walsh, and Morabito (2002), that combines objective and subjective multi-criteria rules by which separate projects may be ranked. Although these methods may be ideal for comparison of projects that have many different non-cost and non-numeric aspects to consider, they are not well suited for the analysis of quantitative investment problems where profit is the clear objective.

value of a set of investment options to extend and refine a given R&D project and formulated an optimal stopping rule for investment. Where exercise of one option is a prerequisite for the next, as in Roberts and Weitzman's model, interdependence between the different stages of the project is direct and the method of compound options can be used to value the project as a whole. More generally, Trigeorgis (1993) and Kulatilaka (1995) have demonstrated that when multiple options are written on the same underlying asset, the potential for interference (substitutability) or reinforcement (complimentarity) may cause the value of the collection of options to either exceed or fall short of the sum of their stand-alone values. Exercising an option to abandon a given project, for example, forecloses the option to expand. Additivity of option values is not assured. Koussis, Martzoukos, and Trigeorgis (2003) have recently formulated a more comprehensive model that allows management to take multiple learning and value-enhancing actions prior to implementing a given project. Again, these actions represent options that are written on a single underlying asset and therefore tend to interact in ways that destroy additivity. The authors argue that the value of the collection of options will generally tend to be less than the sum of their separate values, but the converse may sometimes be true.

Several papers have examined interactions among multiple options written on distinct and separate underlying assets. Keeney (1987), for example, investigated the impact of positive dependence regarding the performance of alternative sites on the value of a portfolio of locations being studied for possible use as a nuclear waste repository. Keeney demonstrated that dependence among sites, plus the ability to process sites in sequence, created an option to truncate investment, and the value of this option contributed significantly to the value to the portfolio. Also like us, Keeney argued that the source of interdependence stemmed (at least in part) from shared geological characteristics. Kester (1993) presents and solves a numerical illustration in which a firm must consider whether or not to launch each of several new products. If the success or failure of each new product would foretell the probability of success or failure of the others, then the optimal sequence of product introductions must take into account the impact of these information spillovers. Childs, Ott, and Triantis (1998) examined the effect of interrelationships between two projects that may be carried out either sequentially or in parallel, and showed that the optimal investment program (and combined value) is highly sensitive to the type of interdependence that links the two projects. Brosch (2001) emphasized the real-world prevalence of firms that hold interrelated options on multiple underlying assets and established by example (involving two projects) that the type of "inter-project compoundness" that exists in such cases may lead to a considerable deviation from value additivity.

These last four papers perhaps come closest to our work, at least in terms of focusing on interactions among multiple options that have been written on distinct and separate underlying assets. In this sense, the problem we examine involves a true portfolio of distinct assets, not simply a collection of options that all impact the same underlying asset. With the exception of Keeney, each of these earlier papers took the composition of the portfolio as given, however, and proceeded to analyze how it could be optimally managed. Like Keeney, we inquire as to management's initial incentive to assemble one type of portfolio rather than another—taking into account the impact of interdependence among assets, the value of real options thereby created, and the degree of risk aversion on the part of the decision-maker.

3. Partially Shared Risks: A Model of Multivariate Dependence

Many distinct notions of multivariate positive dependence have been advanced in the statistical literature, based on different measures of the tendency of random variables to assume concordant values.[8] See Colangelo, Scarsini, and Shaked (2005) for an overview and comparison of alternative concepts. For our purpose, it seems appropriate to treat information spillovers according to the model of "partially shared" risks, which is a probability structure that divides exploration risk into two parts: one that is unique to each prospect and another that is common to all prospects located within the same geological trend or "play." This treatment is common in the petroleum engineering literature and our use follows the standard assumptions.[9] Indeed, White (1992) defines the concept of an exploration play as a group of prospects that share common elements of risk.

Let the random vector $\{Z_0, Z_1, \ldots, Z_N\}$ represent a set of geological factors that collectively determine exploratory success. $Z_i = 1$ denotes the presence of a necessary factor and $Z_i = 0$ denotes its absence. We assume these geological factors are distributed independently, with:

$$p(Z_i=1) = q_i; \quad \text{for } i = 0, 1, \ldots, N;$$

Successful exploration of the i^{th} prospect requires the presence of factor Z_0 (the common factor) and factor Z_i (the factor unique to the i^{th} prospect). The common factor could represent, for example, the original depositional event that created petroliferous

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[8] Examples include positive association, affiliation, positive quadrant dependence, right-tail increasing in sequence, etc.

[9] See, for example, Megill (1979), Stabell (2000), and Wang et. al. (2000).

sediments, whereas the unique factor could represent the existence of a migratory path to the i^{th} prospect and the existence of a trapping structure sufficient to form a reservoir there. This allows us to write: $S_i = Z_0 \times Z_i$; for i = 1, ..., N.[10] Since the factors are assumed to be independent, the marginal probability of success on the i^{th} prospect is given by:

$$p_i = p(S_i{=}1) = q_0 q_i. \qquad (1)$$

Since prospects are assumed to be symmetric, we can suppress the subscript on the prospect-specific risk factor and write $q_i = q$ and thus $p_i = p$, for i = 1, …, N. Note that q is an upper bound for p.

It will be convenient to use "bar notation" to represent conditional probabilities. Thus:

$$p_{i|j} = \Pr(S_i{=}1|S_j{=}1),$$

$$= p(F_0 = 1 \cap F_i = 1 \cap F_j = 1) / p(F_0 = 1 \cap F_j = 1) = q_0 q^2 / q_0 q = q. \qquad (2)$$

Similarly:

$$p_{i|\bar{j}} = \Pr(S_i{=}1|S_j{=}0) = \frac{p(1-q)}{1-p}, \qquad (3)$$

where the last equality follows from the identity: $p = p \times p_{i|j} + (1-p) \times p_{i|\bar{j}}$.

The covariance between any two exploration outcomes is given by p(q-p), and the simple correlation coefficient between any two outcomes takes the form:

[10] Although we focus on petroleum exploration, the partially-shared risk structure is arguably relevant to a broader range of multi-prospect problems. Consider, for example, the problem of introducing a new product in a set of test markets. If we suppose that success in any one market requires validity of the underlying value proposition (presumed common to all markets) plus effective execution of the test program in that particular locale, then the same type of information spillovers would emanate from a series of test marketing results as from a series of exploratory wells. Kester's (1993) example of new product introductions appears to fit this mold. Spillovers of underwriting information in the IPO model of Benveniste et. al. (2003) represent another example of a shared risk that is partially resolved by the first project.

$$r = \frac{q - p}{1 - p}\,. \tag{4}$$

Positive dependence implies q > p, therefore all outcomes are positively correlated. As q varies between p (the marginal probability) and 1, the correlation coefficient varies between zero and unity. Either q or r may be used to indicate the degree of dependence among prospects. Depending on the context, it will sometimes be more convenient to work with one measure of dependence than the other, but any result can easily be restated in terms of the other parameter.

We will have occasion to use two additional properties of the shared-risk information structure (proofs are provided in the appendix):

> (P1) Only one exploratory success is sufficient to confirm the presence of the common factor; Thus, once an exploratory success has occurred, the conditional probability of success on remaining prospects rises to q, and remains there regardless of ensuing outcomes.
>
> (P2) A string of n consecutive failures reduces the conditional probability of success on remaining prospects by at least as much as any other string of n or fewer outcomes. Nothing is more discouraging than a streak of consecutive failures, except an even longer streak of consecutive failures.

4. The Risk-Neutral Case

It follows immediately from Property 1 that the agent would exercise the option to truncate exploration only after experiencing a sequence of some n consecutive failures (and no successes). To reckon the value of the portfolio, then, we must examine the implications of such a stopping rule. For n = 1, …, N-1, we let the random variable $\Pi^{[n]}$ represent the realized value of the portfolio given that exploration will be truncated after a sequence of n failures in n trials. Relative to the passive policy of drilling all prospects, the stopping rule trims branches and outcomes of the investment decision tree. By taking directly into account those branches that would be trimmed under the given stopping rule, we can express the expected value of the portfolio, subject to the given stopping rule, as follows:

$$
\begin{aligned}
E\left[\Pi^{[n]}\right] &= E[\Pi^0] - p_{\bar{1},\ldots,\bar{n}} \times \sum_{j=n+1}^{N} \left(p_{j|\bar{1},\ldots,\bar{n}} V - 1\right) \\
&= E[\Pi^0] - p_{\bar{1},\ldots,\bar{n},n+1} \times (N-n)V + p_{\bar{1},\ldots,\bar{n}}(N-n) \\
&= E[\Pi^0] - p(1-q)^n (N-n)V + p_{\bar{1},\ldots,\bar{n}}(N-n), \qquad (5)
\end{aligned}
$$

where $E[\Pi^0]$ represents the expected value under the passive policy of exploring all prospects, and where we have used symmetry to make the substitution $p_{n+1|\bar{1}\ldots\bar{n}} = p_{j|\bar{1}\ldots\bar{n}}$ for all $j \geq n+1$. The probability of no successes in n trials can be written as (see appendix):

$$
p_{\bar{1}\ldots\bar{n}} = 1 - \frac{p}{q}\left[1 - (1-q)^n\right] \; . \qquad (6)
$$

which is strictly increasing in q. It follows by inspection of (5) that $E[\Pi^{[n]}]$ is strictly increasing in q for fixed n = 1, …, N-1. The policy of truncating after n failures becomes more profitable as the degree of dependence rises.

Recall that for q = p (i.e., independent prospects), the investor would explore all N prospects, even if the first N-1 were unsuccessful. As q rises above p (which means the degree of dependence rises above zero), the value of information spillovers rises too, until at some point a threshold ($q = q^{OV} > p$) is reached, at which point the weight of N-1 previous failures would be just sufficient to dissuade the investor from exploring the N^th prospect. This threshold for invoking the option to truncate exploration (which we call the “option threshold”) is obtained as the solution to the following equation: $E[\Pi^{[N-1]}] = E[\Pi^0]$, which may be expressed using Eq. (5) as follows:

$$p_{N|\bar{1}\ldots\overline{N-1}} \quad = \quad \frac{1}{V}. \tag{7}$$

Note that at q = p, the LHS of (7) equals p, which is greater than 1/V (since pV > 1). And, at q = 1, the LHS equals 0, which is less than 1/V. Moreover, the LHS is strictly decreasing in q, which ensures that a unique solution exists for q^{OV}. To be clear, given q = q^{OV}, it would not be optimal to truncate after any fewer number of failures than N-1 since (by Property 2) $p_{k|\bar{1}\ldots\overline{k-1}} > p_{N|\bar{1}\ldots\overline{N-1}} = 1/V$ for all k < N.

The relationship between the option threshold and N is also of interest. Holding q fixed, the LHS of Eq. (7) is a decreasing function of N (by Property 2), thus q^{OV} must itself be a decreasing function of the number of prospects included in the portfolio. That means the special case of N=2 provides an upper bound on the option threshold for arbitrary N. Given N=2, Eq. (7) reduces to:

$$\frac{1-q^{OV}}{1-p}p = \frac{1}{V},$$

which implies:

$$q^{OV} = 1 - \frac{1}{pV} + \frac{1}{V}. \tag{8}$$

In terms of correlation, the option threshold can then be expressed by substituting from Equation (8) into (4):

$$r^{OV} \quad = \quad \frac{q^{OV} - p}{1-p} \quad = \quad \frac{pV-1}{pV}, \tag{9}$$

which is a particularly intuitive result since the option threshold in this case happens to correspond to the expected profit margin of the prospects under consideration (recall that the cost of exploration is taken to be 1). If prospects offer only a small return over the cost of exploration, then relatively little correlation among prospects is needed for a string of consecutive failures to condemn the last remaining prospect. Figure 1 gives exact values of the option threshold (i.e., the solution to Eq. 7) for a broad range of assumed profit margins and values of N.

Gathering results developed thus far establishes the following:

Proposition 1: for $N \geq 2$, fixed p, and $r \geq r^{OV}$, any increase in dependence among prospects increases the expected value of the portfolio.

Proof: Since the degree of dependence is assumed to exceed the option threshold, the expected value of the portfolio may be written as:

$$E[\Pi^*] = \max \{E[\Pi^{[1]}], \ldots, E[\Pi^{[N-1]}]\}.$$

We have shown already that each term of the set $\{E[\Pi^{[n]}]\}$ is strictly increasing in q. It follows immediately that $E[\Pi^*]$ is itself strictly increasing in q. QED

Discussion: Proposition 1 implies that risk-neutral investors should exhibit "plunging" behavior: once beyond the threshold, more dependence is preferred to less. As long as dependence is high enough to meet the option threshold, the value of the portfolio is maximized by selecting from available prospects those that are most highly correlated. For risk-neutral investors, then, the option threshold is a "plunging" threshold.

We turn to a second threshold that is of some importance. If dependence is high enough, the investor would walk away after failing on the very first trial. The "walk away threshold" (q^{WA}) is defined to be that level of dependence that would make the investor indifferent about exploring a second prospect after failing on the first. Thus, holding p, V, and N constant, q^{WA} is obtained as the root of the equation:

$$E[\Pi^{[2]}] = E[\Pi^{[1]}].$$

After substituting from Eq. (5), and rearranging terms, the condition defining q^{WA} simplifies to:

$$p_{2|\bar{1}} \quad = \quad \frac{1}{V + (N-2)(qV-1)}. \tag{10}$$

The LHS of this equation decreases linearly in q, per Eq. (3), whereas the RHS is decreasing and convex. Thus, at most two roots exist. Moreover, at q = p, the LHS equals p, which exceeds the RHS (since pV > 1), while at q = 1, the LHS equals 0, which is less than the RHS. It follows that a single root exists between q and 1, and q^{WA} is therefore uniquely defined. In addition, for fixed q, the RHS is decreasing in N, whereas the LHS is constant. Thus, q^{WA} is increasing in N. It takes more dependence to walk away on the basis of a single failure from a larger number of unexplored prospects. The case of N=2 provides a lower bound for q^{WA}. But, with only two prospects, by definition the two thresholds correspond: $q^{OV} = q^{WA}$. Thus, for the special case of N = 2, we are able to write (cf. Equation (9)):

$$r^{OV} \quad = \quad \frac{pV-1}{pV} \quad = \quad r^{WA};$$

and for the general case of N > 2:

$$r^{OV} \quad < \quad \frac{pV-1}{pV} \quad < \quad r^{WA}.$$

5. The Impact of Risk Aversion

Although dependence increases the volatility of exploration outcomes and "increases risk" in that sense, risk aversion on the part of the investor does not necessarily imply aversion to dependence. Indeed the tendency for risk averse investors to plunge into dependence can be even greater than for risk neutral investors. The question is whether the option to truncate exploration creates enough value to compensate the investor for the added risk that dependence brings. In some instances, this will depend on the investor's degree of aversion to risk and the answer may go either way. However, in other cases, the option to truncate actually *reduces* the dispersion of monetary returns (overcoming the increase in variance of exploration outcomes), in addition to increasing the mean, and in such cases risk-aversion would necessarily heighten an investor's preference for dependent prospects.

This tradeoff between adding risk to the portfolio via dependence and then truncating downside outcomes via information spillovers cannot be resolved unambiguously without knowing the values of underlying parameters (p, V, and N). Whether a risk-averse investor would prefer dependence at all, or perhaps to an even greater extent than would a risk-neutral investor, depends on the details of the problem. Despite that ambiguity, the impact of risk aversion and other background parameters on portfolio choice is systematic and can be described quite simply with reference to the special case of N = 2. Extensions for the case of N > 2 are presented in the Appendix.

<u>The Two-Prospect Case (N = 2)</u>

With only two prospects, and for given values of p and q, the monetary return to the passive strategy (all prospects being explored regardless) is denoted $\Pi^{\circ}(p,q)$, with probability distribution determined from the decision tree shown in the upper panel of Figure 2. Under the alternative policy of truncating exploration if the first prospect fails, the monetary return is denoted $\Pi^{[1]}(p,q)$, with distribution determined from the decision tree in the lower panel of Figure 2. Given that the investor would elect to truncate after the first failure, but otherwise irrespective of the investor's risk preference, we show that more dependence is preferred to less:

Proposition 2: For N=2, fixed p, and $r^b > r^a$;

$$\Pi^{[1]}(p,q^b) \succ_{sd} \Pi^{[1]}(p,q^a), \tag{11}$$

where $\succ_{sd}$ denotes first-order stochastic dominance.

Proof: Since we assume $r^b > r^a$, it follows that $q^b > q^a$. If we denote the cumulative distribution function of $\Pi^{[1]}(p,q)$ by $G^{[1]}(\cdot|p,q)$, it is then sufficient to show that $G^{[1]}(\cdot\ |p,q^b) \leq G^{[1]}(\cdot|p,q^a)$ for all q^a and q^b such that $p < q^a < q^b$. Since we are evaluating the truncated distributions, returns are determined according to the lower panel of Figure 2. There are three segments to the distribution function:

$G^{[1]}(-1\|p,q)$	=	$1-p$	which is invariant with respect to q
$G^{[1]}(V-2\|p,q)$	=	$1-pq$	which is decreasing in q
$G^{[1]}(2V-2\|p,q)$	=	1	which is invariant with respect to q.

Thus, $G^{[1]}(\cdot|p,q)$ is non-increasing in q. It follows that $G^{[1]}(\cdot|p,p^b) \leq G^{[1]}(\cdot|p,p^a)$. QED[11]

The investor's preference for higher dependence is due to the quality of information that spills over. If the second prospect is condemned after failing on the first, the investor saves the cost of exploration, which is 1; but also foregoes the (diminished) expected revenue that comes from exploring the second. The value of information generated by the first exploration attempt, is just the difference between these two quantities multiplied by the probability that a failure does occur: $(1-p)(1-p_{2|\bar{1}}V)$. The term $p_{2|\bar{1}}$ measures the probability of committing a Type I error: abandoning a good

[11] Proposition 2 generalizes easily to the case of N > 2. A proof of the general case is provided in the appendix.

prospect due to the occurrence of a "false negative." Reducing the probability of false negatives increases the value of information—which in turn increases the value of the portfolio. Using Eq. (3), the probability of a false negative can be expressed in terms of the correlation:

$$p_{2|\bar{1}} \quad = \quad p(1-r). \tag{12}$$

Thus, if the agent is able to assemble prospects with enough dependence to surpass the option threshold, then he would prefer that portfolio of dependent prospects to a comparable portfolio of independent prospects, and would take as much dependence as possible in order to enhance the quality of the information on which he acts.

We have previously characterized r^{OV}, the option threshold for a risk-neutral investor (see Eq. 9). We now let r^{RA} represent the option threshold of the *risk averse* investor; i.e., the degree of dependence just sufficient to render him indifferent about exploring the second prospect after failing on the first. Of course, the numeric value of r^{RA} will depend on the degree of risk aversion, and we will come to that. However, it follows from the results given so far that, compared to the alternative of independent prospects, any amount of dependence *below* r^{RA} is unambiguously bad. Regardless of the degree of risk aversion, the investor would not assemble a portfolio of prospects with $0 < r < r^{RA}$, at least not if it were possible to assemble a similar set of independent prospects instead. Below the option threshold, dependence inflates the variance, but not the mean.

Above the option threshold, more dependence is always preferred to less (see Proposition 2). Thus, for the N = 2 case, regardless of the degree of risk aversion, the investor will exhibit "plunging" behavior: either shunning correlation completely (by pursuing a geologically diversified set of prospects), or maximizing the degree of correlation (by pursuing prospects that are as highly dependent as the geology permits).

Risk-averse and risk-neutral agents are distinguished not by the plunging phenomenon itself, but by the threshold level of correlation that triggers this response. As we show next, the threshold of risk-averse agents may lie either above or below that of risk-neutral agents.

The risk-averse option threshold is derived by comparing financial returns under the alternative truncation policies. Under the passive policy, in which all prospects are explored regardless, the return has mean and variance given by:

$$E[\Pi^{\circ}(p,q)] = 2(pV-1) \qquad (13a)$$

$$Var[\Pi^{\circ}(p,q)] = 2pV^2(1-2p+q). \qquad (13b)$$

With p constant, the mean return is invariant with respect to q, but the variance increases linearly with q, and therefore also with r. With truncation after one failure, the mean and variance are both affected. The mean is:

$$E[\Pi^{[1]}(p,q)] = pV - 1 + p(qV-1), \qquad (14a)$$

which increases linearly with q, and therefore also with r. The variance is:

$$Var[\Pi^{[1]}(p,q)] = pq(3V^2-4V) + p(V^2-4V+3) + 1 - [pV-1+p(qV-1)]^2, \qquad (14b)$$

which may either rise or fall with q, depending on the parameter values. At the option threshold, the investor must be indifferent between the portfolio of independent prospects (Equations 13a and 13b evaluated at q=p), and the portfolio of dependent prospects (Equations 14a and 14b evaluated at $q = q^{RA}$). A comparison of these equations establishes that the option threshold for a risk-averse investor may lie either above or below that of the risk-neutral investor, depending on the characteristics of prospects:

Proposition 3: For N = 2 and fixed values of p and V:

$$r^{RA} \overset{>}{\underset{<}{=}} r^{OV} \quad \Leftrightarrow \quad pV - 1 \overset{>}{\underset{<}{=}} \frac{1}{2}\left(1 - \frac{1}{V}\right) \tag{15}$$

Proof: We first establish that r^{RA} is unique. By definition, at $r = r^{RA}$ the investor is indifferent between the portfolio of independent prospects and the portfolio of dependent prospects. But, regarding the portfolio of dependent prospects, higher values of r stochastically dominate lower values (by Proposition 2). Thus, indifference can be achieved only at one value of r.

Next, consider the value $r = r^{OV}$, which is also unique (as shown previously in Section 3). At r^{OV}, the two portfolios (of independent and dependent prospects) have, by definition, the same expected value. The difference in their variances is given by Δ:

$$\Delta \;=\; \mathrm{Var}\left[\Pi^{[1]}\left(p, q^{OV}\right)\right] - \mathrm{Var}\left[\Pi^{o}\left(p, p\right)\right] \;.$$

Thus, if Δ is greater than (less than) 0, the portfolio of independent prospects would have the same mean but lesser (greater) variance, and therefore would be preferred to (dominated by) the portfolio dependent prospects with $r = r^{OV}$. Since r^{RA} is defined as the point of indifference between these two portfolios, it follows immediately from Proposition 2 (stochastic dominance):

$$r^{RA} - r^{OV} \overset{>}{\underset{<}{=}} 0 \quad \Leftrightarrow \quad \Delta \overset{>}{\underset{<}{=}} 0. \tag{16}$$

Since the two portfolios share the same mean at $q = q^{OV}$, Δ is given by the difference in second moments measured around zero:

$$\Delta \;=\; pq^{OV}(2V-2)^2 + p(1-q^{OV})(V-2)^2 + (1-p)(-1)^2$$

$$-p^2(2V-2)^2 - 2p(1-p)(V-2)^2 - (1-p)^2(-2)^2; \qquad (17)$$

where $q^{OV} = 1 - \frac{1}{pV} + \frac{1}{V}$. After making this substitution and simplifying, we have:

$$\begin{aligned} \Delta &= (1-p)(p-V^{-1})(2V-2)^2 + (1-p)(V^{-1}-2p)(V-2)^2 - (1-p)(3-4p) \\ &\propto (p-V^{-1})(2V-2)^2 + (V^{-1}-2p)(V-2)^2 - (3-4p) \\ &= 2pV^2 - 3V + 1; \end{aligned}$$

which, in view of Eq. (16), leads directly to Eq. (15). QED

Discussion: With N = 2, either type of investor (risk-neutral or risk-averse) has an incentive to plunge into dependence if there is enough geological dependence among available prospects to satisfy the investor's threshold. Other things being equal, the lower the option threshold, the more likely it is that the investor would plunge since any given set of available prospects would be more likely to satisfy the lower threshold. Figure 3 illustrates the difference between risk-neutral and risk-averse investors in terms of the plunging threshold. The diagram partitions the parameter space into regions where r^{RA} is respectively greater than or less than r^{OV}—as determined by Eq. (15). Notice that the LHS of the criterion in (15) is just the intrinsic rate of return (pV-1) for a single prospect; whereas the RHS depends only on V. This accounts for the systematic pattern observed in Figure 3. The combination of a relatively high V (which implies large prospects) but low expected rate of return (which together with high V implies high dry hole risk) pushes r^{RA} below r^{OV}, and therefore makes the risk-averse investor more likely to plunge. Conversely, the combination of smaller prospects with lower dry-hole risk pushes r^{RA} above r^{OV}. Thus, one hypothesis for further empirical research would be that, as the size of remaining prospects has tended to fall over time (due to depletion) and the probability of success has tended to rise due to technical innovation, risk-neutral investors should have become relatively more likely to plunge than risk-averse investors.

6. Summary and Conclusions

Our most basic finding regarding a portfolio of exploration prospects is that the value of the whole is greater than the sum of its parts—at least if the prospects are sufficiently correlated to give value to the options that are inherent in sequential exploration. This super-additivity creates incentives for investors to plunge into correlated assets and resist diversification, regardless of the degree of risk aversion.

Nothing about the intuition behind our results is specific to the "shared risk" information structure we have employed. Although that model mimics (in a crude way) the geological source and pattern of dependence in the case of petroleum deposits, other forms of positive dependence would lead us in the same direction and towards the same types of conclusions. Any investor in assets that may be exploited sequentially faces a tradeoff between: (a) loading his portfolio with assets whose returns are correlated, which will impart a high variance to the total return, and (b) extracting value from the options that naturally arise due to the interdependence among assets. Loosely speaking , we can say that the value of the options increases with the strength of dependence among assets, so it should not come as a surprise that even risk-averse investors might have a preference for dependence.

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Figure 1a: Risk Neutral Option Thresholds

(assuming p = 0.15)

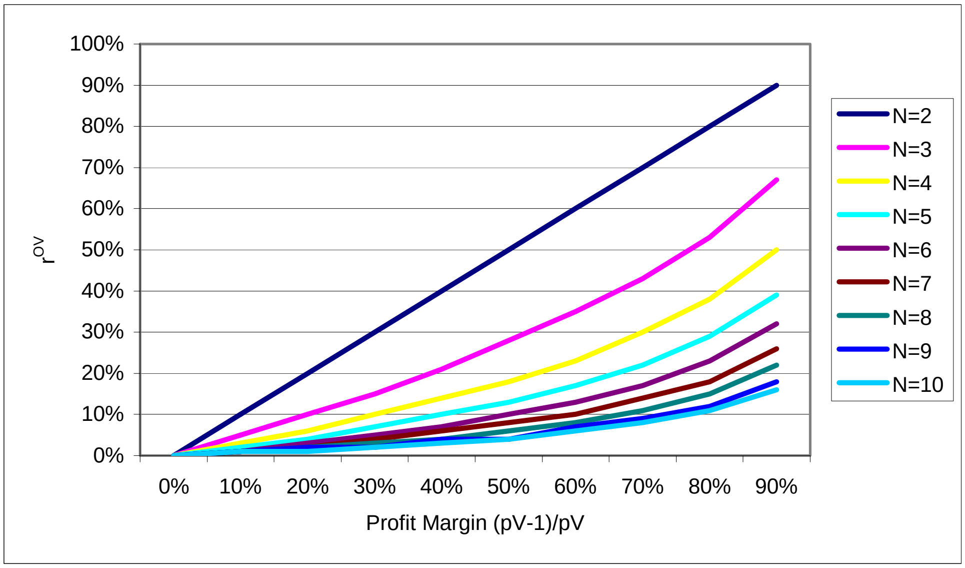


Figure 1b: Risk Neutral Option Thresholds

(assuming p = 0.50)

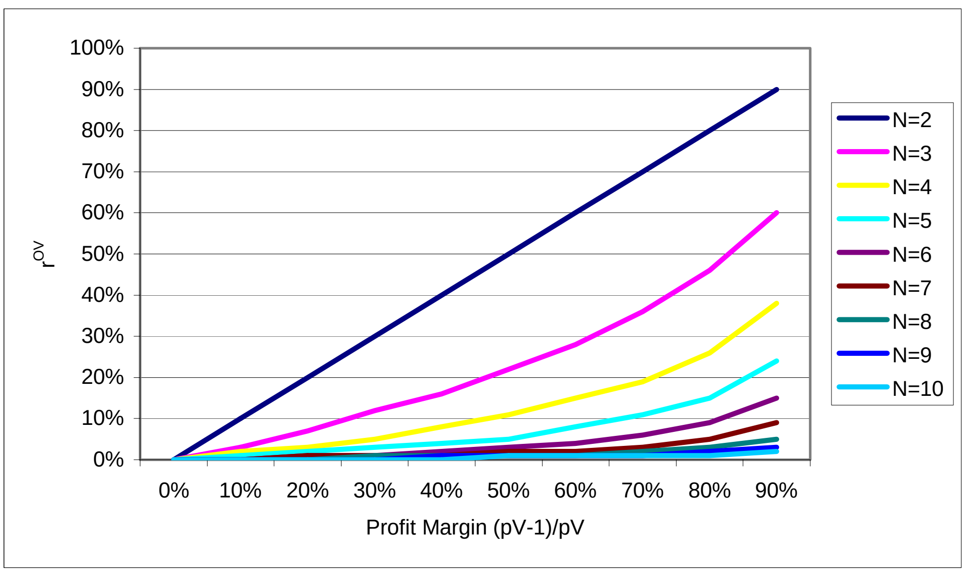
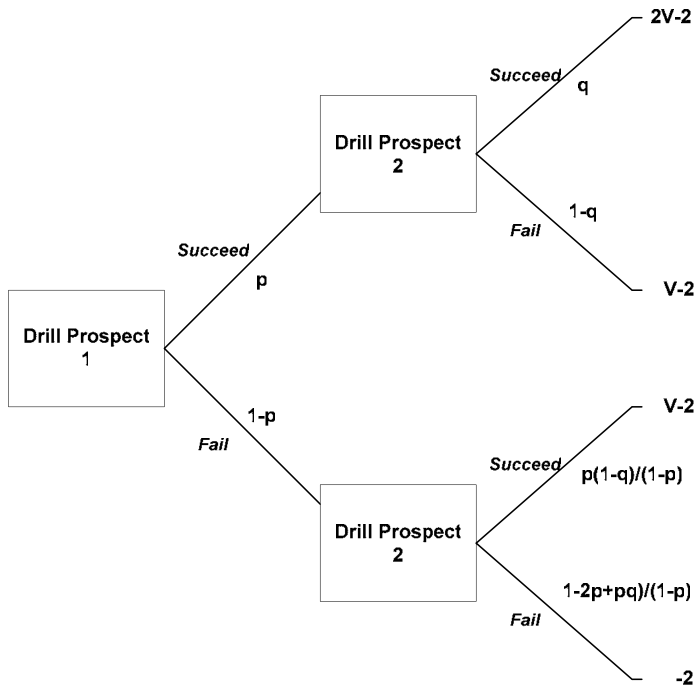


Figure 2: Exploration Decision Tree

a. The Naïve Exploration Program



b. The Truncated Exploration Program

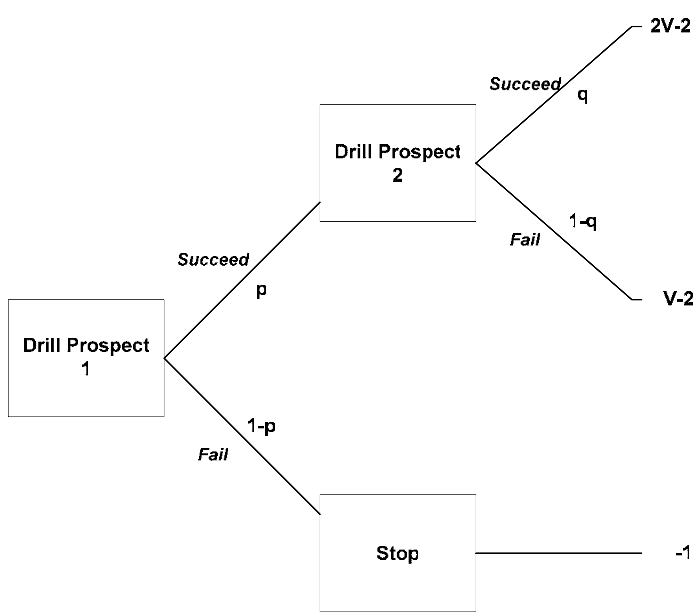
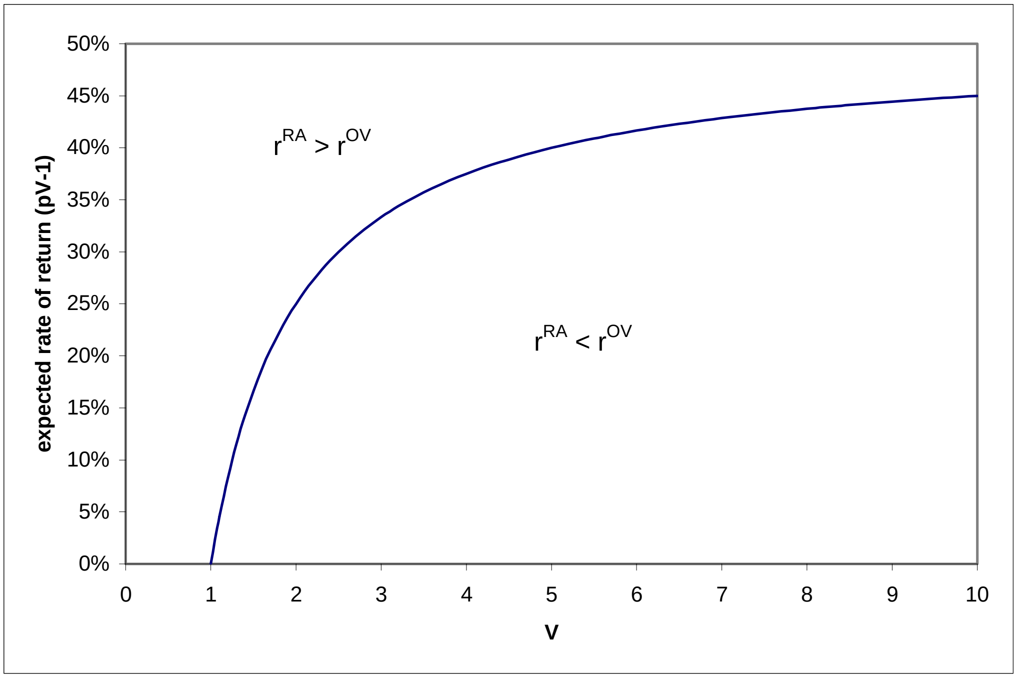


Figure 3: The Option Threshold:

Risk-Averse vs. Risk-Neutral Investors (N = 2)



APPENDIX

Property 1: Once an exploratory success has occurred, the conditional probability of success on remaining prospects rises to q and remains there regardless of ensuing outcomes.

Proof: Consider the probability of success on the n^{th} prospect, conditional on m successes and n-m-1 failures having already occurred, where 1 ≤ m ≤ n-1:

$$p_{n|1\ldots m;\overline{m+1}\ldots\overline{n-1}} = \Pr[S_n = 1 \mid S_1 = 1 \cap \ldots \cap S_m = 1 \cap S_{m+1} = 0 \cap \ldots \cap S_{n-1} = 0]. \qquad \text{(A1)}$$

Since the random variables are assumed to be exchangeable, the conditional probability is invariant with respect to the order of prior outcomes, so for notational convenience (and without loss of generality) we have assumed the successes occur first. The conditional probability would be the same for any permutation of these prior outcomes. Based on the independence of the underlying factors (Z_0, Z_1, …, Z_N), and the conditions for success on each prospect, Equation (A1) can be written as:

$$p_{n|1\ldots m;\overline{m+1}\ldots\overline{n-1}} = \frac{\Pr[Z_0 = 1 \cap Z_1 = 1 \cap \ldots \cap Z_m = 1 \cap Z_{m+1} = 0 \cap \ldots \cap Z_{n-1} = 0 \cap Z_n = 1]}{\Pr[Z_0 = 1 \cap Z_1 = 1 \cap \ldots \cap Z_m = 1 \cap Z_{m+1} = 0 \cap \ldots \cap Z_{n-1} = 0 = 1]}$$

$$= \frac{q_0 \times q_1 \times \ldots \times q_m \times (1 - q_{m+1}) \times \ldots \times (1 - q_{n-1}) \times q_n}{q_0 \times q_1 \times \ldots \times q_m \times (1 - q_{m+1}) \times \ldots \times (1 - q_{n-1})}$$

$$= \frac{q_0 \times q^{m+1} \times (1-q)^{n-m-1}}{q_0 \times q^m \times (1-q)^{n-m-1}} = q, \qquad \text{(A2)}$$

which is independent of m and n-m. QED

Property 2: A string of n consecutive failures reduces the conditional probability of success on remaining prospects by at least as much as any other string of n or fewer outcomes.

Proof: Since the conditional probability of success given any prior success is simply q (see Property 1), it is only necessary to examine the ratio of conditional probabilities given sequences of consecutive failures. For arbitrary k≥2, Bayes Theorem allows us to write:

$$p_{k|\bar{1}...\overline{k-1}} = \frac{\Pr[S_1 = 0 \cap ... \cap S_{k-1} = 0 \mid S_k = 1] \times \Pr[S_k = 1]}{\Pr[S_1 = 0 \cap ... \cap S_{k-1} = 0]}$$

$$= \frac{(1-q)^k \times q \times q_0}{\Pr[S_1 = 0 \cap ... \cap S_{k-1} = 0]}, \tag{A3}$$

where we have used Property 1 to simplify the numerator. Then, by repeating this operation for k+1, and taking the ratio of conditional probabilities, we have:

$$\frac{p_{k+1|\bar{1}...\bar{k}}}{p_{k|\bar{1}...\overline{k-1}}} = \frac{\Pr[S_1 = 0 \cap ... \cap S_{k-1} = 0] \times (1-q)^{k+1} \times q \times q_0}{\Pr[S_1 = 0 \cap ... \cap S_k = 0] \times (1-q)^k \times q \times q_0}$$

$$= \frac{1-q}{1-p_{k|\bar{1}...\overline{k-1}}}, \tag{A4}$$

which will be less than one if and only if: $p_{k|\bar{1}...\overline{k-1}} < q$. For k=2, Bayes Theorem implies:

$p_{2|\bar{1}} = \dfrac{p_{\bar{1}|2}\, p_2}{p_{\bar{1}}} = \dfrac{(1-q)p}{1-p} < q$, where the inequality follows from p < q. Thus,

$p_{3|\bar{1}\bar{2}} < p_{2|\bar{1}} < q$. Higher order comparisons can then be established by recursion. QED

The Probability of No Success in n Trials:

Obtaining no success (in n trials) is complementary to the event of obtaining one or more:

$$p_{\bar{1}...\bar{n}} = 1 - \sum_{j=1}^{n} \binom{n}{j} q_0 q^j (1-q)^{n-j} \tag{A5}$$

$$= 1 - q_0 \sum_{j=1}^{n} \binom{n}{j} q^j (1-q)^{n-j}$$

$$= 1 - q_0 \left[1 - \binom{n}{0} q^0 (1-q)^n \right]$$

$$= 1 - \frac{p}{q} \left[1 - (1-q)^n \right] \ . \qquad \text{(A6)}$$

Proposition 2: (Generalization) For $N \geq 2$, fixed p, and $r^b > r^a$:

$$\Pi^{[1]}(p,q^b) \underset{sd}{\succ} \Pi^{[1]}(p,q^a),$$

where $\underset{sd}{\succ}$ denotes first-order stochastic dominance.

Proof: Since we assume $r^b > r^a$, it follows that $q^b > q^a$. If we denote the cumulative distribution function of $\Pi^{[1]}(p,q)$ by $G^{[1]}(\cdot|p,q)$, it is then sufficient to show that $G^{[1]}(\cdot|p,q^b) \leq G^{[1]}(\cdot|p,q^a)$ for all q^a and q^b such that $q^a < q^b$. $G^{[1]}(\cdot|p,q)$ describes the distribution of returns if exploration is truncated after failing on the first prospect. The probability of this outcome is 1–p, and it generates total payoff equal to –1. If the first prospect is successful then all prospects will be explored, and if there are n successes in total (out of N prospects) the total payoff will amount to nV-N. Given success on the first prospect, the probability of success on each subsequent prospect is simply q. This allows us to write down the entire probability distribution of outcomes, where $g(\Pi)$ represents the probability of outcome Π:

Π	$g(\Pi)$
-1	1-p
V-N	$p \times \binom{N-1}{0} q^0 (1-q)^{N-1}$

2V-N	$p \times \binom{N-1}{1} q^{1} (1-q)^{N-2}$
kV-N	$p \times \binom{N-1}{k-1} q^{k-1} (1-q)^{N-k}$
.	.
.	.
.	.
NV-N	$p \times \binom{N-1}{N-1} q^{N-1} (1-q)^{0}$

Each probability after the first is equal to the probability of success on the first prospect multiplied by the binomial probability of k-1 successes among the following N-1 prospects. For k = 1, …, N, the cumulative distribution function can therefore be written as:

$G(kV\text{-}N|p,q) = (1-p) + p \times B[k-1,N \mid q]$, where $B[\cdot|q]$ represents the cumulative binomial distribution. Since the cumulative binomial distribution is known to exhibit first-degree stochastic dominance in q, then it must also be true that $G(\cdot|p,q)$ exhibits first-degree stochastic dominance in q. QED

Proposition 4: Given N > 2 and fixed p; and if r^{RA} is assumed to be unique, then:

$$r^{RA} - r^{OV} \gtreqless 0 \quad \Leftrightarrow \quad \Delta \gtreqless 0, \tag{A7}$$

where all terms are as defined for the case of N = 2.

Proof: Since it is assumed that r^{RA} is unique, then q^{RA} must also be unique. Consider the values r^{OV} and q^{OV}, which we showed earlier to be unique for all N. Given q^{OV}, by definition the two portfolios (of independent and dependent prospects, respectively) have the same expected value. The difference in their variances is given by Δ:

$$\Delta \quad = \quad \mathrm{Var}\left[\Pi^{[1]}(p, q^{OV})\right] \;-\; \mathrm{Var}\left[\Pi^{o}(p, p)\right] \;.$$

Thus, if Δ is greater than (less than) 0, the portfolio of independent prospects would have the same mean but smaller (greater) variance, and therefore would be preferred to (dominated by) the portfolio of dependent prospects with $q = q^{OV}$. But any investor would prefer $\Pi^0(p,p)$ to $\Pi^{[1]}(p,p)$, and also prefer $\Pi^{[1]}(p,1)$ to $\Pi^0(p,p)$. Thus, if there is a single value q that renders the investor indifferent between $\Pi^0(p,p)$ and $\Pi^{[1]}(p,q)$, then it must be the case that if Δ is greater than (less than) 0, then q^{RA} is greater than (less than) q^{OV}. QED

The principal distinction from the N=2 case is the possibility that, depending on the shape of the utility function, r^{RA} may not be unique, in which case we offer Proposition 5, below.[12] Of course, with N > 2, the partition of the parameter space induced by the condition Δ = 0 generally deviates from that set forth in Eq. (15). Nonetheless, Equation (A7) provides a necessary and sufficient condition for the option threshold of a risk-averse investor to fall below the risk-neutral threshold. We emphasize that Δ depends only on p and V. Therefore, whether r^{RA} lies above or below r^{OV} is determined not by the degree of risk aversion, but only by the fundamental factors (p and V) that determine the intrinsic value of the prospects.

For problems where the risk-averse threshold is not unique, we will define r^{RA} to be the *least* degree of dependence that leaves the risk-averse investor indifferent between

[12] As we showed earlier, the risk-neutral threshold (r^{OV}) is unique for all N.

dependent and independent prospects. I.e., if the prospects were any less correlated, the investor would not truncate exploration even after N-1 consecutive failures. Given this interpretation, we offer a sufficient (not necessary) condition for $r^{RA} < r^{OV}$ (i.e., a sufficient condition for risk-averse investors to have a greater propensity to plunge):

Proposition 5: Given $N > 2$ and fixed p; and if r^{RA} is understood to represent the least degree of dependence that renders the risk-averse investor indifferent between dependent and independent prospects, then:

$$\Delta < 0 \quad \Rightarrow \quad r^{RA} < r^{OV} \quad . \tag{A8}$$

Proof: The proof follows the same lines as for Proposition 4. At q^{OV} the two portfolios by definition have the same expected value. The difference in their variances is Δ:

$$\Delta = \mathrm{Var}\left[\Pi^{[1]}(p, q^{OV})\right] - \mathrm{Var}\left[\Pi^{o}(p, p)\right] \; .$$

Thus, if Δ is less than 0, the portfolio of dependent prospects with $q=q^{OV}$ would have the same mean but smaller variance, and therefore would be preferred to the portfolio of independent prospects. But any investor would prefer $\Pi^0(p,p)$ to $\Pi^{[1]}(p,p)$. Thus, the least value of q that renders the investor indifferent between $\Pi^0(p,p)$ and $\Pi^{[1]}(p,q)$, must lie between p and q^{OV}. QED

Finally, it is worth mentioning that, with $N > 2$, a risk-averse investor's option threshold does not necessarily correspond to his plunging threshold. Whereas the option threshold (r^{RA}) represents the level of dependence below which the investor would prefer a portfolio of independent prospects, it does not follow that all portfolios with greater dependence than r^{RA} would necessarily be preferred to r^{RA}. Compared to the case of $N = 2$, the difference is that whereas $\Pi^{[1]}$ exhibits stochastic dominance in q, $\Pi^{[N-1]}$ does not.

It is the latter that determines the option threshold (indifference regarding the N^{th} prospect after N-1 failures), but in the case of N = 2, the two coincide. Thus, with N = 2, the preference for dependence is increasing beyond the option threshold, which provides the incentive to plunge.

With N > 2, the incentive for risk-averse investors to plunge still exists, but with a potentially higher threshold. Call this plunging theshold r^P. To demonstrate that $r^P < 1$, consider the following. Given r = 1, no investor would continue beyond a first failure, which implies: $E[U(\Pi^{[n]}(p,1))] < E[U(\Pi^{[1]}(p,1))]$ for all n > 1. By the continuity of the utility function in q, it follows that there exists an interval (1-ε,1) for which $E[U(\Pi^{[n]}(p,q)] \leq E[U(\Pi^{[1]}(p,q))]$ for all n > 1 and $q \in (1-\varepsilon,1)$. Moreover, the value r = 1 represents perfect information, which any investor would prefer to r = 0. Thus, $E[U(\Pi^{0}(p,p)] < E[U(\Pi^{[1]}(p,1))]$. By the continuity of the utility function in q, it follows that there exists an interval (1-δ,1) for which $E[U(\Pi^{0}(p,p)] \leq E[U(\Pi^{[1]}(p,q))]$ for all $q \in (1-\delta,1)$. If we let $r^P = \max(1-\delta,1-\varepsilon)$, it then follows that:

Proposition 6: For fixed N > 2, fixed p, and $r^P < q^a < q^b$:

$$E[U(\Pi^*(p,p))] < E[U(\Pi^*(p,q^a))] < E[U(\Pi^*(p,q^b))] \qquad \text{(A9)}$$

Proof: We have established already, for all $q > r^P$, that $E[U(\Pi^*(p,q))] = E[U(\Pi^{[1]}(p,q))]$, and that $E[U(\Pi^{0}(p,p))] = E[U(\Pi^*(p,p))] < E[U(\Pi^{[1]}(p,q))]$. We have also shown that for given N > 2 and fixed p, $\Pi^{[1]}(p,q)$ exhibits first-order stochastic dominance in q. Equation (A9) then follows directly. QED.