Hochschild homology/cohomology of preprojective algebras of ADET quivers

by

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Abstract

Preprojective algebras Π_Q of quivers Q were introduced by Gelfand and Ponomarev in 1979 in order to provide a model for quiver representations (in the special case of finite Dynkin quivers). They showed that in the Dynkin case, the preprojective algebra decomposes as the direct sum of all indecomposable representations of the quiver with multiplicity 1. Since then, preprojective algebras have found many other important applications, see e.g. to Kleinian singularities. In this thesis, I computed the Hochschild homology/cohomology of Π_Q over $\mathbb{C}$ for quivers of type ADET, together with the cup product, and more generally, the calculus structure. It turns out that the Hochschild cohomology also has a Batalin-Vilkovisky structure. I also computed the calculus structure for the centrally extended preprojective algebra, introduced by P. Etingof and E. Rains.

Thesis Supervisor: Pavel Etingof
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Chapter 1

Introduction

Let Q be a finite quiver with vertex set I, and let us write $a \in Q$ to say that a is an arrow in Q. Let $P = \mathbb{C}\bar{Q}$ be the path algebra of the double $\bar{Q}$ of the quiver Q (which is obtained from Q by adding a reverse arrow a^* for any arrow $a \in Q$). We define the *preprojective algebra* Π_Q to be the quotient $\Pi_Q = P/(\sum_{a\in Q}[a, a^*])$. Let e_i, $i \in I$ be the trivial path, starting and ending at the vertex i. We define the ring $R = \bigoplus_{i\in I} \mathbb{C}e_i$. Then Π_Q is naturally an R-bimodule.

Preprojective algebras of quivers were introduced by Gelfand and Ponomarev in 1979 in order to provide a model for quiver representations (in the special case of finite Dynkin quivers). They showed that in the Dynkin case, the preprojective algebra decomposes as the direct sum of all indecomposable representations of the quiver with multiplicity 1. Since then, preprojective algebras have found many other important applications, see e.g. to Kleinian singularities [3].

Ironically, it is exactly in the case of finite Dynkin quivers, originally considered by Gelfand and Ponomarev, that preprojective algebras fail to have certain good properties enjoyed by the preprojective algebras of other connected quivers. In the non-Dynkin case, Π_Q is Koszul and has cohomological dimension 2. The situation is completely different in the case of Dynkin quivers. The preprojective algebras of these quivers are only almost Koszul and cohomology groups $HH^i(\Pi_Q) \neq 0$ for infinitely many i.

As a result of the Schofield resolution [22], the Hochschild cohomology of Π_Q is periodic with period 6. The Hochschild cohomology ring was computed in [12] for quivers of type A. In this thesis, we do the computations for the quivers of type D and E over a field of characteristic zero which yields the complete description of the Hochschild cohomology ring of any quiver (over a field of characteristic zero).

We compute the Hochschild cohomology and homology structure in Chapter 2. For the computation of the additive structure, together with the natural grading (all arrows have degree 1), we use the periodic Schofield resolution (with period 6) and consider the corresponding complex computing Hochschild homology. Using this complex, we find the possible range of degrees in which each particular Hochschild homology space can sit. Then we use this information, as well as the Connes complex for cyclic homology and the formula for the Euler characteristic of cyclic homology to find the exact dimensions of the homogeneous components of the homology groups. Then we show that the same computation actually yields the Hochschild cohomology spaces as well. This work generalizes the results from [12].

The method to compute the cup product is the same one as in [12]: via the isomorphism $HH^i(\Pi_Q) \equiv \underline{Hom}(\Omega^i\Pi_Q, \Pi_Q)$ (where for an Π_Q-bimodule M we write ΩM for the kernel of its projective cover) we identify elements in $HH^i(\Pi_Q)$ with equivalence classes of maps $\Omega^i(\Pi_Q) \to \Pi_Q$. For $[f] \in HH^i(\Pi_Q)$ and $[g] \in HH^j(\Pi_Q)$, the product is $[f][g] := [f \circ \Omega^i g]$ in $HH^{i+j}(\Pi_Q)$. All products $HH^i(\Pi_Q) \times HH^j(\Pi_Q) \to HH^{i+j}(\Pi_Q)$ for $0 \leq i \leq j \leq 5$ are computed. The remaining ones follow from the perodicity of the Schofield resolution and the graded commutativity of the multiplication.

The Hochschild cohomology ring of any associative algebra, together with the Hochschild homology, forms a structure of calculus. This was proved in [6]. In Chapter 3, we compute the calculus structure for the preprojective algebras of Dynkin quivers over a field of characteristic zero, using the Batalin-Vilkovisky structure of the Hochschild cohomology. Together with the results of [2], where the Batalin-Vilkovisky structure is computed for non-ADE quivers (and the calculus can be easily computed from that), this work gives us a complete description of the calculus for any quiver. First, we compute the Connes differential on Hochschild homology by using the Cartan identity. Since it turns out this differential makes the Hochschild cohomology ring a Batalin-Vilkovisky-algebra, this gives us an easy way to compute the Gerstenhaber bracket and the contraction map. Then we use the Cartan identity to compute the Lie derivative.

Another bad property of preprojective algebras in the case of finite of Dynkin quivers is that their deformed versions are not flat. Motivated by this, the paper [10] introduces central extensions of preprojective algebras of finite Dynkin quivers, and shows that they have better properties, in particular their deformed versions are flat. The following paper [9] computes the center Z and the trace space $A/[A, A]$ for the deformed preprojective algebra A; the answer turns out to be related to the structure of the maximal nilpotent subalgebra of the simple Lie algebra attached to the quiver.

In Chapter 4, we generalize the results of [9] by calculating the additive structure of the Hochschild homology and cohomology of Π_Q and the cyclic homology of Π_Q, and to describe the universal deformation of Π_Q. Namely, we show that the (co)homology is periodic with period 4, and compute the first four (co)homology groups in each case.

Quivers of type T are introduced in the paper [20]. It turns out that preprojectve algebras of T-quivers enjoy similar properties as in the ADE case, for example their Hilbert series have the form $h(t) = (1 + Pt^h)(1 - Ct + t^2)^{-1}$, where h is the Coxeter number and P the permutation matrix corresponding to some involution of the vertex set I. In Chapter 5, we compute the calculus structure of the preprojective algebra, together with the Hochschild cohomology and homology structure. It turns out that in the T-case we have a projective resolution of the preprojective algebra which is very similar to the Schofield resolution in the ADE-case which is also periodic with period 6. And the Hochschild cohomology structure, together with its cup product, for quivers of type T_n is very similar to the one for type A_{2n}. But unlike the ADE-case, where $HH_i(\Pi_Q) \cong HH^{6m+2-i}(\Pi_Q)$, we have $HH_i(\Pi_Q) \cong HH^{6m+5-i}(\Pi_Q)$.

1.1 The preprojective algebra

Given a quiver Q, we define the *preprojective algebra* Π_Q to be the quotient of the path algebra $P_{\bar{Q}}$ by the relation $\sum_{a\in Q}[a,a^*]=0$.

Given a monomial $x=a_1a_2\cdots a_n\in P_{\bar{Q}}$, we write x^* to be the monomial $a_n^*\cdots a_2^*a_1^*$, and we extend this definition linearly to all elements in $P_{\bar{Q}}$.

We introduce a grading, such that each trivial path has degree 0 and each arrow in $\bar{Q}$ has degree 1.

From now on, we assume that Q is of ADE type, and we write $A=\Pi_Q$.

1.1.1 Graded spaces and Hilbert series

Let $M=\bigoplus_{d\geq 0} M(d)$ be a $\mathbb{Z}_+$-graded vector space, with finite dimensional homogeneous subspaces. We denote by $M[n]$ the same space with grading shifted by n. The graded dual space M^* is defined by the formula $M^*(n)=M(-n)^*$.

Definition 1.1.1.1. (The Hilbert series of vector spaces)
Let $M=\bigoplus_{d\geq 0} M(d)$ be a $\mathbb{Z}_+$-graded vector space, with finite dimensional homogeneous subspaces. We define the Hilbert series *$h_M(t)$ to be the series*

$$h_M(t)=\sum_{d=0}^{\infty}\dim M(d)t^d.$$

Definition 1.1.1.2. (The Hilbert series of bimodules)
Let $M=\bigoplus_{d\geq 0} M(d)$ be a $\mathbb{Z}_+$-graded bimodule over the ring R, so we can write $M=\bigoplus_{i,j\in I} M_{i,j}$. We define the Hilbert series *$H_M(t)$ to be a matrix valued series with the entries*

$$H_M(t)_{i,j}=\sum_{d=0}^{\infty}\dim M(d)_{i,j}t^d.$$

1.1.2 Frobenius algebras and Nakayama automorphism

Definition 1.1.2.1. *Let $\mathcal{A}$ be a finite dimensional unital $\mathbb{C}-$algebra, let $\mathcal{A}^*=Hom_{\mathbb{C}}(\mathcal{A},\mathbb{C})$. We call it Frobenius if there is a linear function $f:\mathcal{A}\to\mathbb{C}$, such that the form*

$(x, y) := f(xy)$ *is nondegenerate, or, equivalently, if there exists an isomorphism* $\phi : \mathcal{A} \xrightarrow{\cong} \mathcal{A}^*$ *of left* $\mathcal{A}$*–modules: given* f*, we can define* $\phi(a)(b) = f(ba)$*, and given* ϕ*, we define* $f = \phi(1)$.

Remark 1.1.2.2. If $\tilde{f}$ is another linear function satisfying the same properties as f from above, then $\tilde{f}(x) = f(xa)$ for some invertible $a \in \mathcal{A}$. Indeed, we define the form $\{a, b\} = \tilde{f}(ab)$. Then $\{-, 1\} \in \mathcal{A}^*$, so there is an $a \in \mathcal{A}$, such that $\phi(a) = \{-, 1\}$. Then $\tilde{f}(x) = \{x, 1\} = \phi(a)(x) = f(xa)$.

Definition 1.1.2.3. *Given a Frobenius algebra* $\mathcal{A}$ *(with a function* f *inducing a bilinear form* $(-, -)$ *from above), the automorphism* $\eta : \mathcal{A} \to \mathcal{A}$ *defined by the equation* $(x, y) = (y, \eta(x))$ *is called the* Nakayama automorphism *(corresponding to* f*).*

Remark 1.1.2.4. We note that the freedom in choosing f implies that η is uniquely determined up to an inner automorphism. Indeed, let $\tilde{f}(x) = f(xa)$ and define the bilinear form $\{a, b\} = \tilde{f}(ab)$. Then

$$\begin{aligned}\{x, y\} = \tilde{f}(xy) = f(xya) = (x, ya) = (ya, \eta(x)) = f(ya\eta(x)a^{-1}a)\\ = (y, a\eta(x)a^{-1}).\end{aligned}$$

1.1.3 Root system parameters

Let w_0 be the longest element of the Weyl group W of Q. Then we define ν to be the involution of I, such that $w_0(\alpha_i) = -\alpha_{\nu(i)}$ (where α_i is the simple root corresponding to $i \in I$). It turns out that $\eta(e_i) = e_{\nu(i)}$ ([22]; see [12]).

Let m_i, $i = 1, ..., r$, be the exponents of the root system attached to Q, enumerated in increasing order. Let $h = m_r + 1$ be the Coxeter number in Q, i.e. the order of a Coxeter element in W.

Let P be the permutation matrix corresponding to the involution ν. Let $r_+ = \dim\ker(P - 1)$ and $r_- = \dim\ker(P + 1)$. Thus, r_- is half the number of vertices which are not fixed by ν, and $r_+ = r - r_-$.

A is finite dimensional, and the following Hilbert series is known from [20, Theorem 2.3.]:

$$H_A(t) = (1 + Pt^h)(1 - Ct + t^2)^{-1}. \tag{1.1.3.1}$$

It turns out that the top degree of A is $h-2$ (i.e. $A(d)$ vanishes for $d > h-2$), and for the top degree A^{top} part we get the following decomposition in 1-dimensional submodules:

$$A^{top} = A(h-2) = \bigoplus_{i \in I} e_i A(h-2) e_{\nu(i)} \tag{1.1.3.2}$$

It is known that A is a Frobenius algebra (see e.g. [12],[20]).

1.1.4 The symmetric bilinear form, roots and weights

We write $a \in Q$ to say that a is an arrow in Q. Let $h(a)$ denote its *head* and $t(a)$ its *tail*, i.e. for $a : i \to j$, $h(a) = j$ and $t(a) = i$. The *Ringel form* of Q is the bilinear form on $\mathbb{Z}^I$ defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in I} \alpha_i \beta_i - \sum_{a \in Q} \alpha_{t(a)} \beta_{h(a)}$$

for α, $\beta \in \mathbb{Z}^I$. We define the *quadratic form* $q(\alpha) = \langle \alpha, \alpha \rangle$ and the *symmetric bilinear form* $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$. It can be shown that q is positive definite for a finite Dynkin quiver Q.

We define the *set of roots* $\Delta = \{\alpha \in \mathbb{Z}^I | q(\alpha) = 1\}$.

We call the elements of $\mathbb{C}^I$ *weights*. A weight $\mu = (\mu_i)$ is called *regular* if the inner product $(\mu, \alpha) \neq 0$ for all $\alpha \in \Delta$. We call the coordinate vectors $\varepsilon_i \in \mathbb{C}^I$ the *fundamental weights* and define ρ to be the sum of all fundamental weights.

Chapter 2

Hochschild cohomology and homology of ADE quivers

2.1 The main results

2.1.1 Additive structure

Let U be a positively graded vector space with Hilbert series $h_U(t) = \sum_{i, m_i < \frac{h}{2}} t^{2m_i}$. Let Y be a vector space with $\dim Y = r_+ - r_- - \#\{i : m_i = \frac{h}{2}\}$, and let $K = \ker(P+1)$, $L = \langle e_i | \nu(i) = i \rangle$, so that $\dim K = r_-$, $\dim L = r_+ - r_-$ (we agree that the spaces K, L, Y sit in degree zero).

The main results are the following theorems.

Theorem 2.1.1.1. *The Hochschild cohomology spaces of A, as graded spaces, are as follows:*

$$HH^0(A) = U[-2] \oplus L[h-2],$$
$$HH^1(A) = U[-2],$$
$$HH^2(A) = K[-2],$$
$$HH^3(A) = K^*[-2],$$
$$HH^4(A) = U^*[-2],$$
$$HH^5(A) = U^*[-2] \oplus Y^*[-h-2],$$
$$HH^6(A) = U[-2h-2] \oplus Y[-h-2],$$

and $HH^{6n+i}(A) = HH^i(A)[-2nh]\, \forall i \geq 1$.

Corollary 2.1.1.2. *The center* $Z = HH^0(A)$ *of* A *has Hilbert series*

$$h_Z(t) = \sum_{i,\, m_i < \frac{h}{2}} t^{2m_i - 2} + (r_+ - r_-)t^{h-2}.$$

Theorem 2.1.1.3. *The Hochschild homology spaces of A, as graded spaces, are as follows:*

$$HH_0(A) = R,$$
$$HH_1(A) = U,$$
$$HH_2(A) = U \oplus Y[h],$$
$$HH_3(A) = U^*[2h] \oplus Y^*[h],$$
$$HH_4(A) = U^*[2h].$$
$$HH_5(A) = K^*[2h],$$
$$HH_6(A) = K[2h],$$

and $HH_{6n+i}(A) = HH_i(A)[2nh]\, \forall i \geq 1$.

(Note that the equality $HH_0(A) = R$ was established in [20]).

Theorem 2.1.1.4. *The cyclic homology spaces of A, as graded spaces, are as follows:*

$$\begin{aligned} HC_0(A) &= R, \\ HC_1(A) &= U, \\ HC_2(A) &= Y^*[h], \\ HC_3(A) &= U^*[2h], \\ HC_4(A) &= 0, \\ HC_5(A) &= K[2h], \\ HC_6(A) &= 0, \end{aligned}$$

and $HH_{6n+i}(A) = HH_i(A)[2nh]\,\forall i \geq 1$.

The rest of this chapter is devoted to the proof of Theorems 2.1.1.1,2.1.1.3,2.1.1.4

2.1.2 Product structure

From Theorem 2.1.1.1, we already know the additive structure of $HH^*(A)$. As the main result of this paper, we present the product structure in $HH^*(A)$. The rest of the paper is devoted to this computation. Since the product $HH^i(A) \times HH^j(A) \to HH^{i+j}(A)$ is graded-commutative, we can assume $i \leq j$ here.

Let $(U[-2])_+$ be the positive degree part of $U[-2]$ (which lies in non-negative degrees).

We have a decomposition $HH^0(A) = \mathbb{C} \oplus (U[-2])_+ \oplus L[-h-2]$ where we have the natural identification $(U[-2])(0) = \mathbb{C}$.

Let $z_0 = 1 \in \mathbb{C} \subset U[-2] \subset HH^0(A)$ (in lowest degree 0),
θ_0 the corresponding element in $HH^1(A)$ (in lowest degree 0),
ψ_0 the dual element of z_0 in $U^*[-2] \subset HH^5(A)$ (in highest degree -4), i.e. $\psi_0(z_0) = 1$,
ζ_0 the corresponding element in $U^*[-2] \subset HH^4(A)$ (in highest degree -4), that is

the dual element of θ_0, $\zeta_0(\theta_0) = 1$,.
$\varphi_0 : HH^0(A) \to HH^6(A)$ the natural quotient map (which induces the natural isomorphism $U[-2] \to U[-2h-2]$) and
ϕ the quotient map $L \to Y$ induced by φ_0 in Theorem 4.0.8.

Theorem 2.1.2.1. *(The product structure in $HH^*(A)$ for quivers of type A, D and E)*

1. *The multiplication by $\varphi_0(z_0)$ induces the natural isomorphisms $\varphi_i : HH^i(A) \to HH^{i+6}(A)$ $\forall i \geq 1$ and the natural quotient map φ_0. Therefore, it is enough to compute products $HH^i(A) \times HH^j(A) \to HH^{i+j}(A)$ with $0 \leq i \leq j \leq 5$.*

2. *The $HH^0(A)$-action on $HH^i(A)$:*

   (a) *($(U[-2])_+$-action)*
   The action of $(U[-2])_+$ on $U[-2] \subset HH^1(A)$ corresponds to the multiplication

$$\begin{aligned} (U[-2])_+ \times U[-2] &\to U[-2], \\ (u, v) &\mapsto u \cdot v \end{aligned}$$

   in $HH^0(A)$, projected on $U[-2] \subset HH^0(A)$.
   $(U[-2])_+$ acts on $U^*[-2] = HH^4(A)$ and $U^*[-2] \subset HH^5(A)$ the following way:

$$\begin{aligned} (U[-2])_+ \times U^*[-2] &\to U^*[-2], \\ (u, f) &\mapsto u \circ f, \end{aligned}$$

   where $(u \circ f)(v) = f(uv)$.
   $(U[-2])_+$ acts by zero on $L[h-2] \subset HH^0(A)$, $HH^2(A)$, $HH^3(A)$ and $Y^*[-h-2] \subset HH^5(A)$.

(b) *($L[h-2]$-action)*

$L[h-2]$ acts by zero on $HH^i(A)$, $1 \le i \le 4$, and on $U^*[-2] \subset HH^5(A)$. The $L[h-2]$-action on $HH^5(A)$ restricts to

$$\begin{aligned} L[h-2] \times Y^*[-h-2] &\rightarrow U^*[-2], \\ (a, y) &\mapsto y(\phi(a))\psi_0. \end{aligned}$$

3. *(Zero products)*

For quivers of type A_{2n+1}, D, E, all products $HH^i(A) \times HH^j(A) \rightarrow HH^{i+j}(A)$, $1 \le i \le j \le 5$, where $i + j \ge 6$ or i, j are both odd are zero except the pairings

$$HH^1(A) \times HH^5(A) \rightarrow HH^6(A)$$

and

$$HH^5(A) \times HH^5(A) \rightarrow HH^{10}(A).$$

For quivers of type A_{2n}, all products $HH^i(A) \times HH^j(A) \rightarrow HH^{i+j}(A)$, $1 \le i \le j \le 5$, where i, j are both odd are zero.

4. *($HH^1(A)$-products)*

(a) *The multiplication*

$$HH^1(A) \times HH^4(A) = U[-2] \times U^*[-2] \rightarrow HH^5(A)$$

is the same one as the restriction of

$$HH^0(A) \times HH^5(A) \rightarrow HH^5(A)$$

on $U[-2] \times U^*[-2]$.

(b) *The multiplication of the subspace $U[-2]_+ \subset HH^1(A)$ with $HH^i(A)$ where $i = 2, 5$ is zero.*

(c) The multiplication by θ_0 induces a symmetric isomorphism

$$\alpha : HH^2(A) = K[-2] \to K^*[-2] = HH^3(A).$$

On $HH^5(A)$, it induces a skew-symmetric isomorphism

$$\beta : Y^*[-h-2] \to Y[-h-2] \subset HH^6(A),$$

and acts by zero on $U^*[-2] \subset HH^5(A)$. α and β will be given by explicit matrices M_α amd M_β later.

5. *($HH^2(A)$-products)*

$$\begin{aligned} HH^2(A) \times HH^2(A) &\to HH^4(A), \\ (a,b) &\mapsto \langle a,b\rangle \zeta_0 \end{aligned}$$

is given by $\langle -,- \rangle = \alpha$ where α is regarded as a symmetric bilinear form.

$HH^2(A) \times HH^3(A) \to HH^5(A)$ is the multiplication

$$\begin{aligned} K[-2] \times K^*[-2] &\to HH^5(A), \\ (a,y) &\mapsto y(a)\psi_0. \end{aligned}$$

6. *($HH^5(A) \times HH^5(A) \to HH^{10}(A)$)*
The restriction of this product to

$$\begin{aligned} Y^*[-h-2] \times Y^*[-h-2] &\to HH^{10}(A), \\ (a,b) &\mapsto \Omega(a,b)\varphi_4(\zeta_0) \end{aligned}$$

is given by $\Omega(-,-) = -\beta$ where β is regarded as a skew-symmetric bilinear *form.*

The multiplication of the subspace $U^*[-2] \subset HH^5(A)$ *with* $HH^5(A)$ *is zero.*

7. *(Quivers of type* A_{2n}*: Products involving* $U^*[-2]$*).*

   (a) *(($(U_-)^*[-2]$-action).*
   $(U_-)^*[-2] \subset HH^i(A)$, $i = 4,5$ *acts by zero on* $HH^j(A)$, $j = 2,3,4,5$.

   (b) *Let us choose a nonzero* $\zeta' \in (U^{top})^*[-2] \in HH^4(A)$*, and* $z' \in U^{top}[-2] \subset HH^0(A)$*, let* $\theta' = \theta_0 z' \in U^{top}[-2] \subset HH^1(A)$ *and* $\psi' = \theta_0\zeta' \in (U^{top})^*[-2] \subset HH^5(A)$.

      i. $HH^2(A) \times HH^4(A) \to HH^6(A)$. *The multiplication with* $v \in HH^2(A)$ *gives us a map*

      $$\begin{aligned} (U^{top})^*[-2] &\to U^{top}[-2h-2], \\ \zeta' &\mapsto \gamma(v)\varphi_0(z'), \end{aligned}$$

      where $\gamma : HH^2(A) \to \mathbb{C}$ *is a linear function, given in Subsection 5.7.7.*

      ii. $HH^2(A) \times HH^5(A) \to HH^7(A)$. *This pairing*

      $$K[-2] \times U^*[-2] \to U[-2h-2]$$

      is the same as the corresponding pairing

      $$HH^2(A) \times HH^4(A) \to HH^6(A).$$

      iii. $HH^3(A) \times HH^4(A) \to HH^7(A)$. *The multiplication with* $w \in HH^3(A)$ *gives us a map*

      $$\begin{aligned} (U^{top})^*[-2] &\to U^{top}[-2h-2], \\ \zeta' &\mapsto \gamma(\alpha^{-1}(w))\varphi_0(\theta'). \end{aligned}$$

iv. $HH^4(A) \times HH^4(A) \to HH^8(A)$ *and* $HH^4(A) \times HH^5(A) \to HH^9(A)$. ζ'^2 *gives us a nonzero* $v \in HH^8(A)$. *Then* $\zeta'\psi' alpha(v) \in HH^9(A)$. $HH^4(A)$ *annihilates* $(U_-)^*[-2] \subset HH^5(A)$.

2.2 Hochschild (co)homology and cyclic homology of A

2.2.1 The Schofield resolution of A

We want to compute the Hochschild (co)homology of A, by using the Schofield resolution, described in [22].

Define the $A-$bimodule $\mathcal{N}$ obtained from A by twisting the right action by η, i.e., $\mathcal{N} = A$ as a vector space, and $\forall a, b \in A, x \in \mathcal{N} : a \cdot x \cdot b = ax\eta(b)$. Introduce the notation $\epsilon_a = 1$ if $a \in Q$, $\epsilon_a = -1$ if $a \in Q^*$. Let x_i be a homogeneous basis of A and x_i^* the dual basis under the form attached to the Frobenius algebra A. Let V be the bimodule spanned by the edges of $\bar{Q}$. We start with the following exact sequence:

$$0 \to \mathcal{N}[h] \xrightarrow{i} A \otimes_R A[2] \xrightarrow{d_2} A \otimes_R V \otimes_R A \xrightarrow{d_1} A \otimes_R A \xrightarrow{d_0} A \to 0,$$

where

$$\begin{aligned} d_0(x \otimes y) &= xy, \\ d_1(x \otimes v \otimes y) &= xv \otimes y - x \otimes vy, \\ d_2(z \otimes t) &= \sum_{a \in \bar{Q}} \epsilon_a za \otimes a^* \otimes t + \sum_{a \in \bar{Q}} \epsilon_a z \otimes a \otimes a^* t, \\ i(a) &= a \sum x_i \otimes x_i^*. \end{aligned}$$

Since $\eta^2 = 1$, we can make a canonical identification $A = \mathcal{N} \otimes_A \mathcal{N}$ (via $x \mapsto x \otimes 1$), so by tensoring the above exact sequence with $\mathcal{N}$, we obtain the exact sequence

$$0 \to A[2h] \xrightarrow{d_6} A \otimes_R \mathcal{N}[h+2] \xrightarrow{d_5} A \otimes_R V \otimes_R \mathcal{N}[h] \xrightarrow{d_4} A \otimes_R \mathcal{N}[h] \xrightarrow{j} \mathcal{N}[h] \to 0,$$

and by connecting both sequences with $d_3 = ij$ and repeating this process, we obtain the Schofield resolution which is periodic with period 6:

$$\begin{aligned} \ldots \to A \otimes A[2h] &\xrightarrow{d_6} A \otimes_R \mathcal{N}[h+2] \xrightarrow{d_5} A \otimes_R V \otimes_R \mathcal{N}[h] \xrightarrow{d_4} A \otimes_R \mathcal{N}[h] \\ &\xrightarrow{d_3} A \otimes_R A[2] \xrightarrow{d_2} A \otimes_R V \otimes_R A \xrightarrow{d_1} A \otimes_R A \xrightarrow{d_0} A \to 0. \end{aligned}$$

This implies that the Hochschild homology and cohomology of A is periodic with period 6, in the sense that the shift of the (co)homological degree by 6 results in the shift of degree by $2h$ (respectively $-2h$).

2.2.2 The Hochschild homology complex

Let A^{op} be the algebra A with opposite multiplication. We define $A^e = A \otimes_R A^{op}$. Then any $A-$bimodule naturally becomes a left A^e- module (and vice versa).

We make the following identifications (for all integers $m \geq 0$):
$(A \otimes_R A) \otimes_{A^e} A[2mh] = A^R[2mh] : (a \otimes b) \otimes c = bca,$
$(A \otimes_R V \otimes_R A) \otimes_{A^e} A[2mh] = (V \otimes_R A)^R[2mh] : (a \otimes x \otimes b) \otimes c = -x \otimes bca,$
$(A \otimes_R A) \otimes_{A^e} A[2mh+2] = A^R[2mh+2] : (a \otimes b) \otimes c = -bca,$
$(A \otimes_R \mathcal{N}) \otimes_{A^e} A[(2m+1)h] = \mathcal{N}^R[(2m+1)h] : (a \otimes b) \otimes c = -b\eta(ca),$
$(A \otimes_R V \otimes_R \mathcal{N}) \otimes_{A^e} A[(2m+1)h] = (V \otimes_R A)^R[(2m+1)h] :$
$(a \otimes x \otimes b) \otimes c = x \otimes b\eta(ca),$
$(A \otimes_R \mathcal{N}) \otimes_{A^e} A[(2m+1)h+2] = \mathcal{N}^R[(2m+1)h+2] : (a \otimes b) \otimes c = b\eta(ca).$

Now, we apply to the Schofield resolution the functor $- \otimes_{A_e} A$ to calculate the Hochschild homology:

$$\ldots \to \underbrace{A^R[2h]}_{=C_6} \xrightarrow{d_6'} \underbrace{\mathcal{N}^R[h+2]}_{=C_5} \xrightarrow{d_5'} \underbrace{(V \otimes_R \mathcal{N})^R[h]}_{=C_4} \xrightarrow{d_4'}$$

$$\xrightarrow{d_4'} \underbrace{\mathcal{N}^R[h]}_{=C_3} \xrightarrow{d_3'} \underbrace{A^R[2]}_{=C_2} \xrightarrow{d_2'} \underbrace{(V \otimes_R A)^R}_{=C_1} \xrightarrow{d_1'} \underbrace{A^R}_{=C_0} \to 0.$$

We compute the differentials:

$$d_1'(a \otimes b) = d_1(-1 \otimes a \otimes 1) \otimes_{A^e} b = (-a \otimes 1 + 1 \otimes a) \otimes_{A^e} b = [a, b],$$

$$d_2'(x) = d_2(-1 \otimes 1) \otimes_{A^e} x = -(\sum_{a \in \bar{Q}} \epsilon_a a \otimes a^* \otimes 1 + \sum_{a \in \bar{Q}} \epsilon_a 1 \otimes a \otimes a^*) \otimes_{A^e} x$$
$$= -\sum_{a \in \bar{Q}} \epsilon_a a^* \otimes [a, x],$$

$$\begin{aligned} d_3'(x) &= d_3(-1 \otimes 1) \otimes_{A^e} \eta(x) = -\sum (x_i \otimes x_i^*) \otimes_{A^e} \eta(x) = \sum x_i^* \eta(x) x_i \\ &= \sum x_i^* x x_i = \sum x_i x \eta(x_i^*), \end{aligned}$$

the second to last equality is true, since we can assume that each x_i lies in a subspace $e_k A e_{\nu_k}$, and then we see that
$x_i^* \eta(x) x_i = x_i^* x_i = x_i^* x x_i$ if $x = e_k$, $k = \nu(k)$,
and $x_i^* \eta(x) x_i = 0 = x_i^* x x_i$ if $k \neq \nu(k)$ or $x = e_j$, $j \neq k$ or $\deg x > 0$,
and the last equality is true because if (x_i^*) is a dual basis of (x_i), then (x_i) is a dual basis of $\eta(x_i^*)$.

$$d_4'(a \otimes b) = d_4(1 \otimes a \otimes 1) \otimes_{A^e} \eta(b) = (a \otimes 1 - 1 \otimes a) \otimes_{A^e} \eta(b) = ab - b\eta(a),$$

$$d_5'(x) = d_5(1 \otimes 1) \otimes_{A^e} x = (\sum_{a \in \bar{Q}} \epsilon_a a \otimes a^* \otimes 1 + \sum_{a \in \bar{Q}} \epsilon_a 1 \otimes a \otimes a^*) \otimes_{A^e} x$$
$$= \sum_{a \in \bar{Q}} \epsilon_a a^* \otimes (x\eta(a) - ax),$$
$$d_6'(x) = d_6(1 \otimes 1) \otimes_{A^e} x = \sum (x_i \otimes x_i^*) \otimes_{A^e} x = \sum x_i^* \eta(x)\eta(x_i)$$
$$= \sum x_i \eta(x) x_i^* = \sum x_i x x_i^*,$$

the second to last equality is true because if (x_i^*) is a dual basis of (x_i), then (x_i) is a dual basis of $\eta(x_i^*)$, and

the last equality is true because for each $j \in I$, $\sum x_i e_j x_i^* = \sum \dim(e_k A e_j)\omega_j$, where we call ω_j the dual of e_j, and $\dim(e_k A e_j) = \dim(e_k A e_{\nu(j)})$ (given a basis in $e_k A e_j$, the involution which reverses all arrows gives us a basis in $e_j A e_k$, its dual basis lies in $e_k A e_{\nu(j)}$).

Since $A = [A, A] + R$ (see [20]), $HH_0(A) = R$, and $HH_6(A)$ sits in degree $2h$.

Let us define $\overline{HH_i}(A) = HH_i(A)$ for $i > 0$ and $\overline{HH_i}(A) = HH_i(A)/R$ for $i = 0$. Then $\overline{HH_0}(A) = 0$.

The top degree of A is $h - 2$ (since $h_A(t) = \frac{1+Pt^h}{1-Ct+t^2}$ by [20, 2.3.], and A is finite dimensional). Thus we see immediately from the homology complex that $HH_1(A)$ lives in degrees between 1 and $h-1$, $HH_2(A)$ between 2 and h, $HH_3(A)$ between h and $2h-2$, $HH_4(A)$ between $h+1$ and $2h-1$, $HH_5(A)$ between $h+2$ and $2h$ and $HH_6(A)$ in degree $2h$.

2.2.3 Self-duality of the homology complex

The nondegenerate form allows us to make identifications $A = \mathcal{N}^*[h-2]$ and $\mathcal{N} = A^*[h-2]$ via $x \mapsto (-, x)$.

We can define a nondegenerate form on $V \otimes A$ and $V \otimes \mathcal{N}$ by

$$(a \otimes x_a, b \otimes x_b) = \delta_{a,b^*} \epsilon_a (x_a, x_b) \tag{2.2.3.1}$$

where $a, b \in Q$, and $\delta_{x,y}$ is 1 if $x = y$ and 0 else. This allows us to make identifications

$V \otimes_R A = (V \otimes_R \mathcal{N})^*[h]$ and $V \otimes_R \mathcal{N} = (V \otimes_R A)^*[h]$.

Let us take the first period of the Hochschild homology complex, i.e. the part involving the first 6 bimodules:

$$\underbrace{\mathcal{N}^R[h+2]}_{=C_5} \overset{d_5'}{\to} \underbrace{(V \otimes_R \mathcal{N})^R[h]}_{=C_4} \overset{d_4'}{\to} \underbrace{\mathcal{N}^R[h]}_{=C_3} \overset{d_3'}{\to} \underbrace{A^R[2]}_{=C_2} \overset{d_2'}{\to} \underbrace{(V \otimes_R A)^R}_{=C_1} \overset{d_1'}{\to} \underbrace{A^R}_{=C_0} \to 0.$$

By dualizing and using the above identifications, we get the dual complex:

$$\overset{(d_3')^*}{\leftarrow} \underbrace{(A^R[2])^*}_{=C_3[-2h]} \overset{(d_2')^*}{\leftarrow} \underbrace{((V \otimes_R A)^R)^*}_{=C_4[-2h]} \overset{(d_1')^*}{\leftarrow} \underbrace{(A^R)^*}_{=C_5[-2h]} \leftarrow 0.$$

$$\underbrace{(\mathcal{N}^R[h+2])^*}_{=C_0[-2h]} \overset{(d_5')^*}{\leftarrow} \underbrace{((V \otimes_R \mathcal{N})^R[h])^*}_{=C_1[-2h]} \overset{(d_4')^*}{\leftarrow} \underbrace{(\mathcal{N}^R[h])^*}_{=C_2[-2h]} \overset{(d_3')^*}{\leftarrow}$$

We see that $C_i^* = C_{5-i}$. We will now prove that, moreover, $d_i' = \pm(d_{6-i}')^*$, i.e. the homology complex has a self-duality property.

Proposition 2.2.3.2. *One has* $d_i' = \pm(d_{6-i}')^*$.

Proof. $\underline{(d_1')^* = d_5'}$:

We have

$$\begin{aligned}(\sum_{a\in\bar{Q}}(a\otimes x_a), d_5'(y)) &= (\sum_{a\in\bar{Q}}(a\otimes x_a), \sum_{a\in\bar{Q}}\epsilon_a a^*\otimes(y\eta(a)-ay)) \\ &= \sum_{a\in\bar{Q}}(x_a, y\eta(a)-ay) = (\sum_{a\in\bar{Q}}[a,x_a],y) \\ &= (d_1'(\sum_{a\in\bar{Q}} a\otimes x_a), y)\end{aligned}$$

$\underline{(d_2')^* = -d_4'}$:

We have

$$(x, d_4'(\sum_{a\in\bar{Q}} a\otimes x_a)) = (x, \sum_{a\in\bar{Q}} ax_a - x_a\eta(a)) = \sum_{a\in\bar{Q}}(-[a,x], x_a)$$
$$= (\sum_{a\in\bar{Q}} \epsilon_a a^* \otimes [a,x], \sum_{a\in\bar{Q}} a\otimes x_a) = (-d_2'(x), \sum_{a\in\bar{Q}} a\otimes x_a)$$

$(d_3')^* = d_3'$:

We have

$$(x, d_3'(y)) = (x, \sum x_i y\eta(x_i^*)) = (\sum x_i^* x x_i, y) = (d_3'(x), y).$$

□

2.2.4 Cyclic homology

Now we want to introduce the cyclic homology which will help us in computing the Hochschild cohomology of A. We have the Connes exact sequence

$$0 \to \overline{HH_0}(A) \xrightarrow{B_0} \overline{HH_1}(A) \xrightarrow{B_1} \overline{HH_2}(A) \xrightarrow{B_2} \overline{HH_3}(A) \xrightarrow{B_3} \overline{HH_4}(A) \to \dots$$

where the B_i are the Connes differentials (see [19, 2.1.7.]) and the B_i are all degree-preserving. We define the *reduced cyclic homology* (see [19, 2.2.13.])

$$\overline{HC_i}(A) = \ker(B_{i+1} : \overline{HH_{i+1}}(A) \to \overline{HH_{i+2}}(A))$$
$$= \mathrm{Im}(B_i : \overline{HH_i}(A) \to \overline{HH_{i+1}}(A)).$$

The usual cyclic homology $HC_i(A)$ is related to the reduced one by the equality $\overline{HC}_i(A) = HC_i(A)$ for $i > 0$, and $\overline{HC}_0(A) = HC_0(A)/R$.

Let $U = HH_1(A)$. Then by the degree argument and the injectivity of B_1 (which follows from the fact that $\overline{HH}_0(A) = 0$), we have $HH_2(A) = U \oplus Y[h]$ where $Y = HH_2(A)(h)$ (the degree-h-component). Using the duality of the Hochschild homology complex, we find $HH_4(A) = U^*[2h]$ and $HH_3(A) = U^*[2h] \oplus Y^*[h]$. Let us set $K = HH_5(A)[-2h]$.

So we can rewrite the Connes exact sequence as follows:

$$
\begin{array}{lccll}
\text{degree} & & & & \\
 & 0 & & & \\
 & \downarrow & & & \\
1 \le \deg \le h-1 & HH_1(A) & = & U & \overline{HC}_0(A) = 0 \\
 & \downarrow B_1 & & \downarrow \sim & \\
1 \le \deg \le h & HH_2(A) & = & U \quad \oplus Y[h] & HC_1(A) = U \\
 & \downarrow B_2 & & \qquad \downarrow \sim & \\
h+1 \le \deg \le 2h-1 & HH_3(A) & = & U^*[2h] \oplus Y^*[h] & HC_2(A) = Y^*[h] \\
 & \downarrow B_3 & & \downarrow \sim & \\
h+1 \le \deg \le 2h-1 & HH_4(A) & = & U^*[2h] & HC_3(A) = U^*[2h] \\
 & \downarrow B_4 & & \downarrow 0 & \\
2h & HH_5(A) & = & K[2h] & HC_4(A) = 0 \\
 & \downarrow B_5 & & \downarrow \sim & \\
2h & HH_6(A) & = & K[2h] & HC_5(A) = K[2h] \\
 & \downarrow B_6 & & \downarrow 0 & \\
2h+1 \le \deg \le 3h-1 & HH_7(A) & = & U[2h] & \\
 & \downarrow B_7 & & & \\
 & \vdots & & &
\end{array}
$$

From the exactness of the sequence it is clear that B_2 and B_3 restrict to an isomorphism on $Y[h]$ and $U^*[2h]$ respectively and that $B_4 = 0$. $B_6 = 0$ because it preserves degrees, so B_5 is an isomorphism.

An analogous argument applies to the portion of the Connes sequence from homological degree $6n+1$ to $6n+6$ for $n > 0$.

Thus we see that the cyclic homology groups $HC_i(A)$ live in different degrees: $HC_{6n+1}(A)$ between $2hn+1$ and $2hn+h-1$, $HC_{6n+2}(A)$ in degree $2hn+h$, $HC_{6n+3}(A)$ between $2hn+h+1$ and $2hn+2h-1$, and $HC_{6n+5}(A)$ in degree $2hn+2h$. So to prove the main results, it is sufficient to determine the Hilbert series of the cyclic homology spaces.

This is done with the help of the following lemma.

Lemma 2.2.4.1. *The Euler characteristic of the reduced cyclic homology* $\chi_{\overline{HC}(A)}(t) = \sum (-1)^i h_{\overline{HC}_i(A)}(t)$ *is*

$$\sum_{k=0}^{\infty} a_k t^k = \frac{1}{1-t^{2h}}(-\sum t^{2m_i} - r_- t^{2h} + (r_+ - r_-)t^h).$$

Proof. To compute the Euler characteristic, we use the theorem from [8] that

$$\prod_{k=1}^{\infty}(1-t^k)^{-a_k} = \prod_{s=1}^{\infty} \det H_A(t^s).$$

From [20, Theorem 2.3.] we know that

$$H_A(t) = (1+Pt^h)(1-Ct+t^2)^{-1}.$$

Since $r = r_+ + r_-$,

$$\det(1+Pt^h) = (1+t^h)^{r_+}(1-t^h)^{r_-}.$$

From 4.1.4.2 we know that

$$\prod_{s=1}^{\infty} \det(1-Ct^s+t^{2s}) = \prod_{k=1}^{\infty}(1-t^{2k})^{-\#\{i:m_i \equiv k \mod h\}}.$$

So

$$\prod_{k=1}^{\infty}(1-t^k)^{-a_k} = \prod_{s=1}^{\infty}\det H_A(t^s)$$
$$= \prod_{s=1}^{\infty}(1+t^{hs})^{r_+}(1-t^{hs})^{r_-}\det(1-Ct^s+t^{2s})^{-1}$$
$$= \frac{\prod_{s\, even}(1-t^{hs})^{r_-}}{\prod_{s\, odd}(1-t^{hs})^{r_+-r_-}}\prod_{k=1}^{\infty}(1-t^{2k})^{\#\{i: m_i\equiv k \mod h\}}.$$

It follows that

$$\chi_{\overline{HC}(A)}(t) = \sum_{k=0}^{\infty}a_k t^k = (1+t^h+t^{2h}+\ldots)(-\sum t^{2m_i} - r_- t^{2h} + (r_+ - r_-)t^h).$$

This implies the lemma. □

Since all $HC_i(A)$ live in different degrees, we can immediately derive their Hilbert series from the Euler characteristic:

$$h_{HC_1(A)}(t) = \sum_{i,\, m_i<\frac{h}{2}} t^{2m_i},$$
$$h_{HC_2(A)}(t) = (r_+ - r_- - \#\{i : m_i = \frac{h}{2}\})t^h,$$
$$h_{HC_3(A)}(t) = \sum_{i,\, m_i>\frac{h}{2}} t^{2m_i},$$
$$h_{HC_5(A)}(t) = r_- t^{2h}.$$

It follows that $h_U(t) = \sum_{i,\, m_i<\frac{h}{2}} t^{2m_i}$, $\dim Y = r_+ - r_- - \#\{i : m_i = \frac{h}{2}\}$, $\dim K = r_-$, and Y, K sit in degree zero.

This completes the proof of Theorems 2.1.1.3,2.1.1.4.

2.2.5 The Hochschild cohomology complex

Now we would like to prove Theorem 2.1.1.1.

We make the following identifications: $Hom_{A^e}(A \otimes_R A, A) = A^R$ and $Hom_{A^e}(A \otimes_R \mathcal{N}, A) = \mathcal{N}^R$, by identifying ϕ with the image $\phi(1 \otimes 1) = a$ (we write $\phi = a \circ -$), and $Hom_{A^e}(A \otimes_R V \otimes_R A, A) = (V \otimes_R A)^R[-2]$ and $Hom_{A^e}(A \otimes_R V \otimes_R \mathcal{N}, A) = (V \otimes_R \mathcal{N})^R[-2]$, by identifying ϕ which maps $1 \otimes a \otimes 1 \mapsto x_a$ ($a \in \bar{Q}$) with the element $\sum_{a \in \bar{Q}} \epsilon_{a^*} a^* \otimes x_a$ (we write $\phi = \sum_{a \in \bar{Q}} \epsilon_{a^*} a^* \otimes x_a \circ -$).

Now, apply the functor $Hom_{A^e}(-, A)$ to the Schofield resolution to obtain the Hochschild cohomology complex

$$\overset{d_4^*}{\leftarrow} \mathcal{N}^R[-h] \overset{d_3^*}{\leftarrow} A^R[-2] \overset{d_2^*}{\leftarrow} (V \otimes A)^R[-2] \overset{d_1^*}{\leftarrow} A^R \leftarrow 0$$
$$\ldots \leftarrow A^R[-2h] \overset{d_6^*}{\leftarrow} \mathcal{N}^R[-h-2] \overset{d_5^*}{\leftarrow} (V \otimes \mathcal{N})^R[-h-2] \overset{d_4^*}{\leftarrow}$$

Proposition 2.2.5.1. *Using the differentials d_i' from the Hochschild homology complex, we can rewrite the Hochschild cohomology complex in the following way:*

$$\overset{d_5'[-2h-2]}{\longleftarrow} \mathcal{N}^R[-h] \overset{d_6'[-2h-2]}{\longleftarrow} A^R[-2] \overset{d_1'[-2]}{\longleftarrow} (V \otimes A)^R[-2] \overset{d_2'[-2]}{\longleftarrow} A^R \leftarrow 0$$
$$\ldots \leftarrow A^R[-2h] \overset{d_3'[-2h-2]}{\longleftarrow} \mathcal{N}^R[-h-2] \overset{d_4'[-2h-2]}{\longleftarrow} (V \otimes \mathcal{N})^R[-h-2] \overset{d_5'[-2h-2]}{\longleftarrow}.$$

Proof.

$$d_1^*(x)(1 \otimes a \otimes 1) = x \circ d_1(1 \otimes a \otimes 1) = x \circ (a \otimes 1 - 1 \otimes a) = [a, x],$$

so

$$d_1^*(x) = \sum_{a \in \bar{Q}} \epsilon_{a^*} a^* \otimes [a, x] = d_2'(x).$$

$$d_2^*(\sum_{a \in \bar{Q}} a \otimes x_a)(1 \otimes 1) = (\sum_{a \in \bar{Q}} a \otimes x_a) \circ (\sum_{b \in \bar{Q}} \epsilon_b b \otimes b^* \otimes 1 + \sum_{b \in \bar{Q}} \epsilon_b 1 \otimes b \otimes b^*)$$
$$= \sum_{a \in \bar{Q}} (a x_a - x_a a) = \sum_{a \in \bar{Q}} [a, x_a],$$

so

$$d_2^*(\sum_{a\in\bar{Q}} a\otimes x_a) = \sum_{a\in\bar{Q}}[a, x_a] = d_1'(\sum_{a\in\bar{Q}} a\otimes x_a).$$

$$d_3^*(x)(1\otimes 1) = x\circ d_3(1\otimes 1) = x\circ(\sum x_i\otimes x_i^*) = \sum x_i x x_i^*,$$

so

$$d_3^*(x) = \sum x_i x x_i^* = d_6'(x).$$

$$d_4^*(x)(1\otimes a\otimes 1) = x\circ d_1(1\otimes a\otimes 1) = x\circ(a\otimes 1 - 1\otimes a) = ax - x\eta(a),$$

so

$$d_4^*(x) = \sum_{a\in\bar{Q}} \epsilon_{a^*} a^*\otimes(ax - x\eta(a)) = d_5'(x).$$

$$\begin{aligned} d_5^*(\sum_{a\in\bar{Q}} a\otimes x_a)(1\otimes 1) &= (\sum_{a\in\bar{Q}} a\otimes x_a)\circ(\sum_{b\in\bar{Q}}\epsilon_b b\otimes b^*\otimes 1 + \sum_{b\in\bar{Q}}\epsilon_b 1\otimes b\otimes b^*) \\ &= \sum_{a\in\bar{Q}}(ax_a - x_a\eta(a)), \end{aligned}$$

so

$$d_5^*(\sum_{a\in\bar{Q}} a\otimes x_a) = \sum_{a\in\bar{Q}}(ax_a - x_a\eta(a)) = d_4'(\sum_{a\in\bar{Q}} a\otimes x_a).$$

$$d_6^*(x)(1\otimes 1) = x\circ d_6(1\otimes 1) = x\circ(\sum x_i\otimes x_i^*) = \sum x_i x\eta(x_i^*),$$

so

$$d_6^*(x) = \sum x_i x\eta(x_i^*) = d_3'(x).$$

□

Thus we see that each 3-term portion of the cohomology complex can be identified, up to shift in degree, with an appropriate portion of the homology complex.

This fact, together with Theorem 2.1.1.3, implies Theorem 2.1.1.1.

2.3 The deformed preprojective algebra

In this subsection we would like to consider the universal deformation of the preprojective algebra A. If $\nu = 1$, then $P = 1$ and hence by Theorem 2.1.1.1 $HH^2(A) = 0$ and thus A is rigid. On the other hand, if $\nu \neq 1$ (i.e. for types A_n, $n \geq 2$, D_{2n+1}, and E_6), then $HH^2(A)$ is the space K of ν-antiinvariant functions on I, sitting in degree -2.

Proposition 2.3.0.2. *Let λ be a weight (i.e. a complex function on I) such that $\nu\lambda = -\lambda$. Let A_λ be the quotient of P_Q by the relation*

$$\sum_{a \in Q} [a, a^*] = \sum \lambda_i e_i.$$

Then $\mathrm{gr} A_\lambda = A$ *(under the filtration by length of paths). Moreover, A_λ, with λ a formal parameter in K, is a universal deformation of A.*

Proof. To prove the first statement, it is sufficient to show that for generic λ such that $\nu(\lambda) = -\lambda$, the dimension of the algebra A_λ is the same as the dimension of A, i.e. $rh(h+1)/6$. But by Theorem 7.3 of [3], A_λ is Morita equivalent to the preprojective algebra of a subquiver Q' of Q, and the dimension vectors of simple modules over A_λ are known (also from [3]). This allows one to compute the dimension of A_λ for any λ, and after a somewhat tedious case-by-case computation one finds that indeed $\dim A_\lambda = \dim A$ for a generic $\lambda \in K$.

The second statement boils down to the fact that the induced map $\phi : K \to HH^2(A)$ defined by the above deformation is an isomorphism (in fact, the identity). This is proved similarly to the case of centrally extended preprojective algebras, which is considered in chapter 4. □

Remark. For type A_n (but not D and E) the algebra A_λ for generic $\lambda \in K$ is actually semisimple, with simple modules of dimensions $n, n-2, n-4, \ldots$.

2.4 Some basic facts about preprojective algebras

2.4.1 Labeling of quivers

From now on, we use the following labelings for the different types of quivers:

$Q = D_{n+1}$

Figure 2-1: D_{n+1}-quiver

A is the path algebra modulo the relations

$$\begin{aligned} a_1^* a_1 &= 0, \\ a_{i+1}^* a_{i+1} &= a_i a_i^*, \quad 1 \le i \le n-3 \\ a_{n-1} a_{n-1}^* = a_n a_n^* &= 0 \\ a_{n-1}^* a_{n-1} + a_n^* a_n &= a_{n-2} a_{n-2}^*. \end{aligned}$$

$Q = E_6$

Figure 2-2: E_6-quiver

A is the path algebra modulo the relations

$$\begin{aligned} a_1a_1^* = a_4a_4^* = a_5a_5^* &= 0, \\ a_1^*a_1 &= a_2a_2^*, \\ a_4^*a_4 &= a_3a_3^*, \\ a_2^*a_2 + a_3^*a_3 + a_5^*a_5 &= 0. \end{aligned}$$

$Q = E_7$

Figure 2-3: E_7-quiver

A is the path algebra modulo the relations

$$\begin{aligned} a_1a_1^* = a_5a_5^* = a_6a_6^* &= 0, \\ a_1^*a_1 &= a_2a_2^*, \\ a_2^*a_2 &= a_3a_3^*, \\ a_5^*a_5 &= a_4a_4^*, \\ a_3^*a_3 + a_4^*a_4 + a_6^*a_6 &= 0. \end{aligned}$$

Figure 2-4: E_8-quiver

$Q = E_8$

A is the path algebra modulo the relations

$$\begin{aligned} a_0a_0^* = a_5a_5^* = a_6a_6^* &= 0, \\ a_0^*a_0 &= a_1a_1^*, \\ a_1^*a_1 &= a_2a_2^*, \\ a_2^*a_2 &= a_3a_3^*, \\ a_5^*a_5 &= a_4a_4^*, \\ a_3^*a_3 + a_4^*a_4 + a_6^*a_6 &= 0. \end{aligned}$$

2.4.2 The Nakayama automorphism

Recall that A is a Frobenius algebra. The linear function $f : A \to \mathbb{C}$ is zero in the non-top degree part of A. It maps a top degree element $\omega_i \in e_iA^{top}e_{\nu(i)}$ to 1. It is uniquely determined by the choice of one of these ω_i and a Nakayama automorphism.

For each quiver, we define a Nakayama automorphism η and make a choice of one $\omega_i \in e_iA^{top}e_{\nu(i)}$:

$Q = D_{n+1}$, n **odd**

We define η by

$$\eta(a_i) = -a_i, \qquad (2.4.2.1)$$

$$\eta(a_i^*) = a_i^*, \qquad (2.4.2.2)$$

and

$$\omega_1 = a_1^* \dots a_{n-2}^* a_{n-1}^* a_{n-1} a_{n-2} \dots a_1. \qquad (2.4.2.3)$$

Let

$$\overline{a_i} = (-1)^i a_{i-1} \cdots a_1 a_1^* \cdots a_{n-1}^* a_{n-1} \cdots a_{i+1} \qquad \forall 1 \le i \le n-2,$$

$$\overline{a_{n-1}} = a_{n-2} \cdots a_1 a_1^* \cdots a_{n-1}^*,$$

$$\overline{a_n} = -a_{n-2} \cdots a_1 a_1^* \cdots a_{n-2}^* a_n^*,$$

$$\overline{a_i^*} = a_{i+1}^* \cdots a_{n-1}^* a_{n-1} \cdots a_1 a_1^* \cdots a_{i-1}^* \qquad \forall 1 \le i \le n-2,$$

$$\overline{a_{n-1}^*} = a_{n-1} \cdots a_1 a_1^* \cdots a_{n-2}^*,$$

$$\overline{a_n^*} = -a_n a_{n-2} \cdots a_1 a_1^* \cdots a_{n-2}^*,$$

and $\omega_i = a_i^* \overline{a_i^*}$ $\forall 1 \le i \le n-1$ (where ω_1 coincides with the expression in (2.4.2.3)), $\omega_n = a_{n-1}\overline{a_{n-1}}$, $\omega_{n+1} = a_n \overline{a_n}$. Then $\omega_{i+1} = a_i \overline{a_i}$ $\forall 1 \le i \le n-2$, and $\omega_i = \overline{a_i} \cdot (-a_i)$ $\forall 1 \le i \le n-1$, $\omega_n = \overline{a_n} \cdot (-a_n) = \overline{a_{n+1}} \cdot (-a_{n+1})$, $\omega_{i+1} = \overline{a_i^*} a_i^*$ $\forall 1 \le i \le n$.

These ω_i define the function f (and the bilinear form) associated to the Frobenius algebra A. Since $\{\overline{a_1}, \dots, \overline{a_n}, \overline{a_1^*}, \dots, \overline{a_n^*}\}$ in $A(h-3)$ is a dual basis of $\{a_1, \dots, a_n, a_1^*, \dots a_n^*\}$ in $A(1)$ and $\{-a_1, \dots, -a_n, a_1^*, \dots a_n^*\}$ in $A(1)$ is a dual basis to $\{\overline{a_1}, \dots, \overline{a_n}, \overline{a_1^*}, \dots, \overline{a_n^*}\}$ in $A(h-3)$, it follows that the Nakayama automorphism associated to our bilinear form is given by the equations (2.4.2.1) and (2.4.2.2).

$Q = D_{n+1}$, n **even**

We define η by

$$\begin{aligned}\forall i \leq n-2: \quad \eta(a_i) &= -a_i,\\ \forall i \leq n-2: \quad \eta(a_i^*) &= a_i^*,\\ \eta(a_{n-1}) &= -a_n,\\ \eta(a_{n-1}^*) &= a_n^*,\\ \eta(a_n) &= -a_{n-1},\\ \eta(a_n^*) &= a_{n-1}^*,\end{aligned}$$

$$\omega_1 = a_1^* \dots a_{n-2}^* a_{n-1}^* a_{n-1} a_{n-2} \dots a_1. \tag{2.4.2.4}$$

Let

$$\begin{aligned}\overline{a_i} &= (-1)^i a_{i-1} \cdots a_1 a_1^* \cdots a_{n-1}^* a_{n-1} \cdots a_{i+1} \qquad \forall 1 \leq i \leq n-2,\\ \overline{a_{n-1}} &= a_{n-2} \cdots a_1 a_1^* \cdots a_{n-2}^* a_n^*,\\ \overline{a_n} &= -a_{n-2} \cdots a_1 a_1^* \cdots a_{n-1}^*,\\ \overline{a_i^*} &= a_{i+1}^* \cdots a_{n-1}^* a_{n-1} \cdots a_1 a_1^* \cdots a_{i-1}^* \qquad \forall 1 \leq i \leq n-2,\\ \overline{a_{n-1}^*} &= a_{n-1} \cdots a_1 a_1^* \cdots a_{n-2}^*,\\ \overline{a_n^*} &= -a_n a_{n-2} \cdots a_1 a_1^* \cdots a_{n-2}^*\end{aligned}$$

and $\omega_i = a_i^* \overline{a_i^*}$ $\forall 1 \leq i \leq n-1$ (where ω_1 coincides with the expression in (2.4.2.4)), $\omega_n = a_{n-1}\overline{a_{n-1}}$, $\omega_{n+1} = a_n \overline{a_n}$. Then $\omega_{i+1} = a_i \overline{a_i}$ $\forall 1 \leq i \leq n-2$, $\omega_{n-1} = a_n^* \overline{a_n^*}$ and $\omega_{i+1} = \overline{a_i^*} a_i$ $\forall 1 \leq i \leq n-2$, $\omega_n = \overline{a_{n-1}^*} a_n^*$, $\omega_{n+1} = \overline{a_n^*} a_{n-1}^*$, $\omega_i = \overline{a_i} \cdot (-a_i)$ $\forall 1 \leq i \leq n-2$, $\omega_n = \overline{a_5} \cdot (-a_6)$, $\omega_{n+1} = \overline{a_n} \cdot (-a_{n-1})$.

Again, these ω_i define the function f (and the bilinear form) associated to the Frobenius algebra A. Since $\{\overline{a_1}, \dots, \overline{a_n}, \overline{a_1^*}, \dots, \overline{a_n^*}\}$ in $A(h-3)$ is a dual basis of $\{a_1, \dots, a_n, a_1^*, \dots a_n^*\}$ in $A(1)$ and $\{-a_1, \dots, -a_n, -a_{n-1}, a_1^*, \dots a_n^*, a_{n-1}^*\}$ in $A(1)$ is a dual basis to $\{\overline{a_1}, \dots, \overline{a_{n-1}}, \overline{a_n}, \overline{a_1^*}, \dots, \overline{a_{n-1}^*}, \overline{a_n^*}\}$ in $A(h-3)$, it follows that the

Nakayama automorphism associated to our bilinear form is given by η above.

$Q = E_6$

We define η by

$$\begin{aligned}
\eta(a_1) &= -a_4, \\
\eta(a_1^*) &= a_4^*, \\
\eta(a_2) &= -a_3, \\
\eta(a_2^*) &= a_3^*, \\
\eta(a_5) &= -a_5, \\
\eta(a_5^*) &= a_5^*,
\end{aligned}$$

and

$$\omega_3 = a_3^* a_3 (a_2^* a_2 a_3^* a_3)^2. \tag{2.4.2.5}$$

Let

$$\begin{aligned}
\overline{a_1} &= -a_2 a_3^* a_4^* a_4 a_3 a_5^* a_5 a_3^* a_4^*, \\
\overline{a_2} &= a_3^* a_4^* a_4 a_3 a_5^* a_5 a_3^* a_4^* a_4, \\
\overline{a_3} &= a_5^* a_5 a_3^* a_4^* a_4 a_3 a_2^* a_1^* a_1, \\
\overline{a_4} &= -a_3 a_5^* a_5 a_3^* a_4^* a_4 a_3 a_2^* a_1^*, \\
\overline{a_5} &= a_2^* a_1^* a_1 a_2 a_3^* a_4^* a_4 a_3 a_5^*, \\
\overline{a_1^*} &= -a_1 a_2 a_3^* a_4^* a_4 a_3 a_5^* a_5 a_3^*, \\
\overline{a_2^*} &= -a_1^* a_1 a_2 a_3^* a_4^* a_4 a_3 a_5^* a_5, \\
\overline{a_3^*} &= -a_4^* a_4 a_3 a_5^* a_5 a_3^* a_4^* a_4 a_3, \\
\overline{a_4^*} &= -a_4 a_3 a_5^* a_5 a_3^* a_4^* a_4 a_3 a_2^*, \\
\overline{a_5^*} &= a_5 a_2^* a_1^* a_1 a_2 a_3^* a_4^* a_4 a_3
\end{aligned}$$

and $\omega_1 = a_1 \overline{a_1}$, $\omega_2 = a_2 \overline{a_2}$, $\omega_3 = a_2^* \overline{a_2^*}$ (which coincides with the expression in

(2.4.2.5)), $\omega_4 = a_3\overline{a_3}$, $\omega_5 = a_4\overline{a_4}$. $\omega_6 = a_5\overline{a_5}$. Then $\omega_2 = a_1^*\overline{a_1^*}$, $\omega_3 = a_3^*\overline{a_3^*} = a_5^*\overline{a_5^*}$, $\omega_4 = a_4^*\overline{a_4^*}$ and $\omega_1 = \overline{a_1^*}a_4^*$, $\omega_2 = \overline{a_2^*}a_3^* = \overline{a_1}\cdot(-a_4)$, $\omega_3 = \overline{a_2}\cdot(-a_3) = \overline{a_3}\cdot(-a_2) = \overline{a_5}\cdot(-a_5)$, $\omega_4 = \overline{a_3^*}a_2^* = \overline{a_4}\cdot(-a_1)$, $\omega_5 = \overline{a_4^*}a_1^*$, $\omega_6 = \overline{a_5^*}a_5^*$.

Again, these ω_i define the function f (and the bilinear form) associated to the Frobenius algebra A. Since $\{\overline{a_1},\dots,\overline{a_5},\overline{a_1^*},\dots,\overline{a_5^*}\}$ in $A(h-3)$ is a dual basis of $\{a_1,\dots,a_5,a_1^*,\dots a_5^*\}$ in $A(1)$ and $\{-a_4,-a_3,-a_2,-a_1,-a_5,a_4^*,a_3^*,a_2^*,a_1^*,a_5^*\}$ in $A(1)$ is a dual basis to $\{\overline{a_1},\dots,\overline{a_5},\overline{a_1^*},\dots\overline{a_5^*}\}$ in $A(h-3)$, it follows that the Nakayama automorphism associated to our bilinear form is given by η above.

$Q = E_7$

We define η by

$$\begin{aligned}\eta(a_i) &= -a_i,\\ \eta(a_i^*) &= a_i^*,\end{aligned}$$

and

$$\omega_4 = (a_4^*a_4a_3^*a_3)^4. \qquad (2.4.2.6)$$

Given the basis $\{a_1,\dots,a_6,a_1^*,\dots a_6^*\}$ in $A(1)$, we claim that a dual basis $\{\overline{a_1},\dots,\overline{a_6},\overline{a_1^*},\dots,\overline{a_6^*}\}$ in $A(h-3)$ is given by Let

$$\begin{aligned}
\overline{a_1} &= -a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*,\\
\overline{a_2} &= a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_1,\\
\overline{a_3} &= -a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_1a_2,\\
\overline{a_4} &= -a_3^*a_2^*a_1^*a_1a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5,\\
\overline{a_5} &= a_4a_3^*a_2^*a_1^*a_1a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*,\\
\overline{a_6} &= a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_1a_2a_3a_6^*,
\end{aligned}$$

$$\overline{a_1^*} = -a_1a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*,$$
$$\overline{a_2^*} = -a_1^*a_1a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4a_3^*,$$
$$\overline{a_3^*} = -a_2^*a_1^*a_1a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4,$$
$$\overline{a_4^*} = -a_5^*a_5a_4a_3^*a_2^*a_1^*a_1a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3,$$
$$\overline{a_5^*} = -a_5a_4a_3^*a_2^*a_1^*a_1a_2a_3a_6^*a_6a_4^*a_4a_3^*a_3a_4^*,$$
$$\overline{a_6^*} = a_6a_4^*a_4a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_1a_2a_3$$

and $\omega_i = a_i\overline{a_i}\ \forall 1 \le i \le 3$, $\omega_{i+1} = a_i\overline{a_i}\ \forall 4 \le i \le 6$, $\omega_4 = a_3^*\overline{a_3^*}$. Then $\omega_2 = a_1^*\overline{a_1^*}$, $\omega_3 = a_2^*\overline{a_2^*}$, $\omega_4 = a_4^*\overline{a_4^*} = a_6^*\overline{a_6^*}$ (which coincides with the expression (2.4.2.6)), $\omega_5 = a_5^*\overline{a_5^*}$ and $\omega_i = \overline{a_i^*}a_i^*\ \forall 1 \le i \le 3$, $\omega_{i+1} = \overline{a_i^*}a_i^*\ \forall 4 \le i \le 6$, $\omega_{i+1} = \overline{a_i} \cdot (-a_i)\ \forall 1 \le i \le 3$, $\omega_i = \overline{a_i} \cdot (-a_i)\ \forall 4 \le i \le 5$, $\omega_4 = \overline{a_6} \cdot (-a_6)$.

Again, these ω_i define the function f (and the bilinear form) associated to the Frobenius algebra A. Since $\{\overline{a_1}, \dots, \overline{a_6}, \overline{a_1^*}, \dots, \overline{a_6^*}\}$ in $A(h-3)$ is a dual basis of $\{a_1, \dots, a_6, a_1^*, \dots a_6^*\}$ in $A(1)$ and $\{-a_1, \dots, -a_6, a_1^*, \dots, a_6^*\}$ in $A(1)$ is a dual basis to $\{\overline{a_1}, \dots, \overline{a_6}, \overline{a_1^*}, \dots \overline{a_6^*}\}$ in $A(h-3)$, it follows that the Nakayama automorphism associated to our bilinear form is given by η above.

$Q = E_8$

We define η by

$$\eta(a_i) = -a_i,$$
$$\eta(a_i^*) = a_i^*,$$

and

$$\omega_4 = (a_4^*a_4a_3^*a_3)^7 \tag{2.4.2.7}$$

Then

$$\overline{a_0} = a_1a_2a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*,$$

$$\overline{a_1} = -a_2a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*a_0,$$

$$\overline{a_2} = a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*a_0a_1,$$

$$\overline{a_3} = -a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*a_0a_1a_2,$$

$$\overline{a_4} = -a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*a_0a_1a_2a_3a_6^*a_6a_4^*,$$

$$\overline{a_5} = a_4a_3^*a_2^*a_1^*a_0^*a_0a_1a_2a_3a_6^*a_6a_4^*a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*,$$

$$\overline{a_6} = a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*a_0a_1a_2a_3a_6^*,$$

$$\overline{a_0^*} = a_0a_1a_2a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*,$$

$$\overline{a_1^*} = a_0^*a_0a_1a_2a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*,$$

$$\overline{a_2^*} = a_1^*a_0^*a_0a_1a_2a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*,$$

$$\overline{a_3^*} = a_2^*a_1^*a_0^*a_0a_1a_2a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4,$$

$$\overline{a_4^*} = -a_4a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*a_0a_1a_2a_3a_6^*a_6,$$

$$\overline{a_5^*} = a_5a_4a_3^*a_2^*a_1^*a_0^*a_0a_1a_2a_3a_6^*a_6a_4^*a_6^*a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*,$$

$$\overline{a_6^*} = a_6a_3^*a_3a_3^*a_3a_4^*a_4a_3^*a_3a_3^*a_3a_4^*a_5^*a_5a_4a_3^*a_2^*a_1^*a_0^*a_0a_1a_2a_3a_6^*a_6a_4^*a_4$$

and $\omega_i = a_i\overline{a_i}\ \forall 0 \le i \le 3$, $\omega_{i+1} = a_i\overline{a_i}\ \forall 4 \le i \le 6$, $\omega_4 = a_3^*\overline{a_3^*}$. Then $\omega_2 = a_1^*\overline{a_1^*}$, $\omega_3 = a_2^*\overline{a_2^*}$, $\omega_4 = a_4^*\overline{a_4^*} = a_6^*\overline{a_6^*}$ (which coincides with the expression (2.4.2.7)), $\omega_5 = a_5^*\overline{a_5^*}$ and $\omega_i = \overline{a_i^*}a_i^*\ \forall 0 \le i \le 3$, $\omega_{i+1} = \overline{a_i^*}a_i^*\ \forall 4 \le i \le 6$, $\omega_{i+1} = \overline{a_i} \cdot (-a_i)\ \forall 0 \le i \le 3$, $\omega_i = \overline{a_i} \cdot (-a_i)\ \forall 4 \le i \le 5$, $\omega_4 = \overline{a_6} \cdot (-a_6)$.

Again, these ω_i define the function f (and the bilinear form) associated to the Frobenius algebra A. Since $\{\overline{a_0}, \ldots, \overline{a_6}, \overline{a_0^*}, \ldots, \overline{a_6^*}\}$ in $A(h-3)$ is a dual basis of $\{a_0, \ldots, a_6, a_0^*, \ldots a_6^*\}$ in $A(1)$ and $\{-a_0, \ldots, -a_6, a_0^*, \ldots, a_6^*\}$ in $A(1)$ is a dual basis to $\{\overline{a_0}, \ldots, \overline{a_6}, \overline{a_0^*}, \ldots \overline{a_6^*}\}$ in $A(h-3)$, it follows that the Nakayama automorphism associated to our bilinear form is given by η above.

2.4.3 Preprojective algebras by numbers

We summarize useful numbers associated to preprojective algebras, by quiver:

Q		exponents m_i	h	$\deg A^{top}$	degrees $HH^0(A)$
D_{n+1}	n odd	$1, 3, \ldots, 2n-1, n$	$2n$	$2n-2$	$0, 4, \ldots, 2n-6, 2n-2$
	n even				$0, 4, \ldots 2n-4, 2n-2$
E_6		$1, 4, 5, 7, 8, 11$	12	10	$0, 6, 8, 10$
E_7		$1, 5, 7, 9, 11, 13, 17$	18	16	$0, 8, 12, 16$
E_8		$1, 7, 11, 13, 17, 19, 23, 29$	30	28	$0, 12, 20, 24, 28$

We see that for quivers of type D and E, the degrees of the space U (which are $2m_i$, $m_i < \frac{h}{2}$) are even and range from 0 to $h-2$.

We get the following degree ranges for the Hochschild cohomology:

$$\begin{aligned}
HH^0(A) &= U[-2] \oplus L[h-2] & 0 \leq \deg HH^0(A) \leq h-2 \\
HH^1(A) &= U[-2] & 0 \leq \deg HH^1(A) \leq h-4 \\
HH^2(A) &= K[-2] & \deg HH^2(A) = -2 \\
HH^3(A) &= K^*[-2] & \deg HH^3(A) = -2 \\
HH^4(A) &= U^*[-2] & -h \leq \deg HH^4(A) \leq -4 \\
HH^5(A) &= U^*[-2] \oplus Y^*[-h-2] & -h-2 \leq \deg HH^5(A) \leq -4 \\
HH^6(A) &= U[-2h-2] \oplus Y[-h-2] & -2h \leq \deg HH^6(A) \leq -h-2
\end{aligned}$$

2.4.4 Basis of the preprojective algebra for $Q = D_{n+1}$

We need to work with the Hilbert series and with an explicit basis of A. We do this for each type of quiver separately.

We write B for a set of all homogeneous basis elements of A, $B_{i,-}$ for a homogeneous basis of e_iA, $B_{-,j}$ for a homogeneous basis of Ae_j, $B_{i,j}$ for a basis of e_iAe_j and $B_{i,j}(d)$ for a basis of $e_iAe_j(d)$.

A basis of A is given by the following elements:

For $\underline{k, j \leq n-1}$:

$$
\begin{aligned}
B_{k,n} &= \{(a_{k-1}a_{k-1}^*)^l a_k^* \cdots a_{n-2}^* a_{n-1}^* | 0 \leq l \leq k-1\}, \\
B_{k,n+1} &= \{(a_{k-1}a_{k-1}^*)^l a_k^* \cdots a_{n-2}^* a_n^* | 0 \leq l \leq k-1\}, \\
B_{n,n} &= \{(a_{n-1}a_n^* a_n a_{n-1}^*)^l | 0 \leq l \leq \begin{cases} \frac{n-1}{2} & n\, odd, \\ \frac{n-2}{2} & n\, even \end{cases} \}, \\
B_{n+1,n+1} &= \{(a_n a_{n-1}^* a_{n-1} a_n^*)^l | 0 \leq l \leq \begin{cases} \frac{n-1}{2} & n\, odd, \\ \frac{n-2}{2} & n\, even \end{cases} \}, \\
B_{n+1,n} &= \{a_n a_{n-1}^* (a_{n-1}a_n^* a_n a_{n-1}^*)^l | 0 \leq l \leq \begin{cases} \frac{n-3}{2} & n\, odd, \\ \frac{n-2}{2} & n\, even \end{cases} \}, \\
B_{n,n+1} &= \{a_{n-1} a_n^* (a_n a_{n-1}^* a_{n-1} a_n^*)^l | 0 \leq l \leq \begin{cases} \frac{n-3}{2} & n\, odd, \\ \frac{n-2}{2} & n\, even \end{cases} \}, \\
B_{n,j} &= \{a_{n-1}a_{n-2} \cdots a_j (a_{j-1}a_{j-1}^*)^l | 0 \leq l \leq j-1\}, \\
B_{n+1,j} &= \{a_n a_{n-2} \cdots a_j (a_{j-1}a_{j-1}^*)^l | 0 \leq l \leq j-1\}.
\end{aligned}
$$

For $\underline{k \leq j \leq n-1}$,

$$
\begin{aligned}
B_{k,j} &= \{(a_{k-1}a_{k-1}^*)^l a_k^* \cdots a_{j-1}^* | 0 \leq l \leq \min\{k-1, n-j-1\}\} \cup \\
&\quad \{(a_{k-1}a_{k-1}^*)^l a_k^* \cdots a_{n-1}^* a_{n-1} a_{n-2} \cdots a_j | 0 \leq l \leq k-1\} \cup \\
&\quad \{(a_{k-1}a_{k-1}^*)^l a_k^* \cdots a_n^* a_n a_{n-2} \cdots a_j | 0 \leq l \leq k-1+j-n\}.
\end{aligned}
$$

For $\underline{j < k \leq n-1}$,

$$
\begin{aligned}
B_{k,j} &= \{a_{k-1} \cdots a_j (a_j^* a_j)^l | 0 \leq l \leq \min\{n-k-1, j-1\}\} \cup \\
&\quad \{a_k^* \cdots a_{n-2}^* a_{n-1}^* a_{n-1} a_{n-2} \cdots a_j (a_j^* a_j)^l | 0 \leq l \leq j-1\} \cup \\
&\quad \{a_k^* \cdots a_{n-2}^* a_n^* a_n a_{n-2} \cdots a_j (a_j^* a_j)^l | 0 \leq l \leq j-1+k-n\}.
\end{aligned}
$$

2.4.5 Hilbert series of the preprojective algebra for $Q = E_6$

We give the columns of the Hilbert series $H_A(t)$ which can be calculated from (1.1.3.1):

$$(H_A(t)_{i,1})_{1\le i\le 6} = \begin{pmatrix} 1+t^6 \\ t+t^5+t^7 \\ t^2+t^4+t^6+t^8 \\ t^3+t^5+t^9 \\ t^4+t^{10} \\ t^3+t^7 \end{pmatrix},$$

$$(H_A(t)_{i,2})_{1\le i\le 6} = \begin{pmatrix} t+t^5+t^7 \\ 1+t^2+t^4+2t^6+t^8 \\ t+2t^3+2t^5+2t^7+t^9 \\ t^2+2t^4+t^6+t^8+t^{10} \\ t^3+t^5+t^9 \\ t^2+t^4+t^6+t^8 \end{pmatrix},$$

$$(H_A(t)_{i,3})_{1\le i\le 6} = \begin{pmatrix} t^2+t^4+t^6+t^8 \\ t+2t^3+2t^5+2t^7+t^9 \\ 1+2t^2+3t^4+3t^6+2t^8+t^{10} \\ t+2t^3+2t^5+2t^7+t^9 \\ t^2+t^4+t^6+t^8 \\ t+t^3+2t^5+t^7+t^9 \end{pmatrix},$$

$$(H_A(t)_{i,4})_{1\le i\le 6} = \begin{pmatrix} t^3+t^5+t^9 \\ t^2+2t^4+t^6+t^8+t^{10} \\ t+2t^3+2t^5+2t^7+t^9 \\ 1+t^2+t^4+2t^6+t^8 \\ t+t^5+t^7 \\ t^2+t^4+t^6+t^8 \end{pmatrix},$$

$$(H_A(t)_{i,5})_{1\le i\le 6} = \begin{pmatrix} t^4 + t^{10} \\ t^3 + t^5 + t^9 \\ t^2 + t^4 + t^6 + t^8 \\ t + t^5 + t^7 \\ 1 + t^6 \\ t^3 + t^7 \end{pmatrix},$$

$$(H_A(t)_{i,6})_{1\le i\le 6} = \begin{pmatrix} t^3 + t^7 \\ t^2 + t^4 + t^6 + t^8 \\ t + t^3 + 2t^5 + t^7 + t^9 \\ t^2 + t^4 + t^6 + t^8 \\ t^3 + t^7 \\ 1 + t^4 + t^6 + t^{10} \end{pmatrix}.$$

2.5 $HH^0(A) = Z$

From the Hilbert series (Corollary 2.1.1.2) we see that we have one (unique up to a constant factor) central element of degree $2m_i - 2$ for each exponent $m_i < \frac{h}{2}$. We will denote a deg $i (< h-2)$ central element by z_i.

From (1.1.3.2) and from the Hilbert series we can also see that the top degree ($=\deg h-2$) center is spanned by one element ω_i in each e_iAe_i, such that $\nu(i) = i$.

The $\omega_i \in L[h-2]$ are already given in section 1.1.2, and we will find the $z_i \in U[-2]$ for each Dynkin quiver separately.

2.5.1 $Q = D_{n+1}$

We define the nonzero elements

$$\begin{aligned}
b_{i,0} &= e_i,\\
b_{i,j} &= a_i^* \dots a_{i+j-1}^* a_{i+j-1} \dots a_i \text{ (where } 1 \le j \le \min\{i-1, n-1-i\}),\\
c_{i,j} &= a_i^* \dots a_{n-2}^* (a_{n-2}a_{n-2}^*)^j a_{n-2} \dots a_i \text{ (where } 1 \le i \le n-2,\ 1 \le j \le i-1\\
c_{n-1,j} &= (a_{n-2}a_{n-2}^*)^j,\ 1 \le j \le n-2\\
c_i' &= a_i^* \dots a_{n-2}^* a_{n-1}^* a_{n-1} (a_{n-2}a_{n-2}^*)^{i-1} a_{n-2} \dots a_i,\ 1 \le i \le n-1\\
d_0 &= e_n,\\
d_j &= (a_{n-1}a_n^* a_n a_{n-1}^*)^j \text{ for } 1 \le j < \frac{n}{2},\\
d_0' &= e_{n+1},\\
d_j' &= (a_n a_{n-1}^* a_{n-1} a_n^*)^j \text{ for } 1 \le j < \frac{n}{2}
\end{aligned}$$

and extend this notation for any other j, where $b_{i,j}$, $c_{i,j}$, d_j and d_j' are zero.

The exponents m_i are $1, 3, \dots, 2n-1, n$ and $h = 2n$. From Corollary 2.1.1.2 we get the Hilbert series of Z, depending on the parity of n, since $r_+ = n+1$ for n odd and $r_+ = n-1$ for n even:

$$n \text{ odd: } h_Z(t) = 1 + t^4 + t^8 + \ldots + t^{2n-6} + (n+1)t^{2n-2},$$
$$n \text{ even: } h_Z(t) = 1 + t^4 + t^8 \ldots + t^{2n-4} + (n-1)t^{2n-2}.$$

The central elements of degree $4j < 2n-2$ are

$$z_{4j} = \sum_{i=2j+1}^{n-1-2j} b_{i,2j} + \sum_{i=0}^{2j-1} c_{n-1-i,2j-i} + d_j + d'_j.$$

The top degree central elements are $\omega_i = c'_i$ ($1 \le i \le n-1$), and additionally $\omega_n = d_{\frac{n-1}{2}}$, $-\omega_{n+1} = d'_{\frac{n-1}{2}}$ if n is odd.

For $j + k < \frac{n-1}{2}$ we get the following product:

$$z_{4j} z_{4k} = z_{4(j+k)}.$$

If n is odd and $j + k = \frac{n-1}{2}$, the multiplication becomes

$$z_{4j} z_{4k} = d_{\frac{n-1}{2}} + d'_{\frac{n-1}{2}} = \omega_n - \omega_{n+1}.$$

2.5.2 $Q = E_6$

The Coxeter number is $h = 12$, and the exponents $m_i < \frac{h}{2} = 6$ are 1, 4, 5, $r_+ = 2$. For the center, we get the following Hilbert series (from Corollary 2.1.1.2):

$$h_Z(t) = 1 + t^6 + t^8 + 2t^{10}.$$

From the degrees, we see that the product of any two positive degree central elements is always 0. The central elements are $z_0 = 1$, z_6, z_8, ω_3 and ω_6.

We give the central elements z_6 and z_8 explicitly (it can be easily checked that they are central):

Proposition 2.5.2.1. *1. The central element of* deg 6 *is*

$$z_6 = a_1a_2a_3^*a_3a_2^*a_1^* - a_2(a_3^*a_3)^2a_2^* - a_5^*a_5a_3^*a_3a_5^*a_5 + a_3(a_2^*a_2)^2a_3^* - a_4a_3a_2^*a_2a_3^*a_4^*,$$

2. the central element of deg 8 *is*

$$z_8 = -a_2a_5^*a_5a_3^*a_3a_5^*a_5a_2^* - a_5^*a_5(a_3^*a_3)^2a_5^*a_5 - a_3a_5^*a_5a_2^*a_2a_5^*a_5a_3^*.$$

2.5.3 $Q = E_7$

The Coxeter number is $h = 18$, the exponents $m_i < \frac{h}{2} = 9$ are $1, 5, 7$, $r_+ = 7$, and the Hilbert series of the center is (see Corollary 2.1.1.2):

$$h_Z(t) = 1 + t^8 + t^{12} + 7t^{16}$$

The center is spanned by $z_0 = 1$, z_8, z_{12}, $\omega_1, \dots, \omega_7$. The only interesting product to compute is z_8^2 which lies in the top degree.

We give z_8 and z_{12} explicitly:

Proposition 2.5.3.1. *1. The central element of degree* 8 *is*

$$z_8 = -a_1a_2a_3a_6^*a_6a_3^*a_2^*a_1^* - a_2a_3(a_4^*a_4)^2a_3^*a_2^* - a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*$$
$$- a_4^*a_4(a_3^*a_3)^2a_4^*a_4 - a_4a_4^*a_4a_6^*a_6a_4^*a_4a_4^* + a_6a_4^*a_4a_6^*a_6a_4^*a_4a_6^*.$$

2. The central element of degree 12 *is*

$$z_{12} = -a_3(a_4^*a_4a_6^*a_6)^2a_4^*a_4a_3^* - a_4^*a_4a_6^*a_6(a_4^*a_4)^2a_6^*a_6a_4^*a_4$$
$$+ a_4(a_6^*a_6a_4^*a_4)^2a_6^*a_6a_4^* + a_6(a_4^*a_4a_6^*a_6)^2a_4^*a_4a_6^*.$$

Proposition 2.5.3.2. *We get*

$$z_8^2 = \omega_1 + \omega_3 - \omega_7.$$

2.5.4 $Q = E_8$

The Coxeter number $h = 30$, and the exponents $m_i < \frac{h}{2} = 15$ are $1, 7, 11, 13$, $r_+ = 8$. For the center, we get the following Hilbert series (from Corollary 2.1.1.2):

$$h_Z(t) = 1 + t^{12} + t^{20} + t^{24} + 8t^{28}.$$

The center is spanned by $z_0 = 1$, z_{12}, z_{20}, z_{24}, $\omega_1, \dots, \omega_8$. The only interesting product is z_{12}^2.

Proposition 2.5.4.1. *1. The central element of degree* 12 *is*

$$\begin{aligned} z_{12} = {} & a_1a_2a_3a_6^*a_6a_4^*a_4a_6^*a_6a_3^*a_2^*a_1^* + a_2a_3a_4^*a_4(a_3^*a_3)^2a_4^*a_4a_3^*a_2^* \\ & + a_3(a_4^*a_4a_6^*a_6)^2a_4^*a_4a_3^* + (a_3^*a_3a_4^*a_4a_3^*a_3)^2 - a_4(a_6^*a_6a_4^*a_4)^2a_6^*a_6a_4^* \\ & + a_5a_4a_6^*a_6(a_4^*a_4)^2a_6^*a_6a_4^*a_5^* - a_6(a_4^*a_4a_6^*a_6)^2a_4^*a_4a_6^*. \end{aligned}$$

2. The central element of degree 20 *is*

$$\begin{aligned} z_{20} = {} & -a_1a_2a_3(a_4^*a_4)^2(a_3^*a_3)^3(a_4^*a_4)^2a_3^*a_2^*a_1^* - a_2a_3(a_6^*a_6a_4^*a_4)^2(a_4^*a_4a_6^*a_6)^2a_3^*a_2^* \\ & + a_3(a_6^*a_6a_4^*a_4)^4a_6^*a_6a_3^* - (a_4^*a_4a_6^*a_6)^5 + (a_6^*a_6(a_4^*a_4)^2)^3a_6^*a_6 \\ & - (a_6^*a_6a_4^*a_4)^5 - a_4(a_4^*a_4a_6^*a_6a_4^*a_4)^3a_4^* - a_6(a_4^*a_4a_6^*a_6)^4a_4^*a_4a_6^*. \end{aligned}$$

3. The central element of degree 24 *is*

$$z_{24} = z_{12}^2.$$

2.6 $HH^1(A)$

Recall Theorem 2.1.1.1 where we know that $HH^1(A)$ is isomorphic to the non-topdegree part of $HH^0(A)$. In fact, $HH^1(A)$ is generated by the central elements in the following way:

Proposition 2.6.0.2. *$HH^1(A)$ is spanned by maps*

$$\theta_k : (A \otimes V \otimes A) \longrightarrow A,$$
$$\theta_k(1 \otimes a_i \otimes 1) = 0,$$
$$\theta_k(1 \otimes a_i^* \otimes 1) = a_i^* z_k.$$

Proof. These maps clearly lie in $\ker d_2^*$: Recall

$$\begin{array}{ccc} A \otimes A & \xrightarrow{d_2} & A \otimes V \otimes A \\ x \otimes y & \longmapsto & \sum_{a \in \bar{Q}} \epsilon_a x a \otimes a^* \otimes y + \sum_{a \in \bar{Q}} \epsilon_a x \otimes a \otimes a^* y, \end{array}$$

then

$$\begin{aligned} d_2^* \circ \theta_k(1 \otimes 1) &= \theta_k\Big(\sum_{a \in \bar{Q}} \epsilon_a a \otimes a^* \otimes 1 + \sum_{a \in \bar{Q}} \epsilon_a 1 \otimes a \otimes a^*\Big) \\ &= \sum_{i \in I} a_i a_i^* z_k - \sum_{i \in I} a_i^* a_i z_k = \sum_{i \in I} [a_i, a_i^*] z_k = 0. \end{aligned}$$

We will later see in section 2.9 that $HH^4(A)$ is generated by ζ_k where $\zeta_k(\theta_k) = 1$ under the duality $HH^4(A) = (HH^1(A))^*$ (which follows from the self-duality of the Hochschild homology complex and the duality between Hochschild cohomology and homology), so θ_k is nonzero in $HH^1(A)$. □

2.7 $HH^2(A)$

We know from Theorem 2.1.1.1 that $HH^2(A) = K[-2]$ lies in degree -2, i.e. in the lowest degree of $A^R[-2]$ (using the identifications in 2.2.5), that is in $R[-2]$. Since the image of d_2^* lies in degree > -2, $HH^2(A) = \ker d_3^*$.

Proposition 2.7.0.3. *$HH^2(A)$ is given by the kernel of the matrix $H_A(1)$, where we identify $\mathbb{C}^I = R = \bigoplus_{i \in I} Re_i$.*

Proof. Recall

$$d_3^*(y) = \sum_{x_i \in B} x_i y x_i^* = \sum_{j,k \in I} \sum_{x_i \in B_{j,k}} x_i y x_i^*.$$

For each $x_i \in e_k A e_j$, we see that $x_i e_l x_i^* = \delta_{jl} \omega_k$.

It follows that for $y = \sum_{i \in I} \lambda_i e_i$ the map is given by

$$d_3^*(y) = \sum_{i \in I} \mu_i \omega_i,$$

where the vectors $\lambda = (\lambda_i)_{i \in I} \in \mathbb{C}^I$ and $\mu = (\mu_i)_{i \in I} \in \mathbb{C}^I$ satisfy the equation

$$H_A(1)\lambda = \mu. \tag{2.7.0.4}$$

So the kernel of d_3^* is given by the kernel of $H_A(1)$. □

Now, we find the elements in $HH^2(A)$ for the quivers separately.

2.7.1 $Q = D_{n+1}$, n even

$$H_A(1) = \begin{pmatrix} 2 & 2 & 2 & 2 & \dots & \dots & 2 & 1 & 1 \\ 2 & 4 & 4 & 4 & \dots & \dots & 4 & 2 & 2 \\ 2 & 4 & 6 & 6 & \dots & \dots & 6 & 3 & 3 \\ 2 & 4 & 6 & 8 & \dots & \dots & 8 & 4 & 4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 2 & 4 & 6 & 8 & \dots & \dots & 2(n-1) & n-1 & n-1 \\ 1 & 2 & 3 & 4 & \dots & \dots & n-1 & \frac{n}{2} & \frac{n}{2} \\ 1 & 2 & 3 & 4 & \dots & \dots & n-1 & \frac{n}{2} & \frac{n}{2} \end{pmatrix} \tag{2.7.1.1}$$

with kernel $\langle e_n - e_{n+1} \rangle$. So a basis of $HH^2(A)$ is given by

$$\{f_n = [e_n - e_{n+1}]\}.$$

2.7.2 $Q = E_6$

$$H_A(1) = \begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 2 \\ 3 & 6 & 8 & 6 & 3 & 4 \\ 4 & 8 & 12 & 8 & 4 & 6 \\ 3 & 6 & 8 & 6 & 3 & 4 \\ 2 & 3 & 4 & 3 & 2 & 2 \\ 2 & 4 & 6 & 4 & 2 & 4 \end{pmatrix} \tag{2.7.2.1}$$

with kernel $\langle e_1 - e_5, e_2 - e_4 \rangle$. So a basis of $HH^2(A)$ is given by

$$\{f_1 = [e_1 - e_5], f_2 = [e_2 - e_4]\}.$$

2.8 $HH^3(A)$

We know that $HH^3(A)$ lives in degree -2. The kernel of d_4^* has to be the top degree part of $\mathcal{N}^R[-h]$ (since $\operatorname{Im} d_3^*$ lives in degree -2), so

$$HH^3(A) = \mathcal{N}^R[-h](-2)/\operatorname{Im} d_3^*.$$

Proposition 2.8.0.2. *$HH^3(A)$ is given by the cokernel of the matrix $H_A(1)$, where we identify $\mathbb{C}^I = A^{top} = \bigoplus_{i \in I} e_i A^{top} e_{\nu(i)}$.*

Proof. This follows immediately from the discussion in the previous section because d_3^* is given by $H_A(1)$. □

Note that $HH^3(A) = (HH^2(A))^*$ (from the self-duality of the Hochschild homology complex and the duality between Hochschild homology and cohomology). We choose a basis h_i of $HH^3(A)$, so that $h_i(f_j) = \delta_{ij}$.

2.8.1 $Q = D_{n+1}$, n even

From $H_A(1)$ in (2.7.1.1) we see that:

$$\begin{aligned}
d_3^*(2e_1 - e_2) &= 2\omega_1, \\
d_3^*(-e_{i-1} + 2e_i - e_{i+1}) &= 2\omega_i \qquad \forall 2 \leq i \leq n-2, \\
d_3^*((-n-1)e_{n-2} + 2(n-1)e_{n-1} - 2(n-1)e_n) &= (n-1)\omega_{n-1}, \\
d_3^*(2e_n - e_{n-1}) &= \omega_n + \omega_{n+1},
\end{aligned}$$

so

$$HH^3(A) = (\mathcal{N}^R)^{top}[-h]/(\omega_1 = \omega_2 = \ldots = \omega_{n-1} = 0,\, \omega_n + \omega_{n+1} = 0)$$

with basis

$$\{h_n = [\omega_n]\}.$$

2.8.2 $Q = E_6$

From $H_A(1)$ in (2.7.2.1) we see that:

$$\begin{aligned}
d_3^*(2e_1 - e_2) &= \omega_1 + \omega_5, \\
d_3^*(-e_1 + 2e_2 - e_3) &= \omega_2 + \omega_4, \\
d_3^*(-2e_2 + 2e_3 - e_6) &= 2\omega_3, \\
d_3^*(-e_3 + 2e_6) &= 2\omega_6,
\end{aligned}$$

so

$$HH^3(A) = (\mathcal{N}^R)^{top}[-h]/(\omega_3 = \omega_6 = \omega_1 + \omega_5 = \omega_2 + \omega_3 = 0)$$

with basis

$$\{h_1 = [\omega_1],\, h_2 = [\omega_2]\}.$$

2.9 $HH^4(A)$

We have $HH^4(A) = U^*[-2]$, so its top degree is -4, and its generators sit in degrees $-4 - \deg z_k$ for each central element, one in each degree.

Proposition 2.9.0.1. *Let $\zeta_0 \in \ker d_5^*$ be a top degree element in $(V \otimes \mathcal{N})^R[-h-2]$, such that $m(\zeta_0)$ is nonzero, where m is the multiplication map. Then $HH^4(A)$ is generated by elements $\zeta_k \in \ker d_5^*$ which satisfy $\zeta_k z_k = \zeta_0$.*

Proof. If $x \in \mathcal{N}^R[-h]$ lies in degree -4, then $m(d_4^*(x)) = 0$, so ζ_0 is nonzero in $HH^4(A)$.

For every non-topdegree central element z_k we can find a ζ_k satisfying the properties above, which is done for each quiver separately below.

For any central element $z \in A$, we have that $d_4^*(zy) = d_4^*(y)z$. If $\zeta_k = d_4^*(y)$, then by construction $\zeta_0 = \zeta_k z_k = d_4^*(z_k y)$ which is a contradiction.

So these ζ_k are all nonzero in $HH^4(A)$, and also generate this cohomology space.

□

A basis of $HH^4(A)$ is given by these ζ_k, and we choose them so that $\zeta_k(\theta_k) = 1$ under the duality $HH^4(A) = (HH^1(A))^*$.

2.9.1 $Q = D_{n+1}$, n odd

We define

$$\begin{aligned}
\zeta_0 =& [a_{n-1}^* \otimes a_{n-1}a_n^*a_n(a_{n-1}^*a_{n-1}a_n^*a_n)^{\frac{n-3}{2}} \\
&+ a_{n-1} \otimes a_n^*a_na_{n-1}^*(a_{n-1}a_n^*a_na_{n-1}^*)^{\frac{n-3}{2}}], \\
\zeta_{4k} =& \frac{1}{2}[a_{n-1}^* \otimes a_{n-1}a_n^*a_n(a_{n-1}^*a_{n-1}a_n^*a_n)^{\frac{n-3}{2}-k} \\
&+ a_{n-1} \otimes a_n^*a_na_{n-1}^*(a_{n-1}a_n^*a_na_{n-1}^*)^{\frac{n-3}{2}-k} \\
&- a_n^* \otimes a_na_{n-1}^*a_{n-1}(a_n^*a_na_{n-1}^*a_{n-1})^{\frac{n-3}{2}-k} \\
&- a_n \otimes a_{n-1}^*a_{n-1}a_n^*(a_na_{n-1}^*a_{n-1}a_n^*)^{\frac{n-3}{2}-k}].
\end{aligned}$$

2.9.2 $Q = D_{n+1}$, n even

We define

$$\begin{aligned}\zeta_0 =&[a_{n-1}^* \otimes a_{n-1}(a_n^* a_n a_{n-1}^* a_{n-1})^{\frac{n-2}{2}-k}\\ &+ a_{n-1} \otimes a_n^*(a_n a_{n-1}^* a_{n-1} a_n^*)^{\frac{n-2}{2}-k}],\\ \zeta_{4k} =&\frac{1}{2}[a_{n-1}^* \otimes a_{n-1}(a_n^* a_n a_{n-1}^* a_{n-1})^{\frac{n-2}{2}-k}\\ &+ a_{n-1} \otimes a_n^*(a_n a_{n-1}^* a_{n-1} a_n^*)^{\frac{n-2}{2}-k}\\ &- a_n^* \otimes a_n(a_{n-1}^* a_{n-1} a_n^* a_n)^{\frac{n-2}{2}-k}\\ &- a_n \otimes a_{n-1}^*(a_{n-1} a_n^* a_n a_{n-1}^*)^{\frac{n-2}{2}-k}].\end{aligned}$$

2.9.3 $Q = E_6$

We define

$$\begin{aligned}\zeta_0 &= [a_3^* \otimes a_3(a_2^* a_2 a_3^* a_3)^2 + a_3 \otimes a_2^*(a_2 a_3^* a_3 a_2^*)^2],\\ \zeta_6 &= \frac{1}{4}[-a_3^* \otimes a_3 a_2^* a_2 - a_3 \otimes a_2^* a_2 a_2^* + a_2^* \otimes a_2 a_2^* a_2 + a_2 \otimes a_2^* a_2 a_3^*\\ &\quad -a_2^* \otimes a_2 a_3^* a_3 - a_2 \otimes a_3^* a_3 a_3^* + a_3^* \otimes a_3 a_3^* a_3 + a_3 \otimes a_3^* a_3 a_2^*],\\ \zeta_8 &= \frac{1}{2}[a_3^* \otimes a_3 + a_3 \otimes a_2^* - a_2^* \otimes a_2 - a_2 \otimes a_3^*].\end{aligned}$$

2.9.4 $Q = E_7$

We define

$$\begin{aligned}\zeta_0 &= [a_4^* \otimes a_4 a_3^* a_3(a_4^* a_4 a_3^* a_3)^3 + a_4 \otimes a_3^* a_3 a_4^*(a_4 a_3^* a_3 a_4^*)^3],\\ \zeta_8 &= \frac{1}{2}[a_4^* \otimes a_4 a_3^* a_3 a_4^* a_4 a_3^* a_3 + a_4 \otimes a_3^* a_3 a_4^* a_4 a_3^* a_3 a_4^*\\ &\quad -a_3^* \otimes a_3 a_4^* a_4 a_3^* a_3 a_4^* a_4 - a_3 \otimes a_4^* a_4 a_3^* a_3 a_4^* a_4 a_3^*],\\ \zeta_{12} &= \frac{1}{2}[a_4^* \otimes a_4 a_3^* a_3 + a_4 \otimes a_3^* a_3 a_4^* - a_3^* \otimes a_3 a_4^* a_4 - a_3 \otimes a_4^* a_4 a_3^*].\end{aligned}$$

2.9.5 $Q = E_8$

We define

$$\begin{aligned}
\zeta_0 &= [a_4^* \otimes a_4 a_3^* a_3 (a_4^* a_4 a_3^* a_3)^6 + a_4 \otimes a_3^* a_3 a_4^* (a_4 a_3^* a_3 a_4^*)^6], \\
\zeta_{12} &= \frac{1}{2}[a_4^* \otimes a_4 a_3^* a_3 (a_4^* a_4 a_3^* a_3)^3 + a_4 \otimes a_3^* a_3 a_4^* (a_4 a_3^* a_3 a_4^*)^3 \\
&\quad - a_3^* \otimes a_3 a_4^* a_4 (a_3^* a_3 a_4^* a_4)^3 - a_3 \otimes a_4^* a_4 a_3^* (a_3 a_4^* a_4 a_3^*)^3], \\
\zeta_{20} &= \frac{1}{2}[a_4^* \otimes a_4 a_3^* a_3 a_4^* a_4 a_3^* a_3 + a_4 \otimes a_3^* a_3 a_4^* a_4 a_3^* a_3 a_4^* \\
&\quad - a_3^* \otimes a_3 a_4^* a_4 a_3^* a_3 a_4^* a_4 - a_3 \otimes a_4^* a_4 a_3^* a_3 a_4^* a_4 a_3^*], \\
\zeta_{24} &= \frac{1}{2}[a_4^* \otimes a_4 a_3^* a_3 + a_4 \otimes a_3^* a_3 a_4^* - a_3^* \otimes a_3 a_4^* a_4 - a_3 \otimes a_4^* a_4 a_3^*].
\end{aligned}$$

2.10 $HH^5(A)$

We have $HH^5(A) = U^*[-2] \oplus Y^*[-h-2]$. We discuss these two subspaces separately.

2.10.1 $U^*[-2]$

In $U^*[-2]$, like in $HH^4(A)$, we have generators coming from the center in some dual sense.

We have $d_6^*(U^*[-2]) = 0$.

Proposition 2.10.1.1. *Let ψ_0 be a top degree element $[\omega_i]$ in some $e_i \mathcal{N}^R e_i[-h-2]$. Then $HH^5(A)$ is generated by $\psi_k \in \mathcal{N}^R$ which satisfy $\psi_k z_k = \psi_0$.*

Proof. If $\sum_{a \in \bar{Q}} a \otimes x_a \in (V \otimes \mathcal{N})^R$ lies in degree -4, then the image of $d_5^*(x) = \sum_a a x_a - x_a \eta(a)$, under the linear map f (which is associated to A as a Frobenius algebra) is zero where $f(\omega_i) = 1$. So ψ_0 is nonzero in $HH^5(A)$.

For every non-topdegree central element z_k we can find a ζ_k satisfying the properties above, which is done for each quiver separately in subsection 2.10.3.

For any central element $z \in A$, we have that $d_5^*(zy) = d_5^*(y)z$. If $\psi_k = d_5^*(y)$, then by construction $\psi_0 = \psi_k z_k = d_4^*(z_k y)$ which is a contradiction.

So these ψ_k are nonzero in $HH^5(A)$ and generate this cohomology space. □

The relation $ax_a = x_a\eta(a)$ then gives us that all ω_i's are equivalent in $HH^5(A)$.

2.10.2 $Y^*[-h-2]$

We have to introduce some new notations.

Definition 2.10.2.1. *We define F to be the set of vertices in I which are fixed by ν, i.e.*

$$F = \{i \in I | \nu(i) = i\}.$$

Definition 2.10.2.2. *Let η_{ij} be the restriction of η on e_iAe_j $(i,j \in F)$. Let $n_{ij}^+ = \dim\ker(\eta_{ij} - 1)$ and $n_{ij}^- = \dim\ker(\eta_{ij} + 1)$.*

We define the signed truncated dimension matrix $(H_A^\eta)_{i,j\in F}$ *in the following way:*

$$(H_A^\eta)_{ij} = n_{ij}^+ - n_{ij}^-.$$

Now we can make the following statement:

Proposition 2.10.2.3. *$Y^*[-h-2]$ is given by the kernel of the matrix H_A^η, where we identify $\mathbb{C}^F = \bigoplus_{i\in F} Re_i$.*

Proof. $Y^*[-h-2]$ is the kernel of the restriction $d_6^*|_{\mathcal{N}^R[-h-2](-h-2)=R_F[-h-2]} \to A^R[-2h]$, where R_F is the linear span of e_i's, such that i is fixed by ν,

$$d_6^*(y) = \sum_{x_j\in B} x_j y \eta(x_j^*) = \sum_{x_j \in B} \eta(x_j) y x_j^*.$$

then

$$d_6^* : R_F[-h-2] \to (A^{top})^R[-2h]$$

can also be written as a matrix multiplication

$$H_A^\eta : \mathbb{C}^F \to \mathbb{C}^F$$

under the identifications $R_F = \mathbb{C}^F = \bigoplus_{i\in F} e_iA^{top}e_i$. □

We compute the matrices H_A^η and their kernels for each quiver separately.

Recall that $\dim Y = r_+ - r_- - \#\{m_i | m_i = \frac{h}{2}\} = \dim R_F - \#\{m_i | m_i = \frac{h}{2}\}$. We will find Y^* explicitly for each quiver.

$Q = E_6, E_8$

$\frac{h}{2}$ is not an exponent, so $Y^* = R_F$.

$Q = D_{n+1}$, n **odd**

All basis elements of $e_k A e_j$ given in section 2.4.4 are eigenvectors of η_{kj}.

For any of these basis elements x, $\eta(x) = (-1)^{n_x} x$ where n_x is the number of no-star letters in the monomial expression of x. So H_A^η can be computed directly, and we get

$$H_A^\eta = \begin{pmatrix} 2 & 0 & \cdots & 2 & 0 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2 & 0 & \cdots & 2 & 0 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & \cdots & 1 & 0 & \frac{n+1}{2} & -\frac{n-1}{2} \\ 1 & 0 & \cdots & 1 & 0 & -\frac{n+1}{2} & \frac{n+1}{2} \end{pmatrix},$$

and the kernel is given by

$$\langle e_{2k-1} - e_1, e_{2k}, (e_n + e_{n+1}) - e_1 | k \leq \frac{n-1}{2} \rangle.$$

$Q = D_{n+1}$, n **even**

Since $F = \{1, \ldots, n-1\}$, we work only with $e_k A e_j$ for $j, k \leq n-1$, and we have to work with a modified basis, so that they are all eigenvectors of η:

For $\underline{k \leq j \leq n-1}$,

$$\begin{aligned} B_{k,j} &= \{(a_{k-1}a_{k-1}^*)^l a_k^* \cdots a_{j-1}^* | 0 \leq l \leq \min\{k-1, n-j-1\}\} \cup \\ &\quad \{(a_{k-1}a_{k-1}^*)^l a_k^* \cdots (a_{n-1}^* a_{n-1} - a_n^* a_n) a_{n-2} a_j | 0 \leq l \leq k-1\} \cup \\ &\quad \{(a_{k-1}a_{k-1}^*)^l a_k^* \cdots (a_{n-1}^* a_{n-1} + a_n^* a_n) a_{n-2} a_j | 0 \leq l \leq k-1+j-n\}. \end{aligned}$$

For $\underline{j < k \leq n-1}$,

$$\begin{aligned} B_{k,j} &= \{a_{k-1} \cdots a_j (a_j^* a_j)^l | 0 \leq l \leq \min\{n-k-1, j-1\}\} \cup \\ &\quad \{a_k^* \cdots a_{n-2}^* (a_{n-1}^* a_{n-1} - a_n^* a_n) a_{n-2} \cdots a_j (a_j^* a_j)^l | 0 \leq l \leq j-1\} \cup \\ &\quad \{a_k^* \cdots a_{n-2}^* (a_{n-1}^* a_{n-1} + a_n^* a_n) a_{n-2} \cdots a_j (a_j^* a_j)^l | 0 \leq l \leq j-1+k-n\}. \end{aligned}$$

From that, we can calculate the matrix:

$$H_A^\eta = \begin{pmatrix} 2 & 0 & \cdots & 2 & 0 & 2 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 0 & \cdots & 2 & 0 & 2 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 0 & \cdots & 2 & 0 & 2 \end{pmatrix},$$

and we get immediately its kernel

$$\langle e_{2k+1} - e_1, e_{2k} | 1 \leq k \leq \frac{n}{2} \rangle.$$

$Q = E_7$

We don't use an explicit basis of A here. All we have to know is the number of no-star letters in the monomial basis elements which can be directly obtained from the Hilbert series $H_A(t)$ in the following way: given a monomial x of length l in $e_k A e_j$, n_{kj} the number of arrows in Q on the shortest path from j to k of length $d(k,j)$, x contains $n_{k,j} + \frac{l-d(k,j)}{2}$ arrows in Q.

So we obtain the formula

$$(H_A^\eta)_{k,j} = (-1)^n_{k,j} \left. \frac{H_A(t)_{k,j}}{t^{d(k,j)}} \right|_{t=\sqrt{(-1)}},$$

where we can get $H_A(\sqrt{-1})$ from (1.1.3.1) and compute

$$H_A^\eta = \begin{pmatrix} 3 & 0 & 3 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & -3 & 0 & 0 & 0 & 3 \end{pmatrix},$$

and its kernel is

$$\langle e_1 + e_7, e_2, e_3 + e_7, e_4, e_5, e_6 \rangle.$$

2.10.3 Result

Now we give explicit bases for each quiver where $\psi_i \in U^*[-2]$ satisfy the properties given in section 2.10.1 and $\varepsilon_i \in Y^*[-h-2]$ are taken from section 2.10.2.

Note the duality $HH^6(A) = (HH^5(A))^*$, $\phi_0(z_0) \in U[-2h-2]$, $\varphi_0(\omega_i) \in Y[-h-2]$. We choose ψ_0 such that $\psi_0(\varphi_0(z_0)) = 1$ (from that follows $\psi_k(\varphi_0(z_k)) = z_k\psi_k(\varphi_0(z_0)) = \psi_0(\varphi_0(z_0)) = 1$ and ε_i such that $\varepsilon_i(\phi_0(\omega_j)) = \delta_{ij}$.

$Q = D_{n+1}$, n odd

We define

$$\psi_{4k} = [(a_{n-1}^* a_{n-1} a_n^* a_n)^{\frac{n-1}{2}-k}],$$

$$\varepsilon_{2k-1} = [e_{2k-1} - e_1],\ \varepsilon_{2k} = [e_{2k}], \varepsilon_n = [(e_n + e_{n+1}) - e_1], k \leq \frac{n-1}{2}.$$

$Q = D_{n+1}$, n **even**

We define

$$\psi_{4k} = [a_{n-1}^* a_{n-1} (a_n^* a_n a_{n-1}^* a_{n-1})^{\frac{n-2}{2}-k}],$$

$$\varepsilon_{2k+1} = [e_{2k+1} - e_1], \varepsilon_{2k} = [e_{2k}], 1 \leq k \leq \frac{n}{2} - 1.$$

$Q = E_6$

We define

$$\begin{aligned}
\psi_0 &= [a_3^* a_3 (a_2^* a_2 a_3^* a_3)^2], \\
\psi_6 &= [-a_3^* a_3 a_2^* a_2], \\
\psi_8 &= [a_3^* a_3 - a_2^* a_2],
\end{aligned}$$

$$\varepsilon_3 = [e_3], \ \varepsilon_6 = [e_6].$$

$Q = E_7$

We define

$$\begin{aligned}
\psi_0 &= [(a_4^* a_4 a_3^* a_3)^4], \\
\psi_8 &= [(a_4^* a_4 a_3^* a_3)^2], \\
\psi_{12} &= [a_4^* a_4 a_3^* a_3],
\end{aligned}$$

$$\varepsilon_1 = [e_1 + e_7], \varepsilon_2 = [e_2], \varepsilon_3 = [e_3 + e_7], \varepsilon_4 = [e_4], \varepsilon_5 = [e_5], \varepsilon_6 = [e_6].$$

$Q = E_8$

We define

$$\begin{aligned} \psi_0 &= [(a_4^* a_4 a_3^* a_3)^7], \\ \psi_{12} &= [(a_4^* a_4 a_3^* a_3)^4], \\ \psi_{20} &= [(a_4^* a_4 a_3^* a_3)^2], \\ \psi_{24} &= [a_4^* a_4 a_3^* a_3], \end{aligned}$$

$$\varepsilon_1 = [e_1], \varepsilon_2 = [e_2], \varepsilon_3 = [e_3], \varepsilon_4 = [e_4], \varepsilon_5 = [e_5], \varepsilon_6 = [e_6], \varepsilon_7 = [e_7], \varepsilon_8 = [e_8].$$

2.11 $HH^6(A)$

$HH^6(A) = U[-2h-2] \oplus Y[-h-2] = HH^0(A)/\mathrm{Im}(d_6^*)$, and $\mathrm{Im}(d_6^*)$ is spanned by the columns of the matrices H_A^7 which were computed in the previous section.

This gives us the following result:

Proposition 2.11.0.1. *$HH^6(A)$ is a quotient of $HH^0(A)$. In particular,*

$$HH^6(A) = \begin{cases} HH^0(A) & Q = E_6, E_8 \\ HH^0(A)/(\sum_{\substack{i=1 \\ odd}}^{n-2} \omega_i = 0, \omega_n = \omega_{n+1}) & Q = D_{n+1},\ n \text{ odd} \\ HH^0(A)/(\sum_{\substack{i=1 \\ odd}}^{n-1} \omega_i = 0), & Q = D_{n+1},\ n \text{ even} \\ HH^0(A)/(\omega_1 + \omega_3 - \omega_7 = 0) & Q = E_7 \end{cases}$$

2.12 Products involving $HH^0(A) = Z$

Recall the decomposition $HH_0(A) = \mathbb{C} \oplus (U[-2])_+ \oplus L[h-2]$. It is clear that the $\mathbb{C}$-part acts on $HH^i(A)$ as the usual multiplication with $\mathbb{C}$, with z_0 as identity. From the periodicity of the Schofield resolution with period 6, it follows that the multiplication with $\varphi(z_0) \in HH^6(A)$ gives the natural isomorphism $HH^i(A) \to HH^{i+6}(A)$ for $i \geq 1$.

We summarize all products not involving the k-part.

2.12.1 $HH^0(A) \times HH^0(A) \twoheadrightarrow HH^0(A)$

This is already done in the $HH^0(A)$-section of this paper. We state the results:

$Q = D_{n+1}$, n **odd**

The products are

$$z_{4j}z_{4k} = \begin{cases} z_{4(j+k)} & j+k < \frac{n-1}{2} \\ \omega_n - \omega_{n+1} & j+k = \frac{n-1}{2} \\ 0 & j+k > \frac{n-1}{2} \end{cases}.$$

$Q = D_{n+1}$, n **even**

The products are

$$z_{4j}z_{4k} = \begin{cases} z_{4(j+k)} & j+k < \frac{n-1}{2} \\ 0 & j+k \geq \frac{n-1}{2} \end{cases}.$$

E_6

All products are zero.

E_7

The only nonzero product is $z_8^2 = \omega_1 + \omega_3 - \omega_7$.

E_8

The only nonzero product is $z_{12}^2 = z_{24}$.

2.12.2 $HH^0(A) \times HH^1(A) \twoheadrightarrow HH^1(A)$

From the definition of the maps θ_k (which are generated by the central elements z_k), it follows that the Z-action is natural, i.e. the multiplication rule is the same as with the z_k counterpart: $z_k\theta_0 = \theta_k$.

We state the other nonzero products:

$Q = D_{n+1}$

We have $z_{4j}\theta_{4k} = \theta_{4(j+k)}$ if $j + k < \frac{n-1}{2}$.

E_8

We have $z_{12}\theta_{12} = \theta_{24}$.

2.12.3 $HH^0(A) \times HH^i(A) \twoheadrightarrow HH^i(A)$, $i = 2$ or 3

$HH^2(A) = K[-2]$ and $HH^3(A) = K^*[-2]$ live in only one degree, so $(U[-2])_+ \subset HH^0(A)$ acts by zero.

2.12.4 $HH^0(A) \times HH^4(A) \twoheadrightarrow HH^4(A)$

We defined ζ_k, such that $z_k\zeta_k = \zeta_0$ holds. By degree arguments, only these other products are nonzero:

$Q = D_{n+1}$

For $l < k$, $z_{4l}\zeta_{4k} = \zeta_{4(k-l)}$ (since $z_{4(k-l)}(z_{4l}\zeta_{4k}) = (z_{4(k-l)}z_{4l})\zeta_{4k} = \zeta_0$), and $\zeta_{4(k-l)}$ is (up to a multiple) the only one element of degree $-4 - 4(k - l)$ in $HH^4(A)$).

$Q = E_8$

We have $z_{12}\zeta_{24} = \zeta_{12}$ (since $z_{12}(z_{12}\zeta_{24}) = (z_{12}z_{12})\zeta_{24} = \zeta_0$, and ζ_{12} is (up to a multiple) the only element of degree -16 in $HH^4(A)$).

2.12.5 $HH^0(A) \times HH^5(A) \twoheadrightarrow HH^5(A)$

By definition, $z_k\psi_k = \psi_0$ holds. Since $\psi_i \in U^*[-2]$ corresponds to $\zeta_i \in U^*[-2]$ in $HH^4(A)$ with the rule $z_k\psi_k = \psi_0$ corresponding to $z_k\zeta_k = \zeta_0$ above, the multiplication rules of ψ_k with elements in $HH^0(A)$ can be derived from above.

Products involving $\omega_i \in L[h-2] \subset HH^0(A)$ and $\varepsilon_j = \sum_{k\in F} \lambda_k e_k \in Y^*[-h-2]$ are easy to calculate: $\omega_i\varepsilon_j = \lambda_i[\omega_i] = \lambda_i\psi_0$.

Proposition 2.12.5.1. *The multiplication* $((U[-2])_+) \times Y^*[-h-2] \to HH^5(A)$ *is zero.*

We will show this for any quiver separately.

$Q = D_{n+1}$, n **odd**

For $l < k$, $z_{4l}\psi_{4k} = \psi_{4(k-l)}$.

The nonzero products involving $\omega_i \in L[h-2] \subset HH^0(A)$ and $\varepsilon_j \in Y^*[-h-2]$ are

$$\omega_{2k-1}\varepsilon_{2k-1} = \omega_{2k}\varepsilon_{2k} = \omega_n\varepsilon_n = \omega_{n+1}\varepsilon_n = \omega_1\varepsilon_{2k-1} = \omega_1\varepsilon_n = \psi_0,$$

$$\omega_1\varepsilon_{2k-1} = \omega_1\varepsilon_n = -\psi_0.$$

We show $(U[-2])_+ \times Y^*[-h-2] \xrightarrow{0} HH^5(A)$: by degree argument, $z_{4k}\varepsilon_i = \lambda\psi_{2n-2-4k}$. Then $z_{2n-2-4k}(z_{4k}\varepsilon_i) = \lambda z_{2n-2-4k}\psi_{2n-2-4k} = \lambda\psi_0$, and by associativity this equals $(z_{2n-2-4k}z_{4k})\varepsilon_i = (\omega_n - \omega_{n+1})\varepsilon_i = 0$, so $\lambda = 0$.

$Q = D_{n+1}$, n **even**

For $l < k$, $z_{4l}\psi_{4k} = \psi_{4(k-l)}$.

The nonzero products involving $\omega_i \in L[h-2] \subset HH^0(A)$ and $\varepsilon_j \in Y^*[-h-2]$ are

$$\omega_{2k+1}\varepsilon_{2k+1} = \omega_{2k}\varepsilon_{2k} = \psi_0,$$

$$\omega_1\varepsilon_{2k+1} = -\psi_0.$$

We show $(U[-2])_+ \times Y^*[-h-2] \xrightarrow{0} HH^5(A)$: by degree argument, $z_{4k}\varepsilon_i = \lambda\psi_{2n-2-4k}$. Then $z_{2n-2-4k}(z_{4k}\varepsilon_i) = \lambda z_{2n-2-4k}\psi_{2n-2-4k} = \lambda\psi_0$, and this equals $(z_{2n-2-4k}z_{4k})\varepsilon_i = 0$, so $\lambda = 0$.

$Q = E_6$

The nonzero products involving $\omega_i \in L[h-2] \subset HH^0(A)$ and $\varepsilon_j \in Y^*[-h-2]$ are

$$\omega_3\varepsilon_3 = \omega_6\varepsilon_6 = \psi_0.$$

By degree argument, $(U[-2])_+ \times Y^*[-h-2] \xrightarrow{0} HH^5(A)$.

$Q = E_7$

The nonzero products involving $\omega_i \in L[h-2] \subset HH^0(A)$ and $\varepsilon_j \in Y^*[-h-2]$ are

$$\omega_1\varepsilon_1 = \omega_2\varepsilon_2 = \omega_3\varepsilon_3 = \omega_4\varepsilon_4 = \omega_5\varepsilon_5 = \omega_6\varepsilon_6 = \omega_7\varepsilon_1 = \omega_7\varepsilon_3 = \psi_0.$$

We show $(U[-2])_+ \times Y^*[-h-2] \xrightarrow{0} HH^5(A)$: by degree argument, only products involving z_8 may eventually be nontrivial,

$$z_8\varepsilon_i = \lambda\psi_8, \quad \lambda \in \mathbb{C}.$$

Then

$$z_8(z_8\varepsilon_i) = \lambda z_8\psi_8 = \lambda\psi_0,$$

and by associativity this equals

$$z_8^2\varepsilon_i = (\omega_1 + \omega_3 - \omega_7)\varepsilon_i = 0,$$

so $\lambda = 0$.

2.13 Products involving $HH^1(A)$

2.13.1 $HH^1(A) \times HH^1(A) \xrightarrow{0} HH^2(A)$

This follows by degree argument since $\deg HH^1(A) > 0$, $\deg HH^2(A) = -2$.

2.13.2 $HH^1(A) \times HH^2(A) \twoheadrightarrow HH^3(A)$

$HH^2(A)$ and $HH^3(A)$ are trivial for $Q = D_{n+1}$ where n is odd and for $Q = E_7$, E_8.

We know that $HH^1(A)$ is generated by maps θ_k and $HH^2(A)$ by f_i ($i \neq \nu(i)$, and we lift

$$\begin{aligned} f_i : A \otimes A[2] &\longrightarrow A, \\ 1 \otimes 1 &\longmapsto e_i - e_{\nu(i)} \end{aligned}$$

to

$$\begin{aligned} \hat{f}_i : A \otimes A[2] &\longrightarrow A \otimes A, \\ 1 \otimes 1 &\longmapsto e_i \otimes e_i - e_{\nu(i)} \otimes e_{\nu(i)}. \end{aligned}$$

Then

$$\hat{f}_i d_3(1 \otimes 1) = \hat{f}_i(\sum_{x_j \in B} x_j \otimes x_j^*) = \sum_{x_j \in B} x_j e_i \otimes e_i x_j^* - x_j e_{\nu(i)} \otimes e_{\nu(i)} x_j^*.$$

To compute the lift Ωf_i, we need to find out the preimage of $\sum x_j e_i \otimes e_i x_j^* - x_j e_{\nu(i)} \otimes e_{\nu(i)} x_j^*$ under d_1.

Definition 2.13.2.1. *Let $b_1, \ldots, b_k$ be arrows, p the monomial $\pm b_1 \cdots b_k$ and define*

$$v_p := \pm(1 \otimes b_1 \otimes b_2 \cdots b_k + b_1 \otimes b_2 \otimes b_3 \cdots b_k + \ldots + b_1 \cdots b_{k-1} \otimes b_k \otimes 1),$$

and for $i < j$,

$$v_p^{(i,j)} := \pm \sum_{l=i}^{j} b_1 \cdots b_{l-1} \otimes b_l \otimes b_{l+1} \cdots b_k.$$

We will use the following lemma in our computations.

Lemma 2.13.2.2. *In the above setting,*

$$d_1(v_p) = \pm(b_1 \cdots b_k \otimes 1 - 1 \otimes b_1 \cdots b_k).$$

From that, we see immediately that when assuming all x_j are monomials (which we can do), then

$$\hat{f}_i(\sum_{x_j\in B} x_j \otimes x_j^*) = d_1(\sum_{x_j\in B} v^{(1,\deg(x_j))}_{x_je_ix_j^*} - v^{(1,\deg(x_j))}_{x_je_{\nu(i)}x_j^*}) + 1\otimes \underbrace{\sum_{x_j\in B}(x_je_ix_j^* - x_je_{\nu(i)}x_j^*)}_{=0},$$

so we have

$$\begin{aligned}\Omega f_i : \Omega^3(A) &\longrightarrow \Omega(A),\\ 1\otimes 1 &\longmapsto \sum_{x_j\in B} v^{(1,\deg(x_j))}_{x_je_ix_j^*} - v^{(1,\deg(x_j))}_{x_je_{\nu(i)}x_j^*}.\end{aligned}$$

Then

$$\theta_k(\sum_{x_j\in B} v^{(1,\deg(x_j))}_{x_je_ix_j^*} - v^{(1,\deg(x_j))}_{x_je_{\nu(i)}x_j^*}) = z_k(\sum_{x_j\in B_{-,i}} s(x_j)x_jx_j^* - \sum_{x_j\in B_{-,\nu(i)}} s(x_j)x_jx_j^*),$$

where $s(x_j)$ is the number of arrows in Q^* in the monomial expression of x_j.

So we get

$$(\theta_k\circ\Omega f_i)(1\otimes 1) = z_k(\sum_{x_j\in B_{-,i}} s(x_j)x_jx_j^* - \sum_{x_j\in B_{-,\nu(i)}} s(x_j)x_jx_j^*).$$

Under our identification in 2.2.5,

$$\theta_k f_i = [z_k(\sum_{l\in I}\sum_{x_j\in B_{l,i}} s(x_j)\omega_l - \sum_{l\in I}\sum_{x_j\in B_{l,\nu(i)}} s(x_j)\omega_l)] \in HH^3(A).$$

All products are zero if z_k lies in a positive degree, so we only have to calculate the products where $k=0$.

We make the following

Proposition 2.13.2.3. *The multiplication with θ_0 induces a symmetric isomorphism*

$$\alpha : HH^2(A) = K[-2] \overset{\cong}{\to} K^*[-2] = HH^3(A).$$

Now we have to work with explicit basis elements $x_j \in Ae_i$, $i \neq \nu(i)$, so we treat the Dynkin quivers separately and find the matrix M_α which represents this map.

$Q = D_{n+1}$, n even

We can work with the basis given in section 2.4.4 and compute

$$\theta_0 f_n = \frac{n}{2}([\omega_{n+1}] - [\omega_n]) = -nh_n \tag{2.13.2.4}$$

because of the relation $[\omega_n] + [\omega_{n+1}] = 0$ in $HH^3(A)$. α is given by the matrix

$$M_\alpha = (-n).$$

E_6

We will write out the basis elements of Ae_1, Ae_5:

$$\begin{aligned}
B_{1,1} &= \langle e_1, a_1a_2a_5^*a_5a_2^*a_1^*\rangle,\\
B_{2,1} &= \langle a_1^*, a_2a_5^*a_5a_2^*a_1^*, a_2a_3^*a_3a_5^*a_5a_2^*a_1^*\rangle,\\
B_{3,1} &= \langle a_2^*a_1^*, a_3^*a_3a_2^*a_1^*, a_3^*a_3a_3^*a_3a_2^*a_1^*, a_5^*a_5a_3^*a_3a_3^*a_3a_2^*a_1^*\rangle,\\
B_{4,1} &= \langle a_3a_2^*a_1^*, a_3a_5^*a_5a_2^*a_1^*, a_3a_5^*a_5a_3^*a_3a_3^*a_5a_5a_2^*a_1^*\rangle,\\
B_{5,1} &= \langle a_4a_3a_2^*a_1^*, a_4a_3a_5^*a_5a_3^*a_3a_5^*a_5a_2^*a_1^*\rangle,\\
B_{6,1} &= \langle a_5a_2^*a_1^*, a_5a_3^*a_3a_5^*a_5a_2^*a_1^*\rangle,
\end{aligned}$$

and

$$e_iAe_5 = \langle \eta(x) | x \in e_{\nu(i)}Ae_1\rangle,$$

where $\eta(a) = -\epsilon_a\bar{a}$ and for any arrow $a : i \to j$, $\bar{a}$ is the arrow $j :\to i$, so η preserves the number of star letters of a monomial x. From this, we obtain

$$\theta_0 f_1 = -4[\omega_1] - 2[\omega_2] + 2[\omega_4] + 4[\omega_5] = -8h_1 - 4h_2.$$

because of the relations $[\omega_1] + [\omega_4] = [\omega_2] + [\omega_3] = 0$ in $HH^3(A)$.

We do the same thing for Ae_2 and Ae_4:

$$B_{1,2} = \langle a_1, a_1a_2a_5^*a_5a_2^*, a_1a_2a_5^*a_5a_3^*a_3a_2^*\rangle,$$

$$B_{2,2} = \langle e_2, a_2a_2^*, a_2a_5^*a_5a_2^*, a_2a_3^*a_3a_5^*a_5a_2^*, a_2a_5^*a_5a_3^*a_3a_2^*, a_2a_5^*a_5a_3^*a_3a_5^*a_5a_2^*\rangle,$$

$$B_{3,2} = \langle a_2^*, a_5^*a_5a_2^*, a_3^*a_3a_2^*, a_5^*a_5a_3^*a_3a_2^*, a_3^*a_3a_5^*a_5a_2^*,$$
$$a_3^*a_3a_5^*a_5a_3^*a_3a_2^*, a_5^*a_5a_3^*a_3a_5^*a_5a_2^*, a_5^*a_5a_3^*a_3a_5^*a_5a_3^*a_3a_2^*\rangle,$$

$$B_{4,2} = \langle a_3a_2^*, a_3a_5^*a_5a_2^*, a_3a_3^*a_3a_2^*, a_3a_3^*a_3a_5^*a_5a_2^*,$$
$$a_3a_5^*a_5a_3^*a_3a_5^*a_5a_2^*, a_3a_3^*a_3a_5^*a_5a_3^*a_3a_5^*a_5a_2^*\rangle,$$

$$B_{5,2} = \langle a_4a_3a_2^*, a_4a_3a_5^*a_5a_2^*, a_4a_3a_5^*a_5a_3^*a_3a_5^*a_5a_2^*\rangle,$$

$$B_{6,2} = \langle a_5a_2^*, a_5a_3^*a_3a_2^*, a_5a_3^*a_3a_5^*a_5a_2^*, a_5a_3^*a_3a_5^*a_5a_3^*a_3a_2^*\rangle,$$

and we get the basis elements for e_iAe_3 from $\eta(x_j)$ where $x_j \in e_{\nu(i)}Ae_4$. Since η preserves the number of star-letters of a monomial, we can immediately calculate

$$\theta_0 f_2 = -2[\omega_1] - 4[\omega_2] + 4[\omega_4] + 2[\omega_5] = -4h_1 - 8h_2$$

because of the relations $[\omega_1] + [\omega_4] = [\omega_2] + [\omega_3] = 0$ in $HH^3(A)$.

So α is given by the symmetric, nondegenerate matrix

$$M_\alpha = \begin{pmatrix} -8 & -4 \\ -4 & -8 \end{pmatrix}. \qquad (2.13.2.5)$$

2.13.3 $HH^1(A) \times HH^3(A) \xrightarrow{0} HH^4(A)$

This follows by degree argument: $\deg HH^1(A) \geq 0$, $\deg HH^3(A) = -2$, but $\deg HH^4(A) \leq -4$.

2.13.4 $HH^1(A) \times HH^4(A) \to HH^5(A)$

Proposition 2.13.4.1. *Given $\theta_k \in HH^1(A)$ and $\zeta_l \in HH^4(A)$, we get the following cup product:*

$$\theta_k \zeta_l = \psi_l z_k. \tag{2.13.4.2}$$

Proof. It is enough to show $\theta_0\zeta_0 = \psi_0$: $z_l(\theta_0\zeta_l) = \theta_0\zeta_0\psi_0$ implies that $(\theta_0\zeta_l) = \psi_l$, and the equation above follows from $\theta_k = z_k\theta_0$.

Let in general $x = \sum_{a\in\bar{Q}} a \otimes x_a \in HH^4(A)$. Then x represents the map

$$\begin{aligned} x := A \otimes V \otimes \mathcal{N}[h] &\longrightarrow A, \\ 1 \otimes a_i \otimes 1 &\longmapsto -x_{a_i^*} \\ 1 \otimes a_i^* \otimes 1 &\longmapsto x_{a_i}, \end{aligned}$$

and it lifts to

$$\begin{aligned} \hat{x} : A \otimes V \otimes \mathcal{N}[h] &\longrightarrow A \otimes A, \\ 1 \otimes a \otimes 1 &\longmapsto -1 \otimes x_{a^*} \\ 1 \otimes a^* \otimes 1 &\longmapsto 1 \otimes x_a. \end{aligned}$$

Then

$$\begin{aligned} (\hat{x} \circ d_5)(1 \otimes 1) &= \hat{x}(\sum_{a\in\bar{Q}} \epsilon_a a \otimes a^* \otimes 1 + \sum_{a\in\bar{Q}} \epsilon_a 1 \otimes a \otimes a^*) \\ &= \sum_{a\in Q} a \otimes x_a - \sum_{a\in Q} 1 \otimes x_a \eta(a) + \sum_{a\in Q} a^* \otimes x_{a^*} - \sum_{a\in Q} 1 \otimes x_{a^*}\eta(a^*) \\ &= \sum_{a\in Q} a \otimes x_a - \sum_{a\in Q} 1 \otimes a x_a \sum_{a\in Q} a^* \otimes x_{a^*} - \sum_{a\in Q} 1 \otimes a^* x_{a^*} \\ &= d_1(\sum_{a\in Q} 1 \otimes a \otimes x_a + 1 \otimes a^* \otimes x_{a^*}), \end{aligned}$$

so we have

$$\begin{aligned} \Omega x : \Omega^5(A) &\longrightarrow \Omega(A), \\ 1 \otimes 1 &\longmapsto \sum_{a \in Q} 1 \otimes a \otimes x_a + 1 \otimes a^* \otimes x_{a^*}, \end{aligned}$$

and this gives us

$$(\theta_0 \circ x)(1 \otimes 1) = \sum_{a \in Q} a^* x_{a^*},$$

so the cup product is

$$\theta_0 \cdot x = \sum_{a \in Q} a^* x_{a^*}. \tag{2.13.4.3}$$

It can be easily checked by using explicit elements that the RHS is ψ_0 for $x = \zeta_0$, but we the reason here why this is true: for $x = \sum_{a \in Q} a \otimes x_a = \zeta_0$, the RHS becomes

$$\sum_{a \in Q} a^* x_{a^*} = \sum_{a \in Q} (a^*, x_{a^*})[\omega_{t(a)}],$$

where $(-,-) : A \times A \to \mathbb{C}$ is the bilinear form attached to A as a Frobenius algebra (see 1.1.2).

But under the bilinear form on $V \otimes A$, given in Subsection 2.2.3 which induces the duality $HH^4(A) = (HH^1(A))^*$,

$$(a \otimes x_a, b \otimes x_b) = \delta_{a,b^*} \epsilon_a (x_a, x_b),$$

$$\sum_{a \in Q} (a^*, x_{a^*}) = (\theta_0, \zeta_0) = 1.$$

So for $x = \zeta_0$, equation (2.13.4.3) becomes

$$\theta_0 \zeta_0 = (\theta_0, \zeta_0) \psi_0 = \psi_0, \tag{2.13.4.4}$$

because $[\omega_i] = \psi_0$ in $HH^5(A)$ for all $i \in I$. □

2.13.5 $HH^1(A) \times HH^5(A) \to HH^6(A)$

We know that

$$\begin{aligned} 0 &\leq \deg(HH^1(A)) \leq h-4, \\ -h-2 &\leq \deg(HH^5(A)) \leq -2, \\ -2h &\leq \deg(HH^6(A)) \leq -h-2, \end{aligned}$$

so the product is trivial unless we pair the lowest degree parts of $HH^1(A)$ (generated by θ_0) and $HH^5(A)$ (which is $Y^*[-h-2]$). The product will then live in degree $-h-2$ which is the top degree part of $HH^6(A)$, the space $Y[-h-2]$.

Given an element $\psi \in HH^5(A)(-h-2))$ which has the form

$$\begin{aligned} \psi : A \otimes \mathcal{N}[h+2] &\longrightarrow A, \\ 1 \otimes 1 &\longmapsto \sum_{i \in F} \lambda_i e_i \in R, \end{aligned}$$

this lifts to

$$\begin{aligned} \hat{\psi} : A \otimes \mathcal{N}[h+2] &\longrightarrow A \otimes A, \\ 1 \otimes 1 &\longmapsto \sum_{i \in F} \lambda_i e_i \otimes e_i. \end{aligned}$$

Then

$$\begin{aligned} \hat{\psi}(d_6(1 \otimes 1)) &= \hat{\psi}(\sum_{x_j \in B} x_j \otimes x_j^*) = \hat{\psi}(\sum_{x_j \in B} \eta(x_j) \otimes \eta(x_j^*)) \\ &= \sum_{x_j \in B} \sum_{i \in F} \lambda_i \eta(x_j) e_i \otimes e_i x_j^* \\ &= d_1(\sum_{i \in F} \sum_{x_j \in B} \lambda_i v^{(1,\deg(x_j))}_{\eta(x_j) e_i x_j^*}) + \underbrace{1 \otimes \sum_{x_j \in B} \sum_{i \in F} \lambda_i \eta(x_j) e_i x_j^*}_{=0}, \end{aligned}$$

so ψ lifts to

$$\begin{aligned} \Omega\psi : \Omega^6(A) &\longrightarrow \Omega(A), \\ 1 \otimes 1 &\longmapsto \sum_{i \in F} \sum_{x_j \in B} \lambda_i v^{(1,\deg(x_j))}_{\eta(x_j) e_i x_j^*}. \end{aligned}$$

We get

$$(\theta_0 \circ \Omega\psi)(1 \otimes 1) = \sum_{i \in F} \sum_{x_j \in B_{-,i}} \lambda_i s(x_j)\eta(x_j)x_j^*,$$

where $s(x_j)$ is the number of arrows in Q^* in the monomial expression of x_j (or in general if x_j is a homogeneous polynomial where each monomial term has the same number of arrows in Q^*, then $s(x_j)$ is the number of Q^*-arrows in each monomial term).

Under our identifications in Subsection 2.2.5

$$\theta_0\psi = \sum_{i \in F} \sum_{x_j \in B_{-,i}} \lambda_i s(x_j)\eta(x_j)x_j^* = \sum_{i,k \in F} \sum_{x_j \in B_{ki}} \lambda_i s(x_j)\eta(x_j)x_j^*.$$

To simplify this computation, we will choose a basis, such that all $x_j \in e_k A e_l$ for some $k, l \in I$ and that additionally x_j is an eigenvector of η for $k, l \in F$ (since η is an involution on $e_k A e_l$ for $k, l \in F$). Let $B_{k,l}^+$ be a basis of $(e_k A e_l)_+ = \ker(\eta|_{e_k A e_l} - 1)$ and $B_{k,l}^-$ a basis of $(e_k A e_l)_- = \ker(\eta|_{e_k A e_l} + 1)$.

Let us define

$$\kappa_{k,l} = \sum_{x_j \in B_{k,l}^+} s(x_j) - \sum_{x_j \in B_{k,l}^-} s(x_j). \tag{2.13.5.1}$$

Then the above equation becomes

$$\theta_0\psi = \sum_{l \in F} \lambda_l \sum_{k \in F} \kappa_{k,l}\varphi_0(\omega_k). \tag{2.13.5.2}$$

Proposition 2.13.5.3. *The multiplication by θ_0 induces a skew-symmetric isomorphism*

$$\beta : Y^*[-h-2] \xrightarrow{\cong} Y[-h-2].$$

We will treat the Dynkin quivers separately and find the matrix M_β which represents β for each of these quivers.

$Q = D_{n+1}$, n **odd**

We use the same basis as given in section 2.4.4. Recall that these basis elements have the property $\eta(x) = (-1)^{n_x}x$ where n_x is the number of Q-arrows in the monomial expression of x.

We can compute that for $k, l \leq n-1$,

$$\begin{aligned}
\kappa_{k,l} &= \begin{cases} n-k+l-1 & k\, odd, \quad l\, odd \\ l-n & k\, odd, \quad l\, even \\ -k & k\, even, \quad l\, odd \\ 0 & k\, even, \quad l\, even \end{cases}, \\
\kappa_{k,n} = \kappa_{k,n+1} &= \begin{cases} n - \frac{k+1}{2} & k\, odd \\ -\frac{k}{2} & k\, even \end{cases}, \\
\kappa_{n,l} = \kappa_{n+1,l} &= \begin{cases} n - \frac{l-1}{2} & l\, odd \\ \frac{l}{2} & k\, even \end{cases}, \\
\kappa_{n,n} = \kappa_{n+1,n+1} &= \frac{n^2-1}{4}, \\
\kappa_{n+1,n} = \kappa_{n,n+1} &= -\left(\frac{n-1}{2}\right)^2.
\end{aligned}$$

$Y^*[-h-2]$ has basis $\varepsilon_{2k+1} = [e_{2k+1} - e_1]$ $(0 \leq k \leq \frac{n-3}{2})$, $\varepsilon_{2k} = [e_{2k}]$ $(k \leq \frac{n-1}{2})$,

$\varepsilon_n = [e_n + e_{n+1} - e_1]$, and we can calculate the products

$$\begin{aligned}
\theta_0\varepsilon_{2k+1} &= \sum_{i\in F}(\kappa_{i,2k+1} - \kappa_{i,1})\varphi_0(\omega_i) \\
&= 2k\sum_{\substack{i=1\\ odd}}^{n-2}\varphi_0(\omega_i) - n\sum_{\substack{i=2\\ even}}^{2k}\varphi_0(\omega_i) + k\varphi_0(\omega_n + \omega_{n+1}), \\
\theta_0\varepsilon_{2k} &= \sum_{i\in F}(\kappa_{i,2k+1})\varphi_0(\omega_i) \\
&= (2k-n)\sum_{\substack{i=1\\ odd}}^{2k-1}\varphi_0(\omega_i) + 2k\sum_{\substack{i=2k+1\\ odd}}^{n-2}\varphi_0(\omega_i) + k\varphi_0(\omega_n + \omega_{n+1}), \\
\theta_0\varepsilon_n &= \sum_{i\in F}(\kappa_{i,n} + \kappa_{n+1,1} - \kappa_{i,1})\varphi_0(\omega_i) \\
&= (n-1)\sum_{i=1,odd}^{n-2}\varphi_0(\omega_i) - n\sum_{\substack{i=2\\ even}}^{n-1}\varphi_0(\omega_i) + \frac{n-1}{2}\varphi_0(\omega_n + \omega_{n+1}).
\end{aligned}$$

We use the defining relations in $Y[-h-2]$,

$$\begin{aligned}
\varphi_0(\omega_1) &= -\varphi_0(\sum_{\substack{i=3\\ odd}}^{n-2}\varphi_0(\omega_i) - \varphi_0(\omega_n)), \\
\varphi_0(\omega_{n+1}) &= \varphi_0(\omega_n)
\end{aligned}$$

to write the RHS of the above cup product calculations in terms of the basis $(\omega_i)_{2\leq i\leq n}$:

$$\begin{aligned}
\theta_0\varepsilon_{2k+1} &= -n\sum_{\substack{i=2\\ even}}^{2k}\varphi_0(\omega_i), \\
\theta_0\varepsilon_{2k} &= n\sum_{\substack{i=2k+1\\ odd}}^{n-2}\varphi_0(\omega_i) + n\varphi_0(\omega_n), \\
\theta_0\varepsilon_n &= -n\sum_{\substack{i=2\\ even}}^{n-1}\varphi_0(\omega_i).
\end{aligned}$$

β is given by the skew-symmetric, nondegenerate matrix

$$M_\beta = \begin{pmatrix} 0 & -n & 0 & -n & \dots & \dots & -n & 0 & -n & 0 & -n \\ n & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -n & \dots & \dots & -n & 0 & -n & 0 & -n \\ n & 0 & n & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & -n & 0 & -n & 0 & -n \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n & 0 & n & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & -n & 0 & -n \\ n & 0 & n & 0 & \dots & \dots & 0 & n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & -n \\ n & 0 & n & 0 & \dots & \dots & 0 & n & 0 & n & 0 \end{pmatrix}$$

with respect to the chosen basis $\varepsilon_2, \varepsilon_3, \dots \varepsilon_n$ of $Y^*[-h-2]$ and the dual basis $\varphi_0(\omega_2), \varphi_0(\omega_3) \dots \varphi_0(\omega_n)$ of $Y[-h-2]$).

$Q = D_{n+1}$, n **even**

We use the same basis as in section 2.10.2 for our computations.

For $k, l \leq n-1$,

$$\kappa_{k,l} = \begin{cases} n-k+l-1 & k\,odd, \quad l\,odd \\ l-n & k\,odd, \quad l\,even \\ -k & k\,even, \quad l\,odd \\ 0 & k\,even, \quad l\,even \end{cases}.$$

$Y^*[-h-2]$ has basis $\varepsilon_{2k} = [e_{2k}]$, $\varepsilon_{2k+1} = [e_{2k+1} - e_1]$ $(1 \leq k \leq \frac{n-2}{2})$, and we calculate the products

$$\begin{aligned}
\theta_0\varepsilon_{2k+1} &= \sum_{i\in F}(\kappa_{i,2k+1}-\kappa_{i,1})\varphi_0(\omega_i) \\
&= 2k\sum_{\substack{i=1,\\ odd}}^{n-1}\varphi_0(\omega_i) - n\sum_{\substack{i=2\\ even}}^{2k}\varphi_0(\omega_i), \\
\theta_0\varepsilon_{2k} &= \sum_{i\in F}(\kappa_{i,2k})\varphi_0(\omega_i) \\
&= (2k-n)\sum_{\substack{i=1\\ odd}}^{2k-1}[\omega_i] + 2k\sum_{i=2k+1}^{n-2}\varphi_0(\omega_i),
\end{aligned}$$

and we use the defining relation of $Y[-h-2]$,

$$\varphi_0(\omega_1) = -\sum_{\substack{i=3\\ odd}}^{n-2}\varphi(\omega_i)$$

to write the results of the cup product calculations in terms of the basis $\varphi_0(\omega_2), \varphi_0(\omega_3), \ldots, \varphi_0(\omega_{n-1})$. We get

$$\begin{aligned}
\theta_0\varepsilon_{2k+1} &= -n\sum_{\substack{i=2\\ even}}^{2k}\varphi_0(\omega_i) \\
\theta_0\varepsilon_{2k} &= n\sum_{\substack{i=2k+1\\ odd}}^{n-1}\varphi_0(\omega_i).
\end{aligned}$$

β is given by the matrix

$$M_\beta = \begin{pmatrix} 0 & -n & 0 & -n & \dots & \dots & -n & 0 & -n & 0 & -n \\ n & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -n & \dots & \dots & -n & 0 & -n & 0 & -n \\ n & 0 & n & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & -n & 0 & -n & 0 & -n \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n & 0 & n & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & -n & 0 & -n \\ n & 0 & n & 0 & \dots & \dots & 0 & n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & -n \\ n & 0 & n & 0 & \dots & \dots & 0 & n & 0 & n & 0 \end{pmatrix}$$

with respect to the basis $\varepsilon_2, \varepsilon_3, \dots, \varepsilon_{n-1}$ and its dual basis $\varphi_0(\omega_2), \varphi_0(\omega_3), \dots \varphi_0(\omega_{n-1})$.

$Q = E_6$

We work with the bases

$$
\begin{aligned}
B^+_{3,3} &= \{e_3, a_3^*a_3 - a_2^*a_2, (a_3^*a_3 - a_2^*a_2)^2, a_5^*a_5a_3^*a_3a_5^*a_5, \\
&\quad a_5^*a_5a_3^*a_3a_5^*a_5a_3^*a_3, a_3^*a_3a_5^*a_5a_3^*a_3a_5^*a_5a_3^*a_3\}, \\
B^-_{3,3} &= \{a_5^*a_5, a_3^*a_3a_5^*a_5, a_5^*a_5a_3^*a_3, a_3^*a_3a_5^*a_5a_3^*a_3, \\
&\quad a_5^*a_5a_3a_3^*(a_3^*a_3 - a_2^*a_2)^2, a_3^*a_3a_5^*a_5a_3a_3^*(a_3^*a_3 - a_2^*a_2)^2\}, \\
B^+_{6,3} &= \{a_5a_3^*a_3a_5^*a_5, a_5a_3^*a_3a_5^*a_5a_3^*a_3), \\
&\quad a_5a_3^*a_3a_5^*a_5a_3a_3^*(a_3^*a_3 - a_2^*a_2)\}, \\
B^-_{6,3} &= \{a_5, a_5a_3^*a_3, a_5a_3^*a_3(a_3^*a_3 - a_2^*a_2)\}, \\
B^+_{3,6} &= \{a_5^*, a_3^*a_3a_5^*, (a_3^*a_3 - a_2^*a_2)a_3^*a_3a_5^*\}, \\
B^-_{3,6} &= \{a_5^*a_5a_3^*a_3a_5^*, a_3^*a_3a_5^*a_5a_3^*a_3a_5^*, a_3^*a_3a_5^*a_5(a_3^*a_3)^2a_5^*\}, \\
B^+_{6,6} &= \{e_6, a_5a_3^*a_3a_5^*a_5(a_3^*a_3)^2a_5^*\}, \\
B^-_{6,6} &= \{a_5a_3^*a_3a_5^*, a_5(a_3^*a_3)^2a_5^*\}.
\end{aligned}
$$

We immediately get the matrix

$$
M_\beta = \begin{pmatrix} \kappa_{3,3} & \kappa_{3,6} \\ \kappa_{6,3} & \kappa_{6,6} \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ 6 & 0 \end{pmatrix}
$$

which represents the β with respect to the basis $\varepsilon_3, \varepsilon_6$ and dual basis $\varphi_0(\omega_3), \varphi_0(\omega_6)$.

E_7

For E_7 and E_8 we don't have to work with an explicit basis to calculate $\kappa_{k,l}$ since for any basis element x, $\eta(x) = \pm x$. It is enough to know the following: given any monomial $x \in e_kAe_j$ of length l, $n_{k,j}$ the number of arrows $x \in Q$ and $d(k,j)$ the distance between the vertices k, j, we know that x contains $n_{k,j} + \frac{l-d(k,j)}{2}$ arrows in Q and $d(k,j) - n_{k,j} + \frac{l-d(k,j)}{2}$ arrows in $\bar{Q}$.

We can derive the following formula:

$$\kappa_{k,j} = (-1)^{n_{k,j}} \left((d(k,j) - n_{k,j}) \frac{H_A(t)}{t^{d(k,j)}}\bigg|_{t=\sqrt{-1}} + \frac{1}{2} t \frac{d}{dt} \frac{H_A(t)}{t^{d(k,j)}}\bigg|_{t=\sqrt{-1}} \right). \tag{2.13.5.4}$$

The resulting matrix is

$$(\kappa_{k,j})_{k,j} = \begin{pmatrix} 12 & 6 & 9 & 3 & 0 & 3 & -9 \\ -6 & 0 & 3 & 0 & 0 & 0 & -3 \\ 15 & -3 & 12 & 3 & 0 & 3 & -12 \\ -3 & 0 & -3 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 & -9 & 0 \\ -3 & 0 & -3 & 0 & 9 & 0 & -6 \\ -15 & 3 & -12 & 6 & 0 & 6 & 12 \end{pmatrix}.$$

A basis of $Y^*[-h-2]$ is given by

$$\varepsilon_1 = [e_1 + e_7], \varepsilon_2 = [e_2], \varepsilon_3 = [e_3 + e_7], \varepsilon_4 = [e_4], \varepsilon_5 = [e_5], \varepsilon_6 = [e_6],$$

$(\theta_0 \varepsilon_i)_{1 \le 1 \le 6}$ is given by

$$\begin{pmatrix} 3 & 6 & 0 & 3 & 0 & 3 \\ -9 & 0 & 0 & 0 & 0 & 0 \\ 3 & -3 & 0 & 3 & 0 & 3 \\ -9 & 0 & -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9 \\ -9 & 0 & -9 & 0 & 9 & 0 \\ -3 & 3 & 0 & 6 & 0 & 6 \end{pmatrix} \begin{pmatrix} \varphi_0(\omega_1) \\ \varphi_0(\omega_2) \\ \varphi_0(\omega_3) \\ \varphi_0(\omega_4) \\ \varphi_0(\omega_5) \\ \varphi_0(\omega_6) \\ \varphi_0(\omega_7) \end{pmatrix}.$$

Now use the defining relation of $Y[-h-2]$,

$$\varphi_0(\omega_7) = \varphi_0(\omega_1) + \varphi_0(\omega_3)$$

to obtain the matrix

$$M_\beta = \begin{pmatrix} 0 & 9 & 0 & 9 & 0 & 9 \\ -9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 & 9 \\ -9 & 0 & -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9 \\ -9 & 0 & -9 & 0 & 9 & 0 \end{pmatrix}$$

which represents β with respect to the basis $\varepsilon_1 \dots \varepsilon_6$ and its dual basis $\varphi_0(\omega_1), \dots, \varphi_0(\omega_6)$.

E_8

We can use (2.13.5.4) and get the matrix

$$M_\beta = (\kappa_{k,j})_{k,j} = \begin{pmatrix} 0 & 15 & 0 & 15 & 0 & 0 & 0 & -15 \\ -15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 15 & 0 & 0 & 0 & -15 \\ -15 & 0 & -15 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -15 \\ 0 & 0 & 0 & 0 & 0 & 0 & -15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 15 & 0 & -15 \\ 15 & 0 & 15 & 0 & 15 & 0 & 15 & 0 \end{pmatrix}$$

which represents β with respect to the basis $\varepsilon_1, \dots \varepsilon_8$ and its dual basis $\varphi_0(\omega_1), \dots, \varphi_0(\omega_8)$.

Remark 2.13.5.5. With respect to our chosen bases $(\varepsilon_i)_{i\in I'}$ and $\phi_0(\omega_i)_{i\in I'}$, such that the vertex set $I' \subset I$, together with the arrows in I form a connected subquiver $\bar{Q}'$, M_β can be written in this general form:

$$M_\beta = \frac{h}{2} \cdot (C')^\epsilon, \qquad (2.13.5.6)$$

where we call $(C')^{\epsilon}$ the *signed adjacency matrix* of the subquiver $\bar{Q}'$, that is

$$(C')_{ij} = \begin{cases} 0 & \text{if } i,j \text{ are not adjacent,} \\ +1 & \text{if arrow } i \leftarrow j \text{ lies in } Q^*, \\ -1 & \text{if arrow } i \leftarrow j \text{ lies in } Q, \end{cases} \tag{2.13.5.7}$$

In the D_{n+1}-case, we have

$$M_\beta = n \cdot \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 \end{pmatrix}^{-1},$$

in the E_6-case, we have

$$M_\beta = 6 \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1},$$

in the E_7-case, we have

$$M_\beta = 9 \cdot \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}^{-1},$$

and in the E_8-case, we have

$$M_\beta = 15 \cdot \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}^{-1}.$$

2.14 Products involving $HH^2(A)$

We start with $HH^2(A) \times HH^3(A) \to HH^5(A)$ first and then deduce $HH^2(A) \times HH^2(A) \to HH^4(A)$ from associativity.

2.14.1 $HH^2(A) \times HH^3(A) \to HH^5(A)$

We will prove the following general proposition:

Proposition 2.14.1.1. *For the basis elements $f_i \in HH^2(A)$, $h_j \in HH^3(A)$, the cup product is*

$$f_i h_j = \delta_{ij}\psi_0. \tag{2.14.1.2}$$

Proof. Recall the maps

$$\begin{aligned} h_j : A \otimes \mathcal{N} &\to A, \\ 1 \otimes 1 &\mapsto \omega_j \end{aligned}$$

and lift it to

$$\begin{aligned} \hat{h}_j : A \otimes \mathcal{N} &\to A \otimes A, \\ 1 \otimes 1 &\mapsto 1 \otimes \omega_j. \end{aligned}$$

Then

$$\hat{h}_j(d_4(1 \otimes a \otimes 1)) = \hat{h}_j(a \otimes 1 - 1 \otimes a) = a \otimes \omega_j = d_1(1 \otimes a \otimes \omega_j),$$

so

$$\begin{aligned} \Omega h_j : \Omega^4(A) &\to \Omega(A), \\ 1 \otimes a \otimes 1 &\mapsto 1 \otimes a \otimes \omega_j. \end{aligned}$$

Then we have

$$\begin{aligned}\Omega h_j(d_5(1\otimes 1)) &= \Omega h_j(\sum_{a\in\bar{Q}Q}\epsilon_a a\otimes a^*\otimes 1+\sum_{a\in\bar{Q}}\epsilon_a 1\otimes a\otimes a^*)\\ &= \sum_{a\in\bar{Q}}\epsilon_a\otimes a^*\otimes\omega_j = d_2(1\otimes\omega_j),\end{aligned}$$

so

$$\begin{aligned}\Omega^2 h_j:\Omega^5(A) &\to \Omega^2(A),\\ 1\otimes 1 &\mapsto 1\otimes\omega_j.\end{aligned}$$

This gives us

$$f_i(\Omega^2 h_j)(1\otimes 1) = f_i(1\otimes\omega_j) = \delta_{ij}\omega_j,$$

i.e. the cup product

$$f_i h_j = \delta_{ij}[\omega_j] = \delta_{ij}\psi_0.$$

□

2.14.2 $HH^2(A)\times HH^2(A)\to HH^4(A)$

Since $\deg HH^2(A) = -2$, their product has degree -4 (i.e. lies in $span(\zeta_0)$, so it can be written as

$$\begin{aligned}HH^2(A)\times HH^2(A) &\to HH^4(A),\\ (a,b) &\mapsto \langle a,b\rangle\zeta_0,\end{aligned}$$

where $\langle -,-\rangle : HH^2(A)\times HH^2(A)\to\mathbb{C}$ is a bilinear form. We prove the following proposition:

Proposition 2.14.2.1. *The cup product $HH^2(A)\times HH^2(A)\to HH^4(A)$ is given by $\langle -,-\rangle=\alpha$, where α (from Proposition 2.13.2.3) is regarded as a symmetric bilinear form.*

Proof. We use (2.13.4.2) to get

$$\theta_0(f_i f_j) = \theta_0(\langle f_i, f_j\rangle \zeta_0) = \langle f_i, f_j\rangle \psi_0. \tag{2.14.2.2}$$

On the other hand, by Proposition 2.13.2.3 and Proposition 2.14.1.1,

$$(\theta_0 f_i) f_j = \alpha(f_i) f_j = \sum (M_\alpha)_{li} h_l f_j = (M_\alpha)_{ji} \psi_0 = (M_\alpha)_{ij} \psi_0. \tag{2.14.2.3}$$

By associativity of the cup product, we can equate (2.14.2.2) and (2.14.2.3) to get

$$\langle f_i, f_j\rangle = (M_\alpha)_{ij}. \tag{2.14.2.4}$$

□

2.14.3 $HH^2(A) \times HH^4(A) \xrightarrow{0} HH^6(A)$

This computation uses the Batalin-Vilkovisky structure on Hochschild cohomology, introduced later in section 3.3: we have $\deg HH^2(A) = -2$, $\deg HH^4(A) \geq -h$ and $\deg HH^6(A) \leq -h-2$. So we know by degree argument that

$$f_k\zeta_l = \begin{cases} 0 & l > h-4 \\ \sum_s \lambda_s \varphi(\omega_s) & l = h-4 \end{cases}. \tag{2.14.3.1}$$

We use (3.3.0.14) and the isomorphism $HH^i(A) = HH_{6m+2-i}(A)$ to get for the Gerstenhaber bracket on $HH^*(A)$:

$$\begin{aligned} [f_k, \zeta_l] &= \Delta(f_k\zeta_l) - \underbrace{\Delta(f_k)}_{=0}\zeta_l - f_k\underbrace{\Delta(\zeta_l)}_{=0} \\ &= \sum_s \lambda_s(\frac{1}{2}+m)h\beta^{-1}(\varphi(\omega_s)) \end{aligned}$$

The Gerstenhaber bracket has to be independent of the choice of $m \geq 0$. This implies that the RHS has to be zero, so all $\lambda_s = 0$. This shows that

$$f_k\zeta_{h-4} = 0, \tag{2.14.3.2}$$

so we have that the cup product of $HH^2(A)$ with $HH^4(A)$ is zero.

2.14.4 $HH^2(A) \times HH^5(A) \xrightarrow{0} HH^7(A)$

Let $a \in HH^2(A)$ and $b \in HH^5(A)$ be homogeneous elements, then $ab = \lambda\theta_k \in HH^7(A) = U[-2h-2]$, $\lambda \in \mathbb{C}$. Then

$$\lambda\psi_0 = \lambda\psi_k z_k = \lambda\theta_k\zeta_k = \lambda b(a\zeta_k) = 0,$$

the last equality coming from the product $a\zeta_k \in HH^2(A) \cup HH^4(A) = 0$.

2.15 Products involving $HH^3(A)$

2.15.1 $HH^3(A) \times HH^3(A) \xrightarrow{0} HH^6(A)$

This follows by degree argument: $\deg HH^3(A) = -2$, $\deg HH^6(A) \leq -h - 2 < -4$.

2.15.2 $HH^3(A) \times HH^4(A) \xrightarrow{0} HH^7(A)$

This follows by degree argument: $\deg HH^3(A) = -2$, $\deg HH^4(A) \geq -h$, $\deg HH^7(A) \leq -h - 4 < -h - 2$.

2.15.3 $HH^3(A) \times HH^5(A) \xrightarrow{0} HH^8(A)$

This follows by degree argument: $\deg HH^3(A) = -2$, $\deg HH^5(A) \geq -h - 2$, $\deg HH^8(A) = -2h - 2 < -h - 4$.

2.16 Products involving $HH^4(A)$

2.16.1 $HH^4(A) \times HH^4(A) \xrightarrow{0} HH^8(A)$

This follows by degree argument: $\deg HH^4(A) \geq -h$, $\deg HH^8(A) = -2h-2 < -2h$.

2.16.2 $HH^4(A) \times HH^5(A) \xrightarrow{0} HH^9(A)$

This is clear for $Q = D_{n+1}$, n odd, $Q = E_7$, E_8 where $HH^9(A) = K[-2h - 2] = 0$.

Let $Q = D_{n+1}$, n even or $Q = E_6$. Let $a \in HH^4(A)$, $b \in HH^5(A)$.The product $HH^2(A) \times HH^3(A) \to HH^5(A)$, $(x, y) \mapsto \langle x, y \rangle \zeta_0$ induces a nondegenerate bilinear form $\langle -, - \rangle$. If $ab \in HH^9(A) = HH^3(A)[-2h]$ is nonzero, then we can find a $c \in HH^2(A)$, such that $c(ab) = \zeta_0$. But this equals $(ca)b = 0$ since $HH^2(A) \times HH^4(A) \xrightarrow{0} HH^6(A)$ which gives us a contradiction.

2.17 $HH^5(A) \times HH^5(A) \to HH^{10}(A)$

Proposition 2.17.0.1. *The multiplication of the subspace $U[-2]^*$ with $HH^5(A)$ is zero.*

The pairing on $Y^*[-h-2]$ is

$$\begin{aligned} Y^*[-h-2] \times Y^*[-h-2] &\to HH^{10}(A), \\ (a,b) &\mapsto \Omega(a,b)\varphi_4(\zeta_0), \end{aligned} \tag{2.17.0.2}$$

where the skew-symmetric bilinear form $\Omega(-,-)$ is given by the matrix $-M_\beta$ from subsection 2.13.5.

Proof. We have $\deg HH^5(A) \geq -h-2$ and $\deg HH^{10}(A) \leq -2h-4$, so we can get a nonzero multiplication only by pairing bottom degree parts of $HH^5(A)$ which is $Y^*[-h-2]$. The product lies in the top degree part of $HH^{10}(A) = HH^4(A)[-2h]$ which is spanned by $\varphi_4(\zeta_0)$. This gives us the pairing of the form (2.17.0.2).

We want to find the matrix $(\Omega(\varepsilon_i, \varepsilon_j))_{i,j}$ where ε_i are a basis of $Y^*[-h-2]$, given in the section about $HH^5(A)$. Recall that the multiplication $HH^1(A) \times HH^5(A) \to HH^6(A)$ was given by a skew-symmetric matrix $((M_\beta)_{i,j})_{i,j\in F}$, so that $\theta_0\varepsilon_i = \sum\limits_{k\in F}(M_\beta)_{k,i}\varphi_0(\omega_k)$.

We multiply $\varepsilon_i\varepsilon_j = \Omega(\varepsilon_i, \varepsilon_j)\varphi_4(\zeta_0)$ with θ_0 (see 2.13.4.2):

$$\theta_0(\varepsilon_i\varepsilon_j) = \Omega(\varepsilon_i, \varepsilon_j)\varphi_5(\psi_0). \tag{2.17.0.3}$$

Using associativity, this equals

$$(\theta_0\varepsilon_i)\varepsilon_j = \sum_{k\in F}(M_\beta)_{k,i}\varphi_0(\omega_k)\varepsilon_j = (M_\beta)_{j,i}\psi_0 = -(M_\beta)_{i,j}\varphi_5(\psi_0). \tag{2.17.0.4}$$

We see from equations (2.17.0.3) and (2.17.0.4) that

$$\Omega(\varepsilon_i, \varepsilon_j) = -(M_\beta)_{i,j}.$$

This completes the cup product computation of $HH^*(A)$. □

2.18 Presentation of $HH^*(A)$

For each quiver, we give a presentation of $HH^*(A)$ as an algebra over $\mathbb{C}$ by generators and relations. We write X for the element $\phi_0(z_0) \in HH^6(A)$.

2.18.1 $Q = D_{n+1}$, n odd

$HH^*(A)$ is generated by

$$1, z_4, \omega_1, \ldots, \omega_n, \theta_0, \zeta_{2n-6}, \varepsilon_2, \ldots, \varepsilon_n, X$$

with relations ($\forall i, j = 2, \ldots, n$, $\forall k, l = 1, \ldots, n$)

$$(z_4)^{\frac{n+1}{2}} = \theta_0^2 = \zeta_{2n-6}^2 = z_4\varepsilon_i = 0,$$

$$z_4\omega_k = \theta_0\omega_k = \zeta_{2n-6}\omega_k = \omega_l\omega_k = X\sum_{\substack{m=1\\ m\ odd}}^{n-2}\omega_m = Xz_4^{\frac{n-1}{2}} = 0,$$

$$\omega_i\varepsilon_j = \delta_{ij}z_4^{\frac{n-3}{2}}\theta_0\zeta_{2n-6},$$

$$\varepsilon_i\varepsilon_j = -\Omega(\varepsilon_i, \varepsilon_j)Xz_4^{\frac{n-3}{2}}\zeta_{2n-6},$$

where $\Omega(-,-)$ is a skew-symmetric bilinear form given by the matrix

$$\begin{pmatrix}
0 & -n & 0 & -n & \dots & \dots & -n & 0 & -n & 0 & -n \\
n & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -n & \dots & \dots & -n & 0 & -n & 0 & -n \\
n & 0 & n & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & \dots & -n & 0 & -n & 0 & -n \\
\vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n & 0 & n & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & -n & 0 & -n \\
n & 0 & n & 0 & \dots & \dots & 0 & n & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & -n \\
n & 0 & n & 0 & \dots & \dots & 0 & n & 0 & n & 0
\end{pmatrix}.$$

2.18.2 $Q = D_{n+1}$, n even

$HH^*(A)$ is generated by

$$1, z_4, \omega_1, \dots, \omega_{n-1}, \theta_0, f_n, \zeta_{2n-4}, \varepsilon_2, \dots, \varepsilon_{n-1}, X$$

with relations ($\forall i, j = 2, \dots, n-1$, $\forall k, l = 1, \dots, n-1$)

$$(z_4)^{\frac{n}{2}} = \theta_0^2 = z_4 f_n = \zeta_{2n-4}^2 = \zeta_{2n-4} f_n = 0,$$

$$z_4 \varepsilon_i = f_n \varepsilon_i = 0,$$

$$z_4 \omega_k = \theta_0 \omega_k = f_n \omega_k = \zeta_{2n-4} \omega_k = \omega_l \omega_k = X \sum_{\substack{m=1 \\ m \text{ odd}}}^{n-1} \omega_m = 0,$$

$$f_n^2 = -n z_4^{\frac{n-2}{2}} \zeta_{2n-4}$$

$$\omega_i\varepsilon_j = \delta_{ij} z_4^{\frac{n-2}{2}} \theta_0 \zeta_{2n-4},$$

$$\varepsilon_i\varepsilon_j = -\Omega(\varepsilon_i, \varepsilon_j) X z_4^{\frac{n-2}{2}} \zeta_{2n-4},$$

where $\Omega(-,-)$ is a skew-symmetric bilinear form given by the matrix

$$\begin{pmatrix}
0 & -n & 0 & -n & \dots & \dots & -n & 0 & -n & 0 & -n \\
n & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -n & \dots & \dots & -n & 0 & -n & 0 & -n \\
n & 0 & n & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & \dots & -n & 0 & -n & 0 & -n \\
\vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n & 0 & n & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & -n & 0 & -n \\
n & 0 & n & 0 & \dots & \dots & 0 & n & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & -n \\
n & 0 & n & 0 & \dots & \dots & 0 & n & 0 & n & 0
\end{pmatrix}.$$

2.18.3 $Q = E_6$

$HH^*(A)$ is generated by

$$1, z_6, z_8, \omega_3, \omega_6, \theta_0, f_1, f_2, \zeta_6, \zeta_8, \varepsilon_3, \varepsilon_6, X$$

with relations (for $u, v \in \{6, 8\}$, $k, l \in \{3, 6\}$, $i, j \in \{1, 2\}$)

$$z_u z_v = \theta_0^2 = z_u f_i = \zeta_u \zeta_v = \zeta_u f_i = z_u \varepsilon_k = f_i \varepsilon_k = 0,$$

$$z_u \omega_k = \theta_0 \omega_k = f_i \omega_k = \zeta_u \omega_k = \omega_l \omega_k = 0,$$

$$z_8 \zeta_8 = z_6 \zeta_6, \qquad \omega_k \varepsilon_l = \delta_{kl} \theta_0 z_8 \zeta_8,$$

$$f_i f_j = \langle f_i, f_j \rangle z_8 \zeta_8,$$

where $\langle -, - \rangle$ is the symmetric bilinear form, given by the matrix

$$\begin{pmatrix} -8 & -4 \\ -4 & -8 \end{pmatrix},$$

$$\varepsilon_k \varepsilon_l = -\Omega(\varepsilon_k, \varepsilon_l) X z_8 \zeta_8,$$

where $\Omega(-, -)$ is a skew-symmetric bilinear form, given by the matrix

$$\begin{pmatrix} 0 & -6 \\ 6 & 0 \end{pmatrix}.$$

2.18.4 $Q = E_7$

$HH^*(A)$ is generated by

$$1, z_8, z_{12}, \omega_1, \ldots, \omega_6, \theta_0, \zeta_8, \zeta_{12}, \varepsilon_1, \ldots, \varepsilon_6, X$$

with relations (for $u, v \in \{8, 12\}$, $k, l \in \{1, \ldots, 6\}$)

$$z_u z_{12} = \theta_0^2 = z_u^3 = z_u \varepsilon_k = 0,$$

$$z_u \omega_k = \theta_0 \omega_k = \omega_l \omega_k = X z_8^2 = 0,$$

$$z_8 \zeta_8 = z_{12} \zeta_{12}, \qquad \omega_k \varepsilon_l = \delta_{kl} \theta_0 z_{12} \zeta_{12},$$

$$\varepsilon_k \varepsilon_l = -\Omega(\varepsilon_k, \varepsilon_l) X z_{12} \zeta_{12},$$

where $\Omega(-, -)$ is a skew-symmetric bilinear form, given by the matrix

$$\begin{pmatrix} 0 & 9 & 0 & 9 & 0 & 9 \\ -9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 & 9 \\ -9 & 0 & -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9 \\ -9 & 0 & -9 & 0 & 9 & 0 \end{pmatrix}.$$

2.18.5 $Q = E_8$

$HH^*(A)$ is generated by

$$1, z_{12}, z_{20}, \omega_1, \ldots, \omega_8, \theta_0, \zeta_{20}, \zeta_{24}, \varepsilon_1, \ldots, \varepsilon_6, X$$

with relations (for $u, v \in \{12, 20\}$, $k, l \in \{1, \ldots, 8\}$)

$$z_u z_{20} = \theta_0^2 = z_u^3 = z_u \varepsilon_k = z_{12}^3 = 0,$$

$$z_u \omega_k = \theta_0 \omega_k = \omega_l \omega_k = 0,$$

$$z_{12}^2 \zeta_{24} = z_{20} \zeta_{20}, \qquad \omega_k \varepsilon_l = \delta_{kl} \theta_0 z_{20} \zeta_{20},$$

$$\varepsilon_k \varepsilon_l = -\Omega(\varepsilon_k, \varepsilon_l) X z_{20} \zeta_{20},$$

where $\Omega(-,-)$ is a skew-symmetric bilinear form, given by the matrix

$$\begin{pmatrix} 0 & 15 & 0 & 15 & 0 & 0 & 0 & -15 \\ -15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 15 & 0 & 0 & 0 & -15 \\ -15 & 0 & -15 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -15 \\ 0 & 0 & 0 & 0 & 0 & 0 & -15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 15 & 0 & -15 \\ 15 & 0 & 15 & 0 & 15 & 0 & 15 & 0 \end{pmatrix}$$

Chapter 3

The calculus structure of the preprojective algebra

We recall the definition of the calculus.

3.1 Definition of calculus

Definition 3.1.0.1. (Gerstenhaber algebra) *A graded vector space $\mathcal{V}^\bullet$ is a Gerstenhaber algebra if it is equipped with a graded commutative and associative product $\wedge$ of degree 0 and a graded Lie bracket $[,]$ of degree -1. These operations have to be compatible in the sense of the following Leibniz rule*

$$[\gamma, \gamma_1 \wedge \gamma_2] = [\gamma, \gamma_1] \wedge \gamma_2 + (-1)^{k_1(k+1)} \gamma_1 \wedge [\gamma, \gamma_2], \tag{3.1.0.2}$$

where $\gamma \in \mathcal{V}^k$ and $\gamma_1 \in \mathcal{V}^{k_1}$.

We recall from [4] that

Definition 3.1.0.3. (Precalculus) *A precalculus is a pair of a Gerstenhaber algebra $(\mathcal{V}^\bullet, \wedge, [,])$ and a graded vector space $\mathcal{W}^\bullet$ together with*

- *a module structure $\iota_\bullet : \mathcal{V}^\bullet \otimes \mathcal{W}^{-\bullet} \to \mathcal{W}^{-\bullet}$ of the graded commutative algebra $\mathcal{V}^\bullet$ on $\mathcal{W}^{-\bullet}$.*

- *an action* $\mathcal{L}_\bullet : \mathcal{V}^{\bullet+1} \otimes \mathcal{W}^{-\bullet} \to \mathcal{W}^{-\bullet}$ *of the graded Lie algebra* $\mathcal{V}^{\bullet+1}$ *on* $\mathcal{W}^{-\bullet}$ *which are compatible in the sense of the following equations*

$$\iota_a \mathcal{L}_b - (-1)^{|a|(|b|+1)} \mathcal{L}_b \iota_a = \iota_{[a,b]}, \tag{3.1.0.4}$$

and

$$\mathcal{L}_{a \wedge b} = \mathcal{L}_a \iota_b + (-1)^{|a|} \iota_a \mathcal{L}_b. \tag{3.1.0.5}$$

Definition 3.1.0.6. (Calculus) *A calculus is a precalculus* $(\mathcal{V}^\bullet, \mathcal{W}^\bullet, [,], \wedge, \iota_\bullet, \mathcal{L}_\bullet)$ *with a degree 1 differential* d *on* $\mathcal{W}^\bullet$ *such that the* Cartan identity,

$$\mathcal{L}_a = d\iota_a - (-1)^{|a|} \iota_a d, \tag{3.1.0.7}$$

holds.

Let A be an associative algebra. The contraction of the Hochschild cochain $P \in C^k(A, A)$ with the Hochschild chain $(a_0, a_1, \ldots, a_n)$ is defined by

$$I_P(a_0, a_1, \ldots, a_n) = \begin{cases} (a_0 P(a_1, \ldots, a_k), a_{k+1}, \ldots, a_n) & n \geq k, \\ 0 & \text{else.} \end{cases} \tag{3.1.0.8}$$

We have

Proposition 3.1.0.9. (Yu. Daletski, I. Gelfand and B. Tsygan [6]) *The contraction* I_P *together with the Connes differential, the Gerstenhaber bracket, the cup product and the action of cochains on chains ([5, (3.5), page 46]) induce on the pair* $(HH^\bullet(A, A), HH_\bullet(A, A))$ *a structure of calculus.*

3.2 Results about the calculus structure of the Hochschild cohomology/homology of preprojective algebras of Dynkin quivers

Notation 3.2.0.10. For $c_k \in HH^i(A)$, $0 \leq i \leq 5$, we write $c_k^{(s)}$ for the corresponding cocycle in HH^{i+6s}. We write $c_{k,t}$ for the corresponding cycle in HH_{j+6t}, $0 \leq j \leq 5$ (under the isomorphism $\mathbb{D}$.

We state the results in terms of the bases of $HH^\bullet(A)$ and $HH_\bullet(A)$ which were introduced in Chapter 2:

Theorem 3.2.0.11. *The calculus structure is given by tables 3.1, 3.2, 3.3 and the* Connes differential B, *given as follows*

The Connes differential B is given as follows:

$$\begin{aligned}
B_{1+6s}(\theta_{k,s}) &= (1+\frac{k}{2}+sh)z_{k,s},\\
B_{2+6s}(\omega_{k,s}) &= (\frac{1}{2}+s)h\beta^{-1}(\omega_{k,s}),\\
B_{2+6s}(z_{k,s}) &= 0,\\
B_{3+6s}(\psi_{k,s}) &= ((s+1)h-1-\frac{k}{2})\zeta_{k,s},\\
B_{3+6s}(\varepsilon_{k,s}) &= 0,\\
B_{4+6s} &= 0,\\
B_{5+6s}(h_{k,s}) &= (s+1)h\alpha^{-1}(h_{k,s}),\\
B_{6+6s} &= 0.
\end{aligned}$$

$a \backslash b$	$\theta_{l,t}$	$\omega_{l,t}$	$z_{l,t}$	$\psi_{l,t}$	$\varepsilon_{l,t}$	$\zeta_{l,t}$	$h_{l,t}$	$f_{l,t}$
$z_k^{(s)}$	$(z_k\theta_l)_{t-s}$	$\delta_{k0}\omega_{l,t-s}$	$(z_k z_l)_{t-s}$	$(z_k\psi_l)_{t-s}$	$\delta_{k0}\varepsilon_{l,t-s}$	$(z_k\zeta_l)_{t-s}$	$\delta_{k0}h_{l,t-s}$	$\delta_{k0}f_{l,t-s}$
$\omega_k^{(s)}$	0	0	$\delta_{l0}\omega_{k,t-s}$	0	$\delta_{kl}\psi_{0,t-s}$	0	0	0
$\theta_k^{(s)}$	0	0	$(z_l\theta_k)_{t-s}$	$(z_k\psi_l)_{t-s}$	$\delta_{k0}\beta(\varepsilon_{l,t-s})$	0	0	$\delta_{k0}\alpha(f_{l,t})$
$f_k^{(s)}$	$\delta_{l0}\alpha(f_{k,t-s-1})$	0	$\delta_{l0}f_{k,t-s-1}$	$\delta_{l,h-3}(k+1)\cdot$ $\theta_{h-3,t-s-1}$	0	$\delta_{l,h-3}(k+1)\cdot$ $z_{l,t-s}$	$\delta_{kl}\psi_{0,t-s}$	$(M_\alpha)_{kl}\zeta_{0,t-s}$
$h_k^{(s)}$	0	0	$\delta_{l0}h_{k,t-s-1}$	0	0	$\delta_{k,\frac{h-3}{2}}\delta_{l,h-3}\cdot$ $\theta_{h-3,t-s}$	0	$\delta_{kl}\psi_{0,t-s}$
$\zeta_k^{(s)}$	$(z_l\psi_k)_{t-s-1}$	0	$(z_l\zeta_k)_{t-s-1}$	$\delta_{k,h-3}\delta_{l,h-3}\cdot$ $\alpha(f_{\frac{h-3}{2},t-s-1})$	0	$\delta_{k,h-3}\delta_{l,h-3}\cdot$ $f_{\frac{h-3}{2},t-s-1}$	$\delta_{k,h-3}\delta_{l,\frac{h-3}{2}}\cdot$ $\theta_{k,t-s}$	$\delta_{k,h-3}(l+1)\cdot$ $z_{k,t-s}$
$\varepsilon_k^{(s)}$	$-\delta_{l0}\beta(\varepsilon_{k,t-s})$	$\delta_{kl}\psi_{0,t-s-1}$	$\delta_{l,0}\varepsilon_{k,t-s-1}$	0	$-(M_\beta)_{k,l}\cdot$ $\zeta_{0,t-s-1}$	0	0	0
$\psi_k^{(s)}$	0	0	$(z_k\psi_l)_{t-s-1}$	0	0	$\delta_{k,h-3}\delta_{l,h-3}\cdot$ $\alpha(f_{h-3,t-s-1})$	0	$\delta_{k,h-3}(l+1)\cdot$ $\theta_{h-3,t-s}$

Table 3.1: contraction map $\iota_a(b)$

b / a	$z_l^{(t)}$	$\omega_l^{(t)}$	$\theta_l^{(t)}$	$f_l^{(t)}$	$h_l^{(t)}$	$\zeta_l^{(t)}$	$\varepsilon_l^{(t)}$	$\psi_l^{(t)}$
$z_k^{(s)}$	0	$-\delta_{k0}sh\cdot \beta^{-1}(\omega_l^{(s+t)})$	$(\frac{k}{2}-sh)\cdot (z_kz_l)^{(s+t)}$	0	$-\delta_{k0}sh\cdot \alpha^{-1}(h_l^{(s+t)})$	0	0	$(\frac{k}{2}-sh)\cdot (z_k\zeta_l)^{(s+t)}$
$\omega_k^{(s)}$		0	0	0	0	0	$-(\frac{h}{2}+1+th)\cdot \delta_{kl}\zeta_0^{(s+t)}$	0
$\theta_k^{(s)}$			$(\frac{l-k}{2}+(s-t)h)\cdot (z_k\theta_l)^{(s+t)}$	$-(1+th)\cdot \delta_{k0}f_l^{(s+t)}$	$(-1+(s-t)h)\cdot \delta_{k0}h_l^{(s+t)}$	$-(2+\frac{l}{2}+th)\cdot (z_k\zeta_l)^{(s+t)}$	$-(1+th+\frac{h}{2})\cdot \delta_{k0}\varepsilon_l^{(s+t)}$	$-(2+\frac{k+l}{2}+(t-s)h)\cdot (z_k\psi_l)^{(s+t)}$
$f_k^{(s)}$				0	$-(1+sh)\cdot \delta_{kl}\zeta_0^{(s+t)}$	0	0	$-(k+1)\cdot (1+sh)\cdot \delta_{l,h-3}z_{h-3}^{(s+t+1)}$
$h_k^{(s)}$					$(s-t)h\cdot (M_\alpha^{-1})_{kl}\cdot \psi_0^{(s+t)}$	$-(\frac{h+1}{2}+th)\cdot \delta_{k,\frac{h-3}{2}}\delta_{l,h-3}\cdot z_{h-3}^{(s+t+1)}$	0	$((s-t)h-\frac{h-1}{2})\cdot \delta_{k,\frac{h-3}{2}}\delta_{l,h-3}\cdot \theta_{h-3}^{(s+t+1)}$
$\zeta_k^{(s)}$						0	0	$-(sh+\frac{h+1}{2})\cdot \delta_{k,h-3}\delta_{l,h-3}\cdot f_{\frac{h-3}{2}}^{(s+t+1)}$
$\varepsilon_k^{(s)}$							0	0
$\psi_k^{(s)}$								$(s-t)h\cdot \delta_{k,h-3}\delta_{l,h-3}\cdot \alpha(f_{\frac{h-3}{2}}^{(s+t+1)})$

Table 3.2: Gerstenhaber bracket $[a,b]$

$a \backslash b$	$\theta_{l,t}$	$\omega_{l,t}$	$z_{l,t}$	$\psi_{l,t}$	$\varepsilon_{l,t}$	$\zeta_{l,t}$	$h_{l,t}$	$f_{l,t}$
$\theta_k^{(s)}$	$(1+\frac{l}{2}+th)\cdot (z_k\theta_l)_{t-s}$	$(\frac{1}{2}+t)h\cdot \delta_{k0}\omega_{l,t-s}$	$(1+\frac{k+l}{2}+(t-s)h)\cdot (z_k z_l)_{t-s}$	$((t+1)h-1-\frac{l}{2})\cdot (z_k\psi_l)_{t-s}$	$(\frac{1}{2}+(t-s))h\cdot \delta_{k0}\varepsilon_{l,t-s}$	$((t-s+1)h -1-\frac{l-k}{2}) (z_k\zeta_l)_{t-s}$	$(t+1)h\cdot \delta_{k0}h_{l,t-s}$	$(t-s+1)h$ $\delta_{k0}f_{l,t-s}$
$f_k^{(s)}$	$-(1+sh)\cdot \delta_{l0}f_{k,t-s-1}$	0	0	$-(1+sh)\cdot \delta_{l,h-3}z_{h-3,t-s}$	0	0	$-(1+sh)\cdot \delta_{kl}\zeta_{0,t-s}$	0
$h_k^{(s)}$	$(1+th)\cdot \delta_{l0}h_{k,t-s-1}$	0	$\delta_{l0}(t-s)h\cdot \alpha^{-1}(h_{k,t-s-1})$	$(th+\frac{h+1}{2})\cdot \delta_{k,\frac{h-3}{2}}\delta_{l,h-3}\cdot \theta_{h-3,t-s}$	0	$((t-s)h+\frac{h-1}{2})\cdot \delta_{k,\frac{h-3}{2}}\delta_{l,h-3}\cdot z_{h-3,t-s}$	$(t+1)h\cdot (M_\alpha^{-1})_{lk}\cdot \psi_{0,t-s}$	$((t-s+1)h$ - $\delta_{kl}\zeta_{0,t-s}$
$\zeta_k^{(s)}$	$-(2+\frac{k}{2}+sh)\cdot (z_l\zeta_k)_{t-s-1}$	0	0	$-(sh+\frac{h+1}{2})\cdot \delta_{k,h-3}\delta_{l,h-3}\cdot f_{\frac{h-3}{2},t-s-1}$	0	0	$-(sh+\frac{h+1}{2})\cdot \delta_{k,h-3}\delta_{l,\frac{h-3}{2}}\cdot z_{h-3,t-s}$	0
$\varepsilon_k^{(s)}$	$((s+\frac{1}{2})h+1)\cdot \delta_{l0}\varepsilon_{k,t-s-1}$	$-((s+\frac{1}{2})h+1)\cdot \delta_{kl}\zeta_{0,t-s-1}$	0	0	0	0	0	0
$\psi_k^{(s)}$	$(1+\frac{l}{2}+th)\cdot (z_l\psi_k)_{t-s-1}$	0	$((t-s)h -1-\frac{k-l}{2})\cdot (z_l\zeta_k)_{t-s-1}$	$(th+\frac{h+1}{2})\cdot \delta_{k,h-3}\delta_{l,h-3}\cdot \alpha(f_{\frac{h-3}{2},t-s-1})$	0	$(t-s)h\cdot \delta_{k,h-3}\delta_{l,h-3} f_{\frac{h-3}{2},t-s-1}$	$(t+1)h\cdot \delta_{k,h-3}\delta_{l,h-3}\cdot \theta_{h-3,t-s}$	$(l+1)\cdot ((t-s)h +1+\frac{h-3}{2}) \delta_{k,h-3} z_{h-3,t-s}$
$z_k^{(s)}$	$(k-sh) (z_k\theta_l)_{t-s}$	$-\delta_{k0}sh \beta^{-1}(\omega_{l,t-s})$	0	$(\frac{k}{2}-sh)\cdot (z_k\zeta_l)_{t-s}$	0	0	$(k-sh)\cdot \alpha^{-1}(h_{l,t-s})$	0
$\omega_k^{(s)}$	$(1+th)\cdot \delta_{l0}\omega_{k,t-s}$	0	$\delta_{l0}(\frac{1}{2}+t-s)h\cdot \beta^{-1}(\omega_{k,t-s})$	0	$\delta_{kl}\cdot(-1+h +(t-s)h)\cdot \zeta_{0,t-s}$	0	0	0

Table 3.3: Lie derivative $\mathcal{L}_a(b)$

3.3 Batalin-Vilkovisky structure on Hochschild cohomology

In 2.2.5.1, an isomorphism $HH_\bullet(A) = HH^{8-\bullet}(A)[2h+2]$ was introduced. However, because of the periodicity of the Schofield resolution (with period 6), we get for every $m \geq 0$ an isomorphism

$$\mathbb{D} : HH_\bullet(A) \xrightarrow{\sim} HH^{6m+2-\bullet}(A)[2mh+2] \tag{3.3.0.12}$$

It translates the Connes differential $B : HH_\bullet(A) \to HH_{\bullet+1}(A)$ on Hochschild homology into a differential $\Delta : HH^\bullet(A) \to HH^{\bullet-1}(A)$ on Hochschild cohomology, i.e. we have the commutative diagram

$$\begin{array}{ccc} HH_\bullet(A) & \xrightarrow{B} & HH_{\bullet+1}(A) \\ \mathbb{D}\big\downarrow\sim & & \sim\big\downarrow\mathbb{D} \\ HH^{6m+2-\bullet}(A)[2mh+2] & \xrightarrow{\Delta} & HH_{6m+1-\bullet}(A)[2mh+2] \end{array}$$

Theorem 3.3.0.13. (BV structure on Hochschild cohomology) *Δ makes $HH^\bullet(A)$ a Batalin-Vilkovisky algebra, i.e. for the Gerstenhaber bracket we get the following equation:*

$$[a,b] = \Delta(a \cup b) - \Delta(a) \cup b - (-1)^{|a|} a \cup \Delta(b), \qquad \forall a, b \in HH^*(A). \tag{3.3.0.14}$$

The isomorphism $\mathbb{D}$ intertwines contraction and cup-product maps, i.e. we have

$$\mathbb{D}(\iota_\eta c) = \eta \cup \mathbb{D}(c), \qquad \forall c \in HH_\bullet(A), \eta \in HH^\bullet(A). \tag{3.3.0.15}$$

Remark 3.3.0.16. Note that Δ in equation (3.3.0.14) depends on which $m \in \mathbb{N}$ we choose to identify $\mathbb{D} : HH_\bullet(A) \xrightarrow{\sim} HH^{6m+2-\bullet}(A)[2mh+2]$, where the Gerstenhaber bracket does not.

Proof. We apply the functor

$$\begin{aligned}\mathrm{Hom}_{A^e}(-, A \otimes_{\mathbb{C}} A) : A^e - \mathrm{mod} &\to A^e - \mathrm{mod},\\ M &\mapsto M^\vee\end{aligned}$$

on the Schofield resolution:

$$\begin{aligned}(A\otimes A)^\vee &\xrightarrow{d_1^\vee} (A\otimes V\otimes A)^\vee \xrightarrow{d_2^\vee} (A\otimes A[2])^\vee \xrightarrow{d_3^\vee} (A\otimes \mathcal{N}[h])^\vee \xrightarrow{d_4^\vee}\\ &\xrightarrow{d_4^\vee} (A\otimes V\otimes \mathcal{N}[h])^\vee \xrightarrow{d_5^\vee} (A\otimes \mathcal{N}[h+2])^\vee \xrightarrow{d_6^\vee} (A\otimes A[2h])^\vee \xrightarrow{d_7^\vee} \dots\end{aligned} \tag{3.3.0.17}$$

An element in $(A\otimes A)^\vee$ or $(A\otimes\mathcal{N})^\vee$ is determined by the image of $1\otimes 1$,

An element in $(A\otimes V\otimes A)^\vee$ or $(A\otimes V\otimes\mathcal{N})^\vee$ by the images of $1\otimes a\otimes 1$ for all arrows $a\in\bar{Q}$.

Let us define $\sigma = \begin{cases} +1 & Q = A, \\ -1 & Q = D, E \end{cases}$.

We make the following identifications:

<u>$(A\otimes A)[-2mh] = (A\otimes A[2mh])^\vee$:</u>

we identify $x\otimes y$ with the map that sends $1\otimes 1$ to $\sigma^m y\otimes x$,

<u>$(A\otimes V\otimes A)[-2mh-2] = (A\otimes V\otimes A[2mh])^\vee$:</u>

we identify $\sum_{a\in\bar{Q}} \epsilon_a x_a\otimes a^*\otimes y_a$ with the map that sends $1\otimes a\otimes 1$ to $-\sigma^m y_a\otimes x_a$,

<u>$(A\otimes A)[-2mh-2] = (A\otimes A[2mh+2])^\vee$:</u>

we identify $x\otimes y$ with the map that sends $1\otimes 1$ to $-\sigma^m y\otimes x$,

<u>$(A\otimes\mathcal{N})[-(2m+1)h] = (A\otimes\mathcal{N}[(2m+1)h])^\vee$:</u>

we identify $x\otimes y$ with the map that sends $1\otimes 1$ to $-\sigma^m\eta(y)\otimes x$,

<u>$(A\otimes V\otimes\mathcal{N})[-(2m+1)h-2] = (A\otimes V\otimes\mathcal{N}[(2m+1)h])^\vee$:</u>

we identify $\sum_{a\in\bar{Q}} \epsilon_a x_a\otimes\eta(a^*)\otimes y_a$ with the map that sends $1\otimes a\otimes 1$ to $\sigma^{m+1}\eta(y_a)\otimes x_a$,

<u>$(A\otimes\mathcal{N})[-2(m+1)h-2] = (A\otimes A[2(m+1)h+2])^\vee$:</u>

we identify $x\otimes y$ with the map that sends $1\otimes 1$ to $\sigma^{m+1}\eta(y)\otimes x$,

so (3.3.0.17) becomes

$$
\begin{aligned}
(A\otimes A) &\stackrel{d_1^\vee}{\to} (A\otimes V\otimes A) \stackrel{d_2^\vee}{\to} (A\otimes A[-2]) \stackrel{d_3^\vee}{\to} (A\otimes \mathcal{N}[-h]) \stackrel{d_4^\vee}{\to}\\
&\stackrel{d_4^\vee}{\to} (A\otimes V\otimes \mathcal{N}[-h-2]) \stackrel{d_5^\vee}{\to} (A\otimes \mathcal{N}[-h-2]) \stackrel{d_6^\vee}{\to} (A\otimes A[-2h]) \stackrel{d_7^\vee}{\to} \dots
\end{aligned}
\qquad (3.3.0.18)
$$

We show under the identification from above, the differentials $d_i^\vee$ corresponds to the differentials from the Schofield resolution, i.e. (3.3.0.18) can be rewritten in this form:

$$
\begin{aligned}
(A\otimes A) &\stackrel{d_2[-2]}{\to} (A\otimes V\otimes A) \stackrel{d_1[-2]}{\to} (A\otimes A[-2]) \stackrel{d_6[-2h-2]}{\to}\\
&\stackrel{d_6[-2h-2]}{\to} (A\otimes \mathcal{N}[-h]) \stackrel{d_5[-2h-2]}{\to} (A\otimes V\otimes \mathcal{N}[-h-2]) \stackrel{d_4[-2h-2]}{\to}\\
&\stackrel{d_4[-2h-2]}{\to} (A\otimes \mathcal{N}[-h-2]) \stackrel{d_3[-2h-2]}{\to} (A\otimes A[-2h]) \stackrel{d_2[-2h-2]}{\to} \dots
\end{aligned}
\qquad (3.3.0.19)
$$

It is enough to show this for the first period.

$$d_1^\vee(x\otimes y)(1\otimes a\otimes 1) = (x\otimes y)\circ(a\otimes 1 - 1\otimes a) = ay\otimes x - y\otimes xa,$$

so

$$d_1^\vee(x\otimes y) = \sum_{a\in\bar{Q}} \epsilon_a(xa\otimes a^*\otimes y - x\otimes a^*\otimes ay) = \sum_{a\in\bar{Q}} \epsilon_a(xa\otimes a^*\otimes y + x\otimes a\otimes a^* y) = d_2(x\otimes y),$$

$$
\begin{aligned}
d_2^\vee(\sum_{a\in\bar{Q}} \epsilon_a x_a\otimes a^*\otimes y_a)(1\otimes 1) &= (\sum_{a\in\bar{Q}} \epsilon_a x_a\otimes a^*\otimes y_a)\circ(\sum_{b\in\bar{Q}} \epsilon_b b\otimes b^*\otimes 1 + \epsilon_b 1\otimes b\otimes b^*)\\
&= \sum_{a\in\bar{Q}} (\epsilon_a a^* y_a\otimes x_a - \epsilon_a y_a\otimes x_a a^*),
\end{aligned}
$$

so

$$d_2^\vee(\sum_{a\in\bar{Q}} \epsilon_a x_a\otimes a^*\otimes y_a) = \sum_{a\in\bar{Q}} \epsilon_a(x_a a^*\otimes y_a - x_a\otimes a^* y_a) = d_1(\epsilon_a x_a\otimes a^*\otimes y_a),$$

$$d_3^\vee(x\otimes y)(1\otimes 1) = (x\otimes y)\circ(\sum_{x_i\in B} x_i\otimes x_i^*) = -\sum_{x_i\in B} x_i y\otimes x x_i^* = -\sum_{x_i\in B} \eta(x_i^*) y\otimes x x_i,$$

so

$$d_3^\vee(x \otimes y) = \sum_{x_i \in B} xx_i \otimes x_i^* \eta(y) = \sum_{x_i \in B} xyx_i \otimes x_i^* = d_6(x \otimes y)$$

$$d_4^\vee(x \otimes y)(1 \otimes a \otimes 1) = (x \otimes y) \circ (a \otimes 1 - 1 \otimes a) = -a\eta(y) \otimes x + \eta(y) \otimes x\eta(a),$$

so

$$\begin{aligned} d_4^\vee(x \otimes y) &= \sum_{a \in \bar{Q}} \epsilon_a \sigma(-x \otimes \eta(a^*) \otimes \eta(a)y + x\eta(a) \otimes \eta(a^*) \otimes y) \\ &= \sum_{a \in \bar{Q}} (\epsilon_a xa \otimes a^* \otimes y + \epsilon_a x \otimes a \otimes a^* y) = d_5(x \otimes y), \end{aligned}$$

$$\begin{aligned} & d_5^\vee(\sum_{a \in \bar{Q}} \epsilon_a x_a \otimes \eta(a^*) \otimes y_a)(1 \otimes a \otimes 1) \\ &\quad = (\sum_{a \in \bar{Q}} \epsilon_a x_a \otimes \eta(a^*) \otimes y_a) \circ (\sum_{b \in \bar{Q}} (\epsilon_b b \otimes b^* \otimes 1 + \epsilon_b 1 \otimes b \otimes b^*)) \\ &\quad = \sigma \sum_{a \in \bar{Q}} (-\epsilon_a^* \eta(y_a) \otimes x_a + \epsilon_a \eta(y_a) \otimes x_a \eta(a^*)), \end{aligned}$$

so

$$\begin{aligned} d_5^\vee(\sum_{a \in \bar{Q}} \epsilon_a x_a \otimes \eta(a^*) \otimes y_a &= \sum_{a \in \bar{Q}} (-\epsilon_a x_a \otimes \eta(a^*) y_a + \epsilon_a x_a \eta(a^*) \otimes y_a) \\ &= d_4(\sum_{a \in \bar{Q}} \epsilon_a x_a \otimes \eta(a^*) \otimes y_a), \end{aligned}$$

$$\begin{aligned} d_6^\vee(x \otimes y)(1 \otimes 1) &= (x \otimes y) \circ (\sum_{x_i \in B} x_\otimes x_i^*) = \sigma \sum_{x_i \in B} x_i \eta(y) \otimes x\eta(x_i^*) \\ &= \sigma \sum_{x_i \in B} x_i^* \eta(y) \otimes xx_i = \sigma \sum_{x_i \in B} x_i^* \otimes xyx_i, \end{aligned}$$

so

$$d_6^\vee(x \otimes y) = \sum_{x_i \in B} xyx_i \otimes x_i^* = d_3(x \otimes y)$$

Fix $m \geq 0$. The map which shifts the degree by $-2mh-2$ produces the following diagram which commutes by the computations above:

$$\begin{array}{ccccc}
A\otimes A[2mh+2] & \xrightarrow{d_{6m+2}} & A\otimes V\otimes A[2h] & \xrightarrow{d_{6m+1}} & \dots \\
\downarrow & & \downarrow & & \\
(A\otimes A) & \xrightarrow{d_1^\vee} & (A\otimes V\otimes A)[-2] & \xrightarrow{d_2^\vee} & \dots
\end{array}$$

$$\begin{array}{ccccccc}
\dots & \xrightarrow{d_2} & A\otimes V\otimes A & \xrightarrow{d_1} & A\otimes A & \xrightarrow{mult.} & A \\
& & \downarrow & & \downarrow & & \\
\dots & \xrightarrow{d_{6m+1}^\vee} & (A\otimes V\otimes A)[-2mh-2] & \xrightarrow{d_{6m+2}^\vee} & (A\otimes A)[-mh-2] & \xrightarrow{mult.} & A
\end{array}$$

Similarly to the proof of [18, Theorem 3.4.3.], this self-dual morphism of the Schofield resolution $C^\bullet$ into the dual complex $(C^\bullet)^\vee$ can be used to prove (3.3.0.15).

(3.3.0.14) follows, as in the proof of [18, Theorem 3.4.3.], from (3.3.0.15) and the calculus structure.

□

3.3.1 Computation of the calculus structure of the preprojective algebra

Since the calculus structure is defined on Hochschild chains and cochains, we have to work with the on the resolution for computations. It turns out that we only have to compute $\mathcal{L}_{\theta_0}$ directly, the rest can be deduced from formulas given by the calculus and the BV structure.

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{d_3} & A\otimes A[2] & \xrightarrow{d_2} & A\otimes V\otimes A & \xrightarrow{d_1} & A\otimes A & \xrightarrow{d_0} & A & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu_1} & & \| & & \| & & \\
\dots & \xrightarrow{b_3} & A^{\otimes 4} & \xrightarrow{b_2} & A^{\otimes 3} & \xrightarrow{b_1} & A^{\otimes 2} & \xrightarrow{b_0} & A & \longrightarrow & 0
\end{array}$$

These maps ψ_i gives us a chain map between the Schofield and the bar resolution:

$$\begin{aligned}\mu_1(1\otimes y\otimes 1) &= 1\otimes y\otimes 1,\\ \mu_2(1\otimes 1) &= \sum_{a\in\bar{Q}}\epsilon_a 1\otimes a\otimes a^*\otimes 1,\\ \mu_3(1\otimes 1) &= \sum_{a\in\bar{Q}}\sum_{x_i\in B}\epsilon_a 1\otimes x_i\otimes a\otimes a^*\otimes x_i^*,\end{aligned}$$

and

$$\mu_{3+i} = \mu_i\sum_{a\in\bar{Q}}\sum_{x_i\in B} x_i\otimes a\otimes a^*\otimes x_i^*.$$

Now, we apply the functor $-\otimes_{A^e} A$ on the commutative diagram:

$$\begin{array}{ccccccccc}\cdots & \xrightarrow{d_3'} & A^R[2] & \xrightarrow{d_2'} & (V\otimes A)^R & \xrightarrow{d_1} & A^R & \longrightarrow & 0\\ & & \downarrow{\scriptstyle \mu_2'} & & \downarrow{\scriptstyle \mu_1'} & & \| & & \\ \cdots & \xrightarrow{b_3'} & (A^{\otimes 3})^R & \xrightarrow{b_2} & (A^{\otimes 2})^R & \xrightarrow{b_1} & (A^{\otimes 1})^R & \longrightarrow & 0\end{array}$$

where

$$\begin{aligned}\mu_1'(x\otimes y) &= x\otimes y,\\ \mu_2'(x) &= \sum_{a\in\bar{Q}}\epsilon_a a\otimes a^*\otimes x,\\ \mu_3'(x) &= \sum_{a\in\bar{Q}}\sum_{x_i\in B}\epsilon_a x_i\otimes a\otimes a^*\otimes x_i^*\eta(x),\end{aligned}$$

and

$$\mu_{3+i}' = \mu_i'\sum_{a\in\bar{Q}}\sum_{x_i\in B} x_i\otimes a\otimes a^*\otimes x_i^*.$$

Now, we compute $\mathcal{L}_{\theta_0}$:

Lemma 3.3.1.1. *For each $x\in HH_i(A)$,*

$$\mathcal{L}_{\theta_0}(x) = x\frac{\deg(x)}{2}. \tag{3.3.1.2}$$

Proof. Via μ', we already identified $x \in HH_i(A)$ with cycles in the Hochschild chain, but we still have to identify θ_0 with an element in $\mathrm{Hom}_{A^e}(A^{\otimes 3}, A)$:

given any monomial $b = b_1 \dots b_l$, $b_i \in V$, the map

$$\tau(1 \otimes b \otimes 1) = \sum_{i=1}^{l} b_1 \dots b_{i-1} \otimes b_i \otimes b_{i+1} \dots b_l$$

makes the diagram

$$\begin{array}{ccccccc}
A \otimes V \otimes A & \xrightarrow{d_1} & A \otimes A & \xrightarrow{d_0} & A & \longrightarrow & 0 \\
\tau \big\uparrow & & \big\| & & \big\| & & \\
A^{\otimes 3} & \xrightarrow{b_1} & A^{\otimes 2} & \xrightarrow{b_0} & A & \longrightarrow & 0
\end{array}$$

commute.

Applying $\mathrm{Hom}_{A^e}(\_ \otimes A)$, we get a map

$$\tau^* : \mathrm{Hom}_k(V) \to \mathrm{Hom}_k(A),$$

such that

$$(\theta_0 \circ \tau^*)(b_1 \dots b_l) = \sum_{i=1}^{l} b_1 \dots b_{i-1} \theta_0(b_i) b_{i+1} \dots b_l = s(b) \cdot b,$$

where for $b = b_1 \dots b_l$, $s(b)$ is the number of $b_i \in Q^*$.

Recall from [5, (3.5), page 46] that the Lie derivative of $\theta_0 \circ \tau^*$ on Hochschild chains is defined by

$$\begin{aligned}
\mathcal{L}_{\theta_0 \circ \tau^*}(a_1 \otimes \dots \otimes a_k) &= \sum_{i=1}^{k} a_1 \otimes \dots \otimes (\theta_0 \circ \tau^*)(a_i) \otimes \dots \otimes a_k \\
&= \sum_{i=1}^{k} (s(a_1) + \dots s(a_k)) a_1 \otimes \dots \otimes a_k,
\end{aligned}$$

and it can easily be checked that for each $x \in HH_i(A)$, $\mathcal{L}_{\theta_0 \circ \tau^*}$ acts on $\mu'_i(x)$, $x \in HH^i(A)$, by multiplication with $\frac{1}{2}\deg(x)$. ∎

The contraction map

From (3.3.0.15) we know that the contraction map on Hochschild homology is given by the cup product on Hochschild cohomology which was computed in [12] for quivers of type A and in Chapter 2 for type D and E. Table 3.1 contains these results, rewritten in terms of the contraction maps.

The Connes differential

We start with the computation of the Connes differential and recall the diagram from Subsection 2.2.4:

$$
\begin{array}{cccccc}
\text{degree} & & & & & \\
 & 0 & & & & \\
 & \downarrow & & & & \\
2 \leq \deg \leq h-1 & HH_1(A) & = & U & & \\
 & \downarrow B_1 & & \downarrow \sim & & \\
2 \leq \deg \leq h & HH_2(A) & = & U & \oplus & Y[h] \\
 & \downarrow B_2 & & & & \downarrow \sim \\
h \leq \deg \leq 2h-2 & HH_3(A) & = & U^*[2h] & \oplus & Y^*[h] \\
 & \downarrow B_3 & & \downarrow \sim & & \\
h+1 \leq \deg \leq 2h-2 & HH_4(A) & = & U^*[2h] & & \\
 & \downarrow B_4 & & \downarrow 0 & & \\
2h & HH_5(A) & = & K^*[2h] & & \\
 & \downarrow B_5 & & \downarrow \sim & & \\
2h & HH_6(A) & = & K[2h] & & \\
 & \downarrow B_6 & & \downarrow 0 & & \\
2h+2 \leq \deg \leq 3h-1 & HH_7(A) & = & U[2h] & & \\
 & \downarrow B_7 & & & & \\
 & \vdots & & & &
\end{array}
$$

Proposition 3.3.1.3. *The Connes differential B is given as follows:*

$$\begin{aligned}
B_{1+6s}(\theta_k^{(s)}) &= (1+\frac{k}{2}+sh)z_k^{(s)}, \\
B_{2+6s}(\omega_k^{(s)}) &= (\frac{1}{2}+s)h\beta^{-1}(\omega_k^{(s)}), \\
B_{2+6s}(z_k^{(s)}) &= 0, \\
B_{3+6s}(\psi_k^{(s)}) &= ((s+1)h-1-\frac{k}{2})\zeta_k, \\
B_{3+6s}(\varepsilon_k^{(s)}) &= 0, \\
B_{4+6s} &= 0, \\
B_{5+6s}(h_k^{(s)}) &= (s+1)h\alpha^{-1}(h_k^{(s)}), \\
B_{6+6s} &= 0.
\end{aligned}$$

Proof. We use the Cartan identity (3.1.0.7) with $a \in \theta_0$,

$$\mathcal{L}_{\theta_0} = B\iota_{\theta_0} + \iota_{\theta_0}B, \tag{3.3.1.4}$$

where $\mathcal{L}_{\theta_0}$ acts on $x \in HH_i$ by multiplication by $\frac{1}{2}\deg(x)$ (see Lemma (3.3.1.1)). The above identities for the Connes differential follow since ι_{θ_0} acts on $\theta_k^{(t)}$, $\omega_k^{(t)}$, $\psi_k^{(t)}$ and $h_k^{(t)}$ by zero, and $z_k^{(t)}$, $\beta^{-1}(\omega_k^{(t)})$, $\zeta_k^{(t)}$ and $\alpha^{-1}(h_k^{(s)})$ are their unique preimages the contraction with ι_{θ_0}. □

We compute the brackets using the identification
$HH^i(A) = HH_{6m+2-i}(A)[-2mh-2]$ for $m >> 1$ and the BV-identity (3.3.0.14).

Brackets involving $HH^{6s}(A)$:

By degree argument these brackets are zero:
$\omega_k^{(s)}$ with $HH^{1+6t}(A)$, $HH^{2+6t}(A)$, $HH^{3+6t}(A)$, $HH^{4+6t}(A)$, and $\psi_l \in HH^{5+6t}(A)$. From the BV-identity (3.3.0.14), we see that brackets of $z_k^{(s)}$ with $z_l^{(t)} \in HH^{6t}(A)$, $HH^{2+6t}(A)$, $HH^{4+6t}(A)$ and $\varepsilon_l^{(t)} \in HH^{5+6t}(A)$ are zero because Δ acts by zero on $U[-2th-2] \subset HH^{6t}(A)$, $HH^{2+6t}(A)$ and $HH^{4+6t}(A)$.

We compute the remaining brackets:

$$\begin{aligned}
[z_k^{(s)}, \omega_l^{(t)}] &= \Delta(z_k^{(s)} \cup \omega_l^{(t)}) - \underbrace{\Delta(z_k^{(s)})}_{=0} \cup \omega_l^{(t)} - z_k^{(s)} \cup \Delta(\omega_l^{(t)}) \\
&= \delta_{k0}\Delta(\omega_l^{(s+t)}) - (\frac{h}{2} + (m-t)h)z_k^{(s)} \cup \beta^{-1}(\omega_l^{(t)}) \\
&= \delta_{k0}(\frac{h}{2} + (m-s-t)h)\beta^{-1}(\omega_l^{(s+t)}) - (\frac{h}{2} + (m-t)h)\beta^{-1}(\omega_l^{(s+t)}) \\
&= -\delta_{k0} sh\beta^{-1}(\omega_l^{(s+t)}), \\
[z_k^{(s)}, \theta_l^{(t)}] &= \Delta(z_k^{(s)} \cup \theta_l^{(t)}) - \underbrace{\Delta(z_k^{(s)})}_{=0} \cup \theta_l^{(t)} - z_k^{(s)} \cup \Delta(\theta_l^{(t)}) \\
&= \Delta((z_k\theta_l)^{(s+t)}) - (1 + \frac{l}{2} + (m-t)h)z_k^{(s)} z_l^{(t)} \\
&= (1 + \frac{k+l}{2} + (m-s-t)h)(z_k z_l)^{(s+t)} \\
&\quad -(1 + \frac{l}{2} + (m-t)h)(z_k z_l)^{(s+t)} \\
&= (\frac{k}{2} - sh)(z_k z_l)^{(s+t)},
\end{aligned}$$

$$
\begin{aligned}
[z_k^{(s)}, h_l^{(t)}] &= \Delta(z_k^{(s)} \cup h_l^{(t)}) - \underbrace{\Delta(z_k^{(s)})}_{=0} \cup h_l^{(t)} - z_k^{(s)} \cup \Delta(h_l^{(t)}) \\
&= \delta_{k0}\Delta(h_l^{(s+t)}) - (h + (m-t-1)h) z_k^{(s)} \cup \alpha^{-1}(h_l^{(t)}) \\
&= \delta_{k0}(h + (m-s-t-1)h)\alpha^{-1}(h_l^{(s+t)}) \\
&\quad -\delta_{k0}(h + (m-t-1)h)\alpha^{-1}(h_l^{(s+t)}) \\
&= -\delta_{k0} s h \alpha^{-1}(h_l^{(s+t)}), \\
[z_k^{(s)}, \psi_l^{(t)}] &= \Delta(z_k^{(s)} \cup \psi_l^{(t)}) - \underbrace{\Delta(z_k^{(s)})}_{=0} \cup \psi_l^{(t)} - z_k^{(s)} \cup \Delta(\psi_l^{(t)}) \\
&= \Delta((z_k\psi_l)^{(s+t)}) - (h - 1 - \frac{t}{2}) z_k^{(s)} \zeta_l^{(t)} \\
&= ((m-s-t)h - 1 - \frac{l-k}{2})(z_k\zeta_l)^{(s+t)} \\
&\quad -((m-t)h - 1 - \frac{l}{2})(z_k\zeta_l)^{(s+t)} \\
&= (\frac{k}{2} - sh)(z_k\zeta_l)^{(s+t)} \\
[\omega_k^{(s)}, \varepsilon_l^{(t)}] &= \Delta(\omega_k^{(s)} \cup \varepsilon_l^{(t)}) - \Delta(\omega_k^{(s)}) \cup \varepsilon_l^{(t)} - \omega_k^{(s)} \cup \underbrace{\Delta\varepsilon_l^{(t)}}_{=0} \\
&= \Delta(\delta_{kl}\psi_0^{s+t}) - (\frac{h}{2} + (m-s)h)\beta^{-1}(\omega_k^{(s)}) \cup \varepsilon_l^{(t)} \\
&= \delta_{kl}(h - 1 + (m-s-t-1)h)\zeta_0^{(s+t)} - \delta_{kl}(\frac{h}{2} + (m-s)h)\zeta_0 \\
&= \delta_{kl}(-\frac{h}{2} - 1 - th)\zeta_0
\end{aligned}
$$

Brackets involving $HH^{1+6s}(A)$:

$$\begin{aligned}
[\theta_k^{(s)}, \theta_l^{(t)}] &= \Delta(\underbrace{\theta_k^{(s)} \cup \theta_l^{(t)}}_{=0}) - \Delta(\theta_k^{(s)}) \cup \theta_l^{(t)} + \theta_k^{(s)} \cup \Delta(\theta_l^{(t)}) \\
&= -(1 + \frac{k}{2} + (m-s)h) z_k^{(s)} \theta_l^{(t)} + (1 + \frac{l}{2} + (m-t)h)) \theta_k^{(s)} z_l^{(t)} \\
&= (\frac{l-k}{2} + (s-t)h)(z_k \theta_l)^{(s+t)} \\
[\theta_k^{(s)}, f_l^{(t)}] &= \Delta(\theta_k^{(s)} \cup f_l^{(t)}) - \Delta(\theta_k^{(s)}) \cup f_l^{(t)} + \theta_k^{(s)} \cup \underbrace{\Delta(f_l^{(t)})}_{=0} \\
&= \delta_{k0}(\Delta(\alpha(f_l^{(s+t)})) - (1 + (m-s)h) f_l^{(s+t)}) \\
&= \delta_{k0}(h + (m-s-t-1)h) f_l^{(s+t)} - (1 + (m-s)h) f_l^{(s+t)} \\
&= -\delta_{k0}(1 + th) f_l^{(s+t)} \\
[\theta_k^{(s)}, h_l^{(t)}] &= \Delta(\underbrace{\theta_k^{(s)} \cup h_l^{(t)}}_{=0}) - \Delta(\theta_k^{(s)}) \cup h_l^{(t)} + \theta_k^{(s)} \cup \Delta(h_l^{(t)}) \\
&= -(1 + (m-s)h + \frac{k}{2}) z_k^{(s)} \cup h_l^{(t)} + (h + (m-t-1)h) \theta_k^{(s)} \cup \alpha^{-1}(h_l^{(t)}) \\
&= -\delta_{k0}(1 + (m-s)h) h_l^{(s+t)} \\
&\quad +\delta_{k0}(m-t)h h_l^{(s+t)} \\
&= \delta_{k0}(-1 + (s-t)h) h_l^{(s+t)} \\
[\theta_k^{(s)}, \zeta_l^{(t)}] &= \Delta(\theta_k^{(s)} \cup \zeta_l^{(t)}) - \Delta(\theta_k^{(s)}) \cup \zeta_l^{(t)} + \theta_k^{(s)} \cup \underbrace{\Delta(\zeta_l^{(t)})}_{=0} \\
&= \Delta((z_k \psi_l)^{(s+t)}) - (1 + \frac{k}{2} + (m-s)h) z_k^{(s)} \cup \zeta_l^{(t)} \\
&= (h - 1 - \frac{l-k}{2} + (m-s-t-1)h)(z_k \zeta_l)^{(s+t)} \\
&\quad -(1 + \frac{k}{2} + (m-s)h)(z_k \zeta_l)^{(s+t)} \\
&= -(2 + \frac{l}{2} + th)(z_k \zeta_l)^{(s+t)}
\end{aligned}$$

$$
\begin{aligned}
[\theta_k^{(s)}, \psi_l^{(t)}] &= \underbrace{\Delta(\theta_k^{(s)} \cup \psi_l^{(t)})}_{=0} - \Delta(\theta_k^{(s)}) \cup \psi_l^{(t)} + \theta_k^{(s)} \cup \Delta(\psi_l^{(t)}) \\
&= -(1 + \frac{k}{2} + (m-s)h) z_k^{(s)} \psi_l^{(t)} + (h - 1 - \frac{l}{2} + (m-t-1)h) \theta_k^{(s)} \zeta_l^{(t)} \\
&= -(2 + \frac{k+l}{2} + (t-s)h)(z_k \psi_l)^{(s+t)}, \\
[\theta_k^{(s)}, \varepsilon_l^{(t)}] &= \Delta(\theta_k^{(s)} \cup \varepsilon_l^{(t)}) - \Delta(\theta_k^{(s)}) \cup \varepsilon_l^{(t)} + \theta_k^{(s)} \cup \underbrace{\Delta(\varepsilon_l^{(t)})}_{=0} \\
&= \delta_{k0} \Delta(\beta(\varepsilon_l^{(s+t)})) - \delta_{k0}(1 + (m-s)h + \frac{k}{2}) z_k^{(s)} \varepsilon_l^{(t)} \\
&= \delta_{k0}(\frac{h}{2} + (m-s-t-1)h) \varepsilon_l^{(s+t)} - (1 + (m-s)h) \varepsilon_l^{(s+t)} \\
&= -\delta_{k0}(1 + (t + \frac{1}{2})h) \varepsilon_l^{(s+t)}
\end{aligned}
$$

Brackets involving $HH^{2+6s}(A)$:

By degree argument, the bracket of $HH^{2+6s}(A)$ with $HH^{2+6t}(A)$ is zero.

$$\begin{aligned}
[f_k^{(s)}, h_l^{(t)}] &= \Delta(f_k^{(s)} \cup h_l^{(t)}) - \underbrace{\Delta(f_k^{(s)})}_{=0} \cup h_l^{(t)} - f_k^{(s)} \cup \Delta(h_l^{(t)}) \\
&= \Delta(\delta_{kl}\psi_0^{(s+t)}) - (h + (m-t-1)h) f_k^{(s)} \cup \alpha^{-1}(h_l^{(t)}) \\
&= \delta_{kl}(h - 1 + (m-s-t-1)h)\zeta_0 - \delta_{kl}(m-t)h\zeta_0 \\
&= -\delta_{kl}(1+sh)\zeta_0^{(s+t)}, \\
[f_k^{(s)}, \zeta_l^{(t)}] &= \Delta(f_k^{(s)} \cup \zeta_l^{(t)}) - \underbrace{\Delta(f_k^{(s)})}_{=0} \cup \zeta_l^{(t)} - f_k^{(s)} \cup \underbrace{\Delta(\zeta_l^{(t)})}_{=0} \\
&= \delta_{l,h-3}(k+1)\Delta(z_{h-3}^{(s+t)}) = 0, \\
[f_k^{(s)}, \psi_l^{(t)}] &= \Delta(f_k^{(s)} \cup \psi_l^{(t)}) - \underbrace{\Delta(f_k^{(s)})}_{=0} \cup \psi_l^{(t)} - f_k^{(s)} \cup \Delta(\psi_l^{(t)}) \\
&= \Delta(f_k^{(s)} \cup \psi_l^{(t)}) - (h - 1 - \frac{l}{2} + (m-t-1)h) f_k^{(s)} \cup \zeta_l^{(t)} \\
&= \delta_{l,h-3}(k+1)\Delta(\theta_{h-3}^{(s+t+1)}) \\
&\quad -\delta_{l,h-3}((m-t)h - 1 - \frac{h-3}{2})(k+1) z_{h-3}^{(s+t+1)} \\
&= \delta_{l,h-3}(k+1)(1 + \frac{h-3}{2} + (m-s-t-1)h) z_{h-3}^{(s+t+1)} \\
&\quad -\delta_{l,h-3}((m-t)h - 1 - \frac{h-3}{2})(k+1) z_{h-3}^{(s+t+1)} \\
&= -\delta_{l,h-3}(k+1)(1+sh) z_{h-3}^{(s+t+1)}, \\
[f_k^{(s)}, \varepsilon_l^{(t)}] &= \Delta\underbrace{(f_k^{(s)}) \cup \varepsilon_l^{(t)}}_{=0}) - \underbrace{\Delta(f_k^{(s)})}_{=0} \cup \varepsilon_l^{(t)} - \Delta f_k^{(s)} \cup \underbrace{\Delta(\varepsilon_l^{(t)})}_{=0} = 0
\end{aligned}$$

Brackets involving $HH^{3+6s}(A)$:

We have

$$\begin{aligned}
[h_k^{(s)}, h_l^{(t)}] &= \Delta(\underbrace{h_k^{(s)} \cup h_l^{(t)}}_{=0}) - \Delta(h_k^{(s)}) \cup h_l^{(t)} + h_k^{(s)} \cup \Delta(h_l^{(t)}) \\
&= -(h + (m-s-1)h)\alpha^{-1}(h_k^{(s)}) \cup h_l^{(t)} \\
&\quad +(h + (m-t-1)h)h_k^{(s)} \cup \alpha^{-1}(h_l^{(t)}) \\
&= (s-t)h\alpha^{-1}(h_k^{(s)}) \cup h_l^{(t)} = (s-t)h(M_\alpha^{-1})_{kl}\psi_0^{(s+t)}, \\
[h_k^{(s)}, \zeta_l^{(t)}] &= \Delta(h_k^{(s)} \cup \zeta_l^{(t)}) - \Delta(h_k^{(s)}) \cup \zeta_l^{(t)} + \underbrace{h_k^{(s)} \cup \Delta(\zeta_l^{(t)})}_{=0} \\
&= \delta_{k,\frac{h-3}{2}}\delta_{l,h-3}\Delta(\theta_{h-3}^{(s+t+1)}) - (m-s)h\alpha^{-1}(h_k^{(s)}) \cup \zeta_l^{(t)} \\
&= \delta_{k,\frac{h-3}{2}}\delta_{l,h-3}((1 + \frac{h-3}{2} + (m-s-t-1)h)z_{h-3}^{(s+t+1)} \\
&\quad -(m-s)hz_{h-3}^{(s+t+1)}) \\
&= \delta_{k,\frac{h-3}{2}}\delta_{l,h-3}(-\frac{h+1}{2} - th)z_{h-3}^{(s+t+1)}
\end{aligned}$$

We have

$$\begin{aligned}
[h_k^{(s)}, \psi_l^{(t)}] &= \Delta(\underbrace{h_k^{(s)} \cup \psi_l^{(t)}}_{=0}) - \Delta(h_k^{(s)}) \cup \psi_l^{(t)} + h_k^{(s)} \cup \Delta(\psi_l^{(t)}) \\
&= -(m-s)h\alpha^{-1}(h_k^{(s)}) \cup \psi_l^{(t)} + ((m-t)h - 1 - \frac{l}{2})h_k^{(s)} \cup \zeta_l^{(t)} \\
&= -(m-s)h\delta_{k,\frac{h-3}{2}}\delta_{l,h-3}\theta_{h-3}^{(s+t+1)} \\
&\quad +((m-t)h - 1 - \frac{l}{2})\delta_{k,\frac{h-3}{2}}\theta_{h-3}^{(s+t+1)} \\
&= \delta_{l,h-3}\delta_{k,\frac{h-3}{2}}((s-t)h - \frac{h-1}{2})\theta_{h-3}^{(s+t+1)}, \\
[h_k^{(s)}, \varepsilon_l^{(t)}] &= \Delta(\underbrace{h_k^{(s)} \cup \varepsilon_l^{(t)}}_{=0}) - \Delta(h_k^{(s)}) \cup \varepsilon_l^{(t)} - \Delta h_k^{(s)} \cup \underbrace{\Delta(\varepsilon_l^{(t)})}_{=0} \\
&= -(m-s)h\alpha^{-1}(h_k^{(s)}) \cup \varepsilon_l^{(t)} = 0.
\end{aligned}$$

Brackets involving $HH^{4+6s}(A)$:

The bracket $[\zeta_k^{(s)}, \zeta_l^{(t)}] = \Delta(\zeta_k^{(s)} \cup \zeta_l^{(t)}) - \Delta(\zeta_k^{(s)}) \cup \zeta_l^{(t)} - \zeta_k^{(s)} \cup \Delta(\zeta_l^{(t)}) = 0$ because Δ is zero on HH^{2+6s} and HH^{4+6s}.

$$\begin{aligned}
[\zeta_k^{(s)}, \psi_l^{(t)}] &= \Delta(\zeta_k^{(s)} \cup \psi_l^{(t)}) - \underbrace{\Delta(\zeta_k^{(s)})}_{=0} \cup \psi_l^{(t)} - \zeta_k^{(s)} \cup \Delta(\psi_l^{(t)}), \\
&= \delta_{k,h-3}\delta_{l,h-3}\Delta(\alpha(f_{\frac{h-3}{2}}^{(s+t+1)})) - ((m-t)h - 1 - \frac{l}{2})\zeta_k^{(s)} \cup \zeta_l^{(t)} \\
&= \delta_{k,h-3}\delta_{l,h-3}(m-s-t-1)hf_{\frac{h-3}{2}}^{(s+t+1)} - ((m-t)h - 1 - \frac{h-3}{2})f_{\frac{h-3}{2}}^{(s+t+1)} \\
&= \delta_{k,h-3}\delta_{l,h-3}(-sh - \frac{h+1}{2})f_{\frac{h-3}{2}}.
\end{aligned}$$

The bracket of $HH^{5+6s}(A)$ with $HH^{5+6s}(A)$:

$$\begin{aligned}
[\psi_k^{(s)}, \psi_l^{(t)}] &= \underbrace{\Delta(\psi_k^{(s)} \cup \psi_l^{(t)})}_{=0} - \Delta(\psi_k^{(s)}) \cup \psi_l^{(t)} + \psi_k^{(s)} \cup \Delta(\psi_l^{(t)}), \\
&= -((m-s)h - 1 - \frac{k}{2})\zeta_k^{(s)} \cup \psi_l^{(t)} + ((m-t)h - 1 - \frac{l}{2})\psi_k^{(s)} \cup \zeta_l^{(t)} \\
&= \delta_{k,h-3}\delta_{l,h-3}(s-t)h\alpha(f_{\frac{h-3}{2}}^{(s+t+1)}).
\end{aligned}$$

The Lie derivative $\mathcal{L}$

We use the Cartan identity (3.1.0.7) to compute the Lie derivative.

$HH^{1+6s}(A)$-Lie derivatives:

From the Cartan identity, we see that

$$\mathcal{L}_{\theta_k^{(s)}} = B\iota_{\theta_k^{(s)}} + \iota_{\theta_k^{(s)}}B.$$

On $\theta_{l,t}$, $\omega_{l,t}$, $\psi_{l,t}$ and $h_{l,t}$, the Connes differential acts by multiplication with $\frac{1}{2}$ degree and taking the preimage under ι_{θ_0}, and $\iota_{\theta_k^{(s)}}$ acts on them by zero. B acts by zero on $z_l^{(t)}$, $\varepsilon_k^{(t)}$, $\zeta_{l,t}$ and $f_{l,t}$. Since B is degree preserving, this means that $\mathcal{L}_{\theta_k^{(s)}}$ acts on $\theta_{l,t}$, $\omega_{l,t}$, $\psi_{l,t}$ and $h_{l,t}$ by multiplication with $\frac{1}{2}$ their degree times $z_k^{(s)}$, and on $z_{l,t}$, $\varepsilon_{k,t}$, $\zeta_{l,t}$ and $f_{l,t}$ by multiplication with $z_k^{(s)}$ and then multiplication with $\frac{1}{2}$ degree of their product. So we get the following formulas:

$$\begin{aligned}
\mathcal{L}_{\theta_k^{(s)}}(\theta_{l,t}) &= (1+\frac{l}{2}+th)(z_k\theta_l)_{t-s},\\
\mathcal{L}_{\theta_k^{(s)}}(z_{l,t}) &= (1+\frac{k+l}{2}+(t-s)h)(z_kz_l)_{t-s},\\
\mathcal{L}_{\theta_k^{(s)}}(\omega_{l,t}) &= \delta_{k0}(\frac{1}{2}+t)h\omega_{l,t-s},\\
\mathcal{L}_{\theta_k^{(s)}}(\varepsilon_{l,t}) &= \delta_{k0}(\frac{1}{2}+(t-s))h\varepsilon_{l,t-s},\\
\mathcal{L}_{\theta_k^{(s)}}(\psi_{l,t}) &= ((t+1)h-1-\frac{l}{2})(z_k\psi_l)_{t-s},\\
\mathcal{L}_{\theta_k^{(s)}}(\zeta_{l,t}) &= ((t-s+1)h-1-\frac{l-k}{2})(z_k\zeta_l)_{t-s},\\
\mathcal{L}_{\theta_k^{(s)}}(h_{l,t}) &= \delta_{k0}(t+1)hh_{l,t-s},\\
\mathcal{L}_{\theta_k^{(s)}}(f_{l,t}) &= \delta_{k0}(t-s+1)hf_{l,t-s}
\end{aligned}$$

$HH^{2+6s}(A)$**-Lie derivatives**:

We compute $\mathcal{L}_{f_k^{(s)}}$:

$$
\begin{aligned}
\mathcal{L}_{f_k^{(s)}}(\theta_{l,t}) &= B(\iota_{f_k^{(s)}}(\theta_{l,t})) - \iota_{f_k^{(s)}}(B(\theta_{l,t})) \\
&= B(\delta_{l0}\alpha(f_{k,t-s-1})) - (1+\frac{l}{2}+th))\iota_{f_k^{(s)}} z_{l,t} \\
&= \delta_{l0}(t-s)hf_{k,t-s-1} - \delta_{l0}(1+th)f_{k,t-s-1} = -\delta_{l0}(1+sh)f_{k,t-s}, \\
\mathcal{L}_{f_k^{(s)}}(f_{l,t}) &= B(\underbrace{\iota_{f_k^{(s)}}(f_{l,t})}_{\in HH_{4+6(t-s)}}) = 0, \\
\mathcal{L}_{f_k^{(s)}}(z_{l,t}) &= \delta_{l0}B(f_{k,t-s}) = 0, \\
\mathcal{L}_{f_k^{(s)}}(\omega_{l,t}) &= B(\underbrace{\iota_{f_k^{(s)}}\omega_{l,t}}_{=0}) + \iota_{f_k^{(s)}}B(\omega_{l,t}) = (\frac{1}{2}+t)h\iota_{f_k^{(s)}}\beta^{-1}(\omega_{l,t}) = 0, \\
\mathcal{L}_{f_k^{(s)}}(\varepsilon_{l,t}) &= B(\underbrace{\iota_{f_k^{(s)}}(\varepsilon_{l,t})}_{=0}), \\
\mathcal{L}_{f_k^{(s)}}(\psi_{l,t}) &= B(\iota_{f_k^{(s)}}(\psi_{l,t})) - \iota_{f_k^{(s)}}B(\psi_{l,t}) \\
&= B(\delta_{l,h-3}(k+1)\theta_{h-3,t-s}) - ((t+1)h-1-\frac{l}{2})\iota_{f_k^{(s)}}(\zeta_{l,t}) \\
&= \delta_{l,h-3}(k+1)(1+\frac{h-3}{2}+(t-s)h)z_{h-3,t-s} \\
&\quad -\delta_{l,h-3}(k+1)((t+1)h-1-\frac{l}{2})z_{h-3,t-s} \\
&= -\delta_{l,h-3}(k+1)(1+sh)z_{h-3,t-s}, \\
\mathcal{L}_{f_k^{(s)}}(\zeta_{l,t}) &= B(\iota_{f_k^{(s)}}(\zeta_{l,t})) = B(k\delta_{l,h-3}z_{h-3,t-s}) = 0, \\
\mathcal{L}_{f_k^{(s)}}(h_{l,t}) &= B(\iota_{f_k^{(s)}}(h_{l,t})) - \iota_{f_k^{(s)}}B(h_{l,t}) \\
&= B(\delta_{k,l}\psi_{0,t-s}) - (t+1)h\iota_{f_k^{(s)}}\alpha^{-1}(h_{l,t}) \\
&= \delta_{kl}((t-s+1)h-1)\zeta_{0,t-s} - \delta_{kl}(t+1)h\zeta_{0,t-s} \\
&= -\delta_{kl}(sh+1)\zeta_{0,t-s}
\end{aligned}
$$

$HH^{3+6s}(A)$**-Lie derivatives**:

We compute $\mathcal{L}_{h_k^{(s)}}$:

$$\begin{aligned}
\mathcal{L}_{h_k^{(s)}}(\theta_l^{(t)}) &= B(\underbrace{\iota_{h_k^{(s)}}(\theta_l^{(t)})}_{=0}) + \iota_{h_k^{(s)}}B(\theta_{l,t}) = (1+\frac{l}{2}+th)\iota_{h_k^{(s)}}z_{l,t} \\
&= \delta_{l0}(1+th)h_{k,t-s-1}, \\
\mathcal{L}_{h_k^{(s)}}(z_{l,t}) &= B(\delta_{l0}h_{k,t-s-1}) = \delta_{l0}(t-s)h\alpha^{-1}(h_{k,t-s-1}), \\
\mathcal{L}_{h_k^{(s)}}(\omega_{l,t}) &= B\underbrace{\iota_{h_k^{(s)}}(\omega_{l,t})}_{=0} + \underbrace{\iota_{h_k^{(s)}}B(\omega_{l,t})}_{\text{cup product in } HH^3(A)\times HH^5(A)} = 0, \\
\mathcal{L}_{h_k^{(s)}}(\varepsilon_{l,t}) &= B\underbrace{\iota_{h_k^{(s)}}\varepsilon_{l,t}}_{=0} = 0, \\
\mathcal{L}_{h_k^{(s)}}(\psi_{l,t}) &= B(\underbrace{\iota_{h_k^{(s)}}(\psi_{l,t})}_{=0}) + h_k^{(s)}B(\psi_{l,t}) \\
&= ((t+1)h-1-\frac{l}{2})h_k^{(s)}\zeta_{l,t} = \delta_{k,\frac{h-3}{2}}\delta_{l,h-3}(th+\frac{h+1}{2})\theta_{h-3,t-s}, \\
\mathcal{L}_{h_k^{(s)}}(\zeta_{l,t}) &= B(\iota_{h_k^{(s)}}(\zeta_{l,t})) = \delta_{k,\frac{h-3}{2}}B(\delta_{l,h-3}\theta_{h-3,t-s}) \\
&\quad = \delta_{l,h-3}((t-s)h+\frac{h-1}{2})z_{h-3,t-s}, \\
\mathcal{L}_{h_k^{(s)}}(h_{l,t}) &= B(\underbrace{\iota_{h_k^{(s)}}(h_{l,t})}_{=0}) + \iota_{h_k^{(s)}}B(h_{l,t}) = (t+1)h\iota_{h_k^{(s)}}\alpha^{-1}(h_{l,t}) \\
&= (t+1)h(M_\alpha^{-1})_{lk}\psi_{0,t-s} \\
\mathcal{L}_{h_k^{(s)}}(f_{l,t}) &= B(\iota_{h_k^{(s)}}(f_{l,t})) = B(\delta_{kl}\psi_{0,t-s}) = \delta_{kl}((t-s+1)h-1)\zeta_{0,t-s}
\end{aligned}$$

$\underline{HH^{4+6s}(A)\text{-}\mathbf{Lie\ derivatives}}$:

We compute $\mathcal{L}_{\zeta_k^{(s)}}$:

$$\begin{aligned}
\mathcal{L}_{\zeta_k^{(s)}}(\theta_{l,t}) &= B\iota_{\zeta_k^{(s)}}(\theta_{l,t}) - \iota_{\zeta_k^{(s)}}B(\theta_{l,t}) = B((z_l\psi_k)_{t-s-1}) - \iota_{\zeta_k^{(s)}}(1+\frac{l}{2}+th)z_{l,t} \\
&= ((t-s)h-1-\frac{k-l}{2})(z_l\zeta_k)_{t-s-1} - (1+\frac{l}{2}+th)(z_l\zeta_k)_{t-s-1} \\
&= (-sh-2-\frac{k}{2})(z_l\zeta_k)_{t-s-1}, \\
\mathcal{L}_{\zeta_k^{(s)}}(\omega_{l,t}) &= 0, \\
\mathcal{L}_{\zeta_k^{(s)}}(z_{l,t}) &= B\iota_{\zeta_k^{(s)}}(z_{l,t}) - \iota_{\zeta_k^{(s)}}\underbrace{B(z_{l,t})}_{=0} \\
&= B((z_l\zeta_k)_{t-s-1}) = 0, \\
\mathcal{L}_{\zeta_k^{(s)}}(\psi_{l,t}) &= B\iota_{\zeta_k^{(s)}}(\psi_{l,t}) - \iota_{\zeta_k^{(s)}}B(\psi_{l,t}) \\
&= \delta_{k,h-3}\delta_{l,h-3}B(\alpha(f_{\frac{h-3}{2},t-s-1})) - \iota_{\zeta_k^{(s)}}((t+1)h-1-\frac{l}{2})\zeta_{l,t} \\
&= \delta_{k,h-3}\delta_{l,h-3}((t-s)hf_{\frac{h-3}{2},t-s-1} \\
&\quad -((t+1)h-1-\frac{h-3}{2})f_{\frac{h-3}{2},t-s-1}) \\
&= \delta_{k,h-3}\delta_{l,h-3}(-sh-\frac{h+1}{2})f_{\frac{h-3}{2},t-s-1}, \\
\mathcal{L}_{\zeta_k^{(s)}}(\varepsilon_{l,t}) &= B\underbrace{\iota_{\zeta_k^{(s)}}(\varepsilon_{l,t})}_{=0} - \iota_{\zeta_k^{(s)}}\underbrace{B(\varepsilon_{l,t})}_{=0} = 0, \\
\mathcal{L}_{\zeta_k^{(s)}}(\zeta_{l,t}) &= B\iota_{\zeta_k^{(s)}}(\zeta_{l,t}) - \iota_{\zeta_k^{(s)}}\underbrace{B(\zeta_{l,t})}_{=0} \\
&= \delta_{kh-3}\delta_{l,h-3}B(f_{\frac{h-3}{2},t-s-1}) = 0, \\
\mathcal{L}_{\zeta_k^{(s)}}(h_{l,t}) &= B\iota_{\zeta_k^{(s)}}(h_{l,t}) - \iota_{\zeta_k^{(s)}}B(h_{l,t}) \\
&= \delta_{l,\frac{h-3}{2}}\delta_{k,h-3}B(\theta_{h-3,t-s}) - (t+1)h\iota_{\zeta_k^{(s)}}\alpha^{-1}(h_{l,t}), \\
&= \delta_{l,\frac{h-3}{2}}\delta_{k,h-3}z_{h-3,t-s}((1+\frac{h-3}{2}+(t-s)h)-(t+1)h) \\
&= \delta_{l,\frac{h-3}{2}}\delta_{k,h-3}z_{h-3,t-s}(-\frac{h+1}{2}-sh), \\
\mathcal{L}_{\zeta_k^{(s)}}(f_{l,t}) &= B\iota_{\zeta_k^{(s)}}(f_{l,t}) - \iota_{\zeta_k^{(s)}}\underbrace{B(f_{l,t})}_{=0} \\
&= (l+1)\delta_{k,h-3}B(z_{h-3,t-s}) = 0
\end{aligned}$$

$\underline{HH^{5+6s}(A)\textbf{-Lie derivatives}}$:

We compute $\mathcal{L}_{\varepsilon_k^{(s)}}$:

$$\begin{aligned}
\mathcal{L}_{\varepsilon_k^{(s)}}(\theta_{l,t}) &= B(\iota_{\varepsilon_k^{(s)}}(\theta_{l,t})) + \iota_{\varepsilon_k^{(s)}} B(\theta_{l,t}) \\
&= B(-\delta_{l0}\beta(\varepsilon_{k,t-s-1})) + (1 + \frac{l}{2} + th)\iota_{\varepsilon_k^{(s)}}(z_{l,t}) \\
&= -\delta_{l0}(\frac{1}{2} + t - s - 1)h\varepsilon_{k,t-s-1} + (1 + th)\delta_{l0}\varepsilon_{k,t-s-1} \\
&= ((s + \frac{1}{2})h + 1)\delta_{l0}\varepsilon_{k,t-s-1}, \\
\mathcal{L}_{\varepsilon_k^{(s)}}(z_{l,t}) &= B(\iota_{\varepsilon_k^{(s)}}(z_{l,t})) = B(\varepsilon_{k,t-s-1}) = 0, \\
\mathcal{L}_{\varepsilon_k^{(s)}}(\omega_{l,t}) &= B(\iota_{\varepsilon_k^{(s)}}(\omega_{l,t})) + \iota_{\varepsilon_k^{(s)}} B(\omega_{l,t}) \\
&= B(\delta_{kl}\psi_{0,t-s-1}) + (\frac{1}{2} + t)h\iota_{\varepsilon_k^{(s)}}\beta^{-1}(\omega_{l,t}) \\
&= \delta_{kl}((t-s)h - 1)\zeta_{0,t-s-1} - \delta_{kl}(\frac{1}{2} + t)h\zeta_{0,t-s-1} \\
&= -\delta_{kl}(1 + (\frac{1}{2} + s)h)\zeta_{0,t-s-1}, \\
\mathcal{L}_{\varepsilon_k^{(s)}}(\psi_{l,t}) &= B(\underbrace{\iota_{\varepsilon_k^{(s)}}(\psi_{l,t})}_{=0}) + \iota_{\varepsilon_k^{(s)}} B(\psi_{l,t}) \\
&= ((t+1)h - 1 - \frac{l}{2})\iota_{\varepsilon_k^{(s)}}\zeta_{l,t} = 0, \\
\mathcal{L}_{\varepsilon_k^{(s)}}(\varepsilon_{l,t}) &= B(\iota_{\varepsilon_k^{(s)}}(\varepsilon_{l,t})) = B(-(M_\beta)_{kl}\zeta_{0,t-s-1}) = 0, \\
\mathcal{L}_{\varepsilon_k^{(s)}}(\zeta_{l,t}) &= B(\underbrace{\iota_{\varepsilon_k^{(s)}}\zeta_{l,t}}_{=0}) = 0, \\
\mathcal{L}_{\varepsilon_k^{(s)}}(h_{l,t}) &= B(\iota_{\varepsilon_k^{(s)}}(h_{l,t})) + (t+1)h\iota_{\varepsilon_k^{(s)}}\alpha^{-1}(h_{l,t}) = 0, \\
\mathcal{L}_{\varepsilon_k^{(s)}}(f_{l,t}) &= B(\underbrace{\iota_{\varepsilon_k^{(s)}}(f_{l,t})}_{=0}) = 0
\end{aligned}$$

We compute $\mathcal{L}_{\psi_k^{(s)}}$:

$$
\begin{aligned}
\mathcal{L}_{\psi_k^{(s)}}(\theta_{l,t}) &= B\underbrace{\iota_{\psi_k^{(s)}}(\theta_{l,t})}_{=0}+\iota_{\psi_k^{(s)}}B(\theta_{l,t}) \\
&= \iota_{\psi_k^{(s)}}z_{l,t}(1+\frac{l}{2}+th)=(z_l\psi_k)_{t-s-1}(1+\frac{l}{2}+th), \\
\mathcal{L}_{\psi_k^{(s)}}(z_{l,t}) &= B\iota_{\psi_k^{(s)}}(z_{l,t})+\iota_{\psi_k^{(s)}}\underbrace{B(z_{l,t})}_{=0} \\
&= B((z_l\psi_k)_{t-s-1})=((t-s)h-1-\frac{k-l}{2})(z_l\zeta_k)_{t-s-1}, \\
\mathcal{L}_{\psi_k^{(s)}}(\omega_{l,t}) &= B\underbrace{\iota_{\psi_k^{(s)}}(\omega_{l,t})}_{=0}+\iota_{\psi_k^{(s)}}B(\omega_{l,t}) \\
&= (\frac{1}{2}+t)h\iota_{\psi_k^{(s)}}\beta^{-1}(\omega_{l,t})=0, \\
\mathcal{L}_{\psi_k^{(s)}}(\psi_{l,t}) &= B\underbrace{\iota_{\psi_k^{(s)}}(\psi_{l,t})}_{=0}+\iota_{\psi_k^{(s)}}B(\psi_{l,t}) \\
&= ((t+1)h-1-\frac{l}{2})\iota_{\psi_k^{(s)}}\zeta_{l,t} \\
&= \delta_{k,h-3}\delta_{l,h-3}\underbrace{((t+1)h-1-\frac{h-3}{2})}_{=th+\frac{h+1}{2}}\alpha(f_{\frac{h-3}{2},t-s-1}), \\
\mathcal{L}_{\psi_k^{(s)}}(\varepsilon_{l,t}) &= B\underbrace{\iota_{\psi_k^{(s)}}(\varepsilon_{l,t})}_{=0}+\iota_{\psi_k^{(s)}}\underbrace{B(\varepsilon_{l,t})}_{=0}=0 \\
\mathcal{L}_{\psi_k^{(s)}}(\zeta_{l,t}) &= B\iota_{\psi_k^{(s)}}(\zeta_{l,t})+\iota_{\psi_k^{(s)}}\underbrace{B(\zeta_{l,t})}_{=0} \\
&= \delta_{k,h-3}\delta_{l,h-3}B(\alpha(f_{\frac{h-3}{2},t-s-1})) \\
&= \delta_{k,h-3}\delta_{l,h-3}(t-s)hf_{\frac{h-3}{2},t-s-1}, \\
\mathcal{L}_{\psi_k^{(s)}}(h_{l,t}) &= B\underbrace{\iota_{\psi_k^{(s)}}(h_{l,t})}_{=0}+\iota_{\psi_k^{(s)}}B(h_{l,t})=\iota_{\psi_k^{(s)}}\alpha^{-1}(h_{l,t})(t+1)h \\
&= \delta_{k,h-3}\delta_{l,\frac{h-3}{2}}(t+1)h\theta_{h-3,t-s}, \\
\mathcal{L}_{\psi_k^{(s)}}(f_{l,t}) &= B_{\psi_k^{(s)}}(f_{l,t})+\iota_{\psi_k^{(s)}}\underbrace{B(f_{l,t})}_{=0} \\
&= (l+1)\delta_{k,h-3}B(\theta_{h-3,t-s}) \\
&= (l+1)(1+(t-s)h+\frac{h-3}{2})\delta_{k,h-3}z_{h-3,t-s}
\end{aligned}
$$

$HH^{6+6s}(A)$**-Lie derivatives:**

B acts on $\theta_{l,t}$, $\omega_{l,t}$, $\psi_{l,t}$ and $h_{l,t}$ by multiplication with $\frac{1}{2}$ degree and taking the preimage under ι_{θ_0}. On $z_{l,t}$, $\varepsilon_{l,t}$, $\zeta_{l,t}$ and $f_{l,t}$, B acts by zero. Since the spaces U, U^*, K, K^*, Y and Y^* are z_k-invariant and $z_k^{(s)}$ has degree $k-2sh$, $\mathcal{L}_{z_k^{(s)}}$ acts on $\theta_{l,t}$, $\omega_{l,t}$, $\psi_{l,t}$ and $h_{l,t}$ by multiplication with $\frac{k}{2}-sh$ and taking the preimage under ι_{θ_0} and multiplication with $z_k^{(s)}$, and on $z_{l,t}$, $\varepsilon_{l,t}$, $\zeta_{l,t}$ and $f_{l,t}$ it acts by zero. We have the following formulas:

$$
\begin{aligned}
\mathcal{L}_{z_k^{(s)}}(\theta_{l,t}) &= (\frac{k}{2}-sh)(z_k\theta_l)_{t-s},\\
\mathcal{L}_{z_k^{(s)}}(z_{l,t}) &= 0,\\
\mathcal{L}_{z_k^{(s)}}(\omega_{l,t}) &= -\delta_{k0}sh\beta^{-1}(\omega_{l,t-s}),\\
\mathcal{L}_{z_k^{(s)}}(\psi_{l,t}) &= (\frac{k}{2}-sh)(z_k\zeta_l)_{t-s},\\
\mathcal{L}_{z_k^{(s)}}(\varepsilon_{l,t}) &= 0,\\
\mathcal{L}_{z_k^{(s)}}(\zeta_{l,t}) &= 0,\\
\mathcal{L}_{z_k^{(s)}}(h_{l,t}) &= (\frac{k}{2}-sh)\alpha^{-1}(h_{l,t-s}),\\
\mathcal{L}_{z_k^{(s)}}(f_{l,t}) &= 0
\end{aligned}
$$

Now we compute $\mathcal{L}_{\omega_k^{(s)}}$:

We observe that $\iota_{\omega_k^{(s)}}(\varepsilon_{l,t}) = \delta_{kl}\psi_{0,t-s}$, $\iota_{\omega_k^{(s)}}(z_{l,t}) = \delta_{l0}\omega_{k,t-s}$, and

$$\iota_{\omega_k^{(s)}}(\theta_{l,t}) = \iota_{\omega_k^{(s)}}(\omega_{l,t}) = \iota_{\omega_k^{(s)}}(\psi_{l,t}) = \iota_{\omega_k^{(s)}}(\zeta_{l,t}) = \iota_{\omega_k^{(s)}}(h_{l,t}) = \iota_{\omega_k^{(s)}}(f_{l,t}) = 0.$$

Then we have

$$\begin{aligned}
\mathcal{L}_{\omega_k^{(s)}}(\varepsilon_{l,t}) &= B\iota_{\omega_k^{(s)}}(\varepsilon_{l,t}) = \delta_{kl}B(\psi_{0,t-s}) \\
&= \delta_{kl}((t-s+1)h-1)\zeta_{0,t-s}, \\
\mathcal{L}_{\omega_k^{(s)}}(z_{l,t}) &= B\iota_{\omega_k^{(s)}}(z_{l,t}) = \delta_{l0}B(\omega_{k,t-s}) \\
&= \delta_{l0}(\frac{1}{2}+t-s)h\beta^{-1}(\omega_{k,t-s}), \\
\mathcal{L}_{\omega_k^{(s)}}(\theta_{l,t}) &= \iota_{\omega_k^{(s)}}B(\theta_{l,t}) = (1+\frac{l}{2}+th)\iota_{\omega_k^{(s)}}z_{l,t} \\
&= \delta_{l0}(1+th)\omega_{k,t-s},
\end{aligned}$$

and

$$\mathcal{L}_{\omega_k^{(s)}}(\omega_{l,t}) = \mathcal{L}_{\omega_k^{(s)}}(\psi_{l,t}) = \mathcal{L}_{\omega_k^{(s)}}(\zeta_{l,t}) = \mathcal{L}_{\omega_k^{(s)}}(h_{l,t}) = \mathcal{L}_{\omega_k^{(s)}}(f_{l,t}) = 0.$$

Chapter 4

The centrally extended preprojective algebra

4.0.2 Definition

Let $\mu = (\mu_i)$ be a regular weight. We define the *centrally extended preprojective algebra* $A = A^\mu$ to be the quotient of $P[z]$ (z is a central variable) by the relation $\sum_{a\in Q} [a, a^*] = z(\sum_{i\in I} \mu_i e_i)$. By taking the quotient $A/(z)$, we obtain the usual *preprojective algebra* $\Pi_Q = P/(\sum_{a\in Q} [a, a^*])$.

The grading on A is given by $\deg(R) = 0$, $\deg(a) = \deg(a^*) = 1$ and $\deg(z) = 2$.

From now on, we assume μ to be a generic weight or $\mu = \rho$.

4.1 Hochschild homology/cohomology and cyclic homology of A

4.1.1 Periodic projective resolution of A

Let V be the R-bimodule which is generated by the arrows in $\bar{Q}$ (i.e. the degree 1-part of A). For a $\mathbb{Z}$−graded R-bimodule M, we denote $M[i]$ to be the bimodule M, shifted by degree i (i.e. $M(d) = M[i](d + i)$).

We want to compute Hochschild homology and cohomology of A, so we want to find a projective resolution of A.

Let

$$\begin{aligned} C_{-1} &= A, \\ C_0 &= A \otimes_R A, \\ C_1 &= (A \otimes_R V \otimes_R A) \oplus (A \otimes_R A)[2], \\ C_2 &= (A \otimes_R V \otimes_R A)[2] \oplus (A \otimes_R A)[2], \\ C_3 &= A \otimes_R A[4], \\ C_4 &= C_0[2h]. \end{aligned}$$

We define the following A–bimodule-homomorphisms $d_i : C_i \to C_{i-1}$:

$$d_0(b_1 \otimes b_2) = b_1 b_2,$$

$$d_1(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = b_1\alpha \otimes b_2 - b_1 \otimes \alpha b_2 + b_3 z \otimes b_4 - b_3 \otimes z b_4,$$

$$\begin{aligned} d_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = (&-b_1 z \otimes \alpha \otimes b_2 + b_1 \otimes \alpha \otimes z b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_3 a \otimes a^* \otimes b_4 \\ &+ \sum_{a \in \bar{Q}} \epsilon_a b_3 \otimes a \otimes a^* b_4, -b_3\mu \otimes b_4 + b_1\alpha \otimes b_2 - b_1 \otimes \alpha b_2), \end{aligned}$$

where we introduce the notation $\epsilon_a = \begin{cases} +1 & a \in Q \\ -1 & a \in Q^* \end{cases}$,

$$d_3(b_1 \otimes b_2) = (\sum_{a \in \bar{Q}} \epsilon_a b_1 a \otimes a^* \otimes b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_1 \otimes a \otimes a^* b_2, b_1 z \otimes b_2 - b_1 \otimes z b_2),$$

$$d_4(b_1 \otimes b_2) = \sum b_1 x_i \otimes x_i^* b_2,$$

where $\{x_i\}$ is a basis of A and $\{x_i^*\}$ the dual basis under the (symmetric and non-degenerate) trace form $(x, y) = Tr(xy)$ introduced in [9, Section 2.2.]. It is easy to see that d_4 is independent of the choice of the basis $\{x_i\}$. It is clear that all d_i are degree-preserving.

Using the trace form, it is easy to show that $\sum ax_i \otimes x_i^* = \sum x_i \otimes x_i^* a$ for any $a \in A$:

$$\sum ax_i \otimes x_i^* = \sum\sum (ax_i, x_j^*)x_j \otimes x_i^* = \sum\sum x_i \otimes (x_i^* a, x_j)x_j^* = \sum x_i \otimes x_i^* a.$$

This implies

$$d_4(b_1 \otimes b_2) = b_0(b_1 \otimes b_2) \sum x_i \otimes x_i^*$$

Theorem 4.1.1.1. *From the maps d_i we obtain the following projective resolution $C_\bullet$ of A with period 4:*

$$\cdots \xrightarrow{d_3[2h]} C_2[2h] \xrightarrow{d_2[2h]} C_1[2h] \xrightarrow{d_1[2h]} C_0[2h] \xrightarrow{d_4} C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} A \to 0.$$

Proof. Let us first show that these C_i, d_i define a complex. We show that $d_i d_{i+1} = 0$ for $i \leq 3$ and $d_4 d_1[2h] = 0$:

$$d_0 d_1(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = d_0(b_1\alpha \otimes b_2 - b_1 \otimes \alpha b_2 + b_3 z \otimes b_4 - b_3 \otimes z b_4) = 0,$$

$$\begin{aligned}
&d_1 d_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = \\
&\quad = d_1(-b_1 z \otimes \alpha \otimes b_2 + b_1 \otimes \alpha \otimes z b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_3 a \otimes a^* \otimes b_4 \\
&\qquad + \sum_{a \in \bar{Q}} \epsilon_a b_3 \otimes a \otimes a^* b_4, -b_3\mu \otimes b_4 + b_1\alpha \otimes b_2 - b_1 \otimes \alpha b_2) \\
&\quad = -b_1 z\alpha \otimes b_2 + b_1 z \otimes \alpha b_2 + b_1 \alpha \otimes z b_2 - b_1 \otimes \alpha z b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_3 a a^* \otimes b_4 \\
&\qquad - \sum_{a \in \bar{Q}} \epsilon_a b_3 a \otimes a^* b_4 + \sum_{a \in \bar{Q}} \epsilon_a b_3 a \otimes a^* b_4 - \sum_{a \in \bar{Q}} \epsilon_a b_3 \otimes a a^* b_4 - b_3 z\mu \otimes b_4 \\
&\qquad + b_3 \mu \otimes z b_4 + b_1 \alpha z \otimes b_2 - b_1 \alpha \otimes z b_2 - b_1 z \otimes \alpha b_2 + b_1 \otimes z\alpha b_2 = 0
\end{aligned}$$

(since $\sum_{a\in\bar{Q}} \epsilon_a aa^* = z\mu$),

$$d_2d_3(b_1 \otimes b_2) =$$

$$= d_2(\sum_{a\in\bar{Q}} \epsilon_a b_1 a \otimes a^* \otimes b_2 + \sum_{a\in\bar{Q}} \epsilon_a b_1 \otimes a \otimes a^* b_2, b_1 z \otimes b_2 - b_1 \otimes z b_2) =$$

$$= (-\sum_{a\in\bar{Q}} \epsilon_a b_1 a z \otimes a^* \otimes b_2 + \sum_{a\in\bar{Q}} \epsilon_a b_1 a \otimes a^* \otimes z b_2 - \sum_{a\in\bar{Q}} \epsilon_a b_1 z \otimes a \otimes a^* b_2$$

$$+ \sum_{a\in\bar{Q}} \epsilon_a b_1 \otimes a \otimes z a^* b_2 + \sum_{a\in\bar{Q}} \epsilon_a b_1 z a \otimes a^* \otimes b_2 + \sum_{a\in\bar{Q}} \epsilon_a b_1 z \otimes a \otimes a^* b_2$$

$$- \sum_{a\in\bar{Q}} \epsilon_a b_1 a \otimes a^* \otimes z b_2 - \sum_{a\in\bar{Q}} \epsilon_a b_1 \otimes a \otimes a^* z b_2,$$

$$-b_1 z\mu \otimes b_2 + b_1 \otimes z\mu b_2 + \sum_{a\in\bar{Q}} \epsilon_a b_1 a a^* \otimes b_2 - \sum_{a\in\bar{Q}} \epsilon_a b_1 a \otimes a^* b_2$$

$$+ \sum_{a\in\bar{Q}} \epsilon_a b_1 a \otimes a^* b_2 - \sum_{a\in\bar{Q}} \epsilon_a b_1 \otimes a a^* b_2) = 0,$$

$$d_3d_4(b_1 \otimes b_2) = d_3(\sum b_1 x_i \otimes x_i^* b_2) =$$

$$= (\sum_{a\in\bar{Q}} \sum \epsilon_a b_1 x_i a \otimes a^* \otimes x_i^* b_2 + \sum_{a\in\bar{Q}} \sum \epsilon_a b_1 x_i \otimes a \otimes a^* x_i^* b_2,$$

$$\sum b_1 x_i z \otimes x_i^* b_2 - \sum b_1 x_i \otimes z x_i^* b_2).$$

Using the trace form, it is easy to show that $\sum x_i a \otimes x_i^* = \sum x_i \otimes a x_i^*$ for any $a \in A$:

$$\sum x_i a \otimes x_i^* = \sum\sum (x_i a, x_j^*) x_j \otimes x_i^* = \sum\sum x_i \otimes (a x_i^*, x_j) x_j^* = \sum x_i \otimes a x_i^*.$$

It follows that $\sum b_1 x_i z \otimes x_i^* b_2 - \sum b_1 x_i \otimes z x_i^* b_2 = 0$.

Similarly, $\sum x_i a \otimes b \otimes x_i^* = \sum x_i \otimes b \otimes a x_i^*$ for any $a \in A$. Therefore

$$\sum \epsilon_a b_1 x_i a \otimes a^* \otimes x_i^* b_2 = \sum \epsilon_a b_1 x_i \otimes a^* \otimes a x_i^* b_2 = \sum \underbrace{\epsilon_{a^*}}_{=-\epsilon_a} b_1 x_i \otimes a \otimes a^* x_i^* b_2,$$

so $d_3 d_4 = 0$.

$$d_4 d_1[2h](b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) =$$
$$= d_0(b_1\alpha \otimes b_2 - b_1 \otimes \alpha b_2 + b_3 z \otimes b_4 - b_3 \otimes z b_4) \sum x_i \otimes x_i^* = 0.$$

Now we show exactness. Since the complex is periodic, it is enough to show exactness for C_0, C_1, C_2 and C_3.

We recall the definition of Anick's resolution [1]. Denote $T_R W$ to be the tensor algebra of a graded R-bimodule W, $T_R^+ W$ its augmentation ideal. Let $L \subset T_R^+ W$ be an R-graded bimodule and $B = T_R W/(L)$. Then we the following resolution:

$$B \otimes_R L \otimes_R B \xrightarrow{\partial} B \otimes_R W \otimes_R B \xrightarrow{f} B \otimes_R B \xrightarrow{m} B \to 0, \qquad (4.1.1.2)$$

where m is the multiplication map, f is given by

$$f(b_1 \otimes w \otimes b_2) = b_1 w \otimes b_2 - b_1 \otimes w b_2$$

and ∂ is given by

$$\partial(b_1 \otimes l \otimes b_2) = b_1 \cdot D(l) \cdot b_2,$$

$$\begin{aligned} D : T_R^+ W &\to B \otimes_R W \otimes_R B, \\ w_1 \otimes \ldots \otimes w_n &\mapsto \sum_{p=1}^{n} (\overline{w_1 \otimes \ldots \otimes w_{p-1}}) \otimes w_p \otimes (\overline{w_{p+1} \otimes \ldots \otimes w_n}), \end{aligned}$$

where bar stands for the image in B of the projection map.

In our setting, $W = V \oplus Rz$, L the R-bimodule generated by $\sum_{a \in \bar{Q}} \epsilon_a a a^* - \mu z$ and $\alpha z - z\alpha \, \forall \alpha \in \bar{Q}$. Then $B = A$.
In Anick's resolution, $m = d_0$, $A \otimes_R W \otimes_R A$ can be identified with C_1 (via $A \otimes_R A[2] = A \otimes_R Rz \otimes_R A$), so that f becomes d_1. Then Im(∂)=Im(d_2) $\subset C_1$. This implies exactness in C_0 and C_1.

For exactness in 2^{nd} and 3^{rd} term, we show that the complex

$$C_4 = C_0[2h] \xrightarrow{d_4} C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} A = C_{-1} \to 0$$

is selfdual:

By replacing $C_4 = C_0[2h]$ by $\bar{C}_4 = \mathrm{Im}(d_4)$, we get the complex

$$0 \to \bar{C}_4 \xrightarrow{\bar{d}_4} \bar{C}_3 \xrightarrow{\bar{d}_3} \bar{C}_2 \xrightarrow{\bar{d}_2} \bar{C}_1 \xrightarrow{\bar{d}_1} \bar{C}_0 \xrightarrow{\bar{d}_0} A \to 0.$$

Now, the map $\sum b_1 x_i \otimes x_i^* b_2 \mapsto b_1 b_2$ allows us to identify $Im(d_4) \cong A[2h]$ as $A-$bimodules so $\bar{d}_4$ becomes multiplication with $\sum x_i \otimes x_i^*$.

We introduce the following nondegenerate, bilinear forms:
On $A \otimes_R A$, let

$$(x \otimes y, a \otimes b) = Tr(xb)Tr(ya),$$

and on $A \otimes_R V \otimes_R A$, we define

$$(x \otimes \alpha \otimes y, a \otimes \beta \otimes b) = Tr(xb)Tr(ya)(\alpha, \beta),$$

where we define the form on V by

$$(\alpha, \beta) = \epsilon_\beta \delta_{\alpha^* \beta}$$

$(\alpha, \beta \in \bar{Q}$ and $\delta_{ab} = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$).

Via the trace form $(x, y) = Tr(xy)$, we can identify $A \cong A^*$, $x \mapsto (x, -)$, and similarly we can use the forms from above to identify $A \otimes_R A \cong (A \otimes_R A)^*$ and $A \otimes_R V \otimes_R A \cong (A \otimes_R V \otimes_R A)^*$, which induces an identification $\bar{C}_i = \bar{C}_{3-i}^*$.

We claim the following: $\bar{d}_0^* = \bar{d}_4$, $\bar{d}_1^* = -\bar{d}_3$ and $\bar{d}_2^* \iota = \bar{d}_2$, where $\iota(x,y) = (-x,y)$:

$$\begin{aligned}(\bar{d}_4(x),(b_1\otimes b_2)) &= (\sum xx_i\otimes x_i^*, b_1\otimes b_2) = \sum Tr(xx_ib_2)Tr(x_i^*b_1)\\ &= \sum (b_2x,x_i)(x_i^*,b_1) = (b_2x,b_1) = (x,b_1b_2)\\ &= (x,\bar{d}_0(b_1\otimes b_2)).\end{aligned}$$

For $\alpha,\beta\in\bar{Q}$,

$$\begin{aligned}&(-\bar{d}_3(x\otimes y),(b_1\otimes\alpha\otimes b_2, b_3\otimes b_4)) =\\ &\quad= ((-\sum_{a\in\bar{Q}}\epsilon_a xa\otimes a^*\otimes y - \sum_{a\in\bar{Q}}\epsilon_a x\otimes a\otimes a^*y, -xz\otimes y + x\otimes zy),\\ &\qquad (b_1\otimes\alpha\otimes b_2, b_3\otimes b_4))\\ &\quad= -Tr(x\alpha b_2)Tr(yb_1) + Tr(xb_2)Tr(\alpha yb_1) - Tr(xzb_4)Tr(yb_3)\\ &\qquad +Tr(xb_4)Tr(zyb_3)\\ &\quad= Tr(xb_2)Tr(yb_1\alpha) - Tr(x\alpha b_2)Tr(yb_1) + Tr(xb_4)Tr(yb_3z)\\ &\qquad -Tr(xzb_4)Tr(yb_3)\\ &\quad= (x\otimes y, b_1\alpha\otimes b_2 - b_1\otimes\alpha b_2 + b_3z\otimes b_4 - b_3\otimes zb_4)\\ &\quad= (x\otimes y, d_1(b_1\otimes\alpha\otimes b_2, b_3\otimes b_4)),\end{aligned}$$

$$\begin{aligned}&(\bar{d}_2(b_1\otimes\alpha\otimes b_2, b_3\otimes b_4),(c_1\otimes\beta\otimes c_2, c_3\otimes c_4)) =\\ &\quad= ((-b_1z\otimes\alpha\otimes b_2 + b_1\otimes\alpha\otimes zb_2 + \sum_{a\in\bar{Q}}\epsilon_a b_3a\otimes a^*\otimes b_4\\ &\qquad +\sum_{a\in\bar{Q}}\epsilon_a b_3\otimes a\otimes a^*b_4, -b_3\mu\otimes b_4 + b_1\alpha\otimes b_2 - b_1\otimes\alpha b_2),\\ &\qquad (c_1\otimes\beta\otimes c_2,\quad c_3\otimes c_4))\\ &\quad= -Tr(b_1zc_2)Tr(b_2c_1)(\alpha,\beta) + Tr(b_1c_2)Tr(zb_2c_1)(\alpha,\beta)\\ &\qquad +Tr(b_3\beta c_2)Tr(b_4c_1) - Tr(b_3c_2)Tr(\beta b_4c_1)\\ &\qquad -Tr(b_3\mu c_4)Tr(b_4c_3) + Tr(b_1\alpha c_4)Tr(b_2c_3) - Tr(b_1c_4)Tr(\alpha b_2c_3)\end{aligned}$$

$$= Tr(b_1c_2)Tr(b_2c_1z)(\alpha,\beta) - Tr(b_1zc_2)Tr(b_2c_1)(\alpha,\beta)$$
$$-Tr(b_1c_4)Tr(b_2c_3\alpha) + Tr(b_1\alpha c_4)Tr(b_2c_3)$$
$$-Tr(b_3\mu c_4)Tr(b_4c_3) - Tr(b_3c_2)Tr(b_4c_1\beta) + Tr(b_3\beta c_2)Tr(b_4c_1)$$
$$= ((b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4), (c_1z \otimes \beta \otimes c_2 - c_1 \otimes \beta \otimes zc_2$$
$$+\sum_{a\in\bar{Q}} \epsilon_a c_3 a \otimes a^* \otimes c_4 + \sum_{a\in\bar{Q}} \epsilon_a c_3 \otimes a \otimes a^* c_4,$$
$$-c_3 \otimes \mu c_4 - c_1\beta \otimes c_2 + c_1 \otimes \beta c_2))$$
$$= ((b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4), \bar{d}_2(-c_1 \otimes \beta \otimes c_2, c_3 \otimes c_4)).$$

Now, the selfduality of our complex $\bar{C}_\bullet$ and exactness in $\bar{C}_0$ and $\bar{C}_1$ implies exactness in $\bar{C}_2$ and $\bar{C}_3$. □

4.1.2 Computation of Hochschild cohomology/homology

Now we use the projective resolution $C_\bullet$ to compute the Hochschild cohomology and homology groups of A. Let us write $A^e = A \otimes_R A^{op}$.

Theorem 4.1.2.1. *The Hochschild cohomology groups of A are:*

$$\begin{aligned}
HH^0(A) &= Z \text{ (the center of } A), \\
HH^{4n+1}(A) &= (Z \cap \mu^{-1}[A,A])[-2nh-2], \\
HH^{4n+2}(A) &= A/([A,A] + \mu Z)[-2nh-2], \\
HH^{4n+3}(A) &= A_+/[A,A][-2nh-4], \\
HH^{4n+4}(A) &= Z/A_{top}[-2(n+1)h]
\end{aligned}$$

where $n \geq 0$, and A_{top} is the top-degree part of A.

Proof. Apply the functor $Hom_{A^e}(-,A)$ on $C_\bullet$, identify

$$Hom_{A^e}(A \otimes_R A, A) \cong A^R$$

($\phi \in Hom_{A^e}(A \otimes_R A, A)$ is determined by $\phi(1 \otimes 1) = a \in A$ and observe $ra =$

$\phi(r \otimes 1) = \phi(1 \otimes r) = ar$, $\forall r \in R$. We write $a \circ -$ for ϕ) and

$$Hom_{A^e}(A \otimes_R V \otimes_R A, A) \cong (A \otimes_R V)^R[-2]$$

($\sum_{a \in \bar{Q}} x_a \otimes a^*$ is identified with the homomorphism ψ which maps each element $1 \otimes a \otimes 1$ to x_a $(a \in \bar{Q})$, we write $\sum_{a \in \bar{Q}} x_a \otimes a^*$ for $\psi(-)$)
to obtain the Hochschild cohomology complex

$$\cdots \leftarrow A^R[-2h] \overset{d_4^*}{\leftarrow} A^R[-4] \overset{d_3^*}{\leftarrow} \begin{matrix} (A \otimes_R V)^R[-4] \\ \oplus \\ A^R[-2] \end{matrix} \overset{d_2^*}{\leftarrow} \begin{matrix} (A \otimes_R V)^R[-2] \\ \oplus \\ A^R[-2] \end{matrix} \overset{d_1^*}{\leftarrow} A^R \leftarrow 0.$$

$$d_1^*(x)(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = x \circ d_1(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) =$$
$$= b_1 \alpha x b_2 - b_1 x \alpha b_2 + b_3 z x b_4 - b_3 x z b_4 = b_1[\alpha, x]b_2,$$

so

$$d_1^*(x) = (\sum_{a \in \bar{Q}} [a, x] \otimes a^*, 0).$$

Let $\alpha = \sum_{a \in \bar{Q}} r_a a$, $r_a \in R$.

$$d_2^*(\sum_{a \in \bar{Q}} x_a \otimes a^*, 0)(b_1 \alpha \otimes b_2, b_3 \otimes b_4) = (\sum_{a \in \bar{Q}} x_a \otimes a^*) \circ d_2(b_1 \alpha \otimes b_2, b_3 \otimes b_4) =$$
$$= \sum_{a \in \bar{Q}} (x_a \otimes a^*) \circ (-b_1 z \otimes \alpha \otimes b_2 + b_1 \otimes \alpha \otimes z b_2 + \sum_{\beta \in \bar{Q}} \epsilon_\beta b_3 \beta \otimes \beta^* \otimes b_4$$
$$+ \sum_{\beta \in \bar{Q}} \epsilon_\beta b_3 \otimes \beta \otimes \beta^* b_4) =$$
$$= \sum_{a \in \bar{Q}} (-b_1 z r_a x_a b_2 + b_1 r_a x_a z b_2) - \sum_{a \in \bar{Q}} \epsilon_a b_3 a^* x_a b_4 + \sum_{a \in \bar{Q}} \epsilon_a b_3 x_a a^* b_4$$
$$= \sum_{a \in \bar{Q}} \epsilon_a b_3 [x_a, a^*] b_4,$$

so

$$d_2^*(\sum_{a\in\bar{Q}} x_a \otimes a^*, 0) = (0, \sum_{a\in\bar{Q}} \epsilon_a [x_a, a^*]).$$

$$d_2^*(0,y)(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = y \circ d_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) =$$
$$= y \circ (-b_3\mu \otimes b_4 + b_1\alpha \otimes b_2 - b_1 \otimes \alpha b_2) = -b_3\mu y b_4 + b_1 \alpha y b_2 - b_1 y \alpha b_2,$$

so

$$d_2^*(0,y) = (-\sum_{a\in\bar{Q}} [y,a] \otimes a^*, -\mu y).$$

Putting this together, we obtain:

$$d_2^*(\sum_{a\in\bar{Q}} x_a \otimes a^*, y) = (-\sum_{a\in\bar{Q}} [y,a] \otimes a^*, -\mu y + \sum_{a\in\bar{Q}} \epsilon_a [x_a, a^*]).$$

$$d_3^*(\sum_{a\in\bar{Q}} x_a \otimes a^*, 0)(b_1 \otimes b_2) = (\sum_{a\in\bar{Q}} x_a \otimes a^*) \circ d_3(b_1 \otimes b_2) =$$
$$= (\sum_{a\in\bar{Q}} x_a \otimes a^*) \circ (\sum_{\alpha\in\bar{Q}} \epsilon_\alpha b_1 \alpha \otimes \alpha^* \otimes b_2 + \sum_{\alpha\in\bar{Q}} \epsilon_\alpha b_1 \otimes \alpha \otimes \alpha^* b_2,$$
$$b_1 z \otimes b_2 - b_1 \otimes z b_2) =$$
$$= \sum_{a\in\bar{Q}} (-\epsilon_a b_1 a^* x_a b_2 + \epsilon_a b_1 x_a a^* b_2) = \sum_{a\in\bar{Q}} \epsilon_a b_1 [x_a, a^*] b_2,$$
$$d_3^*(0,y)(b_1 \otimes b_2) =$$
$$= y \circ (\sum_{\alpha\in\bar{Q}} \epsilon_\alpha b_1 \alpha \otimes \alpha^* \otimes b_2 + \sum_{\alpha\in\bar{Q}} \epsilon_\alpha b_1 \otimes \alpha \otimes \alpha^* b_2, b_1 z \otimes b_2 - b_1 \otimes z b_2)$$
$$= b_1 z y b_2 - b_1 y z b_2 = 0,$$

so we get

$$d_3^*(\sum_{a\in\bar{Q}} x_a \otimes a^*, y) = \sum_{a\in\bar{Q}} \epsilon_a [x_a, a^*].$$

$$d_4^*(x)(b_1 \otimes b_2) = x \circ (\sum b_1 x_i \otimes x_i^* b_2) = \sum b_1 x_i x x_i^* b_2,$$

so

$$d_4^*(x) = \sum x_i x x_i^*.$$

Now, we want to compute the Hochschild cohomology (since the complex is periodic, $HH^i(A) = HH^{i+4}(A)[2h]\,\forall i \geq 1$, so it is enough to do the calculations until HH^4):

$\underline{HH^0(A) = Z}$ (the center of A), since a cocycle $x \in \ker d_1^*$ lies in A^R and has to satisfy $\sum_{a\in\bar{Q}} [a,x] \otimes a^* = 0$, i.e. commute with all $a \in \bar{Q}$.

$\underline{HH^1(A) = (Z \cap \mu^{-1}[A,A])[-2]:}$ $(\sum_{a\in\bar{Q}} x_a \otimes a^*, y)$ is a cocycle if $\sum_{a\in\bar{Q}} [y,a] \otimes a^* = 0$ (i.e. $y \in Z$) and $y = \mu^{-1} \sum_{a\in\bar{Q}} \epsilon_a [x_a, a^*]$ (since μ is invertible) which implies $y \in \mu^{-1}[A,A]$. Since $\sum_{a\in\bar{Q}} \epsilon_a [a^*, x_a] = 0$ implies that $x_a = [a,x]$ (we refer to [9, Corollary 3.5.] where this statement follows from the exactness of the complex in the 1^{st} term) for some $x \in A$, and $\sum_{a\in\bar{Q}} [a,x] \otimes a^*$ lies in $\mathrm{Im} d_1^*$, $HH^1(A)$ is controlled only by $y \in (Z \cap \mu^{-1}[A,A])[-2]$. Since $[A(1),A] = [A,A]$, any $y \in (Z \cap \mu^{-1}[A,A])[-2]$ also gives rise to a cocycle.

$\underline{HH^2(A) = A/([A,A] + \mu Z)[-2]:}$ An element $(\sum_{a\in\bar{Q}} x_a \otimes a^*, y)$ is a cocycle if $\sum_{a\in\bar{Q}} \epsilon_a [x_a, a^*] = 0$, so $x_a = [x,a]$ for some $x \in A^R$, (where x is unique up to a central element), so cocycles are of the form $(\sum_{a\in\bar{Q}} [x,a] \otimes a^*, y)$. The coboundaries are spanned by $(\sum_{a\in\bar{Q}} [x,a] \otimes a^*, \mu x)$ (where the first component determines x uniquely modulo Z) and $(0, \sum_{a\in\bar{Q}} \epsilon_a [x_a, a^*])$ (where the image is $[A,A]^R$). It follows that

$$HH^2(A) = A^R/([A,A]^R + \mu Z)[-2] = A/([A,A] + \mu Z)[-2].$$

$\underline{HH^3(A) = A_+/[A,A][-4]:}$ We denote A_+ to be the positive degree part of A. $d_4^*(x) = \sum x_i x x_i^*$ is zero if x has positive degree (since $x_i x x_i^*$ exceeds the top degree $2h-4$).

Observe also that d_4^* injects R into A_{top}:

Since $A = \oplus e_k A e_j$, we can choose a basis $\{x_i\}$, such that these elements all belong to a certain subspace $e_k A e_j$ for some k, j. We denote $\{x_{i'}^{j,k}\}$ the subbasis of $\{x_i\}$ which spans e_kAe_j.

Assume that $0 = d_4^*(\sum_{j=1}^{r} \lambda_j e_j)$. Then $\forall k$,

$$\begin{aligned} 0 &= \sum_{j=1}^{r} \lambda_j Tr(\sum_i e_k x_i e_j x_i^*) = \sum_{j=1}^{r} \lambda_j \sum_{i',j,k} \underbrace{(x_{i'}^{j,k}, (x_{i'}^{j,k})^*)}_{=1} = \sum_{j=1}^{r} \lambda_j \dim e_k A e_j \\ &= \sum_j \lambda_j \sum_d \dim e_k A[d] e_j = H_A(1)_{k,j} = \sum_j \lambda_j \left(\frac{h}{2-C}\right)_{k,j}. \end{aligned}$$

The last equality follows from [10, Theorem 3.2.]. Since the matrix $\frac{h}{2-C}$ is nondegenerate, all $\lambda_j = 0$.

So we see that the images $d_4^*(e_j)$ are nonzero and linearly independent. So the cocycles are the elements in A_+^R, and the coboundaries are $\sum_{a\in\bar{Q}} \epsilon_a [x_a, a^*]$ which generate $[A,A]^R$. Therefore $HH^3(A) = A_+^R/[A,A]^R[-4] = A_+/[A,A][-4]$.

$\underline{HH^4(A) = Z/A_{top}[-2h]}$: Since $d_5^* = d_1^*$, the cocycles are the central elements. From the above discussion about the image of d_4^* and the fact that A_{top} is r-dimensional, it follows that the coboundaries are the top degree elements of A. □

Similarly, we compute the Hochschild homology groups of A.

Theorem 4.1.2.2. *The Hochschild homology groups of A are:*

$$\begin{aligned} HH_0(A) &= A/[A,A], \\ HH_{4n+1}(A) &= A/([A,A]+\mu Z)[2nh+2], \\ HH_{4n+2}(A) &= (Z \cap \mu^{-1}[A,A])[2nh+2], \\ HH_{4n+3}(A) &= Z/A_{top}[2nh+4], \\ HH_{4n+4}(A) &= A_+/[A,A][2(n+1)h]. \end{aligned}$$

Proof. Apply the functor $(A \otimes_{A^e} -)$ to $C_\bullet$, identify

$$A \otimes_{A^e} (A \otimes_R A) \cong A^R$$

$(a \otimes (b \otimes c) = cab \otimes 1 \otimes 1 \mapsto cab$ and observe

$\forall a \in A, r \in R : ar = a \otimes (r \otimes 1) = a \otimes (1 \otimes r) = ra)$ and

$$A \otimes_{A^e} (A \otimes_R V \otimes_R A) \cong (A \otimes_R V)^R$$

(via $a \otimes (b \otimes \alpha \otimes c) = cab \otimes (1 \otimes \alpha \otimes 1) \mapsto cab \otimes \alpha$).

We get the following periodic complex for computing the Hochschild homology:

$$\cdots \to A^R[2h] \xrightarrow{d'_4} A^R[4] \xrightarrow{d'_3} \begin{matrix} (A \otimes_R V)^R[2] \\ \oplus \\ A^R[2] \end{matrix} \xrightarrow{d'_2} \begin{matrix} (A \otimes_R V)^R \\ \oplus \\ A^R[2] \end{matrix} \xrightarrow{d'_1} A^R \to 0.$$

The differentials become:

$$\begin{aligned} d'_1(\sum_{a \in \bar{Q}} x_a \otimes a, y) &= 1 \otimes d_1(\sum_{a \in \bar{Q}} x_a \otimes a \otimes 1, y \otimes 1) \\ &= 1 \otimes (\sum_{a \in \bar{Q}} x_a a \otimes 1 - \sum_{a \in \bar{Q}} x_a \otimes a + yz \otimes 1 - y \otimes z) = \sum_{a \in \bar{Q}} [x_a, a], \end{aligned}$$

$$\begin{aligned} d'_2(\sum_{a \in \bar{Q}} x_a \otimes a, y) &= 1 \otimes d_2(\sum_{a \in \bar{Q}} x_a \otimes a \otimes 1, y \otimes 1) = \\ &= 1 \otimes (-\sum_{a \in \bar{Q}} x_a z \otimes a \otimes 1 + \sum_{a \in \bar{Q}} x_a \otimes a \otimes z + \sum_{\alpha \in \bar{Q}} \epsilon_\alpha y\alpha \otimes \alpha^* \otimes 1 \\ &\quad + \sum_{\alpha \in \bar{Q}} \epsilon_\alpha y \otimes \alpha \otimes \alpha^*, -y\mu \otimes 1 + \sum_{a \in \bar{Q}} x_a a \otimes 1 - \sum_{a \in \bar{Q}} x_a \otimes a) \\ &= (-\sum_{a \in \bar{Q}} x_a z \otimes a + \sum_{a \in \bar{Q}} z x_a \otimes a + \sum_{\alpha \in \bar{Q}} \epsilon_\alpha y\alpha \otimes \alpha^* + \sum_{\alpha \in \bar{Q}} \epsilon_\alpha \alpha^* y \otimes \alpha, \\ &\quad -y\mu + \sum_{a \in \bar{Q}} x_a a - \sum_{a \in \bar{Q}} a x_a) = (\sum_{\alpha \in \bar{Q}} \epsilon_\alpha [y, \alpha] \otimes \alpha^*, \sum_{a \in \bar{Q}} [x_a, a] - y\mu), \end{aligned}$$

$$\begin{aligned} d'_3(x) &= 1 \otimes d_3(x \otimes 1) = \\ &= (1 \otimes (\sum_{a \in \bar{Q}} \epsilon_a x a \otimes a^* \otimes 1 + \sum_{a \in \bar{Q}} \epsilon_a x \otimes a \otimes a^*), 1 \otimes (xz \otimes 1 - x \otimes z)) \\ &= (\sum_{a \in \bar{Q}} \epsilon_a x a \otimes a^* + \sum_{a \in \bar{Q}} \epsilon_a a^* x \otimes a, xz - zx) = (\sum_{a \in \bar{Q}} \epsilon_a [x, a] \otimes a^*, 0), \end{aligned}$$

$$d_4'(x) = 1 \otimes d_4(x \otimes 1) = 1 \otimes \sum x x_i \otimes x_i^* = \sum x_i^* x x_i.$$

Now, we compute the homology (and since the complex is periodic, $HH_i(A) = HH_{i+4}(A)$ for $i > 0$, so it is enough to calculate the homology up to HH_4):

$HH_0(A) = A/[A, A]$: the boundaries are of the form $\sum_{a \in \bar{Q}} [x_a, a]$, and they generate $[A, A]^R$. So $HH_0(A) = A^R/[A, A]^R = A/[A, A]$ follows.

$HH_1(A) = A/([A, A] + \mu Z)[2]$: The cycle condition $\sum_{a \in \bar{Q}} [x_a, a] = 0$ implies $x_a = \epsilon_a [x, a^*]$ for some $x \in A$ (again, we refer to the result $H_1 = 0$ in [9, Corollary 3.5.]), so the cycles are $(\sum_{a \in \bar{Q}} \epsilon_a [x, a^*] \otimes a, y)$.
The boundaries are of the form $(\sum_{a \in \bar{Q}} \epsilon_a [x, a^*] \otimes a, \sum_{\alpha \in \bar{Q}} [x_\alpha, \alpha] + \mu x)$ (where the first component determines x uniquely modulo Z. So

$$HH_1(A) = A^R/([A, A]^R + \mu Z)[2] = A/([A, A] + \mu Z)[2].$$

$HH_2(A) = Z \cap \mu^{-1}[A, A][2]$: The cycle conditions are $\sum_{\alpha \in \bar{Q}} \epsilon_\alpha [y, \alpha] \otimes \alpha^* = 0$ (this tells us $y \in Z$) and $\sum_{a \in \bar{Q}} [x_a, a] - y\mu = 0$, so $y \in \mu^{-1}[A, A]$ and x_a unique up to an element of the form $\epsilon_a [x, a^*]$ for some $x \in A$. So the cycles are of the form $(\sum_{a \in \bar{Q}} x_a \otimes a, y)$, $y \in Z \cap \mu^{-1}[A, A]$, x_a uniquely controlled by y (mod $\epsilon_a [x, a^*]$) , and the boundaries have the form $(\sum_{a \in \bar{Q}} \epsilon_a [x, a^*] \otimes a, 0)$, i.e. homology is controlled only by y now. So $HH_2(A) = Z \cap \mu^{-1}[A, A][2]$.

$HH_3(A) = Z/A_{top}[4]$: The cycle condition $\sum_{a \in \bar{Q}} \epsilon_a [x, a] \otimes a^* = 0$ implies that the cycles are the central elements Z. The boundaries $\sum x_i^* x x_i$ consist of the top degree part of A, so $HH_3(A) = Z/A_{top}[4]$.

$HH_4(A) = A_+/[A, A][2h]$: $\ker d_4 = A_+^R$, $\operatorname{Im} d_5 = \operatorname{Im} d_1 = [A, A]^R$, therefore $HH_4(A) = A_+^R/[A, A]^R = A_+/[A, A]$. □

4.1.3 The intersection $Z \cap \mu^{-1}[A, A]$.

We found $Z \cap \mu^{-1}[A, A]$ as the $(4i+2)-$th homology and $(4i+1)-$th cohomology group, so to understand the (co)homology of A better, we are interested in its structure.

Now, we define the following Hilbert series:

$$q(t) = h_{Z\cap\mu^{-1}[A,A]}(t),$$
$$q_*(t) = h_{A/([A,A]+\mu Z)}(t).$$

To relate both to each other, we prove the following

Proposition 4.1.3.1. *The trace form defines a nondegenerate pairing* $(Z\cap\mu^{-1}[A,A]) \times A/([A,A]+\mu Z) \to k$.

Proof. Since the trace form is nondegenerate on A, it is enough to show that $(Z\cap \mu^{-1}[A,A])^{\perp} \subset [A,A]+\mu Z$, or equivalently $([A,A]+\mu Z)^{\perp} \subset Z\cap\mu^{-1}[A,A]$. The latter follows from $[A,A]^{\perp}\subset Z$, since

$$(x,[y_1,y_2]) = Tr(x[y_1,y_2]) = Tr([x,y_1]y_2) = ([x,y_1],y_2) = 0\ \forall y_1,y_2\in A$$

implies $[x,y_1]=0$, i.e. $x\in Z$. □

Corollary 4.1.3.2. *$q(t)$ and $q_*(t)$ are palindromes of each other, i.e.* $q(t) = t^{2h-4}q_*(1/t)$.

Let us define the Hilbert series $p(t) = h_{A/\mu^{-1}[A,A]}(t)$. We recall from [9, end of section 2.2.] that $p(t) = \sum_{i=1}^{r}(1+t^2+\ldots+t^{2(m_i-1)})$ where the m_i are the exponents of the root system. Since the trace form also defines a nondegenerate pairing $Z\times A/[A,A]\to k$ (see [9, Corollary 2.2.]), it follows for the Hilbert series $p_*(t) = h_Z(t)$ that $p(t) = t^{2h-4}p_*(1/t)$. Since $zZ\subset\mu^{-1}[A,A]$ is spanned by even degree elements, we see that Z is generated as a $k[z]$−module by elements of degree $2(m_i-1)$.

Proposition 4.1.3.3. *We have*

$$q_*(t) \geq p(t) - \sum_{i=1}^{r} t^{2(m_i-1)} = \sum_{i=1}^{r}\left(1+t^2+\ldots+t^{2(m_i-2)}\right).$$

Proof. From the exact sequence

$$0\to Z/(Z\cap\mu^{-1}[A,A]) \to A/\mu^{-1}[A,A] \to A/(\mu^{-1}[A,A]+Z)\to 0$$

we obtain the equation

$$q_*(t) = p(t) - h_{Z/(Z\cap\mu^{-1}[A,A])}(t).$$

Since $zZ \subset \mu^{-1}[A,A]$ $(z = \mu^{-1}\sum_{a\in\bar{Q}}[a,a^*] \in \mu^{-1}[A,A])$, we have the inequality

$$h_{Z/(Z\cap\mu^{-1}[A,A])}(t) \leq h_{Z/zZ}(t) = \sum_{i=1}^{r} t^{2(m_i-1)},$$

and our inequality

$$q_*(t) \geq p(t) - \sum_{i=1}^{r} t^{2(m_i-1)}$$

follows. □

Theorem 4.1.3.4. *The inequality from above is an equality:*

$$q_*(t) = p(t) - \sum_{i=1}^{r} t^{2(m_i-1)}.$$

We will prove this in the next section where we compute the cyclic homology groups of A. From this, we get a result for our intersection space:

Corollary 4.1.3.5. $Z \cap \mu^{-1}[A,A] = zZ$.

4.1.4 Cyclic homology of A

The Connes differentials B_i (see [19, 2.1.7.]) give us an exact sequence

$$R \xrightarrow{B_{-1}} HH_0(A) \xrightarrow{B_0} HH_1(A) \xrightarrow{B_1} HH_2(A) \xrightarrow{B_2} HH_3(A) \xrightarrow{B_3} HH_4(A) \xrightarrow{B_4} \ldots.$$

In our case, we have the following exact sequence:

$$R \xrightarrow{B_{-1}} A/[A,A] \xrightarrow{B_0} A/([A,A]+\mu Z)[2] \xrightarrow{B_1} Z\cap\mu^{-1}[A,A][2] \xrightarrow{B_2} Z/A_{top}[4]$$
$$\xrightarrow{B_3} A_+/[A,A][2h] \xrightarrow{B_4} \ldots,$$

and the B_i are all degree-preserving.

We define the *reduced cyclic homology* (see [19, 2.2.13.])

$$\overline{HC_i}(A) = \ker(B_{i+1} : HH_{i+1}(A) \to HH_{i+2}(A))$$
$$= \mathrm{Im}(B_i : HH_i(A) \to HH_{i+1}(A)).$$

Theorem 4.1.4.1. *We get the following cyclic homology groups:*

$$\begin{aligned} \overline{HC_{4n}}(A) &= A_+/[A,A][2nh], \\ \overline{HC_{4n+1}}(A) &= 0, \\ \overline{HC_{4n+2}}(A) &= Z/A_{top}[2nh+4], \\ \overline{HC_{4n+3}}(A) &= 0. \end{aligned}$$

Proof. First we observe that $B_{4n+3} = 0$, since the elements of $Z/A_{top}[4]$ have degree $\leq (2h-6)+4 = 2h-2$ and the elements in $A_+/[A,A]\,[2h]$ have degree $\geq 2h+1$. So we have for each n the exact sequences

$$0 \to \frac{A_+}{[A,A][2nh]} \stackrel{B_{4n}}{\to} \frac{A}{[A,A]+\mu Z}[2nh+2] \stackrel{B_{4n+1}}{\to} (Z \cap \mu^{-1}[A,A])[2nh+2]$$
$$\stackrel{B_{4n+2}}{\to} \frac{Z}{A_{top}}[2nh+4] \to 0.$$

The only thing to show is that $W := \overline{HC_{4n+1}}(A) = \mathrm{Im}B_{4n+1} = 0$. We will use the following theorem from [8]:

Theorem 4.1.4.2. *Let* $\chi_{\overline{HC}(A)}(t) = \sum a_k t^k$*, the Euler characteristic of* $\overline{HC}(A)$*. Then*

$$\prod_{k=1}^{\infty}(1-t^k)^{-a_k} = \prod_{s=1}^{\infty} \det H_A(t^s) = \prod_{s=1}^{\infty}\left(\frac{1-t^{2hs}}{1-t^{2s}}\right)^r \frac{1}{\det(1-Ct^s+t^{2s})},$$

where C is the adjacency matrix of the quiver Q.

Since

$$\chi_{\overline{HC}(A)}(t) = \frac{1}{1-t^{2h}}(h_{A_+/[A,A]}(t) - h_W(t) + h_{Z/A_{top}}(t)t^4),$$

to show $W=0$, it is enough to show that if we set

$$\frac{1}{1-t^{2h}}(h_{A_+/[A,A]}(t)+h_{Z/A_{top}}(t)t^4)=\sum b_k t^k,$$

then

$$\prod_{k=1}^{\infty}(1-t^k)^{b_k}=\prod_{s=1}^{\infty}\left(\frac{1-t^{2s}}{1-t^{2hs}}\right)^r \det(1-Ct^s+t^{2s}).$$

We have

$$\begin{aligned} h_{A_+/[A,A]}(t) &= p(t)-r=\sum_{i=1}^{r}\frac{t^2-t^{2m_i}}{1-t^2} \text{ and} \\ h_{Z/A_{top}}(t)t^4 &= \sum_{i=1}^{r}\frac{t^{2(m_i-1)}-t^{2h-4}}{1-t^2}t^4=\sum_{i=1}^{r}\frac{t^{2m_i+2}-t^{2h}}{1-t^2}. \end{aligned}$$

From these, we get that

$$\sum_{k=1}^{\infty} b_k t^k=(1+t^{2h}+t^{4h}+\ldots)\sum_{i=1}^{r}(t^2+t^4+\ldots+t^{2m_i-2}+0+t^{2m_i+2}+\ldots+t^{2h-2}),$$

$$b_k=0 \text{ if } k \text{ is odd}$$

$$b_{2k}=\begin{cases} 0 & \text{if } k \text{ is divisible by } h \\ r-\#\{i:m_i=p\} & \text{if } k\equiv p \mod h \end{cases}.$$

$$\prod_{k=1}^{\infty}(1-t^k)^{b_k}=\prod_{n\not\equiv 0 \mod h}(1-t^{2n})^r/\prod_{\substack{n\geq 0\\ i\in I}}(1-t^{2(m_i+nh)})$$

Now, it comes down to showing that

$$\prod_{s=1}^{\infty}\det(1-Ct^s+t^{2s})=\prod_{k=1}^{\infty}(1-q^k)^{n_k},$$

where $q=t^2$ and $n_k=\begin{cases} 0 & \text{if } n \text{ is divisible by } h \\ -\#\{i:m_i=p\} & \text{if } n\equiv p \mod h \end{cases}$
(recall that the m_i are the exponents of our root system), for the different Dynkin

quivers of type A_{n-1}, D_{n+1}, E_6, E_7 and E_8. Here we will use the identities for $\det(1-Ct+t^2) = \prod_{j=1}^{r}(t^2 - e^{2\pi i m_j/h})$ from [21, Corollary 4.5.].

Case 1: $Q = A_{n-1}$

The exponents are $1, \ldots, n-1$ and the Coxeter number is $h = n$.

$$\det(1-Ct+t^2) = \frac{1-t^{2n}}{1-t^2},$$

so if we set

$$\prod_{k=1}^{\infty}(1-q^k)^{n_k} = \prod_{s=1}^{\infty}\det(1-Ct^s+t^{2s}) = \prod_{s=1}^{\infty}\frac{1-q^{ns}}{1-q^s}$$

then

$$n_k = \begin{cases} 0 & \text{if } n|k \\ -1 & \text{if } n \not| k \end{cases}.$$

Case 2: $Q = D_{n+1}$

The exponents are $1, 3, \ldots, 2n-1, n$ and the Coxeter number is $h = 2n$.

$$\det(1-Ct+t^2) = \frac{(1-t^4)(1-t^{4n})}{(1-t^2)(1-t^{2n})},$$

so

$$\prod_{k=1}^{\infty}(1-q^k)^{n_k} = \prod_{s=1}^{\infty}\det(1-Ct^s+t^{2s}) = \prod_{s=1}^{\infty}\frac{(1-q^{2s})(1-q^{2ns})}{(1-q^s)(1-q^{ns})}$$

implies that

$$n_k = \mathrm{div}(k,2n) - \mathrm{div}(k,n) + \mathrm{div}(k,2) - 1,$$

where we denote $\mathrm{div}(p,q) = \begin{cases} 1 & \text{if } q|p \\ 0 & \text{if } q \not| p \end{cases}.$

$$n_k = \begin{cases} 0-0+0-1=-1 & k \text{ odd}, k \not\equiv 0, n \mod 2n \\ 0-0+1-1=0 & k \text{ even}, k \not\equiv 0, n \mod 2n \\ 0-1+1-1=-1 & k \text{ even}, k \equiv n \mod 2n \\ 0-1+0-1=-2 & k \text{ odd}, k \equiv n \mod 2n \\ 1-1+1-1=0 & k \equiv 0 \mod 2n \end{cases}.$$

Case 3: $Q = E_6$

The exponents are $1, 4, 5, 7, 8, 11$ and the Coxeter number is $h = 12$.

$$\det(1 - Ct + t^2) = \frac{(1-t^{24})(1-t^4)(1-t^6)}{(1-t^{12})(1-t^8)(1-t^2)},$$

then

$$\prod_{k=1}^{\infty}(1-q^k)^{n_k} = \prod_{s=1}^{\infty}\det(1 - Ct^s + t^{2s}) = \prod_{s=1}^{\infty}\frac{(1-q^{12s})(1-q^{2s})(1-q^{3s})}{(1-q^{6s})(1-q^{4s})(1-q^s)}$$

implies

$$n_k = \operatorname{div}(k,12) + \operatorname{div}(k,2) + \operatorname{div}(k,3) - \operatorname{div}(k,6) - \operatorname{div}(k,4) - 1.$$

Observe that if we have a prime factorization $q = a^2 b$ (a, b distinct), then

$$\operatorname{div}(k,q) + \operatorname{div}(k,a) + \operatorname{div}(k,b) - \operatorname{div}(k,ab) - \operatorname{div}(k,a^2) - 1$$

is -1 if k and q are relatively prime or if $k \equiv la^2 \mod q$ ($l \neq 0$) and 0 else.

This proves our case for $12 = 2^2 \cdot 3$.

Case 4: $Q = E_7$

The exponents are $1, 5, 7, 9, 11, 13, 17$ and the Coxeter number is

$h = 18$.

$$\det(1 - Ct + t^2) = \frac{(1-t^{36})(1-t^6)(1-t^3)}{(1-t^{18})(1-t^{12})(1-t^2)},$$

so

$$\prod_{k=1}^{\infty}(1-q^k)^{n_k} = \prod_{s=1}^{\infty}\det(1 - Ct^s + t^{2s}) = \prod_{s=1}^{\infty}\frac{(1-q^{18s})(1-q^{3s})(1-q^{2s})}{(1-q^{9s})(1-q^{6s})(1-q^s)}$$

implies

$$n_k = \mathrm{div}(k,18) + \mathrm{div}(k,3) + \mathrm{div}(k,2) - \mathrm{div}(k,9) - \mathrm{div}(k,6) - 1.$$

We use the same argument as above, for $18 = 2 \cdot 3^2$.

Case 5: $Q = E_8$

The exponents are $1, 7, 11, 13, 17, 19, 23, 29$ and the Coxeter number is $h = 30$.

$$\det(1 - Ct + t^2) = \frac{(1-t^{60})(1-t^{10})(1-t^6)(1-t^4)}{(1-t^{30})(1-t^{20})(1-t^{12})(1-t^2)},$$

then

$$\prod_{k=1}^{\infty}(1-q^k)^{n_k} = \prod_{s=1}^{\infty}\det(1 - Ct^s + t^{2s}) = \prod_{s=1}^{\infty}\frac{(1-q^{30s})(1-q^{5s})(1-q^{3s})(1-q^{2s})}{(1-q^{15s})(1-q^{10s})(1-q^{6s})(1-q^s)}$$

implies

$$n_k = \mathrm{div}(k,30) + \mathrm{div}(k,5) + \mathrm{div}(k,3) + \mathrm{div}(k,2)$$
$$- \mathrm{div}(k,15) - \mathrm{div}(k,10) - \mathrm{div}(k,6) - 1.$$

We use a similar argument here: If we have a prime factorization $q = abc$ (a, b, c distinct), then

$$\mathrm{div}(k,q) + \mathrm{div}(k,a) + \mathrm{div}(k,b) + \mathrm{div}(k,c) - \mathrm{div}(k,ab) - \mathrm{div}(k,bc) - \mathrm{div}(k,ac) - 1$$

is -1 if k and q are relatively prime and 0 else.

This proves our case for $30 = 2 \cdot 3 \cdot 5$. $\square$

Proof. (of Theorem 4.1.3.4):

From the isomorphism

$$(Z \cap \mu^{-1}[A, A])[2] \xrightarrow{B_2} Z/A_{top}$$

we obtain the equation $t^2 q(t) = t^4 \sum_{i=1}^{r} (t^{2(m_i-1)} + \ldots + t^{2h-6})$, so

$$q(t) = \sum_{i=1}^{r} (t^{2m_i} + \ldots + t^{2h-4}).$$

Recall the duality of exponents, i.e. $m_{r+1-i} = h - m_i$. Then we get

$$\begin{aligned} q_*(t) &= t^{2h-4} q(1/t) = t^{2h-4} \sum_{i=1}^{r} (t^{-2m_i} + \ldots + t^{-2h+4}) \\ &= t^{2h-4} \sum_{i=1}^{r} (t^{-2(h-m_i)} + \ldots + t^{-2h+4}) \\ &= \sum_{i=1}^{r} (1 + \ldots + t^{2(m_i-2)}) = p(t) - \sum_{i=1}^{r} t^{2(m_i-1)}. \end{aligned}$$

$\square$

4.2 Universal deformation of A

Definition 4.2.0.3. *For any weight $\lambda = (\lambda_i)$, we define the algebra*

$$A(\lambda) = P[z] / \left(\sum_{a \in \bar{Q}} [a, a^*] = z\mu + \sum_{i=1}^{r} \lambda_i e_i \right)$$

and introduce a deformation parametrized by formal variables c_i^j, $1 \leq i \leq r$, $1 \leq j \leq h-1$:

$$A(\lambda)_c = P[z][[c]] / \left(\sum_{a \in \bar{Q}} [a, a^*] = z\mu + \sum_{i=1}^{r} \lambda_i e_i + \sum_{i=1}^{r} \sum_{j=1}^{h-1} c_i^j z^j e_i \right).$$

Theorem 4.2.0.4. *This deformation is flat $\forall \lambda \in R$, i.e. $A(\lambda)_c$ is free over $\mathbb{C}[[c]]$, and*

$$A(\lambda)_c/(c) = A(\lambda).$$

Proof. It is sufficient to check flatness for generic λ. From [10, end of section 3.2.], we know that for generic λ, $A(\lambda) = \oplus \mathrm{End} V_\alpha$ is a semisimple algebra. So it suffices to show that the representation V_α can be deformed to a representation of $A(\lambda)_c$ for all λ.

We recall from [3, Theorem 4.3.] that $\forall \beta \in R$, such that $\beta \cdot \alpha = 0$, it exists an $\alpha-$dimensional irreducible representation V_α of P, such that

$$\sum_{a \in \bar{Q}} [a, a^*] = \sum_{i=1}^{r} \beta_i e_i.$$

If we set $z = \gamma \in \mathbb{C}$ in $A(\lambda)_c$, then the relation becomes

$$\sum_{a \in \bar{Q}} [a, a^*] = \sum_{i=1}^{r} e_i(\lambda_i + \gamma(\mu_i + c_i^1) + \gamma^2 c_i^2 + \ldots).$$

Then for $\alpha = \sum_{i=1}^{r} \alpha_i \epsilon_i$, since the trace of $[a, a^*]$ is zero, the condition to have an $\alpha-$dimensional representation of $A(\lambda)_c$ (i.e. a representation of P satisfying the above relation) is

$$\sum_{i=1}^{r} \alpha_i(\lambda_i + \gamma(\mu_i + c_i^1) + \gamma^2 c_i^2 + \ldots) = 0.$$

By Hensel's lemma, this equation in $\mathbb{C}[[c]]$ has a unique solution γ, such that its constant term $\gamma_0 \in \mathbb{C}$ satisfies $\sum_{i=1}^{r} \alpha_i(\lambda + \gamma_0) = 0 \Rightarrow \gamma_0 = -\frac{\sum \alpha_i \lambda_i}{\sum \alpha_i}$. □

In particular, if we treat λ as formal parameter, then $A(\lambda)_c$ is a flat deformation of $A(0)$.

Let E be the linear span of $z^j e_i$, $0 \le j \le h-2$, $1 \le i \le r$. From [9, Proposition 2.4.] we know that the projection map $E \longrightarrow A/[A,A]$ is surjective. Then the deformation $A(\lambda)_c$ is parametrized by E which gives us a natural map $\eta : E \longrightarrow HH^2(A)$. On the other hand, the isomorphism $HH^2(A) = A/([A,A] + \mu Z)$ in Theorem 4.1.2.1 induces a projection map $\theta : E \longrightarrow HH^2(A)$.

Proposition 4.2.0.5. *The maps $\theta, \eta : E \longrightarrow HH^2(A)$ are identical.*

Proof. We have the following commutative diagram which connects our periodic projective resolution with the bar resolution of A,

$$\begin{array}{ccccccc}
\begin{matrix}(A \otimes V \otimes A[2]) \\ \oplus \\ (A \otimes A[2])\end{matrix} & \xrightarrow{d_2} & \begin{matrix}(A \otimes V \otimes A) \\ \oplus \\ (A \otimes A[2])\end{matrix} & \xrightarrow{d_1} & A \otimes A & \xrightarrow{d_0} & A \\
\downarrow f_2 & & \downarrow f_1 & & \| & & \| \\
A^{\otimes 4} & \xrightarrow[\tilde{d}_2]{} & A^{\otimes 3} & \xrightarrow[\tilde{d}_1]{} & A^{\otimes 2} & \xrightarrow[\tilde{d}_0]{} & A,
\end{array}$$

where we define

$$f_1(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = b_1 \otimes \alpha \otimes b_2 + b_3 \otimes z \otimes b_4$$

and

$$f_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = -b_1 \otimes z \otimes \alpha \otimes b_2 + b_1 \otimes \alpha \otimes z \otimes b_2 + \sum_{a \in Q} \epsilon_a b_3 \otimes a \otimes a^* \otimes b_4.$$

Let us check the commutativity of the diagram:

$$\tilde{d}_1 f_1(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) =$$
$$= b_1\alpha \otimes b_2 - b_1 \otimes \alpha b_2 + b_3 z \otimes b_4 - b_3 \otimes z b_4 = d_1(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4),$$

$$
\begin{aligned}
& f_1 d_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = \\
&\quad = f_1(-b_1 z \otimes \alpha \otimes b_2 + b_1 \otimes \alpha \otimes z b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_3 a \otimes a^* \otimes b_4 \\
&\qquad + \sum_{a \in \bar{Q}} \epsilon_a b_3 \otimes a \otimes a^* b_4, -b_3 \mu \otimes b_4 + b_1 \alpha \otimes b_2 - b_1 \otimes \alpha b_2) \\
&\quad = -b_1 z \otimes \alpha \otimes b_2 - b_1 \otimes z \otimes \alpha b_2 + b_1 \alpha \otimes z \otimes b_2 + b_1 \otimes \alpha \otimes z b_2 \\
&\qquad + \sum_{a \in \bar{Q}} \epsilon_a b_3 a \otimes a^* \otimes b_4 - b_3 \otimes z\mu \otimes b_4 + \sum_{a \in \bar{Q}} \epsilon_a b_3 \otimes a \otimes a^* b_4 \\
&\quad = \tilde{d}_2 f_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4).
\end{aligned}
$$

We apply $Hom_{A^e}(-, A)$ to the above diagram:

$$
\begin{array}{ccccc}
\begin{array}{c}(A \otimes V)^R[2] \\ \oplus \\ (A^R[2])\end{array} & \xleftarrow{d_2^*} & \begin{array}{c}(A \otimes V)^R[2] \\ \oplus \\ A^R\end{array} & \xleftarrow{d_1^*} & A^R \\
{\scriptstyle f_2^*}\uparrow & & \uparrow{\scriptstyle f_1^*} & & \| \\
Hom_{A^e}(A^{\otimes 4}, A) & \xleftarrow[(\tilde{d}_2)^*]{} & Hom_{A^e}(A^{\otimes 3}, A) & \xleftarrow[(\tilde{d}_1)^*]{} & A^R.
\end{array}
$$

The map f_2^* induces a natural isomorphism on $HH^2(A)$, so via this identification we want to prove that $f_2^* \eta = \theta$.

The element $\gamma := \sum \gamma_i^j z^j e_i$, $\gamma_i^j \in \mathbb{C}$ defines the 1-parameter deformation

$$
A^\gamma = A[[\hbar]] / \sum_{a \in Q} [a, a^*] = z + \hbar(\sum_{i=1}^{r} \gamma_i^0 e_i + \sum_{i=1}^{r} \sum_{j=1}^{h-2} \gamma_i^j z^j e_i),
$$

so the cocycle $\eta(\gamma)$ is defined to be a bilinear map m on $A \times A$ (where we identify

$$
Hom_{A^e}(A^{\otimes 4}, A) = Hom_k(A \otimes A, A)
$$

here), such that for $a, b \in A$,

$$
a * b \equiv ab + \hbar m(a, b) \mod \hbar^2
$$

where $''*''$ is the product in A^γ. This gives us:

$$f_2^*\eta(\gamma)(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = \eta(\gamma) f_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) =$$
$$= \eta(\gamma)(-b_1 \otimes z \otimes \alpha \otimes b_2 + b_1 \otimes \alpha \otimes z \otimes b_2 + \sum_{a \in Q} \epsilon_a b_3 \otimes a \otimes a^* \otimes b_4)$$
$$= b_1(m(z,\alpha) - m(\alpha, z))b_2 + b_3(\sum_{a\in Q} m(a,a^*) - m(a^*,a))b_4$$
$$= b_3(\sum_{i=1}^{r} \gamma_i^0 e_i + \sum_{i=1}^{r}\sum_{j=1}^{h-2} \gamma_i^j z^j e_i)b_4 = \theta(\gamma)(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4).$$

We obtain the second to last equality by:
$0 + \hbar(m(z,\alpha) - m(\alpha, z)) = z * \alpha - \alpha * z = 0$ and

$$z\mu + \hbar(\sum_{a\in Q} m(a,a^*) - m(a^*,a)) = \sum_{a\in Q}(a * a^* - a^* * a)$$
$$= z\mu + \hbar\left(\sum_{i=1}^{r} \gamma_i^0 e_i + \sum_{i=1}^{r}\sum_{j=1}^{h-2} \gamma_i^j z^j e_i\right).$$

This finishes our proof that $f_2^*\eta = \theta$. □

We see that the map $E \longrightarrow HH^2(A)$ induced by the deformation $A(\lambda)_c$ is just the projection map. From this we can derive the universal deformation of A very easily.

Let $E' \subset E$ be the subspace which is complimentary to $\ker(\theta : E \longrightarrow A/([A,A] + \mu Z))$ with basis $w_i, \dots, w_s$, and choose formal parameters $t_i, \dots, t_s$. The subdeformation A' of A, parametrized by $E' \subset E$ is:

$$A' = P[z][[t_1, \dots, t_s]] / \left(\sum_{a \in \bar{Q}} [a, a^*] = \mu z + \sum_{i=1}^{s} t_i w_i\right).$$

Theorem 4.2.0.6. *A' is the universal deformation of A.*

Proof. $\eta : E' \longrightarrow HH^2(A)$ is the map induced by the deformation A'. Since θ induces an isomorphism $E' \longrightarrow A/([A,A] + \mu Z) = HH^2(A)$, by Proposition 4.2.0.5 η is an isomorphism and therefore induces a universal deformation. □

4.3 Results about the calculus structure of the Hochschild cohomology/homology of the centrally extended preprojective algebras of Dynkin quivers

Theorem 4.3.0.7. *The calculus structure os given by tables 4.1, 4.2 and 4.3 and the Connes differential as follows:*

$$
\begin{aligned}
B_{4s}((1^*/c)_s) &= (2h-4-\deg(c)+2nh)Eu^*/c,\\
B_{4s+1} &= 0,\\
B_{4s+2}((c\cdot Eu)_s) &= (\deg(c)+4+2sh)c,\\
B_{4s+3} &= 0.
\end{aligned}
$$

a \ b	$(1^*/c')_t$	$(Eu^*/c')_t$	$(c'\cdot Eu)_t$	$(c')_t$
$c^{(s)}$	$(c1^*/c')_{t-s}$	$(c\cdot Eu^*/c')_{t-s}$	$(cc'\cdot Eu)_{t-s}$	$(cc')_{t-s}$
$(c\cdot Eu)^{(s)}$	0	$(c1^*/c')_{t-s}$	0	$(cc'\cdot Eu)_{t-s}$
$(Eu^*/c)^{(s)}$	0	0	$(c'1^*/c)_{t-s}$	$(c'\cdot Eu^*/c)_{t-s}$
$(1^*/c)^{(s)}$	0	0	0	0

Table 4.1: contraction map $\iota_a(b)$

b / a	$(c')^{(t)}$	$(c' \cdot Eu)^{(t)}$	$(Eu^*/c')^{(t)}$	$(1^*/c')^{(t)}$
$c^{(s)}$	0	$(\deg(c) - 2sh)(cc')^{(s+t)}$	0	$(\deg(c) - 2sh)(c \cdot Eu^*/c$
$(c \cdot Eu)^{(s)}$		$(\deg(c') - \deg(c) + 2(s-t)h)(cc' \cdot Eu)^{(s+t)}$	$(2(1-t)h - 8 - \deg(c'))(c \cdot Eu^*/c')^{(s+t)}$	$(2(s-t+1)h - 8 - \deg(c) -$
$(Eu^*/c)^{(s)}$	0	0	0	
$(1^*/c)^{(s)}$	0	0	0	0

Table 4.2: Gerstenhaber bracket $[a, b]$

$a \backslash b$	$(1^*/c')_t$	$(Eu^*/c')_t$	$(c' \cdot Eu)_t$	$(c')_t$
$c^{(s)}$	$(\deg(c) - 2sh)(c \cdot Eu^*/c')_{t-s}$	0	$(\deg(c) - 2th)(cc')_{t-s}$	0
$(c \cdot Eu)^{(s)}$	0	$(c1^*/c')_{t-s}$	0	$(cc' \cdot Eu)_{t-s}$
$(Eu^*/c)^{(s)}$	0	0	$(2(1-s)h - 8 - \deg(c))(c' \cdot Eu^*/c)_{t-s}$	0
$(1^*/c)^{(s)}$	0	0	0	0

Table 4.3: Lie derivative $\mathcal{L}_a(b)$

4.4 Batalin-Vilkovisky structure on Hochschild cohomology

We recall [17, Section 4] the following: we have an isomorphism $\mathbb{D} : HH_\bullet(A) \to HH^{6m+5-\bullet}(A)\ \forall m \geq 0$. It translates the Connes differential $B : HH_\bullet(A) \to HH_{\bullet+1}(A)$ on Hochschild homology into a differential $\Delta : HH^\bullet(A) \to HH^{\bullet-1}(A)$ on Hochschild cohomology, i.e. we have the commutative diagram

$$\begin{array}{ccc} HH_\bullet(A) & \xrightarrow{B} & HH_{\bullet+1}(A) \\ \mathbb{D}\downarrow\sim & & \sim\downarrow\mathbb{D} \\ HH^{4m+3-\bullet}(A)[2mh-4] & \xrightarrow{\Delta} & HH^{4m+2-\bullet}(A)[2mh-4] \end{array}$$

Theorem 4.4.0.8. (BV structure on Hochschild cohomology) Δ *makes* $HH^\bullet(A)$ *a Batalin-Vilkovisky algebra (defined in Theorem 3.3.0.13)*

Proof. We refer to [17, Theorem 2.4.65]. □

Remark 4.4.0.9. Note that Δ depends on which $m \in \mathbb{N}$ we choose to identify $\mathbb{D} : HH_\bullet(A) \xrightarrow{\sim} HH^{4m+3-\bullet}(A)[2mh-4]$, where the Gerstenhaber bracket does not.

4.4.1 Computation of the calculus structure of the centrally extended preprojective algebra

Cup product

As described in [17, Section 4], we fix an isomorphism $\mathbb{D} : HH_i(A) \cong HH_{3-i}(A)^*[2h]$ and use the elements $Eu \in HH^1(A)$ (where Eu is the Euler vector field), $Eu^* \in HH^2(A)$ and $1^* \in HH^3(A)$. Then we can describe all elements in $HH^1(A)$, $HH^2(A)$ and $HH^3(A)$ by $c \cdot Eu$, Eu^*/c, $1^*/c$ for $c \in HH^0(A)$, where $c \cup Eu^*/c = Eu^*$ and $c \cup 1^*/c = 1^*$. We have $Eu \cup Eu = 0$ by graded commutativity and $Eu \cup Eu^*$ from [17, Theorem 2.4.27]. Cup products $HH^i(A) \cup HH^j(A)$ for $i + j \geq 4$, $i, j \leq 3$ are zero.

Notation 4.4.1.1. For $c_k \in HH^i(A)$, $0 \leq i \leq 3$, we write $c_k^{(s)}$ for the corresponding cocycle in HH^{i+4s}. We write $c_{k,t}$ for the corresponding cycle in HH_{j+4t}, $0 \leq j \leq 3$ (under the isomorphism $\mathbb{D}$.

The contraction map

From (3.3.0.15), we know that the contraction map on Hochschild homology is given by the cup product on Hochschild cohomology. Table 4.1 contains the results, rewritten in terms of contraction maps.

The Connes differential

Proposition 4.4.1.2. *The Connes differential B is given by*

$$\begin{aligned} B_{4s}((1^*/c)_s) &= (2h-4-\deg(c)+2nh)Eu^*/c, \\ B_{4s+1} &= 0, \\ B_{4s+2}((c\cdot Eu)_s) &= (\deg(c)+4+2sh)c, \\ B_{4s+3} &= 0. \end{aligned}$$

Similar to Subsection 3.3.1, we can see that for any $x \in HH_i(A)$, $\mathcal{L}_{Eu}(x) = x\deg(x)$. We use the Cartan identity (3.1.0.7),

$$\mathcal{L}_{Eu} = B\iota_{Eu} + \iota_{Eu}B. \tag{4.4.1.3}$$

We compute

$$(2h-4-\deg(c)+2sh)(1^*/c)_s = \mathcal{L}_{Eu}((1^*/c)_s) = B(\underbrace{\iota_{Eu}((1^*/c)_s)}_{=0} + \iota_{Eu}(B(1^*/c)_s),$$

so

$$B((1^*/c)_s) = (2h-4-\deg(c)+2sh)(Eu^*/c)_s.$$

Since $B^2=0$, it follows that

$$B((Eu^*/c)_s) = 0.$$

We compute

$$(\deg(c) + 4 + 2sh)(c \cdot Eu)_s = \mathcal{L}_{Eu}((c\dot{E}u)_s) = B \underbrace{\iota_{Eu}((c\dot{E}u)_s)}_{=0} + \iota_{Eu} B((c\dot{E}u)_s),$$

so

$$B((c\dot{E}u)_s) = (\deg(c) + 4)c_s.$$

Since $B^2 = 0$, it follows that

$$B(c_s) = 0.$$

The Gerstenhaber bracket

We compute the brackets using the identification $HH^i(A) = HH_{4m+3-i}[-2mh+4]$ for $m >> 1$ and the BV identity (3.3.0.14). We rewrite the results from Proposition 4.4.1.2:

$$\begin{aligned}
\Delta(c^{(s)}) &= 0, \\
\Delta((c \cdot Eu)^{(s)}) &= (\deg(c) + 4 + 2(m-s)h), \\
\Delta((Eu^*/c)^{(s)}) &= 0, \\
\Delta((1^*/c)^{(s)}) &= (2h - 4 - \deg(c) + 2(m-n)h)(Eu^*/c)^{(n)}.
\end{aligned}$$

Brackets involving $HH^{4s}(A)$:

We have

$$
\begin{aligned}
[c^{(s)}, c'^{(t)}] &= \Delta((cc')^{(s+t)}) - \Delta(c^{(s)}) \cup c'^{(t)} - c^{(s)} \cup \Delta(c'^{(t)}) = 0, \\
[c^{(s)}, (c' \cdot Eu)^{(t)}] &= \Delta((cc' \cdot Eu)^{(s+t)}) - \underbrace{\Delta(c^{(s)})}_{=0} \cup (c' \cdot Eu)^{(t)} \\
&\quad - c^{(s)} \cdot \Delta((c' \cdot Eu)^{(t)}) \\
&= (\deg(cc') + 4 + 2(m - s - t)h) cc'^{(s+t)} \\
&\quad - c^{(s)} \cup (\deg(c') + 4 + 2(m - t)h) c'^{(t)} \\
&= (\deg(c) - 2sh) cc'^{(s+t)}, \\
[c^{(s)}, (Eu^*/c')^{(t)}] &= \Delta((cEu^*/c')^{(s+t)}) - \Delta(c^{(s)}) \cup (Eu^*/c')^{(t)} \\
&\quad - c^{(s)} \cup \Delta((Eu^*/c')^{(t)}) = 0, \\
[c^{(s)}, (1^*/c')^{(t)}] &= \Delta((c1^*/c')^{(s+t)}) - \Delta(c^{(s)}) \cup (1^*/c')^{(t)} \\
&\quad - c^{(s)} \cup \Delta((1^*/c')^{(t)}) \\
&= (2h - 4 + \deg(c) - \deg(c') + 2(m - s - t)h)(c \cdot Eu^*/c')^{(s+t)} \\
&\quad - (2h - 4 - \deg(c') + 2(m - t)h)(c \cdot Eu^*/c')^{(s+t)} \\
&= (\deg(c) - 2sh)(c \cdot Eu^*/c')^{(s+t)}
\end{aligned}
$$

Brackets involving $HH^{1+4s}(A)$:

We have

$$\begin{aligned}
[(c\cdot Eu)^{(s)},(c'\cdot Eu)^{(t)}] &= \Delta((c\cdot Eu)^{(s)}\cdot(c'\cdot Eu)^{(t)}) \\
&\quad -(\deg(c)+4+2(m-s)h)(cc'\cdot Eu)^{(s+t)} \\
&\quad +(\deg(c')+4+2(m-t)h)(cc'\cdot Eu)^{(s+t)} \\
&= (\deg(c')-\deg(c)+2(s-t)h)(cc'\cdot Eu)^{(s+t)}, \\
[(c\cdot Eu)^{(s)},(Eu^*/c')^{(t)}] &= \Delta((c1^*/c')^{(s+t)})-\Delta((c\cdot Eu)^{(s)})\cup(Eu^*/c')^{(t)} \\
&\quad +(c\cdot Eu)^{(s)}\cup\Delta((Eu^*/c')^{(t)}) \\
&= (2h-4+\deg(c)-\deg(c')+2(m-s-t)h)(c\cdot Eu^*/c')^{(s+t)} \\
&\quad -(\deg(c)+4+2(m-s)h)(c\cdot Eu^*/c')^{(s+t)} \\
&= (2(1-t)h-8-\deg(c'))(c\cdot Eu^*/c')^{(s+t)}, \\
[(c\cdot Eu)^{(s)},(1^*/c')^{(t)}] &= \Delta((c\cdot Eu)^{(s)})\cup(1^*/c')^{(t)}+c\cdot Eu^{(s)}\cdot\Delta((1^*/c')^{(t)}) \\
&= -(\deg(c)+4+2(m-s)h)(c1^*/c')^{(s+t)} \\
&\quad +(2h-4-\deg(c')+2(m-t)h)(c1^*/c')^{(s+t)} \\
&= (2(s-t+1)h-8-\deg(c)-\deg(c'))(c1^*/c')^{(s+t)}
\end{aligned}$$

Brackets involving $HH^{2+4s}(A)$ or $HH^{3+4s}(A)$:

We have

$$\begin{aligned}
[(Eu^*/c)^{(s)},(Eu^*/c')^{(t)}] &= \Delta((Eu^*/c)^{(s)}\cup(Eu^*/c')^{(t)})-\Delta((Eu^*/c)^{(s)})\cup(Eu^*/c')^{(t)} \\
&\quad -(Eu^*/c)^{(s)}\cup\Delta((Eu^*/c')^{(t)})=0, \\
[(Eu^*/c)^{(s)},(1^*/c')^{(t)}] &= \Delta((Eu^*/c)^{(s)}\cup(1^*/c')^{(t)})-\Delta((Eu^*/c)^{(s)})\cup(1^*/c')^{(t)} \\
&\quad -(2h-4-\deg(c')+2(m-t)h)(Eu^*/c)^{(s)}\cup(Eu^*/c')^{(t)}=0, \\
[(1^*/c)^{(s)},(1^*/c')^{(t)}] &= \Delta((1^*/c)^{(s)}\cup(1^*/c')^{(t)})-\Delta((1^*/c)^(s))\cup(1^*/c')^{(t)} \\
&\quad +(1^*/c)^{(s)}\cup\Delta(1^*/c')^{(t)}=0
\end{aligned}$$

The Lie derivative $\mathcal{L}$

We use the Cartan identity (3.1.0.7) to compute the Lie derivative.

$\underline{HH^{1+4s}(A)\text{-}\mathbf{Lie\ derivatives}}$:

$$\begin{aligned}
\mathcal{L}_{c\cdot Eu^{(s)}}((1^*/c')_t) &= B\iota_{c\cdot Eu^{(s)}}((1^*/c')_t) + \iota_{c\cdot Eu^{(s)}}B((1^*/c')_t) \\
&= (2h - 4 - \deg(c') + 2th)\iota_{c\cdot Eu^{(s)}}(Eu^*/c')_t \\
&= (2h - 4 - \deg(c') + 2th)(c1^*/c')_{t-s} \\
\mathcal{L}_{c\cdot Eu^{(s)}}((Eu^*/c')_t) &= B\iota_{c\cdot Eu^{(s)}}((Eu^*/c')_t) + \iota_{c\cdot Eu^{(s)}}B((Eu^*/c')_t) \\
&= B((c1^*/c')_{t-s}) \\
&= (2h - 4 + \deg(c) - \deg(c') + 2(t-s)h)(Eu^*/c')_{t-s}, \\
\mathcal{L}_{c\cdot Eu^{(s)}}((c'\cdot Eu)_t) &= B\iota_{c\cdot Eu^{(s)}}((c'\cdot Eu)_t) + \iota_{c\cdot Eu^{(s)}}B((c'\cdot Eu)_t) \\
&= (\deg(c') + 4 + 2th)\iota_{c\cdot Eu^{(s)}}((c')_t) \\
&= (\deg(c') + 4 + 2th)(cc'\cdot Eu)_{t-s}, \\
\mathcal{L}_{c\cdot Eu^{(s)}}((c')_t) &= B\iota_{c\cdot Eu^{(s)}}((c')_t) + \iota_{c\cdot Eu^{(s)}}B((c')_t) \\
&= B((cc'\cdot Eu)_{t-s}) \\
&= (\deg(cc') + 4 + 2(t-s)h)(cc')_{t-s}
\end{aligned}$$

$\underline{HH^{2+4s}(A)\text{-}\textbf{Lie derivatives}}$:

$$\begin{aligned}
\mathcal{L}_{(Eu^*/c)^{(s)}}((1^*/c')_t) &= B\iota_{(Eu^*/c)^{(s)}}((1^*/c')_t) + \iota_{(Eu^*/c)^{(s)}}B((1^*/c')_t) \\
&= -(2h-4-\deg(c')-2th)\iota_{(Eu^*/c)^{(s)}}((Eu^*/c')_t) = 0, \\
\mathcal{L}_{(Eu^*/c)^{(s)}}((Eu^*/c')_t) &= B\iota_{(Eu^*/c)^{(s)}}((Eu^*/c')_t) + \iota_{(Eu^*/c)^{(s)}}B((Eu^*/c')_t) = 0 \\
\mathcal{L}_{(Eu^*/c)^{(s)}}((c'\cdot Eu)_t) &= B\iota_{(Eu^*/c)^{(s)}}((c'\cdot Eu)_t) - \iota_{(Eu^*/c)^{(s)}}B((c'\cdot Eu)_t) \\
&= B((c'1^*/c)_{t-s}) - (\deg(c')+4+2th)\iota_{(Eu^*/c)^{(s)}}(c')_t \\
&= (2h-4+\deg(c')-\deg(c)+2(t-s)h)(c'\cdot Eu^*/c)_{t-s} \\
&\quad -(\deg(c)+4+2th)(c'\cdot Eu^*/c)_{t-s} \\
&= (2h-8-\deg(c)-2sh)(c'\cdot Eu^*/c)_{t-s}, \\
\mathcal{L}_{(Eu^*/c)^{(s)}}((c')_t) &= B\iota_{(Eu^*/c)^{(s)}}((c')_t) - \iota_{(Eu^*/c)^{(s)}}B((c'\cdot Eu)_t) \\
&= B((c'\cdot Eu^*/c)_{t-s}) = 0
\end{aligned}$$

$\underline{HH^{3+4s}(A)\text{-}\mathbf{Lie\ derivatives}}$: Since $\iota_{(1^*/c)^{(s)}} = 0$, it follows that

$$\mathcal{L}_{(1^*/c)^{(s)}} = B\iota_{(1^*/c)^{(s)}} + \iota_{(1^*/c)^{(s)}}B.$$

$\underline{HH^{4+4s}(A)\text{-}\mathbf{Lie\ derivatives}}$: We have

$$\begin{aligned}
\mathcal{L}_{c^{(s)}}((1^*/c')_t) &= B\iota_{c^{(s)}}((1^*/c')_t) - \iota_{c^{(s)}}B((1^*/c')_t) \\
&= B((c1^*/c')_{t-s}) - (2h-4-\deg(c')+2th)(c\cdot Eu^*/c')_{t-s} \\
&= (2h-4+\deg(c)-\deg(c')+2(t-s)h)(c\cdot Eu^*/c')_{t-s} \\
&\quad -(2h-4+\deg(c')+2th)(c\cdot Eu^*/c')_{t-s} \\
&= (\deg(c)-2sh)(c\cdot Eu^*/c')_{t-s}, \\
\mathcal{L}_{c^{(s)}}((Eu^*/c')_t) &= B\iota_{c^{(s)}}((Eu^*/c')_t) - \iota_{c^{(s)}}B((Eu^*/c')_t) \\
&= B((c\cdot Eu^*/c')_{t-s}) = 0, \\
\mathcal{L}_{c^{(s)}}((c'\cdot Eu)_t) &= B\iota_{c^{(s)}}((c'\cdot Eu)_t) - \iota_{c^{(s)}}B((c'\cdot Eu)_t) \\
&= B((cc'\cdot Eu)_{t-s}) - \iota_{c^{(s)}}(\deg(c')+4+2th)(c')_t \\
&= (\deg(c'c)+4+2(t-s)h)c'c - (\deg(c')+4+2th)(c'c)_{t-s} \\
&= (\deg(c)-2th)(cc')_{t-s} \\
\mathcal{L}_{c^{(s)}}((c')_t) &= B\iota_{c^{(s)}}((c')_t) - \iota_{c^{(s)}}B((c')_t) = 0
\end{aligned}$$

Chapter 5

Hochschild cohomology/homology and calculus structure of the preprojective algebra of type T

5.1 The preprojective algebra

Let Q be a quiver of type T. We call the loop b. Let $Q' = Q \setminus \{b\}$.

We define $(Q')^*$ to be the quiver obtained from Q' by reversing all of its arrows. We call $\bar{Q}' = Q' \cup (Q')^*$ the *double* of Q'. Let C be the adjacency matrix corresponding to the quiver $\bar{Q} = \bar{Q}' \cup \{b\}$.

We define the *preprojective algebra* Π_Q to be the quotient of the path algebra $P_{\bar{Q}}$ by the relation $\sum\limits_{a \in Q'} [a, a^*] = b^2$. It is known that Π_Q is a Frobenius algebra (see [20]).

From now on, we write $A = \Pi_Q$.

5.2 The main results

Definition 5.2.0.1. *We define the spaces*

$$U = \bigoplus_{d<h-2} HH^0(A)(d)[2] \quad \text{and}$$
$$K = HH^2(A)[2].$$

Theorem 5.2.0.2. *The spaces U and K have the following properties:*

(a) *U has Hilbert series*

$$h_U(t) = \sum_{i=0}^{n-1} t^{2i}. \tag{5.2.0.3}$$

(b) *K is n-dimensional and sits in degree zero.*

Theorem 5.2.0.4 (Hochschild cohomology). *The Hochschild cohomology spaces are given by*

$$\begin{aligned}
HH^0(A) &= U[-2] \oplus R^*[h-2], \\
HH^1(A) &= U[-2], \\
HH^2(A) &= K[-2], \\
HH^3(A) &= K^*[-2], \\
HH^4(A) &= U^*[-2], \\
HH^5(A) &= U^*[-2], \\
HH^6(A) &= U[-2h-2], \\
HH^{6k+i}(A) &= HH^i(A)[-2kh] \quad \forall i \geq 1.
\end{aligned}$$

Theorem 5.2.0.5 (Hochschild homology). *The Hochschild homology spaces are* *given by*

$$
\begin{aligned}
HH_0(A) &= U^*[h] \oplus R, \\
HH_1(A) &= U^*[h], \\
HH_2(A) &= K^*[h], \\
HH_3(A) &= K[h], \\
HH_4(A) &= U[h], \\
HH_5(A) &= U[h], \\
HH_6(A) &= U^*[3h], \\
HH_{6k+i}(A) &= HH_i(A)[2kh] \quad \forall i \geq 1.
\end{aligned}
$$

Theorem 5.2.0.6 (Cyclic homology). *The cyclic homology spaces are given by*

$$
\begin{aligned}
HC_0(A) &= U^*[h] \oplus R, \\
HC_1(A) &= 0, \\
HC_2(A) &= K^*[h], \\
HC_3(A) &= 0, \\
HC_4(A) &= U[h], \\
HC_5(A) &= 0, \\
HC_6(A) &= U^*[3h], \\
HC_{6k+i}(A) &= HH_i(A)[2kh] \quad \forall i \geq 1.
\end{aligned}
$$

Let $(U[-2])_+$ be the positive degree part of $U[-2]$ (which lies in non-negative degrees).

We have a decomposition $HH^0(A) = \mathbb{C} \oplus (U[-2])_+ \oplus L[-h-2]$ where we have the natural identification $(U[-2])(0) = \mathbb{C}$. This identification also gives us a decomposition $HH^*(A) = \mathbb{C} \oplus HH^*(A)_+$.
We also decompose $U = U^{top} \oplus U_-$, where U^{top} is the top degree part of U and a one-dimesional space.

We give a brief description of the product structure in $HH^*(A)$ which will be computed in this paper. Since the product $HH^i(A) \times HH^j(A) \to HH^{i+j}(A)$ is graded-commutative, we can assume $i \leq j$ here.

Let $z_0 = 1 \in \mathbb{C} \subset U[-2] \subset HH^0(A)$ (in lowest degree 0),
θ_0 the corresponding element in $HH^1(A)$ (in lowest degree 0),
ψ_0 the dual element of z_0 in $U^*[-2] \subset HH^5(A)$ (in highest degree -4), i.e. $\psi_0(z_0) = 1$,
ζ_0 the corresponding element in $U^*[-2] \subset HH^4(A)$ (in highest degree -4), that is the dual element of θ_0, $\zeta_0(\theta_0) = 1$,
$\varphi_0 : HH^0(A) \to HH^6(A)$ the natural quotient map (which induces the natural isomorphism $U[-2] \to U[-2h-2]$).

Theorem 5.2.0.7 (Cup product). *1. The multiplication by $\varphi_0(z_0)$ induces the natural isomorphisms*
$\varphi_i : HH^i(A) \to HH^{i+6}(A)$ $\forall i \geq 1$ and the natural quotient map φ_0. Therefore, it is enough to compute products $HH^i(A) \times HH^j(A) \to HH^{i+j}(A)$ with $0 \leq i \leq j \leq 5$.

2. The $HH^0(A)$-action on $HH^i(A)$.

(a) ($(U[-2])_+$-action).
The action of $(U[-2])_+$ on $U[-2] \subset HH^1(A)$ corresponds to the multiplication

$$\begin{aligned}(U[-2])_+ \times U[-2] &\to U[-2],\\ (u,v) &\mapsto u \cdot v\end{aligned}$$

in $HH^0(A)$, projected on $U[-2] \subset HH^0(A)$.
$(U[-2])_+$ acts on $U^*[-2] = HH^4(A)$ and $U^*[-2] \subset HH^5(A)$ the following

way:

$$\begin{aligned}(U[-2])_+ \times U^*[-2] &\rightarrow U^*[-2], \\ (u,f) &\mapsto u \circ f,\end{aligned}$$

where $(u \circ f)(v) = f(uv)$.

$(U[-2])_+$ *acts by zero on* $R^*[h-2] \subset HH^0(A)$, $HH^2(A)$ *and* $HH^3(A)$.

(b) ($R^*[h-2]$*-action).*

$R^*[h-2]$ *acts by zero on* $HH^*(A)_+$.

3. *(Zero products).*

For all odd i, j, *the cup product* $HH^i(A) \cup HH^j(A)$ *is zero.*

4. *(*$HH^1(A)$*-products).*

(a) *The multiplication*

$$HH^1(A) \times HH^4(A) = U[-2] \times U^*[-2] \rightarrow HH^5(A)$$

is the same one as the restriction of

$$HH^0(A) \times HH^5(A) \rightarrow HH^5(A)$$

on $U[-2] \times U^*[-2]$.

(b) *The multiplication of the subspace* $U[-2]_+ \subset HH^1(A)$ *with* $HH^2(A)$ *is zero.*

(c) *The multiplication by* θ_0 *induces a symmetric isomorphism*

$$\alpha : HH^2(A) = K[-2] \rightarrow K^*[-2] = HH^3(A),$$

given by the matrix $(2n+1)(2-C')^{-1}$, *where* C' *is obtained from the adjacency matrix by changing the sign on the diagonal.*

5. *($HH^2(A)$-products).*

$$\begin{aligned} HH^2(A) \times HH^2(A) &\rightarrow HH^4(A), \\ (a,b) &\mapsto \langle a,b\rangle \zeta_0 \end{aligned}$$

is given by $\langle -,- \rangle = \alpha$ *where* α *is regarded as a symmetric bilinear form.*

$HH^2(A) \times HH^3(A) \rightarrow HH^5(A)$ *is the multiplication*

$$\begin{aligned} K[-2] \times K^*[-2] &\rightarrow HH^5(A), \\ (a,y) &\mapsto y(a)\psi_0. \end{aligned}$$

6. *(Products involving* $U^*[-2]$*).*

   (a) *($(U_-)^*[-2]$-action).*
   $(U_-)^*[-2] \subset HH^i(A)$, $i = 4,5$ *acts by zero on* $HH^j(A)$, $j = 2,3,4,5$.

   (b) *Let us choose a nonzero* $\zeta' \in (U^{top})^*[-2] \in HH^4(A)$, *and* $z' \in U^{top}[-2] \subset HH^0(A)$, *let* $\theta' = \theta_0 z' \in U^{top}[-2] \subset HH^1(A)$ *and* $\psi' = \theta_0\zeta' \in (U^{top})^*[-2] \subset HH^5(A)$.

      i. $HH^2(A) \times HH^4(A) \rightarrow HH^6(A)$. *The multiplication with* $v \in HH^2(A)$ *gives us a map*

$$\begin{aligned} (U^{top})^*[-2] &\rightarrow U^{top}[-2h-2], \\ \zeta' &\mapsto \gamma(v)\varphi_0(z'), \end{aligned}$$

where $\gamma : HH^2(A) \rightarrow \mathbb{C}$ *is a linear function, given in Subsection 5.7.7.*

ii. $HH^2(A) \times HH^5(A) \to HH^7(A)$. *This pairing*

$$K[-2] \times U^*[-2] \to U[-2h-2]$$

is the same as the corresponding pairing

$$HH^2(A) \times HH^4(A) \to HH^6(A).$$

iii. $HH^3(A) \times HH^4(A) \to HH^7(A)$. *The multiplication with* $w \in HH^3(A)$ *gives us a map*

$$\begin{aligned}(U^{top})^*[-2] &\to U^{top}[-2h-2],\\ \zeta' &\mapsto \gamma(\alpha^{-1}(w))\varphi_0(\theta').\end{aligned}$$

iv. $HH^4(A) \times HH^4(A) \to HH^8(A)$ *and* $HH^4(A) \times HH^5(A) \to HH^9(A)$. ζ'^2 *gives us a nonzero* $v \in HH^8(A)$. *Then* $\zeta'\psi' alpha(v) \in HH^9(A)$. $HH^4(A)$ *annihilates* $(U_-)^*[-2] \subset HH^5(A)$.

Comparing this theorem (with the results of the explicit compoutation of the T-case later in this chapter) with the results about the A-case in [12], we get the following:

Corollary 5.2.0.8 (Relation to the A-case). *Let* $\omega_1, \ldots, \omega_n$ *be a basis of* $R^*[h-2] \subset HH^0(A)$. *Then we have*

$$HH^*(\Pi_{T_n}) = HH^*(\Pi_{A_{2n}})[\omega_1, \ldots, \omega_n]/(R^*[h-2]HH^*(\Pi_{A_{2n}})_+). \tag{5.2.0.9}$$

We can write $HH^*(\Pi_{A_{2n}})$ *as a quotient*

$$HH^*(\Pi_{A_{2n}}) = HH^*(\Pi_{T_n})/(R^*[h-2]). \tag{5.2.0.10}$$

5.3 Results about the Calculus

We will introduce for every $m \geq 0$ an isomorphism

$$\mathbb{D} : HH_m(A) \xrightarrow{\sim} HH^{6m+5}(A)[(2m+1)h+2] \qquad (5.3.0.11)$$

which intertwines contraction and cup-product maps.

In Section 5.6, we will introduce basis elements $z_k \in U[-2] \subset HH^0(A)$, $\theta_k \in HH^1(A)$, $f_k \in HH^2(A)$, $h_k \in HH^3(A)$, $\zeta_k \in HH^4(A)$ and $\psi_k \in HH^5(A)$.

For $c_k \in HH^i(A)$, $0 \leq i \leq 5$, we write $c_k^{(s)}$ for the corresponding cocycle in HH^{i+6s}. We write $c_{k,t}$ for a cycle in HH_{j+6t}, $1 \leq j \leq 6$ which equals $\mathbb{D}^{-1}(c_k^{(s)})$.

The map $\alpha : K \to K^*$, given by a matrix M_α, is introduced in Subsection 5.7.3.

We state the results in terms of these bases of $HH^\bullet(A)$ and $HH_\bullet(A)$.

Theorem 5.3.0.12. *The calculus structure is given by tables 5.1, 5.2, 5.3 and the* Connes differential B, *given by*

$$\begin{aligned}
B_{6s}(\psi_{k,s}) &= ((2s+1)h-2-k)\zeta_{k,s}, \\
B_{1+6s} &= 0, \\
B_{2+6s}(h_{k,s}) &= (2s+1)h\alpha^{-1}(h_{k,s}), \\
B_{3+6s} &= 0, \\
B_{4+6s}(\theta_{k,s}) &= ((2s+1)h+2+k)z_{k,s}, \\
B_{5+6s} &= 0.
\end{aligned}$$

a \ b	$\psi_{l,t}$	$\zeta_{l,t}$	$h_{l,t}$	$f_{l,t}$	$\theta_{l,t}$	$z_{l,t}$
$z_k^{(s)}$	$(z_k\psi_l)_{t-s}$	$(z_k\zeta_l)_{t-s}$	$\delta_{k0}h_{l,t-s}$	$\delta_{k0}f_{l,t-s}$	$(z_k\theta_l)_{t-s}$	$(z_k z_l)_{t-s}$
ω_k	0	0	0	0	0	0
$\theta_k^{(s)}$	0	$(z_k\psi_l)_{t-s}$	0	$\delta_{k0}\alpha(f_{l,t-s})$	0	$(z_l\theta_k)_{t-s}$
$f_k^{(s)}$	$\delta_{l,h-3}k\cdot$ $\theta_{h-3,t-s-1}$	$\delta_{l,h-3}k\cdot$ $z_{l,t-s-1}$	$\delta_{kl}\psi_{0,t-s}$	$(M_\alpha)_{kl}\zeta_{0,t-s}$	$\delta_{l0}\alpha(f_{k,t-s})$	$\delta_{l0}f_{k,t-s}$
$h_k^{(s)}$	0	$\delta_{k,n}\delta_{l,h-3}\cdot$ $\theta_{h-3,t-s-1}$	0	$\delta_{kl}\psi_{0,t-s}$	0	$\delta_{l0}h_{k,t-s}$
$\zeta_k^{(s)}$	$\delta_{k,h-3}\delta_{l,h-3}\cdot$ $\alpha(f_{n,t-s-1})$	$\delta_{k,h-3}\delta_{l,h-3}\cdot$ $f_{n,t-s-1}$	$\delta_{k,h-3}\delta_{l,n}\cdot$ $\theta_{k,t-s-1}$	$\delta_{k,h-3}l\cdot$ $z_{k,t-s-1}$	$(z_l\psi_k)_{t-s}$	$(z_l\zeta_k)_{t-s}$
$\psi_k^{(s)}$	0	$\delta_{k,h-3}\delta_{l,h-3}\cdot$ $\alpha(f_{n,t-s-1})$	0	$\delta_{k,h-3}l\cdot$ $\theta_{h-3,t-s-1}$	0	$(z_k\psi_l)_{t-s}$

Table 5.1: contraction map $\iota_a(b)$

$a \backslash b$	$z_l^{(t)}$	ω_l	$\theta_l^{(t)}$	$f_l^{(t)}$	$h_l^{(t)}$	$\zeta_l^{(t)}$	$\psi_l^{(t)}$
$z_k^{(s)}$	0	0	$(k-2sh)\cdot (z_k z_l)^{(s+t)}$	0	$-2\delta_{k0} sh\cdot \alpha^{-1}(h_l^{(s+t)})$	0	$(k-2sh)\cdot (z_k \zeta_l)^{(s+t)}$
ω_k		0	0	0	0	0	0
$\theta_k^{(s)}$			$(l-k+2(s-t)h)\cdot (z_k \theta_l)^{(s+t)}$	$-2(1+th)\cdot \delta_{k0} f_l^{(s+t)}$	$2(-1+(s-t)h)\cdot \delta_{k0} h_l^{(s+t)}$	$-(4+l+2th)\cdot (z_k \zeta_l)^{(s+t)}$	$-(4+k+l+2(t-s)h)\cdot (z_k \psi_l)^{(s+t)}$
$f_k^{(s)}$				0	$-2(1+sh)\cdot \delta_{kl} \zeta_0^{(s+t)}$	0	$-2(k+1)\cdot (1+sh)\cdot \delta_{l,h-3} z_{h-3}^{(s+t+1)}$
$h_k^{(s)}$					$2(s-t)h\cdot (M_\alpha^{-1})_{kl}\cdot \psi_0^{(s+t)}$	$-(h+1+2th)\cdot \delta_{k,\frac{h-3}{2}} \delta_{l,h-3}\cdot z_{h-3}^{(s+t+1)}$	$(2(s-t)h-(h-1))\cdot \delta_{k,\frac{h-3}{2}} \delta_{l,h-3}\cdot \theta_{h-3}^{(s+t+1)}$
$\zeta_k^{(s)}$						0	$-(2sh+h+1)\cdot \delta_{k,h-3} \delta_{l,h-3}\cdot f_{\frac{h-3}{2}}^{(s+t+1)}$
$\psi_k^{(s)}$							$2(s-t)h\cdot \delta_{k,h-3} \delta_{l,h-3}\cdot \alpha(f_{\frac{h-3}{2}}^{(s+t+1)})$

Table 5.2: Gerstenhaber bracket $[a, b]$

$a \backslash b$	$\psi_{l,t}$	$\zeta_{l,t}$	$h_{l,t}$	$f_{l,t}$	$\theta_{l,t}$	$z_{l,t}$
$\theta_k^{(s)}$	$((2t+1)h-2-l)\cdot (z_k\psi_l)_{t-s}$	$((2(t-s)+1)h -2-l+k) (z_k\zeta_l)_{t-s}$	$(2t+1)h\cdot \delta_{k0}h_{l,t-s}$	$(2(t-s)+1)h\cdot \delta_{k0}f_{l,t-s}$	$((2t+1)h+2+l)\cdot (z_k\theta_l)_{t-s}$	$((2(t-s)+1)h+2+k+l)\cdot (z_kz_l)_{t-s}$
$f_k^{(s)}$	$-2k(1+sh)\cdot \delta_{l,h-3}z_{h-3,t-s-1}$	0	$-2(1+sh)\cdot \delta_{kl}\zeta_{0,t-s}$	0	$-2(1+sh)\cdot \delta_{l0}f_{k,t-s}$	0
$h_k^{(s)}$	$(2th+1)\cdot \delta_{k,n}\delta_{l,h-3}\cdot \theta_{h-3,t-s-1}$	$(2(t-s)h-1)\cdot \delta_{k,n}\delta_{l,h-3}\cdot z_{h-3,t-s-1}$	$(2t+1)h\cdot (M_\alpha^{-1})_{lk}\cdot \psi_{0,t-s}$	$(2(t-s+1)h-2)\cdot \delta_{kl}\zeta_{0,t-s}$	$((2t+1)h+2)\cdot \delta_{l0}h_{k,t-s}$	$\delta_{l0}(2(t-s)+1)h\cdot \alpha^{-1}(h_{k,t-s})$
$\zeta_k^{(s)}$	$-((2s+1)h+1)\cdot \delta_{k,h-3}\delta_{l,h-3}\cdot f_{n,t-s-1}$	0	$-((2s+1)h+1)\cdot \delta_{k,h-3}\delta_{l,n}\cdot z_{h-3,t-s-1}$	0	$-(2sh+4+k)\cdot (z_l\zeta_k)_{t-s}$	0
$\psi_k^{(s)}$	$(2th+1)\cdot \delta_{k,h-3}\delta_{l,h-3}\cdot \alpha(f_{n,t-s-1})$	$(2(t-s)-1)h\cdot \delta_{k,h-3}\delta_{l,h-3}\cdot f_{n,t-s-1}$	$(2t+1)h\cdot \delta_{k,h-3}\delta_{l,n}\cdot \theta_{h-3,t-s-1}$	$l\cdot (2(t-s)h -1)\cdot \delta_{k,h-3} z_{h-3,t-s-1}$	$((2t+1)h +2+l)\cdot (z_l\psi_k)_{t-s}$	$((2(t-s)+1)h -2-k+l)\cdot (z_l\zeta_k)_{t-s}$
$z_k^{(s)}$	$(k-2sh)\cdot (z_k\zeta_l)_{t-s}$	0	$(k-2sh)\cdot \alpha^{-1}(h_{l,t-s})$	0	$(k-2sh) (z_k\theta_l)_{t-s}$	0

Table 5.3: Lie derivative $\mathcal{L}_a(b)$

5.4 Properties of A

5.4.1 Labeling

We choose a labeling of the quiver T_n.

Figure 5-1: T_n-quiver

5.4.2 Bases and Hilbert series

From [20], we know that $H_A(t) = (1+t^h)(1-Ct+t^2)^{-1}$, where C is the adjacency matrix of the underlying graph. By choosing the labeling of the quiver above, we get

$$(\dim e_i A e_j)_{i,j\in I} = H_A(1) = 2 \cdot \begin{bmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & \cdots & n-1 & n-1 \\ 1 & 2 & 3 & \cdots & \cdots & n-1 & n \end{bmatrix}. \tag{5.4.2.1}$$

We will work with explicit bases $\mathcal{B}_i$ of e_iAe_i. The i^{th} diagonal entry of $H_A(t)$ is $\sum_{j=0}^{i-1} t^{2j} + t^{2n-1-2j}$, and since in A all paths starting and ending in the same vertex with the same length are equivalent, we can say that bases of $\mathcal{B}_i$ are given by paths of length $2j$, $0 \le j \le i-1$ and $2n-1-2j$, $0 \le j \le i-1$ (one of each length). We call $c_{i,k}$ to be a nonzero element in A, represented by a path of length k, starting and ending at i.

5.4.3 The trace function

For the T-quiver, the Nakayama automorphism is trivial. The bilinear form $(-,-)$ which comes with our Frobenius algebra A is given by a trace function $\mathrm{Tr} : A \to \mathbb{C}$ of degree $-(2n-1)$ by
$(x,y) = \mathrm{Tr}(xy)$. We work with an explicit trace function which maps a polynomial of degree $2n-1$ to the sum of its coefficients. Given the basis $(e_i)_{i\in I}$, we denote $(\omega_i)_{i\in I}$ as its dual basis, i.e. ω_i is the monomial of top degree in e_iAe_i with coefficient 1.

5.4.4 The quotient $A/[A,A]$

The quotient $A/[A,A]$ turns out to be different than in the ADE-case.

Proposition 5.4.4.1. *The quotient is*

$$A/[A,A] = R \oplus \langle b^i \,|\, i\, odd \rangle.$$

Proof. The commutator $[A,A]$ is the linear span of

- paths $p_{ki} = [p_{ki}, e_i]$ from i to k, $i \neq k$ and
- $p_{ii} - p_{jj} = [p_{ij}, p_{ji}]$, where $p_{kl} \in e_kAe_l$, i.e. all differences $p_{ii} - p_{jj}$ where p_{ii} and p_{jj} have same degree > 0. Since all paths in e_1Ae_1 of degree > 0 give us a zero element in A, this gives us that
  $e_iAe_i(d) \subset [A,A]\ \forall i \in I$ and even $d > 0$.

From above, we get $e_iAe_k \subset [A,A]$. Since all paths in e_1Ae_1 of even degree > 0 give us a zero element in A and $p_{ii} - p_{jj} \in [A,A]$ for any pair of paths of same degree, this implies that $e_iAe_i(d) \subset [A,A]\ \forall i \in I$ and for all even $d > 0$.

$R \cap [A,A] = 0$ since R is a commutative ring.

So the quotient $A/[A,A]$ is spanned only by R and by odd degree paths $p_{ii} \in e_iAe_i$, and the only relations involving those is $p_{ii} = p_{jj}$. Let p_{ii} have odd degree $d \leq 2n-1$, then $(a_i^*a_i)^{\frac{2n-1-d}{2}} p_{ii} = \omega_i$, so $p_{ii} \neq 0$ in A. So if take one path $p_{ii} \in e_iAe_i$ (for some $i \in I$) in each odd degree $\leq 2n-1$, we get a basis in $A/([A,A]+R)$. Specifically we can choose odd powers of b as a basis. □

5.5 Hochschild and cyclic (co)homology of A

In this section, we prove Theorems 5.2.0.4 and 5.2.0.5: we construct a projective resolution of A, prove duality theorems and compute the Hochschild and cyclic cohomology/homology spaces.

5.5.1 A periodic projective resolution of A

Let ϕ be an automorphism of A, such that $\phi(a) = a \,\forall a \in \bar{Q}'$ and $\phi(b) = -b$. Note that $\phi = -\mathrm{Id}$ on A^{top}. Define the $A-$bimodule ${}_1A_\phi$ obtained from A by twisting the right action by ϕ, i.e. ${}_1A_\phi = A$ as a vector space, and $\forall x, z \in A, y \in {}_1A_\phi : x \cdot y \cdot z = xy\phi(z)$. Introduce the notation $\epsilon_a = 1$ if $a \in Q'$, $\epsilon_a = -1$ if $a \in (Q')^*$, and let $\epsilon_b = 1$. Let x_i be a homogeneous basis $\mathcal{B}$ of A and x_i^* the dual basis under the form attached to the Frobenius algebra A. Let V be the bimodule spanned by the edges of $\bar{Q}$.

We start with the following complex:

$$S_\bullet : \quad 0 \to {}_1A_\phi[h] \xrightarrow{i} A \otimes_R A[2] \xrightarrow{d_2} A \otimes_R V \otimes_R A \xrightarrow{d_1} A \otimes_R A \xrightarrow{d_0} A \to 0,$$

where

$$\begin{aligned}
d_0(x \otimes y) &= xy, \\
d_1(x \otimes v \otimes y) &= xv \otimes y - x \otimes vy, \\
d_2(z \otimes t) &= \sum_{i=1}^{n} \epsilon_{a_i} z a_i \otimes a_i^* \otimes t + \sum_{i=1}^{n} \epsilon_{a_i} z \otimes a_i \otimes a_i^* t \\
&\quad - zb \otimes b \otimes t - z \otimes b \otimes bt, \\
i(x) &= x \sum_{x_i \in \mathcal{B}} \phi(x_i) \otimes x_i^*.
\end{aligned}$$

$d_i d_{i+1} = 0$ for $i = 0, 1, 2$ is obvious. We show $d_2 i = 0$: We have

$$\begin{aligned} d_2(i(1)) &= d_2(\sum_{x_i \in \mathcal{B}} \phi(x_i) \otimes x_i^*) \\ &= \sum_{x_i \in \mathcal{B}} \sum_{j=1}^{n} \epsilon_{a_j} \phi(x_i) a_j \otimes a_j^* \otimes x_i^* + \sum_{x_i \in \mathcal{B}} \sum_{j=1}^{n} \epsilon_{a_j} \phi(x_i) \otimes a_j \otimes a_j^* x_i^* \\ &\quad - \sum_{x_i \in \mathcal{B}} \phi(x_i) b \otimes b \otimes x_i^* - \sum_{x_i \in \mathcal{B}} \phi(x_i) \otimes b \otimes b x_i^*. \end{aligned}$$

The first two terms cancel since

$$\begin{aligned} \forall a \in \bar{Q}' : \quad \sum_{x_i \in \mathcal{B}} \phi(x_i) a \otimes a^* \otimes x_i^* &= \sum_{x_i, x_j \in \mathcal{B}} (\phi(x_i) a, -\phi(x_j^*)) \phi(x_j) \otimes a^* \otimes x_i^* \\ &= \sum_{x_i, x_j \in \mathcal{B}} \phi(x_j) \otimes a^* \otimes (a x_j^*, x_i) x_i^* \\ &= \sum_{x_i \in \mathcal{B}} \phi(x_i) \otimes a^* \otimes a x_i^*. \end{aligned}$$

The last two terms cancel since

$$\begin{aligned} \sum_{x_i \in \mathcal{B}} \phi(x_i) b \otimes b \otimes x_i^* &= \sum_{x_i, x_j \in \mathcal{B}} (\phi(x_i) b, -\phi(x_j^*)) \phi(x_j) \otimes b \otimes x_i^* \\ &= \sum_{x_i, x_j \in \mathcal{B}} \phi(x_j) \otimes b \otimes (-b x_j^*, x_i) x_i^* \\ &= -\sum_{x_i \in \mathcal{B}} \phi(x_i) \otimes b \otimes b x_i^*. \end{aligned}$$

Lemma 5.5.1.1. *$S_\bullet$ is self dual.*

Proof. We introduce the nondegenerate forms

- $(x, y)_\phi = \mathrm{Tr}(x\phi(y))$ on A,
- $(x \otimes x', y \otimes y')_\phi = \mathrm{Tr}(x\phi(y'))\mathrm{Tr}(x'y)$ on $A \otimes_R A$ and
- $(x \otimes \alpha \otimes x', y \otimes \beta \otimes y')_\phi = \mathrm{Tr}(x\phi(y'))\mathrm{Tr}(x'y)(\alpha, \beta)$ on $A \otimes_R V \otimes_R A$, where we define the form on V to be $(\alpha, \beta) = \delta_{\alpha^*, \beta} \epsilon_\beta$.

We apply the functor $(-)^\star = \mathrm{Hom}_{\mathbb{C}}(-,\mathbb{C})$ and make the identifications $A^\star \simeq A$, $(A\otimes_R A)^\star \simeq A\otimes_R A$ and $(A\otimes_R V\otimes_R A)^\star \simeq A\otimes_R V\otimes_R A$ by the map $x\mapsto (-,x)_\phi$.

We have

$$\begin{aligned}(i(x), y\otimes z)_\phi &= (\sum_{x_i\in\mathcal{B}} x\phi(x_i)\otimes x_i^*, y\otimes z)_\phi = \sum_{x_i\in\mathcal{B}} \mathrm{Tr}(x\phi(x_i)\phi(z))\mathrm{Tr}(x_i^* y)\\ &= -\sum_{x_i\in\mathcal{B}} \mathrm{Tr}(\phi(x)x_i z)\mathrm{Tr}(x_i^* y) = -\sum_{x_i\in\mathcal{B}} \mathrm{Tr}(z\phi(x)x_i)\mathrm{Tr}(x_i^* y)\\ &= -\mathrm{Tr}(z\phi(x)y) = \mathrm{Tr}(x\phi(yz)) = (x, yz)_\phi\\ &= (x, d_0(y\otimes z)),\end{aligned}$$

so $i = d_0^\star$.

We have

$$\begin{aligned}(x\otimes v\otimes y, d_2(z\otimes t))_\phi = (x\otimes v\otimes y, &\sum_{i=1}^n \epsilon_{a_i} z a_i\otimes a_i^*\otimes t + \sum_{i=1}^n \epsilon_{a_i} z\otimes a_i\otimes a_i^* t\\ &- zb\otimes b\otimes t - z\otimes b\otimes bt)_\phi,\end{aligned}$$

which gives us $\forall a\in\bar{Q}'$

$$(x\otimes a\otimes y, d_2(z\otimes t))_\phi = -\mathrm{Tr}(x\phi(t))\mathrm{Tr}(yza) + \mathrm{Tr}(x\phi(at))\mathrm{Tr}(yz)$$

and

$$(x\otimes b\otimes y, d_2(z\otimes t))_\phi = -\mathrm{Tr}(x\phi(t))\mathrm{Tr}(yzb) - \mathrm{Tr}(x\phi(bt))\mathrm{Tr}(yz),$$

i.e. for $v\in V$,

$$\begin{aligned}(x\otimes v\otimes y, d_2(z\otimes t))_\phi &= -\mathrm{Tr}(x\phi(t))\mathrm{Tr}(yzv) + \mathrm{Tr}(x\phi(vt))\mathrm{Tr}(yz)\\ &= \mathrm{Tr}(xv\phi(t))\mathrm{Tr}(yz) - \mathrm{Tr}(x\phi(t))\mathrm{Tr}(vyz)\\ &= (xv\otimes y - x\otimes vy, z\otimes t)_\phi = (d_1(x\otimes v\otimes y), z\otimes y)_\phi,\end{aligned}$$

so $d_2 = d_1^\star$. □

Proposition 5.5.1.2. *$S_\bullet$ is an exact sequence.*

Proof. We recall the definition of Anick's resolution [1]. Denote $T_R W$ to be the tensor algebra of a graded R-bimodule W, $T_R^+ W$ its augmentation ideal. Let $L \subset T_R^+ W$ be an R-graded bimodule and $A' = T_R W/(L)$. Then we have the following resolution:

$$A' \otimes_R L \otimes_R A' \xrightarrow{\partial} A' \otimes_R W \otimes_R A' \xrightarrow{f} A' \otimes_R A' \xrightarrow{m} A' \to 0, \qquad (5.5.1.3)$$

where m is the multiplication map, f is given by

$$f(a_1' \otimes w \otimes a_2') = a_1' w \otimes a_2' - a_1' \otimes w a_2'$$

and ∂ is given by

$$\partial(a_1' \otimes l \otimes a_2') = a_1' \cdot D(l) \cdot a_2',$$

$$\begin{aligned} D : T_R^+ W &\to A' \otimes_R W \otimes_R A', \\ w_1 \otimes \ldots \otimes w_n &\mapsto \sum_{p=1}^{n} \overline{(w_1 \otimes \ldots \otimes w_{p-1})} \otimes w_p \otimes \overline{(w_{p+1} \otimes \ldots \otimes w_n)}, \end{aligned}$$

where bar stands for the image in B of the projection map.

In our setting, $W = V$, L the R-bimodule generated by $\sum_{i=1}^{n} \epsilon_{a_i} a_i a_i^* - b^2$. Then $A' = A$.

It is also clear that Im(∂)=Im(d_2) $\subset A \otimes_R V \otimes_R A$, so from Anick's resolution we know that the part

$$A \otimes_R A[2] \xrightarrow{d_2} A \otimes_R V \otimes_R A \xrightarrow{d_1} A \otimes_R A \xrightarrow{d_0} A \to 0$$

is exact. Exactness of the whole complex $S_\bullet$ follows from its self duality. □

Since $\phi^2 = 1$, we can make a canonical identification $A = {}_1A_\phi \otimes_A {}_1A_\phi$ (via $x \mapsto x \otimes 1$), so by tensoring $S_\bullet$ with ${}_1A_\phi$, we obtain the exact sequence

$$0 \to A[2h] \xrightarrow{i'} A \otimes_R {}_1A_\phi[h+2] \xrightarrow{d_5} A \otimes_R V \otimes_R {}_1A_\phi[h] \xrightarrow{d_4} A \otimes_R {}_1A_\phi[h] \xrightarrow{j} {}_1A_\phi[h] \to 0.$$

By connecting this sequence to $S_\bullet$ with $d_3 = ij$ and repeating this process, we obtain the periodic Schofield resolution with period 6:

$$\begin{aligned} \ldots \to A \otimes A[2h] \xrightarrow{d_6} A \otimes_R {}_1A_\phi[h+2] \xrightarrow{d_5} A \otimes_R V \otimes_R {}_1A_\phi[h] \xrightarrow{d_4} A \otimes_R \mathcal{N}[h] \\ \xrightarrow{d_3} A \otimes_R A[2] \xrightarrow{d_2} A \otimes_R V \otimes_R A \xrightarrow{d_1} A \otimes_R A \xrightarrow{d_0} A \to 0. \end{aligned}$$

This implies that the Hochschild homology and cohomology of A is periodic with period 6, in the sense that the shift of the (co)homological degree by 6 results in the shift of degree by $2h$ (respectively $-2h$).

From that we get the periodicities for the Hochschild homology/cohomology

$$HH_{j+6i}(A) \cong HH_j(A)[2ih], \qquad HH^{j+6i}(A) \cong HH^j(A)[-2ih], \qquad \forall j \geq 1. \tag{5.5.1.4}$$

5.5.2 Calabi-Yau Frobenius algebras

Let us define the functor

$$\begin{aligned} \mathrm{Hom}_{A^e}(-, A \otimes_{\mathbb{C}} A) : A^e - \mathrm{mod} &\to A^e - \mathrm{mod}, \\ M &\mapsto M^\vee. \end{aligned}$$

We recall the definition of the Calabi-Yau algebras from [17].

Definition 5.5.2.1. *A Frobenius algebra A is called* ***Calabi-Yau Frobenius of dimension*** m *if*

$$A^\vee \simeq \Omega^{m+1} A \tag{5.5.2.2}$$

If there is more than one such m, then we pick the smallest one.

If additionally A has a grading, such that the above isomorphism is a graded isomorphism when composed with some shift, then we say that A is a ***graded Calabi-Yau Frobenius algebra****. More precisely, if $A^\vee[m'] \simeq \Omega^{m+1}A$ is a graded isomorphism, where $[l]$ is the shift by l with the new grading, then one says that A is* ***graded Calabi-Yau Frobenius with dimension** m **of shift** m'.*

Proposition 5.5.2.3. *A is Calabi-Yau Frobenius with dimension 5 of shift $h+2$, i.e.*

$$A^\vee[h+2] \simeq \Omega^6 A. \tag{5.5.2.4}$$

Proof. This follows from [17] since A is symmetric and periodic with period 6. □

From [17], we can deduce the dualities

$$\begin{aligned} HH_i(A) &\cong HH_{5-i}(A)^*[2h], && (5.5.2.5)\\ HH^i(A) &\cong HH_{5-i}(A)[-h-2], && (5.5.2.6)\\ HH^i(A) &\cong HH^{11-i}(A)^*[-2h-4] \cong HH^{5-i}(A)^*[-4]. && (5.5.2.7) \end{aligned}$$

5.5.3 Hochschild homology of A

Let A^{op} be the algebra A with opposite multiplication. We define $A^e = A \otimes_R A^{op}$. Then any $A-$bimodule naturally becomes a left A^e- module (and vice versa).

Now, we apply to the Schofield resolution the functor $- \otimes_{A_e} A$ to get the Hochschild homology complex

$$\begin{aligned} \ldots \to A^R[2h] \xrightarrow{d_6'} {}_1A_\phi^R[h+2] \xrightarrow{d_5'} (V \otimes_R {}_1A_\phi)^R[h] \xrightarrow{d_4'} \\ \xrightarrow{d_4'} {}_1A_\phi^R[h] \xrightarrow{d_3'} A^R[2] \xrightarrow{d_2'} (V \otimes_R A)^R \xrightarrow{d_1'} A^R \to 0. \end{aligned}$$

Let $\overline{HH_i(A)}$ be HH_0/R for $i=0$ and $HH_i(A)$ otherwise. We have the Connes exact sequence

$$0 \to \overline{HH_0}(A) \overset{B_0}{\to} \overline{HH_1}(A) \overset{B_1}{\to} \overline{HH_2}(A) \overset{B_2}{\to} \overline{HH_3}(A) \overset{B_3}{\to} \overline{HH_4}(A) \to \dots, \qquad (5.5.3.1)$$

where the B_i are the Connes differentials (see [19, 2.1.7.]) and the B_i are all degree-preserving.

In our case, $\overline{HH_0}(A) = A/([A, A] + R) = U^*[h]$ (see Proposition 5.4.4.1), where $U^*[h] = \langle b^i | i \,\text{odd}\rangle$. From (5.5.3.1), we know that $U^*[h] \subset \overline{HH_1(A)}$. Denote $X = \overline{HH_1(A)}/U^*[h]$. Since $\deg HH_2(A) \leq 2h$, $HH_3(A) = HH_2(A)^*[2h]$ and the Connes differential maps $HH_2(A)/X$ isomorphically to its image in $HH_3(A)$, $HH_2(A)/X$ sits in degree h. We call this space $K^*[h]$, where K^* sits in degree 0. $HH_3(A) = X^*[2h] \oplus K[h]$ and $HH_4(A) = U[2h] \oplus X^*[2h]$ follow from the duality (5.5.2.5). The Connes differential maps $HH_5(A)/U[2h]$ isomrphically into its image in $HH_6(A)$. Since $\deg HH_5(A) \leq 2h$ and $HH_6(A) = HH_5(A)^*[4h]$ (5.5.2.5), $HH_5(A)/U[2h]$ sits in degree $2h$. We call that space $Y[2h]$ where Y sits in degree 0.

From our discussion, we get the homology spaces

$$\begin{aligned}
HH_0(A) &= U^*[h] \oplus R, \\
HH_1(A) &= U^*[h] \oplus X, \\
HH_2(A) &= K^*[h] \oplus X, \\
HH_3(A) &= K[h] \oplus X^*[2h], \\
HH_4(A) &= U[h] \oplus X^*[2h], \\
HH_5(A) &= U[h] \oplus Y[3h], \\
HH_6(A) &= U^*[3h] \oplus Y^*[3h], \\
HH_{6k+i}(A) &= HH_i(A)[2kh] \quad \forall i \geq 1.
\end{aligned}$$

5.5.4 Hochschild cohomology of A

We make the identifications
$\mathrm{Hom}_{A^e}(A \otimes_R A, A) = A^R = \mathrm{Hom}_{A^e}(A \otimes_R {}_1A_\phi, A)$ by identifying φ with the image $\varphi(1 \otimes 1) = a$ (we write $\varphi = a \circ -$) and
$\mathrm{Hom}_{A^e}(A \otimes_R V \otimes_R A, A) = (V \otimes_R A)^R[-2] = \mathrm{Hom}_{A^e}(A \otimes_R V \otimes_R {}_1A_\phi, A)$ by identifying φ which maps $1 \otimes a \otimes 1 \mapsto x_a$ $(a \in \bar{Q})$ with the element $\sum_{i=1}^{n} \epsilon_{a_i^*} a_i^* \otimes x_{a_i}$ (we write $\varphi = (\sum_{i=1}^{n} \epsilon_{a_i^*} a_i^* \otimes x_{a_i} + b \otimes x_b) \circ -)$.

Now, apply the functor $\mathrm{Hom}_{A^e}(-, A)$ to the Schofield resolution to obtain the Hochschild cohomology complex

$$\overset{d_4^*}{\longleftarrow} A^R[-h] \overset{d_3^*}{\longleftarrow} A^R[-2] \overset{d_2^*}{\longleftarrow} (V \otimes A)^R[-2] \overset{d_1^*}{\longleftarrow} A^R \leftarrow 0$$
$$\ldots \leftarrow A^R[-2h] \overset{d_6^*}{\longleftarrow} A^R[-h-2] \overset{d_5^*}{\longleftarrow} (V \otimes A)^R[-h-2] \overset{d_4^*}{\longleftarrow}$$

and compute the differentials. We have

$$d_1^*(x)(1 \otimes y \otimes 1) = x \circ d_1(1 \otimes y \otimes 1) = x \circ (y \otimes 1 - 1 \otimes y) = [y, x],$$

so

$$d_1^*(x) = \sum_{i=1}^{n} \epsilon_{a_i^*} a_i^* \otimes [a_i, x] + b \otimes [b, x].$$

We have

$$\begin{aligned} d_2^*(\sum_{i=1}^{n} a_i \otimes x_{a_i} + b \otimes x_b)(1 \otimes 1) &= (\sum_{i=1}^{n} a_i \otimes x_{a_i} + b \otimes x_b) \circ (\sum_{j=1}^{n} \epsilon_{a_j} a_j \otimes a_j^* \otimes 1 \\ &\quad + \sum_{j=1}^{n} \epsilon_{a_j} 1 \otimes a_j \otimes a_j^* - b \otimes b \otimes 1 - 1 \otimes b \otimes b) \\ &= \sum_{i=1}^{n} (a_i x_{a_i} - x_{a_i} a_i) - (b x_b + x_b b) \\ &= \sum_{i=1} [a_i, x_{a_i}] - (b x_b + x_b b), \end{aligned}$$

so

$$d_2^*(\sum_{i=1}^{n} a_i \otimes x_{a_i} + b \otimes x_b) = \sum_{i=1}^{n} [a_i, x_{a_i}] - (bx_b + x_b b).$$

We have

$$d_3^*(x)(1 \otimes 1) = x \circ d_3(1 \otimes 1) = x \circ (\sum_{x_i \in \mathcal{B}} \phi(x_i) \otimes x_i^*) = \sum_{x_i \in \mathcal{B}} \phi(x_i) x x_i^* = 0,$$

so

$$d_3^*(x) = \sum_{x_i \in \mathcal{B}} \phi(x_i) x x_i^*,$$

and we evaluate this sum: let $\rho = \sum_{i=1}^{n} (-1)^{n+i} e_i$, then $\rho^2 = \sum_{i=1}^{n} e_i = 1$, and for all monomials $x \in A$, $\rho\phi(x)\rho = (-1)^{\deg(x)} x$. We write $x = \rho y$ (where $y = \rho x$), then

$$d_3^*(x) = \sum_{x_i \in \mathcal{B}} \phi(x_i) x x_i^* = \rho \sum_{x_i \in \mathcal{B}} (\rho\phi(x_i)\rho) y x_i^* = \rho \sum_{x_i \in \mathcal{B}} (-1)^{\deg(x_i)} x_i y x_i^*.$$

The map $y \mapsto \sum_{x_i \in \mathcal{B}} (-1)^{\deg(x_i)} x_i y x_i^*$ is zero in positive degree, and the restriction to $\deg y = 0$ is a map $\bigoplus_{i \in I} \mathbb{C} e_i \to \bigoplus_{i \in I} \mathbb{C} \omega_i$, given by the matrix $H_A(-1) = (1 + (-1)^h)(2 + C) = 0$, since the Coxeter number $h = 2n + 1$ is odd. This implies that

$$d_3^* = 0.$$

We have

$$d_4^*(x)(1 \otimes a_i \otimes 1) = x \circ d_1(1 \otimes a_i \otimes 1) = x \circ (a_i \otimes 1 - 1 \otimes a_i) = a_i x - x\phi(a_i) = [a, x],$$

and

$$d_4^*(x)(1 \otimes b \otimes 1) = x \circ d_1(1 \otimes b \otimes 1) = x \circ (b \otimes 1 - 1 \otimes b) = bx - x\phi(b),$$

so

$$d_4^*(x) = \sum_{i=1}^{n} \epsilon_{a_i^*} a_i^* \otimes [a_i, x] + b \otimes (xb + bx),$$

We have

$$d_5^*(\sum_{i=1}^{n} a_i \otimes x_{a_i} + b \otimes x_b)(1 \otimes 1) = (\sum_{i=1}^{n} a_i \otimes x_{a_i} + b \otimes x_b) \circ (\sum_{i=1}^{n} \epsilon_{a_i} a_i \otimes a_i^* \otimes 1 + \sum_{i=1}^{n} \epsilon_{a_i} 1 \otimes a_i \otimes a_i^*)$$
$$= \sum_{i=1}^{n} (a_i x_{a_i} - x_{a_i}\phi(a_i)) + (b x_b + x_b \phi(b)),$$

so

$$d_5^*(x, y) = [x, y].$$

We have

$$d_6^*(x)(1 \otimes 1) = x \circ d_6(1 \otimes 1) = x \circ (\sum_{x_i \in \mathcal{B}} \phi(x_i) \otimes x_i^*) = \sum_{x_i \in \mathcal{B}} \phi(x_i) x \phi(x_i^*) = -\sum_{x_i \in \mathcal{B}} x_i x x_i^*,$$

so

$$d_6^*(x) = -\sum_{x_i \in \mathcal{B}} x_i x x_i^*.$$

From our results about Hochschild homology and the dualities (5.5.2.6), we obtain the following spaces for the Hochschild cohomology (for $HH^0(A)$, keep in mind that we get $HH^6(A) = HH^0(A)[-2h]/\mathrm{Im} d_6^*$, and the image of d_6^* lies in top degree). The cohomology spaces are

$$\begin{aligned}
HH^0(A) &= U[-2] \oplus L[h-2], \\
HH^1(A) &= U[-2] \oplus X^*[h-2], \\
HH^2(A) &= K[-2] \oplus X^*[h-2], \\
HH^3(A) &= K^*[-2] \oplus X[-h-2], \\
HH^4(A) &= U^*[-2] \oplus X[-h-2], \\
HH^5(A) &= U^*[-2] \oplus Y^*[-h-2], \\
HH^6(A) &= U[-2h-2] \oplus Y[-h-2], \\
HH^{6k+i}(A) &= HH^i(A)[-2kh] \quad \forall i \geq 1.
\end{aligned}$$

We have $L[h-2] = R^*[h-2]$. Since there is no non-top degree element in A which commutes with all $a \in \bar{Q}'$ and anticommutes with b, $\ker d_4^*$ lies in top degree -2 which implies that the space X has to be zero.

From the discussion in Subsection 5.5.3, we know that K is a degree-zero space, so $HH^2(A)$ sits entirely in degree -2. Since $d_3^* = 0$ and the image of d_4^* lies in degree > -2, $K = \bigoplus_{i\in I} \mathbb{C}e_i$, so K is n-dimensional. This proves Theorem 5.2.0.2 (b).

The map d_6^* can be viewed as a map $\bigoplus_{i\in I} \mathbb{C}e_i \to \bigoplus_{i\in I} \mathbb{C}\omega_i$, given by the matrix $-H_A(1) = -2(2-C)^{-1}$. Since it is nondegenerate, the space Y is also zero.

Theorems 5.2.0.4 and 5.2.0.5 follow.

5.5.5 Cyclic homology of A

The *Connes exact sequence* and *reduced homology* were defined in Subsection (2.2.4).

We write down Connes exact sequence, together with the Hochschild homology spaces:

$$\begin{array}{ccccl}
0 & & & & \\
\downarrow & & & & \\
\overline{HH_0}(A) & = & U^*[h] & & \\
{\scriptstyle B_1}\downarrow & & \sim\downarrow & & \\
\overline{HH_1}(A) & = & U^*[h] & & \overline{HC_0}(A) = U^*[h] \\
{\scriptstyle B_1}\downarrow & & 0\downarrow & & \\
\overline{HH_2}(A) & = & K^*[h] & & HC_1(A) = 0 \\
{\scriptstyle B_2}\downarrow & & \sim\downarrow & & \\
\overline{HH_3}(A) & = & K[h] & & HC_2(A) = K^*[h] \\
{\scriptstyle B_3}\downarrow & & 0\downarrow & & \\
\overline{HH_4}(A) & = & U[h] & & HC_3(A) = 0 \\
{\scriptstyle B_4}\downarrow & & \sim\downarrow & & \\
\overline{HH_5}(A) & = & U[h] & & HC_4(A) = U[h] \\
{\scriptstyle B_5}\downarrow & & 0\downarrow & & \\
\overline{HH_6}(A) & = & U^*[3h] & & HC_5(A) = 0 \\
{\scriptstyle B_6}\downarrow & & 0\downarrow & & \\
\overline{HH_7}(A) & = & U^*[3h] & & HC_6(A) = U^*[3h] \\
{\scriptstyle B_7}\downarrow & & & & \\
\vdots & & & &
\end{array}$$

This proves Theorem 5.2.0.6.

5.6 Basis of $HH^*(A)$

Now we construct a basis of $HH^*(A)$.

5.6.1 $HH^0(A) = Z$

We compute the structure of the center.

Proposition 5.6.1.1. *The non-topdegree central elements lie in even degrees, one in each degree (up to scaling). They are given by*

$$z_{2k} = \sum_{i=k+1}^{n} c_{i,2k}, \qquad 0 \leq k \leq \frac{h-3}{2} \tag{5.6.1.2}$$

Proof. First we prove that a degree $2k$-element z is a multiple of z_{2k}: z commutes with all e_i, hence lies in $\bigoplus_{i \in A} e_i A e_i$. From the discussion, in Subsection (5.4.2), we can write

$$z = \sum_{i=k+1}^{n} \lambda_i c_{i,2k} = \sum_{i=k+1}^{n} \lambda_i (a_i^* a_i)^k.$$

Now,

$$\forall j \geq k+1, \quad \lambda_j \underbrace{a_j (a_j^* a_j)^k}_{\neq 0} = a_j z = z a_j = \lambda_{j+1} (a_j a_j^*)^k a_j$$

which implies that all λ_i are equal. So each even degree central is a multiple of z_{2k}.

Since $z_{2k}^* = z_{2k}$ and z_{2k} commutes with all a_j, z_{2k} also commutes with all a_j^*. Commutativity with b is clear, since each element in $e_n A e_n$ can be expressed as a polynomial in b.

So $z_{2k} = \sum\limits_{i=k+1}^{n} c_{i,2k}$ is the central element in degree $2k$.

Now, let z be of odd degree $< h-2$. From Subsection (5.4.2), we can write

$$z = \sum_{i=n-k}^{n} \lambda_i c_{i,2k+1},$$

where

$$c_{i,2k+1} = a_i^* \cdot \ldots \cdot a_{n-1}^* b^{2i-2n+2k} a_{n-1} \cdot \ldots \cdot a_i$$

and the law $\forall i \geq n-k$,

$$(0 \neq) a_i c_{i,2k+1} = a_{i+1}^* \cdot \ldots \cdot a_{n-1}^* b^{2i-2n+2k+2} a_{n-1} \cdot \ldots \cdot a_i = c_{i+1,2k+1} a_i,$$

so we have

$$\lambda_{n-k} \underbrace{c_{n-k,2k+1} a_{n-k-1}}_{\neq 0} = z a_{n-k-1} = a_{n-k-1} z = 0$$

and

$$\forall j \geq n-k, \quad \lambda_j a_j c_{j,2k+1} = a_j z = z a_j = \lambda_{j+1} c_{i+1,2k+1} a_j = \lambda a_j c_{j,2k+1}.$$

This implies that all $\lambda_j = 0$, so we have no non-top odd degree central elements.

$\square$

Theorem 5.2.0.2 (a) follows.

5.6.2 $HH^1(A)$

Since $HH^1(A) = U[-2]$, we know from the previous subsection that the Hilbert series of $HH^1(A)$ is $\sum_{i=0}^{\frac{h-3}{2}} t^{2i}$. It is easy to see that

$$\theta_{2k} := \sum_{i=1}^{n} a_i \otimes a_i^* z_{2k} - \sum_{i=1}^{n} a_i^* \otimes a_i z_{2k} + b \otimes b z_{2k}, \qquad 0 \leq k \leq \frac{h-3}{2}$$

lie in $\ker d_1^*$. The cup product calculation $HH^1(A) \cup HH^4(A)$ will show that each θ_{2k} is nonzero (since the product with ζ_{2k} is nonzero).

5.6.3 $HH^2(A)$ and $HH^3(A)$

$HH^2(A)$ and $HH^3(A)$ sit in degree -2 and both are n-dimensional. So $HH^2(A)$ is the bottom degree part of $A^R[-2]$ and $HH^3(A)$ the top degree part of $A^R[-h]$. Denote $f_i = [e_i] \in HH^2(A)$ and $h_i = [\omega_i] \in HH^3(A)$, we have

$$HH^2(A) = \bigoplus_{i=1}^{n} \mathbb{C} f_i, \qquad HH^3(A) = \bigoplus_{i=1}^{n} \mathbb{C} h_i.$$

5.6.4 $HH^4(A)$

The Hilbert series of $HH^4(A)$ is $t^{-4}\sum_{i=0}^{\frac{h-3}{2}} t^{-2i}$. We claim that a basis is given by

$$\zeta_{2i} := [-b \otimes b^{h-3-2i}].$$

It is clear that ζ_{2i} all lie in $\ker d_5^*$. Since the image of d_4^* has zero trace, ζ_0 is nonzero in $HH^4(A)$. And $\zeta_{2i} \neq 0$ follows from $z_{2i}\zeta_{2i} = \zeta_0$.

5.6.5 $HH^5(A)$

From Proposition 5.4.4.1, we know that the space
$HH^5(A) = A^R/([A,A]^R + R)[-h-2] = A/([A,A]+R)[-h-2]$ is spanned by

$$\psi_{2i} := [b^{h-3-2i}].$$

5.7 The Hochschild cohomology ring $HH^*(A)$

The degree ranges of the Hochschild cohomology spaces are

$$\begin{aligned}
0 &\leq \deg HH^0(A) &&\leq h-2,\\
0 &\leq \deg HH^1(A) &&\leq h-3,\\
-2 &= \deg HH^2(A), &&\\
-2 &= \deg HH^3(A), &&\\
-h-1 &\leq \deg HH^4(A) &&\leq -4,\\
-h-1 &\leq \deg HH^5(A) &&\leq -4,\\
-2h &\leq \deg HH^6(A) &&\leq -h-3.
\end{aligned}$$

We compute the cup product in terms of our constructed basis in $HH^*(A)$ from the last section.

5.7.1 The Z-module structure of $HH^*(A)$

$HH^0(A)$ is a local ring, with radical generated by z_2. In $HH^0(A)$, we have $z_{2i}z_{2j} = z_{2(i+j)}$ for $2i + 2j \leq h - 3$, and the product is 0 otherwise. $HH^i(A)$ are cyclic Z-modules for $i = 1, 4, 5$, generated by θ_0, ζ_{h-3}, ψ_{h-3} respectively. The Z-modules $HH^2(A)$ and $HH^3(A)$ are annihilated by the radical of Z.

5.7.2 $HH^i(A) \cup HH^j(A)$ for i, j odd

All cup products $HH^i(A)$ with $HH^j(A)$ for i, j odd are zero by degree argument.

5.7.3 $HH^1(A) \cup HH^2(A)$

By degree argument, $\theta_i f_j = 0$ for $i \neq 0$.

Proposition 5.7.3.1. *The multiplication with θ_0 gives us a map*

$$HH^2(A) = K[-2] \xrightarrow{\alpha} K^*[-2] = HH^3(A),$$

given by the matrix

$$h \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 3 \end{bmatrix}^{-1}.$$

Proof. Let $x \in K[-2]$, represented by the map

$$\begin{aligned} f_x : A \otimes A[2] &\longrightarrow A, \\ 1 \otimes 1 &\longmapsto x, \end{aligned}$$

which we lift to

$$\begin{aligned} \hat{f}_x : A \otimes A[2] &\longrightarrow A \otimes A, \\ 1 \otimes 1 &\longmapsto 1 \otimes x. \end{aligned}$$

Then we have

$$\hat{f}_x d_3(1 \otimes 1) = \hat{f}_x(\sum_{x_j \in \mathcal{B}} \phi(x_j) \otimes x_j^*) = \sum_{x_j \in \mathcal{B}} \phi(x_j) \otimes x x_j^*.$$

To compute the lift Ωf_i, we need to find out the preimage of $\sum_{x_j \in \mathcal{B}} \phi(x_j) \otimes x x_j^*$ under d_1.

Definition 5.7.3.2. *Let $b_1, \ldots, b_k$ be arrows, p the monomial $b_1 \cdots b_k$ and define*

$$v_p := (1 \otimes b_1 \otimes b_2 \cdots b_k + b_1 \otimes b_2 \otimes b_3 \cdots b_k + \ldots + b_1 \cdots b_{k-1} \otimes b_k \otimes 1).$$

We will use the following lemma in our computations.

Lemma 5.7.3.3. *In the above setting,*

$$d_1(v_p) = (b_1 \cdots b_k \otimes 1 - 1 \otimes b_1 \cdots b_k).$$

From that, we immediately see that

$$\sum_{x_j \in \mathcal{B}} \phi(x_j) \otimes x x_j^* = d_1(\sum_{x_j \in \mathcal{B}} v_{\phi(x_j)} x x_j^*) + 1 \otimes \underbrace{\sum_{x_j \in \mathcal{B}} \phi(x_j) x x_j^*}_{=0},$$

so we have

$$\begin{aligned} \Omega f_x : \Omega^3(A) &\rightarrow \Omega(A), \\ 1 &\mapsto \sum_{x_j \in \mathcal{B}} v_{\phi(x_j)} x x_j^*. \end{aligned}$$

Then we have

$$\theta_0(\sum_{x_j \in \mathcal{B}} v_{\phi(x_j)} x x_j^*) = \sum_{x_j \in \mathcal{B}} \deg(x_j)\phi(x_j) x x_j^*.$$

So we get

$$\theta_0 f_x = \sum_{x_j \in \mathcal{B}} \deg(x_j)\phi(x_j) x x_j^* = \sum_{k,l=1}^{n} \sum_{x_j \in \mathcal{B}_{kl}} \deg(x_j)\phi(x_j) x x_j^*.$$

$\phi(x_j)$ is x_j if the number of $b's$ in x_j is even and $-x_j$ if it is odd. Observe that the number of $b's$ in x_j and $\deg(x_j) - d(k,l)$ (where $d(k,l)$ is the distance between the vertices k and l) have the same parity. So $\phi(x_j) = (-1)^{\deg(x_j)-d(k,l)} x_j$, and so the multiplication with θ_0 induces a map

$$HH^2(A) = K[-2] \xrightarrow{\alpha} K^*[-2] = HH^3(A), \tag{5.7.3.4}$$

given by the matrix

$$\left(H_A^\phi\right)_{k,l} = \sum_{h_j \in \mathcal{B}_{k,l}} (-1)^{\deg(x_j)-d(k,l)} \deg(x_j) = (-1)^{d(i,j)} \left(\frac{d}{dt} H_A(t)_{k,l}\right)\Bigg|_{t=-1}. \tag{5.7.3.5}$$

Let us define

$$H_A^\delta := \left(\frac{d}{dt} H_A(t)\right)\Bigg|_{t=-1}.$$

Then we have

$$\begin{aligned} H_A^\delta &= \left((1+t^h)\frac{d}{dt}(1 - Ct + t^2)^{-1} + ht^{h-1}(1 - Ct + t^2)^{-1}\right)\Bigg|_{t=-1} \\ &= h(2+C)^{-1}. \end{aligned}$$

For any nondegenerate matrix M, call M_- the matrix obtained from M by changing all signs in the (i,j)-entry whenever $d(i,j)$ is odd. It is easy to see that for matrices $M = N^{-1}$, $M_- = (N_-)^{-1}$. In our case, we have $H_A^\phi = (H_A^\delta)_-$. This implies

$$H_A^\phi = h((2+C)_-)^{-1}. \tag{5.7.3.6}$$

□

5.7.4 $HH^1(A) \cup HH^4(A)$

Since $HH^1(A) = Z\theta_0$ and $HH^4(A) = Z\zeta_{h-3}$, it is enough to compute $\theta_0\zeta_{h-3}$.

Proposition 5.7.4.1. *Given $\theta_0 \in HH^1(A)$ and $\zeta_{h-3} \in HH^4(A)$, we get the cup product*

$$\theta_0\zeta_{h-3} = \psi_{h-3}. \tag{5.7.4.2}$$

Proof. ζ_{h-3} represents the map

$$\begin{aligned}
\zeta_{h-3} : A \otimes V \otimes {}_1A_\phi[h] &\rightarrow A, \\
1 \otimes b \otimes 1 &\mapsto -e_n, \\
1 \otimes a_i \otimes 1 &\mapsto 0, \\
1 \otimes a_i^* \otimes 1 &\mapsto 0,
\end{aligned}$$

and it lifts to

$$\begin{aligned}
\hat{\zeta}_{h-3} : A \otimes V \otimes {}_1A_\phi[h] &\rightarrow A \otimes A, \\
1 \otimes b \otimes 1 &\mapsto -e_n \otimes e_n, \\
1 \otimes a_i \otimes 1 &\mapsto 0, \\
1 \otimes a_i^* \otimes 1 &\mapsto 0.
\end{aligned}$$

Then

$$\begin{aligned}
(\hat{\zeta}_{h-3} \circ d_5)(1 \otimes 1) &= \hat{\zeta}_{h-3}(\sum_{i=1} \epsilon_{a_i} a_i \otimes a_i^* \otimes 1 + \sum_{i=1} \epsilon_{a_i} \otimes a_i \otimes a_i^* \\
&\quad -b \otimes b \otimes 1 - 1 \otimes b \otimes b) \\
&= b \otimes 1 - 1 \otimes b = d_1(1 \otimes b \otimes 1),
\end{aligned}$$

so we have

$$\begin{aligned}\Omega\zeta_{h-3} : \Omega^5(A) &\rightarrow \Omega(A),\\ 1 \otimes 1 &\mapsto 1 \otimes b \otimes 1,\end{aligned}$$

and this gives us

$$(\theta_0 \circ \zeta_{h-3})(1 \otimes 1) = b,$$

so the cup product is

$$\theta_0\zeta_{h-3} = [b] = \psi_{h-3}. \tag{5.7.4.3}$$

□

5.7.5 $HH^2(A) \cup HH^3(A)$

We compute the cup product in the following proposition.

Proposition 5.7.5.1. *For the basis elements $f_i \in HH^2(A)$, $h_j \in HH^3(A)$, the cup product is*

$$f_i h_j = \delta_{ij}\psi_0. \tag{5.7.5.2}$$

Proof. Recall the maps

$$\begin{aligned}h_j : A \otimes {}_1A_\phi &\rightarrow A,\\ 1 \otimes 1 &\mapsto \omega_j\end{aligned}$$

and lift them to

$$\begin{aligned}\hat{h}_j : A \otimes {}_1A_\phi &\rightarrow A \otimes A,\\ 1 \otimes 1 &\mapsto 1 \otimes \omega_j.\end{aligned}$$

Then $\forall\, a \in \bar{Q}$ we have

$$\hat{h}_j(d_4(1 \otimes a \otimes 1)) = \hat{h}_j(a \otimes 1 - 1 \otimes a) = a \otimes \omega_j = d_1(1 \otimes a \otimes \omega_j),$$

so

$$\begin{aligned} \Omega h_j : \Omega^4(A) &\to \Omega(A), \\ 1 \otimes a \otimes 1 &\mapsto 1 \otimes a \otimes \omega_j. \end{aligned}$$

Then we have

$$\begin{aligned} \Omega h_j(d_5(1 \otimes 1)) &= \Omega h_j(\sum_{i=1}^{n} \epsilon_{a_i} a_i \otimes a_i^* \otimes 1 + \sum_{i=1}^{n} \epsilon_{a_i} 1 \otimes a_i \otimes a_i^* \\ &\quad - b \otimes b \otimes 1 - 1 \otimes b \otimes b) \\ &= (\sum_{i=1}^{n} \epsilon_{a_i} a_i \otimes a_i^* - b \otimes b \otimes \omega_j = d_2(1 \otimes \omega_j), \end{aligned}$$

so

$$\begin{aligned} \Omega^2 h_j : \Omega^5(A) &\to \Omega^2(A), \\ 1 \otimes 1 &\mapsto 1 \otimes \omega_j. \end{aligned}$$

This gives us

$$f_i(\Omega^2 h_j)(1 \otimes 1) = f_i(1 \otimes \omega_j) = \delta_{ij}\omega_j,$$

i.e. the cup product

$$f_i h_j = \delta_{ij}[\omega_j] = \delta_{ij}\psi_0.$$

□

5.7.6 $HH^2(A) \cup HH^2(A)$

Since $\deg HH^2(A) = -2$, their product has degree -4 (i.e. lies in $\mathrm{span}(\zeta_0)$), so it can be written as

$$
\begin{aligned}
HH^2(A) \times HH^2(A) &\rightarrow HH^4(A), \\
(x, y) &\mapsto \langle -, - \rangle \zeta_0,
\end{aligned}
$$

where $\langle -, - \rangle : HH^2(A) \times HH^2(A) \rightarrow \mathbb{C}$ is a bilinear form.

Proposition 5.7.6.1. *The cup product $HH^2(A) \times HH^2(A) \rightarrow HH^4(A)$ is given by $\langle -, - \rangle = \alpha$, where α (from Proposition 5.7.3.1) is regarded as a symmetric bilinear form.*

Proof. We use (5.7.4.2) to get

$$
\theta_0(f_i f_j) = \theta_0(\langle f_i, f_j \rangle \zeta_0) = \langle f_i, f_j \rangle \psi_0. \tag{5.7.6.2}
$$

On the other hand, by Proposition 5.7.3.1 and Proposition 5.7.5.1,

$$
(\theta_0 f_i) f_j = \alpha(f_i) f_j = \sum_{l=1}^{n} \left(H_A^{\phi}\right)_{li} h_l f_j = \left(H_A^{\phi}\right)_{ji} \psi_0 = \left(H_A^{\phi}\right)_{ij} \psi_0. \tag{5.7.6.3}
$$

By associativity of the cup product, we can equate (5.7.6.2) and (5.7.6.3) to get

$$
\langle f_i, f_j \rangle = \left(H_A^{\phi}\right)_{ij}. \tag{5.7.6.4}
$$

□

5.7.7 $HH^2(A) \cup HH^4(A)$

By degree argument, $f_i \zeta_j = 0$ for $j < h - 3$ and $f_i \zeta_{h-3} = \lambda_i \varphi_0(z_{h-3})$ for some $\lambda_i \in \mathbb{C}$.

Proposition 5.7.7.1. *We have*

$$
f_i \zeta_{h-3} = i \cdot z_{h-3}. \tag{5.7.7.2}
$$

Proof. Let $x \in HH^2(A)$. x is represented by a map f_x,

$$
\begin{aligned}
f_x : A \otimes A[2] &\rightarrow A, \\
1 \otimes 1 &\mapsto x,
\end{aligned}
$$

and we lift it to

$$
\begin{aligned}
f_x : A \otimes A[2] &\rightarrow A, \\
1 \otimes 1 &\mapsto 1 \otimes x.
\end{aligned}
$$

We know that for $h_j \in HH^3(A)$ and $x = \sum_{i=1}^{n} r_i f_i$ the cup product is $xh_j = r_j f_j$. This determines the lift

$$
\begin{aligned}
\Omega^3 f_x : \Omega^5(A) &\rightarrow \Omega^3(A), \\
1 \otimes 1 &\mapsto x.
\end{aligned}
$$

Then

$$
\begin{aligned}
\Omega^4 f_x d_6(1 \otimes 1) &= \Omega^4 f_x(\sum_{x_j \in \mathcal{B}} \phi(x_j) \otimes x_j^*) = \sum_{x_j \in \mathcal{B}} \phi(x_j) \otimes x\phi(x_j^*) \\
&= -\sum_{x_j \in \mathcal{B}} x_j \otimes xx_j^* = d_4(-\sum_{x_j \in \mathcal{B}} v_{x_j} x\phi(x_j^*)).
\end{aligned}
$$

For each term $v_{x_j} x\phi(x_j^*)$,

$$
\zeta_{h-3}(v_{x_j} x\phi(x_j^*)) = \begin{cases} 0 & \text{if } x_j \text{ contains even number of } b's \\ -\frac{x_j}{b} xx_j^* & \text{if } x_j \text{ contains odd number of } b's, \end{cases}
$$

where for a monomial x_j, the expression "$\frac{x_j}{b}$" means removing one letter b (and it doesn't matter which one you remove). Denote $\mathcal{B}^{odd}$ (resp. $\mathcal{B}^{even}$ a basis of $e_k A e_l$

which have odd (resp. even) number of b's in their monomial expression. Then

$$\zeta_{h-3} \circ \Omega^4 f_x(1 \otimes 1) = \sum_{x_j \in \mathcal{B}^{odd}} \frac{x_j x x_j^*}{b}.$$

The automorphism γ which reverses all arrows of a path is the identity on A^{top}. Let (x_i) be a basis of A, (x_i^*) its dual basis. Then $(\gamma(x_i))$ is a basis and $(\gamma(x_i^*))$ its dual basis. This shows that

$$\sum_{x_j \in \mathcal{B}^{odd}} x_j x x_j^* = \sum_{x_j \in \mathcal{B}^{odd}} \frac{x_j^* x x_j}{b} = \sum_{x_j \in \mathcal{B}^{even}} \frac{x_j x x_j^*}{b},$$

so

$$\sum_{x_j \in \mathcal{B}^{odd}} x_j x x_j^* = \frac{1}{2} \sum_{x_j \in \mathcal{B}} \frac{x_j x x_j^*}{b}.$$

The $(h-3)$-degree part of A lies in $e_n A e_n$ and is spanned by $z_{h-3} b^{h-3}$. This means that $\frac{\omega_n}{b} = z_{h-3}$ and $\frac{\omega_i}{b} = 0$ for $i < n$. We get

$$\zeta_{h-3} f_l = \frac{1}{2} (H_A(1))_{nl} \, \varphi_0(z_{h-3}) = l \cdot \varphi_0(z_{h-3}). \tag{5.7.7.3}$$

□

5.7.8 $HH^2(A) \cup HH^5(A)$

By degree argument, $f_i \psi_j = 0$ for $j \neq h-3$.

Proposition 5.7.8.1. *We have*

$$f_i \zeta_{h-3} = i \cdot \varphi_0(\theta_{h-3}). \tag{5.7.8.2}$$

Proof. Since $\psi_{h-3} = \theta_0 \zeta_{h-3}$, we have

$$f_i \psi_{h-3} = (f_i \zeta_{h-3})\theta_0 = i \cdot \varphi_0(z_{h-3})\theta_0 = i \cdot \varphi_0(\theta_{h-3}).$$

□

5.7.9 $HH^3(A) \cup HH^4(A)$

$h_i\zeta_j = 0$ for $j < h-3$ and $h_i\zeta_{h-3} = \lambda_i\varphi_0(\theta_{h-3})$ for some $\lambda_i \in \mathbb{C}$.

Proposition 5.7.9.1. *We have*

$$h_i\zeta_{h-3} = \delta_{in}\varphi_0(\theta_{h-3}). \tag{5.7.9.2}$$

Proof. Let λ_i be from above. From (5.7.7.2), we get

$$\theta_0 f_i \zeta_{h-3} = i \cdot \varphi_0(\theta_{h-3}),$$

and we use (5.7.3.1) to see that

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \vdots \\ \vdots \\ \lambda_n \end{pmatrix} = \frac{1}{2n+1} \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 3 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ \vdots \\ \vdots \\ \vdots \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{pmatrix}.$$

□

5.7.10 $HH^4(A) \cup HH^4(A)$

By degree argument, $\zeta_i\zeta_j = 0$ if $i < h-3$ or $j < h-3$ and $\zeta_{h-3}^2 = \sum_{k=1}^{n} \lambda_k\varphi_0(f_k)$.

Proposition 5.7.10.1. *We have*

$$\zeta_{h-3}^2 = \varphi_0(f_n). \tag{5.7.10.2}$$

Proof. Let λ_k be from above. Then we have, using (5.7.5.1),

$$h_l\zeta_{h-3}^2 = \lambda_l\psi_0.$$

Using (5.7.9.2), the LHS becomes

$$\delta_{ln}\theta_{h-3}\zeta_{h-3} = \delta_{ln}\psi_0,$$

so

$$\lambda_l = \delta_{ln}.$$

□

5.7.11 $HH^4(A) \cup HH^5(A)$

By degree argument, $\zeta_i\psi_j = 0$ if $i < h-3$ or $j < h-3$.

Proposition 5.7.11.1. *We have*

$$\zeta_{h-3}\psi_{h-3} = \sum_{i=1}^{n} i\varphi_0(h_i). \tag{5.7.11.2}$$

Proof. We use (5.7.3.1), (5.7.4) and (5.7.10.2) to obtain

$$\zeta_{h-3}\psi_{h-3} = \zeta_{h-3}^2\theta_0 = f_n\theta_0 = \sum_{i=1}^{n} i\varphi_0(h_i).$$

The last equality follows from

$$(2n+1)\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 3 \end{bmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ \vdots \\ \vdots \\ n \end{pmatrix}.$$

□

5.8 Batalin-Vilkovisky structure on Hochschild cohomology

From general theory, we have an isomorphism $\mathbb{D} : HH_\bullet(A) \to HH^{6m+5-\bullet}(A)$ $\forall m \geq 0$. It translates the Connes differential $B : HH_\bullet(A) \to HH_{\bullet+1}(A)$ on Hochschild homology into a differential $\Delta : HH^\bullet(A) \to HH^{\bullet-1}(A)$ on Hochschild cohomology, i.e. we have the commutative diagram

$$\begin{array}{ccc} HH_\bullet(A) & \xrightarrow{B} & HH_{\bullet+1}(A) \\ \mathbb{D}\downarrow\wr & & \wr\downarrow\mathbb{D} \\ HH^{6m+5-\bullet}(A)[(2m+1)h+2] & \xrightarrow{\Delta} & HH^{6m+4-\bullet}(A)[(2m+1)h+2] \end{array}$$

Theorem 5.8.0.3. (BV structure on Hochschild cohomology) *Δ makes $HH^\bullet(A)$ a Batalin-Vilkovisky algebra, defined in Theorem 3.3.0.13*

Proof. We refer to [17, Theorem 2.4.65]. □

Remark 5.8.0.4. Note that Δ in equation (3.3.0.14) depends on which $m \in \mathbb{N}$ we choose to identify $\mathbb{D} : HH_\bullet(A) \xrightarrow{\sim} HH^{6m+5-\bullet}(A)[(2m+1)h+2]$, where the Gerstenhaber bracket does not.

5.8.1 Computation of the calculus structure of the preprojective algebra

Since the calculus structure is defined on Hochschild chains and cochains, we have to work with the on the resolution for computations. It turns out that we only have to compute $\mathcal{L}_{\theta_0}$ directly, the rest can be deduced from formulas given by the calculus and the BV structure.

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{d_3} & A\otimes A[2] & \xrightarrow{d_2} & A\otimes V\otimes A & \xrightarrow{d_1} & A\otimes A & \xrightarrow{d_0} & A & \longrightarrow & 0 \\ & & \mu_2\downarrow & & \mu_1\downarrow & & \parallel & & \parallel & & \\ \cdots & \xrightarrow{b_3} & A^{\otimes 4} & \xrightarrow{b_2} & A^{\otimes 3} & \xrightarrow{b_1} & A^{\otimes 2} & \xrightarrow{b_0} & A & \longrightarrow & 0 \end{array}$$

These maps ψ_i give us a chain map between the Schofield and the bar resolution:

$$\begin{aligned}
\mu_1(1\otimes y\otimes 1) &= 1\otimes y\otimes 1,\\
\mu_2(1\otimes 1) &= \sum_{a\in\bar{Q}}\epsilon_a 1\otimes a\otimes a^*\otimes 1 - 1\otimes b\otimes b\otimes 1,\\
\mu_3(1\otimes 1) &= \sum_{a\in\bar{Q}}\sum_{x_i\in\mathcal{B}}\epsilon_a 1\otimes\phi(x_i)\otimes a\otimes a^*\otimes x_i^*\\
&\quad -\sum_{x_i\in\mathcal{B}} 1\otimes\phi(x_i)\otimes b\otimes b\otimes x_i^*,
\end{aligned}$$

and

$$\mu_{3+i} = \mu_i\left(\sum_{a\in\bar{Q}}\sum_{x_i\in\mathcal{B}}\epsilon_a\phi(x_i)\otimes a\otimes a^*\otimes x_i^* - \sum_{x_i\in\mathcal{B}}\phi(x_i)\otimes b\otimes b\otimes x_i^*\right).$$

Now, we apply the functor $-\otimes_{A^e} A$ on the commutative diagram:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_3'} & A^R[2] & \xrightarrow{d_2'} & (V\otimes A)^R & \xrightarrow{d_1} & A^R & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle \mu_2'} & & \Big\downarrow{\scriptstyle \mu_1'} & & \Big\| & & \\
\cdots & \xrightarrow{b_3'} & (A^{\otimes 3})^R & \xrightarrow{b_2} & (A^{\otimes 2})^R & \xrightarrow{b_1} & (A^{\otimes 1})^R & \longrightarrow & 0
\end{array}$$

where

$$\begin{aligned}
\mu_1'(x\otimes y) &= x\otimes y,\\
\mu_2'(x) &= \sum_{a\in\bar{Q}}\epsilon_a a\otimes a^*\otimes x - b\otimes b\otimes x,\\
\mu_3'(x) &= \sum_{a\in\bar{Q}}\sum_{x_i\in\mathcal{B}}\epsilon_a\phi(x_i)\otimes a\otimes a^*\otimes x_i^*x - \phi(x_i)\otimes b\otimes b\otimes x_i^*x,
\end{aligned}$$

and

$$\mu_{3+i}' = \mu_i'\left(\sum_{a\in\bar{Q}}\sum_{x_i\in\mathcal{B}}\phi(x_i)\otimes a\otimes a^*\otimes x_i^* - \sum_{x_i\in\mathcal{B}}\phi(x_i)\otimes b\otimes b\otimes x_i^*\right).$$

Now, we compute $\mathcal{L}_{\theta_0}$:

Lemma 5.8.1.1. *For each $x \in HH_i(A)$,*

$$\mathcal{L}_{\theta_0}(x) = \deg(x)x \tag{5.8.1.2}$$

Proof. Via μ', we already identified $x \in HH_i(A)$ with cycles in the Hochschild chain, but we still have to identify θ_0 with an element in $\mathrm{Hom}_{A^e}(A^{\otimes 3}, A)$:

given any monomial $b = b_1 \dots b_l$, $b_i \in V$, the map

$$\tau(1 \otimes b \otimes 1) = \sum_{i=1}^{l} b_1 \dots b_{i-1} \otimes b_i \otimes b_{i+1} \dots b_l$$

makes the diagram

$$\begin{array}{ccccccc} A \otimes V \otimes A & \xrightarrow{d_1} & A \otimes A & \xrightarrow{d_0} & A & \longrightarrow & 0 \\ {\scriptstyle\tau}\uparrow & & \| & & \| & & \\ A^{\otimes 3} & \xrightarrow{b_1} & A^{\otimes 2} & \xrightarrow{b_0} & A & \longrightarrow & 0 \end{array}$$

commute.

Applying $\mathrm{Hom}_{A^e}(\_ \otimes A)$, we get a map

$$\tau^* : \mathrm{Hom}_k(V) \to \mathrm{Hom}_k(A),$$

such that

$$(\theta_0 \circ \tau^*)(b_1 \dots b_l) = \sum_{i=1}^{l} b_1 \dots b_{i-1}\theta_0(b_i)b_{i+1} \dots b_l = \deg(b) \cdot b,$$

Recall from [5, (3.5), page 46] that the Lie derivative of $\theta_0 \circ \tau^*$ on Hochschild chains is defined by

$$\begin{aligned} \mathcal{L}_{\theta_0 \circ \tau^*}(a_1 \otimes \cdots \otimes a_k) &= \sum_{i=1}^{k} a_1 \otimes \cdots \otimes (\theta_0 \circ \tau^*)(a_i) \otimes \cdots \otimes a_k \\ &= \sum_{i=1}^{k} (\deg(a_1) + \cdots + \deg(a_k)) a_1 \otimes \cdots \otimes a_k, \end{aligned}$$

and it can easily be checked that for each $x \in HH_i(A)$, $\mathcal{L}_{\theta_0 \circ T^*}$ acts on $\mu_i'(x)$, $x \in HH^i(A)$, by multiplication with $\deg(x)$.

□

Notation 5.8.1.3. For $c_k \in HH^i(A)$, $0 \leq i \leq 5$, we write $c_k^{(s)}$ for the corresponding cocycle in HH^{i+6s}. We write $c_{k,t}$ for the corresponding cycle in HH_{j+6t}, $0 \leq j \leq 5$ (under the isomorphism $\mathbb{D}$.

The contraction map

From (3.3.0.15) we know that the contraction map on Hochschild homology is given by the cup product on Hochschild cohomology. Table 5.1 contains these results, rewritten in terms of the contraction maps.

The Connes differential

We start with the computation of the Connes differential and refer the reader to the Subsection 5.2.0.6.

Proposition 5.8.1.4. *The Connes differential B is given by*

$$\begin{aligned}
B_{6s}(\psi_{k,s}) &= ((2s+1)h - 2 - k)\zeta_{k,s}, \\
B_{1+6s} &= 0, \\
B_{2+6s}(h_{k,s}) &= (2s+1)h\alpha^{-1}(h_{k,s}), \\
B_{3+6s} &= 0, \\
B_{4+6s}(\theta_{k,s}) &= ((2s+1)h + 2 + k)z_{k,s}, \\
B_{5+6s} &= 0.
\end{aligned}$$

Proof. We use the Cartan identity (3.1.0.7) with $a \in \theta_0$,

$$\mathcal{L}_{\theta_0} = B\iota_{\theta_0} + \iota_{\theta_0}B, \tag{5.8.1.5}$$

where $\mathcal{L}_{\theta_0}$ acts on $x \in HH_i$ by multiplication by $\deg(x)$ (see Lemma (5.8.1.1)). The above identities for the Connes differential follow since ι_{θ_0} acts on $\theta_{k,s}$, $\psi_{k,s}$ and $h_{k,s}$ by zero, and $z_{k,s}$, $\zeta_{k,s}$ and $\alpha^{-1}(h_{k,s})$ are their unique preimages the contraction with ι_{θ_0}. □

The Gerstenhaber bracket

We compute the brackets using the identification
$HH^i(A) = HH_{6m+5-i}(A)[-2(m+1)h-2]$ for $m >> 1$ and the BV-identity (3.3.0.14). We rewrite the results from Proposition 5.8.1.4:

$$\begin{aligned}
\Delta(\theta_k^{(s)}) &= ((1+2(m-s))h+k+2)z_k^{(s)},\\
\Delta(f_k^{(s)}) &= 0,\\
\Delta(h_k^{(s)}) &= (1+2(m-s))h\alpha^{-1}(h_k^{(s)}),\\
\Delta(\zeta_k^{(s)}) &= 0,\\
\Delta(\psi_k^{(s)}) &= ((1+2(m-s)h-k-2)\zeta_k^{(s)},\\
\Delta(z_k^{(s)}) &= 0.
\end{aligned}$$

The cup products relations involving our basis of $HH^*(\Pi_{T_n})$ are the same ones as the relations in the A_{2n}-case. When comparing the differential Δ with the one in the A_{2n}-case where we identify $HH^i(\Pi_{A_{2n}}) = HH_{6m+2-i}(\Pi_{A_{2n}})[-2mh-2]$ for $m >> 1$, we have to multiply the coefficients by 2 and add h. In the BV-identity (3.3.0.14), we use only cup product and Δ to compute the Gerstenhaber bracket. In these computations, when comparing to the A_{2n}-case, we get the same results with the factor 2. So using the results from Table 3.2, we get Table 5.2.

The Lie derivative $\mathcal{L}$

We use the Cartan identity (3.1.0.7) to compute the Lie derivative.

<u>$HH^{1+6s}(A)$**-Lie derivatives:**</u>

From the Cartan identity, we see that

$$\mathcal{L}_{\theta_k^{(s)}} = B\iota_{\theta_k^{(s)}} + \iota_{\theta_k^{(s)}}B.$$

On $\theta_{l,t}$, $\psi_{l,t}$ and $h_{l,t}$, the Connes differential acts by multiplication with its degree and taking the preimage under ι_{θ_0}, and $\iota_{\theta_k^{(s)}}$ acts on them by zero. B kills $z_{l,t}$, $\zeta_{l,t}$ and $f_{l,t}$. Since B is degree preserving, this means that $\mathcal{L}_{\theta_k^{(s)}}$ acts on $\theta_{l,t}$, $\psi_{l,t}$ and $h_{l,t}$ by multiplication with their degree times $z_k^{(s)}$, and on $z_{l,t}$, $\zeta_{l,t}$ and $f_{l,t}$ by multiplication with $z_k^{(s)}$ and then multiplication with the degree of their product. So we get the following formulas:

$$\begin{aligned}
\mathcal{L}_{\theta_k^{(s)}}(\psi_{l,t}) &= ((2t+1)h-2-l)(z_k\psi_l)_{t-s},\\
\mathcal{L}_{\theta_k^{(s)}}(\zeta_{l,t}) &= ((2(t-s)+1)h-2-l+k)(z_k\zeta_l)_{t-s},\\
\mathcal{L}_{\theta_k^{(s)}}(h_{l,t}) &= \delta_{k0}(2t+1)hh_{l,t-s},\\
\mathcal{L}_{\theta_k^{(s)}}(f_{l,t}) &= \delta_{k0}(2(t-s)+1)hf_{l,t-s},\\
\mathcal{L}_{\theta_k^{(s)}}(\theta_{l,t}) &= ((2t+1)h+2+l)(z_k\theta_l)_{t-s},\\
\mathcal{L}_{\theta_k^{(s)}}(z_{l,t}) &= ((2(t-s)+1)h+2+l+k)(z_kz_l)_{t-s}.
\end{aligned}$$

$\underline{HH^{2+6s}(A)\textbf{-Lie derivatives}}$:

We compute $\mathcal{L}_{f_k^{(s)}}$:

$$\begin{aligned}
\mathcal{L}_{f_k^{(s)}}(\psi_{l,t}) &= B(\iota_{f_k^{(s)}}(\psi_{l,t})) - \iota_{f_k^{(s)}}B(\psi_{l,t}) \\
&= B(\delta_{l,h-3}k\theta_{h-3,t-s-1}) - ((2t+1)h-2-l)\iota_{f_k^{(s)}}(\zeta_{l,t}) \\
&= \delta_{l,h-3}k((2(t-s-1)+2)h-1)z_{h-3,t-s-1} \\
&\quad -\delta_{l,h-3}k((2t+1)h-2-l)z_{h-3,t-s-1} \\
&= -2\delta_{l,h-3}k((1+sh)z_{h-3,t-s-1}, \\
\mathcal{L}_{f_k^{(s)}}(\zeta_{l,t}) &= B(\iota_{f_k^{(s)}}(\zeta_{l,t})) = B(k\delta_{l,h-3}z_{h-3,t-s-1}) = 0, \\
\mathcal{L}_{f_k^{(s)}}(h_{l,t}) &= B(\iota_{f_k^{(s)}}(h_{l,t})) - \iota_{f_k^{(s)}}B(h_{l,t}) \\
&= B(\delta_{k,l}\psi_{0,t-s}) - (2t+1)h\iota_{f_k^{(s)}}\alpha^{-1}(h_{l,t}) \\
&= \delta_{kl}((2(t-s)+1)h-2)\zeta_{0,t-s} - \delta_{kl}(2t+1)h\zeta_{0,t-s} \\
&= -2\delta_{kl}(sh+1)\zeta_{0,t-s}, \\
\mathcal{L}_{f_k^{(s)}}(f_{l,t}) &= B(\underbrace{\iota_{f_k^{(s)}}(f_{l,t})}_{\in HH_{1+6(t-s)}}) = 0, \\
\mathcal{L}_{f_k^{(s)}}(\theta_{l,t}) &= B(\iota_{f_k^{(s)}}(\theta_{l,t})) - \iota_{f_k^{(s)}}(B(\theta_{l,t})) \\
&= B(\delta_{l0}\alpha(f_{k,t-s})) - ((2t+1)h+2+l)\iota_{f_k^{(s)}}z_{l,t} \\
&= \delta_{l0}(2(t-s)+1)hf_{k,t-s} - \delta_{l0}((2t+1)h+2)f_{k,t-s} \\
&= -2\delta_{l0}(1+sh)f_{k,t-s}, \\
\mathcal{L}_{f_k^{(s)}}(z_{l,t}) &= \delta_{l0}B(f_{k,t-s}) = 0,
\end{aligned}$$

$HH^{3+6s}(A)$**-Lie derivatives**:

We compute $\mathcal{L}_{h_k^{(s)}}$:

$$\begin{aligned}
\mathcal{L}_{h_k^{(s)}}(\psi_{l,t}) &= B(\underbrace{\iota_{h_k^{(s)}}(\psi_{l,t})}_{=0}) + \iota_{h_k^{(s)}} B(\psi_{l,t}) \\
&= ((2t+1)h - 2 - l)\iota_{h_k^{(s)}}\zeta_{l,t} = \delta_{k,n}\delta_{l,h-3}(2th+1)\theta_{h-3,t-s-1}, \\
\mathcal{L}_{h_k^{(s)}}(\zeta_{l,t}) &= B(\iota_{h_k^{(s)}}(\zeta_{l,t})) = \delta_{k,n}B(\delta_{l,h-3}\theta_{h-3,t-s-1}) \\
&= \delta_{kn}\delta_{l,h-3}((2(t-s-1)+1)h + 2 + h - 3)z_{h-3,t-s-1} \\
&= \delta_{kn}\delta_{l,h-3}((2(t-s)h - 1)z_{h-3,t-s-1}, \\
\mathcal{L}_{h_k^{(s)}}(h_{l,t}) &= B(\underbrace{\iota_{h_k^{(s)}}(h_{l,t})}_{=0}) + \iota_{h_k^{(s)}} B(h_{l,t}) = (2t+1)h\iota_{h_k^{(s)}}\alpha^{-1}(h_{l,t}) \\
&= (2t+1)h(M_\alpha^{-1})_{lk}\psi_{0,t-s} \\
\mathcal{L}_{h_k^{(s)}}(f_{l,t}) &= B(\iota_{h_k^{(s)}}(f_{l,t})) = B(\delta_{kl}\psi_{0,t-s}) = \delta_{kl}(2(t-s+1)h - 2)\zeta_{0,t-s}, \\
\mathcal{L}_{h_k^{(s)}}(\theta_{l,t}) &= B(\underbrace{\iota_{h_k^{(s)}}(\theta_l^{(t)})}_{=0}) + \iota_{h_k^{(s)}} B(\theta_{l,t}) = ((2t+1)h + 2 + l)\iota_{h_k^{(s)}} z_{l,t} \\
&= \delta_{l0}((2t+1)h + 2)h_{k,t-s}, \\
\mathcal{L}_{h_k^{(s)}}(z_{l,t}) &= B(\delta_{l0}h_{k,t-s}) = \delta_{l0}(2(t-s)+1)h\alpha^{-1}(h_{k,t-s}).
\end{aligned}$$

$\underline{HH^{4+6s}(A)\textbf{-Lie derivatives}}$:

We compute $\mathcal{L}_{\zeta_k^{(s)}}$:

$$\begin{aligned}
\mathcal{L}_{\zeta_k^{(s)}}(\psi_{l,t}) &= B\iota_{\zeta_k^{(s)}}(\psi_{l,t}) - \iota_{\zeta_k^{(s)}}B(\psi_{l,t}) \\
&= \delta_{k,h-3}\delta_{l,h-3}B(\alpha(f_{n,t-s-1})) - \iota_{\zeta_k^{(s)}}((2t+1)h-2-l)\zeta_{l,t} \\
&= \delta_{k,h-3}\delta_{l,h-3}((2(t-s-1)+1)hf_{n,t-s-1} \\
&\quad -((2t+1)h-h+1)f_{n,t-s-1}) \\
&= \delta_{k,h-3}\delta_{l,h-3}((-2s-1)h-1)f_{n,t-s-1}, \\
\mathcal{L}_{\zeta_k^{(s)}}(\zeta_{l,t}) &= B\iota_{\zeta_k^{(s)}}(\zeta_{l,t}) - \iota_{\zeta_k^{(s)}}\underbrace{B(\zeta_{l,t})}_{=0} \\
&= \delta_{k,h-3}\delta_{l,h-3}B(f_{n,t-s-1}) = 0, \\
\mathcal{L}_{\zeta_k^{(s)}}(h_{l,t}) &= B\iota_{\zeta_k^{(s)}}(h_{l,t}) - \iota_{\zeta_k^{(s)}}B(h_{l,t}) \\
&= \delta_{l,n}\delta_{k,h-3}B(\theta_{h-3,t-s-1}) - (2t+1)h\iota_{\zeta_k^{(s)}}\alpha^{-1}(h_{l,t}), \\
&= \delta_{l,n}\delta_{k,h-3}z_{h-3,t-s-1}((2(t-s-1)+1)h+2+h-3-(2t+1)h) \\
&= \delta_{l,n}\delta_{k,h-3}z_{h-3,t-s-1}(-(2s+1)h-1), \\
\mathcal{L}_{\zeta_k^{(s)}}(f_{l,t}) &= B\iota_{\zeta_k^{(s)}}(f_{l,t}) - \iota_{\zeta_k^{(s)}}\underbrace{B(f_{l,t})}_{=0} \\
&= l\delta_{k,h-3}B(z_{h-3,t-s}) = 0, \\
\mathcal{L}_{\zeta_k^{(s)}}(\theta_{l,t}) &= B\iota_{\zeta_k^{(s)}}(\theta_{l,t}) - \iota_{\zeta_k^{(s)}}B(\theta_{l,t}) = B((z_l\psi_k)_{t-s}) - \iota_{\zeta_k^{(s)}}((2t+1)h+2+l)z_{l,t} \\
&= ((2(t-s)+1)h-2-(k-l))(z_l\zeta_k)_{t-s} - ((2t+1)h+2+l)(z_l\zeta_k)_{t-s-1} \\
&= (-2sh-4-k)(z_l\zeta_k)_{t-s}, \\
\mathcal{L}_{\zeta_k^{(s)}}(z_{l,t}) &= B\iota_{\zeta_k^{(s)}}(z_{l,t}) - \iota_{\zeta_k^{(s)}}\underbrace{B(z_{l,t})}_{=0} \\
&= B((z_l\zeta_k)_{t-s}) = 0.
\end{aligned}$$

$\underline{HH^{5+6s}(A)\text{-}\mathbf{Lie\ derivatives}}$:

We compute $\mathcal{L}_{\psi_k^{(s)}}$:

$$\begin{aligned}
\mathcal{L}_{\psi_k^{(s)}}(\psi_{l,t}) &= B\underbrace{\iota_{\psi_k^{(s)}}(\psi_{l,t})}_{=0}+\iota_{\psi_k^{(s)}}B(\psi_{l,t}) \\
&= ((2t+1)h-2-l)\iota_{\psi_k^{(s)}}\zeta_{l,t} \\
&= \delta_{k,h-3}\delta_{l,h-3}(\underbrace{(2t+1)h-2-(h-3)}_{=2th+1}\alpha(f_{n,t-s-1}), \\
\mathcal{L}_{\psi_k^{(s)}}(\zeta_{l,t}) &= B\iota_{\psi_k^{(s)}}(\zeta_{l,t})+\iota_{\psi_k^{(s)}}\underbrace{B(\zeta_{l,t})}_{=0} \\
&= \delta_{k,h-3}\delta_{l,h-3}B(\alpha(f_{n,t-s-1})) \\
&= \delta_{k,h-3}\delta_{l,h-3}(2(t-s-1)+1)hf_{n,t-s-1} \\
&= \delta_{k,h-3}\delta_{l,h-3}(2(t-s)-1)hf_{n,t-s-1}, \\
\mathcal{L}_{\psi_k^{(s)}}(h_{l,t}) &= B\underbrace{\iota_{\psi_k^{(s)}}(h_{l,t})}_{=0}+\iota_{\psi_k^{(s)}}B(h_{l,t}) \\
&= \iota_{\psi_k^{(s)}}\alpha^{-1}(h_{l,t})(2t+1)h \\
&= \delta_{k,h-3}\delta_{l,n}(2t+1)h\theta_{h-3,t-s-1}, \\
\mathcal{L}_{\psi_k^{(s)}}(f_{l,t}) &= B\iota_{\psi_k^{(s)}}(f_{l,t})+\iota_{\psi_k^{(s)}}\underbrace{B(f_{l,t})}_{=0} \\
&= l\delta_{k,h-3}B(\theta_{h-3,t-s-1}) \\
&= l((2(t-s-1)+1)h+2+(h-3))\delta_{k,h-3}z_{h-3,t-s} \\
&= l(2(t-s)h-1)\delta_{k,h-3}z_{h-3,t-s-1}, \\
\mathcal{L}_{\psi_k^{(s)}}(\theta_{l,t}) &= B\underbrace{\iota_{\psi_k^{(s)}}(\theta_{l,t})}_{=0}+\iota_{\psi_k^{(s)}}B(\theta_{l,t}) \\
&= \iota_{\psi_k^{(s)}}z_{l,t}((2t+1)h+2+l)=(z_l\psi_k)_{t-s}((2t+1)h+2+l), \\
\mathcal{L}_{\psi_k^{(s)}}(z_{l,t}) &= B\iota_{\psi_k^{(s)}}(z_{l,t})+\iota_{\psi_k^{(s)}}\underbrace{B(z_{l,t})}_{=0} \\
&= B((z_l\psi_k)_{t-s})=((2(t-s)+1)h-2-(k-l))(z_l\zeta_k)_{t-s}.
\end{aligned}$$

$HH^{6+6s}(A)$**-Lie derivatives**:

B acts on $\theta_{l,t}, \psi_{l,t}$ and $h_{l,t}$ by multiplication with its degree and taking the preimage under ι_{θ_0}. On $z_{l,t}, \zeta_{l,t}$ and $f_{l,t}$, B acts by zero. Since the spaces U, U^*, K and K^* are z_k-invariant and $z_k^{(s)}$ has degree $k-2sh$, $\mathcal{L}_{z_k^{(s)}}$ acts on $\theta_{l,t}, \psi_{l,t}$ and $h_{l,t}$ by multiplication with $k-2sh$ and taking the preimage under ι_{θ_0} and multiplication with $z_k^{(s)}$, and on $z_{l,t}, \zeta_{l,t}$ and $f_{l,t}$ it acts by zero. We have the following formulas:

$$
\begin{aligned}
\mathcal{L}_{z_k^{(s)}}(\psi_{l,t}) &= (k-2sh)(z_k\zeta_l)_{t-s},\\
\mathcal{L}_{z_k^{(s)}}(\zeta_{l,t}) &= 0,\\
\mathcal{L}_{z_k^{(s)}}(h_{l,t}) &= (k-2sh)\alpha^{-1}(h_{l,t-s}),\\
\mathcal{L}_{z_k^{(s)}}(f_{l,t}) &= 0,\\
\mathcal{L}_{z_k^{(s)}}(\theta_{l,t}) &= (k-2sh)(z_kz_l)_{t-s},\\
\mathcal{L}_{z_k^{(s)}}(z_{l,t}) &= 0.
\end{aligned}
$$

This concludes the computation of the calculus structure for quivers of type T.

Appendix A

Correction to [12]

We want to make a correction to the $HH^2(A) \cup HH^2(A)$-computation in [12]: the calculation of $HH^2(A) \cup HH^2(A)$ in (5.7.7.2) shows that the bilinear form on K is given by the matrix M_α, defined in Subsection 5.7.3. This is a general computation which also applies to quivers of type A. But the results in [12] suggest that the bilinear form on K is given by a matrix different from M_α which is incorrect.

I verified that the matrix M_α from $HH^1(A) \cup HH^2(A)$ in [12] correct, therefore the result of $HH^2(A) \cup HH^2(A)$ is wrong: similarly to the computation in Subsection 5.7.3 of this paper, you can calculate the matrix M_α by using the derivative of $H_A(t)$. Then you get (by labeling the A-quiver as in [12])

$$M_\alpha = h \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}^{-1}$$

for type A_{2n+1} (and also D_n, E_n) and

$$M_\alpha = h \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 3 \end{bmatrix}^{-1}$$

for type A_{2n}.

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